13. Stability and KAM Theory

Questions of stability of orbits have been of interest since Newton first set down the laws that govern the motion of the celestial bodies. “Is the universe stable?” is almost a theological question. Even though the question is old and important, very little is known about the problem, and much of what is known is difficult to come by.

This chapter contains an introduction to the question of the stability and instability of orbits of Hamiltonian systems and in particular the classical Lyapunov theory and the celebrated KAM theory. This subject could be the subject of a complete book; so, the reader will find only selected topics presented here. The main example is the stability of the libration points of the restricted problem, but other examples are touched.

Consider the differential equation

$$\dot{z} = f(z), \tag{13.1}$$

where f is a smooth function from the open set $O \subset \mathbb{R}^m$ into $\mathbb{R}^m$. Let the equation have an equilibrium point at $\zeta_0 \in O$; so, $f(\zeta_0) = 0$. Let $\phi(t, \zeta)$ be the general solution of (13.1). The equilibrium point ζ_0 is said to be positively (respectively, negatively) stable, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(t,\zeta) - \zeta_0\| < \epsilon$ for all $t \geq 0$ (respectively, $t \leq 0$) whenever $\|\zeta - \zeta_0\| < \delta$. The equilibrium point ζ_0 is said to be stable if it is both positively and negatively stable. In many books “stable” means positively stable, but the above convention is the common one in the theory of Hamiltonian differential equations. The equilibrium ζ_0 is unstable if it is not stable. The adjectives “positively” and “negatively” can be used with “unstable” also. The equilibrium ζ_0 is asymptotically stable, if it is positively stable, and there is an $\eta > 0$ such that $\phi(t, \zeta) \to \zeta_0$ as $t \to +\infty$ for all $\|\zeta - \zeta_0\| < \eta$.

Recall the one result already given on stability in Theorem 1.3.2, which states that a strict local minimum or maximum of a Hamiltonian is a stable equilibrium point. So for a general Newtonian system of the form $H = p^T M p/2 + U(q)$, a strict local minimum of the potential U is a stable equilibrium point because the matrix M is positive definite. It has been stated many times that an equilibrium point of U that is not a minimum is unstable. Laloy (1976) showed that for

$$U(q_1, q_2) = \exp(-1/q_1^2)\cos(1/q_1) - \exp(-1/q_2^2)\{\cos(1/q_2) + q_2^2\},$$

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the origin is a stable equilibrium point, and yet the origin is not a local minimum for U. See Taliaferro (1980) for some positive results along these lines.

Henceforth, let the equilibrium point be at the origin. A standard approach is to linearize the equations; i.e., write (13.1) in the form

$$\dot{z} = Az + g(z),$$

where $A = \partial f(0)/\partial z$ and $g(z) = f(z) - Az$; so, $g(0) = \partial g(0)/\partial z = 0$. The eigenvalues of A are called the exponents (of the equilibrium point). If all the exponents have negative real parts, then a classical theorem of Lyapunov states that the origin is asymptotically stable. By Proposition 3.3.1, the eigenvalues of a Hamiltonian matrix are symmetric with respect to the imaginary axis; so, this theorem never applies to Hamiltonian systems. In fact, because the flow defined by a Hamiltonian system is volume-preserving, an equilibrium point can never be asymptotically stable.

Lyapunov also proved that if one exponent has positive real part then the origin is unstable. Thus for the restricted 3-body problem the Euler collinear libration points, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, are always unstable, and the Lagrange triangular libration points, $\mathcal{L}_4$ and $\mathcal{L}_5$, are unstable for $\mu_1 < \mu < 1 - \mu_1$ by the results of Section 4.1.

Thus a necessary condition for stability of the origin is that all the eigenvalues be pure imaginary. It is easy to see that this condition is not sufficient in the non-Hamiltonian case. For example, the exponents of

$$\dot{z_1} = z_2 + z_1(z_1^2 + z_2^2),$$

$$\dot{z_2} = -z_1 + z_2(z_1^2 + z_2^2)$$

are $\pm i$, and yet the origin is unstable. (In polar coordinates, $\dot{r} = r^3 > 0$.) However, this equation is not Hamiltonian.

In the second 1917 edition of Whittaker's book on dynamics, the equations of motion about the Lagrange point $\mathcal{L}_4$ are linearized, and the assertion is made that the libration point is stable for $0 < \mu < \mu_1$ on the basis of this linear analysis. In the third edition of Whittaker (1937) this assertion was dropped, and an example due to Cherry (1928) was included. A careful look at Cherry's example shows that it is a Hamiltonian system of two degrees of freedom, and the linearized equations are two harmonic oscillators with frequencies in a ratio of 2:1. The Hamiltonian is in the normal form given in Theorem 10.4.1; i.e., in action–angle variables, Cherry's example is

$$H = 2I_1 - I_2 + I_1 I_2^{1/2} \cos(\phi_1 + 2\phi_2). \tag{13.2}$$

Cherry explicitly solves this system, but we show the equilibrium is unstable as a consequence of Chetaev's theorem 13.1.2.

13.1 Lyapunov and Chetaev's Theorems

In this section we present the parts of classical Lyapunov stability theory as it pertains to Hamiltonian systems. Consider the differential equation (13.1).

Return to letting ζ_0 be the equilibrium point. Let $V : O \to \mathbb{R}$ be smooth where O is an open neighborhood of the equilibrium point ζ_0. One says that V is positive definite (with respect to ζ_0) if there is a neighborhood $Q \subset O$ of ζ_0 such that $V(\zeta_0) < V(z)$ for all $z \in O \setminus \{\zeta_0\}$. That is, ζ_0 is a strict local minimum of V. Define $\dot{V} : O \to \mathbb{R}$ by $\dot{V}(z) = \nabla V(z) \cdot f(z)$.

Theorem 13.1.1 (Lyapunov's Stability Theorem). *If there exists a function V that is positive definite with respect to ζ_0 and such that $\dot{V} \leq 0$ in a neighborhood of ζ_0 then the equilibrium ζ_0 is positively stable.*

Proof. Let $\epsilon > 0$ be given. Without loss of generality assume that $\zeta_0 = 0$ and $V(0) = 0$. Because $V(0) = 0$ and 0 is a strict minimum for V, there is an $\eta > 0$ such that $V(z)$ is positive for $0 < \|z\| \leq \eta$. By taking η smaller if necessary we can ensure that $\dot{V}(z) \leq 0$ for $\|z\| \leq \eta$ and that $\eta < \epsilon$ also.

Let $M = \min\{V(z) : \|z\| = \eta\}$. Because $V(0) = 0$ and V is continuous, there is a $\delta > 0$ such that $V(z) < M$ for $\|z\| < \delta$ and $\delta < \eta$. We claim that if $\|\zeta\| < \delta$ then $\|\phi(t, \zeta)\| \leq \eta < \epsilon$ for all $t \geq 0$.

Because $\|\zeta\| < \delta < \eta$ there is a t^* such that $\|\phi(t, \zeta)\| < \eta$ for all $0 \leq t < t^*$ and t^* is the smallest such number. Assume t^* is finite and so $\|\phi(t^*, \zeta)\| = \eta$. Define $v(t) = V(\phi(t, \zeta))$ so $v(0) < M$ and $\dot{v}(t) \leq 0$ for $0 \leq t \leq t^*$ and so $v(t^*) < M$. But $v(t^*) = V(\phi(t^*, \zeta)) \geq M$ which is a contradiction and so t^* is infinite.

Consider the case when (13.1) is Hamiltonian; i.e. of the form

$$\dot{z} = J\nabla H(z), \tag{13.3}$$

where H is a smooth function from $O \subset \mathbb{R}^{2n}$ into $\mathbb{R}$. Again let $z_0 \in O$ be an equilibrium point and let $\phi(t, \zeta)$ be the general solution.

Corollary 13.1.1 (Dirichlet's stability theorem 1.3.2). *If z_o is a strict local minimum or maximum of H, then z_0 is a stable equilibrium for* (13.3).

Proof. Because $\pm H$ is an integral we may assume that H has a minimum. Because $\dot{H} = 0$ the system is positively stable. Reverse time by replacing t by $-t$. In the new time $\dot{H} = 0$, so the system is positively stable in the new time or negatively stable in the original time.

For the moment consider a Hamiltonian system of two degrees of freedom that has an equilibrium at the origin and is such that the linearized equations look like two harmonic oscillators with distinct frequencies ω_1, ω_2, $\omega_i \neq 0$. The quadratic terms of the Hamiltonian can be brought into normal form by a linear symplectic change of variables so that the Hamiltonian is of the form

$$H = \pm\frac{\omega_1}{2}(x_1^2 + y_1^2) \pm \frac{\omega_2}{2}(x_2^2 + y_2^2) + \cdots.$$

If both terms have the same sign then the equilibrium is stable by Dirichlet's Theorem. However, in the restricted problem at Lagrange triangular libration points $\mathcal{L}_4$ and $\mathcal{L}_5$ for $0 < \mu < \mu_1$ the Hamiltonian is of the above form, but the signs are opposite.

Theorem 13.1.2 (Chetaev's theorem). *Let $V : O \to \mathbb{R}$ be a smooth function and Ω an open subset of O with the following properties.*

- $\zeta_0 \in \partial\Omega$.
- $V(z) > 0$ *for* $z \in \Omega$.
- $V(z) = 0$ *for* $z \in \partial\Omega$.
- $\dot{V}(z) = V(z) \cdot f(z) > 0$ *for* $z \in \Omega$.

Then the equilibrium solution ζ_0 of (13.1) *is unstable. In particular, there is a neighborhood Q of the equilibrium such that all solutions which start in $Q \cap \Omega$ leave Q in positive time.*

Proof. Again we can take $\zeta_0 = 0$. Let $\epsilon > 0$ be so small that the closed ball of radius ϵ about 0 is contained in the domain O and let $Q = \Omega \cap \{\|z\| < \epsilon\}$. We claim that there are points arbitrarily close to the equilibrium point which move a distance at least ϵ from the equilibrium.

Q has points arbitrarily close to the origin, so for any $\delta > 0$ there is a point $p \in Q$ with $\|p\| < \delta$ and $V(p) > 0$.

Let $v(t) = V(\phi(t, p))$. Either $\phi(t, p)$ remains in Q for all $t \geq 0$ or $\phi(t, p)$ crosses the boundary of Q for the first time at a time $t^* > 0$.

If $\phi(t, p)$ remains in Q then $v(t)$ is increasing because $\dot{v} > 0$ and so $v(t) \geq v(0) > 0$ for $t \geq 0$. The closure of $\{\phi(t, p) : t \geq 0\}$ is compact and $\dot{v} > 0$ on this set so $\dot{v}(t) \geq \kappa > 0$ for all $t \geq 0$. Thus $v(t) \geq v(0) + \kappa t \to \infty$ as $t \to \infty$. This is a contradiction because $\phi(t, p)$ remains in an ϵ neighborhood of the origin and v is continuous.

If $\phi(t, p)$ crosses the boundary of Q for the first time at a time $t^* > 0$, $\dot{v}(t) > 0$ for $0 \leq t < t^*$ and so $v(t^*) \geq v(0) > 0$. Because the boundary of Q consist of the points q where $V(q) = 0$ or where $\|q\| = \epsilon$, it follows that $\|v(t^*)\| = \epsilon$.

Cherry's counterexample in action–angle coordinates is

$$H = 2I_1 - I_2 + I_1^{1/2} I_2 \cos(\phi_1 + 2\phi_2). \tag{13.4}$$

To see that the origin is unstable, consider the Chetaev function

$$W = -I_1^{1/2} I_2 \sin(\phi_1 + 2\phi_2),$$

and compute

$$\dot{W} = 2I_1 I_2 + \frac{1}{2} I_2^2.$$

Let Ω be the region where $W > 0$. In Ω, $I_2 \neq 0$; so, $\dot{W} > 0$ in Ω. Ω has points arbitrarily close to the origin, so Chetaev's theorem show that the origin is unstable even though the linearized system is stable.

Theorem 13.1.3 (Lyapunov's instability theorem). *If there is a smooth function $V : O \to \mathbb{R}$ that takes positive values arbitrarily close to ζ_0 and is such that $\dot{V} = V \cdot f$ is positive definite with respect to ζ_0 then the equilibrium ζ_0 is unstable.*

Proof. Let $\Omega = \{z : V(z) > 0\}$ and apply Chetaev's theorem.

As the first application consider a Hamiltonian system of two degrees of freedom with an equilibrium point and the exponents of this system at the equilibrium point are $\pm\omega i, \pm\lambda$, $\omega \neq 0$, $\lambda \neq 0$; i.e. one pair of pure imaginary exponents and one pair of real exponents. For example, the Hamiltonian of the restricted problem at the Euler collinear libration points $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ is of this type. We show that the equilibrium point is unstable. Specifically, consider the system

$$H = \frac{\omega}{2}(x_1^2 + y_1^2) + \lambda x_2 y_2 + H^\dagger(x, y) \tag{13.5}$$

where $H^\dagger$ is real analytic in a neighborhood of the origin in $\mathbb{R}^4$ in its displayed arguments and of at least third degree. Note that we have assumed that the equilibrium is at the origin and that the quadratic terms are already in normal form. As we have already seen, Lyapunov's center theorem 9.2.1 implies that the system admits an analytic surface called the Lyapunov center filled with periodic solutions.

Theorem 13.1.4. *The equilibrium at the origin for the system with Hamiltonian* (13.5) *is unstable. In fact, there is a neighborhood of the origin such that any solution which begins off the Lyapunov center leaves the neighborhood in both positive and negative time. In particular, the small periodic solutions given on the Lyapunov center are unstable.*

Proof. There is no loss in generality by assuming λ is positive. The equations of motion are

$$\dot{x_1} = \omega y_1 + \frac{\partial H^\dagger}{\partial y_1} \qquad \dot{y_1} = -\omega x_1 - \frac{\partial H^\dagger}{\partial x_1}$$

$$\dot{x_2} = \lambda x_2 + \frac{\partial H^\dagger}{\partial y_2} \qquad \dot{y_2} = -\lambda y_2 - \frac{\partial H^\dagger}{\partial x_2}.$$

We may assume that the Lyapunov center has been transformed to the coordinate plane $x_2 = y_2 = 0$; i.e. $\dot{x_2} = \dot{y_2} = 0$ when $x_2 = y_2 = 0$. That means that $H^\dagger$ does not have a term of the form $x_2(x_1^n y_1^m)$ or of the form $y_2(x_1^n y_1^m)$.

Consider the Chetaev function $V = \frac{1}{2}(x_2^2 - y_2^2)$ and compute

$$\dot{V} = \lambda(x_2^2 + y_2^2) + x_2 \frac{\partial H^\dagger}{\partial y_2} - y_2 \frac{\partial H^\dagger}{\partial y_2}$$

$$= \lambda(x_2^2 + y_2^2) + W(x, y).$$

We claim that in a sufficiently small neighborhood Q of the origin $\|W(x,y)\| \leq (\lambda/2)(x_2^2 + y_2^2)$ and so $\dot{V} > 0$ on $Q \backslash \{x_2 = y_2 = 0\}$; i.e. off the Lyapunov center. Let $H^\dagger = H_0^\dagger + H_2^\dagger + H_3^\dagger$ where $H_0^\dagger$ is independent of x_2, y_2, $H_2^\dagger$ is quadratic in x_2, y_2, and $H_3^\dagger$ is at least cubic in x_2, y_2. $H_0^\dagger$ contributes nothing to W; $H_2^\dagger$ contributes to W a function that is quadratic in x_2, y_2 and at least linear in x_1, y_1, and so can be estimated by $O(\{x_1^2 + y_1^2\}^{1/2})O(\{x_2^2 + y_2^2\})$; and $H_3^\dagger$ contributes to W a function that is cubic in x_2, y_2 and so is $O(\{x_2^2 + y_2^2\}^{3/2})$. These estimates prove the claim.

Let $\Omega = \{x_2^2 > y_2^2\} \cap Q$ and apply Chetaev's theorem to conclude that all solutions which start in Ω leave Q in positive time. If you reverse time you will conclude that all solutions which start in $\Omega^- = \{x_2^2 < y_2^2\} \cap Q$ leave Q in negative time.

Proposition 13.1.1. *The Euler collinear libration points $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ of the restricted 3-body problem are unstable. There is a neighborhood of these points such that there are no invariant sets in this neighborhood other than the periodic solutions on the Lyapunov center manifold.*

As the second application consider a Hamiltonian system of two degrees of freedom with an equilibrium point and the exponents of this system at the equilibrium point are $\pm\alpha \pm \beta i$, $\alpha \neq 0$; i.e., two exponents with positive real parts and two with negative real parts. For example, the Hamiltonian of the restricted problem at the Lagrange triangular points $\mathcal{L}_4$ and $\mathcal{L}_5$ is of this type when $\mu_1 < \mu < 1 - \mu_1$. We show that the equilibrium point is unstable. Specifically, consider the system

$$H = \alpha(x_1 y_1 + x_2 y_2) + \beta(y_1 x_2 - y_2 x_1) + H^\dagger(x, y), \tag{13.6}$$

where $H^\dagger$ is real analytic in a neighborhood of the origin in $\mathbb{R}^4$ in its displayed arguments and of at least third degree. Note that we have assumed that the equilibrium is at the origin and that the quadratic terms are already in normal form.

Theorem 13.1.5. *The equilibrium at the origin for the system with Hamiltonian* (13.6) *is unstable. In fact, there is a neighborhood of the origin such that any nonzero solution leaves the neighborhood in either positive or negative time.*

Proof. We may assume $\alpha > 0$. The equations of motion are

$$\dot{x}_1 = \alpha x_1 + \beta x_2 + \frac{\partial H^\dagger}{\partial y_1}, \qquad \dot{x}_2 = -\beta x_1 + \alpha x_2 + \frac{\partial H^\dagger}{\partial y_2},$$

$$\dot{y}_1 = -\alpha y_1 + \beta y_2 - \frac{\partial H^\dagger}{\partial x_1}, \quad \dot{y}_2 = -\beta y_1 - \alpha y_2 + \frac{\partial H^\dagger}{\partial x_2}.$$

Consider the Lyapunov function

$$V = \frac{1}{2}(x_1^2 + x_2^2 - y_1^2 - y_2^2)$$

and compute

$$\dot{V} = \alpha(x_1^2 + x_2^2 + y_1^2 + y_2^2) + W.$$

where W is at least cubic. Clearly V takes on positive values close to the origin and $\dot{V}$ is positive definite, so all solutions in $\{(x, y) : V(x, y) > 0\}$ leave a small neighborhood in positive time. Reversing time shows that all solutions in $\{(x, y) : V(x, y) < 0\}$ leave a small neighborhood in positive time.

Proposition 13.1.2. *The triangular equilibrium points $\mathcal{L}_4$ and $\mathcal{L}_5$ of the restricted 3-body problem are unstable for $\mu_1 < \mu < 1 - \mu_1$. There is a neighborhood of these points such that there are no invariant sets in this neighborhood other than the equilibrium point itself.*

The classical references on stability are Lynpunov (1892) and Chetaev (1934), but very readable account can be found in LaSalle and Lefschetz (1961). The text by Markeev (1978) contains many of the stability results for the restricted problem given here and below plus a discussion of the elliptic restrict problem and other systems.

13.2 Moser's Invariant Curve Theorem

We return to questions about the stability of equilibrium points later, but now consider the corresponding question for maps. Let

$$F(z) = Az + f(z) \tag{13.7}$$

be a diffeomorphism of a neighborhood of a fixed point at the origin in $\mathbb{R}^m$; so, $f(0) = 0$ and $\partial f(0)/\partial z = 0$. The eigenvalues of A are the multipliers of the fixed point.

The fixed point 0 is said to be stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|F^k(z)\| < \epsilon$ for all $\|z\| < \delta$ and $k \in \mathbb{Z}$.

We reduce several of the stability questions for equilibrium points of a differential equation to the analogous question for fixed points of a diffeomorphism. Let us specialize by letting the fixed point be the origin in $\mathbb{R}^2$ and by letting (13.7) be area-preserving (symplectic). Assume that the origin is an elliptic fixed point; so, A has eigenvalues λ and $\lambda^{-1} = \bar{\lambda}$, $\mid \lambda \mid = 1$. If $\lambda = 1$, -1, $\sqrt[3]{1} = e^{2\pi i/3}$, or $\sqrt[4]{1} = i$ then typically the origin is unstable; see Meyer (1971) and the Problems.

Therefore, let us consider the case when λ is not an mth root of unity for $m = 1, 2, 3, 4$. In this case, the map can be put into normal form up through terms of order three; i.e., there are symplectic action–angle coordinates, I, ϕ, such that in these coordinates, $F : (I, \phi) \to (I', \phi')$, where

$$\begin{aligned} I' &= I + c(I, \phi), \\ \phi' &= \phi + \omega + \alpha I + d(I, \phi), \end{aligned} \tag{13.8}$$

and $\lambda = \exp(\omega i)$, and c, d are $O(I^{3/2})$. We do not need the general results because we construct the maps explicitly in the applications given below.

For the moment assume c and d are zero; so, the map (13.8) takes circles $I = I_0$ into themselves, and if $\alpha \neq 0$, each circle is rotated by a different amount. The circle $I = I_0$ is rotated by an amount $\omega + \alpha I_0$. When $\omega + \alpha I_0 = 2\pi p/q$, where p and q are relatively prime integers, then each point on the circle $I = I_0$ is a periodic point of period q.

If $\omega + \alpha I_0 = 2\pi\delta$, where δ is irrational, then the orbits of a point on the circle $I = I_0$ are dense ($c = d = 0$ still). One of the most celebrated theorems in Hamiltonian mechanics states that many of these circles persist as invariant curves. In fact, there are enough invariant curves encircling the fixed point that they assure the stability of the fixed point. This is the so called "invariant curve theorem".

Theorem 13.2.1 (The invariant curve theorem). *Consider the mapping* $F : (I, \phi) \to (I', \phi')$ *given by*

$$\begin{aligned} I' &= I + \epsilon^{s+r} c(I, \phi, \epsilon), \\ \phi' &= \phi + \omega + \epsilon^s h(I) + \epsilon^{s+r} d(I, \phi, \epsilon), \end{aligned} \tag{13.9}$$

where (i) c and d are smooth for $0 \leq a \leq I < b < \infty$, $0 \leq \epsilon \leq \epsilon_0$, and all ϕ, (ii) c and d are 2π-periodic in ϕ, (iii) r and s are integers $s \geq 0, r \geq 1$, (iv) h is smooth for $0 \leq a \leq I < b < \infty$, (v) $dh(I)/dI \neq 0$ for $0 \leq a \leq I < b < \infty$, and (vi) if Γ is any continuous closed curve of the form $\Xi = \{(I, \phi) : I = \Theta(\phi), \Theta : \mathbb{R} \to [a, b]$ continuous and 2π-periodic $\}$, then $\Xi \cap F(\Xi) \neq \emptyset$.

Then for sufficiently small ϵ, there is a continuous F-invariant curve Γ of the form $\Gamma = \{(I, \phi) : I = \Phi(\phi), \Phi : \mathbb{R} \to [a, b]$ continuous and $2\pi-$ periodic $\}$.

Remarks.

1. The origin of this theorem was in the announcements of Kolmogorov who assumed analytic maps, and the analog of the invariant curve was shown

to be analytic. In the original paper by Moser (1962), where this theorem was proved, the degree of smoothness required of c, d, h was very large, C^{333}, and the invariant curve was shown to be continuous. This spread led to a great deal of work to find the least degree of differentiability required of c, d, and h to get the most differentiability for the invariant curve. However, in the interesting examples, c, d, and h are analytic, and the existence of a continuous invariant curve yields the necessary stability.

2. The assumption (v) is the twist assumption discussed above, and the map is a perturbation of a twist map for small ϵ.
3. Assumption (vi) rules out the obvious example where F maps every point radially out or radially in. If F preserves the inner boundary $I = a$ and is area-preserving, then assumption (vi) is satisfied.
4. The theorem can be applied to any subinterval of $[a, b]$, therefore the theorem implies the existence of an infinite number of invariant curves. In fact, the proof shows that the measure of the invariant curves is positive and tends to the measure of the full annulus $a \le I \le b$ as $\epsilon \to 0$.
5. The proof of this theorem is quite technical. See Siegel and Moser (1971) and Herman (1983) for a complete discussion of this theorem and related results.

The following is a slight modification of the invariant curve theorem that is needed later on.

Corollary 13.2.1. *Consider the mapping* $F : (I, \phi) \to (I', \phi')$ *given by*

$$\begin{aligned} I' &= I + \epsilon c(I, \phi, \epsilon), \\ \phi' &= \phi + \epsilon h(\phi) I + \epsilon^2 d(I, \phi, \epsilon), \end{aligned} \tag{13.10}$$

where (i) c *and* d *are smooth for* $0 \le a \le I < b < \infty$, $0 \le \epsilon \le \epsilon_0$, *and all* ϕ, *(ii)* c *and* d *are* 2π*-periodic in* ϕ, *(iii)* $h(\phi)$ *is smooth and* 2π*-periodic in* ϕ, *and (iv) if* Γ *is any continuous closed curve of the form* $\Xi = \{(I, \phi) : I = \Theta(\phi), \Theta : \mathbb{R} \to [a, b]$ *continuous and* 2π*-periodic, then* $\Xi \cap F(\Xi) \ne \emptyset$.

If $h(\phi)$ *is nonzero for all* ϕ *then for sufficiently small* ϵ, *there is a continuous* F*-invariant curve* Γ *of the form* $\Gamma = \{(I, \phi) : I = \Phi(\phi), \Phi : \mathbb{R} \to [a, b]$ *continuous and* $2\pi - periodic\}$.

Proof. Consider the symplectic change of variables from the action–angle variables I, ϕ to the action–angle variables J, ψ defined by the generating function

$$S(J, \phi) = J M^{-1} \int_0^{\phi} \frac{d\tau}{h(\tau)}, \qquad M = \int_0^{2\pi} \frac{d\tau}{h(\tau)}.$$

So

$$\psi = \frac{\partial S}{\partial J} = M^{-1} \int_0^{\phi} \frac{d\tau}{h(\tau)}, \qquad I = \frac{\partial S}{\partial \phi} = \frac{MJ}{h(\phi)},$$

and the map in the new coordinates becomes

$$J' = J + O(\epsilon), \qquad \psi' = \psi + \epsilon MJ + O(\epsilon^2).$$

The theorem applies in the new coordinates.

13.3 Arnold's Stability Theorem

The invariant curve theorem can be used to establish a stability result for equilibrium points as well. In particular, we prove Arnold's stability theorem using Moser's invariant curve theorem.

As discussed above, the only way an equilibrium point can be stable is if the eigenvalues of the linearized equations (the exponents) are pure imaginary. Arnold's theorem addresses the case when exponents are pure imaginary, and the Hamiltonian is not positive definite.

Consider the two degree of freedom case, and assume the Hamiltonian has been normalized a bit. Specifically, consider a Hamiltonian H in the symplectic coordinates x_1, x_2, y_1, y_2 of the form

$$H = H_2 + H_4 + \cdots + H_{2N} + H^{\dagger}, \tag{13.11}$$

where

1. H is real analytic in a neighborhood of the origin in $\mathbb{R}^4$.
2. H_{2k}, $1 \le k \le N$, is a homogeneous polynomial of degree k in I_1, I_2, where $I_i = (x_i^2 + y_i^2)/2$, $i = 1, 2$.
3. $H^{\dagger}$ has a series expansion that starts with terms at least of degree $2N+1$.
4. $H_2 = \omega_1 I_1 - \omega_2 I_2$, ω_i nonzero constants;
5. $H_4 = \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2)$, A, B, C, constants.

There are several implicit assumptions in stating that H is of the above form. Because H is at least quadratic, the origin is an equilibrium point. By (4), H_2 is the Hamiltonian of two harmonic oscillators with frequencies ω_1 and ω_2; so, the linearized equations of motion are two harmonic oscillators. The sign convention is to conform with the sign convention at $\mathcal{L}_4$. It is not necessary to assume that ω_1 and ω_2 are positive, but this is the interesting case when the Hamiltonian is not positive definite. H_{2k}, $1 \le k \le N$, depends only on I_1 and I_2; so, H is assumed to be in Birkhoff normal form (Corollary 10.4.1) through terms of degree $2N$. This usually requires the nonresonance condition $k_1\omega_1 + k_2\omega_2 \neq 0$ for all integers k_1, k_2 with $\mid k_1 \mid + \mid k_2 \mid \le 2N$, but it is enough to assume that H is in this normal form.

Theorem 13.3.1 (Arnold's stability theorem). *The origin is stable for the system whose Hamiltonian is* (13.11)*, provided for some* k*,* $1 \le k \le N$*,* $D_{2k} = H_{2k}(\omega_2, \omega_1) \neq 0$ *or, equivalently, provided* H_2 *does not divide* H_{2k}*. In particular, the equilibrium is stable if*

$$D_4 = \frac{1}{2}\{A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2\} \neq 0. \tag{13.12}$$

Moreover, arbitrarily close to the origin in $\mathbb{R}^4$, there are invariant tori and the flow on these invariant tori is the linear flow with irrational slope.

Proof. Assume that $D_2 = \cdots = D_{2N-2} = 0$ but $D_{2N} \neq 0$; so, there exist homogeneous polynomials F_{2k}, $k = 2, \ldots, N-1$, of degree $2k$ such that $H_{2k} = H_2 F_{2k-2}$. The Hamiltonian (13.11) is then

$$H = H_2(1 + F_2 + \cdots + F_{2N-4}) + H_{2N} + H^\dagger.$$

Introduce action–angle variables $I_i = (x_i^2 + y_i^2)/2$, $\phi_i = \arctan(y_i/x_i)$, and scale the variables by $I_i = \epsilon^2 J_i$, where ϵ is a small scale variable. This is a symplectic change of coordinates with multiplier ϵ^{-2}; so, the Hamiltonian becomes

$$H = H_2 F + \epsilon^{2N-2} H_{2N} + O(\epsilon^{2N-1}),$$

where

$$F = 1 + \epsilon^2 F_2 + \cdots + \epsilon^{2N-4} F_{2N-4}.$$

Fix a bounded neighborhood of the origin, say $| J_i | \leq 4$, and call it O so that the remainder term is uniformly $O(\epsilon^{2N+1})$ in O. Restrict your attention to this neighborhood henceforth. Let h be a new parameter that will lie in the bounded interval $[-1, 1]$. Because $F = 1 + \cdots$, one has

$$H - \epsilon^{2N-1} h = KF,$$

where

$$K = H_2 + \epsilon^{2N-2} H_{2N} + O(\epsilon^{2N-1}).$$

Because $F = 1 + \cdots$ the function F is positive on O for sufficiently small ϵ so the level set when $H = \epsilon^{2N-1} h$ is the same as the level set when $K = 0$. Let $z = (J_1, J_2, \phi_1, \phi_2)$, and let ∇ be the gradient operator with respect to these variables. The equations of motion are

$$\dot{z} = J\nabla H = (J\nabla K)F + K(J\nabla F).$$

On the level set when $K = 0$, the equations become

$$\dot{z} = J\nabla H = (J\nabla K)F.$$

For small ϵ, F is positive; so, reparameterize the equation by $d\tau = F dt$, and the equation becomes

$$z' = J\nabla K(z),$$

where $' = d/d\tau$.

In summary, it has been shown that in O for small ϵ, the flow defined by H on the level set $H = \epsilon^{2N-1} h$ is a reparameterization of the flow defined by

K on the level set $K = 0$. Thus it suffices to consider the flow defined by K. To that end, the equations of motion defined by K are

$$\begin{aligned} J_i' &= O(\epsilon^{2N-1}), \\ \phi_1' &= \omega_1 - \epsilon^{2N-2}\frac{\partial H_{2N}}{\partial J_1} + O(\epsilon^{2N-1}), \\ \phi_2' &= +\omega_2 - \epsilon^{2N-2}\frac{\partial H_{2N}}{\partial J_2} + O(\epsilon^{2N-1}). \end{aligned} \tag{13.13}$$

From these equations, the Poincaré map of the section $\phi_2 \equiv 0 \bmod 2\pi$ in the level set $K = 0$ is computed, and then the invariant curve theorem can be applied.

From the last equation in (13.13), the first return time T required for ϕ_2 to increase by 2π is given by

$$T = \frac{2\pi}{\omega_2}\left(1 + \frac{\epsilon^{2N-2}}{\omega_2}\frac{\partial H_{2N}}{\partial J_2}\right) + O(\epsilon^{2N-1}).$$

Integrate the ϕ_1 equation in (13.13) from $\tau = 0$ to $\tau = T$, and let $\phi_1(0) = \phi_0$, $\phi_1(T) = \phi^*$ to get

$$\begin{aligned} \phi^* &= \phi_0 + \left(-\omega_1 - \epsilon^{2N-2}\frac{\partial H}{\partial J_1}\right) T + O(\epsilon^{2N-1}) \\ &= \phi_0 - 2\pi\left(\frac{\omega_1}{\omega_2}\right) - \epsilon^{2N-2}\left(\frac{2\pi}{\omega_2}\right)\left(\omega_2\frac{\partial H_{2N}}{\partial J_1} + \omega_1\frac{\partial H_{2N}}{\partial J_2}\right) + O(\epsilon^{2N-1}). \end{aligned} \tag{13.14}$$

In the above, the partial derivatives are evaluated at (J_1, J_2). From the relation $K = 0$, solve for J_2 to get $J_2 = (\omega_1/\omega_2)J_1 + O(\epsilon^2)$. Substitute this into (13.14) to eliminate J_2, and simplify the expression by using Euler's theorem on homogeneous polynomials to get

$$\phi^* = \phi_0 + \alpha + \epsilon^{2N-2}\beta J_1^{N-1} + O(\epsilon^{2N-1}), \tag{13.15}$$

where $\alpha = -2\pi(\omega_1/\omega_2)$ and $\beta = -2\pi(N/\omega_2^{N+1})H_{2N}(\omega_2, \omega_1)$. By assumption, $D_{2N} = H_{2N}(\omega_2, \omega_1) \neq 0$; so, $\beta \neq 0$. Along with (13.15), the equation $J_1 \to J_1 + O(\epsilon^{2N-1})$ defines an area-preserving map of an annular region, say $1/2 \leq J_1 \leq 3$ for small ϵ. By the invariant curve theorem for sufficiently small ϵ, $0 \leq \epsilon \leq \epsilon_0$, there is an invariant curve for this Poincaré map of the form $J_1 = \rho(\phi_1)$, where ρ is continuous, 2π periodic, and $1/2 \leq \rho(\phi_1, \epsilon) \leq 3$ for all ϕ_1. For all ϵ, $0 \leq \epsilon \leq \epsilon_0$, the solutions of (13.13) which start on $K = 0$ with initial condition $J_1 < 1/2$ must have J_1 remaining less than 3 for all τ. Because on $K = 0$ one has that $J_2 = (\omega_1/\omega_2)J_1 + \cdots$, a bound on J_1 implies a bound on J_2. Thus there are constants c and k such that if $J_1(\tau), J_2(\tau)$

satisfy the equations (13.13), start on $K = 0$, and satisfy $| J_i(0) | \leq c$, then $| J_i(\tau) | \leq k$ for all τ and for all $h \in [-1, 1], 0 \leq \epsilon \leq \epsilon_0$.

Going back to the original variables $(I_1, I_2, \phi_1, \phi_2)$, and the original Hamiltonian H, this means that for $0 \leq \epsilon \leq \epsilon_0$, all solutions of the equations defined by the Hamiltonian (13.11) which start on $H = \epsilon^{2N-1}h$ and satisfy $| I_i(0) | \leq \epsilon^2 c$ must satisfy $| I_i(t) | \leq \epsilon^2 k$ for all t and all $h \in [-1, 1], 0 \leq \epsilon \leq \epsilon_0$. Thus the origin is stable. The invariant curves in the section map sweep out an invariant torus under the flow.

Arnold's theorem was originally proved independent of the invariant curve theorem; see Arnold (1963a,b), and the proof given here is taken from Meyer and Schmidt (1986). Actually, in Arnold's original works the stability criterion was $AC - B^2 \neq 0$ which implies a lot of invariant tori, but is not sufficient to prove stability; see the interesting example in Bruno (1987).

The coefficients A, B, and C of Arnold's theorem for the Hamiltonian of the restricted 3-body problem were computed by Deprit and Deprit-Bartholomê (1967) specifically to apply Arnold's theorem. These coefficients were given in Section 10.5. For $0 < \mu < \mu_1, \mu \neq \mu_2, \mu_3$ they found

$$D_4 = -\frac{36 - 541\omega_1^2\omega_2^2 + 644\omega_1^4\omega_2^4}{8(1 - 4\omega_1^2\omega_2^2)(4 - 25\omega_1^2\omega_2^2)},$$

which is nonzero except for one value $\mu_c \approx 0.010, 913, 667$ which seems to have no mathematical significance (it is not a resonance value), and has no astronomical significance (it does not correspond to the earth–moon system, etc.)

In Meyer and Schmidt (1986), the normalization was carried to sixth-order using an algebraic processor, and $D_6 = P/Q$ where

$$\begin{aligned} P = &-\frac{3105}{4} + \frac{1338449}{48}\sigma - \frac{48991830}{1728}\sigma^2 + \frac{7787081027}{6912}\sigma^3 \\ &- \frac{2052731645}{1296}\sigma^4 - \frac{1629138643}{324}\sigma^5 \\ &+ \frac{1879982900}{81}\sigma^6 + \frac{368284375}{81}\sigma^7, \\ Q = &\ \omega_1\omega_2(\omega_1^2 - \omega_2^2)^5(4 - 25\sigma)^3(9 - 100\sigma), \end{aligned}$$

$$\sigma = \omega_1^2\omega_2^2,$$

From this expression one can see that $D_6 \neq 0$ when $\mu = \mu_c$ ($D_6 \approx 66.6$). So by Arnold's theorem and these calculations we have the following.

Proposition 13.3.1. *In the restricted 3-body problem the libration points* $\mathcal{L}_4$ *and* $\mathcal{L}_5$ *are stable for* $0 < \mu < \mu_1, \mu \neq \mu_2, \mu_3$.

13.4 1:2 Resonance

In this section we consider a system when the linear system is in 1:2 resonance; i.e., when the linearized system has exponents $\pm i\omega_1$ and $\pm i\omega_2$ with $\omega_1 = 2\omega_2$. Let $\omega = \omega_2$. By the discussion in Section 10.5 the normal form for the Hamiltonian is a function of I_1, I_2 and the single angle $\phi_1 + 2\phi_2$. Assume the system has been normalized through terms of degree three; i.e., assume the Hamiltonian is of the form

$$H = 2\omega I_1 - \omega I_2 + \delta I_1^{1/2} I_2 \cos\psi + H^\dagger, \tag{13.16}$$

where $\psi = \phi_1 + 2\phi_2$, $H^\dagger(I_1, I_2, \phi_1, \phi_2) = O((I_1 + I_2)^2)$. Notice this Hamiltonian is just a perturbation of Cherry's example. Lyapunov's center theorem assures the existence of one family of periodic solutions emanating from the origin, the short period family with period approximately $\pi/2\omega$.

Theorem 13.4.1. *If in the presence of* 1:2 *resonance, the Hamiltonian system is in the normal form* (13.16) *with* $\delta \neq 0$ *then the equilibrium is unstable. In fact, there is a neighborhood* O *of the equilibrium such that any solution starting in* O *and not on the Lyapunov center leaves* O *in either positive or negative time. In particular, the small periodic solutions of the short period family are unstable.*

Remark. If $\delta = 0$ then the Hamiltonian can be put into normal form to the next order and the stability of the equilibrium may be decidable on the bases of Arnold's theorem, Theorem 13.3.1.

Proof. The equations of motion are

$$\dot{I}_1 = -\delta I_1^{1/2} I_2 \sin\psi + \frac{\partial H^\dagger}{\partial \phi_1}, \qquad \dot{\phi}_1 = -2\omega - \frac{\delta}{2} I_1^{-1/2} I_2 \cos\psi - \frac{\partial H^\dagger}{\partial I_1},$$

$$\dot{I}_2 = -2\delta I_1^{1/2} I_2 \sin\psi + \frac{\partial H^\dagger}{\partial \phi_2}, \qquad \dot{\phi}_2 = \omega - \delta I_1^{1/2} \cos\psi - \frac{\partial H^\dagger}{\partial I_2}.$$

Lyapunov's center theorem ensures the existence of the short period family with period approximately $\pi/2\omega$. We may assume that this family has been transformed to the plane where $I_2 = 0$. So $\partial H^\dagger/\partial\phi_2 = 0$ when $I_2 = 0$. The Hamiltonian (13.16) is a real analytic system written in action–angle variables thus the terms in $H^\dagger$ must have the d'Alembert character; i.e., a term of the form $I_1^{\alpha/2} I_2^{\beta/2} \cos k(\phi_1 + 2\phi_2)$ must have $\beta \geq 2k$ and $\beta \equiv 2k \bmod 2$ so in particular β must be even. Thus I_2 does not appear with a fractional exponent and because $\partial H^\dagger/\partial\phi_2 = 0$ when $I_2 = 0$ this means that $\partial H^\dagger/\partial\phi_2$ contains a factor I_2. Let $\partial H^\dagger/\partial\phi_2 = I_2 U_1(I_1, I, 2, \psi)$ where $U_1 = O(I_1 + I_2)$.

Consider the Chetaev function

$$V = -\delta I_1^{1/2} I_2 \sin\psi$$

and compute

$$\dot{V} = \delta^2 \left\{ \frac{1}{2} I_2^2 + 2 I_1 I_2 \right\} + W,$$

where

$$W = -\delta \left\{ \frac{1}{2} I_1^{-1/2} I_2 \sin\psi \frac{\partial H^\dagger}{\partial \psi_1} + I_1^{1/2} \sin\psi \frac{\partial H^\dagger}{\partial \psi_2} \right.$$

$$\left. - I_1^{1/2} I_2 \cos\psi \frac{\partial H^\dagger}{\partial I_1} - 2 I_1^{1/2} I_2 \cos\psi \frac{\partial H^\dagger}{\partial I_2} \right\}$$

Because $\partial H^\dagger / \partial \phi_2 = I_2 U_1$, $W = I_2 U_2$ where $U_2 = O((I_1 + I_2)^{3/2}$ and

$$\dot{V} = \delta^2 I_2 (\frac{1}{2} I_2 + 2 I_1 + U_2).$$

Thus there is a neighborhood O where $\dot{V} > 0$ when $I_2 \neq 0$. Apply Chetaev's theorem with $\Omega = O \cap \{V > 0\}$ to conclude that all solutions which start in Ω leave O in positive time. By reversing time we can conclude that all solutions which start in $\Omega' = O \cap \{V < 0\}$ leave O in negative time.

When

$$\mu = \mu_2 = \frac{1}{2} - \frac{1}{30} \sqrt{\frac{611}{3}} \approx 0.0242939$$

the exponents of the Lagrange equilateral triangle libration point $\mathcal{L}_4$ of the restricted 3-body problem are $\pm 2\sqrt{5} i/5$, $\pm \sqrt{5} i/5$ and so the ratio of the frequencies ω_1/ω_2 is 2. Expanding the Hamiltonian about $\mathcal{L}_4$ when $\mu = \mu_2$ in a Taylor series through cubic terms gives

$$H = \tfrac{1}{14} \left\{ 5x_1^2 - 2\sqrt{611} x_1 x_2 - 25 x_2^2 - 40 x_1 y_2 + 40 x_2 y_1 + 20 y_1^2 + 20 y_2^2 \right\}$$

$$\tfrac{1}{240\sqrt{3}} \left\{ -7\sqrt{611} x_1^3 + 135 x_1^2 x_2 + 33\sqrt{611} x_1 x_2^2 + 135 x_2^3 \right\} + \cdots .$$

Using Mathematica we can put this Hamiltonian into the normal form (13.16) with

$$\omega = \frac{\sqrt{5}}{5} \approx 0.447213, \qquad \delta = \frac{11\sqrt{11}}{18\sqrt[4]{5}} \approx 1.35542,$$

and so we have the following.

Proposition 13.4.1. *The libration point $\mathcal{L}_4$ of the restricted 3-body problem is unstable when $\mu = \mu_2$.*

13.5 1:3 Resonance

In this section we consider a system when the linear system is in 1:3 resonance; i.e., $\omega_1 = 3\omega_2$. Let $\omega = \omega_2$. By the discussion in Section 10.5 the normal form for the Hamiltonian is a function of I_1, I_2 and the single angle $\phi_1 + 3\phi_2$. Assume the system has been normalized through terms of degree four; i.e., assume the Hamiltonian is of the form

$$H = 3\omega I_1 - \omega I_2 + \delta I_1^{1/2} I_2^{3/2} \cos\psi + \frac{1}{2}\{AI_1^2 + 2BI_1I_2 + CI_2^2\} + H^\dagger, \quad (13.17)$$

where $\psi = \phi_1 + 3\phi_2$, $H^\dagger = O((I_1 + I_2)^{5/2})$. Let

$$D = A + 6B + 9C, \quad (13.18)$$

and recall from Arnold's theorem the important quantity $D_4 = \frac{1}{2}D\omega^2$.

Theorem 13.5.1. *If in the presence of* 1:3 *resonance, the Hamiltonian system is in the normal form* (13.17) *and if* $6\sqrt{3}|\delta| > |D|$ *then the equilibrium is unstable, whereas, if* $6\sqrt{3}|\delta| < |D|$ *then the equilibrium is stable.*

Proof. Introduce the small parameter ϵ by scaling the variables $I_i \to \epsilon I_i$, $i = 1, 2$ which is symplectic with multiplier ϵ^{-1}, the Hamiltonian becomes

$$H = 3\omega I_1 - \omega I_2 + \epsilon\{\delta I_1^{1/2} I_2^{3/2} \cos\psi + \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2)\} + O(\epsilon^2),$$

and the equations of motion are

$$\dot{I}_1 = -\epsilon\delta I_1^{1/2} I_2^{3/2} \sin\psi + O(\epsilon^2),$$

$$\dot{I}_2 = -3\epsilon\delta I_1^{1/2} I_2^{3/2} \sin\psi + O(\epsilon^2),$$

$$\dot{\phi}_1 = -3\omega - \epsilon\{\frac{1}{2}\delta I_1^{-1/2} I_2^{3/2} \cos\psi + (AI_1 + BI_2)\} + O(\epsilon^2)$$

$$\dot{\phi}_2 = \omega - \epsilon\{\frac{3}{2}\delta I_1^{1/2} I_2^{1/2} \cos\psi + (BI_1 + CI_2)\} + O(\epsilon^2).$$

Instability. Consider the Chetaev function

$$V = -\delta I_1^{1/2} I_2^{3/2} \sin\psi$$

and compute

$$\dot{V} = \epsilon\left\{\delta^2\left(\frac{1}{2}I_2^3 + \frac{9}{2}I_1 I_2^2\right)\right. \\ \left. - \delta I_1^{1/2} I_2^{3/2}\left(AI_1 + BI_2 + 3BI_1 + 3CI_2\right)\cos\psi\right\} + O(\epsilon^2).$$

Consider the flow in the $H = 0$ surface. Solve $H = 0$ for I_2 as a function of I_1, ϕ_1, ϕ_2 to find $I_2 = 3I_1 + O(\epsilon)$. On the $H = 0$ surface we find

$$V = -3\sqrt{3}\delta I_1^2 \sin\psi + O(\epsilon)$$

and

$$\dot{V} = \epsilon\{\delta^2 I_1^3(54 - \delta^{-1}3^{3/2}(A + 6B + 9C)\cos\psi)\} + O(\epsilon^2).$$

If $54 > |\delta^{-1}3^{3/2}D|$ or $6\sqrt{3}|\delta| > |D|$ the function $\dot{V}$ is positive definite in the level set $H = 0$. Because V takes positive and negative values close to the origin in the level set $H = 0$, Chetaev's theorem implies that the equilibrium is unstable.

Stability. Now we compute the cross section map in the level set $H = \epsilon^2 h$ where $-1 \le h \le 1$ and the section is defined by $\phi_2 \equiv 0 \bmod 2\pi$. We use (I_1, ϕ_1) as coordinates in this cross-section. From the equation $H = \epsilon^2 h$ we can solve for I_2 to find that $I_2 = 3I_1 + O(\epsilon)$. Integrating the equation for ϕ_2 we find that the return time T is

$$\begin{aligned} T &= \frac{2\pi}{\omega - \epsilon\{3\sqrt{3}/2\delta\cos\psi + (B + 3C)\}I_1} + \cdots \\ &= \frac{2\pi}{\omega}\left\{1 + \frac{\epsilon}{\omega}\left(\frac{3\sqrt{3}}{2}\delta\cos\psi + (B + 3C)\right)I_1\right\} + \cdots. \end{aligned}$$

Integrating the ϕ_1 equation from $t = 0$ to $t = T$ gives the cross-section map of the form $P : (I_1, \phi_1) \to (I_1', \phi_1')$, where

$$\begin{aligned} I_1' &= I_1 + O(\epsilon), \\ \phi_1' &= \phi_1 + \frac{2\pi\epsilon}{\omega}\{(A + 6B + 9C)I - 6\sqrt{3}\delta\cos 3\phi_1\} + O(\epsilon^2). \end{aligned} \tag{13.19}$$

By hypothesis the coefficient of I_1 in (13.19) is nonzero and so Corollary 13.2.1 implies the existence of invariant curves for the section map. The stability of the equilibrium follows now by the same argument as found in the proof of Arnold's stability theorem 13.3.1.

When

$$\mu = \mu_3 = \frac{1}{2} - \frac{\sqrt{213}}{30} \approx 0.0135160$$

the exponents of the Lagrange equilateral triangle libration point $\mathcal{L}_4$ of the restricted 3-body problem are $\pm 3\sqrt{10}i/10$, $\pm\sqrt{10}i/10$ and so the ratio of the frequencies ω_1/ω_2 is 3.

Using Mathematica we can put this Hamiltonian into the normal form (13.17) with

$$\omega = \frac{\sqrt{10}}{10} \approx 0.316228, \qquad \delta = \frac{3\sqrt{14277}}{80} \approx 4.48074$$

$$A = \frac{309}{1120}, \qquad B = -\frac{1219}{560}, \qquad C = \frac{79}{560}.$$

From this we compute

$$6\sqrt{3}|\delta| \approx 46.5652 > |D| \approx 8.34107,$$

and so we have the following.

Proposition 13.5.1. *The libration point $\mathcal{L}_4$ of the restricted 3-body problem is unstable when $\mu = \mu_3$.*

That the Lagrange point $\mathcal{L}_4$ is unstable when $\mu = \mu_2, \mu_3$ was established in Markeev (1966) and Alfriend (1970, 1971). Hagel (1996) analytically and numerically studied the stability of $\mathcal{L}_4$ in the restricted problem not only at μ_2 but near μ_2 also.

13.6 1:1 Resonance

The analysis of the stability of an equilibrium in the case of 1:1 resonance is only partially complete even in the generic case. In a one-parameter problem such as the restricted 3-body problem generically an equilibrium point has exponents with multiplicity two, but in this case the matrix of the linearized system is not diagonalizable. Thus the equilibrium at $\mathcal{L}_4$ when $\mu = \mu_1$ is typical of an equilibrium in a one-parameter family. An equilibrium with exponents with higher multiplicity or an equilibrium such that the linearized system is diagonalizable is degenerate in a one-parameter family.

Consider a system in the case when the exponents of the equilibrium are $\pm i\omega$ with multiplicity two and the linearized system is not diagonalizable. The normal form for the quadratic part of such a Hamiltonian was given as

$$H_2 = \omega(x_2 y_1 - x_1 y_2) + \frac{\delta}{2}(x_1^2 + x_2^2), \tag{13.20}$$

where $\omega \neq 0$ and $\delta = \pm 1$. The linearized equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ -\delta & 0 & 0 & \omega \\ 0 & -\delta & -\omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}.$$

Recall that the normal form in this case depends on the four quantities

$$\begin{aligned} \Gamma_1 &= x_2 y_1 - x_1 y_2, & \Gamma_2 &= \frac{1}{2}(x_1^2 + x_2^2), \\ \Gamma_3 &= \tfrac{1}{2}(y_1^2 + y_2^2), & \Gamma_4 &= x_1 y_1 + x_2 y_2, \end{aligned}$$

and that $\{\Gamma_1, \Gamma_2\} = 0$ and $\{\Gamma_1, \Gamma_3\} = 0$. The system is in Sokol'skii normal form if the higher-order terms depend on the two quantities Γ_1 and Γ_3 only; that is, the Hamiltonian is of the form

$$H = \omega(x_2 y_1 - x_1 y_2) + \frac{\delta}{2}(x_1^2 + x_2^2) + \sum_{k=2}^{\infty} H_{2k}(x_2 y_1 - x_1 y_2, y_1^2 + y_2^2), \quad (13.21)$$

where here H_{2k} is a polynomial of degree k in two variables.

Consider a system which is in Sokol'skii's normal form up to order four; i.e., consider the system

$$\begin{aligned} H &= \omega(x_2 y_1 - x_1 y_2) + \frac{1}{2}\delta(x_1^2 + x_2^2) \\ &\quad + \left\{A(y_1^2 + y_2^2)^2 + B(x_2 y_1 - x_1 y_2)(y_1^2 + y_2^2) + C(x_2 y_1 - x_1 y_2)^2\right\} \\ &\quad + H^\dagger(x_1, x_2, y_1, y_2) \end{aligned} \quad (13.22)$$

where A, B, and C are constants and $H^\dagger$ is at least fifth order in its displayed arguments.

Theorem 13.6.1 (Sokol'skii's instability theorem). *If in the presence of* 1:1 *resonance the system is reduced to the form* (13.22) *with* $\delta A < 0$ *then the equilibrium is unstable. In fact, there is a neighborhood* Q *of the equilibrium such that any solution other than the equilibrium solution leaves the neighborhood in either positive or negative time.*

Proof. Introduce a small parameter ϵ by the scaling

$$\begin{aligned} x_1 &\to \epsilon^2 x_1, \qquad x_2 \to \epsilon^2 x_2, \\ y_1 &\to \epsilon y_1, \qquad y_2 \to \epsilon y_2, \end{aligned} \quad (13.23)$$

which is symplectic with multiplier ϵ^{-3} so the Hamiltonian (13.21) is

$$H = \omega(x_2 y_1 - x_1 y_2) + \epsilon\left\{\frac{\delta}{2}(x_1^2 + x_2^2) + A(y_1^2 + y_2^2)^2\right\} + O(\epsilon^2). \quad (13.24)$$

The equations of motion are

$$\begin{aligned} \dot{x}_1 &= \omega x_2 + \epsilon 4A(y_1^2 + y_2^2)y_1 + \cdots, \\ \dot{x}_2 &= -\omega x_1 + \epsilon 4A(y_1^2 + y_2^2)y_2 + \cdots, \\ \dot{y}_1 &= \omega y_2 - \epsilon\delta x_1 + \cdots, \\ \dot{y}_2 &= -\omega y_1 - \epsilon\delta x_2 + \cdots. \end{aligned}$$

Consider the Lyapunov function

$$V = \delta\Gamma_4 = \delta(x_1y_1 + x_2y_2),$$

and compute

$$\dot{V} = \epsilon\{-\delta^2(x_1^2 + x_2^2) + 4\delta A(y_1^2 + y_2^2)^2\} + O(\epsilon^2).$$

So V takes on positive and negative values and $\dot{V}$ is negative on $Q' = \{0 < x_1^2+x_2^2+y_1^2+y_2^2 < 1\}$ and for some $\epsilon = \epsilon_0 > 0$. Thus by Lyapunov's instability theorem 13.1.3 all solutions in $\{V > 0\} \cap Q'$ leave the Q' in positive time. By reversing time we see that all solutions $\{V < 0\} \cap Q'$ leave Q' in negative time.

In the original unscaled variables all solutions that start in

$$Q = \{0 < \epsilon_0^{-2}(x_1^2 + x_2^2) + \epsilon_0^{-1}(y_1^2 + y_2^2 < 1\}$$

leave Q is either positive or negative time.

The best we can say at this point in the case of 1:1 stability is formal stability.

Theorem 13.6.2 (Sokol'skii's formal stability theorem). *If in the presence of* 1:1 *resonance the system is reduced to the form* (13.22) *with* $\delta A > 0$ *then the equilibrium is formally stable. That is, the truncated normal form at any finite order has a positive definite Lyapunov function that satisfies the hypothesis of Lyapunov's stability theorem* 13.1.1.

Proof. Given any $N > 2$ the system with Hamiltonian (13.21) can be normalized by a convergent symplectic transformation up to order $2n$; i.e., the system can be transformed to

$$\begin{aligned} H = {}& \omega(x_2y_1 - x_1y_2) + \frac{\delta}{2}(x_1^2 + x_2^2) \\ & + \left\{A(y_1^2 + y_2^2)^2 + B(x_2y_1 - x_1y_2)(y_1^2 + y_2^2) + C(x_2y_1 - x_1y_2)^2\right\} \\ & + \textstyle\sum_{k=3}^{N} H_{2k}(x_2y_1 - x_1y_2, y_1^2 + y_2^2) + H^\dagger(x_1, x_2, y_1, y_2) \end{aligned} \tag{13.25}$$

where H_{2k} is a polynomial of degree k in two variables and now $H^\dagger$ is analytic and of order at least $2k + 3$. Let H^T be the truncated system obtained from the H in (13.25) by setting $H^\dagger = 0$. We claim that the system defined by H^T is stable. Because H^T depends only on Γ_1, Γ_2, and Γ_3 and $\{\Gamma_1, \Gamma_i\} = 0$ for $i = 1, 2, 3$ we see that $\{\Gamma_1, H^T\} = 0$. Thus $\Gamma_1 = x_2y_1 - x_1y_2$ is an integral for the truncated system.

Let $V = 2\delta(H^T - \omega\Gamma_1)$ so $\dot{V} = \{V, H^T\} = 0$ and scale the variables by (13.23) so that

$$V = \epsilon^4\{\delta^2(x_1^2 + x_2^2) + 2\delta A(y_1^2 + y_2^2)^2\} + O(\epsilon^5),$$

so V is positive definite. Thus by Lyapunov's stability theorem 13.1.1 the origin is a stable equilibrium point for the truncated system.

When

$$\mu = \mu_1 = \frac{1}{2}(1 - \sqrt{69}/9) \approx 0.0385209$$

the exponents of the libration point $\mathcal{L}_4$ of the restricted 3-body problem are two pair of pure imaginary numbers. Schmidt (1990) put the Hamiltonian of the restricted 3-body problem at $\mathcal{L}_4$ into the normal form (13.21) with

$$\omega = \frac{\sqrt{2}}{2}, \qquad \delta = 1, \qquad A = \frac{59}{864}.$$

The value for A agrees with the independent calculations in Niedzielska (1994) and Goździewski and Maciejewski (1998). It differs from the numeric value given in Markeev (1978). These quantities in a different coordinate system were also computed by Deprit and Henrard (1968). By these considerations and calculations we have the following.

Proposition 13.6.1. *The libration point $\mathcal{L}_4$ of the restricted 3-body problem is formally stable when $\mu = \mu_1$.*

Sokol'skii (1977) and Kovalev and Chudnenko (1977) announce that they can prove that the equilibrium is actually stable in this case. The proof in Sokol'skij (1977) is wrong and the proof in Kovalev and Chudnenko (1977) is unconvincing, typical Doklady papers! It would be interesting to give a correct proof of stability in this case, because the linearized system is not simple, and so the linearized equations are unstable.

13.7 Stability of Fixed Points

The study of the stability of a periodic solution of a Hamiltonian system of two degrees of freedom can be reduced to the study of the Poincaré map in an energy level (i.e., level surface of the Hamiltonian). We summarize some results and refer the reader to the Problems or Meyer (1971) or Cabral and Meyer (1999) for the details. The proofs for the results given below are similar to the proofs given above.

We consider diffeomorphisms of the form

$$F : N \subset \mathbb{R}^2 \to \mathbb{R}^2 : z \to f(z), \tag{13.26}$$

where N is a neighborhood of the origin in $\mathbb{R}^2$, and F is a smooth function such that

$$F(0) = 0, \qquad \det \frac{\partial F}{\partial z}(z) \equiv 1.$$

The origin is a fixed point for the diffeomorphism because $F(0) = 0$, and it is orientation-preserving and area-preserving because $\det \partial F/\partial z \equiv 1$. This map should be considered as the Poincaré map associated with a periodic solution of a two degree of freedom Hamiltonian system.

The fixed point 0 is stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|F^k(z)| \le \epsilon$ for all $k \in \mathbb{Z}$ whenever $|z| \le \delta$. The fixed point is unstable if it is not stable.

The linearization of this map about the origin is $z \to Az$ where A is the 2×2 matrix $(\partial f/\partial x)(0)$. The eigenvalues λ, λ^{-1} of A are called the multipliers of the fixed point. There are basically four cases: (i) hyperbolic fixed point with multipliers real and $\lambda \ne \pm 1$, (ii) elliptic fixed point with multipliers complex conjugates and $\lambda \ne \pm 1$, (iii) shear fixed point with $\lambda = +1$ and A is not diagonalizable, (iv) flip fixed point with $\lambda = -1$ and A is not diagonalizable.

Proposition 13.7.1. *A hyperbolic fixed point is unstable.*

In the hyperbolic case one need only to look at the linearization; in the other case one must look at higher-order terms. In the elliptic case we can change to action–angle coordinates (I, ϕ) so that the map $F : (I, \phi) \to (I', \phi')$ is in normal form up to some order. In the elliptic case the multipliers are complex numbers of the form $\lambda^{\pm 1} = \exp \pm \omega i \ne \pm 1$.

Proposition 13.7.2. *If* $\lambda^{\pm 1} = \exp \pm 2\pi/3$ *(the multipliers are cube roots of unity) the normal form begins*

$$I' = I + 2\alpha I^{3/2} \sin(3\phi) + \cdots, \phi' = \phi \pm (\pi/3) + \alpha I^{1/2} \cos(3\phi) + \cdots.$$

If $\alpha \ne 0$ *the fixed point is instable.*

If $\lambda^{\pm 1} = \pm i$ *(the multipliers are fourth roots of unity) the normal form begins*

$$I' = I + 2\alpha I^2 \sin(4\phi) + \cdots, \phi' = \phi \pm \pi/2 + \{\alpha \cos(4\phi) + \beta\} I + \cdots.$$

If $\alpha > \beta$ *the fixed point is unstable, but if* $\alpha < \beta$ *the fixed point is stable.*

If λ *is not a cube or fourth root of unity then the normal form begins*

$$I' = I + \cdots, \phi' = \phi \pm \omega + \beta I + \cdots.$$

If $\beta \ne 0$ *then the fixed point is stable.*

Proposition 13.7.3. *For a shear fixed point the multipliers are both* $+1$ *and* A *is not diagonalizable. The first few terms of the normal form* $F : (u, v) \to (u', v')$ *are*

$$u' = u \pm v - \cdots, \quad v' = v - \beta u^2 + \cdots.$$

If $\beta \ne 0$ *then the fixed point is unstable.*

Proposition 13.7.4. *For a flip fixed point the multipliers are both* -1 *and* A *is not diagonalizable. The first few terms of the normal form* $F : (u, v) \to (u', v')$ *are*

$$u' = -u - v + \cdots, \quad v' = -v + \beta u^3 + \cdots.$$

If $\beta > 0$ *the fixed point is stable and if* $\beta < 0$ *the fixed point is unstable.*

13.8 Applications to the Restricted Problem

In Chapter 9, a small parameter was introduced into the restricted problem in three ways. First the small parameter was the mass ratio parameter μ; second the small parameter section was a distance to a primary; and third the small parameter was the reciprocal of the distance to the primaries.

In all three cases an application of the invariant curve theorem can be made. Only the first and third are given here, inasmuch as the computations are easy in these cases.

13.8.1 Invariant Curves for Small Mass

The Hamiltonian of the restricted problem (2.29) for small μ is

$$H = \frac{\|y\|^2}{2} - x^T K y - \frac{1}{\|x\|} + O(\mu).$$

For $\mu = 0$ this is the Hamiltonian of the Kepler problem in rotating coordinates. Be careful that the $O(\mu)$ term has a singularity at the primaries. When $\mu = 0$ and Delaunay coordinates are used, this Hamiltonian becomes

$$H = -\frac{1}{2L^3} - G$$

and the equations of motion become

$$\dot{\ell} = 1/L^3, \qquad \dot{L} = 0,$$

$$\dot{g} = -1, \qquad \dot{G} = 0.$$

The variable g, the argument of the perihelion, is an angular variable. $\dot{g} = -1$ implies that g is steadily decreasing from 0 to -2π and so $g \equiv 0 \bmod 2\pi$ defines a cross-section. The first return time is 2π. Let ℓ, L be coordinates in the intersection of the cross-section $g \equiv 0$ and the level set $H =$ constant. The Poincaré map in these coordinates is

$$\ell' = \ell + 2\pi/L^3, \qquad L' = L.$$

Thus when $\mu = 0$ the Poincaré map in the level set is a twist map. By the invariant curve theorem some of these invariant curves persist for small μ.

13.8.2 The Stability of Comet Orbits

Consider the Hamiltonian of the restricted problem scaled as was done in Section 9.5 in the discussion of comet orbits; i.e., the Hamiltonian 9.7. In Poincaré variables it is

$$H = -P_1 + \frac{1}{2}(Q_2^2 + P_2^2) - \epsilon^3 \frac{1}{2P_1^2} + O(\epsilon^5),$$

where Q_1 is an angle defined modulo 2π, P_1 is a radial variable, and Q_1, P_1 are rectangular variables. For typographical reasons drop, but don't forget, the $O(\epsilon^5)$. The equations of motion are

$$\dot{Q}_1 = -1 + \epsilon^3/P_1^3, \qquad \dot{P}_1 = 0,$$

$$\dot{Q}_2 = P_2, \qquad \dot{P}_2 = -Q_2.$$

The circular solutions are $Q_2 = P_2 = 0 + O(\epsilon^5)$ in these coordinates. Translate the coordinates so that the circular orbits are exactly $Q_2 = P_2 = 0$; this does not affect the displayed terms in the equations. The solutions of the above equations are

$$Q_1(t) = Q_{10} + t(-1 + \epsilon^3/P_1^3), \qquad P_1(t) = P_{10},$$

$$Q_2(t) = Q_{20}\cos t + P_{20}\sin t, \qquad P_2(t) = -Q_{20}\sin t + P_{20}\cos t.$$

Work near $P_1 = 1, Q_2 = P_2 = 0$ for ϵ small. The time for Q_1 to increase by 2π is

$$T = 2\pi / \mid -1 + \epsilon^3/P_1^3 \mid = 2\pi(1 + \epsilon^3 P_1^{-3}) + O(\epsilon^6).$$

Thus

$$\begin{aligned} Q' = Q_2(T) &= Q\cos 2\pi(1 + \epsilon^3 P_1^{-3}) + P\sin 2\pi(1 + \epsilon^3 P_1^{-3}) \\ &= Q + \nu P P_1^{-3} + O(\nu^2), \end{aligned}$$

$$\begin{aligned} P' = P_2(T) &= -Q\sin 2\pi(1 + \epsilon^3 P_1^{-3}) + P\cos 2\pi(1 + \epsilon^3 P_1^{-3}) \\ &= -\nu Q P_1^{-3} + P + O(\nu^2), \end{aligned}$$

where $Q = Q_{20}, P = P_{20}$, and $\nu = 2\pi\epsilon^3$. Let $H = 1$, and solve for P_1 to get

$$P_1 = -1 + \frac{1}{2}(Q^2 + P^2) + O(\nu),$$

and hence

$$P_1^{-3} = -1 - \frac{3}{2}(Q^2 + P^2) + O(\nu),$$

Substitute this back to get

$$Q' = Q + \nu P(-1 - \tfrac{3}{2}(Q^2 + P^2)) + O(\nu^2)$$

$$P' = P - \nu Q(-1 - \tfrac{3}{2}(Q^2 + P^2)) + O(\nu^2).$$

This is the section map in the energy surface $H = 1$. Change to action–angle variables, $I = (Q^2 + P^2)/2$, $\phi = \tan^{-1}(P/Q)$, to get

$$I' = I + O(\nu^2), \qquad \phi' = \phi + \nu(-1 - 3I) + O(\nu^2).$$

This is a twist map. Thus the continuation of the circular orbits into the restricted problem is stable.

Problems

1. Let F be a diffeomorphism defined in a neighborhood O of the origin in $\mathbb{R}^m$, and let the origin be a fixed point for F. Let V be a smooth real-valued function defined on O, and define $\Delta V(x) = V(F(x)) - V(x)$.
   a) Prove that if the origin is a minimum for V and $\Delta V(x) \le 0$ on O, then the origin is a stable fixed point.
   b) Prove that if the origin is a minimum for V and $\Delta V(x) < 0$ on $O\backslash\{0\}$, then the origin is an asymptotically stable fixed point.
   c) State and prove the analog of Chetaev's theorem.
   d) State and prove the analog of Lyapunov's instability theorem.
2. Let $F(x) = Ax$ and $V(x) = x^T Sx$, where A and S are $n \times n$ matrices, and S is symmetric.
   a) Show that $\Delta V(x) = x^T Rx$, where $R = A^T SA - S$.
   b) Let $\mathcal{S}$ be the linear space on all $m \times m$ symmetric matrices and $\mathcal{L} = \mathcal{L}_A : \mathcal{S} \to \mathcal{S}$ be the linear map $\mathcal{L}(S) = A^T SA - S$. Show that $\mathcal{L}$ is invertible if and only if $\lambda_i \lambda_j \ne 1$ for all $i, j = 1, \dots, m$, where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A. (Hint: First prove the result when $A = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$. Then prove the result when $A = D + \epsilon N$, where D is simple (diagonalizable), and N is nilpotent, $N^m = 0$, $SN = NS$, and ϵ is small. Use the Jordan canonical form theorem to show that A can be assumed to be $A = D + \epsilon N$.)
   c) Let A have all eigenvalues with absolute value less than 1. Show that $S = \sum_0^\infty (A^T)^i RA^i$ converges for any fixed R. Show S is symmetric if R is symmetric. Show S is positive definite if R is positive definite. Show that $\mathcal{L}(S) = -R$; so, $\mathcal{L}^{-1}$ has a specific formula when all the eigenvalues of A have absolute value less than 1.
3. Let $F(x) = Ax + f(x)$, where $f(0) = \partial f(0)/\partial x = 0$.
   a) Show that if all the eigenvalues of A have absolute value less than 1, then the origin is asymptotically stable. (Hint: Use Problems 1 and 2.)
   b) Show that if A has one eigenvalue with absolute value greater than 1 then the origin is a positively unstable fixed point.
4. Let $r = 1, s = 0$, and $h(I) = \beta I$, $\beta \ne 0$ in formulas of the invariant curve theorem.

a) Compute F^q, the qth iterate of F, to be of the form $(I, \phi) \to (I", \phi")$ where
$$I'' = I + O(\epsilon), \qquad \phi'' = \phi + q\omega + q\beta I + O(\epsilon).$$
b) Let $2\pi p/q$ be any number between $\omega + \beta a$ and $\omega + \beta b$, so $2\pi p/q = \omega + \beta I_0$ where $a < I_0 <$ b. Show that there is a smooth curve $\Gamma_\epsilon = \{(I, \phi) : I = \Phi(\phi, \epsilon) = I_0 + \cdots\}$ such that F^q moves points on Γ only in the radial direction; i.e., $\Phi(\phi)$ satisfies $\phi'' - \phi - 2\pi p = 0$. (Hint: Use the implicit function theorem.)
c) Show that because F^q is area-preserving, $\Gamma \cap F^q(\Gamma)$ is nonempty, and the points of this intersection are fixed points of F^q or q-periodic points of F.

5. Consider the forced Duffing's equation with Hamiltonian
$$H = \frac{1}{2}(q^2 + p^2) + \frac{\gamma}{4}q^4 + \gamma^2 \cos \omega t,$$
where ω is a constant and $\gamma \neq 0$ is considered as a small parameter. This Hamiltonian is periodic with period $2\pi/\omega$ for small ϵ. If $\omega \neq 1, 2, 3, 4$, the system has a small (order γ^2) $2\pi/\omega$ periodic solution, called the harmonic. The calculations in Section 10.3 show the period map was shown to be
$$I' = I + O(\gamma),$$
$$\phi' = \phi - 2\pi/\omega - (3\pi\gamma/2\omega)I + O(\gamma^2),$$
where the fixed point corresponding to the harmonic has been moved to the origin. Show that the harmonic is stable.
6. Using Poincaré elements show that the continuation of the circular orbits established in Section 6.2 (Poincaré orbits) are of twist type and hence stable.
7. Consider the various types of fixed points discussed in Section 11.1 and prove the propositions in 13.7. That is:
a) Show that extremal points are unstable.
b) Show that 3-bifurcation points are unstable.
c) Show that k-bifurcation points are stable if $k \geq 5$.
d) Transitional and 4-bifurcation points can be stable or unstable depending on the case. Figure out which case is unstable. (The stability conditions are a little harder.) See Meyer (1971) or Cabral and Meyer (1999).