Chapter 3
A Pure Decrement Service Geom/G/1 Queue with Multiple Adaptive Vacations

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Abstract In this chapter, a Geom/G/1 queue model with a pure decrement service policy and multiple adaptive vacations is analyzed. The Probability Generation Function (P.G.F.) of the queue length is obtained by using an embedded Markov chain method. The P.G.F. of the waiting time is then derived based on the independence between the arrival process and the waiting time. The probabilities for the system being in various states of busy, vacation, or idle, respectively, are also derived. Finally, some special cases for the Geom/G/1 queue model with a pure decrement service policy and multiple adaptive vacations are given to demonstrate the general properties of the queue models.

3.1 Introduction

Tian [1] introduced a multiple adaptive vacation policy, and studied a multiple adaptive vacation M/G/1 queue model with an exhaustive service rule, and through this, queue models with multiple vacations and single vacation were extended. Zhang and Tian studied the discrete time queue model with multiple adaptive vacations, and obtained the P.G.F. of the queue length and waiting time in [2].

However, they only researched the queue models with the exhaustive service polity. Many researchers have studied discrete time queue models with some

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vacation policy. For example, M/G/1 queues with multiple types of feedback and gated vacations were studied, and some important results were derived in [3]. A number of discrete time queue models were studied in [4] and [5]. Also, Wu and Takagi studied the queue model with working vacations in [6], and extended the general vacation polity.

The new queue model enriched the theory of the queue with vacations, and involved many queue models as special cases. A discrete time GI/Geo/1 queue model with multiple vacations was studied in [7]. A discrete time queue model with timed vacations was analyzed in [8]–[10].

Bischof studied the queue model with vacations under six different service disciplines in [11], which expanded the research of the nonexhaustive service disciplines. Performance evaluations of SVC-Based IP-Over-ATM networks were given using discrete time queueing theory in [12]. However, these papers did not integrate multiple adaptive vacations with nonexhaustive service disciplines. The authors' purpose for studying a new queueing model was to promote this integration.

In this chapter, we analyze a general Geom/G/1 queueing model with a pure decrement service strategy and multiple adaptive vacations. We show that the pure decrement service systems analyzed in [5] and [13] are special cases of our model presented in this chapter. Furthermore, we compare the system performance for pure decrement service strategies with multiple vacations and single vacation.

The chapter is organized as follows. Section 3.2 describes the analysis model in detail. Section 3.3 presents analysis of system performance. Some special cases are presented in Sect. 3.4. In Sect. 3.5, we discuss some numerical results. Concluding remarks are given in Sect. 3.6.

3.2 Model Description

Based on the classical Geom/G/1 queueing model, we introduce the strategies of a pure decrement service and the multiple adaptive vacations [5], [13].

A pure decrement service strategy can be described as follows. Once the service period starts, the server will keep on working until the number of customers in the system is one less than the number of customers at the start instant of the service period. The server will then enter a new vacation period. If there are some customers waiting at a vacation completion instant, the server will complete the vacation period and start a new service period. Otherwise, the server will take some vacations consecutively according to the assistant workload completed at that time.

The maximum number of vacations during a vacation period is denoted by H. H is a positive integer random variable with the probability distribution h_j and the P.G.F. $H(z)$ as follows:

$$P(H=j)=h_j, \quad j \geq 1, \quad H(z)=\sum_{j=1}^{\infty} h_j z^j.$$

Let V_k $(k=1,2,\ldots,H)$ be the time length for the kth vacation. V_k is an independently identically distributed (i.i.d.) random variable. If there is no customer in the system at the Hth vacation completion instant, the system will enter an idle period and wait for a new customer to arrive. If a customer arrives during the idle period, the server will enter a service period immediately, and continue until there are waiting customers in the system, before taking a vacation again at the completion instant of the service. The system will continually repeat the above processes.

Specifically, (1) if $H \to \infty$, the model corresponds to a pure decrement service Geom/G/1 queueing model with multiple vacations [5], [10], [13]. (2) If $H = 1$, the model corresponds to a pure decrement service Geom/G/1 queueing model with a single vacation [5], [13]. (3) If H follows another distribution, the model corresponds to another special queueing model.

The basic assumptions of the new model presented in this chapter are given as follows.

(1) In order to describe the system states in the nth discrete time instants, we assume that customer arrivals can only occur at discrete time instants $t = n^-$, $n=0,\ 1,\ \ldots$, the service starts and ends can only occur at discrete time instants $t=n^+, n=1,2,\ldots$. The model is called a late arrival system. The interarrival time, denoted by T, is supposed to be an i.i.d. discrete random variable following a geometric distribution with parameter p $(0 < p < 1)$. We can write the probability distribution of T as follows:

$$P(T=j) = p\bar{p}^{j-1}, \qquad j=1,2,\ldots,$$

where $\bar{p} = 1-p$. We denote by C_n the number of customers arriving during the interval $[0,n]$; then C_n follows a binomial distribution,

$$P(C_n = j) = \binom{n}{j} p^j \bar{p}^{n-j}, \qquad j=0,1,\ldots,n.$$

(2) The service time S of a customer is supposed to be an i.i.d. discrete random variable with a general distribution; the probability distribution s_j and the P.G.F. $S(z)$ of S are given as follows:

$$P(S_i = j) = s_j, \quad j \geq 1, \quad S(z) = \sum_{j=1}^{\infty} s_j z^j.$$

Let $\mathrm{E}[S]$ and $\mathrm{E}[S(S-1)]$ be the mean and the second factorial moment of S; then we have

$$\frac{1}{\mu} = \mathrm{E}[S] = \sum_{i=0}^{\infty} i s_i, \quad \mathrm{E}[S(S-1)] = \left.\frac{d^2 S(z)}{dz^2}\right|_{z=1}.$$

(3) The time length V of a vacation is a nonnegative i.i.d. discrete random variable with general probability distribution v_j and the P.G.F. $V(z)$ given by

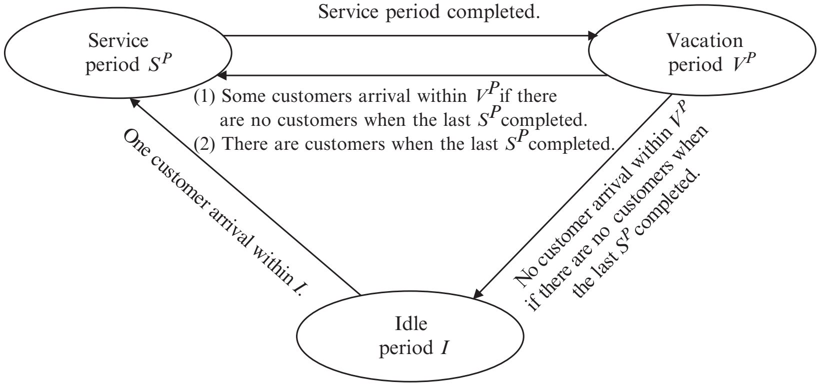

Fig. 3.1 State transition diagram of the model.

$$P(V=j)=v_j, \quad j\geq 1, \quad V(z)=\sum_{j=1}^{\infty} v_j z^j,$$

where the mean $\mathrm{E}[V]$ and the second factorial moment $\mathrm{E}[V(V-1)]$ of V exist.

Suppose that there is a single server in this system, and its buffer capacity is infinite. The interarrival time, the service time, and the time length of a vacation are mutually independent. The service order is First-Come First-Served (FCFS). The model is denoted by Geom/G/1 (PD, MAVs), where PD and MAVs represent the Pure Decrement and the Multiple Adaptive Vacations, respectively. Let S^P represent the service period, V^P represent the vacation period, and I represent the idle period, respectively. The state transition diagram of the model is shown in Fig. 3.1.

Let L_v represent the stationary queue length at the departure instant of a customer, and let $Q_b^{(n)}$ represent the number of customers in the system at the nth vacation completion instant, the P.G.F. of $Q_b^{(n)}$ is denoted by $Q_b^{(n)}(z)$. L_v is supposed to follow an identical distribution for the new model in the service orders of FCFS or Last-Come First-Served (LCFS). For simplification, we assume that the service of the model presented in this chapter follows a LCFS strategy.

3.3 Analysis of System Performance Measures

3.3.1 Number of Customers at the Beginning of a Service Period

Let J be the number of consecutive vacations taken by the server after the end of a service period when the system is empty. J is a random variable, and we have

$$J=\min\{H,k:V_1+\cdots+V_{k-1}<T<V_1+\cdots+V_k\}.$$

We define two events as follows:

$$A_I = \{ \text{ Service period starts with the end of an idle period if there are no customers at the end of the last } S^P \};$$
$$A_v = \{ \text{ Service period starts with the end of a vacation period if there are no customers at the end of the last } S^P \}.$$

Then, we have $P(A_I)$, the probability of A_I and $P(A_v)$, the probability of A_v as

$$P(A_I) = Q_b^{(n)}(0)H(V(\bar{p})), \qquad P(A_v) = Q_b^{(n)}(0)(1-H(V(\bar{p}))).$$

According to a pure decrement service order, if a service period with zero duration is allowed, $Q_b^{(n)}$ is the number of customers in the system at the next start instant of the service period. If $Q_b^{(n)}$ is greater than zero, the service period starts immediately and keeps on working until the number of customers in the system is one less than the number of customers at the start instant of the service period. Then the server will take a vacation. $Q_b^{(n+1)}$ is equal to the sum of $Q_b^{(n)} - 1$ plus the number of customers arriving during the vacation. If $Q_b^{(n)} = 0$, there are two cases as follows:

(1) If there are customer arrivals during the kth $(1 \le k \le H)$ vacation, a service period starts at the instant where the kth $(1 \le k \le H)$ vacation completes. The number of customers in the system at the start instant of the service period $Q_b^{(n+1)}$ is equal to the number of customers arriving during the vacation.
(2) If no customers arrive during the Hth vacation, an idle period will begin at the end of the vacation and continue until a new customer arrives. In this case, $Q_b^{(n+1)}$ is equal to 1. Therefore,

$$Q_b^{(n+1)}(z) = Q_b^{(n)}(0)H(V(\bar{p}))z + \frac{Q_b^{(n)}(z) - Q_b^{(n)}(0)}{z}V(1-p(1-z)) + Q_b^{(n)}(0)(1-H(V(\bar{p})))V(1-p(1-z)). \tag{3.1}$$

If the system is in a steady state, the P.G.F. $Q_b(z)$ of $Q_b^{(n+1)}$ does not depend on n in (3.1). If we let $\lim_{n\to\infty} Q_b^{(n+1)}(z) = Q_b(z)$, we can obtain $Q_b(z)$ as follows:

$$Q_b(z) = \frac{Q_b(0)\Big((1-z(1-H(V(\bar{p})))) \times V(1-p(1-z)) - H(V(\bar{p}))z^2\Big)}{V(1-p(1-z)) - z}. \tag{3.2}$$

Because $Q_b(1) = 1$, we have that

$$Q_b(0) = \frac{1-p\mathrm{E}[V]}{1+H(V(\bar{p}))(1-p\mathrm{E}[V])}. \tag{3.3}$$

According to the Foster rule (see Tian and Zhang [13]), we can prove that if $\rho = p/\mu < 1$ and $p\mathrm{E}[V] < 1$, the system can reach a steady state.

3.3.2 Stationary Queue Length and Waiting Time

Theorem 3.1. *If $\rho = p/\mu < 1$ and $pE[V] < 1$, the stationary queue length L_v in the Geom/G/1 (PD, MAVs) queue can be decomposed into three independent random variables:*

$$L_v = L + L_d + L_r,$$

where L is the stationary queue length in a classical Geom/G/1 queue [5], [13]. The P.G.F. $L(z)$ of L is

$$L(z) = \frac{(1-\rho)(1-z)S(1-p(1-z))}{S(1-p(1-z))-z}. \tag{3.4}$$

The additional queue length L_d is the number of customers arriving during a vacation or is equal to zero, and the additional queue length L_r is the number of customers in the system at the start instant of a vacation. P.G.F.s $L_d(z)$ and $L_r(z)$ of additional queue lengths L_d and L_r are given by

$$\begin{aligned} L_d(z) &= \frac{1 - V(1-p(1-z)) + H(V(\bar{p}))V(1-p(1-z)) - H(V(\bar{p}))z}{(H(V(\bar{p})) + p\mathrm{E}[V](1-H(V(\bar{p}))))(1-z)}, \\ L_r(z) &= \frac{(1-p\mathrm{E}[V])(1-z)}{V(1-p(1-z))-z}. \end{aligned} \tag{3.5}$$

Proof. Q_b is the number of customers in the system at the start instant of a service. In the pure decrement service rule and LCFS order, a nonzero service period in the system is exactly the same as a standard busy period in a Geom/G/1 queue. So there are two kinds of customers in the system at a departure instant as follows:

(1) If $Q_b > 0$, only the customer who initiates the new service period can be served, and the residual $Q_b - 1$ customers wait to be served during the next service period. The P.G.F. of the number of these customers is given by

$$\frac{Q_b(z) - Q_b(0)}{(1-Q_b(0))z}. \tag{3.6}$$

(2) The number of customers (sub generation) who arrive during the service period and cannot be served is equivalent to the number of customers in a classical Geom/G/1 queue; the P.G.F. is given by

$$\frac{(1-\rho)(1-z)S(1-p(1-z))}{S(1-p(1-z))-z}. \tag{3.7}$$

Because the two kinds of customers are mutually independent, we have that

$$L_v(z) = \frac{(1-\rho)(1-z)S(1-p(1-z))}{S(1-p(1-z))-z} \times \frac{Q_b(z) - Q_b(0)}{(1-Q_b(0))z}. \tag{3.8}$$

Substituting (3.2) and (3.3) into (3.8), we have that

$$
\begin{aligned}
L_v(z) &= \frac{(1-\rho)(1-z)S(1-p(1-z))}{S(1-p(1-z))-z} \\
&\times \frac{1-V(1-p(1-z))+H(V(\bar{p}))V(1-p(1-z))-H(V(\bar{p}))z}{(H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p}))))(1-z)} \\
&\times \frac{(1-p\mathrm{E}[V])(1-z)}{V(1-p(1-z))-z} \\
&= L(z)L_d(z)L_r(z).
\end{aligned} \tag{3.9}
$$

Therefore, $L_v(z)$ is also the P.G.F. of the system queue length in the FCFS service strategy. □

Simplifying $L_d(z)$ in Theorem 3.1, we have that

$$
\begin{aligned}
L_d(z) &= \frac{H(V(\bar{p}))}{H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p})))} \\
&+ \frac{p\mathrm{E}[V](1-H(V(\bar{p})))}{H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p})))} \times \frac{1-V(1-p(1-z))}{p\mathrm{E}[V](1-z)}.
\end{aligned} \tag{3.10}
$$

Therefore, the additional queue length L_d is equal to zero with the following probability,

$$
\frac{H(V(\bar{p}))}{H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p})))}
$$

and is equal to the number of customers arriving before an arbitrary time instant during a vacation with the following probability,

$$
\frac{p\mathrm{E}[V](1-H(V(\bar{p})))}{H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p})))}.
$$

Differentiating the two sides of (3.9) and using L'Hospital's rule, we can obtain the mean $\mathrm{E}[L_v]$ of the number of customers at steady state for a Geom/G/1 (PD, MAVs) queue system as follows:

$$
\begin{aligned}
\mathrm{E}[L_v] &= \rho + \frac{p^2\mathrm{E}[S(S-1)]}{2(1-\rho)} + \frac{p^2\mathrm{E}[V(V-1)](1-H(V(\bar{p})))}{2(H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p}))))} \\
&+ \frac{p^2\mathrm{E}[V(V-1)]}{2(1-p\mathrm{E}[V])}.
\end{aligned} \tag{3.11}
$$

Theorem 3.2. *If $\rho = p/\mu < 1$ and $pE[V] < 1$, the stationary waiting time W_v of a customer can be decomposed into three independent random variables in a Geom/G/1 (PD, MAVs) queue as follows:*

$$
W_v = W + W_d + W_r,
$$

where W is the stationary waiting time in a classical Geom/G/1 queue [5], [13]. The P.G.F. $W(z)$ of the stationary waiting time W is given by

$$W(z) = \frac{(1-\rho)(1-z)}{(1-z)-p(1-S(z))}. \tag{3.12}$$

The additional delay W_d is a vacation or is equal to zero, and the additional delay W_r is the time delay caused by the existing customers at the start instant of a vacation. P.G.F.s $W_d(z)$ and $W_r(z)$ of additional delays W_d and W_r are given by

$$\begin{aligned} W_d(z) &= \frac{p(1-H(V(\bar{p})))(1-V(z))+H(V(\bar{p}))(1-z)}{(H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p}))))(1-z)}, \\ W_r(z) &= \frac{(1-p\mathrm{E}[V])(1-z)}{(1-z)-p(1-V(z))}. \end{aligned} \tag{3.13}$$

Proof. In a Geom/G/1 (PD, MAVs) queue, the waiting time is independent of the customers' inputting process after the arrival instant of the customers in the FCFS service strategy. The queue length of the system at a customer's service completion instant is composed of the number of other customers arriving during the waiting time and the service time of the customer. Therefore, we have that

$$L_v(z) = W_v(1-p(1-z))S(1-p(1-z)). \tag{3.14}$$

Substituting the result of Theorem 3.1 into (3.14), we have that

$$\begin{aligned} W_v(z) &= \frac{(1-\rho)(1-z)}{(1-z)-p(1-S(z))} \\ &\quad \times \frac{p(1-H(V(\bar{p})))(1-V(z))+H(V(\bar{p}))(1-z)}{(H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p}))))(1-z)} \\ &\quad \times \frac{(1-p\mathrm{E}[V])(1-z)}{(1-z)-p(1-V(z))} \\ &= W(z)W_d(z)W_r(z). \end{aligned} \tag{3.15}$$

□

From Theorem 3.2, we can obtain the P.G.F. $W_d(z)$ of the stationary waiting time W_d as follows:

$$\begin{aligned} W_d(z) &= \frac{H(V(\bar{p}))}{H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p})))} \\ &\quad + \frac{p\mathrm{E}[V](1-H(V(\bar{p})))}{H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p})))} \times \frac{1-V(z)}{\mathrm{E}[V](1-z)}. \end{aligned} \tag{3.16}$$

Therefore, the additional delay W_d is equal to zero with the following probability,

$$\frac{H(V(\bar{p}))}{H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p})))}$$

and is equal to a vacation time with the following probability,

$$\frac{p\mathrm{E}[V](1-H(V(\bar{p})))}{H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p})))}.$$

Differentiating the two sides of (3.15) and using L'Hospital's rule, we can get the mean waiting time $\mathrm{E}[W_v]$ of a customer during a steady state for a Geom/G/1 (PD, MAVs) queue system as follows:

$$\mathrm{E}[W_v] = \frac{p\mathrm{E}[S(S-1)]}{2(1-\rho)} + \frac{p\mathrm{E}[V(V-1)](1-H(V(\bar{p})))}{2(H(V(\bar{p}))+p\mathrm{E}[V](1-H(V(\bar{p}))))} + \frac{p\mathrm{E}[V(V-1)]}{2(1-p\mathrm{E}[V])}.$$

3.3.3 Analysis of Service Cycle

According to the number J of consecutive vacations [1], [13], we have

$$\begin{aligned} &P(J \geq 1) = 1, \\ &P(J \geq j) = P(H \geq j)P(V_1 + \cdots + V_{j-1} < T) = (V(\bar{p}))^{j-1} \sum_{k=j}^{\infty} h_k, \quad j \geq 2; \end{aligned} \tag{3.17}$$

thus the P.G.F. $J(z)$ of J can be given as

$$J(z) = 1 - \frac{1-z}{1-V(\bar{p})z}(1-H(V(\bar{p})z)). \tag{3.18}$$

Vacation time lengths in the following two cases are: (1) if a customer is present at a vacation start instant, the total time length is the time of a vacation; (2) if there are no customers present at a vacation start instant, the total vacation time length is the sum of the time lengths of a random number of vacations. Concluding from the two cases above, we can get P.G.F. $V_G(z)$ of the total time length V_G for consecutive vacations as follows:

$$\begin{aligned} V_G(z) = &\frac{1-(1-H(V(\bar{p})))(1-p\mathrm{E}[V])}{1+H(V(\bar{p}))(1-p\mathrm{E}[V])}V(z) + \frac{1-p\mathrm{E}[V]}{1+H(V(\bar{p}))(1-p\mathrm{E}[V])} \\ &\times \left(1 - \frac{1-V(z)}{1-V(\bar{p})V(z)}(1-H(V(\bar{p})V(z)))\right). \end{aligned} \tag{3.19}$$

Therefore, the mean total time length of a vacation can be obtained as

$$\begin{aligned} \mathrm{E}[V_G] = &\frac{1-(1-H(V(\bar{p})))(1-p\mathrm{E}[V])}{1+H(V(\bar{p}))(1-p\mathrm{E}[V])}\mathrm{E}[V] \\ &+ \frac{1-p\mathrm{E}[V]}{1+H(V(\bar{p}))(1-p\mathrm{E}[V])} \times \frac{1-H(V(\bar{p}))}{1-V(\bar{p})}\mathrm{E}[V]. \end{aligned} \tag{3.20}$$

In a Geom/G/1 (PD, MAVs) queue model, the server is usually in an idle state. If there are customers in the system at the start instant of the vacation, the idle period will be zero after the completion of a vacation. If there are no customers in the system at the start instant of the vacation, and there are still no customers at the Jth vacation completion instant, the time length I_v of the server's idle period is the inter-arrival time following a nonnegative exponential distribution. We can give the mean $\mathrm{E}[I_v]$ of I_v as

$$\mathrm{E}[I_v] = \frac{1}{p}\frac{(1-p\mathrm{E}[V])H(V(\bar{p}))}{1+H(V(\bar{p}))(1-p\mathrm{E}[V])}. \tag{3.21}$$

According to a pure decrement service strategy, we know that the service period in the new model presented in this chapter is identical to the busy period in a classical Geom/G/1 queue system. This means the P.G.F. $S_p(z)$ of the service period S_p in the queue models of [5] and [13] satisfies the following and the mean length of the service period is given by

$$S_p(z) = S(zS_p((1-p(1-z)))), \qquad \mathrm{E}[S_p] = \frac{1}{\mu - p}.$$

We call the intermediate time between two continuous start instants of the service a service cycle, denoted by C. The mean of the service cycle $\mathrm{E}[C]$ can thus be obtained as follows:

$$\begin{aligned}\mathrm{E}[C] &= \mathrm{E}[S_p] + \mathrm{E}[V_G] + \mathrm{E}[I_v] \\ &= \frac{1-V(\bar{p})+V(\bar{p})(1-p\mathrm{E}[V])(1-H(V(\bar{p})))}{(1+H(V(\bar{p})))(1-p\mathrm{E}[V])(1-V(\bar{p}))}\mathrm{E}[V] \\ &\quad + \frac{1}{p}\frac{(1-p\mathrm{E}[V])H(V(\bar{p}))}{1+H(V(\bar{p}))(1-p\mathrm{E}[V])} + \frac{1}{\mu-p}. \end{aligned} \tag{3.22}$$

Let p_B, p_V, and p_I be the probabilities that the server is in a busy, vacation, or idle state, respectively. We can give that

$$\begin{aligned} p_B &= \frac{\mathrm{E}[S_p]}{\mathrm{E}[C]} = \frac{1}{\mathrm{E}[C](\mu-p)}, \\ p_V &= \frac{\mathrm{E}[V](1-V(\bar{p})+V(\bar{p})(1-p\mathrm{E}[V])(1-H(V(\bar{p}))))}{\mathrm{E}[C](1+H(V(\bar{p})))(1-p\mathrm{E}[V])(1-V(\bar{p}))}, \\ p_I &= \frac{(1-p\mathrm{E}[V])H(V(\bar{p}))}{p\mathrm{E}[C](1+H(V(\bar{p}))(1-p\mathrm{E}[V]))}. \end{aligned} \tag{3.23}$$

3.4 Special Cases

If the random variable H is supposed to have different probability distributions, we can derive some vacation queueing systems with a pure decrement service as special cases of the model presented in this chapter as follows:

Example 3.1. Pure decrement service Geom/G/1 queue with multiple vacations—Geom/G/1 (PD, MV).

If $H \to \infty$, the queue turns into a pure decrement service Geom/G/1 queue with multiple vacations. There is no idle state in the system, and $H(z) = 0$. Then the P.G.F.s of the additional queue lengths L_d, L_r and the additional delays W_d, W_r are respectively given by

$$L_d(z) = \frac{1 - V(1 - p(1-z))}{p\mathrm{E}[V](1-z)}, \quad L_r(z) = \frac{(1 - p\mathrm{E}[V])(1-z)}{V(1-p(1-z)) - z},$$
$$W_d(z) = \frac{1 - V(z)}{p\mathrm{E}[V](1-z)}, \quad W_r(z) = \frac{(1 - p\mathrm{E}[V])(1-z)}{(1-z) - p(1 - V(z))}. \tag{3.24}$$

Equation (3.24) corresponds with the results given in [5], [10] and [13].

Example 3.2. Pure decrement service Geom/G/1 queue with single vacation—Geom/ G/1 (PD, SV).

If $H = 1$, the system turns into a pure decrement service Geom/G/1 queue with a single vacation. There is an idle state in the system, and $H(z) = z$. Then the P.G.Fs. of the additional queue lengths L_d, L_r and the additional delays W_d, W_r are respectively given by

$$L_d(z) = \frac{1 - V(\bar{p})z - (1 - V(\bar{p}))V(1 - p(1-z))}{(V(\bar{p}) + p\mathrm{E}[V](1 - V(\bar{p})))(1-z)},$$
$$L_r(z) = \frac{(1 - p\mathrm{E}[V])(1-z)}{V(1-p(1-z)) - z},$$
$$W_d(z) = \frac{p(1 - V(\bar{p}))(1 - V(z)) + V(\bar{p})(1-z)}{(V(\bar{p}) + p\mathrm{E}[V](1 - V(\bar{p})))(1-z)},$$
$$W_r(z) = \frac{(1 - p\mathrm{E}[V])(1-z)}{(1-z) - p(1 - V(z))}. \tag{3.25}$$

Equation (3.25) corresponds with the results given in [5] and [13].

Example 3.3. The number of vacations H follows a Poisson distribution in a Geom/G/1 queue with a pure decrement service strategy—Geom/G/1 (PD, PV).

If the number of vacations follows a Poisson distribution with a parameter λ, namely $P(H = i) = (\lambda^i / i!)e^{-\lambda}, \lambda > 0, i = 0, 1, 2, \ldots$, then $H(z) = e^{\lambda(z-1)}$. Substituting $H(V(\bar{p})) = e^{\lambda(V(\bar{p})-1)}$ into (3.5) and (3.13), the P.G.Fs. of the additional queue lengths L_d, L_r and the additional delays W_d, W_r are given by

$$L_d(z) = \frac{1 - V(1-p(1-z)) + e^{\lambda(V(\bar{p})-1)}V(1-p(1-z)) - e^{\lambda(V(\bar{p})-1)}z}{(e^{\lambda(V(\bar{p})-1)} + p\mathrm{E}[V](1 - e^{\lambda(V(\bar{p})-1)}))(1-z)},$$
$$L_r(z) = \frac{(1 - p\mathrm{E}[V])(1-z)}{V(1-p(1-z)) - z},$$
$$W_d(z) = \frac{p(1 - e^{\lambda(V(\bar{p})-1)})(1 - V(z)) + e^{\lambda(V(\bar{p})-1)}(1-z)}{(e^{\lambda(V(\bar{p})-1)} + p\mathrm{E}[V](1 - e^{\lambda(V(\bar{p})-1)}))(1-z)},$$
$$W_r(z) = \frac{(1 - p\mathrm{E}[V])(1-z)}{(1-z) - p(1 - V(z))}. \tag{3.26}$$

The special cases mentioned above correspond to different probability distributions of H, and we can obtain different pure decrement service queue models with vacations. From these analyses, we can conclude that the model presented in this chapter is a general model including many special queue models.

3.5 Numerical Results

In this section, we present some numerical results that provide insight into the system behavior. Using the equations presented in Sect. 3.3, we can numerically compare the performance measures of the systems for three different Geom/G/1 (PD, MAVs) queue models: the pure decrement service Geom/G/1 queue with multiple vacations, the pure decrement service Geom/G/1 queue with single vacation and the model where the number of vacations H follows a Poisson distribution in Geom/G/1 queue with a pure decrement service strategy.

Here we assume that the service time S and the time length V of a vacation follow geometric distributions; that is, S follows a geometric distribution with mean $1/\mu = 10$. V follows a geometric distribution with mean $\mathrm{E}[V] = 10$. As we presented in Sect. 3.2, if $H \to \infty$, the model corresponds to a Geom/G/1 (PD, MV) queue. If $H = 1$, the model corresponds to Geom/G/1 (PD, SV) queue. If H follows a Poisson distribution, the model corresponds to a Geom/G/1 (PD, PV) queue. Parameter $\lambda = 0.1$, traffic intensity ρ range from 0.1 to 0.8.

Figure 3.2 shows the mean queue length $\mathrm{E}[L_v]$ as a function of the the traffic intensity ρ with three cases of H; that is, $H \to \infty$ for a Geom/G/1 (PD, MV) queue, $H = 1$ for a Geom/G/1 (PD, SV) queue, and H follows a Poisson distribution for a Geom/G/1 (PD, PV) queue. We can find that when ρ increases, $\mathrm{E}[L_v]$ increases to a

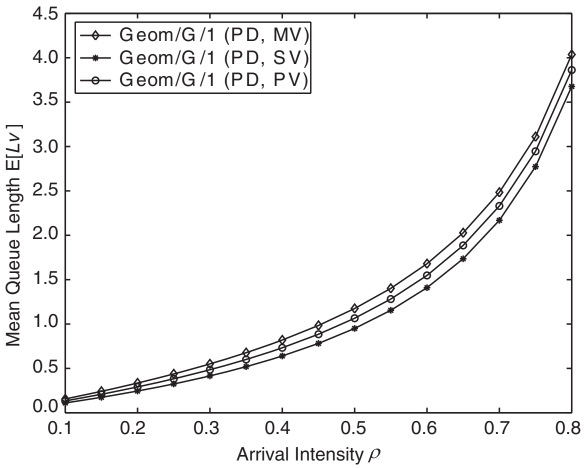

Fig. 3.2 Mean queue length $\mathrm{E}[L_v]$ versus traffic intensity ρ.

high level for all the cases. This is because the larger ρ is, the higher the possibility that there will be customers arriving during the server cycle C. We also note that the mean queue length $E[L_v]$ of Geom/G/1 (PD, MV) is larger than that of Geom/G/1 (PD, SV) and Geom/G/1 (PD, PV). This is because the longer the vacation times are, the larger the mean queue length $E[L_v]$ will be.

Figure 3.3 shows how the mean waiting time $E[W_v]$ changes with the traffic intensity ρ for the three different cases of H; that is, $H \to \infty$ for a Geom/G/1 (PD, MV) queue, $H = 1$ for a Geom/G/1 (PD, SV) queue, and H follows a Poisson distribution for a Geom/G/1 (PD, PV) queue. We can find that when ρ increases, $E[W_v]$ increases to a high level. This is because the greater ρ is, the higher the possibility that there will be customers arriving during the server cycle C; then the mean waiting time will be greater. We also note that the mean waiting time $E[W_v]$ of Geom/G/1 (PD, MV) is longer than that of Geom/G/1 (PD, SV) and Geom/G/1 (PD, PV). This is because the longer the vacation time lengths are, the greater the mean waiting time $E[W_v]$ will be.

In Fig. 3.4, we can observe that, for the Geom/G/1 (PD, MV) queue, when ρ increases, the mean service cycle $E[C]$ of Geom/G/1 (PD, MV) increases, too. It can also be noted that the curves of the mean service cycle $E[C]$ for the Geom/G/1 (PD, SV) queue and Geom/G/1 (PD, PV) queue follow two stages. In the first stage, the heavier the traffic intensity ρ is, the lower the mean service cycle $E[C]$ will be. In the second stage, the heavier the traffic intensity ρ is, the higher the mean service cycle $E[C]$ will be.

In Fig. 3.5, we plot the probability for the system being at the various states as a function of the traffic intensity ρ in Geom/G/1 (PD, PV). It can be observed that when ρ increases, the probability for the system being either in a busy or vacation state increases, whereas the probability of the system being in an idle state decreases and limits to zero. This is because the greater ρ is, the more customers will arrive,

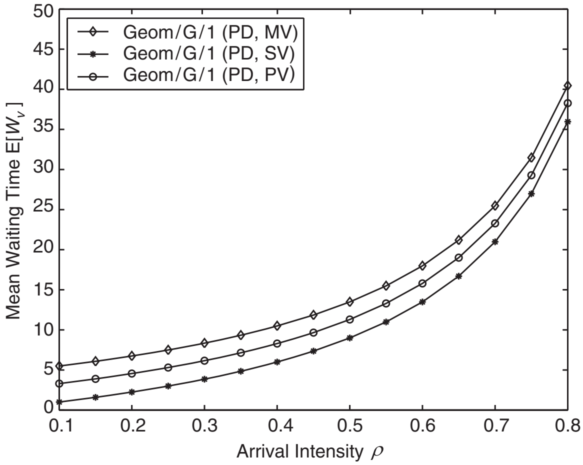

Fig. 3.3 Mean waiting time $E[W_v]$ versus traffic intensity ρ.

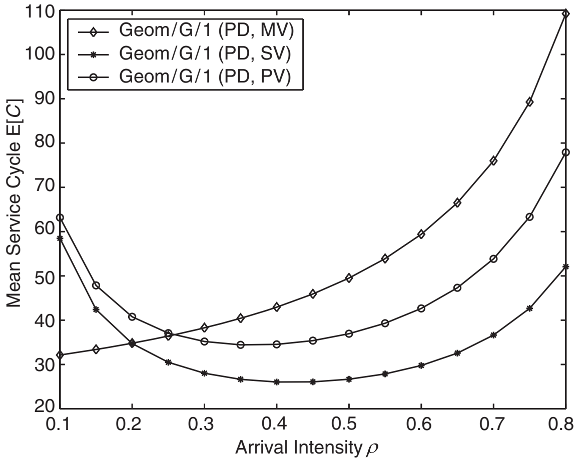

Fig. 3.4 Mean server cycle E$[C]$ versus traffic intensity ρ.

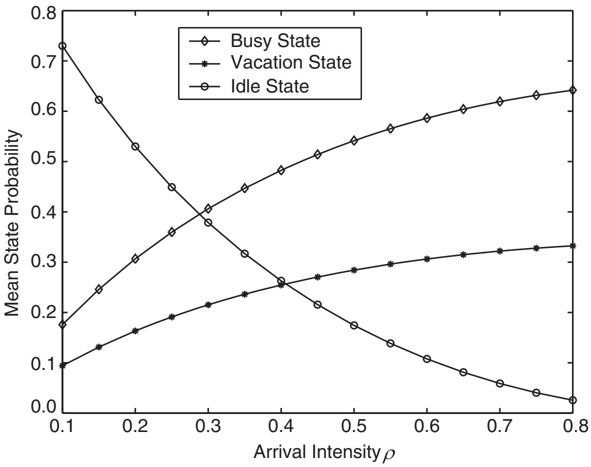

Fig. 3.5 Mean state probability versus traffic intensity ρ.

and so the probability of the system being in a busy or a vacation state will increase, whereas the probability for the system being in an idle state will be smaller.

3.6 Conclusions

In this chapter, we presented a detailed description of a Geom/G/1 queue model with a pure decrement service strategy and multiple adaptive vacations. By using the method of an embedded Markov chain, we derived the P.G.F.s of the queue length and the customers' waiting time. Furthermore, we presented the stochastic decompositions for the additional queue length and the additional delay. Lastly, we obtained the probabilities of the server being in the various states of busy, vacation, or idle, respectively. The model is an extension for many special multiple adaptive vacation queue models with a pure decrement service strategy. When applying to communication networks, it is especially useful for solving problems associated with network flow.

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References

1. N. Tian, Multi-stage adaptive vacation policies in an M/G/1 queueing system, *Applied Mathematics*, vol. 5, no. 4, pp. 12–18, 1992 (in Chinese).
2. G. Zhang and N. Tian, Discrete time Geo/G/1 queue with multiple adaptive vacations, *Queueing Systems*, vol. 38, no. 4, pp. 419–429, 2001.
3. O. Boxma and U. Yechiali, An M/G/1 queue with multiple types of feedback and gated vacations, *Journal of Applied Probability*, vol. 34, no. 3, pp. 773–784, 1997.
4. H. Takagi, Mean message waiting time in a symmetric polling system, in *Proc. Performance'84, E. Gelenbe (Editor)*, 1985.
5. H. Takagi, *Queueing Analysis, Volume 3: Discrete-Time Systems*. Amsterdam: Elsevier Science, 1993.
6. D. Wu and H. Takagi, M/G/1 queue with multiple working vacations, *Performance Evaluation*, vol. 63, no. 7, pp. 654–681, 2006.
7. N. Tian and G. Zhang, A discrete-time GI/Geo/1 queue with multiple vacations, *Queueing Systems*, vol. 40, no. 3, pp. 283–294, 2002.
8. D. Fiems and H. Bruneel, Analysis of a discrete time queueing system with timed vacations, *Queueing Systems*, vol. 42, no. 3, pp. 243–254, 2002.
9. J. Cohen, *The Single Server Queue*. Amsterdam: North-Holland, 1982.
10. D. Gross and C. M. Harris, *Fundamentals of Queueing Theory (Second edition)*. New York: John Wiley & Sons, 1985.
11. W. Bischof, Analysis of M/G/1 queues with setup times and vacations under six different service disciplines, *Queueing Systems*, vol. 39, no. 4, pp. 265–301, 2001.
12. Z. Niu, Y. Takahashi, and N. Endo, Performance evaluation of SVC-based IP-over-ATM network, *IEICE Transactions on Communications*, vol. E81-B, pp. 948–957, 1998.
13. N. Tian and G. Zhang, *Vacation Queueing Models-Theory and Applications*. New York: Springer-Verlag, 2006.