Chapter 9
Analysis of an M/M/*c*/N Queueing System with Balking, Reneging, and Synchronous Vacations

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Abstract In this chapter, we present an analysis for an M/M/*c*/N queueing system with simultaneous balking, reneging, and synchronous vacations of servers. By using the blocked matrix method, we obtain the steady-state probability vector presented by the inverses of two matrices. The computing of the inverses of the two matrices is discussed. Then, we calculate the steady-state probabilities by using the elements of the inverses of the two matrices. We also derive the conditional stationary distribution of the queue length and waiting time.

9.1 Introduction

Many practical queueing systems, especially those with balking and reneging, have been widely applied to many real-life problems such as situations involving impatient telephone switchboard customers, hospital emergency rooms' handling of critical patients, and perishable goods storage inventory systems. Balking and reneging are not only common phenomena in queues arising in daily activities, but also in telecommunication networks and in various machine repair models.

Ke [1] gave an example of the occurrence of balking in the operational model of WWW servers. An interesting example of the occurrence of balking and reneging in air defense systems was given in Ancker and Gafarian [2]. For other examples of articles that address queueing systems which use balking and reneging, interested readers may refer to [1]– [3], and the references therein.

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Haghighi, Medhi, and Mohanty [4] derived the steady-state probabilities for multiserver M/M/*c* queues with balking and reneging. Abou-El-Ata and Hariri [5] analyzed multiserver M/M/*c*/N queues where balking and reneging were applied and derived the steady-state probabilities. Wang and Chang [6] extended this work to study an M/M/*c*/N queue with balking, reneging, and server breakdowns. They derived the steady-state probabilities in matrix form and developed a cost model to determine the optimal number of servers.

In many real-world queueing systems, servers may become unavailable for a random period of time when there are no customers waiting in line at a service completion instant. This random period of server absence, often called a server vacation, can represent the time when the server is performing some secondary task. Single-server queueing models with vacations have been studied by many researchers and have been found to be applicable in analyzing numerous real-world queueing situations, such as flexible manufacturing systems, service systems, and telecommunication systems. Several excellent surveys on these vacation models have been done by Doshi [7], [8] and Takagi [9].

Multiple-server vacation models are more flexible and applicable in practice than their single-server counterparts. However, there are only a few studies on multiple-server vacation models in the vacation model literature due to the complexity of the systems. The M/M/*c* queue with exponentially distributed vacations was first studied by Levy and Yechiali [10]. In the system of [10], all the servers take a vacation together when the system is completely empty. Because all these servers take vacations simultaneously, these vacations are called "synchronous vacations".

Tian, Li, and Cao [11] modeled the M/M/*c* vacation systems of [10] as a quasi birth-and-death (QBD) process, and presented a more detailed analysis. They proved several conditional stochastic decomposition results for the queue length and the customer waiting time. Recently, Zhang and Tian [12] extended the model presented in [11] by studying an M/M/*c* queueing system with synchronous vacations of partial servers. In the system of [12], some servers take vacations when they become idle and other servers are always available for serving arriving customers. They call this type of model the "partial server vacation model".

It may be remarked here that all the studies on multiple-server vacation models mentioned above assume availability of infinite buffer space in front of the servers. However, finite buffer queues are more common in certain practical applications. Yue, Yue, and Sun [13] considered the balking and reneging phenomena in a finite buffer M/M/*c*/N queueing system with the same vacation policy as in [12]. They obtained the steady-state probability vector presented by the inverses of three matrices. However, they did not obtain the explicit expressions for the inverses of these three matrices.

In this chapter, we consider a special case of the partial-server vacation model in [13]. We study a finite buffer M/M/*c*/N queueing system with balking, reneging, and the same synchronous vacation policy as in [11]. The Markov chain underlying the queueing system in this chapter is a level-dependent quasi birth-and-death (LDQBD) process. The matrix-geometric solution method applied in [11] and [12] cannot be used to obtain the stationary probabilities of the system in this chapter. The prevailing method applied to obtain the stationary probabilities of a LDQBD process is to develop some approximations to diminish the level dependence at higher levels. However, in this chapter, we present a different approach to obtain the stationary probabilities of the system.

The rest of this chapter is organized as follows. In Sect. 9.2, we give a description of the queueing model. In Sect. 9.3, we derive the steady-state equations and obtain the steady-state probability vector presented by the inverses of two matrices with the blocked matrix method. We also discuss the computing of the inverses of the two matrices. Then, we calculate the steady-state probabilities by using the elements of the inverses of the two matrices. In Sect. 9.4, we derive the conditional stationary distribution of the queue length and waiting time. Conclusions are given in Sect. 9.5.

9.2 System Model

In this chapter, we consider a finite buffer M/M/c/N queueing system with balking, reneging, and synchronous vacations in all servers. The system capacity is finite N. The assumptions of the system model are as follows:

(1) Customers arrive according to a Poisson process with arrival rate λ. There are c servers in the system. The service time for each server is assumed to be distributed according to an exponential distribution with service rate μ.
(2) If some servers are busy, and some servers are idle, then a customer who on arrival joins the system will be serviced immediately. If all servers are either busy or taking a vacation, then a customer who on arrival finds n customers in the system, either decides to enter the queue with probability b_n or balks with probability $1-b_n$, $0 \leq b_{n+1} \leq b_n < 1$, $0 \leq n \leq N-1$, $b_N = 0$.
(3) All servers take synchronous vacations when the system is completely empty at a service completion instant. At a vacation completion instant, if the system is still empty, all the servers take another vacation together; otherwise, they return to serve the queue. The vacation time is assumed to be exponentially distributed with mean $1/\eta$.
(4) After joining the queue, in the case where all the servers are occupied each customer will wait a certain length of time T_r for service to begin before he gets impatient and leaves the queue without receiving service. This time T_r is assumed to be distributed according to an exponential distribution with mean $1/\alpha$.
(5) The service order is assumed to be on a First-Come First-Served (FCFS) basis and the interarrival times, service times, and vacations are mutually independent.

9.3 Steady-State Probability

In this section, we first develop steady-state probability equations by using the Markov process. Then, we derive the steady-state probabilities by using the blocked matrix method.

9.3.1 Steady-State Equations

Let $L(t)$ be the number of customers in the system at time t and let

$$J(t) = \begin{cases} 0, & \text{servers are on vacation at time } t \\ 1, & \text{servers are not on vacation at time } t. \end{cases}$$

Then, $\{L(t), J(t)\}$ is a Markov process with state space:

$$\Omega = \{(i,0) : i = 0,1,\ldots,N\} \cup \{(i,1) : i = 1,2,\ldots,N\}.$$

The steady-state probabilities of the system are defined as follows:

$$P_0(n) = \lim_{t\to\infty} P\{L(t) = n, J(t) = 0\}, \qquad n = 0,1,\ldots,N,$$
$$P_1(n) = \lim_{t\to\infty} P\{L(t) = n, J(t) = 1\}, \qquad n = 1,2,\ldots,N.$$

By applying the Markov process theory, we can obtain the following set of steady-state probability equations:

$$\begin{aligned}
s_1 P_1(1) + v_1 P_0(1) &= u_0 P_0(0), \\
u_{n-1} P_0(n-1) + v_{n+1} P_0(n+1) &= w_n P_0(n), \qquad n = 1,2,\ldots,N-1, \\
u_{N-1} P_0(N-1) &= w_N P_0(N), \\
\eta P_0(1) + s_2 P_1(2) &= (s_1 + t_1) P_1(1), \\
\eta P_0(n) + t_{n-1} P_1(n-1) + s_{n+1} P_1(n+1) &= (s_n + t_n) P_1(n), \qquad n = 2,3,\ldots,N-1, \\
\eta P_0(N) + t_{N-1} P_1(N-1) &= s_N P_1(N), \\
\sum_{n=0}^{N} P_0(n) + \sum_{n=1}^{N} P_1(n) &= 1,
\end{aligned}$$

where

$$u_i = \lambda b_i, \qquad i = 0,1,\ldots,N-1,$$

$$v_i = i\alpha, \qquad i = 1,2,\ldots,N,$$

$$w_i = \begin{cases} v_i + \eta + u_i, & i = 1,2,\ldots,N-1 \\ \eta + v_N, & i = N, \end{cases}$$

$$s_i = \begin{cases} i\mu, & i = 1,2,\ldots,c \\ c\mu + (i-c)\alpha, & i = c+1, c+2,\ldots,N, \end{cases}$$

$$t_i = \begin{cases} \lambda, & i = 1,2,\ldots,c-1 \\ \lambda b_i, & i = c, c+1,\ldots,N-1. \end{cases}$$

9.3.2 Matrix Solution

In the following, we derive the steady-state probabilities by using the blocked matrix method. Let

$$\mathbf{P} = (P_0(0), P_0(1),\ldots,P_0(N), P_1(1), P_1(2),\ldots,P_1(N))$$

be the steady-state probability vector. Then, the steady-state probability equations above can be rewritten in matrix form as follows:

$$\begin{cases} \mathbf{PQ} = 0 \\ \mathbf{Pe} = 1, \end{cases} \tag{9.1}$$

where $\mathbf{e} = (1,1,\ldots,1)^T$ is a $(2N+1)\times 1$ vector, and the transition rate matrix $\mathbf{Q}$ of the Markov process has the blocked matrix structure:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix}.$$

Each matrix $\mathbf{Q}_{lk}$ $(l,k = 1,2,3)$ is given as follows:

$$\mathbf{Q}_{11} = (-u_0, v_1, 0,\ldots,0)^T, \qquad \mathbf{Q}_{31} = (s_1, 0,\ldots,0)^T,$$
$$\mathbf{Q}_{22} = (0,0,\ldots,v_N,-w_N), \qquad \mathbf{Q}_{23} = (0,0,\ldots,0,\eta),$$

$$\mathbf{Q}_{12} = \begin{pmatrix} u_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -w_1 & u_1 & 0 & \cdots & 0 & 0 & 0 \\ v_2 & -w_2 & u_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -w_{N-2} & u_{N-2} & 0 \\ 0 & 0 & 0 & \cdots & v_{N-1} & -w_{N-1} & u_{N-1} \end{pmatrix},$$

$$\mathbf{Q}_{13} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \eta & 0 & 0 & \cdots & 0 & 0 \\ 0 & \eta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \eta & 0 \end{pmatrix},$$

$$\mathbf{Q}_{33} = \begin{pmatrix} -(s_1+t_1) & t_1 & 0 & \cdots & 0 & 0 & 0 \\ s_2 & -(s_2+t_2) & t_2 & \cdots & 0 & 0 & 0 \\ 0 & s_3 & -(s_3+t_3) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{N-1} & -(s_{N-1}+t_{N-1}) & t_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & s_N & -s_N \end{pmatrix},$$

where $\mathbf{Q}_{11}$ and $\mathbf{Q}_{31}$ are $N \times 1$ vectors, $\mathbf{Q}_{12}$, $\mathbf{Q}_{13}$, and $\mathbf{Q}_{33}$ are $N \times N$ matrices, $\mathbf{Q}_{22}$ and $\mathbf{Q}_{23}$ are $1 \times N$ vectors, $\mathbf{Q}_{21} = 0$ is a constant, and $\mathbf{Q}_{32} = \mathbf{0}$ is an $N \times N$ matrix.

The four submatrices $\mathbf{Q}_{11}$, $\mathbf{Q}_{12}$, $\mathbf{Q}_{21}$, and $\mathbf{Q}_{22}$ give the transition rates during the vacation period. For example, the submatrix $\mathbf{Q}_{12}$ gives the transition rates from vacation state $(0,i)$ to vacation state $(0,j)$, $i = 0,1,\ldots,N-1$, $j = 1,2,\ldots,N$. The two submatrices $\mathbf{Q}_{13}$ and $\mathbf{Q}_{23}$ give the transition rates from a vacation state to a busy state. For example, the submatrix $\mathbf{Q}_{13}$ gives the transition rates from vacation state $(0,i)$ to busy state $(1,j)$, $i = 0,1,\ldots,N-1$, $j = 1,2,\ldots,N$. The two submatrices $\mathbf{Q}_{31}$ and $\mathbf{Q}_{32}$ give the transition rates from a busy state to a vacation state. The submatrix $\mathbf{Q}_{33}$ gives the transition rates during the busy period.

In order to solve (9.1) by using the blocked matrix method, we consider computing the inverses of the matrices $\mathbf{Q}_{12}$ and $\mathbf{Q}_{33}$.

Let c_{ij} be the (ij) element of the inverse matrix $\mathbf{Q}_{12}^{-1}$, $i,\ j = 1,2,\ldots,N$. Let d_{ij} be the (ij) element of the inverse matrix $\mathbf{Q}_{33}^{-1}$, $i,\ j = 1,2,\ldots,N$. We then have the following lemmas.

Lemma 9.1. *The matrix $\mathbf{Q}_{12}$ is invertible. For $j = 1,2,\ldots,N$, the elements of the inverse matrix $\mathbf{Q}_{12}^{-1}$ are given by*

$$c_{ij} = \begin{cases} 0, & i = 1,2,\ldots,j-1 \\ \dfrac{1}{u_{j-1}}, & i = j \\ k_{ij}\dfrac{1}{u_{j-1}}, & i = j+1, j+2, \ldots, N, \end{cases} \tag{9.2}$$

where k_{ij} is given by the following recursive relations

$$k_{ij} = \frac{w_{i-1}}{u_{i-1}} k_{i-1j} - \frac{v_{i-1}}{u_{i-1}} k_{i-2j}, \qquad i = j+1, j+2, \ldots, N, \tag{9.3}$$

where $k_{jj} = 1$ and $k_{j-1j} = 0$.

Proof. See Appendix. □

Remark 1. For the special case where $\alpha = 0$ (i.e., no reneging occurs in the system) the closed-form expression for the $\mathbf{Q}_{12}^{-1}$ can be obtained from Lemma 9.1. Let $\alpha = 0$ in Lemma 9.1; then we have the following recursive relation:

$$k_{ij} = \frac{w_{i-1}}{u_{i-1}} k_{i-1j}, \qquad i = j+1, j+2, \ldots, N.$$

Hence, we get the closed-form expression for k_{ij} as follows:

$$k_{ij} = \frac{w_{i-1}w_{i-2}\cdots w_j}{u_{i-1}u_{i-2}\cdots u_j}, \qquad i = j+1, j+2, \ldots, N.$$

Lemma 9.2. *The matrix $\mathbf{Q}_{33}$ is invertible. For $j = 1,2,\ldots,N$, the elements of the inverse matrix $\mathbf{Q}_{33}^{-1}$ are given by*

$$d_{ij} = \begin{cases} -\displaystyle\sum_{k=1}^{i} \frac{t_k t_{k+1}\ldots t_{j-1}}{s_k s_{k+1}\ldots s_j}, & i = 1,2,\ldots,j-1 \\ -\displaystyle\sum_{k=1}^{j-1} \frac{t_k t_{k+1}\ldots t_{j-1}}{s_k s_{k+1}\cdots s_j} - \frac{1}{s_j}, & i = j, j+1, \ldots, N. \end{cases} \tag{9.4}$$

The empty summation $\sum_{k=1}^{0}$ is defined to be zero.

Proof. See Appendix. □

In the following, we derive the steady-state probabilities from (9.1). To accommodate the partitioned blocked structure of $\mathbf{Q}$, we partition the steady-state probability vector into segments accordingly as follows:

$$\mathbf{P} = (\mathbf{P}_0, P_0(N), \mathbf{P}_1),$$

where

$$\mathbf{P}_0 = (P_0(0), P_0(1), \ldots, P_0(N-1)),$$

$$\mathbf{P}_1 = (P_1(1), P_1(2), \ldots, P_1(N)).$$

Theorem 9.1. *The segments of the steady-state probability vector are given by*

$$\mathbf{P}_0 = -P_0(N)\mathbf{Q}_{22}\mathbf{Q}_{12}^{-1}, \tag{9.5}$$

$$\mathbf{P}_1 = -P_0(N)(\mathbf{Q}_{23} - \mathbf{Q}_{22}\mathbf{Q}_{12}^{-1}\mathbf{Q}_{13})\mathbf{Q}_{33}^{-1}, \tag{9.6}$$

where

$$P_0(N) = \left\{1 - \mathbf{Q}_{22}\mathbf{Q}_{12}^{-1}\mathbf{e}_N - (\mathbf{Q}_{23} - \mathbf{Q}_{22}\mathbf{Q}_{12}^{-1}\mathbf{Q}_{13})\mathbf{Q}_{33}^{-1}\mathbf{e}_N\right\}^{-1} \tag{9.7}$$

and $\mathbf{e}_N = (1,1,\dots,1)^T$ *is an* $N \times 1$ *vector.*

Proof. Based on the partitions of the vector $\mathbf{P}$, (9.1) can be rewritten as

$$\mathbf{P}_0\mathbf{Q}_{11} + \mathbf{P}_1\mathbf{Q}_{31} = 0, \tag{9.8}$$

$$\mathbf{P}_0\mathbf{Q}_{12} + P_0(N)\mathbf{Q}_{22} = 0, \tag{9.9}$$

$$\mathbf{P}_0\mathbf{Q}_{13} + P_0(N)\mathbf{Q}_{23} + \mathbf{P}_1\mathbf{Q}_{33} = 0, \tag{9.10}$$

$$\mathbf{P}_0\mathbf{e}_N + P_0(N) + \mathbf{P}_1\mathbf{e}_N = 1. \tag{9.11}$$

From Lemma 9.1 and (9.9), we have

$$\mathbf{P}_0 = -P_0(N)\mathbf{Q}_{22}\mathbf{Q}_{12}^{-1}. \tag{9.12}$$

Substituting (9.12) into (9.10), from Lemma 9.2, we have

$$\mathbf{P}_1 = -P_0(N)(\mathbf{Q}_{23} - \mathbf{Q}_{22}\mathbf{Q}_{12}^{-1}\mathbf{Q}_{13})\mathbf{Q}_{33}^{-1}, \tag{9.13}$$

where $P_0(N)$ can be obtained as the expression given in (9.7) by substituting (9.12) and (9.13) into (9.11). This completes the proof of Theorem 9.1. □

Theorem 9.2. *The steady-state probabilities are given by*

$$P_0(j) = \frac{-\beta_{j+1}}{\triangle}, \qquad j = 0,1,\dots,N-1, \tag{9.14}$$

$$P_0(N) = -\frac{1}{\triangle}, \tag{9.15}$$

$$P_1(j) = -\frac{\eta}{\triangle}\left(d_{Nj} - \sum_{i=1}^{N-1} d_{ij}\beta_{i+1}\right), \qquad j = 1,2,\dots,N, \tag{9.16}$$

where

$$\Delta = 1 - \sum_{j=1}^{N}\beta_j - \eta\sum_{j=1}^{N}\left(d_{Nj} - \sum_{i=1}^{N-1} d_{ij}\beta_{i+1}\right), \tag{9.17}$$

$$\beta_j = \begin{cases} v_N c_{N-1j} - w_N c_{Nj}, & j = 1,2,\ldots,N-1 \\ -w_N c_{NN}, & j = N, \end{cases} \tag{9.18}$$

c_{ij} and d_{ij} are given by Lemma 9.1 and Lemma 9.2.

Proof. Define

$$\mathbf{Q}_{22}\mathbf{Q}_{12}^{-1} = (\beta_1, \beta_2, \ldots, \beta_N). \tag{9.19}$$

Then, from Lemma 9.1, β_j $(j = 1,2,\ldots,N)$ can be obtained as the expression given in (9.18). Note that

$$\mathbf{Q}_{22}\mathbf{Q}_{12}^{-1}\mathbf{Q}_{13} = \eta(\beta_2, \beta_3, \ldots, \beta_N, 0);$$

we have

$$\mathbf{Q}_{23} - \mathbf{Q}_{22}\mathbf{Q}_{12}^{-1}\mathbf{Q}_{13} = -\eta(\beta_2, \beta_3, \ldots, \beta_N, -1). \tag{9.20}$$

Then, from Theorem 9.1, (9.19) and (9.20), we can derive (9.14)–(9.17). This completes the proof of Theorem 9.2. □

9.3.3 Some Special Cases

In the following, we present some special cases of our model. Some of them are existing models in the literature.

(1) If $\eta = \infty$ (i.e., the servers do not take vacations) and

$$b_n = \begin{cases} 1, & 0 \le n \le c \\ \dfrac{\beta\left(1 - \dfrac{1}{N}(n-c+1)\right)}{(n-c+2)^m}, & c \le n \le N, \end{cases}$$

then our model becomes the model studied by Abou-El-Ata and Hariri [5]: M/M/c/N queue with balking and reneging.

(2) If $\alpha = 0$ (i.e., customers do not renege), then our model becomes the model M/M/c/N queue with balking and synchronous vacation of all servers.

(3) If $N = \infty$, $\alpha = 0$, and $b_i = 1$, $i = 0, 1, \ldots$ (i.e., customers do not balk or renege), then our model becomes the model studied by Tian et al. [11]: M/M/c/∞ queue with synchronous vacation of all servers.

(4) If $c = 1$, then our model becomes the model studied by Yue et al. [14]: M/M/1/N queue with balking, reneging, and multiple vacations.

9.4 Conditional Distributions of Queue Length and Waiting Time

In an M/M/c multiple-server vacation system, Tian et al. [11] investigated the conditional stationary distribution of the queue length and waiting time under the condition when all servers are busy. They presented in such a system a conditional stochastic decomposition property for steady-state queue length and waiting time. In this section, we derive the conditional stationary distribution of the queue length and waiting time for the system studied in this chapter.

Let $P(Q_c = j)$, $j = 0, 1, \ldots, N-c$, represent the conditional stationary distribution of the queue length given that all servers are busy. It is given in the following theorem.

Theorem 9.3. *The conditional stationary distribution of the queue length is given by*

$$P(Q_c = j) = \frac{d_{Nj+c} - \sum_{i=1}^{N-1} d_{ij+c}\beta_{i+1}}{\sum_{j=c}^{N}\left(d_{Nj} - \sum_{i=1}^{N-1} d_{ij}\beta_{i+1}\right)}, \qquad j = 0, 1, \ldots, N-c, \tag{9.21}$$

where d_{ij} and β_j are given in Lemma 9.2 and (9.18), respectively.

Proof. From Theorem 9.2, the probability that all servers are busy is

$$\sum_{j=c}^{N} P_1(j) = -\frac{\eta}{\Delta}\sum_{j=c}^{N}\left(d_{Nj} - \sum_{i=1}^{N-1} d_{ij}\beta_{i+1}\right). \tag{9.22}$$

Note that

$$P(Q_c = j) = \frac{P_1(j+c)}{\sum_{j=c}^{N} P_1(j)}, \qquad j = 0, 1, \ldots, N-c \tag{9.23}$$

and substituting the probability given in (9.22) and the probability $P_1(j)$ given by Theorem 9.2 into (9.23), we can get the conditional distribution of the queue length given by (9.21). □

In the following, we consider the conditional distribution of the waiting time under the condition that all servers are busy when a customer on arrival joins the queue.

Let B_j represent the event that there are j customers in front of the new customer who on arrival joins the queue, and all the servers are busy. Under the assumption B_j, the c customers are in service and the other $j-c$ customers are waiting for service. Let T_j be the time remaining until the number of customers j diminishes by $j-1$ because of the completion of a customer's service or a customer's reneging, $j = c$,

$c+1, \ldots, N-1$. Because both the service time and the waiting time of a customer before he reneges are exponentially distributed, T_j is exponentially distributed with the distribution function given by

$$H_j(t) = 1 - e^{-\theta_j t}, \qquad t \geq 0, \qquad j = c, c+1, \ldots, N-1 \tag{9.24}$$

and the Laplace-Stieltjes transformation (LST) given by

$$H_j^*(s) = \frac{\theta_j}{\theta_j + s}, \quad s \geq 0, \qquad j = c, c+1, \ldots, N-1, \tag{9.25}$$

where $\theta_j = c\mu + (j-c)\alpha$, $j = c, c+1, \ldots, N-1$. It is easy to see that the random variables $T_c, T_{c+1}, \ldots, T_{N-1}$ are mutually independent because of the "no memory" property of the exponential distribution.

Let $\gamma_j = P(T_r > T_j + T_{j-1} + \cdots + T_c)$ and $\Phi_j(t) = P(T_j + T_{j-1} + \cdots + T_c \leq t)$, $j = c, c+1, \ldots, N-1$. Then, γ_j is the probability that the new customer on arrival joins the queue and waits in the queue until he acquires service under the condition B_j. We then have the following lemma.

Lemma 9.3.

$$\gamma_j = \frac{c\mu}{c\mu + (j+1-c)\alpha}, \qquad j = c, c+1, \ldots, N-1 \tag{9.26}$$

and

$$\Phi_j(t) = 1 - \sum_{k=c}^{j} \delta_{jk} e^{-\delta_{jk} t}, \qquad j = c, c+1, \ldots, N-1, \quad t \geq 0, \tag{9.27}$$

where

$$\delta_{jk} = \prod_{i=c, i \neq k}^{j} \frac{\theta_i}{\theta_i - \theta_k}, \quad k = c, c+1, \ldots, j, \quad j = c, c+1, \ldots, N-1. \tag{9.28}$$

Proof.

$$\begin{aligned} \gamma_j &= P(T_r > T_j + T_{j-1} + \cdots + T_c) \\ &= P(T_r > T_j) P(T_r - T_j > T_{j-1} + T_{j-2} + \cdots + T_c | T_r > T_j) \\ &= P(T_r > T_j) P(\tilde{T}_r > T_{j-1} + T_{j-2} + \cdots + T_c), \quad j = c, c+1, \ldots, N-1, \end{aligned} \tag{9.29}$$

where $\tilde{T}_r = [T_r - T_j | T_r > T_j]$ has the same exponential distribution as T_r because of the "no memory" property of the exponential distribution. It is easy to see that

$$P(T_r > T_j) = \frac{\theta_j}{\theta_j + \alpha}. \tag{9.30}$$

Hence, by the recursive relation of (9.29), we get the first result of Lemma 9.3.

Note that the random variables T_c, $T_{c+1},\ldots,T_{N-1}$ are mutually independent. $\Phi_j(t)$ has the LST as follows:

$$\Phi_j^*(s) = \prod_{k=c}^{j} H_k^*(s). \tag{9.31}$$

Substituting (9.25) into (9.31), we get

$$\begin{aligned}\Phi_j^*(s) &= \prod_{k=c}^{j} \frac{\theta_k}{\theta_k + s} \\ &= \sum_{k=c}^{j} \delta_{jk} \frac{\theta_k}{\theta_k + s}, \qquad j = c, c+1, \ldots, N-1. \end{aligned} \tag{9.32}$$

Taking the reverse of the LST for the two sides of (9.32), we get the second result of Lemma 9.3. □

Let $W_c(t)$ represent the distribution of the conditional waiting time given that all the servers are busy when a customer on arrival joins the queue. Let q_j be the stationary probability that there are j customers in the system under the condition that all the servers are busy when a customer on arrival joins the queue. Note that $b_j P_1(j)$ represents the probability that there are j customers in the system when a customer on arrival joins the queue. It is easy to see that

$$q_j = \frac{b_j P_1(j)}{\sum_{j=c}^{N-1} b_j P_1(j)}, \qquad j = c, c+1, \ldots, N-1, \tag{9.33}$$

where $P_1(j)$ is given by Theorem 9.2.

Next, we have the following theorem.

Theorem 9.4. *The distribution of the conditional waiting time is given by*

$$W_c(t) = 1 - \sum_{j=c}^{N-1} q_j \gamma_j \sum_{k=c}^{j} \delta_{jk} e^{-\delta_{jk} t} - \sum_{j=c}^{N-1} q_j (1-\gamma_j) e^{-\alpha t}, \tag{9.34}$$

where γ_j, δ_{jk}, and q_j are given by (9.26), (9.28), and (9.33), respectively.

Proof. The conditional waiting time has the following distribution:

$$W_c(t) = \sum_{j=c}^{N-1} q_j P(W \le t | B_j), \tag{9.35}$$

where W represents the waiting time and B_j represents the event that there are j customers in front of the new customer who on arrival joins the queue, and all the servers are busy. Let F_1 and F_2 be the events that the customer either reneges or does not renege when the customer on arrival joins the queue, respectively. Then, we have

$$P(W \le t|B_j) = P(F_1|B_j)P(W \le t|B_j,F_1) + P(F_2|B_j)P(W \le t|B_j,F_2)$$
$$= (1-\gamma_j)(1-e^{-\alpha t}) + \gamma_j \Phi_j(t). \tag{9.36}$$

Thus, by Lemma 9.3, we get the result of Theorem 9.4. □

Remark 2. Based on Theorem 9.2, we can obtain some other performance measures such as the expected number of customers in the system, the expected number of servers that are busy, the average rate of customer loss due to impatience, and so on. The stationary distribution of waiting time can also be obtained from conditioning on every state $(i, j) \in \Omega$. However, these performance measures and the stationary distribution have very complex expressions. Hence, we have omitted the details from this discussion.

9.5 Conclusions

In this chapter, we studied a finite buffer M/M/*c*/N queueing system with balking, reneging, and the synchronous vacations of all servers. By using the blocked-matrix method, we obtained the steady-state probabilities by using the elements of the inverses of two matrices and derived the conditional stationary distribution of the queue length and waiting time.

Tian et al. [11] and Zhang and Tian [12] proved several conditional stochastic decomposition results for the queue length and customer waiting time. These results can be used to compare the M/M/*c* vacation system with its classical M/M/*c* queueing system. Due to the complexity of the formulas, at present, we have not investigated the conditional stochastic decomposition for the queue length and customer waiting time for the model in this chapter.

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Appendix

Proof of Lemma 9.1. Let $\mathbf{X}_j = (c_{1j}, c_{2j}, \ldots, c_{Nj})^T$, $j = 1,2,\ldots,N$, be the jth column vector of the inverse matrix $\mathbf{Q}_{12}^{-1}$, and let $\varepsilon_j = (0,\ldots,1,\ldots,0)^T$ be the jth unit column vector; then we have

$$\mathbf{Q}_{12}\mathbf{X}_j = \varepsilon_j, \qquad j = 1,2,\ldots,N. \tag{9.37}$$

For $j = 1, 2, \ldots, N$, (9.37) can be rewritten as the following set of equations,

$$v_{i-1}c_{i-2j} - w_{i-1}c_{i-1j} + u_{i-1}c_{ij} = 0, \qquad i \neq j, \quad i = 1, 2, \ldots, N, \tag{9.38}$$

$$v_{i-1}c_{i-2j} - w_{i-1}c_{i-1j} + u_{i-1}c_{ij} = 1, \qquad i = j, \tag{9.39}$$

where c_{0j} and c_{-1j} are defined to be zero. Repeating the use of (9.38) gives

$$c_{ij} = 0, \qquad i = 1, 2, \ldots, j-1. \tag{9.40}$$

Substituting (9.40) into (9.39) yields

$$c_{jj} = \frac{1}{u_{j-1}}. \tag{9.41}$$

From (9.38), we have

$$c_{ij} = \frac{w_{i-1}}{u_{i-1}}c_{i-1j} - \frac{v_{i-1}}{u_{i-1}}c_{i-2j}, \qquad i = j+1, \quad j+2, \ldots, N. \tag{9.42}$$

In (9.42), we let

$$c_{ij} = k_{ij}\frac{1}{u_{j-1}}, \qquad i = j+1, \quad j+2, \ldots, N, \tag{9.43}$$

and substitute (9.43) into (9.42), so we get the recursive relations given by (9.3) for k_{ij}. This completes the proof of Lemma 9.1. □

Proof of Lemma 9.2. Let $Y_j = (d_{1j}, d_{2j}, \ldots, d_{Nj})$ be the jth column vector of the inverse matrix $\mathbf{Q}_{33}^{-1}$, then we have

$$\mathbf{Q}_{33}\mathbf{Y}_j = \varepsilon_j, \qquad j = 1, 2, \ldots, N. \tag{9.44}$$

For $j = 1, 2, \ldots, N$, (9.44) can be rewritten as the following set of equations:

$$-s_1 d_{1j} - t_1(d_{1j} - d_{2j}) = 0, \tag{9.45}$$

$$s_i(d_{i-1j} - d_{ij}) - t_i(d_{ij} - d_{i+1j}) = 0, \qquad i = 1, 2, \ldots, N-1, \quad i \neq j, \tag{9.46}$$

$$s_j(d_{j-1j} - d_{jj}) - t_j(d_{jj} - d_{j+1j}) = 1, \tag{9.47}$$

$$s_N(d_{N-1j} - d_{Nj}) = 0. \tag{9.48}$$

Equation (9.46) can be rewritten as the following recursive relation:

$$d_{i-1j} - d_{ij} = \frac{t_i}{s_i}(d_{ij} - d_{i+1j}), \qquad i = 1, 2, \ldots, N-1, \quad i \neq j. \tag{9.49}$$

From (9.48) and (9.49), we get

$$d_{ij} = d_{jj}, \qquad i = j+1, \quad j+2, \ldots, N. \tag{9.50}$$

In (9.50), we let $i = j+1$ and then substitute it into (9.47), so we get

$$d_{j-1j} - d_{jj} = \frac{1}{s_j}. \tag{9.51}$$

Using (9.51) and repeating the use of the recursive relation (9.49) gives

$$d_{i-1j} - d_{ij} = \frac{t_i t_{i+1} \cdots t_{j-1}}{s_i s_{i+1} \cdots s_j}, \qquad i = 2,3,\ldots,j-1. \tag{9.52}$$

In (9.52), we let $i = 2$ and then substitute it into (9.45), so we get

$$d_{1j} = -\frac{t_1 t_2 \cdots t_{j-1}}{s_1 s_2 \cdots s_j}. \tag{9.53}$$

Note that

$$d_{ij} = d_{1j} - \sum_{k=2}^{i} (d_{k-1j} - d_{kj}), \qquad i = 2,3,\ldots,j-1, \tag{9.54}$$

and then substituting (9.52) and (9.53) into (9.54), we get

$$d_{ij} = -\sum_{k=1}^{i} \frac{t_k t_{k+1} \cdots t_{j-1}}{s_k s_{k+1} \cdots s_j}, \qquad i = 2,3,\ldots,j-1. \tag{9.55}$$

In (9.55), we let $i = j-1$ and then substitute it into (9.51) and use (9.50), so we get the results of Lemma 9.2. □

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