Chapter 5
Modeling of Production System with Nonrenewal Batch Input, Early Setup, and Extra Jobs

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Abstract In this chapter, we model and solve a very general single-machine production system with early setup, bilevel threshold control, and extra job operations. The first threshold is used to control the setup starting time and the second threshold is used to control the production starting time. The system is modeled by the BMAP/G/1 queue and the manufacturing lead time is analyzed. The factorization principle is used to derive the distribution of the manufacturing lead time and the mean value. A numerical example is provided.

5.1 Introduction

Industrial engineers have long been interested in analyzing the trade-offs between the system setup and work-in-process (WIP) inventory in order to provide the conditions under which the system operates most economically in the long run. Usually the system setup increases the work-in-process inventory which results in a higher holding cost. But when the system setup cost is very high, this increased holding cost may offset the setup cost because the setup increases the manufacturing cycle

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time which will result in reduced long-run operating cost per unit time. Hence the system setup and WIP inventory are the two most important factors in the cost-effective operation of a production system. Queueing models have played important roles in their analytical efforts along this line.

In most studies on production systems, it has been assumed that the feed process into the production system follows the Poisson process, mainly due to its analytical tractability. But in many real production settings, the interarrival times of the raw materials are correlated, and independently identically distributed (i.i.d.) exponential interarrival times are rarely found. Also, in many production systems, setup operations take several days and are very costly. One way to reduce the setup cost per unit time is to delay the production until some number of raw materials accumulates and this is the well-known N-policy in a queueing context. The N-policy results in a longer cycle length which means fewer cycles per unit time. But at the same time, the average WIP inventory level becomes larger. Thus, in real production settings, the N-policy is used to reduce the overall average cost per unit time when the setup cost is extremely high compared to the WIP holding cost.

In this chapter, we model and solve a very general single-machine production system with early setup, bilevel threshold control, and extra job operations. The first threshold is used to control the setup starting time and the second threshold is used to control the production starting time. The system can be modeled by the BMAP/G/1 queue with bilevel thresholds, setup time, and multiple vacations. We are especially interested in the manufacturing lead time (MLT), which is defined as the time from the arrival of an order till the time the ordered production is finished. The MLT is an important measure of the performance of the production system because whether the manufacturer can meet the due date of an order is one of the most important success indicators of the production system.

Because the MLT corresponds to the system sojourn time (waiting time + processing time) of a queueing system, our objective is to derive the waiting time distribution of the BMAP/G/1 queueing system under the above-mentioned mixed control policy. The idea and basic methods that are employed in this chapter can be applied to many exhaustive BMAP/G/1 systems with more variability.

The N-policy system was first studied by Yadin and Naor [1]. For other works on N-policy queues, see Hersh and Brosh [2], Hofri [3], Kella [4], Lee and Srinivasan [5], Takagi [6], Lee, and chae [7], and Lee and Ahn [8], to list a few.

Lee and Park [9] showed that the double threshold (α, N)-policy is better than the single threshold N-policy when the setup cost is extremely high compared to the WIP holding cost. We note Lee, Park, and Jeon [10] applied the factorization property of the queue length to the analysis of the WIP inventory of a production system with maintenance, setup, and thresholds.

The chapter is organized as follows. In Sects. 5.2 and 5.3, the system model is described and some notation definitions are given. In Sects. 5.4 and 5.5, the waiting time distribution and the mean waiting time are derived. Numerical examples are shown in Sect. 5.6 and conclusions are drawn in Sect. 5.7.

5.2 System Model

Our queueing system operates as follows (see Fig. 5.1). As soon as the system empties, the server leaves for a vacation of random length V with distribution function (DF) $V(x)$ and the Laplace–Stieltjes transform (LST) $V^*(\theta)$ (the server attends to extra jobs during the vacation). After it returns from the vacation, if it finds α or more customers, it immediately starts a setup of random length H with DF $H(x)$ and the LST $H^*(\theta)$. Otherwise, it takes repeated i.i.d. vacations until it finds α or more customers to start a setup. After the setup is finished, if the total number of customers in the system (queue length) is greater than or equal to N, the server immediately begins to serve the customers. If not, the server waits in the system until the queue length reaches or exceeds N.

In our system, customers arrive according to a BMAP (Batch Markovian Arrival Process) with parameter matrices $(D_0, D_1, D_2, \ldots)$ with $D(z) = \sum_{n=0}^{\infty} D_n z^n$ as the matrix generating function (GF) where $D = D(1) = \sum_{n=0}^{\infty} D_n$ is the infinitesimal generator of the underlying Markov chain (UMC). We assume that the service times are i.i.d. random variables with DF $S(x)$ and the LST $S^*(\theta)$. We also assume that the service times, the vacation times, the setup time, and the arrival process are independent of each other.

An excellent treatment of the BMAP and BMAP/G/1 queues can be found in Lucantoni [11], [12]. For computational algorithms concerning BMAP queues, see Lucantoni [11], [12], Ramaswami [13], and Latouche and Ramaswami [14].

Chang, Takine, and Chae et al. [15] studied the factorization property for a BMAP/G/1 queue with generalized vacations. Lee, Park, and Jeon [16] applied the factorization property to the Park, and Jeon BMAP/G/1 queue with early setup and bilevel threshold policy.

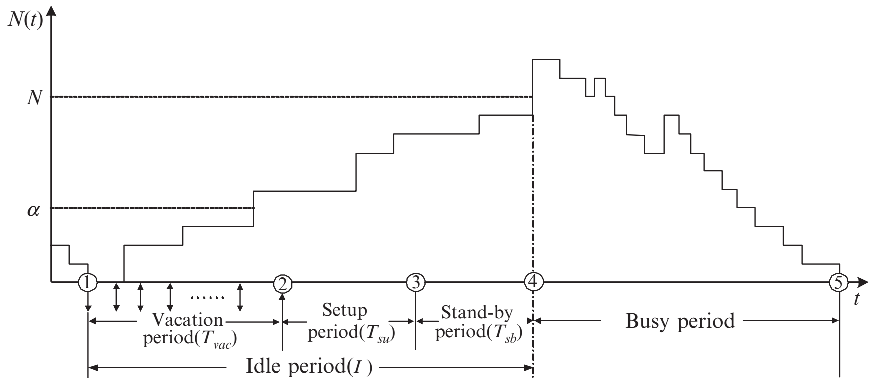

Fig. 5.1 The system.

5.3 Preliminaries

Let π be the stationary vector of the UMC. Then, π can be obtained from

$$\pi D = 0, \quad \pi e = 1,$$

where e is the column vector of 1s with appropriate dimension.

Let λ_g be the group arrival rate. Then, we have

$$\lambda_g = \pi \sum_{n=1}^{\infty} D_n e = \pi(D - D_0)e = -\pi D_0 e.$$

The total customer arrival rate λ becomes

$$\lambda = \pi \sum_{n=1}^{\infty} n D_n e.$$

Let Γ be the size of an arbitrary arrival group with $\gamma_k = Pr(\Gamma = k)$. Then, we have

$$\gamma_k = \frac{\pi D_k e}{\pi \sum_{n=1}^{\infty} D_n e} = \frac{\pi D_k e}{\lambda_g} \tag{5.1}$$

and

$$\mathrm{E}[\Gamma] = \lambda / \lambda_g. \tag{5.2}$$

Let δ_k be the probability that the test customer belongs to a group of size k. From the theory of discrete-time renewal theory, we have, after using (5.1) and (5.2),

$$\delta_k = \frac{k \cdot \gamma_k}{\mathrm{E}[\Gamma]} = \frac{k \pi D_k e}{\lambda}.$$

Now, let us consider a "virtual customer" who arrives at an arbitrary point of time during the busy period and sees the system state (n,i) where n is the queue length (i.e., the number of customers including the one in service) and i is the phase of the UMC at the arrival instance. Let the time-average probability of this state be $y_{busy,n,i}$ with vector $y_{busy,n} = (y_{busy,n,1}, \ldots, y_{busy,n,m})$ and the vector GF $Y_{busy}(z) = \sum_{n=1}^{\infty} y_{busy,n} z^n$. Now, let us consider an arbitrary "actual customer" who arrives during the busy period. If he belongs to a group of size k (with probability δ_k), and is ith within his group (with probability $1/k$), he has $(i-1)$ customers preceding him in his group. Thus, the vector GF $Y^+_{busy}(z)$ of the number of customers just after his arrival becomes

$$Y^+_{busy}(z) = \sum_{k=1}^{\infty} \sum_{i=1}^{k} \delta_k \frac{1}{k} z^{i-1} Y_{busy}(z) \frac{D_k}{\pi D_k e} = Y_{busy}(z) \frac{D - D(z)}{\lambda(1-z)}, \tag{5.3}$$

where $D_k / \pi D_k e$ is multiplied to convert the virtual joint probability of the queue length and the UMC phase to the actual joint probability (note that our test customer belongs to a group of size k). Equation (5.3) was already stated in Lucantoni [11], [12].

5.4 Waiting Time Distribution

In order to obtain the vector Laplace–Stieltjes transform (LST) $w_A^*(\theta)$ of the waiting time of an actual test customer, the first step is to find the vector LSTs $w_{vac,V}^*(\theta)$, $w_{su,V}^*(\theta)$, $w_{sb,V}^*(\theta)$, and $w_{busy,V}^*(\theta)$ of the waiting time of the virtual customer who arrives at an arbitrary time in each period. Once we obtain these quantities, we can obtain the vector LSTs $w_{vac,A}^*(\theta)$, $w_{su,A}^*(\theta)$, $w_{sb,A}^*(\theta)$, and $w_{busy,A}^*(\theta)$ of the waiting time of an actual test customer by postmultiplying appropriate quantities to convert the virtual probabilities to actual probabilities.

To obtain $w_{busy,V}^*(\theta)$, we need $Y_{busy}^*(z,\theta)$ which is the joint transform of the queue length and the remaining service time at the arrival instance of the virtual customer. Then we get

$$w_{busy,V}^*(\theta) = \left[\frac{Y_{busy}^*(z,\theta)}{z}\right]_{z=S^*(\theta)} = \frac{Y_{busy}^*[S^*(\theta),\theta]}{S^*(\theta)}.$$

Then, in the analogous manner as in (5.3), we get

$$w_{busy,A}^*(\theta) = \frac{Y_{busy}^*[S^*(\theta),\theta]}{S^*(\theta)} \frac{D - D(S^*(\theta))}{\lambda(1-S^*(\theta))}. \tag{5.4}$$

Now, if we let $Y_{idle}(z)$ be the vector GF of the queue length at an arbitrary idle time in a BMAP/G/1 queue with generalized vacations, it is proven by Chang et al. [15] that $Y_{busy}^*(z,\theta)$ is given by

$$Y_{busy}^*(z,\theta)[\theta I + D(z)] = (1-\rho)Y_{idle}(z)zD(z)[A(z) - S^*(\theta)I][zI - A(z)], \tag{5.5}$$

where $\rho = \lambda \mathrm{E}[S]$ is the server utilization and $A(z)$ is the matrix GF of the number of customers that arrive during the service time which is given by $A(z) = \int_0^\infty e^{D(z)x} dS(x)$ (Lucantoni [12]). Thus, our temporary objective is to obtain $Y_{idle}(z)$.

5.4.1 Obtaining $Y_{idle}(z)$

In this subsection, we derive the vector GF $Y_{idle}(z)$ of the queue length at an arbitrary idle time. To this end, we first find p_{vac}, p_{su}, and p_{sb} which are time-average probabilities that the system is in a vacation period, in a setup period, and in a standby period, respectively, under the condition that the system is idle (see Fig. 5.1). Let $\mathrm{E}[T_{vac}]$, $\mathrm{E}[H]$, and $\mathrm{E}[T_{sb}]$ be the mean length of each period. Then, we get

$$\mathrm{E}[I] = \mathrm{E}[T_{vac}] + \mathrm{E}[H] + \mathrm{E}[T_{sb}]$$

and

$$p_{vac} = \frac{\mathrm{E}[T_{vac}]}{\mathrm{E}[I]}, \quad p_{su} = \frac{\mathrm{E}[H]}{\mathrm{E}[I]}, \quad p_{sb} = \frac{\mathrm{E}[T_{sb}]}{\mathrm{E}[I]}. \tag{5.6}$$

In the sequel, we denote $(F)_{ij}$ as the (i,j)-element of a matrix F.

We first derive $\mathrm{E}[T_{vac}]$. Let us define a grand vacation process as in Lee et al. [16]. A grand vacation (GV) is the sum of i.i.d. individual vacations until there is a change in queue length upon a return from a vacation. The first grand vacation (GV) G_1 starts from ① (see Fig. 5.1) and lasts until the queue length differs from 0 upon a return from a vacation. At this point, if the queue length is less than α, the second GV G_2 starts and lasts until there is a change in the queue length upon a return from a vacation. The GV process continues in this manner until the queue length upon return from a vacation is greater than or equal to α.

We note that a GV is equivalent to the vacation period in the simple BMAP/G/1 queue with multiple vacations. Let $(R_n)_{ij}$ be the probability that the GV process visits level (queue length) n and the UMC phase is j just after the visit given that the UMC phase is i at ①. It was proven in Lee et al. [16] that R_n can be computed from the following recursion,

$$R_0 = I, \quad R_n = \sum_{i=1}^{n} R_{n-i}(I-V_0)^{-1}V_i, \quad (n \geq 1),$$

where V_i is the matrix probability that i customers arrive during a vacation.

Because $[(I-V_0)^{-1}]_{ij}$ is the mean number of vacations (within a GV) that starts with phase j under the condition that the GV started with phase i, we have

$$\mathrm{E}[T_{vac}] = \left[\kappa \sum_{n=0}^{\alpha-1} R_n(I-V_0)^{-1}e\right]\mathrm{E}[V], \tag{5.7}$$

where κ is the phase probability vector at ①. Obtaining κ is discussed later.

To derive $\mathrm{E}[T_{sb}]$, let us define $(\Phi_k^{sb})_{ij}, (\alpha \leq k \leq N-1)$ as follows:

$(\Phi_k^{sb})_{ij} = Pr$ (the stand-by process visits level k and the phase of UMC is j just after the visit | UMC phase is i at ①).

Noting that (i,j)-element of the matrix $(-D_0)^{-1}$ is the mean time the UMC stays in phase j until the next arrival given that the current phase is in i (see, e.g., Latouche and Ramaswami [14]), we have

$$\mathrm{E}[T_{sb}] = \kappa \sum_{k=\alpha}^{N-1} \Phi_k^{sb}(-D_0)^{-1}e. \tag{5.8}$$

Thus, the mean length of an arbitrary idle period is given by

$$\mathrm{E}[I] = \kappa\left[\sum_{n=0}^{\alpha-1} R_n(I-V_0)^{-1}\mathrm{E}[V] + \mathrm{E}[H]I + \sum_{k=\alpha}^{N-1} \Phi_k^{sb}(-D_0)^{-1}\right]e. \tag{5.9}$$

Then p_{vac}, p_{su}, and p_{sb} can be obtained from (5.6)–(5.9).

Computation of κ and $\{\Phi_k^{sb}, (\alpha \leq k \leq N-1)\}$ is discussed later.

Let $p_{vac}(z)$, $p_{su}(z)$, and $p_{sb}(z)$ be the vector GFs of the queue length at an arbitrary epoch in each period under the condition that the system is idle. We first obtain

$p_{vac}(z)$. Consider an arbitrary time point t^* during the vacation period. At the start of the vacation that contains t^*, the queue length is n and the UMC phase is j with probability

$$\frac{[\kappa R_n (I - V_0)^{-1}]_j}{\kappa \sum_{n=0}^{\alpha-1} R_n (I - V_0)^{-1} e},$$

where the denominator is the mean number of individual vacations during the vacation period. Now, the matrix GF $V^*(z)$ of the number of customers that arrive during the elapsed vacation is given by

$$V^*(z) = \int_0^\infty e^{D(z)x} \left[\frac{1 - V(x)}{\mathrm{E}[V]} \right] dx = \frac{[V(z) - I]}{\mathrm{E}[V]} D(z)^{-1},$$

where $V(z)$ is the GF of $\{V_i\}$. Thus, we get

$$p_{vac}(z) = p_{vac} \frac{\kappa \sum_{n=0}^{\alpha-1} R_n [I - V_0]^{-1} z^n}{\kappa \sum_{n=0}^{\alpha-1} R_n [I - V_0]^{-1} e} \frac{[V(z) - I]}{\mathrm{E}[V]} D(z)^{-1}. \tag{5.10}$$

Now, to derive $p_{su}(z)$, let us define $H_\alpha^-(z) = \sum_{k=\alpha}^\infty H_{k(\alpha)}^- z^k$ as the GF of the matrix probability $H_{k(\alpha)}^-$ that there are k customers at the start of the setup period (point ②). Noticing that $H_\alpha^-(z)$ is equivalent to the queue length GF at the start of the busy period in the simple BMAP/G/1 queue with α-policy and multiple vacation, we have from Lee et al. [16],

$$H_\alpha^-(z) = I + \sum_{j=0}^{\alpha-1} R_j [I - V_0]^{-1} z^j [V(z) - I]. \tag{5.11}$$

Then, we get

$$p_{su}(z) = p_{su} \cdot \kappa H_\alpha^-(z) H^*(z), \tag{5.12}$$

where

$$H^*(z) = \frac{[H(z) - I]}{\mathrm{E}[H]} D(z)^{-1}$$

is the GF of the number of customers that arrive during the elapsed setup time in which $H(z)$ is the matrix GF of the number of customers that arrive during a setup time.

Under the condition that the system is in a stand-by period, the queue length is k and the UMC phase is j with probability

$$\frac{\left(\kappa \Phi_k^{sb}(-D_0)^{-1}1\right)_j}{\kappa \sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1}e}.$$

Thus we get

$$p_{sb}(z) = p_{sb} \cdot \frac{\kappa \sum_{k=\alpha}^{N-1} \Phi_k^{sb}(-D_0)^{-1}z^k}{\kappa \sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1}e}. \tag{5.13}$$

Combining (5.10), (5.12), and (5.13), we get

$$\begin{aligned} Y_{idle}(z) &= p_{vac}(z) + p_{su}(z) + p_{sb}(z) \\ &= \frac{\kappa}{\mathrm{E}[I]} \left\{ \sum_{n=0}^{\alpha-1} R_n [I - V_0]^{-1} z^n [V(z) - I] D(z)^{-1} \right. \\ &\quad \left. + H_\alpha^-(z)[H(z) - I] D(z)^{-1} + \sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1} z^n \right\}. \end{aligned} \tag{5.14}$$

Now, we need to devise a scheme to compute the probability $\Phi_k^{sb}, (\alpha \le k \le N-1)$ that the stand-by process visits level k. This depends on the queue length probability at ③. By conditioning on the queue length at ②, the probability $H_{k(\alpha)}^+$ at the end of the setup period becomes

$$H_{k(\alpha)}^+ = \sum_{i=\alpha}^{k} H_{i(\alpha)}^- H_{k-i} \tag{5.15}$$

and

$$\Phi_k^{sb} = \sum_{i=0}^{k} H_{i(\alpha)}^+ D_{k-i}^*, \quad (\alpha \le k \le N-1),$$

where D_n^* is the probability matrix that the idle period process of the BMAP/G/1/α-policy queueing system (without vacations and setup) visits level n and H_k is the probability that k customers arrive during a setup time. We note, by conditioning on the level visited prior to level n, that we have a recursion,

$$D_0^* = I, \;\; D_n^* = \sum_{l=0}^{n-1} D_l^*(-D_0)^{-1} D_{n-l}.$$

Now, κ can be computed from

$$\kappa K = \kappa, \quad \kappa e = 1,$$

where K is the phase transition probability between ① and ⑤ and can be obtained from

$$K = K(z)|_{z=1},$$

in which $K(z)$ is the matrix GF of the mean number of customers that are served between ① and ⑤. To obtain $K(z)$, we need the GF $Q_{(\alpha,N)}(z)$ of the queue length at the start of the busy period (④). We can show that (see Appendix 3):

$$Q_{(\alpha,N)}(z) = H_{\alpha}^{-}(z)H(z) + \left[\sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1}z^n\right] D(z), \tag{5.16}$$

where $H(z)$ is the matrix GF of the number of customers that arrive during the setup time. Using (5.11) in (5.16), we get

$$\begin{aligned} K(z) = Q_{(\alpha,N)}(z)|_{z=G(z)} &= \sum_{n=0}^{\alpha-1} R_n[I - V_0]^{-1}[G(z)]^n[V(G(z)) - I]H(G(z)) \\ &\quad + H(G(z)) + \sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1}[G(z)]^n D(G(z)). \end{aligned}$$

Thus we have

$$\begin{aligned} K = K(z)|_{z=1} &= \sum_{n=0}^{\alpha-1} R_n[I - V_0]^{-1}G^n[V(G) - I]H(G) \\ &\quad + H(G) + \sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1}G^n D(G). \end{aligned}$$

Using (5.14) in (5.5), we get

$$\begin{aligned} Y_{busy}^*(z,\theta)[\theta I + D(z)] = &\left\{ \sum_{n=0}^{\alpha-1} R_n[I - V_0]^{-1}z^n[V(z) - I]H(z) \right. \\ &\left. + \sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1}z^n D(z) + H(z) - I \right\} \\ &\cdot \left\{[z - S^*(\theta)]A(z)[zI - A(z)]^{-1} - S^*(\theta)I\right\}. \end{aligned}$$

Then, we can obtain $w_{busy,A}^*(\theta)$ from (5.4).

5.4.2 Obtaining the LST of the Waiting Time of the Customer Who Arrives During the Idle Period

Now to find the vector LST $w_{vac,A}^*(\theta)$ of the waiting time of the actual test customer that arrives during a vacation, we first need to know the number of customers that arrive during the time period from the end of the current vacation to the start of the setup period because this determines the remaining vacation period and thereby the remaining idle period. For this purpose, let us define the notation as follows:

$T^v_{\alpha-k}$: The remaining time until the setup starts from the end of the current vacation at which there are k customers
$A(T^v_{\alpha-k})$: The number of customers that arrive during $T^v_{\alpha-k}$
J_1: The UMC phase at the end of the current vacation
J_2: The UMC phase at the start of the setup time

Let us define the (i,j)-element of the matrix transform $T^{V*}_{\alpha-k}(\theta,n)$ as follows:

$$\left[T^{V*}_{\alpha-k}(\theta,n)\right]_{ij} = \int_0^\infty e^{-\theta t} Pr(t < T^v_{\alpha-k} \le t+dt,\, A(T^v_{\alpha-k}) = n,\, J_2 = j | J_1 = i).$$

Then, we have

$$T^{V*}_{\alpha-k}(0,n) = H^-_{n(\alpha-k)}, \quad (n \ge \alpha - k).$$

If the test customer who arrives during a vacation belongs to a group of size j and stands ith in her group, she first has to wait that:

(i) The service times of the customers at the start of the current vacation
(ii) The service times of the customers that arrive during the elapsed vacation time
(iii) The time until the end of the current vacation
(iv) The service times of those $(i-1)$ customers who precede her in her group
(v) The remaining vacation period (from the end of the current vacation)
(vi) The time until the busy period starts.

These quantities are dependent on each other. Let us define ψ^V_n as

$$\psi^V_n = \frac{\kappa R_n [I - V_0]^{-1}}{\kappa \sum_{k=0}^{\alpha-1} R_k [I - V_0]^{-1} e},$$

which is the vector probability that the queue length at the start of the current vacation is n. Then the LST of the waiting time above ((ii)–(v)) contribution is as follows:

$$\psi^V_n [S^*(\theta)]^n \Omega^*_V(a,j,b,\theta)[S^*(\theta)]^a [S^*(\theta)]^{i-1},$$

where $\Omega^*_V(a,j,b,\theta)$ is given in (5.34) in Appendix 1 and represents the remaining vacation time including the probability that a customers arrive during the elapsed vacation time; the test customer belongs to a group of size j (the virtual phase is converted to the actual phase at this point. See (5.25) in Appendix 1. See also Kasahara et al. [17]), and b customers arrive during the remaining vacation time.

Now, additional waiting time depends on the situation at the end of the current vacation. Consider the group G^* to which the test customer belongs. Let us define the following quantities:

$Q^-(G^*)$: The number of customers just before G^* arrives
$Q^+(G^*)$: The number of customers just after G^* arrives
Q_V^+: The number of customers at the end of the current vacation
Q_H^-: The number of customers at the start of the setup period
Q_H^+: The number of customers at the end of the setup period

Then we have different cases as follows:

(Case 1) $Q^-(G^*) < \alpha,\ \ Q^+(G^*) \le \alpha$
(case 1-1) $Q^-(G^*) < \alpha,\ \ Q^+(G^*) \le \alpha,\ \ Q_V^+ \le \alpha,\ \ Q_H^- \le N,\ \ Q_H^+ \le N,$
(case 1-2) $Q^-(G^*) < \alpha,\ \ Q^+(G^*) \le \alpha,\ \ Q_V^+ \le \alpha,\ \ Q_H^- \le N,\ \ Q_H^+ > N,$
(case 1-3) $Q^-(G^*) < \alpha,\ \ Q^+(G^*) \le \alpha,\ \ Q_V^+ \le \alpha,\ \ Q_H^- > N,$
(case 1-4) $Q^-(G^*) < \alpha,\ \ Q^+(G^*) \le \alpha,\ \ \alpha < Q_V^+ \le N,\ \ Q_H^+ \le N,$
(case 1-5) $Q^-(G^*) < \alpha,\ \ Q^+(G^*) \le \alpha,\ \ \alpha < Q_V^+ \le N,\ \ Q_H^+ > N,$
(case 1-6) $Q^-(G^*) < \alpha,\ \ Q^+(G^*) \le \alpha,\ \ Q_V^+ > N.$

(Case 2) $Q^-(G^*) < \alpha,\ \ \alpha < Q^+(G^*) \le N$
(case 2-1) $Q^-(G^*) < \alpha,\ \ \alpha < Q^+(G^*) \le N,\ \ < Q_V^+ \le N,\ \ Q_H^+ \le N,$
(case 2-2) $Q^-(G^*) < \alpha,\ \ \alpha < Q^+(G^*) \le N,\ \ < Q_V^+ \le N,\ \ Q_H^+ > N,$
(case 2-3) $Q^-(G^*) < \alpha,\ \ \alpha < Q^+(G^*) \le N,\ \ < Q_V^+ > N.$

(Case 3) $Q^-(G^*) < \alpha,\ \ Q^+(G^*) > N.$

(Case 4) $\alpha < Q^-(G^*) < N$
(case 4-1) $\alpha < Q^-(G^*) < N,\ \ Q^+(G^*) \le N,\ \ Q_H^- \le N,\ \ < Q_H^+ \le N,$
(case 4-2) $\alpha < Q^-(G^*) < N,\ \ Q^+(G^*) \le N,\ \ Q_H^- \le N,\ \ < Q_H^+ > N,$
(case 4-3) $\alpha < Q^-(G^*) < N,\ \ Q^+(G^*) \le N,\ \ Q_H^- > N,$
(case 4-4) $\alpha < Q^-(G^*) < N,\ \ Q^+(G^*) > N.$

(Case 5) $\alpha < Q^-(G^*) \ge N.$

Now, the waiting times in (case 1-1) and (case 1-2) are as follows:

$$
\begin{aligned}
B_1 = & \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=0}^{\alpha-n-1} \sum_{j=1}^{\alpha-n-a} \sum_{b=0}^{\alpha-n-a-j} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1} \\
& \cdot \sum_{c=\alpha-n-a-j-b}^{N-n-a-j-b} T_{\alpha-n-a-j-b}^{V*}(\theta,c) \\
& \cdot \left[\sum_{k=0}^{N-n-a-j-b-c} H_k^*(\theta) T_{N-n-a-j-b-c-k}^*(\theta) + \sum_{k=N-n-a-j-b-c+1}^{\infty} H_k^*(\theta) \right],
\end{aligned}
$$

where $H_k^*(\theta)$ is the matrix LST of the length of the setup time including the probability that k customers arrive during the setup, and $T_n^*(\theta)$ is the matrix LST of the idle period in the single-threshold BMAP/G/1 queue under n-policy (without vacations and setup) which becomes, conditioning on the first group size,

$$T_n^*(\theta) = [\theta I - D_0]^{-1}\left[\sum_{k=1}^{n-1} D_k T_{n-k}^*(\theta) + \sum_{k=n}^{\infty} D_k\right]$$
$$= [\theta I - D_0]^{-1}\left[\sum_{k=1}^{n-1} D_k [T_{n-k}^*(\theta) - I] + D - D_0\right]$$

with $T_0^*(0) = I$. For the remaining cases, we have

(Case 1-3)

$$B_2 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=0}^{\alpha-n-1} \sum_{j=1}^{\alpha-n-a} \sum_{b=0}^{\alpha-n-a-j} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1}$$
$$\cdot \sum_{c=N-n-a-j-b+1}^{\infty} T_{\alpha-n-a-j-b}^{V*}(\theta,c) H^*(\theta).$$

(Case 1-4) and (Case 1-5)

$$B_3 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=0}^{\alpha-n-1} \sum_{j=1}^{\alpha-n-a} \sum_{b=\alpha-n-a-j+1}^{N-n-a-j} \Omega_V^*(a,j,b,\theta)$$
$$\cdot [S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1}$$
$$\cdot \left[\sum_{k=0}^{N-n-a-j-b} H_k^*(\theta) T_{N-n-a-j-b-k}^*(\theta) + \sum_{k=N-n-a-j-b+1}^{\infty} H_k^*(\theta)\right].$$

(Case 1-6)

$$B_4 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=0}^{\alpha-n-1} \sum_{j=1}^{\alpha-n-a} \sum_{b=N-n-a-j+1}^{\infty} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j}$$
$$\cdot \sum_{i=1}^{j} [S^*(\theta)]^{i-1} H^*(\theta).$$

(Case 2-1) and (Case 2-2)

$$B_5 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=0}^{\alpha-n-1} \sum_{j=\alpha-n-a+1}^{N-n-a} \sum_{b=0}^{N-n-a-j} \Omega_V^*(a,j,b,\theta)$$
$$\cdot [S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1}$$
$$\cdot \left[\sum_{k=0}^{N-n-a-j-b} H_k^*(\theta) T_{N-n-a-j-b-k}^*(\theta) + \sum_{k=N-n-a-j-b+1}^{\infty} H_k^*(\theta)\right].$$

(Case 2-3)

$$B_6 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=0}^{\alpha-n-1} \sum_{j=\alpha-n-a+1}^{N-n-a} \sum_{b=N-n-a-j+1}^{\infty} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \cdot \sum_{i=1}^{j} [S^*(\theta)]^{i-1} H^*(\theta).$$

(Case 3)

$$B_7 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=0}^{\alpha-n-1} \sum_{j=\alpha-n-a+1}^{\infty} \sum_{b=0}^{\infty} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \cdot \sum_{i=1}^{j} [S^*(\theta)]^{i-1} H^*(\theta).$$

(Case 4-1) and (Case 4-2)

$$B_8 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=\alpha-n}^{N-n-1} \sum_{j=1}^{N-n-a} \sum_{b=0}^{N-n-a-j} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1} \cdot \left[\sum_{k=0}^{N-n-a-j-b} H_k^*(\theta) T_{N-n-a-j-b-k}^*(\theta) + \sum_{k=N-n-a-j-b+1}^{\infty} H_k^*(\theta) \right].$$

(Case 4-3)

$$B_9 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=\alpha-n}^{N-n-1} \sum_{j=1}^{N-n-a} \sum_{b=N-n-a-j+1}^{\infty} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \cdot \sum_{i=1}^{j} [S^*(\theta)]^{i-1} H^*(\theta).$$

(Case 4-4)

$$B_{10} = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=\alpha-n}^{N-n-1} \sum_{j=N-n-a+1}^{\infty} \sum_{b=0}^{\infty} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \cdot \sum_{i=1}^{j} [S^*(\theta)]^{i-1} H^*(\theta).$$

(Case 5)

$$B_9 = \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \sum_{a=N-n}^{\infty} \sum_{j=1}^{\infty} \sum_{b=0}^{\infty} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1} H^*(\theta).$$

Now, combining all these, we get

$$w^*_{vac,A}(\theta) = (1-\rho)p_{vac}\sum_{n=1}^{11} B_n. \tag{5.17}$$

In the similar way, we can obtain the waiting time of the actual customer who arrives during the setup time and we get

$$\begin{aligned}
&w^*_{su,A}(\theta)\\
&= (1-\rho)p_{su}\kappa\left[\sum_{n=\alpha}^{N-1} H^-_{n(\alpha)}[S^*(\theta)]^n \left\{\sum_{a=0}^{N-n-1}\sum_{j=1}^{N-n-a}\sum_{b=0}^{N-n-a-j}\Omega^*_V(a,j,b,\theta)\right.\right.\\
&\quad \cdot\ [S^*(\theta)]^a \frac{1}{j}\sum_{i=1}^{j}[S^*(\theta)]^{i-1}\left[T^*_{N-n-a-j-b}(\theta)-I\right] \qquad (5.18)\\
&\quad \left. +\ \sum_{a=0}^{\infty}\sum_{j=1}^{\infty}\sum_{b=0}^{\infty}\Omega^*_V(a,j,b,\theta)[S^*(\theta)]^a\frac{1}{j}\sum_{i=1}^{j}[S^*(\theta)]^{i-1}\right\}\\
&\quad \left. +\ \sum_{n=N}^{\infty} H^-_{n(\alpha)}[S^*(\theta)]^n\sum_{a=0}^{\infty}\sum_{j=1}^{\infty}\sum_{b=0}^{\infty}\Omega^*_V(a,j,b,\theta)[S^*(\theta)]^a\frac{1}{j}\sum_{i=1}^{j}[S^*(\theta)]^{i-1}\right].
\end{aligned}$$

For the actual customer who arrives during the standby period, we get

$$\begin{aligned}
w^*_{sb,A}(\theta) &= (1-\rho)p_{sb}\sum_{n=\alpha}^{N-1}\psi^{sb}_n[S^*(\theta)]^n\\
&\quad \cdot\left\{\sum_{j=1}^{N-k}\frac{D_j}{\lambda}\sum_{i=1}^{j}[S^*(\theta)]^{i-1}(T^*_{N-k-j}(\theta)-I)+\frac{D-D(S^*(\theta))}{\lambda[1-S^*(\theta)]}\right\}, \qquad (5.19)
\end{aligned}$$

where

$$\psi^{sb}_k = \frac{\kappa\Phi^s_k b(-D_0)^{-1}}{\kappa\sum_{n=\alpha}^{N-1}\Phi^{sb}_n(-D_0)^{-1}e}$$

is the vector probability that there are k customers under the condition that system is in a standby period.

Finally the LST of the actual waiting customer can be obtained from (5.17)–(5.19), and we get

$$W^*_q(\theta) = w^*_A(\theta)e = w^*_{vac,A}(\theta)e + w^*_{su,A}(\theta)e + w^*_{sb,A}(\theta)e + w^*_{busy,A}(\theta)e.$$

For the simplicity of the subsequent analysis, let us write the LST of the waiting time of an arbitrary actual waiting customer as

$$W^*_q(\theta) = w^*_N(\theta)e + w^*_1(\theta)e, \tag{5.20}$$

where

$$w_N^*(\theta)e = (1-\rho)\left[p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V[S^*(\theta)]^n\frac{1-V^*(\theta)}{\mathrm{E}[V]\theta}H^*(\theta) + p_{su}\kappa\frac{1-H^*(\theta)}{\mathrm{E}[H]\theta} + p_{sb}\sum_{n=\alpha}^{N-1}\psi_n^{sb}[S^*(\theta)]^n\right] \cdot \theta[\theta I - D(S^*(\theta))]^{-1}\frac{D-D(S^*(\theta))}{\lambda[1-S^*(\theta)]}e$$

and

$$w_1^*(\theta)e = (1-\rho)p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V[S^*(\theta)]^n \cdot \sum_{k=1}^{5}C_k e + 1-\rho)p_{su}\kappa\sum_{n=\alpha}^{N-1}H_{n(\alpha)}^-[S^*(\theta)]^n \cdot C_6 e + (1-\rho)p_{sb}\sum_{n=\alpha}^{N-1}\psi_n^{sb}[S^*(\theta)]^n\sum_{j=1}^{N-n}\frac{D_j}{\lambda}\sum_{i=1}^{j}[S^*(\theta)]^{i-1} \cdot [T_{N-n-j}^*(\theta)-I]e,$$

where

$$C_1 = \sum_{a=0}^{\alpha-n-1}\sum_{j=1}^{\alpha-n-a}\sum_{b=0}^{\alpha-n-a-j}\Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a\frac{1}{j}\sum_{i=1}^{j}[S^*(\theta)]^{i-1} \cdot \sum_{c=\alpha-n-a-j-b}^{N-n-a-j-b}T_{\alpha-n-a-j-b}^V*(\theta,c)\sum_{k=0}^{N-n-a-j-b-c}H_k^*(\theta) \cdot [T_{N-n-a-j-b-c-k}^*(\theta)-I],$$

$$C_2 = \sum_{a=0}^{\alpha-n-1}\sum_{j=1}^{\alpha-n-a}\sum_{b=0}^{\alpha-n-a-j}\Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a\frac{1}{j}\sum_{i=1}^{j}[S^*(\theta)]^{i-1} \cdot [T_{N-n-a-j-b-c-k}^*(\theta)-I]H^*(\theta),$$

$$C_3 = \sum_{a=0}^{\alpha-n-1}\sum_{j=1}^{\alpha-n-a}\sum_{b=\alpha-n-a-j+1}^{N-n-a-j}\Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a\frac{1}{j}\sum_{i=1}^{j}[S^*(\theta)]^{i-1} \cdot \sum_{k=0}^{N-n-a-j-b}H_k^*(\theta)[T_{N-n-a-j-b-c-k}^*(\theta)-I],$$

$$C_4 = \sum_{a=0}^{\alpha-n-1} \sum_{j=\alpha-n-a+1}^{N-n-a} \sum_{b=0}^{N-n-a-j} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1}$$
$$\cdot \sum_{k=0}^{N-n-a-j-b} H_k^*(\theta)[T^*_{N-n-a-j-b-c-k}(\theta)-I],$$

$$C_5 = \sum_{a=\alpha-n}^{\alpha-n-1} \sum_{j=1}^{N-n-a} \sum_{b=0}^{N-n-a-j} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1}$$
$$\cdot \sum_{k=0}^{N-n-a-j-b} H_k^*(\theta)[T^*_{N-n-a-j-b-c-k}(\theta)-I],$$

$$C_6 = \sum_{a=0}^{N-n-1} \sum_{j=1}^{N-n-a} \sum_{b=0}^{N-n-a-j} \Omega_V^*(a,j,b,\theta)[S^*(\theta)]^a \frac{1}{j} \sum_{i=1}^{j} [S^*(\theta)]^{i-1}$$
$$\cdot [T^*_{N-n-a-j-b-c-k}(\theta)-I].$$

5.5 Mean Waiting Time

From (5.20), the mean actual waiting time becomes

$$W_q = -W_q^{*(1)}(0) = -w_N^{*(1)}(0)e - w_1^{*(1)}(0)e,$$

where

$$-w_1^{*(1)}(0)e = (1-\rho)p_{vac} \sum_{n=0}^{\alpha-1} \psi_n^V \sum_{k=1}^{5} E_k$$
$$+ 1-\rho)p_{su}\kappa \sum_{n=\alpha}^{N-1} H^-_{n(\alpha)} \sum_{a=0}^{N-n-1} \sum_{j=1}^{N-n-a} \sum_{b=0}^{N-n-a-j} \Omega_V(a,j,b)$$
$$\cdot \tau_{N-n-a-j-b} + (1-\rho)p_{sb} \sum_{n=\alpha}^{N-1} \psi_n^{sb} \sum_{j=1}^{N-n} \frac{jD_j}{\lambda} \tau_{N-n-j},$$

where

$$\Omega_V(a,j,b) = \Omega_V^*(a,j,b,\theta)|_{\theta=0},$$

$$\tau_n = -\frac{d}{d\theta} T_n^*(\theta)\Big|_{\theta=0} e = \sum_{k=0}^{n-1} D_k^*(-D_0)^{-1} e,$$

$$E_1 = \sum_{a=0}^{\alpha-n-1} \sum_{j=1}^{\alpha-n-a} \sum_{b=0}^{\alpha-n-a-j} \Omega_V(a,j,b) \sum_{c=\alpha-n-a-j-b}^{N-n-a-j-b} H^{-}_{c(\alpha-n-a-j-b)} \cdot \sum_{k=0}^{N-n-a-j-b-c} H_k \tau_{N-n-a-j-b-c-k},$$

$$E_2 = \sum_{a=0}^{\alpha-n-1} \sum_{j=1}^{\alpha-n-a} \sum_{b=0}^{\alpha-n-a-j} \Omega_V(a,j,b) \tau^V_{\alpha-n-a-j-b},$$

$$E_3 = \sum_{a=0}^{\alpha-n-1} \sum_{j=1}^{\alpha-n-a} \sum_{b=\alpha-n-a-j+1}^{N-n-a-j} \Omega_V(a,j,b) \sum_{k=0}^{N-n-a-j-b-c} H_k \tau_{N-n-a-j-b-c-k},$$

$$E_4 = \sum_{a=0}^{\alpha-n-1} \sum_{j=\alpha-n-a+1}^{N-n-a} \sum_{b=0}^{N-n-a-j} \Omega_V(a,j,b) \sum_{k=0}^{N-n-a-j-b-c} H_k \tau_{N-n-a-j-b-c-k},$$

$$E_5 = \sum_{a=\alpha-n}^{N-n-1} \sum_{j=1}^{N-n-a} \sum_{b=0}^{N-n-a-j} \Omega_V(a,j,b) \sum_{k=0}^{N-n-a-j-b-c} H_k \tau_{N-n-a-j-b-c-k}$$

and

$$H_k = H_k^*(\theta)\big|_{\theta=0}.$$

Now we need to determine $w_N^{*(1)}(0)e$. Let us rewrite $w_N^*(\theta)e$ as

$$w_N^*(\theta)e = z^*(\theta) \frac{D - D(S^*(\theta))}{\lambda[1 - S^*(\theta)]} e, \tag{5.21}$$

where

$$\begin{aligned} z^*(\theta) = (1-\rho) \Bigg[& p_{vac} \sum_{n=0}^{\alpha-1} \psi_n^V [S^*(\theta)]^n \frac{1 - V^*(\theta)}{\mathrm{E}[V]\theta} H^*(\theta) \\ & + p_{su} \kappa \frac{1 - H^*(\theta)}{\mathrm{E}[H]\theta} + p_{sb} \sum_{n=\alpha}^{N-1} \psi_n^{sb} [S^*(\theta)]^n \Bigg] \\ & \cdot \theta [\theta I - D(S^*(\theta))]^{-1}. \end{aligned} \tag{5.22}$$

Taking the derivative of (5.21) with respect to θ we get

$$-w_N^{*(1)}(\theta)\Big|_{\theta=0} e = -\frac{z^{*(1)}(0) D^{(1)} e}{\lambda} + \frac{z^*(0) \mathrm{E}[S] D^{(2)} e}{2\lambda}, \tag{5.23}$$

where

$$D^{(n)} = \frac{d^n}{dz^n} D^{(n)}(z)\Big|_{z=1}.$$

The derivation of (5.23) is given in Appendix 2. Now we can show that

$$z^*(0) = \pi. \tag{5.24}$$

Then from (5.23) and (5.24), the mean actual waiting time becomes

$$\begin{aligned}
W_q = {} & (1-\rho)p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V\sum_{k=1}^{5}E_k \\
& +1-\rho)p_{su}\kappa\sum_{n=\alpha}^{N-1}H^-_{n(\alpha)}\sum_{a=0}^{N-n-1}\sum_{j=1}^{N-n-a}\sum_{b=0}^{N-n-a-j}\Omega_V(a,j,b)\tau_{N-n-a-j-b} \\
& +(1-\rho)p_{sb}\sum_{n=\alpha}^{N-1}\psi_n^{sb}\sum_{j=1}^{N-n}\frac{jD_j}{\lambda}\tau_{N-n-j} \\
& -\frac{1}{\lambda}\left[p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V+p_{su}\kappa+p_{sb}\sum_{n=\alpha}^{N-1}\psi_n^{sb}\right](D+e\pi)^{-1}D^{(1)}e \\
& +p_{vac}\sum_{n=0}^{\alpha-1}n\psi_n^V\mathrm{E}[S]e+p_{vac}\mathrm{E}[H]+p_{vac}\frac{\mathrm{E}[V^2]}{2\mathrm{E}[V]} \\
& +p_{sb}\sum_{n=\alpha}^{N-1}n\psi_n^{sb}\mathrm{E}[S]e+p_{su}\frac{\mathrm{E}[H^2]}{2\mathrm{E}[H]}+\frac{\lambda\mathrm{E}[S^2]}{2(1-\rho)}+\frac{\pi\mathrm{E}^2[S]D^{(2)}e}{2\rho(1-\rho)} \\
& +\frac{1}{1-\rho}-\frac{\pi\mathrm{E}[S]D^{(1)}(D+e\pi)^{-1}D^{(1)}e}{\lambda(1-\rho)}.
\end{aligned}$$

5.6 Numerical Example

In this section, we present a numerical example. We consider the parameter matrices as follows:

$$D_0 = \begin{pmatrix} -2.05 & 0.1 & 0.45 \\ 0.4 & -2.65 & 1.05 \\ 0.25 & 0.1 & -1.85 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.2 & 0.4 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.3 & 0.6 & 0.1 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 0.15 & 0.1 & 0.15 \\ 0.1 & 0.1 & 0 \\ 0.05 & 0.1 & 0.05 \end{pmatrix}, \quad D_3 = \begin{pmatrix} .0.1 & 0 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.3 & 0.6 & 0.1 \end{pmatrix}.$$

Then, we get

$$D = \sum_{j=0}^{3} D_j = \begin{pmatrix} 0.15 & 0.1 & 0.15 \\ 0.1 & 0.1 & 0 \\ 0.05 & 0.1 & 0.05 \end{pmatrix}.$$

Table 5.1 Comparison of mean performance measures with simulation.

(α,N)	ρ	Measure	Theoretical Value	Simulation	RPE
(3,5)	0.4311	L	5.0550	5.0517	0.065
(3,5)	0.4311	W_q	2.1456	2.1461	−0.023
(3,5)	0.8622	L	11.6486	11.6616	−0.112
(3,5)	0.8622	W_q	5.0047	5.0116	−0.138
(3,7)	0.4311	L	5.2116	5.2051	0.125
(3,7)	0.4311	W_q	2.2182	2.2166	0.298
(3,7)	0.8622	L	11.8049	11.8050	0.000
(3,7)	0.8622	W_q	5.0771	5.0812	−0.081

From $\pi D = 0,\ \pi e = 1$, and $\lambda = \pi \sum_{n=1}^{\infty} nD_n e$, we get

$$\pi = (0.35326, 0.23913, 0.40761), \quad \lambda = 2.1554348.$$

We consider two cases of thresholds: $(\alpha,N) = (3,5)$ and $(\alpha,N) = (3,7)$. For both cases we assume that the setup time and the vacation time follow the exponential distribution with mean 1.0. For each case, we assume two Erlang service times of order 2 with different mean service times: $\mathrm{E}[S] = 0.2$ and $\mathrm{E}[S] = 0.4$. Table 5.1 shows the comparison of the mean waiting times and the mean queue lengths that can be obtained from Little's law $L = \lambda\{W_q + \mathrm{E}[S]\}$ with those obtained from simulation estimates. The relative percentage error (RPE) is defined by

$$\frac{\text{Theoretical value} - \text{Simulation estimate}}{\text{Theoretical value}}.$$

5.7 Conclusions and Summary

In this chapter, we applied the BMAP/G/1 queue with early setup and multiple vacation to the analysis of the manufacturing lead time of a production system with extra jobs and bilevel threshold control. We employed the factorization principle to derive the distribution of the manufacturing lead time and the mean value.

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Appendix 1

We define the joint matrix transform by

$$\Omega_V^*(z_1, j, z_2, \theta) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\int_0^{\infty} z_1^a z_2^b e^{-\theta y} d\Omega_V(a, j, b, y).$$

Then, we have

$$\Omega_V^*(z_1, j, z_2, \theta) = \int_0^{\infty} e^{-\theta y}\int_0^{x} e^{D(z_1)(x-y)}\frac{j\pi D_j \mathrm{e}}{\lambda}\frac{D_j}{\pi D_j \mathrm{e}} e^{D(z_2)y}\frac{x\cdot dV(x)}{\mathrm{E}[V]}\frac{1}{x}dy, \quad (5.25)$$

which is equivalent to

$$\Omega_V^*(z_1, j, z_2, \theta) = \int_0^{\infty}\int_0^{x} e^{D(z_1)t}\frac{jD_j}{\lambda} e^{D(z_2)(x-t)} e^{-\theta(x-t)}\frac{dV(x)}{\mathrm{E}[V]}dt.$$

Our temporary objective is to obtain the coefficient matrix $\Omega_V^*(a, j, b, \theta)$ of $\Omega_V^*(z_1, j, z_2, \theta)$ such that

$$\Omega_V^*(z_1, j, z_2, \theta) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} z_1^a z_2^b \Omega_V^*(a, j, b, \theta). \quad (5.26)$$

To this end, we apply the well-known uniformization technique. Let us define Θ such that $\Theta = \max_i(-D_0)_{ii}$. First, $e^{D(z_1)t}$ and $e^{D(z_2)(x-t)}$ can be written as

$$e^{D(z_1)t} = e^{-\Theta t} e^{\Theta(I+\Theta^{-1}D(z_1))t} = \sum_{k=0}^{\infty}\frac{e^{-\Theta t}(\Theta t)^k}{k!}(I+\Theta^{-1}D(z_1))^k \quad (5.27)$$

and

$$\begin{aligned} e^{D(z_2)(x-t)} &= e^{-\Theta(x-t)} e^{\Theta(I+\Theta^{-1}D(z_2))(x-t)} \\ &= \sum_{k=0}^{\infty}\frac{e^{-\Theta(x-t)}[\Theta(x-t)]^k}{k!}(I+\Theta^{-1}D(z_2))^k. \end{aligned} \quad (5.28)$$

Using (5.27) and (5.28) in (5.26) yields

$$\begin{aligned} \Omega_V^*(z_1, j, z_2, \theta) = \int_0^{\infty} e^{-(\Theta+\theta)x}\int_0^{x} e^{\theta t}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{t^k(x-t)^l}{k!l!}(\Theta I+D(z_1))^k\frac{jD_j}{\lambda} \\ \cdot(\Theta I+D(z_2))^l\frac{dV(x)}{\mathrm{E}[V]}dt. \end{aligned} \quad (5.29)$$

In (5.29), only $(\Theta I+D(z_1))^k(jD_j/\lambda)(\Theta I+D(z_2))^l$ contains z_1 and z_2. To evaluate this matrix, we define $F_{k,l}(a, j, b),\ (k, l, a, b = 0, 1, \ldots; j = 1, 2, \ldots)$ such that

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} z_1^a z_2^b F_{k,l}(a,j,b) = (\Theta I + D(z_1))^k \frac{jD_j}{\lambda}(\Theta I + D(z_2))^l, \tag{5.30}$$

where $F_{0,0}(0,j,0) = jD_j/\lambda$, and $F_{0,0}(a,j,b) = 0$, $(a \geq 1, b \geq 1)$. $F_{k,l}(a,j,b)$ represents the situation in which a jobs arrive from k Poisson events (with rate Θ) during the elapsed vacation time and b jobs arrive from l Poisson events during the remaining vacation time. Then, $F_{k,l}(a,j,b)$ satisfies the following recursions:

$$F_{k+1,l}(a,j,b) = \begin{cases} (\Theta I + D_0)F_{k,l}(a,j,b), & (a=0) \\ \sum_{i=0}^{a-1} D_{a-i}F_{k,l}(a,j,b) + (\Theta I + D_0)F_{k,l}(a,j,b), & (a \geq 1), \end{cases} \tag{5.31}$$

$$F_{k,l+1}(a,j,b) = \begin{cases} F_{k,l}(a,j,b)(\Theta I + D_0), & (b=0) \\ \sum_{i=0}^{b-1} F_{k,l}(a,j,i)D_{b-i} + F_{k,l}(a,j,b)(\Theta I + D_0), & (b \geq 1). \end{cases} \tag{5.32}$$

Using (5.31) and (5.32) in (5.30), we get

$$\begin{aligned} &\Omega_V^*(z_1,j,z_2,\theta) \\ &= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} z_1^a z_2^b \int_0^{\infty} e^{-(\Theta+\theta)x} \int_0^x e^{\theta t} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{t^k (x-t)^l}{k!l!} F_{k,l}(a,j,b) \frac{dV(x)}{\mathrm{E}[V]} dt. \end{aligned} \tag{5.33}$$

The coefficient matrix of $z_1^a z_2^b$ in (5.33) is given by

$$\Omega_V^*(a,j,b,\theta) = \int_0^{\infty} e^{-(\Theta+\theta)x} \int_0^x e^{\theta t} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{t^k (x-t)^l}{k!l!} F_{k,l}(a,j,b) \frac{dV(x)}{\mathrm{E}[V]} dt. \tag{5.34}$$

If we disregard the length of the remaining vacation time, we have

$$\Omega_V(a,j,b) = \Omega_V^*(a,j,b,\theta)\big|_{\theta=0} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} f_{k,l} F_{k,l}(a,j,b),$$

where

$$f_{k,l} = \frac{1}{\mathrm{E}[V](k+l+1)!} \int_0^{\infty} x^{k+l+1} e^{-\Theta x} dV(x).$$

Appendix 2: Derivation of (5.23)

Taking a derivative of (5.22) with respect to θ, using $\theta = 0$ and adding $z^*(\theta)e\pi$ to both sides yields

$$z^{*(1)}(0) = z^{*(1)}(0)e\pi(D+e\pi)^{-1} - z^*(0)[I - \mathrm{E}[S]D^{(1)}](D+e\pi)^{-1} + (1-\rho)\left[p_v \sum_{n=0}^{\alpha-1} \psi_n^V + p_{su}\kappa + p_{sb}\sum_{n=\alpha}^{N-1}\psi_n^{sb}\right](D+e\pi)^{-1}. \tag{5.35}$$

Taking the second derivative of (5.22), using $\theta = 0$ and postmultiplying e yields

$$\begin{aligned} & z^{*(1)}(0)[I - \mathrm{E}[S]D^{(1)}]e \\ &= -\ (1-\rho)\left\{p_{vac}\sum_{n=0}^{\alpha-1} n\psi_n^V \mathrm{E}[S] + p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V \mathrm{E}[H]\right. \\ &\quad \left. +\ p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V \frac{\mathrm{E}[V^2]}{2\mathrm{E}[V]} + p_{su}\kappa\frac{\mathrm{E}[H^2]}{2\mathrm{E}[H]} + p_{sb}\sum_{n=\alpha}^{N-1} n\psi_n^{sb}\mathrm{E}[S]\right\}e \\ &\quad -\ \frac{\pi}{2}\left[\mathrm{E}[S^2]D^{(1)} + \mathrm{E}^2[S]D^{(2)}\right]e. \end{aligned} \tag{5.36}$$

From (5.36), we get

$$\begin{aligned} z^{*(1)}(0)e &= z^{*(1)}(0)\mathrm{E}[S]D^{(1)}e \\ &= -(1-\rho)\left\{p_{vac}\sum_{n=0}^{\alpha-1} n\psi_n^V \mathrm{E}[S] + p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V \mathrm{E}[H]\right. \\ &\quad \left. + p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V \frac{\mathrm{E}[V^2]}{2\mathrm{E}[V]} + p_{su}\kappa\frac{\mathrm{E}[H^2]}{2\mathrm{E}[H]} + p_{sb}\sum_{n=\alpha}^{N-1} n\psi_n^{sb}\mathrm{E}[S]\right\}e \\ &\quad - \frac{\pi}{2}\left[\mathrm{E}[S^2]D^{(1)} + \mathrm{E}^2[S]D^{(2)}\right]e. \end{aligned} \tag{5.37}$$

Postmultiplying both sides of (5.35) by $D^{(1)}e$, we get

$$z^{*(1)}(0)D^{(1)}e = \lambda z^{*(1)}(0)e - z^*(0)[I - \mathrm{E}[S]D^{(1)}](D+e\pi)^{-1}D^{(1)}e + (1-\rho)\left[p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V + p_{su}\kappa + p_{sb}\sum_{n=\alpha}^{N-1}\psi_n^{sb}\right](D+e\pi)^{-1}D^{(1)}e. \tag{5.38}$$

Using (5.37) in (5.38), we get

$$\begin{aligned}\frac{z^{*(1)}(0)D^{(1)}e}{\lambda} &= -p_{vac}\sum_{n=0}^{\alpha-1} n\psi_n^V \mathrm{E}[S]e + p_{vac}\mathrm{E}[H] + p_{vac}\frac{\mathrm{E}[V^2]}{2\mathrm{E}[V]} \\ &+ p_{su}\frac{\mathrm{E}[H^2]}{2\mathrm{E}[H]} + p_{sb}\sum_{n=\alpha}^{N-1} n\psi_n^{sb}\mathrm{E}[S]e \\ &- \pi 2(1-\rho)[\mathrm{E}[S^2]D^{(1)} + \mathrm{E}^2[S]D^{(2)}]e \\ &+ \frac{1}{\lambda}\left[p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V + p_{su}\kappa + p_{sb}\sum_{n=\alpha}^{N-1}\psi_n^{sb}\right](D+e\pi)^{-1}D^{(1)}e \\ &- \pi\lambda(1-\rho)[I-\mathrm{E}[S]D^{(1)}](D+e\pi)^{-1}D^{(1)}e.\end{aligned}$$

Thus, we get

$$\begin{aligned}-w_N^{*(1)}(\theta)\Big|_{\theta=0} e &= -\frac{z^{*(1)}(0)D^{(1)}e}{\lambda} + \frac{\mathrm{E}[S]z^*(0)D^{(2)}e}{2\lambda} \\ &= -\frac{1}{\lambda}\left[p_{vac}\sum_{n=0}^{\alpha-1}\psi_n^V + p_{su}\kappa + p_{sb}\sum_{n=\alpha}^{N-1}\psi_n^{sb}\right](D+e\pi)^{-1}D^{(1)}e \\ &+ p_{vac}\sum_{n=0}^{\alpha-1} n\psi_n^V\mathrm{E}[S]e + p_{vac}\mathrm{E}[H] + p_{vac}\frac{\mathrm{E}[V^2]}{2\mathrm{E}[V]} \\ &+ p_{sb}\sum_{n=\alpha}^{N-1} n\psi_n^{sb}\mathrm{E}[S]e + p_{su}\frac{\mathrm{E}[H^2]}{2\mathrm{E}[H]} + \frac{\lambda\mathrm{E}[S^2]}{2(1-\rho)} \\ &+ \frac{\pi\mathrm{E}^2[S]D^{(2)}e}{2\rho(1-\rho)} + \frac{1}{1-\rho} - \frac{\pi\mathrm{E}[S]D^{(1)}(D+e\pi)^{-1}D^{(1)}e}{\lambda}(1-\rho).\end{aligned}$$

Appendix 3: Derivation of (5.16)

Let $Q_n^{(\alpha,N)}$ be the matrix probability that there are n customers at the start of the busy period. Then, we have

$$Q_n^{(\alpha,N)} = \sum_{n=N}^{\infty} H_{n(\alpha)}^{+} + \sum_{n=N}^{\infty}\sum_{k=\alpha}^{N-1}\Phi_k^{sb}(-D_0)^{-1}D_{n-k}.$$

Taking GF and using (5.15), we have

$$\begin{aligned}Q_{(\alpha,N)}(z) = \sum_{n=N}^{\infty} H_{n(\alpha)}^{+}z^n + \sum_{k=\alpha}^{N-1}\Phi_k^{sb}(-D_0)^{-1}z^kD(z) - \sum_{k=\alpha}^{N-1}\Phi_k^{sb}(-D_0)^{-1}z^k \\ \cdot\sum_{i=0}^{N-k-1} D_iz^i.\end{aligned} \tag{5.39}$$

The last term in (5.39) becomes

$$\begin{aligned}
&\sum_{k=\alpha}^{N-1} \Phi_k^{sb}(-D_0)^{-1} z^k \sum_{i=0}^{N-k-1} D_i z^i \\
&\quad = \sum_{k=\alpha}^{N-1} \Phi_k^{sb}(-D_0)^{-1} z^k \sum_{i=1}^{N-k-1} D_i z^i + \sum_{k=\alpha}^{N-1} \Phi_k^{sb}(-D_0)^{-1} z^k D_0 \qquad (5.40)\\
&\quad = \sum_{k=\alpha+1}^{N-1} \sum_{i=\alpha}^{k-1} \Phi_i^{sb}(-D_0)^{-1} D_{k-i} z^k - \sum_{k=\alpha}^{N-1} \Phi_k^{sb} z^k,
\end{aligned}$$

where we used

$$\sum_{k=\alpha}^{N-1} \Phi_k^{sb}(-D_0)^{-1} z^k \sum_{i=1}^{N-k-1} D_i z^i = \sum_{k=\alpha+1}^{N-1} \sum_{i=\alpha}^{k-1} \Phi_i^{sb}(-D_0)^{-1} D_{k-i} z^k.$$

Using $\Phi_k^{sb} = \sum_{i=\alpha}^{k} H_{i(\alpha)}^{+} D_{k-i}^{*}$ in (5.40), we get

$$\begin{aligned}
&\sum_{k=\alpha+1}^{N-1} \sum_{i=\alpha}^{k-1} \Phi_i^{sb}(-D_0)^{-1} D_{k-i} z^k - \sum_{k=\alpha}^{N-1} \Phi_k^{sb} z^k \\
&\quad = \sum_{k=\alpha+1}^{N-1} \sum_{i=\alpha}^{k-1} \sum_{n=\alpha}^{i} H_{n(\alpha)}^{+} D_{i-n}^{*}(-D_0)^{-1} D_{k-i} z^k - \sum_{k=\alpha}^{N-1} \sum_{i=\alpha}^{k} H_{i(\alpha)}^{+} D_{k-i}^{*} z^k. \qquad (5.41)
\end{aligned}$$

Let us simplify (5.41). For convenience, let us define

$$\sum_{k=\alpha+1}^{N-1} \Gamma_k z^k = \sum_{k=\alpha+1}^{N-1} \sum_{i=\alpha}^{k-1} \sum_{n=\alpha}^{i} H_{n(\alpha)}^{+} D_{i-n}^{*}(-D_0)^{-1} D_{k-i} z^k. \qquad (5.42)$$

Then, using $D_k^{*} = \sum_{l=0}^{k-1} D_l^{*}(-D_0)^{-1} D_{k-l}$, we get

$$\sum_{k=\alpha+1}^{N-1} \Gamma_k z^k = \sum_{k=\alpha+1}^{N-1} \sum_{i=\alpha}^{k-1} H_{i(\alpha)}^{+} D_{k-i}^{*} z^k. \qquad (5.43)$$

Using (5.43) in (5.42), we get

$$\begin{aligned}
&\sum_{k=\alpha+1}^{N-1} \sum_{i=\alpha}^{k-1} \Phi_i^{sb}(-D_0)^{-1} D_{k-i} z^k - \sum_{k=\alpha}^{N-1} \Phi_k^{sb} z^k \\
&\quad = \sum_{k=\alpha+1}^{N-1} \sum_{i=\alpha}^{k-1} H_{i(\alpha)}^{+} D_{k-i}^{*} z^k - \left[\sum_{k=\alpha} {+1}^{N-1} \sum_{i=\alpha}^{k-1} H_{i(\alpha)}^{+} D_{k-i}^{*} z^k + \sum_{k=\alpha}^{N-1} H_{k(\alpha)}^{+} z^k \right] \qquad (5.44)\\
&\quad = -\sum_{k=\alpha}^{N-1} H_{k(\alpha)}^{+} z^k.
\end{aligned}$$

Using (5.44) in (5.40), we have

$$\sum_{k=\alpha}^{N-1} \Phi_k^{sb}(-D_0)^{-1} z^k \sum_{i=0}^{N-k-1} D_i z^i = -\sum_{k=\alpha}^{N-1} H_{k(\alpha)}^{+} z^k.$$

Thus, (5.39) becomes

$$\begin{aligned} Q_{(\alpha,N)}(z) &= \sum_{n=N}^{\infty} H_{n(\alpha)}^{+} z^n + \sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1} z^n D(z) + \sum_{n=\alpha}^{N-1} H_{n(\alpha)}^{+} z^n \\ &= \sum_{n=\alpha}^{\infty} H_{n(\alpha)}^{+} z^n + \sum_{n=\alpha}^{N-1} \Phi_n^{sb}(-D_0)^{-1} z^n D(z). \end{aligned}$$

Now, using $\sum_{n=\alpha}^{\infty} H_{n(\alpha)}^{+} z^n = \sum_{n=\alpha}^{\infty} H_{n(\alpha)}^{-} z^n H(z)$ finishes the proof.

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