Theme 2.4
The Balance of Teacher Knowledge: Mathematics and Pedagogy

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1. Introduction

Teacher education and the professional development of practicing teachers need to provide a sound basis of knowledge for teaching, theoretically but also with strong ties to issues of practice. Although this seems like a common-sense statement, it is harder to make a reality than expected. At least three factors could account for this difficulty: the sheer complexity of the knowledge required for teaching, the interconnectedness of knowledge, and the fact that teachers' knowledge comes from different and in certain cases even contradictory sources. Consequently, (a) there is still a lack of comprehensive and categorical descriptions that frame teachers' knowledge, particularly for content-oriented viewpoints, and (b) there is apparently no broad consensus about the status of that knowledge—is it private knowledge, based on personal experience and only in the personal realm of thinking and acting, or is it knowledge coming from and staying in practice, or is it discursively generated, shared, and general knowledge?

In this chapter we describe aspects of the research on the relationship between teachers' content knowledge and pedagogical practices from various perspectives to address the question, "Is there evidence for a systematic interdependent relationship of content and pedagogy?" Such evidence comes from different sources. One can measure both knowledge facets by the means of questionnaires, by directly observing the teaching practice, and by case studies of selected teachers. Learning about both knowledge facets can occur within the teaching practice as such but also from prospective and practicing teacher education. Moreover, teachers learn from practice—from within the teacher's own practice and from the practice of others as well as from student oral discourse and written productions in their classes.

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2. Domains of Teacher Knowledge

Content knowledge in mathematics is understood to be knowledge of concepts and a fluency of the procedures; however, content knowledge in Shulman's (1986) description goes far beyond knowledge of relevant facts in the domain.

> It requires understanding the structures of the subject matter.. . . The structures of a subject include both the substantive and the syntactic structures. The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established (p. 9).

Content knowledge for teachers in mathematics thus contains all the "five strands" contended as the basis of students' mathematical proficiency by Kilpatrick, Swafford, & Findell (2001), which are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Also, argumentation and proving is a specific means in mathematics, "to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions" (Shulman, 1986, p. 9). However, these activities must equally be set into the broader contexts of explaining, communicating, and even modeling (Hanna, 1983; Hanna & Jahnke, 1993) to become an essential and inevitable part of teachers' mathematical content knowledge. In a similar way, basic insights into the history and epistemology of mathematics are necessary ingredients of the content knowledge of mathematics teachers (Fauvel & van Maanen, 2000).

As any description of teacher knowledge must necessarily be broad and multifaceted, beyond teacher content knowledge, two other major domains of knowledge for teaching commonly accepted are *pedagogical content knowledge* (PCK), as described by Shulman (1986), and the more recent revision of PCK as *mathematics knowledge for teaching*, or MKT (Ball & Bass, 2000, 2003). Shulman's conception of pedagogical content knowledge once more underlines the point that teachers' knowledge must be more than just being able to conduct a lesson; he comments that

> within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in other words, the ways of representing and formulating the subject that make it comprehensible to others (p. 9).

PCK therefore is closely connected to content knowledge (in this case mathematics), because the teacher consciously must choose between all the possible representations the subject of teaching provides. It is the focus on representing mathematical knowledge that transforms content knowledge into PCK. However, it is still an open question "whether specialized knowledge for teaching mathematics exists independently from common content knowledge. . ." (Ball, Hill, & Bass, 2005, p. 45). The MKT is viewed as the mathematical knowledge that is specific to teaching and different from the knowledge needed by other professions, such as research mathematics, engineering, or financial modeling (Ball & Bass, 2000). For example, while the research mathematician strives for mathematical elegance and compression, the teacher focuses on how to unpack mathematical ideas to make them more accessible to students. Examples of mathematical practices that are specific to teaching include examining alternative solution methods, analyzing their mathematical structure and principles, and judging whether they can be generalized (Ball & Bass, 2003; Ball, Hill, & Bass, 2005; Ferrini-Mundy et al., 2004; Ma, 1999). Recent research suggests that this claim for specialized MKT does exists independently from common content knowledge for primary-grade teachers (Hill & Ball, 2004) and for secondary-level teachers (Brunner et al., 2006; Krauss et al., in press).

4.2 Relationship Between Content Knowledge and Pedagogy

Research is beginning to emerge that extends beyond Shulman's categories to examine the relationship between teachers' mathematical content knowledge and their pedagogical practices. Leikin and colleagues' research findings show that teachers' mathematical knowledge supports better development of their own mathematical and pedagogical knowledge through teaching (Leikin, Levav-Waynberg, Gurevich, & Mednikov, 2006). The results of a German study show that the relationship between content knowledge and PCK are highly correlated (Brunner et al., 2006). In a study of Colombian mathematics teachers' conceptions of their own teaching practices of beginning algebra, Agudelo-Valderrama (2004a) demonstrates the existence of the mutual influence of the teachers' ways of knowing beginning algebra and their conceptions of the crucial determinants of their teaching in their pedagogical practices (see also Agudelo-Valderrama, Clarke, & Bishop, 2007).

A study by Leikin & Dinur (2003) describes and characterizes the factors that affect teacher flexibility in mathematics classes. It confirms that teacher mathematical and pedagogical knowledge and conceptions are the main factors influencing teacher flexibility (consistent with Simon, 1997). In addition, this study shows that other important factors are the teacher's "noticing" and awareness (Mason, 1998; Sherin & van Es, 2005), along with the teacher's beliefs and emotions (Thompson, 1992; Cooney & Shealy, 1997; Sullivan & Mousley, 2001).

Leikin and Dinur differentiate between preliminary and momentary factors that influence teachers' mathematical and pedagogical knowledge. Preliminary factors include primarily the teacher's knowledge and skills—mathematical and PCK (Shulman, 1987), awareness (Mason, 1998), and teachers' beliefs about mathematics and mathematics teaching (Thompson, 1992; Sullivan & Mousley, 2001). For example, a teacher who believes that considering different solutions to a problem is confusing to students will act inflexibly when a solution different from one that the teacher planned is suggested by a student (Leikin et al., 2006; Leikin & Levav-Waynberg, 2007). On the contrary, a teacher aware of the importance of different mathematical solutions to a problem as a means for the development of students' mathematical reasoning may increase a teacher's flexibility during a lesson. Momentary factors are the aspects of the teacher's reasoning and affective reactions in the ongoing moments of teaching, such as the teacher's confusion or curiosity vis-à-vis the students' unexpected answers and the teacher's ability to understand students' language and notice the potential of their answers (Leikin & Dinur, 2003). The preliminary and the momentary factors are clearly interrelated. For instance, teachers' knowledge may determine and be affected by their ability to notice, the teachers' beliefs may influence affective reactions during the lessons or their momentary decisions about continuing the discussion in a particular direction.

A unique chance to investigate teacher knowledge systematically occurred in Germany due to a national option in the 2003 Programme for International Student Assessment study (PISA) (OECD, 2004). To reduce the explanatory gap between student performance and system variables, a study on the professional knowledge of the teachers of mathematics was initiated (Baumert, Blum, & Neubrand, 2004; Blum et al., 2005; Krauss et al., 2004). Teachers' knowledge was conceptualized in this study, called COACTIV (Cognitive Activation in the Classroom: Learning Opportunities for the Enhancement of Mindful Mathematics Learning), as content knowledge and PCK with specific foci. Content knowledge and PCK were highly correlated (more than 0.60 being a typical correlation coefficient), however, one could empirically separate the two facets, content knowledge and PCK, by the test administered (Krauss et al., in press; Kunter et al., 2007).

Under content knowledge a profound understanding of the topics of school mathematics was perceived (e.g., "Explain why $0.999\ldots = 1$!"). Using Shulman's general ideas, three facets of PCK were contained in this study:

1. A teacher should know about the cognitive potential of mathematical tasks (because mathematical tasks are the most commonly used media to carry mathematical content in the classroom): knowing about students' strategies to solve mathematical tasks, to be able to judge the mathematical and cognitive relevance of the tasks, and having multiple solution paths at hand is crucial for teaching. A sample item is: "Show in as many ways as you can give reasons for: The square of an integer is always 1 bigger than the product of the two adjacent numbers."
2. Knowledge about *students' mathematical cognitions* is necessary for adaptive teaching. Errors and difficulties then can be productive sources for concept building and learning. However, the teacher must be able to recognize these errors and points of difficulty.
3. Knowledge of *mathematics-specific methods of teaching* is necessary, since explanation and simplification are teachers' activities strongly bound to the content itself (Kirsch, 2000). A sample item from COACTIV is: "A student says: 'I don't understand why $(-1)(-1) = 1$'. Please outline as many different ways as possible of explaining this mathematical fact to that student."

The results of a German study show the high correlation between content knowledge and PCK (Brunner et al., 2006). However, it was possible to empirically separate content knowledge from PCK (Krauss et al., in press). The teachers in the academic tracks of the German school system (Gymnasium) seem to have higher degrees of integrated knowledge than the teachers in the non-academic tracks, who showed more separation between content and pedagogical aspects of their knowledge. The influence of teachers' knowledge on students' achievement gains were found to be completely mediated by the PCK of the teachers, and there was a positive effect; the students' progress in learning was measured by students' results in a German longitudinal component of PISA. The most influential factor of how teachers' knowledge fostered students' achievement was the selection of cognitively activating tasks. Thus, the use of task characteristics proved again to its decisive role as an appropriate means of analysis of teacher knowledge (J.. Neubrand, 2006). See also the discussion of tasks in Chapter 3, Theme 2.3.

In two case studies Agudelo-Valderrama exhibits two Columbian novice teachers conceptions of their practices of beginning algebra (Agudelo-Valderrama, 2004a; Agudelo-Valderrama & Clarke, 2005; Agudelo-Valderrama, Clarke, & Bishop, 2007). She found that the teachers' ways of knowing beginning algebra represented the basis for their pedagogical approaches (i.e., ideas about the why, the what, and the how of their teaching acts). The teachers provided clear-cut evidence not only of their conceptions of beginning algebra, but also of their conceptions of their roles as teachers and the determinants of their practices. Examination of the cases of two novice teachers showed a mutual influence of the teachers' ways of knowing beginning algebra and their conceptions of the crucial determinants of their teaching in their pedagogical practices. One case-study teacher, Alex, who focused on "giving clear explanations of procedures to be followed in the manipulation of given symbolic expressions" (throughout the six months of data collection) attributed "the unsatisfactory results" of his teaching to external factors mainly related to the pupils. In contrast, Pablo, whose concern was "not to tell" and to teach for understanding by promoting pupils' creation of their mathematical ideas, attributed "the lack of success" of some of his pupils to his inadequate knowledge of the teaching of beginning algebra (i.e., internal factors). However, as his knowledge of contextual factors of teaching increased (e.g., knowledge of the expectations of the pupils and their "powerful parents" to cover a list of content items, or the requirements of the school's assessment-report scheme), he started to restructure his teaching in order to align with the institution. His teaching expertise was of less importance compared to his knowledge of how the school functioned. According to Pablo, the parents' requirements were not sensible, but they were powerful people in the school because they paid high school fees, so he believed he had to comply with the parents' expectations. Pablo's knowledge of the structure and functioning of Colombian society and his social dispositions played a strong role in his teaching decisions and his conceptions of his teaching practice. Pablo's knowledge of how Colombian society and the specific school where he taught was more influential on his teaching decisions than his knowledge of beginning algebra and its pedagogy in restructuring his teaching (see also Agudelo-Valderrama, 2004b; Agudelo-Valderrama & Clarke, 2005).

In both cases the teachers' conceptions of the crucial determinants of their teaching greatly influenced their pedagogical decisions. In Alex's case, his conceptions of the role of social/institutional factors acted as justifications that reinforced his conceptions of his teaching of beginning algebra. However, in Pablo's case, his conceptions of social/institutional factors of teaching were more influential in restructuring his teaching than his strong conceptions (knowledge, beliefs, and attitudes) of mathematics. This observation is further evidence of Wilson & Lloyd's (2000) observation that teachers find it difficult to bring about change despite their strong subject-matter knowledge and commitment.

Agudelo-Valderrama's findings call attention to the critical need to consider teachers' social knowledge as a key component of their practical knowledge, which plays a strong structuring power in their thinking and, therefore, in their pedagogical decisions. She argues that it is necessary to pay increasing attention to the centrality of teachers' social conceptions (i.e., knowledge, beliefs and attitudes) in their thought structures if we are to gain some understanding of the barriers and possibilities of teacher change in pedagogical practices. Agudelo-Valderrama further contends that, in general, the teachers' knowledge and beliefs of the social and educational systems had a strong structuring power in their thinking. Their conceptions (i.e., their knowledge, beliefs, and attitudes) of the social/institutional factors of their teaching represent a key component of their thought structures impinging on their pedagogical decisions, a fact that supports McEwan & Bull's (1991) claim that all knowledge is pedagogical in varying ways. However, "we cannot merely append 'social knowledge' to a growing list of categories" of PCK "because of the fundamentally constitutive nature of social knowledge" and beliefs (Gates, 2001, p. 21).

The findings from this research point to the fundamental importance of sound PCK, which is explicitly bound to the mathematical issues that arise in the class. An ongoing task of teacher education and development programs is to provide knowledge that is properly combined of both components: content knowledge related to the background of the topics taught at school and pedagogical knowledge related to the cognitive aspects of the subject, that is, mathematics. However, Agudelo-Valderrama points out a third component, the crucial role that teachers' social knowledge of institutional and societal expectations plays in the development of their pedagogical knowledge.

3. Learning from Practice

The studies by Leikin, Brunner, and Agudelo-Valderrama all point to how teachers' ways of knowing mathematics affect their pedagogical practices and lead us to consider how practicing teachers learn from their own (Leikin, 2006; DeBlois, 2006) or others' practice (Seago & Goldsmith, 2006; Wood, 2005). See Chapter 2, Theme 2.2, for more discussion about learning from practice.

The main source of teachers' learning through teaching (LTT) is their interactions with students and learning materials (Leikin 2005a, 2006). Leikin (2005b) claims that it is the quality of instructional interactions that exist in the classroom which determine the potential of the lesson to promote both students' and teachers' learning. In this context initiation of interaction by the teacher or by the students, as well as motives for interacting, determine learning processes in the classroom. The motives may be external if they are prescribed by the given educational system, or internal, being mostly psychological, including cognitive conflict, uncertainty, disagreement, or curiosity. Piagetian disequilibration is the main driving force in intellectual growth or learning. For teachers, unexpected, unforeseen, or unplanned situations are the cause of disequilibration and are the sources for learning. These sources surface via interaction with students and via reflection on this interaction (Leikin & Zakis, 2007). Teachers learn mainly in unpredictable (surprising) situations. As Atkinson & Claxton (2000) contend, many of the teachers' actions when teaching are intuitive and unplanned. Teachers' craft knowledge develops from the transformation of their intuitive reactions into formal knowledge or into beliefs.

Development of new mathematical knowledge takes place at all the stages of teachers' work—planning, conducting, and analyzing a lesson. When planning the lesson, teachers clearly express their "need to know the material well enough" and their "need to predict students' possible difficulties, answers, and questions". At the planning stage the teachers are involved in designing activities that allow them to reach new insights. Hence *new pieces of information* are sometimes collected, and some *familiar ideas are refined* (Leikin, 2005a, 2006). The need to "know better than the students" stimulates teachers' thinking about students' possible difficulties. When predicting during lesson planning, teachers reflect on their own uncertainties and thus resolve their own questions. While conducting a lesson, it is through the interaction with students that teachers become aware of new—for them—solutions to known problems, new properties (theorems) of the mathematical objects, and new questions that may be asked about mathematical objects, and in this way they develop new mathematical connections (Leikin, 2006; Leikin & Zakis, 2007).

DeBlois and colleagues conducted three research studies of six professional development seminars in schools using collaborative research methodology (Desgagné, 1997; Erikson, 1989, 1991). The seminars were held to study pupils' mathematics productions in the context of class history, curriculum, and development of a series of mathematical concepts. Data analysis compared the partners' referents, which revealed the influence of interpretation on the interventions. The first study involved four special-education teachers participating in six seminars over the course of one year (DeBlois & Squalli, 2002). During these seminars, the discussions of special education teachers oscillated between mathematical concepts and the procedural aspects of a mathematical concept (e.g., written number and number) and between judgment of the pupil and evaluation of the pupil's production. When teachers identified a mathematical concept and the possible reasoning of the pupil, their teaching strategies connected to the pupil's reasoning. However, when the reasoning of the pupil, without a conceptual analysis, preoccupied them, a particular teaching strategy appeared; this was trial and error. This kind of analysis did not connect mathematics and pedagogy. This teaching strategy led to a cycle of interventions with no connection to the first pupil's errors. In summary, two aspects emerged as important in this research: creating an understanding of the pupil's reasoning and using error as a component for student learning.

A second study involved three special-education teachers during six seminars (DeBlois, 2003a). For this study, DeBlois proposed an interpretative model of pupils' cognitive activity (DeBlois, 2000) to structure the discussion of pupils' mathematics productions. Teachers were invited to experiment with and adapt the model. This model, inspired from Piaget's, (1977) reflective abstractive model, identified a variety of components that are coordinated as students work through a solution (DeBlois, 1996, 1997a, 1997b). These components are the *representations* students use to solve problems or complete projects, the *role* they adopt for themselves, the *procedures* they prefer, and the *reflections* that subsequently emerge (DeBlois, 2000, 2003b; Piaget, 1977). The interpretation of pupils' cognitive activities thus works from a number of hypotheses that are developed in relation to these components and to the (reciprocal) coordinations occurring as a result. This model aims to provide some scaffolding to help teachers understand the cognitive activity of the pupils when there is conceptual analysis of the mathematical notion. DeBlois postulates that the ability to understand students' reasoning contributes to the ability to see error as a component of learning and could help teachers gain strategies for working with students who have some difficulties in learning mathematics. This kind of analysis focuses on the interconnection between mathematics and pedagogy.

DeBlois continued to examine the transformation of teaching strategies with a third experiment, in which she studied six seminars with twenty typical teachers in a primary school (DeBlois, 2006; DeBlois & Maheux, 2005). The analysis focused on pedagogy and epistemologies as teachers were investigated using the construct of "sensibility" that emerges from the distinction between situation and environment (Brousseau, 1986; Maturana & Varela, 1994; René de Cotret, 1999). DeBlois observed teachers that were asked to interpret their pupils' errors in mathematics in order to examine the interpretative process and its influence on the choice of teaching strategies in a mathematics class. Sessions were held to study pupils' mathematics productions in the context of class history, curriculum, and development of a series of mathematical concepts. Data analysis compared the partners' referents, which revealed the influence of interpretation on teaching strategy previews. The results provided insight into the preferred teaching strategies with pupils who experience learning difficulties, as well as suggestions on how changes in interpretation transform teaching interventions. When teachers established a link between the task and pupils' procedures, they were sensitive to the familiarity of the pupils' task. Then they considered error as an extension of earlier knowledge and tried to create a gap with original habits. However, when they looked for the gap between pupils' results and results expected, they showed a sensibility to the teaching given and concluded that the pupils had a problem of attention. Thus, they either explained the task again or asked the pupils to read it again. Finally, when teachers identified a gap between what they knew about their pupils and the pupils' production, they were sensitive to the curriculum or to certain elements of the task (type of numbers, type of relation between numbers). In this last case, the error was seen as the product of the interaction between the task and the pupils. At that moment, they considered learning as an interaction between pupils and task. They desired to understand the situation in which their pupils understood the task and they wanted to know the pupils' representations of the task. It is thought that this kind of analysis allowed for an interconnection between mathematics and pedagogy.

DeBlois's research led to a theoretical framework which identifies four components contributing to the interpretation of the logic of pupils (results, pupils' procedures, progression in pupils' procedures, a gap between what they know about their pupils and the pupils' production), four sensibilities of teachers (teaching, familiarity of the pupil with the task, pupils' understanding, curriculum and characteristics of the task), four kinds of interpretation of the pupils' productions (attention, extension of pupils' procedures, pupils' abilities, product of an interaction between pupils and the task). From this, four kinds of teaching strategies appear: method of working (e.g., present clear directions, grant traces of resolutions, ask to read again the task, circle important words in the task); create a gap with the habits; reconsider certain components of teaching (use a diagram or modeling, exercises, manipulatives); and play with didactic variables (type of numbers).

In a project, Learning and Teaching Linear Functions (LTLF), Seago, Mumme, & Branca (2004) designed video case materials for professional development of mathematics teachers. These materials used other teachers' practice as a basis for examining the mathematics of teaching. The major goals of these materials were to deepen teachers' MKT, specifically related to linear functions. As part of the research and development process in creating these materials, two separate evaluation efforts assessed various aspects of teacher learning. Hill & Collopy (2002) developed and used an external assessment to measure growth in teachers' content knowledge and PCK. They administered pre- and post-seminar assessment using the instrument with a group of twelve LTLF teachers and a comparison sample of ten teachers. LTLF teachers improved in their abilities to algebraically represent problems involving geometric patterns, connect their algebraic representation to the geometric pattern, and compare and link alternative representations of the same linear function. They also were better able to identify potential student misunderstandings that involved using a recursive method for predicting the next term in a sequence. Given the small sample size (N=12), teachers' growth between the pre- and post-tests was not statistically significant. However, equivalent growth did not appear in the comparison group, which provides some assurance that the improvement did not result from retaking the same items over a relatively short time span.

Horizon Research, under the guidance of Weiss and Heck, developed embedded assessments to measure the impact of LTLF materials on teachers' content knowledge and PCK. Embedded assessments in this study meant that the instruments were connected to the actual work teachers do within seminar sessions. In this case, for the embedded assessment instrument administered at the first seminar session teachers were asked to solve a mathematics problem, they reflected on their approach to solving the task and predicted approaches students might use to solve the task. Near the end of the eight-session seminars, teachers were asked to complete a similar process with another mathematics task. The second embedded assessment involved a pre- and post-video analysis task. A comparison group of similar teachers responded to the same tasks. In terms of exhibiting mathematical knowledge pertinent to the teaching of mathematics, LTLF participants were statistically more likely than control-group teachers to increase in their ability/propensity to connect their work on the mathematics tasks back to the pictorial representations in which the task originated (Heck, 2003).

A recent study that investigated teachers' learning from two approaches to mathematics professional development that use classroom artifacts (e.g., student work, transcripts, video clips of classroom practice) to help teachers gain new ways of thinking about their students' algebraic understanding (Seago & Goldsmith, 2006). This work was guided by the formulation of MKT offered by Ball & Bass (2000) and Ma (1999). This perspective focused on the considerable mathematical demands placed on classroom teachers and the kinds of particular mathematical knowledge teachers need to employ in their work. Among the tasks of teaching that require this kind of specialized knowledge include "decompressing" (unpacking) mathematical ideas, analyzing student thinking, choosing representations to effectively convey mathematical ideas, and negotiating mathematically productive discussions. This study focused on changes in teachers' analysis of student thinking, use of representations, and unpacking mathematical ideas.

Goldsmith and Seago (in press) examined the impact of two commercially available professional development programs focused on algebraic thinking: *Fostering Algebraic Thinking Toolkit* (*AT*) (Driscoll et al., 2001) and *Learning and Teaching Linear Functions: VideoCases for Mathematics Professional Development* (*LF*) (Seago, Mumme, & Branca, 2004). These two programs were designed to make use of classroom artifacts to help teachers examine issues related to algebra: *AT* centers on the exploration of algebraic habits of mind, and *LF* focuses on linear functions. Four professional development seminars were conducted, two *AT* groups (both facilitated by Driscoll) and two *LF* seminars (both facilitated by Seago). Seventy-four U.S. middle and high school teachers participated in this study: 49 in the experimental groups and twenty-five as a comparison group. Sixteen case-study teachers (four from each site) were followed more closely.

Results indicate that teachers across both sets of materials learned to be more analytical about student thinking, as evidenced by performance on post-program written assessments. The participants who conducted analyses of video and written student work were more grounded in evidence, more focused on the specific mathematics captured in the artifacts, and more attentive to the mathematical potential of students' ideas (instead of just the correctness of the work) than those of the comparison group (Goldsmith et al., 2006). Analysis of seminar discourse indicates that participants' analysis of mathematical thinking became more sustained, extensive, and nuanced over time, and they developed more differentiated, representation-rich, and flexible approaches to mathematics (Goldsmith et al., 2006; Seago & Goldsmith, 2006) than did the comparison group.

4. Implications for Practicing Mathematics Teachers' Development

The emerging research on the relationship between teachers' content knowledge and pedagogical practices is promising. The studies in this chapter show that teachers can learn PCK in and from practice. Yet more systematic research is needed to understand the conditions under which teachers learn and how it affects their practice and ultimately their students' learning.

For further research on the relationship of content and PCK for teaching, some issues still remain, even if we already have encountered in this article some indications towards answers. What can one say, theoretically and with empirical evidence, about the structural characteristics of teachers' knowledge? Are there researchable units of analysis which can serve as tools indicating certain specific aspects of teachers' knowledge? What are the ways teachers' knowledge influences their teaching practice? How strong is the evidence about the impact and effects of teachers' knowledge on students' achievement? Finally, there are two questions on the possibilities of teachers' further development or even changing their behavior: What are the essential places where teachers learn; is it teacher education and/or the practice itself? What must occur so that the actual teaching in a class is affected by the knowledge that the teacher has acquired?

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