5. Exterior Algebra and Differential Forms

Differential forms play an important part in the theory of Hamiltonian systems, but this theory is not universally known by scientists and mathematicians. It gives the natural higher-dimensional generalization of the results of classical vector calculus. We give a brief introduction with some, but not all, proofs and refer the reader to Flanders (1963) for another informal introduction but a more complete discussion with many applications, or to Spivak (1965) or Abraham and Marsden (1978) for a more complete mathematical discussion. The reader conversant with the theory of differential forms can skip this chapter, and the reader not conversant with the theory should realize that what is presented here is not meant to be a complete development but simply an introduction to a few results that are used sparingly later.

In this chapter we introduce and use the notation of classical differential geometry by using superscripts and subscripts to differentiate between a vector space and its dual. This convention helps sort out the multitude of different types of vectors encountered.

5.1 Exterior Algebra

Let $\mathbb{V}$ be a vector space of dimension m over the real numbers $\mathbb{R}$. The best examples to keep in mind are the space of directed line segments in Euclidean 3-space, $\mathbb{E}^3$, or the space of all forces that can act at a point. Let $\mathbb{V}^k$ denote k copies of $\mathbb{V}$; i.e., $\mathbb{V}^k = \mathbb{V} \times \cdots \times \mathbb{V}$ (k times). A function $\phi : \mathbb{V}^k \longrightarrow \mathbb{R}$ is called k-multilinear if it is linear in each argument; so,

$$\begin{aligned}&\phi(a_1, \ldots, a_{r-1}, \alpha u + \beta v, a_{r+1}, \ldots, a_k)\\ &\qquad = \alpha\phi(a_1, \ldots, a_{r-1}, u, a_{r+1}, \ldots, a_k) + \beta\phi(a_1, \ldots, a_{r-1}, v, a_{r+1}, \ldots, a_k)\end{aligned}$$

for all $a_1, \ldots, a_k, u, v \in \mathbb{V}$, all $\alpha, \beta \in \mathbb{R}$, and all arguments, $r = 1, \ldots, k$. A 1-multilinear map is a linear functional that we sometimes call a covector or 1-form. In $\mathbb{R}^m$ the scalar product $(a, b) = a^T b$ is 2-multilinear, in $\mathbb{R}^{2n}$ the symplectic product $\{a, b\} = a^T J b$ is 2-multilinear, and the determinant of an $m \times m$ matrix is m-multilinear in its m rows (or columns). A k-multilinear function ϕ is skew-symmetric or alternating if interchanging any two arguments changes its sign. For a skew-symmetric k-multilinear ϕ,

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$$\phi(a_1, \dots, a_r, \dots, a_s, \dots, a_k) = -\phi(a_1, \dots, a_s, \dots, a_r, \dots, a_k)$$

for all $a_1, \dots, a_k \in \mathbb{V}$ and all $r, s = 1, \dots, k, r \neq s$. Thus ϕ is zero if two of its arguments are the same. We call an alternating k-multilinear function a k-linear form or k-form for short. The symplectic product $\{a, b\} = a^T Jb$ and the determinant of an $m \times m$ matrix are alternating. Let $\mathbb{A}^0 = \mathbb{R}$ and $\mathbb{A}^k = \mathbb{A}^k(\mathbb{V})$ be the space of all k-forms for $k \geq 1$. It is easy to verify that $\mathbb{A}^k$ is a vector space when using the usual definition of addition of functions and multiplication of functions by a scalar.

In $\mathbb{E}^3$, as we have seen, a linear functional (a 1-form or an alternating 1-multilinear function) acting on a vector v can be thought of as the scalar project of v in a particular direction. A physical example is work. The work done by a uniform force is a linear functional on the displacement vector of a particle.

Given two vectors in $\mathbb{E}^3$, they determine a plane through the origin and a parallelogram in that plane. The oriented area of this parallelogram is a 2-form. Two vectors in $\mathbb{E}^3$ determine (i) a plane, (ii) an orientation in the plane, and (iii) a magnitude, the area of the parallelogram. Physical quantities that also determine a plane, an orientation, and a magnitude are torque, angular momentum, and magnetic field.

Three vectors in $\mathbb{E}^3$ determine a parallelepiped, and its oriented volume is a 3-form. The flux of a uniform vector field, v, crossing a parallelogram determined by two vectors a and b is a 3-form.

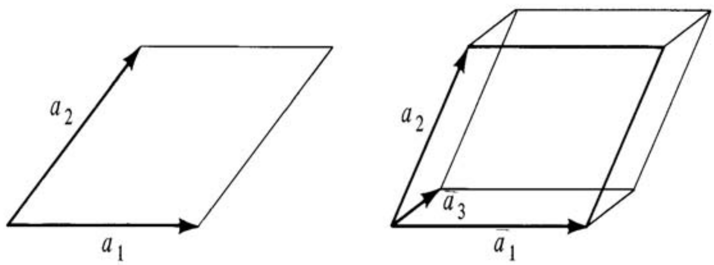

Figure 5.1. Multilinear functions.

If ψ is a 2-multilinear function, then ϕ defined by $\phi(a, b) = \{\psi(a, b) - \psi(b, a)\}/2$ is alternating and is sometimes called the alternating part of ψ. If ψ is already alternating, then $\phi = \psi$. If α and β are 1-forms, then $\phi(a, b) = \alpha(a)\beta(b) - \alpha(b)\beta(a)$ is a 2-form. This construction can be generalized. Let P_k be the set of all permutations of the k numbers $1, 2, \dots, k$ and sign:$P_k \longrightarrow \{+1, -1\}$ the function that assigns $+1$ to an even permutation and -1 to an odd permutation. So if ϕ is alternating, $\phi(a_{\sigma(1)}, \ldots, a_{\sigma(k)}) = \operatorname{sign}(\sigma)\phi(a_1, \ldots, a_k)$. If ψ is a k-multilinear function, then ϕ defined by

$$\phi(a_1, \ldots, a_k) = \frac{1}{k!} \sum_{\sigma \in P} \operatorname{sign}(\sigma)\psi(a_{\sigma(1)}, \ldots, a_{\sigma(k)})$$

is alternating. We write $\phi = \operatorname{alt}(\psi)$. If ψ is already alternating, then $\psi = \operatorname{alt}(\psi)$. If $\alpha \in \mathbb{A}^k$ and $\beta \in \mathbb{A}^r$, then define $\alpha \wedge \beta \in \mathbb{A}^{k+r}$ by

$$\alpha \wedge \beta = \frac{(k+r)!}{k!r!} \operatorname{alt}(\alpha\beta)$$

or

$$\alpha \wedge \beta(a_1, \ldots, a_{k+r})$$
$$= \textstyle\sum_{\sigma \in P} \operatorname{sign}(\sigma)\alpha(a_{\sigma(1)}, \ldots, a_{\sigma(k)})\beta(a_{\sigma(k+1)}, \ldots, a_{\sigma(k+r)}).$$

The operator $\wedge : \mathbb{A}^k \times \mathbb{A}^r \longrightarrow \mathbb{A}^{k+r}$ is called the exterior product or wedge product.

Lemma 5.1.1. *For all k-forms α, r-forms β and δ, and s-forms γ:*

1. $\alpha \wedge (\beta + \delta) = \alpha \wedge \beta + \alpha \wedge \delta$.
2. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.
3. $\alpha \wedge \beta = (-1)^{kr}\beta \wedge \alpha$.

Proof. The first two parts are fairly easy and are left as exercises. Let τ be the permutation $\tau : (1, \ldots, k, k+1, \ldots, k+r) \longrightarrow (k+1, \ldots, k+r, 1, \ldots, k)$; i.e τ interchanges the first k entries and the last r entries. By thinking of τ as being the sequence

$$(1, \ldots, k, k+1, \ldots, k+r) \longrightarrow (k+1, 1, \ldots, k, k+2, \ldots, k+r)$$
$$\longrightarrow (k+1, k+2, 1, \ldots, k+3, \ldots, k+r) \longrightarrow \cdots \longrightarrow (k+1, \ldots, k+r, 1, \ldots, k),$$

it is easy to see that $\operatorname{sign}(\tau) = (-1)^{\mathrm{rk}}$. Now

$$\alpha \wedge \beta(a_1, \ldots, a_{k+r})$$
$$= \textstyle\sum_{\sigma \in P} \operatorname{sign}(\sigma)\alpha(a_{\sigma(1)}, \ldots, a_{\sigma(k)})\beta(a_{\sigma(k+1)}, \ldots, a_{\sigma(k+r)})$$
$$= \textstyle\sum_{\sigma \in P} \operatorname{sign}(\sigma \circ \tau)\alpha(a_{\sigma\circ\tau(1)}, \ldots, a_{\sigma\circ\tau(k)})\beta(a_{\sigma\circ\tau(k+1)}, \ldots, a_{\sigma\circ\tau(k+r)})$$
$$= \textstyle\sum_{\sigma \in P} \operatorname{sign}(\sigma)\operatorname{sign}(\tau)\beta(a_{\sigma(1)}, \ldots, a_{\sigma(r)})\alpha(a_{\sigma(r+1)}, \ldots, a_{\sigma(k+r)})$$
$$= (-1)^{\mathrm{rk}}\beta \wedge \alpha.$$

Let $e_1, \ldots, e_m$ be a basis for $\mathbb{V}$ and $f^1, \ldots, f^m$ be the dual basis for the dual space $\mathbb{V}^*$; so, $f^i(e_j) = \delta^i_j$ where

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

This is our first introduction to the subscript-superscript convention of differential geometry and classical tensor analysis.

Lemma 5.1.2. $\dim \mathbb{A}^k = \binom{m}{k}$. *In particular a basis for* $\mathbb{A}^k$ *is*

$$\{f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq m\}.$$

Proof. Let I denote the set $\{(i_1, \ldots, i_k) : i_j \in \mathbb{Z}, 1 \leq i_1 < \cdots < i_k \leq m\}$ and $f^i = f^{i_1} \wedge \cdots \wedge f^{i_k}$ when $i \in I$. From the definition, $f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k}(e_{j_1}, \ldots, e_{j_k})$ equals 1 if $i, j \in I$ and $i = j$ and equals 0 otherwise; in short, $f^i(e_j) = \delta^i_j$.

Let ϕ be a k-form and define

$$\psi = \sum_{i \in I} \phi(e_{i_1}, \ldots, e_{i_k}) f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} = \sum_{i \in I} \phi(e_i) f^i.$$

Let $v_i = \sum a^j_i e_j, i = 1, \ldots, k$, be k arbitrary vectors. By the multilinearity of ϕ and ψ, one sees that $\phi(v_1, \ldots, v_k) = \psi(v_1, \ldots, v_k)$; so, they agree on all vectors and, therefore, are equal. Thus the set $\{f^i : i \in I\}$ spans $\mathbb{A}^k$.

Assume that

$$\sum_{i \in I} a_{i_1 \ldots i_k} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} = 0.$$

For a fixed set of indices $s_1, \ldots, s_k$, let $r_{k+1}, \ldots, r_m$ be a complementary set; i.e., $s_1, \ldots, s_k, r_{k+1}, \ldots, r_m$ is just a permutation of the integers $1, \ldots, m$. Take the wedge product of (5.1) with $f^{r_{k+1}} \wedge \cdots \wedge f^{r_m}$ to get

$$\sum_{i \in I} a_{i_1 \ldots i_k} f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} \wedge f^{r_{k+1}} \wedge \cdots \wedge f^{r_m} = 0. \tag{5.1}$$

The only term in the above sum without a repeated f in the wedge is the one with $i_1 = s_1, \ldots, i_k = s_k$, and so it is the only nonzero term. Because $s_1, \ldots, s_k, r_{k+1}, \ldots, r_m$ is just a permutation of the integers $1, \ldots, m$, $f^{s_1} \wedge f^{s_2} \wedge \cdots \wedge f^{s_k} \wedge f^{r_{k+1}} \wedge \cdots \wedge f^{r_m} = \pm f^1 \wedge \cdots \wedge f^m$. Thus applying the sum in (5.1) to $e_1, \ldots, e_m$ gives $\pm a_{s_1 \ldots s_k} = 0$. Thus the $f^i, i \in I$, are independent.

In particular, the dimension of $\mathbb{V}^m$ is 1, and the space has as a basis the single element $f^1 \wedge \cdots \wedge f^m$.

Lemma 5.1.3. *Let* $g^1, \ldots, g^r \in \mathbb{V}^*$. *Then* $g^1, \cdots, g^r$ *are linearly independent if and only if* $g^1 \wedge \cdots \wedge g^r \neq 0$.

Proof. If the gs are dependent, then one of them is a linear combination of the others, say $g^r = \sum_{s=1}^{r-1} \alpha_s g^s$. Then $g^1 \wedge \cdots \wedge g^r = \sum_{s=1}^{r-1} \alpha_s g^1 \wedge \cdots \wedge g^{r-1} \wedge g^s$. Each term in this last sum is a wedge product with a repeated entry, and so by the alternating property, each term is zero. Therefore $g^1 \wedge \cdots \wedge g^r = 0$.

Conversely, if $g^1, \dots, g^r$ are linearly independent, then extend them to a basis $g^1, \dots, g^r, \dots, g^m$. By Lemma 5.1.2, $g^1 \wedge \cdots \wedge g^r \wedge \cdots \wedge g^m \neq 0$, so $g^1 \wedge \cdots \wedge g^r \neq 0$.

A linear map $L : \mathbb{V} \longrightarrow \mathbb{V}$ induces a linear map $L_k : \mathbb{A}^k \longrightarrow \mathbb{A}^k$ by the formula $L_k\phi(a_1, \dots, a_k) = \phi(La_1, \dots, La_k)$. If M is another linear map of $\mathbb{V}$ onto itself, then $(LM)_k = M_k L_k$, because $(LM)_k\phi(a_1, \dots, a_k) = \phi(LMa_1, \dots, LMa_k) = L_k\phi(Ma_1, \dots, Ma_k) = M_k L_k \phi(a_1, \dots, a_k)$. Recall that $\mathbb{A}^1 = \mathbb{V}^*$ is the dual space, and $L_1 = L^*$ is called the dual map.

If $\mathbb{V} = \mathbb{R}^m$ (column vectors), then we can identify the dual space $\mathbb{V}^* = \mathbb{A}^1$ with $\mathbb{R}^m$ by the convention $f \longleftrightarrow \hat{f}$, where $f \in \mathbb{V}^*$, $\hat{f} \in \mathbb{R}^m$, and $f(x) = \hat{f}^T x$. In this case, L is an $m \times m$ matrix, and Lx is the usual matrix product. $L_1 f$ is defined by $L_1 f(x) = f(Lx) = \hat{f}^T Lx = (L^T \hat{f})^T x$; so, the matrix representation of L_1 is the transpose of L; i.e., $L_1(f) = L^T \hat{f}$. The matrix representation of L_k is discussed in Flanders (1963).

By Lemma 5.1.2, $\dim \mathbb{A}^m = 1$, and so every element in $\mathbb{A}^m$ is a scalar multiple of a single element. L_m is a linear map; so, there is a constant ℓ such that $L_m f = \ell f$ for all $f \in \mathbb{A}^m$. Define the determinant of L to be this constant ℓ, and denote it by $\det(L)$; so, $L_m f = \det(L) f$ for all $f \in \mathbb{A}^m$.

Lemma 5.1.4. *Let L and $M : \mathbb{V} \longrightarrow \mathbb{V}$ be linear. Then*

1. $\det(LM) = \det(L)\det(M)$.
2. $\det(I) = 1$, *where $I : \mathbb{V} \longrightarrow \mathbb{V}$ is the identity map.*
3. *L is invertible if and only if* $\det(L) \neq 0$, *and, if L is invertible,* $\det(L^{-1}) = \det(L)^{-1}$.

Proof. Part (1) follows from $(LM)_m = M_m L_m$ which was established above. (2) follows from the definition. Let L be invertible; so, $LL^{-1} = I$, and by (1) and (2), $\det(L)\det(L^{-1}) = 1$; so, $\det(L) \neq 0$ and $\det(L^{-1}) = 1/\det(L)$. Conversely assume L is not invertible so there is an $e \in \mathbb{V}$ with $e \neq 0$ and $Le = 0$. Extend e to a basis,$e_1 = e, e_2, \dots, e_m$. Then for any m-form ϕ, $L_m\phi(e_1, \dots, e_m) = \phi(Le_1, \dots, Le_m) = \phi(0, \dots, Le_m) = 0$. So $\det(L) = 0$.

Let $\mathbb{V} = \mathbb{R}^m$, $e_1, e_2, \dots, e_m$ be the standard basis of $\mathbb{R}^m$, and let L be the matrix $L = (L_i^j)$; so, $Le_i = \sum_j L_i^j e_j$. Let ϕ be a nonzero element of $\mathbb{A}^m$.

$$
\begin{aligned}
\det(L)\phi(e_1, \dots, e_m) &= L_m\phi(e_1, \dots, e_m) = \phi(Le_1, \dots, Le_m) \\
&= \sum_{j_1} \cdots \sum_{j_m} \phi(L_1^{j_1} e_{j_1}, \dots, L_m^{j_m} e_{j_m}) \\
&= \sum_{j_1} \cdots \sum_{j_m} L_1^{j_1} \cdots L_m^{j_m} \phi(e_{j_1}, \dots, e_{j_m}) \\
&= \sum_{\sigma \in P} \operatorname{sign}(\sigma) L_1^{\sigma(1)} \cdots L_m^{\sigma(m)} \phi(e_1, \dots, e_m).
\end{aligned}
$$

In the second to last sum above the only nonzero terms are the ones with distinct es. Thus the sum over the nonzero terms is the sum over all permutations of the es. From the above,

$$\det(L) = \sum_{\sigma \in P} \operatorname{sign}(\sigma) L_1^{\sigma(1)} \cdots L_m^{\sigma(m)},$$

which is one of the classical formulas for the determinant of a matrix.

5.2 The Symplectic Form

In this section, let $(\mathbb{V}, \omega)$ be a symplectic space of dimension 2n. Recall that in Chapter 3 a symplectic form ω (on a vector space $\mathbb{V}$) was defined to be a nondegenerate, alternating bilinear form on $\mathbb{V}$, and the pair $(\mathbb{V}, \omega)$ was called a symplectic space.

Theorem 5.2.1. *There exists a basis $f^1, \ldots, f^{2n}$ for $\mathbb{V}^*$ such that*

$$\omega = \sum_{i=1}^{n} f^i \wedge f^{n+i}. \tag{5.2}$$

Proof. By Corollary 3.2.1, there is a symplectic basis $e_1, \ldots, e_{2n}$ so that the matrix of the form ω is the standard $J = (J)$ or $J_{ij} = \omega(e_i, e_j)$. Let $f^1, \ldots, f^{2n} \in \mathbb{V}^*$ be the basis dual to the symplectic basis $e_1, \ldots, e_{2n}$. The 2-form given on the right in (5.2) above agrees with ω on the basis $e_1, \ldots, e_{2n}$.

The basis $f^1, \ldots, f^{2n}$ is a symplectic basis for the dual space $\mathbb{V}^*$. By the above, $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ (n times) $= \pm n! f^1 \wedge f^2 \wedge \cdots \wedge f^{2n}$, where the sign is plus if n is even and minus if n is odd. Thus ω^n is a nonzero element of $\mathbb{A}^{2n}$. Because a symplectic linear transformation preserves ω, it preserves ω^n, and therefore, its determinant is $+1$. (This is the second of four proofs of this fact.)

Corollary 5.2.1. *The determinant of a symplectic linear transformation (or matrix) is $+1$.*

Actually, using the above arguments and the full statement of Theorem 3.2.1, we can prove that a 2-form ν on a linear space of dimension $2n$ is nondegenerate if and only if ν^n is nonzero.

5.3 Tangent Vectors and Cotangent Vectors

Let $\mathcal{O}$ be an open set in an m-dimensional vector space $\mathbb{V}$ over $\mathbb{R}$, $e_1, \ldots, e_m$ a basis for $\mathbb{V}$, and $f^1, \ldots, f^m$ the dual basis. Let $x = (x^1, \ldots, x^m)$ be coordinates in $\mathbb{V}$ relative to $e_1, \ldots, e_m$ and also coordinates in V^* relative to the dual basis. Let $\mathbb{I} = (-1, 1) \subset \mathbb{R}^1$, and let t be a coordinate in $\mathbb{R}^1$. Think of $\mathbb{V}$ as $\mathbb{R}^m$. (We use the more general notation because it is helpful to keep a space and its dual distinct.) $\mathbb{R}^m$ and its dual are often identified with each other which can lead to confusion.

Much of analysis reduces to studying maps from an interval in $\mathbb{R}^1$ into $\mathcal{O}$ (curves, solutions of differential equations, etc.) and the study of maps from $\mathcal{O}$ into $\mathbb{R}^1$ (differentials of functions, potentials, etc.). The linear analysis of these two types of maps is, therefore, fundamental. The linearization of a curve at a point gives rise to a tangent vector, and the linearization of a function at a point gives rise to a cotangent vector. These are the concepts of this section.

A tangent vector at $p \in \mathcal{O}$ is to be thought of as the tangent vector to a curve through p. Let $g, g' : \mathbb{I} \longrightarrow \mathcal{O} \subset \mathbb{V}$ be smooth curves with $g(0) = g'(0) = p$. We say g and g' are equivalent at p if $Dg(0) = Dg'(0)$. Because $Dg(0) \in \mathcal{L}(\mathbb{R}, \mathbb{V})$, we can identify $\mathcal{L}(\mathbb{R}, \mathbb{V})$ with $\mathbb{V}$ by letting $Dg(0)(1) = dg(0)/dt \in \mathbb{V}$. Being equivalent at p is an equivalence relation on curves, and an equivalence class (a maximal set of curves equivalent to each other) is defined to be a tangent vector or a vector to $\mathcal{O}$ at p. That is, a tangent vector, $\{g\}$, is the set of all curves equivalent to g at p; i.e., $\{g\} = \{g' : \mathbb{I} \longrightarrow \mathcal{O} : g'(0) = p \text{ and } dg(0)/dt = dg'(0)/dt\}$. In the x coordinates, the derivative is $dg(0)/dt = (dg^1(0)/dt, \dots, dg^m(0)/dt) = (\gamma^1, \dots, \gamma^m)$; so, $(\gamma^1, \dots, \gamma^m)$ are coordinates for the tangent vector $\{g\}$ relative to the x coordinates. The set of all tangent vectors to $\mathcal{O}$ at p is called the tangent space to $\mathcal{O}$ at p and is denoted by $T_p\mathcal{O}$. This space can be made into a vector space by using the coordinate representation given above. The curve $\xi_i : t \longrightarrow p + te_i$ has $d\xi_i(0)/dt = e_i$ which is $(0, \dots, 0, 1, 0, \dots, 0)$ (1 in the ith position) in the x coordinates. The tangent vector consisting of all curves equivalent to ξ_i at p is denoted by $\partial/\partial x^i$. The vectors $\partial/\partial x^1, \dots, \partial/\partial x^m$ form a basis for $T_p\mathcal{O}$. A typical vector $v_p \in T_p\mathcal{O}$ can be written $v_p = \gamma^1 \partial/\partial x_1 + \cdots + \gamma^m \partial/\partial x_m$. In classical tensor notation, one writes $v_p = \gamma^i \partial/\partial x_i$; it was understood that a repeated index, one as a superscript and one as a subscript, was to be summed over from 1 to m. This was called the Einstein convention or summation convention.

A cotangent vector (or covector for short) at p is to be thought of as the differential of a function at p. Let $h, h' : \mathcal{O} \longrightarrow \mathbb{R}^1$ be two smooth functions. We say h and h' are equivalent at p if $Dh(p) = Dh'(p)$. ($Dh(p)$ is the same as the differential $dh(p)$.) This is an equivalence relation. A cotangent vector or a covector to $\mathcal{O}$ at p is by definition an equivalence class of functions. That is, a covector $\{h\}$ is the set of functions equivalent to h at p; i.e., $\{h\} = \{h' : \mathcal{O} \longrightarrow \mathbb{R}^1 : Dh'(p) = Dh(p)\}$. In the x coordinate, $Dh(p) = (\partial h(p)/\partial x^1, \dots, \partial h(p)/\partial x^m) = (\eta_1, \dots, \eta_m)$; so, $(\eta_1, \dots, \eta_m)$ are coordinates for the covector $\{h\}$. The set of all covectors at p is called the cotangent space to $\mathcal{O}$ at p and is denoted by $T_p^*\mathcal{O}$. This space can be made into a vector space by using the coordinate representation given above. The function

$x^i : \mathcal{O} \longrightarrow \mathbb{R}^1$ defines a cotangent vector at p, which is $(0, \dots, 1, \dots 0)$ (1 in the ith position). The covector consisting of all functions equivalent to x^i at p is denoted by dx^i. The covectors $dx^1, \dots, dx^m$ form a basis for $T_p^*\mathcal{O}$. A typical covector $v^p \in T_p^*\mathcal{O}$ can be written $\eta_1 dx^1 + \cdots + \eta_m dx^m$ or $\eta_i dx^i$ using the Einstein convention.

In the above two paragraphs there is clearly a parallel construction being carried out. If fact they are dual constructions. Let g and h be as above; so, $h \circ g : I \subset \mathbb{R}^1 \longrightarrow \mathbb{R}^1$. By the chain rule, $D(h \circ g)(0)(1) = Dh(p) \circ Dg(0)(1)$ which is a real number; so, $Dh(p)$ is a linear functional on tangents to curves. In coordinates, if

$$\{g\} = v_p = \frac{dg^1}{dt}(0)\frac{\partial}{\partial x_1} + \cdots + \frac{dg^m}{dt}(0)\frac{\partial}{\partial x_m} = \gamma^1 \frac{\partial}{\partial x_1} + \cdots + \gamma^m \frac{\partial}{\partial x_m}$$

and

$$\{h\} = v^p = \frac{\partial h}{\partial x_1}(p) dx^1 + \cdots + \frac{\partial h}{\partial x_m}(p) dx^m = \eta_1 dx^1 + \cdots + \eta_m dx^m,$$

then

$$\begin{aligned} v^p(v_p) &= D(h \circ g)(0)(1) \\ &= \frac{dg^1}{dt}(0)\frac{\partial h}{\partial x_1}(p) + \cdots + \frac{dg^m}{dt}(0)\frac{\partial h}{\partial x_m}(p) \\ &= \gamma^1 \eta_1 + \cdots + \gamma^m \eta_m \\ &= \gamma^i \eta_i \qquad (\text{ Einstein convention}). \end{aligned}$$

Thus $T_p\mathcal{O}$ and $T_p^*\mathcal{O}$ are dual spaces.

At several points in the above discussion the coordinates $x^1, \dots, x^m$ were used. The natural question to ask is to what extent do these definitions depend on the choice of coordinates. Let $y^1, \dots, y^m$ be another coordinate system that may not be linearly related to the xs. Assume that we can change coordinates by $y = \phi(x)$ and back by $x = \psi(y)$, where ϕ and ψ are smooth functions with nonvanishing Jacobians, $D\phi$ and $D\psi$. In classical notation, one writes $x^i = x^i(y)$, $y^j = y^j(x)$, and $D\phi = \{\partial y^j / \partial x^i\}, D\psi = \{\partial x^i / \partial y^j\}$.

Let $g : \mathbb{I} \longrightarrow \mathcal{O}$ be a curve. In x coordinates let $g(t) = (a^1(t), \dots, a^m(t))$ and in y coordinates let $g(t) = (b^1(t), \dots, b^m(t))$. The x coordinate for the tangent vector $v_p = \{g\}$ is $\mathbf{a} = (da^1(0)/dt, \dots, da^m(0)/dt) = (\alpha^1, \dots, \alpha^m)$, and the y coordinate for $v_p = \{g\}$ is $\mathbf{b} = (db^1(0)/dt, \dots, db^m(0)/dt) = (\beta^1, \dots, \beta^m)$. Recall that we write vectors in the text as row vectors, but they are to be considered as column vectors. Thus $\mathbf{a}$ and $\mathbf{b}$ are column vectors. By the change of variables, $a(t) = \psi(b(t))$; so, differentiating gives $\mathbf{a} = D\psi(p)\mathbf{b}$. In classical notation $a^i(t) = x^i(b(t))$; so, $da^i/dt = \sum_i (\partial x^i / \partial y^j) db^j/dt$ or

$$\alpha^i = \sum_{j=1}^{m} \frac{\partial x^i}{\partial y^j} \beta^j \quad \left(= \frac{\partial x^i}{\partial y^j} \beta^j \text{ Einstein convention}\right). \tag{5.3}$$

This formula tells how the coordinates of a tangent vector are transformed. In classical tensor jargon, this is the transformation rule for a contravariant vector.

Let $h : \mathcal{O} \longrightarrow \mathbb{R}^1$ be a smooth function. Let h be $a(x)$ in x coordinates and $b(y)$ in y coordinates. The cotangent vector $v^p = \{h\}$ in x coordinates is $\mathbf{a} = (\partial a(p)/\partial x^1, \ldots, \partial a(p)/\partial x^m) = (\alpha_1, \ldots, \alpha_m)$ and in y coordinates it is $\mathbf{b} = (\partial b(p)/\partial y^1, \ldots, \partial b(p)/\partial y^m) = (\beta_1, \ldots, \beta_m)$. By the change of variables $a(x) = b(\phi(x))$; so, differentiating gives $\mathbf{a} = D\phi(p)^T\mathbf{b}$. In classical notation $a(x) = b(y(x))$; so, $\alpha_i = \partial a/\partial x^i = \sum_j (\partial b/\partial y^j)(\partial y^j/\partial x^i) = \sum_j \beta_j(\partial y^j/\partial x^i)$ or

$$\alpha_i = \sum_{j=1}^{m} \frac{\partial y^j}{\partial x^i}\beta_j \quad (= \frac{\partial y^j}{\partial x^i}\beta_j \text{ Einstein convention}). \tag{5.4}$$

This formula tells how the coordinates of a cotangent vector are transformed. In classical tensor jargon this is the transformation rule for a covariant vector.

5.4 Vector Fields and Differential Forms

Continue the notation of the last section. A tangent (cotangent) vector field on $\mathcal{O}$ is a smooth choice of a tangent (cotangent) vector at each point of $\mathcal{O}$. That is, in coordinates, a tangent vector field, V, can be written in the form

$$V = V(x) = \sum_{i=1}^{m} v^i(x)\frac{\partial}{\partial x^i} \quad (= v^i(x)\frac{\partial}{\partial x^i}), \tag{5.5}$$

where the $v^i : \mathcal{O} \longrightarrow \mathbb{R}^1$, $i = 1, \ldots, m$, are smooth functions, and a cotangent vector field U can be written in the form

$$U = U(x) = \sum_{i=1}^{m} u_i(x)dx^i \quad (= u_i(x)dx^i), \tag{5.6}$$

where $u_i : \mathcal{O} \longrightarrow \mathbb{R}^1, i = 1, \ldots, m$, are smooth functions.

A tangent vector field V gives a tangent vector $V(p) \in T_p\mathcal{O}$ which was defined as the tangent vector of some curve. A different curve might be used for each point of $\mathcal{O}$; so, a natural question to ask is whether there exist a curve $g : \mathbb{I} \subset \mathbb{R} \longrightarrow \mathcal{O}$ such that $dg(t)/dt = V(g(t))$. In coordinates this is

$$\frac{dg^i(t)}{dt} = v^i(g(t)).$$

This is the same as asking for a solution of the differential equation $\dot{x} = V(x)$. Thus a tangent vector field is an ordinary differential equation. In classical tensor jargon it is also called a contravariant vector field.

A cotangent vector field U gives a cotangent vector $U(p) \in T_p^*\mathcal{O}$ which was defined as the differential of a function at p. A different function might be used for each point of $\mathcal{O}$; so, a natural question to ask is whether there exists a function $h : \mathcal{O} \longrightarrow \mathbb{R}^1$ such that $dh(x) = U(x)$. The answer to this question is no in general. Certain integrability conditions discussed below must be satisfied before a cotangent vector field is a differential of a function. If this cotangent vector field is a field of work elements; i.e., a field of forces, then if $dh = -U$, the h would be a potential and the force field would be conservative. But, as we show, not all forces are conservative.

Let $p \in \mathcal{O}$, and denote by $\mathbb{A}_p^k\mathcal{O}$ the space of k-forms on the tangent space $T_p\mathcal{O}$. A k-differential form or k-form on $\mathcal{O}$ is a smooth choice of a k-linear form in $\mathbb{A}_p^k\mathcal{O}$ for all $p \in \mathcal{O}$. That is, a k-form, F, can be written

$$\begin{aligned} F &= \sum_{1 \le i_1 < \cdots < i_k \le m} f_{i_1 i_2 \dots i_k}(x_1, x_2, \dots, x_m) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{i \in I} f_i(x) dx^i, \end{aligned} \tag{5.7}$$

where the functions $f_{i_1 \dots i_k} : \mathcal{O} \longrightarrow \mathbb{R}$ are smooth. In the last expression in (5.7), I denotes the set $\{(i_1, \dots, i_k) : i_j \in \mathbb{Z}, 1 \le i_1 < \cdots < i_k \le m\}$, and $dx^i = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Because $\mathbb{A}_p^0\mathcal{O} = \mathbb{R}$, 0-forms are simply smooth functions, and because $\mathbb{A}_p^1\mathcal{O} = T_p^*\mathcal{O}$, 1-forms are covector fields.

In classical analysis, everything was a vector. In $\mathbb{R}^3$, 1-forms are often identified with (or confused with) vector fields. For example, the differential of a function, $df = f_x dx + f_y dy + f_z dz$, is treated as a vector field by writing $\nabla f = \operatorname{grad} f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$. That is why one calls a force a vector and not a covector even when it is the gradient of a potential function.

Also, because the dimension of the space of 2-linear forms in a 3 dimensional space is

$$\binom{3}{2} = 3$$

classically 2-forms in $\mathbb{R}^3$ were identified with (or confused with) vector fields. Usually one identifies $a(\mathbf{j} \wedge \mathbf{k}) + b(\mathbf{k} \wedge \mathbf{i}) + c(\mathbf{i} \wedge \mathbf{j})$ with the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Think about the cross product of vectors. This is why angular momentum and magnetic fields are sometimes misrepresented as vectors.

Given a 0-form F, (i.e., a function) dF is a 1-form. The natural generalization is the exterior derivative operator d which converts a k-form F as given in (5.7) into a $(k+1)$-form dF by the formula

$$\begin{aligned} dF &= \sum_{j=1}^{m} \sum_{1 \le i_1 < \cdots < i_k \le m} \frac{\partial f_{i_1 \cdots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \textstyle\sum_{i \in I} df_i \wedge dx^i. \end{aligned} \tag{5.8}$$

Lemma 5.4.1. *Let F and G be smooth forms defined on an open set $\mathcal{O}$. Then*

1. *$d(F+G) = dF + dG$.*
2. *$d(F \wedge G) = dF \wedge G + (-1)^{\deg(F)} F \wedge dG$.*
3. *$d(dF) = 0$ for all F.*
4. *If F is a function, then dF agrees with the standard definition of the differential of F,*
5. *The operator d is uniquely defined by the properties given above.*

Proof. Part (4) is obvious, and parts (1), (2), and (5) are left as exercises. Only part (3) is proved here. Let i be a multiple index, and so the summations on i range over I. Let $F = \sum_i f_i dx^i$. Then

$$\begin{aligned} d(dF) &= \sum_i \sum_{j=1}^m \sum_{k=1}^m \left(\frac{\partial^2 f_i}{\partial x_j \partial x_k} \right) dx^k \wedge dx^j \wedge dx^i \\ &= \sum_i \sum_{1 \le j < k \le m} \left(\frac{\partial^2 f_i}{\partial x_j \partial x_k} - \frac{\partial^2 f_i}{\partial x_k \partial x_j} \right) dx^k \wedge dx^j \wedge dx^i \\ &= 0. \end{aligned}$$

The last sum is zero by the equality of mixed partial derivatives.

Remark: The first four parts of this lemma can be used as a coordinate-free definition of the operator d. Formula (5.8) shows its existence, and part (v) shows its uniqueness.

Let (x, y, z) be the standard coordinates in $\mathbb{R}^3$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the usual unit vectors. If $F(x, y, z)$ is a function, then

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

is the usual differential. The classical approach is to make the differential a vector field by defining

$$\nabla F = \operatorname{grad} F = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}.$$

Next consider a 1-form $F = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz$; then

$$dF = \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z} \right) dy \wedge dz + \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x} \right) dz \wedge dx + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx \wedge dy.$$

The classical approach is to make this F a vector field $F = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and to define a new vector field by

$$\nabla \times F = \operatorname{curl} F = \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z} \right) \mathbf{i} + \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x} \right) \mathbf{j} + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) \mathbf{k}.$$

Now let F be a 2-form so $F = a(x,y,z)dy \wedge dz + b(x,y,z)dz \wedge dx + c(x,y,z)dx \wedge dy$ and

$$dF = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \right) dx \wedge dy \wedge dz.$$

The classical approach would have considered F as a vector field $F = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and defined a scalar function

$$\nabla \cdot F = \operatorname{div} F = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \right).$$

The statement that $d(dF) = 0$, or $d^2 = 0$, contains the two classical statements $\operatorname{curl}(\operatorname{grad} F) = 0$ and $\operatorname{div}(\operatorname{curl} F) = 0$.

A k-form F is closed if $dF = 0$. A k-form F is exact if there is a $(k-1)$-form G such that $F = dG$. Part (iii) of Lemma 5.4.1 says that an exact form is closed. A partial converse is also true.

Theorem 5.4.1 (Poincaré's lemma). *Let $\mathcal{O}$ be a ball in $\mathbb{R}^m$ and F a k -form such that $dF = 0$. Then there is a $(k-1)$-form g on $\mathcal{O}$ such that $F = dg$.*

Remark: This is a partial converse to $d(dg) = 0$. Note that the domain of definition, $\mathcal{O}$, of the form F is required to be a ball. The theorem says that in a ball, a closed form is exact. The 1-form, $F = (xdy - ydx)/(x^2 + y^2)$, satisfies $dF = 0$, but there does not exist a function, g, defined on all of $\mathbb{R}^2 \backslash (0,0)$ such that $dg = F$. The form F is the differential of the polar angle $\theta = \arctan(y/x)$ that is not a single-valued function defined on all of $\mathbb{R}^2 \backslash (0,0)$. However, it can be made single valued in a neighborhood of any point in $\mathbb{R}^2 \backslash (0,0)$, e.g., for any point not on the negative x-axis, one can take $-\pi < \theta < \pi$, and for points on the negative x-axis, one can take $0 < \theta < 2\pi$. Because F locally defines a function we have $dF = 0$.

Poincaré's lemma contains classical theorems: (i) if F is a vector field defined on a ball in $\mathbb{R}^3$ with $\operatorname{curl} F = 0$, then there is a smooth function g such that $F = \operatorname{grad}(g)$, and (ii) if F is a smooth vector field defined on a ball such that $\operatorname{div} F = 0$, then there is a smooth vector field g such that $F = \operatorname{curl} g$.

Proof. The full statement of the Poincaré lemma is not needed here; only the case when $k = 1$ is used in subsequent chapters. Therefore, only that case is proved here. The proof of the full theorem can be found in Flanders (1963), or Spivak (1965) or Abraham and Marsden (1978).

Let $F = \sum_i f_i(x) dx^i$ be a given 1-form.

$$dF = \sum_i \sum_j \left(\frac{\partial f_i}{\partial x^j} \right) dx^j \wedge dx^i = \sum_{i<j} \left(\frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i} \right) dx^j \wedge dx^i.$$

So $dF = 0$ if and only if $\partial f_i/\partial x^j = \partial f_j/\partial x^i$. Define

$$g(x) = \int_0^1 \sum_i f_i(tx)x^i dt.$$

So

$$\begin{aligned}\frac{\partial g(x)}{\partial x^j} &= \int_0^1 \left\{ \sum_i \frac{\partial f_i(tx)}{\partial x^j} tx^i + f_j(tx) \right\} dt \\ &= \int_0^1 \left\{ \frac{tdf_j(tx)}{dt} + f_j(tx) \right\} dt \\ &= \int_0^1 \frac{d}{dt} \{t f_j(tx)\} \, dt \\ &= t f_j(tx) \Big|_0^1 = f_j(x).\end{aligned}$$

Thus $dg =$ F.

Note that the function g defined in the proof given above is a line integral and the condition $dF = 0$ is the condition that a line integral be independent of the path.

Corollary 5.4.1. *Let $F = (F^1, \ldots, F^m)$ be a vector valued function defined in a ball $\mathcal{O}$ in $\mathbb{R}^m$. Then a necessary and sufficient condition for F to be the gradient of a function $g : \mathcal{O} \longrightarrow \mathbb{R}$ is that the Jacobian matrix $(\partial F^i/\partial x^j)$ be symmetric.*

Proof. First, to see that it is a corollary, consider F as the differential form $F = F^1 dx^1 + \cdots + F^m dx^m$. Then by the above,

$$dF = \sum_{i<j} \left(\frac{\partial F^i}{\partial x^j} - \frac{\partial F^j}{\partial x^i} \right) dx^i \wedge dx^j.$$

So $dF = 0$ if and only if the Jacobian $(\partial F^i/\partial x^j)$ is symmetric. The corollary follows from part (iii) of Lemma 5.4.1 and Theorem 5.4.1.

5.5 Changing Coordinates and Darboux's Theorem

To change coordinates for vector fields or differential forms, simply transform the coordinates as was done in Section 5.3 using the Jacobian of the transformation. In particular, let x and y be coordinates on $\mathcal{O}$, and assume that the change of coordinates is given by $x = \psi(y)$ and the change back by $y = \phi(x)$, or in classical notation $x = x(y)$ and $y = y(x)$. Assume the Jacobians, $D\phi = (\partial y^j/\partial x^i)$ and $D\psi = (\partial x^i/\partial y^j)$, are nonsingular.

If a vector field V is given by

$$V = \sum_{i=1}^{m} \alpha^i(x)\frac{\partial}{\partial x^i} = \sum_{i=1}^{m} \beta^i(x)\frac{\partial}{\partial y^i},$$

and we set $\mathbf{a}(x) = (\alpha^1(x), \ldots, \alpha^m(x))$, $\mathbf{b}(y) = (\beta^1(y), \ldots, \beta^m(y))$, then

$$\mathbf{a} = D\psi(\mathbf{b}) \quad \text{or} \quad \alpha^i = \sum_{j=1}^{m} \frac{\partial x^i}{\partial y^j}\beta^j. \tag{5.9}$$

If a differential 1-form is given by

$$F = \sum_{i=1}^{m} \alpha_i(x)dx^i = \sum_{i=1}^{m} \beta_i(y)dy^i,$$

and we set $\mathbf{a}(x) = (\alpha_1(x), \ldots, \alpha_m(x))$, $\mathbf{b}(y) = (\beta_1(y), \ldots, \beta_m(y))$, then

$$\mathbf{a} = \mathbf{b}D\phi \qquad \text{or} \qquad \alpha_i = \sum_{j=1}^{m} \frac{\partial y^j}{\partial x^i}\beta_j. \tag{5.10}$$

If a differentiable 2-form F is given by

$$F = \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_{ij}(x)dx^i \wedge dx^j = \sum_{i=1}^{m}\sum_{j=1}^{m} \beta_{ij}(y)dy^i \wedge dy^j, \tag{5.11}$$

and we set $\mathbf{A} = (\alpha_{ij})$, $\mathbf{B} = (\beta_{ij})$ ($\mathbf{A}$ and $\mathbf{B}$ are skew-symmetric matrices), then

$$\mathbf{A} = D\psi^T\mathbf{B}D\psi \quad \text{or} \quad \alpha_{ij} = \sum_{s=1}^{m}\sum_{r=1}^{m} \frac{\partial y^s}{\partial x^i}\frac{\partial y^r}{\partial x^j}\beta_{sr}. \tag{5.12}$$

Let $\mathcal{O}$ be an open set in $\mathbb{R}^{2n}$. A 2-form F on $\mathcal{O}$ is nondegenerate if $F^n = F \wedge F \wedge \cdots \wedge F$ (n times) is nonzero. As we saw above, the coefficients in a coordinate system of a 2-form can be represented as a skew-symmetric matrix. As we saw in Section 5.2, a linear 2-form is nondegenerate if and only if the coefficient matrix is nonsingular. Thus the 2-form F in (5.11) is nondegenerate if and only if $\mathbf{A}$ (or $\mathbf{B}$) is nonsingular on all of $\mathcal{O}$. A symplectic structure or symplectic form on $\mathcal{O}$ is a closed nondegenerate 2-form. The standard symplectic structure in $\mathbb{R}^{2n}$ is

$$\Omega = \sum_{i=1}^{n} dz^i \wedge dz^{i+n} = \sum_{i=1}^{n} dq^i \wedge dp^i. \tag{5.13}$$

where $z = (z^1, \dots, z^{2n}) = (q^1, \dots, q^n, p^1, \dots, p^n)$ are coordinates in $\mathbb{R}^{2n}$. The coefficient matrix of Ω is just J. By Corollary 3.2.1, there is a linear change of coordinates so that the coefficient matrix of a nondegenerate 2-form is J at one point. A much more powerful result that is not needed in the subsequent chapters is the following.

Theorem 5.5.1 (Darboux's theorem). *If F is a symplectic structure on an open ball in $\mathbb{R}^{2n}$, then there exists a coordinate system z such that F in this coordinate system is the standard symplectic structure Ω.*

Proof. See Abraham and Marsden (1978).

A coordinate system for which a symplectic structure is Ω is called a symplectic coordinate (for this form). A symplectic transformation, ϕ, is one that preserves the form Ω or preserves the coefficient matrix J; i.e., $D\phi^T J D\phi = J$.

5.6 Integration and Stokes' Theorem

We do not need any result from integration theory on manifolds, and so we do not develop the theory here. To tease the reader into learning more about this subject, consider a weak form of the general Stokes' theorem. It illustrates the power and beauty of differential forms. Let M be an n-dimensional oriented manifold with an $(n-1)$-dimensional boundary ∂M. Let the boundary ∂M be oriented consistently with M. Let ω be an $(n-1)$-form on M; so, $d\omega$ is an n-form on M. One can define the integral of an n-form on an n-manifold in a logical way, and then one has:

$$\int_{\partial M} \omega = \int_M d\omega \qquad \text{(Stokes' theorem)}.$$

This one general theorem contains Green's theorem, the divergence theorem, and the classical Stokes' theorem of classical vector calculus. See Spivak (1965) for a complete discussion of the general Stokes' theorem and all its ramifications.

Problems

1. Show that if $f^1, \dots, f^k$ are 1-forms, then

$$f^1 \wedge \cdots \wedge f^k(a_1, \dots, a_k) = \det \begin{bmatrix} f^1(a_1) & \cdots & f^k(a_1) \\ \vdots & & \vdots \\ f^1(a_k) & \cdots & f^k(a_k) \end{bmatrix}.$$

2. Show that the mapping $(f^1, f^2) \longrightarrow f^1 \wedge f^2$ is a skew-symmetric bilinear map from $\mathbb{V}^* \times \mathbb{V}^* \longrightarrow \mathbb{A}^2$.
3. Let F and G be 0-, 1- or 2-forms in $\mathbb{R}^3$. Verify Lemma 5.4.1 in this case.
4. a) Let $F = adx + bdy + cdz$ be a 1-form in $\mathbb{R}^3$ such that $dF = 0$. Verify that $\partial a/\partial y = \partial b/\partial x$, $\partial a/\partial z = \partial c/\partial x$, $\partial c/\partial y = \partial b/\partial z$. Also verify that if

$$f(x,y,z) = \int_0^1 (a(tx,ty,tz)x + b(tx,ty,tz)y + c(tx,ty,tz)z)dt,$$

then $F = df$.

   b) Let F be a 2-form in $\mathbb{R}^3$ such that $dF = 0$. Verify that if $F = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$, then $\partial a/\partial x + \partial b/\partial y + \partial c/\partial z = 0$. Also verify that $F = df$ where

$$\begin{aligned} f &= \left(\textstyle\int_0^1 a(tx,ty,tz)tdt\right)(ydz - zdy) \\ &+ \left(\textstyle\int_0^1 b(tx,ty,tz)tdt\right)(zdx - xdz) \\ &+ \left(\textstyle\int_0^1 c(tx,ty,tz)tdt\right)(xdy - ydx). \end{aligned}$$

5. Prove that the $\wedge$ operator is bilinear and associative. (See Lemma 5.1.1.)
6. a) Show that the operator d which operates on smooth forms is linear, i.e., $d(F+G) = dF + dG$.
   b) Show that d satisfies a product rule, $d(F \wedge G) = dF \wedge G + (-1)^{\deg(F)} F \wedge dG$.
   c) Show that if δ is a mapping which takes smooth k-forms to $(k+1)$-forms and satisfies
      i. $\delta(F+G) = \delta F + \delta G$,
      ii. $\delta(F \wedge G) = \delta F \wedge G + (-1)^{\deg(F)} F \wedge \delta G$,
      iii. $\delta(\delta F) = 0$ for all F,
      iv. If F is a function, then δF agrees the standard definition of the differential of F, then δ is the same as the operator d given by the formula in (5.8).
7. Let $Q(q,p)$ and $P(q,p)$ be smooth functions defined on an open set in $\mathbb{R}^2$. Consider the four differential forms $\Omega_1 = PdQ - pdq$, $\Omega_2 = PdQ + qdp$, $\Omega_3 = QdP + pdq$, $\Omega_4 = QdP - qdp$.
   a) Show that Ω_i is exact if and only if Ω_j is exact for $i \neq j$.
   b) Show that Ω_i is closed if and only if Ω_j is closed for $i \neq j$.
   c) Show that if Ω_i is exact (or closed) then so is $\Theta = (Q-q)d(P+p) - (P-p)d(Q+q)$. (Hint: $d(qp) = qdp + pdq$ is exact.)