6

Analysis

In this chapter, we present analysis only for some representative cases of MsFEMs from Chapters 2, 3, and 4. We consider simpler cases to convey the main difficulties that arise in the analysis of MsFEMs. Some of the technical details are avoided to keep the presentation simple and make it accessible to a broader audience.

In Section 6.1, the convergence analysis of MsFEMs for linear elliptic problems is presented. In this chapter, the MsFEM using local information is studied. First, we present a basic convergence analysis of the MsFEM which demonstrates the resonance errors. In Section 6.1.2, the analysis of MsFEMs with oversampling is studied. This analysis shows that an oversampling technique reduces the resonance errors. In Section 6.1.3, the analysis of mixed MsFEMs using local information is presented. The results obtained in Section 6.1 use homogenization theory.

In Section 6.2, the convergence analysis of MsFEM for nonlinear problems is considered. We show the convergence results only for nonlinear elliptic equations with periodic spatial heterogeneities. The proof relies on homogenization theory and uses a number of auxiliary results that can be found in [104].

In Section 6.3, the analysis of MsFEMs using limited global information is presented. We study the convergence of mixed MsFEM (Section 6.3.1) and a Galerkin MsFEM (Section 6.3.2). The convergence analysis is carried out under some suitable assumptions. We show that MsFEMs using global information converge independent of resonance errors.

Although only some representative cases of MsFEMs are analyzed here, we have attempted to illustrate basic mathematical tools and ideas used in the analysis of multiscale methods. We hope the analysis presented in this chapter will help the reader to understand essential error sources that arise in multiscale algorithms and guide them in estimating these errors. This will further help to design more efficient numerical methods for real-life multiscale processes.

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6.1 Analysis of MsFEMs for linear problems (from Chapter 2)

For the analysis here, we restrict ourselves to a periodic case $k(x) = (k_{ij}(x/\varepsilon))$. We assume $k_{ij}(y)$, $y = x/\epsilon$ are smooth periodic functions in y in a unit cube Y. We assume that $f \in L^2(\Omega)$. The assumptions on k_{ij} can be relaxed and one can extend the analysis to the locally periodic case, $k = k(x, x/\epsilon)$, random homogeneous case, and other cases. For simplicity, we consider the analysis in two dimensions. Denote L introduced in (2.1) by L_ϵ. Let p_0 be the solution of the homogenized equation (see Appendix B for the background material on homogenization)

$$L_0 p_0 := -\text{div}(k^* \nabla p_0) = f \ \text{ in } \Omega, \quad p_0 = 0 \ \text{ on } \partial\Omega, \tag{6.1}$$

where

$$k^*_{ij} = \frac{1}{|Y|} \int_Y k_{il}(y)(\delta_{lj} + \frac{\partial \chi^j}{\partial y_l})\, dy,$$

and $\chi^j(y)$ is the periodic solution of the cell problem in the period Y

$$\text{div}_y(k(y) \nabla_y \chi^j) = -\frac{\partial}{\partial y_i} k_{ij}(y) \ \text{ in } Y, \quad \int_Y \chi^j(y)\, dy = 0.$$

We note that $p_0 \in H^2(\Omega)$ because Ω is a convex polygon. Denote by $p_1(x, y) = \chi^j(y)(\partial p_0(x)/\partial x_j)$ and let θ_ε be the solution of the problem

$$L_\varepsilon \theta_\varepsilon = 0 \ \text{ in } \Omega, \quad \theta_\varepsilon(x) = -p_1(x, x/\epsilon) \ \text{ on } \partial\Omega. \tag{6.2}$$

For simplicity of presentation, we denote by $\|\cdot\|_{\alpha,\beta,\cdot}$ and $|\cdot|_{\alpha,\beta,\cdot}$, the norm and semi-norm in $W^{\alpha,\beta}(\cdot)$. If only one subscript is used, for example, $\|\cdot\|_{\alpha,\cdot}$, then the norm or semi-norm in H^α is assumed. Also, for simplicity, we consider when $\mathcal{T}_h$ is a triangular partition. Our analysis of the multiscale finite element method relies on the following homogenization result obtained by Moskow and Vogelius [204].

Lemma 6.1. *Let $p_0 \in H^2(\Omega)$ be the solution of* (6.1), *$\theta_\varepsilon \in H^1(\Omega)$ be the solution to* (6.2) *and $p_1(x) = \chi^j(x/\varepsilon)\partial p_0(x)/\partial x_j$. Then there exists a constant C independent of p_0, ε and Ω such that*

$$\| p - p_0 - \varepsilon(p_1 + \theta_\varepsilon) \|_{1,\Omega} \le C\varepsilon(| p_0 |_{2,\Omega} + \| f \|_{0,\Omega}).$$

6.1.1 Analysis of conforming multiscale finite element methods

The analysis of conforming multiscale finite element methods uses Cea's lemma [55].

Lemma 6.2. *Let p be the solution of* (2.1) *and p_h be the solution of* (2.3). *Then we have*

$$\| p - p_h \|_{1,\Omega} \le C \inf_{v_h \in \mathcal{P}_h} \| p - v_h \|_{1,\Omega}.$$

Error Estimates ($h < \varepsilon$)

Let $\Pi_h : C(\bar{\Omega}) \to W_h \subset H_0^1(\Omega)$ be the usual Lagrange interpolation operator:

$$\Pi_h p(x) = \sum_{j=1}^{J} p(x_j)\phi_i^0(x) \quad \forall p \in C(\bar{\Omega})$$

and $I_h : C(\bar{\Omega}) \to \mathcal{P}_h$ be the corresponding interpolation operator defined through the multiscale basis function ϕ_i,

$$I_h p(x) = \sum_{j=1}^{J} p(x_j)\phi_j(x) \quad \forall p \in C(\bar{\Omega}).$$

From the definition of the basis function ϕ_i in (2.2) we have

$$L_\varepsilon(I_h p) = 0 \text{ in } K, \quad I_h p = \Pi_h p \text{ on } \partial K, \tag{6.3}$$

for any $K \in \mathcal{T}_h$.

Lemma 6.3. *Let $p \in H^2(\Omega)$ be the solution of* (2.1)*. Then there exists a constant C independent of h, ε such that*

$$\| p - I_h p \|_{0,\Omega} + h\| p - I_h p \|_{1,\Omega} \le Ch^2(| p |_{2,\Omega} + \| f \|_{0,\Omega}). \tag{6.4}$$

Proof. At first it is known from standard finite element interpolation theory that

$$\| p - \Pi_h p \|_{0,\Omega} + h\| p - \Pi_h p \|_{1,\Omega} \le Ch^2(| p |_{2,\Omega} + \| f \|_{0,\Omega}). \tag{6.5}$$

On the other hand, because $\Pi_h p - I_h p = 0$ on ∂K, the standard scaling argument yields

$$\| \Pi_h p - I_h p \|_{0,K} \le Ch|\Pi_h p - I_h p|_{1,K} \quad \forall K \in \mathcal{T}_h. \tag{6.6}$$

To estimate $|\Pi_h p - I_h p|_{1,K}$ we multiply the equation in (6.3) by $I_h p - \Pi_h p \in H_0^1(K)$ to get

$$\int_K k(\tfrac{x}{\varepsilon})\nabla I_h p \cdot \nabla(I_h p - \Pi_h p)dx = 0.$$

Thus, upon using the equation in (2.1), we get

$$\begin{aligned}
&\int_K k(\frac{x}{\varepsilon})\nabla(I_h p - \Pi_h p) \cdot \nabla(I_h p - \Pi_h p)dx \\
&= \int_K k(\frac{x}{\varepsilon})\nabla(p - \Pi_h p) \cdot \nabla(I_h p - \Pi_h p)dx - \int_K k(\frac{x}{\varepsilon})\nabla p \cdot \nabla(I_h p - \Pi_h p)dx \\
&= \int_K k(\frac{x}{\varepsilon})\nabla(p - \Pi_h p) \cdot \nabla(I_h p - \Pi_h p)dx - \int_K f(I_h p - \Pi_h p)dx.
\end{aligned}$$

This implies that

$$|I_h p - \Pi_h p|_{1,K} \le Ch|\, p\,|_{2,K} + \|\, I_h p - \Pi_h p\,\|_{0,K}\|\, f\,\|_{0,K}.$$

Hence

$$|I_h p - \Pi_h p|_{1,K} \le Ch(|\, p\,|_{2,K} + \|\, f\,\|_{0,K}), \tag{6.7}$$

where we have used (6.6). Now the lemma follows from (6.5)–(6.7). □

In conclusion, we have the following standard estimate by using Lemmas 6.2 and 6.3.

Theorem 6.4. *Let $p \in H^2(\Omega)$ be the solution of* (2.1) *and $p_h \in \mathcal{P}_h$ be the solution of* (2.3)*. Then we have*

$$\|\, p - p_h\,\|_{1,\Omega} \le Ch(|\, p\,|_{2,\Omega} + \|\, f\,\|_{0,\Omega}). \tag{6.8}$$

Note that the estimate (6.8) blows up as does h/ϵ as $\varepsilon \to 0$ because $|\, p\,|_{2,\Omega} = O(1/\epsilon)$. This is insufficient for practical applications. In the next subsection, we derive an error estimate which is uniform as $\varepsilon \to 0$.

Error Estimates ($h > \varepsilon$)

In this section, we show that the MsFEM gives a convergence result uniform in ϵ as ϵ tends to zero. This is the main feature of the MsFEM over the traditional finite element method. The main result in this subsection is the following theorem.

Theorem 6.5. *Let $p \in H^2(\Omega)$ be the solution of* (2.1) *and $p_h \in \mathcal{P}_h$ be the solution of* (2.3)*. Then we have*

$$\|\, p - p_h\,\|_{1,\Omega} \le C(h+\varepsilon)\|\, f\,\|_{0,\Omega} + C\left(\frac{\varepsilon}{h}\right)^{1/2}\|\, p_0\,\|_{1,\infty,\Omega}, \tag{6.9}$$

where $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ is the solution of the homogenized equation (6.1)*.*

To prove the theorem, we first denote

$$p_{\mathrm{I}}(x) = I_h p_0(x) = \sum_j p_0(x_j)\phi_j(x) \in \mathcal{P}_h.$$

From (6.3) we know that $L_\epsilon p_{\mathrm{I}} = 0$ in K and $p_{\mathrm{I}} = \Pi_h p_0$ on ∂K for any $K \in \mathcal{T}_h$. The homogenization theory implies that

$$\|\, p_{\mathrm{I}} - p_{\mathrm{I}0} - \varepsilon(p_{\mathrm{I}1} - \theta_{\mathrm{I}\varepsilon})\,\|_{1,K} \le C\varepsilon(\|\, f\,\|_{0,K} + |\, p_{\mathrm{I}0}\,|_{2,K}), \tag{6.10}$$

where $p_{\mathrm{I}0}$ is the solution of the homogenized equation on K:

$$L_0 p_{I0} = 0 \;\text{ in } K, \quad p_{I0} = \Pi_h p_0 \;\text{ on } \partial K, \tag{6.11}$$

p_{I1} is given by the relation

$$p_{I1}(x, y) = \chi^j(y) \frac{\partial p_{I0}}{\partial x_j} \;\text{ in } K, \tag{6.12}$$

and $\theta_{I\varepsilon} \in H^1(K)$ is the solution of the problem:

$$L_\varepsilon \theta_{I\varepsilon} = 0 \;\text{ in } K, \quad \theta_{I\varepsilon}(x) = -p_{I1}(x, x/\epsilon) \;\text{ on } \partial K. \tag{6.13}$$

It is obvious from (6.11) that

$$p_{I0} = \Pi_h p_0 \;\text{ in } K, \tag{6.14}$$

because $\Pi_h p_0$ is linear on K. From (6.10) and Lemma 6.1 we obtain that

$$\begin{aligned} \| p - p_I \|_{1,\Omega} &\le \| p_0 - p_{I0} \|_{1,\Omega} + \| \varepsilon(p_1 - p_{I1}) \|_{1,\Omega} \\ &\quad + \| \varepsilon(\theta_\varepsilon - \theta_{I\varepsilon}) \|_{1,\Omega} + C\varepsilon \| f \|_{0,\Omega}, \end{aligned} \tag{6.15}$$

where we have used the regularity estimate $\| p_0 \|_{2,\Omega} \le C \| f \|_{0,\Omega}$. Now it remains to estimate the terms on the right-hand side of (6.15). We show that the dominating resonance error is due to $\theta_{I\varepsilon}$.

Lemma 6.6. *We have*

$$\| p_0 - p_{I0} \|_{1,\Omega} \le Ch \| f \|_{0,\Omega}, \tag{6.16}$$

$$\| \varepsilon(p_1 - p_{I1}) \|_{1,\Omega} \le C(h + \varepsilon) \| f \|_{0,\Omega}. \tag{6.17}$$

Proof. The estimate (6.16) is a direct consequence of standard finite element interpolation theory because $p_{I0} = \Pi_h p_0$ by (6.14). Next we note that $\chi^j(x/\varepsilon)$ satisfies

$$\| \chi^j \|_{0,\infty,\Omega} + \varepsilon \| \nabla \chi^j \|_{0,\infty,\Omega} \le C \tag{6.18}$$

for some constant C independent of h and ε. Thus we have, for any $K \in \mathcal{T}_h$,

$$\| \varepsilon(p_1 - p_{I1}) \|_{0,K} \le C\varepsilon \| \chi^j \frac{\partial}{\partial x_j}(p_0 - \Pi_h p_0) \|_{0,K} \le Ch\varepsilon | p_0 |_{2,K},$$

$$\begin{aligned} \| \varepsilon \nabla(p_1 - p_{I1}) \|_{0,K} &= \varepsilon \| \nabla(\chi^j \frac{\partial(p_0 - \Pi_h p_0)}{\partial x_j}) \|_{0,K} \\ &\le C \| \nabla(p_0 - \Pi_h p_0) \|_{0,K} + C\varepsilon | p_0 |_{2,K} \\ &\le C(h + \varepsilon) | p_0 |_{2,K}. \end{aligned}$$

This completes the proof. □

Lemma 6.7. *We have*

$$\| \varepsilon \theta_\varepsilon \|_{1,\Omega} \le C\sqrt{\varepsilon} \| p_0 \|_{1,\infty,\Omega} + C\varepsilon | p_0 |_{2,\Omega}. \tag{6.19}$$

Proof. Let $\zeta \in C_0^\infty(\mathbb{R}^2)$ be the cut-off function that satisfies $\zeta \equiv 1$ in $\Omega \backslash \Omega_{\delta/2}$, $\zeta \equiv 0$ in Ω_δ, $0 \le \zeta \le 1$ in $\mathbb{R}^2$, and $|\nabla \zeta| \le C/\delta$ in Ω, where for any $\delta > 0$ sufficiently small, we denote Ω_δ as

$$\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \ge \delta\}.$$

With this definition, it is clear that $\theta_\varepsilon + \zeta p_1 = \theta_\varepsilon + \zeta(\chi^j \partial p_0/\partial x_j) \in H_0^1(\Omega)$. Multiplying the equation in (6.2) by $\theta_\varepsilon + \zeta p_1$, we get

$$\int_\Omega k(\frac{x}{\varepsilon})\nabla\theta_\varepsilon \cdot \nabla(\theta_\varepsilon + \zeta\chi^j \frac{\partial p_0}{\partial x_j})dx = 0,$$

which yields, by using (6.18),

$$\begin{aligned} \| \nabla\theta_\varepsilon \|_{0,\Omega} &\le C\| \nabla(\zeta\chi^j \partial p_0/\partial x_j) \|_{0,\Omega} \\ &\le C\| \nabla\zeta \cdot \chi^j \partial p_0/\partial x_j \|_{0,\Omega} + C\| \zeta\nabla\chi^j \partial p_0/\partial x_j \|_{0,\Omega} \\ &\quad + C\| \zeta\chi^j \partial^2 p_0/\partial^2 x_j \|_{0,\Omega} \\ &\le C\sqrt{|\partial\Omega| \cdot \delta}\frac{D}{\delta} + C\sqrt{|\partial\Omega| \cdot \delta}\frac{D}{\varepsilon} + C| p_0 |_{2,\Omega}, \end{aligned} \tag{6.20}$$

where $D = \| p_0 \|_{1,\infty,\Omega}$ and the constant C is independent of the domain Ω. From (6.20) we have

$$\begin{aligned} \| \varepsilon\theta_\varepsilon \|_{0,\Omega} &\le C(\frac{\varepsilon}{\sqrt{\delta}} + \sqrt{\delta})\| p_0 \|_{1,\infty,\Omega} + C\varepsilon| p_0 |_{2,\Omega} \\ &\le C\sqrt{\varepsilon}\| p_0 \|_{1,\infty,\Omega} + C\varepsilon| p_0 |_{2,\Omega}, \end{aligned} \tag{6.21}$$

where we have taken $\delta = \epsilon$. Moreover, by applying the maximum principle to (6.2), we get

$$\| \theta_\varepsilon \|_{0,\infty,\Omega} \le \| \chi^j \partial p_0/\partial x_j \|_{0,\infty,\partial\Omega} \le C\| p_0 \|_{1,\infty,\Omega}. \tag{6.22}$$

Combining (6.21) and (6.22), we complete the proof. □

Lemma 6.8. *We have*

$$\| \varepsilon\theta_{\mathrm{I}\varepsilon} \|_{1,\Omega} \le C\Big(\frac{\varepsilon}{h}\Big)^{1/2} \| p_0 \|_{1,\infty,\Omega}. \tag{6.23}$$

Proof. First we remember that for any $K \in \mathcal{T}_h$, $\theta_{\mathrm{I}\varepsilon} \in H^1(K)$ satisfies

$$L_\varepsilon\theta_{\mathrm{I}\varepsilon} = 0 \text{ in } K, \quad \theta_{\mathrm{I}\varepsilon} = -\chi^j(\frac{x}{\varepsilon})\frac{\partial(\Pi_h p_0)}{\partial x_j} \text{ on } \partial K. \tag{6.24}$$

By applying the maximum principle and (6.18) we get

$$\| \theta_{\mathrm{I}\varepsilon} \|_{0,\infty,K} \le \| \chi^j \partial(\Pi_h p_0)/\partial x_j \|_{0,\infty,\partial K} \le C\| p_0 \|_{1,\infty,K}.$$

Thus we have

$$\| \varepsilon\theta_{\mathrm{I}\varepsilon} \|_{0,\Omega} \le C\varepsilon \| p_0 \|_{1,\infty,\Omega}. \tag{6.25}$$

On the other hand, because the constant C in (6.20) is independent of Ω, we can apply the same argument leading to (6.20) to obtain

$$\begin{aligned}\| \varepsilon\nabla\theta_{\mathrm{I}\varepsilon} \|_{0,K} &\le C\varepsilon \| \Pi_h p_0 \|_{1,\infty,K} (\sqrt{|\partial K|}/\sqrt{\delta} + \sqrt{|\partial K|\delta}/\varepsilon) + C\varepsilon | \Pi_h p_0 |_{2,K} \\ &\le C\sqrt{h} \| p_0 \|_{1,\infty,K} (\frac{\varepsilon}{\sqrt{\delta}} + \sqrt{\delta}) \\ &\le C\sqrt{h\varepsilon} \| p_0 \|_{1,\infty,K},\end{aligned}$$

which implies that

$$\| \varepsilon\nabla\theta_{\mathrm{I}\varepsilon} \|_{0,\Omega} \le C\Big(\frac{\varepsilon}{h}\Big)^{1/2} \| p_0 \|_{1,\infty,\Omega}.$$

This completes the proof. □

Proof. Theorem 6.5 is now a direct consequence of (6.15) and Lemmas 6.6–6.8 and the regularity estimate $\| p_0 \|_{2,\Omega} \le C\| f \|_{0,\Omega}$. □

Remark 6.9. As we pointed out earlier, the MsFEM indeed gives a correct homogenized result as ϵ tends to zero. This is in contrast to the traditional FEM which does not give the correct homogenized result as $\epsilon \to 0$. The L_2 error would grow as $O(h^2/\epsilon^2)$. On the other hand, we also observe that when $h \sim \epsilon$, the multiscale method attains a large error in both H^1 and L^2 norms. This is called the *resonance* effect between the coarse-grid scale (h) and the small scale (ϵ) of the problem. This estimate reflects the intrinsic scale interaction between the two scales in the discrete problem. Extensive numerical experiments confirm that this estimate is indeed generic and sharp. From the viewpoint of practical applications, it is important to reduce or completely remove the resonance error for problems with many scales because the chance of hitting a resonance sampling is high.

Remark 6.10. It can be shown that [147]

$$\|p - p_h\|_{0,\Omega} \le C\left(h + \frac{\epsilon}{h}\right).$$

6.1.2 Analysis of nonconforming multiscale finite element methods

Let ϕ_i be multiscale basis functions obtained using the oversampling technique on K as introduced in Section 2.3.2 and ϕ_i^0 (piecewise linear function if $\mathcal{T}_h$ is a triangulation) be its homogenized part. We keep the same notation for the space spanned by multiscale basis functions as in the conforming case; that is $\mathcal{P}_h = \mathrm{span}\{\phi_i\}$. The analysis follows the proof presented in [143].

The Petrov–Galerkin formulation of the original problem is to seek $p_h \in \mathcal{P}_h$ such that

$$k_h(p_h, v_h) = f(v_h), \quad \forall v_h \in W_h,$$

where

$$k_h(p, v) = \sum_{K \in \mathcal{T}_h} \int_K \nabla p \cdot k(\frac{x}{\epsilon}) \nabla v dx, \quad f(v) = \int_\Omega f v dx.$$

Define $\| \cdot \|_{h,\Omega}$ to be the discrete H^1 semi-norm as

$$\|v\|_{h,\Omega} = \left(\sum_{K \in \mathcal{T}_h} \int_K |\nabla v|^2 dx \right)^{1/2}.$$

We use the following result [107].

Lemma 6.11. *Assume that $K \subset K_E$ is at least a distance of h away from ∂K_E. Then*

$$\|\nabla \eta^i\|_{L^\infty(K)} \leq C/h, \tag{6.26}$$

where C is a constant that is independent of ϵ and h. Here, η^i is the solution of $L_\epsilon \eta^i = 0$ in K, $\eta^i = -\chi^i$ on ∂K.

Theorem 6.12. *Let p_h be the Petrov–Galerkin MsFEM solution. Assume Lemma 6.11 holds and ϵ/h is sufficiently small. If the homogenized part of p, p_0, is in $H^2(\Omega)$, we have*

$$\|p_h - p\|_{h,\Omega} \leq C_1 h + C_2 \frac{\epsilon}{h} + C_3 \sqrt{\epsilon}. \tag{6.27}$$

Proof. To estimate $\|p_h - p\|_{h,\Omega}$, we first show that the following inf-sup condition or coercivity condition of the bilinear form $k_h(\cdot,\cdot)$ holds for sufficiently small ϵ. There exists $C > 0$, independent of ϵ and h such that

$$\sup_{v \in W_h} \frac{|k_h(p_h, v)|}{\|v\|_{1,\Omega}} \geq C \|p_h\|_{h,\Omega}, \quad \forall p_h \in \mathcal{P}_h. \tag{6.28}$$

Define

$$\tilde{k}_{ij}(y) = k_{il}(y) \left(\delta_{lj} + \frac{\partial \chi^j(y)}{\partial y_l} \right)$$

and

$$\tilde{k}(u, v) = \sum_{K \in \mathcal{T}_h} \int_K \nabla v \cdot \tilde{k}(\frac{x}{\epsilon}) \nabla u dx, \quad v \in W_h.$$

Thus, by the expansion $p_h = p_h^0 + \epsilon \chi(x/\epsilon) \cdot \nabla p_h^0 + \epsilon \theta_\epsilon^h$, we have

$$k_h(p_h, v_h) = \tilde{k}(p_h^0, v_h) + \epsilon k_h(\theta_\epsilon^h, v_h) = f(v_h), \quad \forall v_h \in W_h. \tag{6.29}$$

Taking $v_h = p_h^0 \in W_h$ in (6.29), we get

$$k_h(p_h, p_h^0) = \tilde{k}(p_h^0, p_h^0) + \epsilon k_h(\theta_\epsilon^h, p_h^0). \tag{6.30}$$

Moreover, using $\|\theta_\epsilon^h\|_{h,\Omega} \leq (C/h)\|\nabla p_h^0\|_{0,\Omega}$ (which follows from Lemma 6.11), we obtain that

$$|k_h(\theta_\epsilon^h, p_h^0)| \leq C\|\theta_\epsilon^h\|_{h,\Omega}\|\nabla p_h^0\|_{0,\Omega} \leq \frac{C}{h}\|\nabla p_h^0\|_{0,\Omega}^2. \tag{6.31}$$

Next, we note that $\|p_h\|_{h,\Omega} \leq C(1+\epsilon/h)\|\nabla p_h^0\|_{0,\Omega}$ (see [143]) and $\tilde{k}(p_h^0, p_h^0)$ is bounded below and bounded above uniformly when $\epsilon/h \leq C$ (see (3.5) in [143]). Consequently, (6.30) and (6.31) imply that when ϵ/h is sufficiently small

$$\begin{aligned} |k_h(p_h, p_h^0)| \geq |\tilde{k}(p_h^0, p_h^0)| - \epsilon|k_h(\theta_\epsilon^h, p_h^0)| &\geq C(1-\frac{\epsilon}{h})\|\nabla p_h^0\|_{0,\Omega}^2 \\ &\geq C\|\nabla p_h^0\|_{0,\Omega}\|p_h\|_{h,\Omega}. \end{aligned}$$

Thus, (6.28) holds.

Let $p_I \in \mathcal{P}_h$ be the interpolation from $\mathcal{P}_h$. Using inf-sup condition (6.28) we have

$$\begin{aligned} \|p_h - p\|_{h,\Omega} &\leq \|p_I - p\|_{h,\Omega} + \|p_h - p_I\|_{h,\Omega} \\ &\leq \|p_I - p\|_{h,\Omega} + C \sup_{v_h \in W_h} \frac{|k_h(p_h - p_I, v_h)|}{\|v_h\|_{1,\Omega}} \\ &= \|p_I - p\|_{h,\Omega} + C \sup_{v_h \in W_h} \frac{|k_h(p_I - p, v_h)|}{\|v_h\|_{1,\Omega}} \\ &\leq (1+C)\|p_I - p\|_{h,\Omega}. \end{aligned} \tag{6.32}$$

Here, we have used the fact

$$k_h(p_h - p, v_h) = 0, \quad \forall v_h \in W_h.$$

Following the derivation of the proof of Theorem 3.1 in [107] (where $p_I = \sum p_0(x_i)\phi_i(x)$ is chosen) and using Lemma 6.11, we can easily show that

$$\|p_I - p\|_{h,\Omega} \leq C_1 h + C_2\frac{\epsilon}{h} + C_3\sqrt{\epsilon}.$$

Therefore, (6.27) follows from (6.32).

6.1.3 Analysis of mixed multiscale finite element methods

In this section, we present the analysis of mixed multiscale finite element methods. We slightly modify the problem and consider a more general case with varying smooth mobility $\lambda(x)$. We consider the elliptic equation

$$\begin{aligned} -\text{div}(\lambda(x)k_\epsilon(x)\nabla p) &= f \ \text{ in } \quad \Omega \\ -\lambda(x)k_\epsilon(x)\nabla p \cdot n = g(x) \ \text{ on } \partial\Omega, &\quad \int_\Omega p dx = 0, \end{aligned}$$

where $\lambda(x)$ is a positive smooth function and $k_\epsilon(x) = k(x/\epsilon)$ is a symmetric positive and definite periodic tensor with periodicity ϵ. We note that $\lambda(x)$

appears in two-phase flows (see (2.40)). Under the assumption that $\lambda(x)$ is sufficiently smooth, one can analyze the convergence (dominant resonance error) of MsFEMs with basis functions constructed with $\lambda = 1$. The basis functions are constructed with $\lambda = 1$ and satisfy (2.16).

Let $\psi_i^K = k_\epsilon(x)\nabla\phi_i^K$ and the basis function space for the velocity field be defined by

$$\mathcal{V}_h = \bigoplus_K \{\psi_i^K\} \subset H(\text{div}, \Omega),$$

where $H(\text{div}, \Omega)$ is the space of functions such that $\|\cdot\|_{0,\Omega} + \|\text{div}(\cdot)\|_{0,\Omega}$ is bounded. The variational problem is to find $\{v, p\} \in H(\text{div}, \Omega) \times L^2(\Omega)/\mathbb{R}$ such that $v \cdot n = g$ on $\partial\Omega$ and they solve the following variational problem,

$$\begin{aligned} \int_\Omega (\lambda k_\epsilon)^{-1} v \cdot w dx - \int_\Omega \text{div}(w)\ p dx = 0 \quad \forall w \in H_0(\text{div}, \Omega) \\ \int_\Omega \text{div}(v)\ q dx = f \quad \forall q \in L^2(\Omega)/\mathbb{R}, \end{aligned} \tag{6.33}$$

where $H_0(\text{div}, \Omega)$ is the subspace of $H(\text{div}, \Omega)$ which consists of functions with homogeneous Neumann boundary conditions.

Set $Q_h = \oplus_K P_0(K) \cap L^2(\Omega)/\mathbb{R}$, a set of piecewise constant functions. The approximation problem is to find $\{v_h, p_h\} \in \mathcal{V}_h \times Q_h$ such that $v_h \cdot n = g_h$ on $\partial\Omega$

$$\begin{aligned} \int_\Omega (\lambda k_\epsilon)^{-1} v_h \cdot w_h dx - \int_\Omega \text{div}(w_h)\ p_h dx = 0 \quad \forall w_h \in \mathcal{V}_h^0 \\ \int_\Omega \text{div}(v_h)\ q_h dx = f \quad \forall q_h \in Q_h. \end{aligned} \tag{6.34}$$

We state the convergence theorem as the following.

Theorem 6.13. *Let* $\{v, p\} \in H(\text{div}, \Omega) \times L^2(\Omega)/\mathbb{R}$ *solve variational problem* (6.33) *and* $\{v_h, p_h\} \in \mathcal{V}_h \times Q_h$ *solve the discrete variational problem* (6.34)*. If the homogenized solution* $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$*, then*

$$\begin{aligned} \|v - v_h\|_{H(\text{div},\Omega)} + \|p - p_h\|_{0,\Omega} \le C_1(p_0, \lambda)\epsilon \\ + C_2(p_0, f, \lambda, g)h + C_3(p_0, \lambda)\sqrt{\epsilon h} + C_4(p_0, \lambda)\sqrt{\frac{\epsilon}{h}}, \end{aligned} \tag{6.35}$$

where the coefficients are defined in (6.38)*,* (6.41)*,* (6.39)*, and* (6.40)*.*

First, we state a stability estimate [71].

Lemma 6.14. *If* $\{v, p\}$ *and* $\{v_h, p_h\}$*, respectively, solve the continuous variational problem* (6.33) *and the discrete variational problem* (6.34)*, then*

$$\begin{aligned} \|v - v_h\|_{H(\text{div},\Omega)} + \|p - p_h\|_{0,\Omega} \\ \le C\Big(\inf_{\substack{u_h \in \mathcal{V}_h \\ u_h - g_{0,h} \in \mathcal{V}_h^0}} \|v - u_h\|_{H(\text{div},\Omega)} + \inf_{q_h \in Q_h} \|p - q_h\|_{0,\Omega}\Big). \end{aligned} \tag{6.36}$$

The well-posedness of the discrete problem is verified in [71]. To obtain the convergence rate, we need to estimate the right-hand side of (6.36). The following proposition is used in the proof.

Proposition 6.15. *Let p and p_h be the solutions of* (6.33) *and* (6.34), *respectively; then*

$$\inf_{q_h \in Q_h} \|p - q_h\|_{0,\Omega} \le Ch\|g\|_{H^{-1/2}(\partial\Omega)}.$$

Proof. Define $\bar{q}_h = (1/|K|)\int_K p dx$ in each coarse block K. Furthermore, we apply the Poincaré inequality and standard regularity estimate for elliptic equations to obtain

$$\inf_{q_h \in Q_h} \|p - q_h\|_{0,\Omega} \le \|p - \bar{q}_h\|_{0,\Omega} \le Ch\|\nabla p\|_{0,\Omega} \le Ch\|g\|_{H^{-1/2}(\partial\Omega)}.$$

Next, we define the interpolation operator $\Pi_h : H(\mathrm{div}, \Omega) \bigcap H^1(\Omega) \longrightarrow \mathcal{V}_h$ by

$$\Pi_h v|_K = (\int_{e_i^K} v \cdot n ds)\psi_i^K.$$

Let $RT_0 = \mathrm{span}\{R_i^K, i = 1, 2, ..., n;\ K \in \mathcal{T}_h\}$ be the lowest–order Raviart–Thomas finite element space and define the interpolation operator P_h : $H(\mathrm{div}, \Omega) \bigcap H^1(\Omega) \longrightarrow RT_0$ by

$$P_h v|_K = (\int_{e_i^K} v \cdot n ds)R_i^K.$$

It is easy to check that $\mathrm{div}\Pi_h v = \mathrm{div} P_h v$ and $\Pi_h v \cdot n = P_h v \cdot n$.

Next, we need to estimate the first term on the right-hand side of (6.36). The basic idea is to choose a particular u_h approximating v. Let the homogenized flux $v_0 = \lambda k^* \nabla p_0$ and choose $t_h|_K = \Pi_h v_0$. Then we have $t_h - g_{0,h} \in \mathcal{V}_h^0$, where $g_{0,h} = \sum_{e \in \partial\Omega}(\int_e g ds)\psi_i^K$. Consequently, it remains to estimate $\|v - t_h\|_{H(\mathrm{div},\Omega)}$. From the definition of t_h, an easy calculation gives rise to $\mathrm{div}(t_h|_K) = \langle f \rangle_K$ and $\mathrm{div}(v) = f$, where $\langle f \rangle_K = (1/|K|)\int_K f dx$. Therefore, we have

$$\|\mathrm{div}(v) - \mathrm{div}(t_h)\|_{0,\Omega} \le C|f|_{1,\Omega}.$$

The next step is to estimate $\|v - t_h\|_{0,\Omega}$. We use the homogenization technique for this purpose. Set $\phi^K = \alpha_i^K \phi_i^K$, where $\alpha_i^K = \int_{e_i^K} v_0 \cdot n ds$. Then $t_h = k_\epsilon \nabla \phi^K$ and $\mathrm{div}(k_\epsilon \nabla \phi^K) = \mathrm{div}(P_h v_0) = 0$ in K, where $\phi^K \in H^1(K)/\mathbb{R}$ satisfies the following equation

$$\begin{aligned} \mathrm{div}(k_\epsilon \nabla \phi^K) = 0 \qquad & \text{in } K \\ k_\epsilon \nabla \phi^K \cdot n = P_h v_0 \cdot n \qquad & \text{on } e_i^K. \end{aligned}$$

Let ϕ_0^K be the solution of the corresponding homogenization equation,

$$\begin{aligned} \operatorname{div}(k^* \nabla \phi_0^K) &= 0 \qquad \text{in } K \\ k^* \nabla \phi_0^K \cdot n &= P_h v_0 \cdot n \qquad \text{on } e_i^K. \end{aligned}$$

To complete the estimation of $\|v - t_h\|_{0,\Omega}$, we need the following lemma.

Lemma 6.16. *Let* $p_1 = p_0 + \epsilon\chi \cdot \nabla p_0$ *and* $\phi_1^K = \phi_0^K + \epsilon\chi \cdot \nabla \phi_0^K$. *Then*

$$\begin{aligned} |\phi_0^K - p_0|_{1,K} &\le Ch\|\lambda^{-1} - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\ |p_1 - \phi_1^K|_{1,K} &\le C(h\|\lambda^{-1} - 1\|_{0,\infty,K} + \epsilon)\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\ |\phi_0^K|_{1,\infty,K} &\le Ch^{-\frac{d}{2}+1}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} + C\|\lambda\|_{0,\infty,K}|p_0|_{1,\infty,K}. \end{aligned} \tag{6.37}$$

Proof. It is easy to prove that $k^*\nabla\phi_0^K = P_h v_0 \in L^\infty(K)$. Then we have $\phi_0^K \in H^2(K) \cap W^{1,\infty}(K)$. Applying the interpolation estimate of Raviart–Thomas finite elements, we obtain

$$\begin{aligned} |\phi_0^K - p_0|_{1,K} &= \|(k^*)^{-1}P_h v_0 - (\lambda k^*)^{-1}v_0\|_{0,K} \\ &\le C\|\lambda^{-1} - 1\|_{0,\infty,K}\|P_h v_0 - v_0\|_{0,K} \\ &\le Ch\|\lambda^{-1} - 1\|_{0,\infty,K}|v_0|_{1,K} \\ &\le Ch\|\lambda^{-1} - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K}. \end{aligned}$$

Because $\nabla\phi_0^K = (k^*)^{-1}P_h v_0$ and P_h is a bounded operator, it is easy to show that

$$\begin{aligned} |\phi_0^K|_{1,K} &\le C|\lambda|_{0,\infty,K}|p_0|_{1,K} \\ |\phi_0^K|_{2,K} &\le C\|\lambda\|_{1,\infty,K}|p_0|_{2,K}. \end{aligned}$$

Applying the above estimates, we obtain

$$\begin{aligned} |p_1 - \phi_1^K|_{1,K} &\le |p_0 - \phi_0^K|_{1,K} + \|(\nabla_y \cdot \chi)\nabla(p_0 - \phi_0^K)\|_{0,K} \\ &\quad + \epsilon\|\chi(\nabla^2 p_0 - \nabla^2\phi_0^K)\|_{0,K} \le Ch\|\lambda^{-1} - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\ &\quad + C\epsilon\|\lambda\|_{1,\infty,K}|p_0|_{2,K}. \end{aligned}$$

As for the estimation of (6.37), we invoke the inverse inequality of finite elements and get

$$\begin{aligned} |\phi_0^K|_{1,\infty,K} &\le C\|P_h v_0 - \langle v_0\rangle_K\|_{0,\infty,K} + C\|\langle v_0\rangle_K\|_{0,\infty,K} \\ &\le Ch^{-d/2}\|P_h v_0 - \langle v_0\rangle_K\|_{0,K} + C\|\langle v_0\rangle_K\|_{0,\infty,K} \\ &\le Ch^{-d/2+1}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} + C\|\lambda\|_{0,\infty,K}|p_0|_{1,\infty,K}, \end{aligned}$$

where $d = 2$. The proof of the lemma is complete.

Next, we return to estimate $\|v - t_h\|_{0,\Omega}$. Applying the definitions of v and t_h and the Lemma 6.16, we obtain that

$$\begin{aligned}\|v - t_h\|_{0,K} &\leq C\|\lambda - 1\|_{0,\infty,K}\|\nabla p - \nabla \phi^K\|_{0,K} \\ &\leq C\|\lambda - 1\|_{0,\infty,K}(\|\nabla p - \nabla p_1\|_{0,K} + \|\nabla p_1 - \nabla \phi_1^K\|_{0,K} \\ &+ \|\nabla \phi_1^K - \nabla \phi^K\|_{0,K}) \leq C\|\lambda - 1\|_{0,\infty,K}[\epsilon(\|\lambda\|_{0,\infty,K}\|p_0\|_{2,K} \\ &+ \|\phi_0^K\|_{2,K}) + \sqrt{\epsilon h^{d-1}}(\|\lambda\|_{0,\infty,K}|p_0|_{1,\infty,K} + |\phi_0^K|_{1,\infty,K})] \\ &+ C\|\lambda - 1\|_{0,\infty,K}[(h\|\lambda^{-1} - 1\|_{0,\infty,K} + \epsilon)\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K}] \\ &\leq C_{K,1}(p_0, \lambda)\epsilon + C_{K,2}(p_0, \lambda)h + C_{K,3}(p_0, \lambda)\sqrt{\epsilon h} \\ &+ C_{K,4}(p_0, \lambda)\sqrt{\epsilon h^{d-1}},\end{aligned}$$

where d refers to the dimension of the space $\mathbb{R}^d$ ($d = 2$ for simplicity). Here we have used the corrector estimates (see Appendix B for discussions on corrector estimates for the Dirichlet problem and [71] for the corrector results that are used in the Neumann problem). Note that the constants in the above inequality are given by

$$\begin{aligned}C_{K,1}(p_0, \lambda) &= C\|\lambda - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\ C_{K,2}(p_0, \lambda) &= C\|\lambda - 1\|_{0,\infty,K}\|\lambda^{-1} - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\ C_{K,3}(p_0, \lambda) &= C\|\lambda - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\ C_{K,4}(p_0, \lambda) &= C\|\lambda - 1\|_{0,\infty,K}(1 + \|\lambda\|_{0,\infty,K})\|p_0\|_{1,\infty,K}.\end{aligned}$$

Taking the summation all over K, we have

$$\|v - t_h\|_{0,\Omega} \leq C_1(p_0, \lambda)\epsilon + \tilde{C}_2(p_0, \lambda)h + C_3(p_0, \lambda)\sqrt{\epsilon h} + C_4(p_0, \lambda)\sqrt{\frac{\epsilon}{h}}.$$

Here we have used the assumption that the triangulation is quasi-uniform, and the notations of the above coefficients are

$$C_1(p_0, \lambda) = C\|\lambda - 1\|_{0,\infty,\Omega}\|\lambda\|_{1,\infty,\Omega}\|p_0\|_{2,\Omega} \tag{6.38}$$
$$\tilde{C}_2(p_0, \lambda) = C\|\lambda - 1\|_{0,\infty,\Omega}\|\lambda^{-1} - 1\|_{0,\infty,\Omega}\|\lambda\|_{1,\infty,\Omega}\|p_0\|_{2,\Omega}$$
$$C_3(p_0, \lambda) = C\|\lambda - 1\|_{0,\infty,\Omega}\|\lambda\|_{1,\infty,\Omega}\|p_0\|_{2,\Omega} \tag{6.39}$$
$$C_4(p_0, \lambda) = C\|\lambda - 1\|_{0,\infty,\Omega}(1 + \|\lambda\|_{0,\infty,\Omega})\|p_0\|_{1,\infty,\Omega}. \tag{6.40}$$

Finally, applying Proposition 6.15, we get

$$\begin{aligned}\|v - v_h\|_{H(\mathrm{div},\Omega)} + \|p_\epsilon - p_h\|_{0,\Omega} &\leq C_1(p_0, \lambda)\epsilon + C_2(p_0, \lambda, g)h \\ &+ C_3(p_0, \lambda)\sqrt{\epsilon h} + C_4(p_0, \lambda)\sqrt{\frac{\epsilon}{h}},\end{aligned}$$

where

$$C_2(p_0, f, \lambda, g) = \tilde{C}_2(p_0, \lambda) + C\|g\|_{-1/2,\partial\Omega} + C|f|_{1,\Omega}. \tag{6.41}$$

Remark 6.17. From the proof, we see that the resonance term $O(\sqrt{\epsilon/h})$ comes from the terms estimated by $|p_0|_{1,\infty,K}$. If the p_0 can be exactly solved by some finite element method on the coarse grid, then we can use an inverse inequality to improve the convergence to $O(\epsilon + h + \sqrt{\epsilon h})$.

Remark 6.18. From the proof of the convergence theorem, one can see that it is sufficient to require $\lambda \in W^{1,\infty}(\Omega)$ and $\lambda^{-1} \in L^\infty(\Omega)$.

Remark 6.19. If the oversampling technique is used to approximate the flux v (see [71]), the resonance error can be reduced to $O(\epsilon/h)$.

6.2 Analysis of MsFEMs for nonlinear problems (from Chapter 3)

For the analysis of MsFEMs, we assume the following conditions for $k(x,\eta,\xi)$ and $k_0(x,\eta,\xi)$, $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$.

$$|k(x,\eta,\xi)| + |k_0(x,\eta,\xi)| \leq C\,(1 + |\eta|^{\gamma-1} + |\xi|^{\gamma-1}), \tag{6.42}$$

$$(k(x,\eta,\xi_1) - k(x,\eta,\xi_2)) \cdot (\xi_1 - \xi_2) \geq C\,|\xi_1 - \xi_2|^\gamma, \tag{6.43}$$

$$k(x,\eta,\xi)\cdot\xi + k_0(x,\eta,\xi)\eta \geq C|\xi|^\gamma. \tag{6.44}$$

Denote

$$H(\eta_1,\xi_1,\eta_2,\xi_2,r) = (1 + |\eta_1|^r + |\eta_2|^r + |\xi_1|^r + |\xi_2|^r), \tag{6.45}$$

for arbitrary $\eta_1,\ \eta_2 \in \mathbb{R}$, $\xi_1,\ \xi_2 \in \mathbb{R}^d$, and $r > 0$. We further assume that

$$\begin{aligned} &|k(x,\eta_1,\xi_1) - k(x,\eta_2,\xi_2)| + |k_0(x,\eta_1,\xi_1) - k_0(x,\eta_2,\xi_2)| \\ &\leq C\,H(\eta_1,\xi_1,\eta_2,\xi_2,\gamma-1)\,\nu(|\eta_1-\eta_2|) \\ &+ C\,H(\eta_1,\xi_1,\eta_2,\xi_2,\gamma-1-s)\,|\xi_1-\xi_2|^s, \end{aligned} \tag{6.46}$$

where $s > 0$, $\gamma > 1$, $s \in (0, \min(\gamma-1, 1))$ and ν is the modulus of continuity, a bounded, concave, and continuous function in $\mathbb{R}_+$ such that $\nu(0) = 0$, $\nu(t) = 1$ for $t \geq 1$, and $\nu(t) > 0$ for $t > 0$. Throughout, γ' is defined by $1/\gamma + 1/\gamma' = 1$, $y = x/\epsilon$. In further analysis $K \in \mathcal{T}_h$ is referred to simply by K. Inequalities (6.42)-(6.46) are the general conditions that guarantee the existence of a solution and are used in homogenization of nonlinear operators [220]. Here γ represents the rate of the polynomial growth of the fluxes with respect to the gradient and, consequently, it controls the summability of the solution. We do not assume any differentiability with respect to η and ξ in the coefficients. Our objective is to present a MsFEM and study its convergence for general nonlinear equations, where the fluxes can be discontinuous functions in space. These kinds of equations arise in many applications such as nonlinear heat conduction, flow in porous media, and so on. (see, e.g., [207, 244, 245]).

We present the main part of the analysis. For additional proofs of some auxiliary lemma, we refer to [104]. The analysis is presented for problems with scale separation. For this reason, we assume that the smallest scale is ϵ and denote the coefficients by $k(x,\cdot,\cdot) = k_\epsilon(x,\cdot,\cdot)$ and $k_0(x,\cdot,\cdot) = k_{0,\epsilon}(x,\cdot,\cdot)$.

In [111] we have shown using G-convergence theory that

$$\lim_{h\to 0}\lim_{\epsilon\to 0}\|p_h - p_0\|_{W_0^{1,\gamma}(\Omega)} = 0, \tag{6.47}$$

(up to a subsequence) where p_0 is a solution of (3.26) and p_h is a MsFEM solution given by (3.6). This result can be obtained without any assumption on the nature of the heterogeneities and cannot be improved because there could be infinitely many scales $\alpha(\epsilon)$ present such that $\alpha(\epsilon)\to 0$ as $\epsilon\to 0$.

Next we present the convergence results for MsFEM solutions. In the proof of this theorem we show the form of the truncation error (in a weak sense) in terms of the resonance errors between the mesh size and small-scale ϵ. The resonance errors are derived explicitly. To obtain the convergence rate from the truncation error, one needs some lower bounds. Under the general conditions, such as (6.42)–(6.46), one can prove strong convergence of MsFEM solutions without an explicit convergence rate (cf. [245]). To convert the obtained convergence rates for the truncation errors into the convergence rate of MsFEM solutions, additional assumptions, such as monotonicity, are needed.

Theorem 6.20. *Assume $k_\epsilon(x,\eta,\xi)$ and $k_{0,\epsilon}(x,\eta,\xi)$ are periodic functions with respect to x, let p_0 be a solution of* (3.26)*, and p_h is a MsFEM solution given by* (3.6)*. Moreover, we assume that ∇p_h is uniformly bounded in $L^{\gamma+\alpha}(\Omega)$ for some $\alpha>0$*[1]*. Then*

$$\lim_{\epsilon\to 0}\|p_h - p_0\|_{W_0^{1,\gamma}(\Omega)} = 0, \tag{6.48}$$

where $h=h(\epsilon)\gg\epsilon$ and $h\to 0$ as $\epsilon\to 0$ (up to a subsequence).

Theorem 6.21. *Let p_0 and p_h be the solutions of the homogenized problem* (3.26) *and MsFEM* (3.6)*, respectively, with the coefficient $k_\epsilon(x,\eta,\xi)=k(x/\epsilon,\xi)$ and $k_{0,\epsilon}=0$. Then*

$$\|p_h - p_0\|^{\gamma}_{W_0^{1,\gamma}(\Omega)} \le C\left(\left(\frac{\epsilon}{h}\right)^{s/((\gamma-1)(\gamma-s))} + \left(\frac{\epsilon}{h}\right)^{\gamma/(\gamma-1)} + h^{\gamma/(\gamma-1)}\right). \tag{6.49}$$

We first prove Theorem 6.20. Then, using the estimates obtained in the proof of this theorem, we show (6.49). The main idea of the proof of Theorem 6.20 is the following. First, the boundedness of the discrete solutions independent of ϵ and h is shown. This allows us to extract a weakly converging subsequence. The next task is to prove that a limit is a solution of the homogenized equation. For this reason correctors for $v_{r,h}$ (see (3.2)) are used and their convergence is demonstrated. We would like to note that the known convergence results for the correctors assume a fixed (given) homogenized solution, whereas the correctors for $v_{r,h}$ are defined for only a uniformly bounded sequence v_h, that is, the homogenization limits of $v_{r,h}$ (with respect to ϵ) depend on h, and are only uniformly bounded. Because of this, more

[1] Please see Remark 6.28 at the end of the proof of Theorem 6.20 for more discussions and partial results regarding this assumption.

precise corrector results need to be obtained where the homogenized limit of the solution is tracked carefully in the analysis. Note that to prove (6.47) (see [112]), one does not need correctors and can use the fact of the convergence of fluxes, and, thus, the proof of the periodic case differs from the one in [112]. Some results (Lemmas 6.22, 6.23, and their proofs) do not require periodicity assumptions. For these results we use the notations $k_\epsilon(x,\eta,\xi)$ and $k_{0,\epsilon}(x,\eta,\xi)$ to distinguish the two cases. The rest of the proofs require periodicity, and we use $k(x/\epsilon,\eta,\xi)$ and $k_0(x/\epsilon,\eta,\xi)$ notations.

Lemma 6.22. *There exists a constant $C>0$ such that for any $v_h \in W_h$*

$$\langle k_{r,h} v_h, v_h\rangle \geq C \|\nabla v_h\|^\gamma_{L^\gamma(\Omega)},$$

for sufficiently small h.

The proof of this lemma is provided in [104]. The following lemma is used in the proof of Lemma 6.24.

Lemma 6.23. *Let $v_\epsilon - v_0 \in W^{1,\gamma}_0(K)$ and $w_\epsilon - w_0 \in W^{1,\gamma}_0(K)$ satisfy the following problems, respectively,*

$$-\operatorname{div} k_\epsilon(x,\eta,\nabla v_\epsilon) = 0 \ \ in\ K \tag{6.50}$$

$$-\operatorname{div} k_\epsilon(x,\eta,\nabla w_\epsilon) = 0 \ \ in\ K, \tag{6.51}$$

where η is constant in K. Then the following estimate holds:

$$\|\nabla(v_\epsilon - w_\epsilon)\|_{L^\gamma(K)} \leq C\, H_0\, \|\nabla(v_0 - w_0)\|^{\gamma/(\gamma-s)}_{L^\gamma(K)}, \tag{6.52}$$

where

$$H_0 = \left(|K| + \|\eta\|^\gamma_{L^\gamma(K)} + \|\nabla v_0\|^\gamma_{L^\gamma(K)} + \|\nabla w_0\|^\gamma_{L^\gamma(K)}\right)^{(\gamma-s-1)/(\gamma-s)},$$

where $s \in (0, \min(1,\gamma-1))$, $\gamma > 1$.

For the proof of this lemma, we refer to [104].

Next, we introduce, as before, the fast variable $y = x/\epsilon$. Regarding η^{v_h}, where $\eta^{v_h} = (1/|K|)\int_K v_h dx$ in each K, we note that Jensen's inequality implies

$$\|\eta^{v_h}\|_{L^\gamma(\Omega)} \leq C\|v_h\|_{L^\gamma(\Omega)}. \tag{6.53}$$

In addition, the following estimates hold for η^{v_h},

$$\|v_h - \eta^{v_h}\|_{L^\gamma(K)} \leq C\, h\, \|\nabla v_h\|_{L^\gamma(K)}. \tag{6.54}$$

At this stage we define a numerical corrector associated with $v_{r,h} = E^{MsFEM} v_h$, $v_h \in W_h$. First, let

$$P_{\eta,\xi}(y) = \xi + \nabla_y N_{\eta,\xi}(y), \tag{6.55}$$

for $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$, where $N_{\eta,\xi} \in W^{1,\gamma}_{per}(Y)$ is the periodic solution (with average zero) of

$$-\operatorname{div}(k(y,\eta,\xi + \nabla_y N_{\eta,\xi}(y))) = 0 \text{ in } Y, \tag{6.56}$$

where Y is a unit period. The homogenized fluxes are defined as follows:

$$k^*(\eta,\xi) = \int_Y k(y,\eta,\xi + \nabla_y N_{\eta,\xi}(y))\, dy, \tag{6.57}$$

$$k_0^*(\eta,\xi) = \int_Y k_0(y,\eta,\xi + \nabla_y N_{\eta,\xi}(y))\, dy, \tag{6.58}$$

where k^* and k_0^* satisfy the conditions similar to (6.42)–(6.46). We refer to [220] for further details. Using (6.55), we denote our numerical corrector by $P_{\eta^{v_h},\nabla v_h}$ which is defined as

$$P_{\eta^{v_h},\nabla v_h} = \nabla v_h + \nabla_y N_{\eta^{v_h},\nabla v_h}(y). \tag{6.59}$$

Here η^{v_h} is a piecewise constant function defined in each $K \in \mathcal{T}_h$ by $\eta^{v_h} = (1/|K|)\int_K v_h dx$. Consequently, $P_{\eta^{v_h},\nabla v_h}$ is defined in Ω by using (6.59) in each $K \in \mathcal{T}_h$. For the linear problem $P_{\eta^{v_h},\nabla v_h} = \nabla v_h + N(y) \cdot \nabla v_h$. Our goal is to show the convergence of these correctors for the uniformly bounded family of v_h in $W^{1,\gamma}(\Omega)$. We note that the corrector results known in the literature are for a fixed homogenized solution.

Lemma 6.24. *Let $v_{r,h}$ satisfy* (3.2)*, where $k_\epsilon(x,\eta,\xi)$ is a periodic function with respect to x, and assume that v_h is uniformly bounded in $W_0^{1,\gamma}(\Omega)$. Then*

$$\begin{aligned}\|\nabla v_{r,h} - &P_{\eta^{v_h},\nabla v_h}\|_{L^\gamma(\Omega)} \\ &\le C\left(\frac{\epsilon}{h}\right)^{1/(\gamma(\gamma-s))}\left(|\Omega| + \|v_h\|^\gamma_{L^\gamma(\Omega)} + \|\nabla v_h\|^\gamma_{L^\gamma(\Omega)}\right)^{1/\gamma}.\end{aligned} \tag{6.60}$$

We note that here $s \in (0, \min(\gamma-1,1))$, $\gamma > 1$. For the proof of this lemma, we need the following proposition.

Proposition 6.25. *For every $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ we have*

$$\|P_{\eta,\xi}\|^\gamma_{L^\gamma(Y_\epsilon)} \le c\,(1 + |\eta|^\gamma + |\xi|^\gamma)\,|Y_\epsilon|, \tag{6.61}$$

where Y_ϵ is a period of size ϵ.

An easy consequence of this proposition is the following estimate for $N_{\eta,\xi}$ (see (6.56)).

Corollary 6.26. *For every $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ we have*

$$\|\nabla_y N_{\eta,\xi}\|^\gamma_{L^\gamma(Y_\epsilon)} \le c\,(1 + |\eta|^\gamma + |\xi|^\gamma)\,|Y_\epsilon|. \tag{6.62}$$

The proof of Proposition 6.25 is presented in [104].

Proof. (Lemma 6.24) Recall that by definition

$$P_{\eta^{v_h},\nabla v_h} = \nabla v_h + \nabla_y N_{\eta^{v_h},\nabla v_h}(y) = \nabla v_h + \epsilon\,\nabla N_{\eta^{v_h},\nabla v_h}(x/\epsilon),$$

where by using (6.56) $N_{\eta^{v_h},\nabla v_h}(y)$ is a zero-mean periodic function satisfying the following,

$$-\operatorname{div}(k(x/\epsilon, \eta^{v_h}, \nabla v_h + \epsilon\nabla\, N_{\eta^{v_h},\nabla v_h})) = 0 \text{ in } K. \tag{6.63}$$

We expand $v_{r,h}$ as

$$v_{r,h} = v_h(x) + \epsilon\, N_{\eta^{v_h},\nabla v_h}(x/\epsilon) + \theta(x, x/\epsilon). \tag{6.64}$$

We note that here $\theta(x, x/\epsilon)$ is similar to the correction terms that arise in linear problems because of the mismatch between linear boundary conditions and the oscillatory corrector, $N_{\eta^{v_h},\nabla v_h}(x/\epsilon) = N(x/\epsilon)\cdot\nabla v_h$. Next we denote by $w_{r,h} = v_h(x) + \epsilon\, N_{\eta^{v_h},\nabla v_h}(x/\epsilon)$. Clearly $w_{r,h}$ satisfies (6.63). Taking all these into account, the claim in the lemma is the same as proving

$$\begin{aligned} &\|\nabla\theta\|_{L^\gamma(\Omega)} = \|\nabla(v_{r,h} - w_{r,h})\|_{L^\gamma(\Omega)} \\ &\le C\left(\frac{\epsilon}{h}\right)^{1/(\gamma(\gamma-s))}\left(|\Omega| + \|v_h\|^\gamma_{L^\gamma(\Omega)} + \|\nabla v_h\|^\gamma_{L^\gamma(\Omega)}\right)^{1/\gamma}. \end{aligned} \tag{6.65}$$

Here we may write $w_{r,h}$ as a solution of the following boundary value problem:

$$-\operatorname{div}(k(x/\epsilon, \eta^{v_h}, \nabla w_{r,h})) = 0 \text{ in } K \text{ and } w_{r,h} = v_h + \epsilon\,\tilde{N}_{\eta^{v_h},\nabla v_h} \text{ on } \partial K,$$

with $\tilde{N}_{\eta^{v_h},\nabla v_h} = \zeta\, N_{\eta^{v_h},\nabla v_h}$, where ζ is a sufficiently smooth function whose value is 1 on a strip of width ϵ adjacent to ∂K and 0 elsewhere. We denote this strip by S_ϵ. This idea has been used in [164]. By Lemma 6.23 we have the following estimate:

$$\begin{aligned} \|\nabla\theta\|^\gamma_{L^\gamma(K)} &= \|\nabla(v_{r,h} - w_{r,h})\|^\gamma_{L^\gamma(K)} \\ &\le C\,H_0\,\|\nabla(v_h - v_h - \epsilon\,\tilde{N}_{\eta^{v_h},\nabla v_h})\|^{\gamma/(\gamma-s)}_{L^\gamma(K)} \\ &\le C\,H_0\,\|\epsilon\,\nabla\tilde{N}_{\eta^{v_h},\nabla v_h}\|^{\gamma/(\gamma-s)}_{L^\gamma(K)}, \end{aligned} \tag{6.66}$$

where

$$H_0 = \left(|K| + \|\eta^{v_h}\|^\gamma_{L^\gamma(K)} + \|\nabla v_h\|^\gamma_{L^\gamma(K)} + \|\nabla(v_h + \epsilon\,\tilde{N}_{\eta^{v_h},\nabla v_h})\|^\gamma_{L^\gamma(K)}\right)^{\frac{(\gamma-s-1)}{(\gamma-s)}}. \tag{6.67}$$

We need to show that H_0 is bounded and $\|\epsilon\,\nabla\tilde{N}_{\eta^{v_h},\nabla v_h}\|^\gamma_{L^\gamma(\Omega)}$ uniformly vanishes as $\epsilon \to 0$. For this purpose, we use the following notations. Let $J^K_\epsilon = \{i \in \mathbb{Z}^d : Y^i \bigcap K \neq 0, K\backslash Y^i \neq 0\}$ and $F^K_\epsilon = \cup_{i\in J^K_\epsilon} Y^i$. In other words,

F_ϵ^K is the union of all periods Y^i that cover the strip S_ϵ. Using these notations and because ζ is zero everywhere in K, except in the strip S_ϵ, we may write the following

$$\begin{aligned}
\|\epsilon\,\nabla\tilde{N}_{\eta^{v_h},\nabla v_h}\|^\gamma_{L^\gamma(K)} &= \epsilon^\gamma \int_K |\nabla(\zeta\, N_{\eta^{v_h},\nabla v_h})|^\gamma\,dx\\
&= \epsilon^\gamma \int_{S_\epsilon} |\nabla(\zeta\, N_{\eta^{v_h},\nabla v_h})|^\gamma\,dx\\
&\le \epsilon^\gamma \int_{F_\epsilon^K} |\nabla(\zeta\, N_{\eta^{v_h},\nabla v_h})|^\gamma\,dx\\
&= \epsilon^\gamma \sum_{i\in J_\epsilon^K} \int_{Y_\epsilon^i} |\nabla(\zeta\, N_{\eta^{v_h},\nabla v_h})|^\gamma\,dx\\
&\le \epsilon^\gamma \sum_{i\in J_\epsilon^K} \int_{Y_\epsilon^i} \left(|\nabla\, N_{\eta^{v_h},\nabla v_h}|^\gamma|\zeta|^\gamma + |N_{\eta^{v_h},\nabla v_h}|^\gamma|\nabla\zeta|^\gamma\right)\,dx,
\end{aligned} \tag{6.68}$$

where we have used the product rule on the partial derivative in the last line of (6.68). Our aim now is to show that the sum of integrals in the last line of (6.68) is uniformly bounded. We note that (see Corollary 6.26)

$$\|\nabla_y\, N_{\eta^{v_h},\nabla v_h}\|^\gamma_{L^\gamma(Y_\epsilon^i)} \le C(1+|\eta^{v_h}|^\gamma + |\nabla v_h|^\gamma)\,|Y_\epsilon^i|, \tag{6.69}$$

from which, using the Poincaré-Friedrich inequality we have

$$\|N_{\eta^{v_h},\nabla v_h}\|^\gamma_{L^\gamma(Y_\epsilon^i)} \le C(1+|\eta^{v_h}|^\gamma + |\nabla v_h|^\gamma)\,|Y_\epsilon^i|. \tag{6.70}$$

We note also that η^{v_h} and ∇v_h are constant in K. Because ζ is sufficiently smooth, and whose value is one on the strip S_ϵ and zero elsewhere, we know that $|\nabla\zeta| \le C/\epsilon$ (cf. [164]). Applying all these facts to (6.68) we have

$$\begin{aligned}
\|\epsilon\,\nabla\tilde{N}_{\eta^{v_h},\nabla v_h}\|^\gamma_{L^\gamma(K)} &\le C\,\epsilon^\gamma\,(1+|\eta^{v_h}|^\gamma + |\nabla v_h|^\gamma) \sum_{i\in J_\epsilon^K} (1+\epsilon^{-\gamma})\,|Y_\epsilon^i|\\
&= C\,(\epsilon^\gamma+1)\,(1+|\eta^{v_h}|^\gamma + |\nabla v_h|^\gamma) \sum_{i\in J_\epsilon^K} |Y_\epsilon^i|\\
&\le C\,(1+|\eta^{v_h}|^\gamma + |\nabla v_h|^\gamma) \sum_{i\in J_\epsilon^K} |Y_\epsilon^i|.
\end{aligned}$$

Moreover, because all Y_ϵ^i, $i\in J_\epsilon^K$, cover the strip S_ϵ, we know that $\sum_{i\in J_\epsilon^K} |Y_\epsilon^i| \le C\,h^{d-1}\,\epsilon$. Hence, we have

$$\begin{aligned}
\|\epsilon\,\nabla\tilde{N}_{\eta^{v_h},\nabla v_h}\|^\gamma_{L^\gamma(K)} &\le C\,\frac{h^d}{h^d}\,(1+|\eta^{v_h}|^\gamma + |\nabla v_h|^\gamma)\,h^{d-1}\,\epsilon\\
&\le C\,\frac{\epsilon}{h}\left(|K| + \|\eta^{v_h}\|^\gamma_{L^\gamma(K)} + \|\nabla v_h\|^\gamma_{L^\gamma(K)}\right).
\end{aligned} \tag{6.71}$$

Furthermore, using this estimate and noting that $\epsilon/h < 1$, we obtain from (6.67) that

$$H_0 \le C \left(|K| + \|\eta^{v_h}\|^\gamma_{L^\gamma(K)} + \|v_h\|^\gamma_{L^\gamma(K)} + \|\nabla v_h\|^\gamma_{L^\gamma(K)} \right)^{(\gamma-s-1)/(\gamma-s)} . \tag{6.72}$$

Summarizing the results from (6.66) combined with (6.72) and (6.71), we get

$$\begin{aligned} \|\nabla\theta\|^\gamma_{L^\gamma(K)} &\le C\, H_0\, \|\epsilon\, \nabla \tilde{N}_{\eta^{v_h}, \nabla v_h}\|^{\gamma/(\gamma-s)}_{L^\gamma(K)} \\ &\le C \left(\frac{\epsilon}{h}\right)^{1/(\gamma-s)} \left(|K| + \|\eta^{v_h}\|^\gamma_{L^\gamma(K)} + \|v_h\|^\gamma_{L^\gamma(K)} + \|\nabla v_h\|^\gamma_{L^\gamma(K)} \right). \end{aligned}$$

Finally summing over all $K \in \mathcal{T}_h$ and applying (6.53) to $\sum_{K\in\mathcal{T}_h} \|\eta^{v_h}\|^\gamma_{L^\gamma(K)}$, we obtain

$$\begin{aligned} \|\nabla\theta\|^\gamma_{L^\gamma(\Omega)} &= \sum_K \|\nabla\theta\|^\gamma_{L^\gamma(K)} \\ &\le C \left(\frac{\epsilon}{h}\right)^{1/(\gamma-s)} \sum_K \left(|K| + \|v_h\|^\gamma_{L^\gamma(K)} + \|\nabla v_h\|^\gamma_{L^\gamma(K)} \right) \qquad (6.73) \\ &= C \left(\frac{\epsilon}{h}\right)^{1/(\gamma-s)} \left(|\Omega| + \|v_h\|^\gamma_{L^\gamma(\Omega)} + \|\nabla v_h\|^\gamma_{L^\gamma(\Omega)} \right). \end{aligned}$$

The last inequality uniformly vanishes as ϵ approaches zero, thus we have completed the proof of Lemma 6.24.

The next lemma is crucial for the proof of Theorem 6.20 and it guarantees the convergence of MsFEM solutions to a solution of the homogenized equation. This lemma also provides us with the estimate for the truncation error (in a weak sense).

Lemma 6.27. *Suppose $v_h, w_h \in W_h$ where ∇v_h and ∇w_h are uniformly bounded in $L^{\gamma+\alpha}(\Omega)$ and $L^\gamma(\Omega)$, respectively, for some $\alpha > 0$. Let κ^* be the operator associated with the homogenized problem* (3.26)*, such that*

$$\langle \kappa^*\, v_h, w_h\rangle = \sum_{K\in\mathcal{T}_h} \int_K (k^*(v_h, \nabla v_h)\cdot\nabla w_h + k_0^*(v_h, \nabla v_h) w_h)\, dx, \quad \forall v_h, w_h \in W_h. \tag{6.74}$$

Then we have

$$\lim_{\epsilon\to 0} \langle \kappa_{r,h}\, v_h - \kappa^*\, v_h, w_h\rangle = 0.$$

The proof of this lemma is presented in [104]. Now we are ready to prove Theorem 6.20.

Proof. (Theorem 6.20) Because $\kappa_{r,h}$ is coercive, it follows that p_h is bounded, which implies that it has a subsequence (which we also denote by p_h) such that $p_h \rightharpoonup \tilde{p}$ in $W^{1,\gamma}(\Omega)$ as $\epsilon \to 0$. Because the operator κ^* is of type S_+ (see, e.g., [245], page 3, for the definition), then by its definition, the strong convergence would be true if we can show that $\limsup_{\epsilon\to 0} \langle \kappa^* p_h, p_h - \tilde{p}\rangle \to 0$. Moreover, by adding and subtracting the term, we have the following equality

$$\begin{aligned}\langle \kappa^* p_h, p_h - \tilde{p}\rangle &= \langle \kappa^* p_h - \kappa_{r,h} p_h, p_h - \tilde{p}\rangle + \langle \kappa_{r,h} p_h, p_h - \tilde{p}\rangle \\ &= \langle \kappa^* p_h - \kappa_{r,h} p_h, p_h\rangle - \langle \kappa^* p_h - \kappa_{r,h} p_h, \tilde{p}\rangle + \int_\Omega f(p_h - \tilde{p}) dx.\end{aligned} \tag{6.75}$$

Lemma 6.27 implies that the first and second term vanish as $\epsilon \to 0$ provided ∇p_h is uniformly bounded in $L^{\gamma+\alpha}$ for $\alpha > 0$, and the last term vanishes as $\epsilon \to 0$ (up to a subsequence) by the weak convergence of p_h. One can assume additional mild regularity assumptions [201] for input data and obtain Meyers type estimates, $\|\nabla p_0\|_{L^{\gamma+\alpha}(\Omega)} \leq C$, for the homogenized solutions. In this case it is reasonable to assume that the discrete solutions are uniformly bounded in $L^{\gamma+\alpha}(\Omega)$. We have obtained results on Meyers type estimates for our approximate solutions in the case $\gamma = 2$ [114]. Finally, because κ^* is also of type M (see, e.g., [244], page 38, for the definition), all these conditions imply that $\kappa^* \tilde{p} = f$, which means that $\tilde{p} = p_0$.

Remark 6.28. We would like to point out that for the proof of Theorem 6.20 it is assumed that ∇p_h is uniformly bounded in $L^{\gamma+\alpha}(\Omega)$ for some $\alpha > 0$ (see discussions after (6.75)). This has been shown for $\gamma = 2$ in [114]. To avoid this assumption, one can impose additional restrictions on $k^*(\eta, \xi)$ (see, [112], pages 254, 255). We note that the assumption, ∇p_h is uniformly bounded in $L^{\gamma+\alpha}(\Omega)$, is not used for the estimation of the resonance errors.

Next we present some explicit estimates for the convergence rates of MsFEM. First, we note that from the proof of the Lemma 6.27 it follows that the truncation error of MsFEM (in a weak sense) is given by

$$\begin{aligned}&\langle k_{r,h} p_h - \kappa^* p_h, w_h\rangle = \langle f - A^* p_h, w_h\rangle \\ &\leq C\left(\frac{\epsilon}{h}\right)^{s/(\gamma(\gamma-s))} \left(|\Omega| + \|p_h\|^\gamma_{L^\gamma(\Omega)} + \|\nabla p_h\|^\gamma_{L^\gamma(\Omega)}\right)^{(1/\gamma')} \|\nabla w_h\|_{L^\gamma(\Omega)} \\ &+ C\frac{\epsilon}{h}\left(|\Omega| + \|p_h\|^\gamma_{L^\gamma(\Omega)} + \|\nabla p_h\|^\gamma_{L^\gamma(\Omega)}\right)^{1/\gamma'} \|\nabla w_h\|_{L^\gamma(\Omega)} + e(h)\|\nabla w_h\|_{L^\gamma(\Omega)} \\ &= C\left(\left(\frac{\epsilon}{h}\right)^{s/(\gamma(\gamma-s))} + \frac{\epsilon}{h}\right)\left(|\Omega| + \|p_h\|^\gamma_{L^\gamma(\Omega)} + \|\nabla p_h\|^\gamma_{L^\gamma(\Omega)}\right)^{1/\gamma'} \|\nabla w_h\|_{L^\gamma(\Omega)} \\ &+ e(h)\|\nabla w_h\|_{L^\gamma(\Omega)},\end{aligned} \tag{6.76}$$

where $e(h)$ is a generic sequence independent of small-scale ϵ, such that $e(h) \to 0$ as $h \to 0$. We note that the first term on the right side of (6.76) is the leading order resonance error caused by the linear boundary conditions imposed on ∂K, and the second term is due to the mismatch between the mesh size and the small scale of the problem. These resonance errors are also present in the linear case as we discussed in Section 6.1. If one uses the periodic solution of the auxiliary problem for constructing the solutions of the local problems, then the resonance error can be removed. To obtain explicit convergence rates, we first derive upper bounds for $\langle \kappa^* p_h - \kappa^* P_h p_0, p_h - P_h p_0\rangle$, where $P_h u$ denotes a finite element projection of u onto W_h; that is,

$$\langle \kappa^* P_h p_0, v_h\rangle = \int_\Omega f v_h dx, \quad \forall v_h \in W_h,$$

and $\langle \kappa^* p_h, v_h\rangle$ is defined by (6.74). Then using estimate (6.76), we have

$$\begin{aligned}
&\langle \kappa^* p_h - \kappa^* P_h p_0, p_h - P_h u\rangle = \langle \kappa^* p_h - k_{r,h} p_h, p_h - P_h p_0\rangle \\
&+ \langle k_{r,h} p_h - \kappa^* P_h p_0, p_h - P_h p_0\rangle = \langle \kappa^* p_h - k_{r,h} p_h, p_h - P_h p_0\rangle \\
&+ \langle f - \kappa^* P_h p_0, p_h - P_h p_0\rangle = \langle \kappa^* p_h - k_{r,h} p_h, p_h - P_h p_0\rangle \\
&\le C \left(\left(\frac{\epsilon}{h}\right)^{s/(\gamma(\gamma-s))} + \frac{\epsilon}{h} \right) \left(|\Omega| + \|p_h\|^\gamma_{L^\gamma(\Omega)} + \|\nabla p_h\|^\gamma_{L^\gamma(\Omega)} \right)^{1/\gamma'} \times \\
&\|\nabla (p_h - P_h p_0)\|_{L^\gamma(\Omega)} + e(h) \|\nabla (p_h - P_h p_0)\|_{L^\gamma(\Omega)}.
\end{aligned} \tag{6.77}$$

The estimate (6.77) does not allow us to obtain an explicit convergence rate without some lower bound for the left side of the expression. In the proof of Theorem 6.20, we only use the fact that κ^* is the operator of type S_+, which guarantees that the convergence of the left side of (6.77) to zero implies the convergence of the discrete solutions to a solution of the homogenized equation. Explicit convergence rates can be obtained by assuming some kind of an inverse stability condition, $\|\kappa^* u - \kappa^* v\| \ge C\|u - v\|$. In particular, we may assume that κ^* is a monotone operator; that is,

$$\langle \kappa^* u - \kappa^* v, u - v\rangle \ge C \|\nabla (u - v)\|^\gamma_{L^\gamma(\Omega)}. \tag{6.78}$$

A simple way to achieve monotonicity is to assume $k_\epsilon(x, \eta, \xi) = k_\epsilon(x, \xi)$ and $k_{0,\epsilon}(x, \eta, \xi) = 0$, although one can impose additional conditions on $k_\epsilon(x, \eta, \xi)$ and $k_{0,\epsilon}(x, \eta, \xi)$, such that monotonicity condition (6.78) is satisfied. For our further calculations, we only assume (6.78). Then from (6.77) and (6.78), and using the Young inequality, we have

$$\|\nabla (p_h - P_h p_0)\|^\gamma_{L^\gamma(\Omega)} \le C \left(\left(\frac{\epsilon}{h}\right)^{s/((\gamma-1)(\gamma-s))} + \left(\frac{\epsilon}{h}\right)^{\gamma/(\gamma-1)} \right) + e(h).$$

Next taking into account the convergence of standard finite element solutions of the homogenized equation we write

$$\|\nabla P_h p - \nabla p_0\|_{L^\gamma(\Omega)} \le e_1(h),$$

where $e_1(h) \to 0$ (as $h \to 0$) is independent of ϵ. Consequently, using the triangle inequality we have

$$\|\nabla (p_h - p_0)\|^\gamma_{L^\gamma(\Omega)} \le C \left(\left(\frac{\epsilon}{h}\right)^{s/((\gamma-1)(\gamma-s))} + \left(\frac{\epsilon}{h}\right)^{\gamma/(\gamma-1)} \right) + e(h) + e_1(h).$$

Proof. (Theorem 6.21).

For monotone operators, $k_\epsilon(x,\eta,\xi) = k_\epsilon(x,\xi)$ and $k_{0,\epsilon}(x,\eta,\xi) = 0$, $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$, the estimates for $e(h)$ and $e_1(h)$ can be easily derived. In particular, because of the absence of η in k_ϵ, $e(h) = 0$, and $e_1(h) \leq Ch^{1/(\gamma-1)}$ (see, e.g., [75]). Combining these estimates we have

$$\|\nabla(p_h - p_0)\|^\gamma_{L^\gamma(\Omega)} \leq C\left(\left(\frac{\epsilon}{h}\right)^{s/((\gamma-1)(\gamma-s))} + \left(\frac{\epsilon}{h}\right)^{\gamma/(\gamma-1)} + h^{\gamma/(\gamma-1)}\right).$$

From here one obtains (6.49).

Remark 6.29. One can impose various conditions on the operators to obtain different kinds of convergence rates. For example, under the additional assumptions (cf. [207])

$$|\frac{\partial k^*(\eta,\xi)}{\partial \eta}| + |\frac{\partial k^*(\eta,\xi)}{\partial \xi}| \leq C, \quad \frac{\partial k_i^*(\eta,\xi)}{\partial \xi_j}\beta_i\beta_j \geq C|\beta|^2,$$

where $\beta \in \mathbb{R}^d$ is an arbitrary vector, and $\gamma = 2$, following the analysis presented in [207] (pages 51, 52), the convergence rate in terms of the L^γ-norm of $p_h - P_h p$ can be obtained,

$$\begin{aligned}\|\nabla(p_h - P_h p_0)\|^\gamma_{L^\gamma(\Omega)} \leq C&\left(\left(\frac{\epsilon}{h}\right)^{s/((\gamma-1)(\gamma-s))} + \left(\frac{\epsilon}{h}\right)^{\gamma/(\gamma-1)}\right)\\ &+e(h) + C\|p_h - P_h p_0\|^\gamma_{L^\gamma(\Omega)},\end{aligned} \tag{6.79}$$

where $s \in (0,1)$, $\gamma = 2$.

Remark 6.30. For the linear operators ($\gamma = 2$, $s = 1$), we recover the convergence rate $Ch + C_1\sqrt{\epsilon/h}$.

Remark 6.31. We have shown that the MsFEM for nonlinear problems has the same error structure as for linear problems. In particular, our studies revealed two kinds of resonance errors for nonlinear problems with the same nature as those that arise in linear problems.

6.3 Analysis for MsFEMs with limited global information (from Chapter 4)

6.3.1 Mixed finite element methods with limited global information

Elliptic case

We begin by restating the main assumption in a rigorous way.

Assumption A1. There exist functions $v_1, ..., v_N$ and sufficiently smooth $A_1(x), ..., A_N(x)$ such that

$$v(x) = \sum_{i=1}^{N} A_i(x) v_i, \tag{6.80}$$

where $v_i = k\nabla p_i$ and p_i solves $\mathrm{div}(k(x)\nabla p_i) = 0$ in Ω with appropriate boundary conditions.

For our analysis, we assume $A_i(x) \in W^{1,\xi}(\Omega)$, and $v_i = k(x)\nabla p_i \in L^\eta(\Omega)$ for some ξ and η, $i = 1, ..., N$. *Throughout this section, we do not use the Einstein summation convention.*

Remark 6.32. As an example of two global fields in $\mathbb{R}^2$ (similar results hold in $\mathbb{R}^d$; see [218] for details), we use the results of Owhadi and Zhang [218]. Let $v_i = k(x)\nabla p_i$ $(i = 1, 2)$ be defined by the elliptic equation

$$\begin{aligned} \mathrm{div}(k(x)\nabla p_i) = 0 \ \text{ in } \ \Omega \\ p_i = x_i \ \text{ on } \ \partial\Omega, \end{aligned} \tag{6.81}$$

where $x = (x_1, x_2)$. In the harmonic coordinate (p_1, p_2), $p = p(p_1, p_2) \in W^{2,s}$ $(s \geq 2)$. Consequently, $v = \lambda(x)k(x)\nabla p = \sum_i \lambda(\partial p/\partial p_i)k\nabla p_i := \sum_i A_i(x)v_i$, where $A_i(x) = \lambda(\partial p/\partial p_i) \in W^{1,s}$.

To avoid the possibility that $\int_{e_l} v_i \cdot n ds$ is zero or unbounded, we make the following assumption for our analysis.

Assumption A2. There exist positive constants C such that

$$\int_{e_l} |v_i \cdot n| ds \leq C h^{\beta_1} \quad \textit{and} \quad \| \frac{v_i \cdot n}{\int_{e_l} v_i \cdot n ds} \|_{L^r(e_l)} \leq C h^{-\beta_2 + 1/r - 1} \tag{6.82}$$

uniformly for all edges e_l, where $\beta_1 \leq 1$, $\beta_2 \geq 0$, and $r \geq 1$.

Remark 6.33. The second part of Assumption A2 is to assure $|\int_{e_l} v_i \cdot n ds|$ remains positive. It can be also written as

$$\| \frac{v_i \cdot n}{\int_{e_l} v_i \cdot n ds} - \langle \frac{v_i \cdot n}{\int_{e_l} v_i \cdot n ds} \rangle_{e_l} \|_{L^r(e_l)} \leq C h^{-\beta_2 + 1/r - 1},$$

where $\langle \cdot \rangle = (1/|e_l|) \int_{e_l} (\cdot) ds$, which is used to estimate the velocity basis function. If v_i are bounded, then $\beta_2 = 0$. Note that

$$\| \frac{v_i \cdot n}{\int_{e_l} v_i \cdot n ds} - \langle \frac{v_i \cdot n}{\int_{e_l} v_i \cdot n ds} \rangle_{e_l} \|_{L^r(e_l)} = 0$$

if $v_i|_K$ is an RT_0 basis function or standard mixed MsFEM basis functions introduced in [71]. Finally, we note that if $r = 1$ and $|\int_{e_l} v_i \cdot n ds| \geq C h^{\beta_1}$, then $\beta_2 = 0$.

We recall the definition of basis functions $\psi_{ij}^K = k(x)\nabla\phi_{ij}^K$ and

$$\mathcal{V}_h = \bigoplus_K \{\psi_{ij}^K\} \bigcap H(\text{div}, \Omega), \quad \mathcal{V}_h^0 = \bigoplus_K \{\psi_{ij}^K\} \bigcap H_0(\text{div}, \Omega).$$

Let $Q_h = \oplus_K P_0(K) \subset L^2(\Omega)/\mathbb{R}$ (i.e., piecewise constants), be the basis function for the pressure. We define

$$g_{0,h} = \sum_{e\in\{\partial K \bigcap \partial\Omega, K\in\tau_h\}} (\int_e g ds)\psi_{i,e}$$

for some fixed $i \in \{1, 2, ..., N\}$, where $\psi_{i,e}$ is the corresponding multiscale basis function to the edge e. Let $g_h = g_{0,h} \cdot n$ on $\partial\Omega$. The numerical mixed formulation is to find $\{v_h, p_h\} \in \mathcal{V}_h \times Q_h$ which satisfies (4.7) and $v_h \cdot n = g_h$ on $\partial\Omega$.

First, we note the following result.

Lemma 6.34.

$$v_i|_K \in \text{span}\{\psi_{ij}^K\}, \quad i = 1, .., N; \quad j = 1, 2, 3.$$

Proof. First, we prove the lemma for v_1. For this proof, we would like to find constants β_{ij}^Ks such that $\sum_{i,j} \beta_{ij}^K \psi_{ij}^K = v_1$. That is,

$$\begin{aligned}
&\sum_{i,j} \beta_{ij}^K \text{div}(k(x)\nabla\phi_{ij}^K) = \frac{1}{|K|}\sum_{i,j} \beta_{ij}^K = 0 \\
&\sum_{i,j} \beta_{ij}^K k(x)\nabla\phi_{ij}^K \cdot n_{e_l} = \sum_{i,j} \beta_{ij}^K \delta_{jl} \frac{v_i \cdot n_{e_l}}{\int_{e_l} v_i \cdot n ds} = v_1 \cdot n_{e_l}.
\end{aligned} \tag{6.83}$$

Noticing that $v_i = k(x)\nabla p_i$ and $\text{div}(k(x)\nabla p_i) = 0$, we have $p_i = \sum_{i,j} \beta_{ij}^K \phi_{ij}^K + C$ for some constant C because p_i and $\sum_{i,j} \beta_{ij}^K \phi_{ij}^K$ satisfy the same elliptic equation with Neumann boundary condition as p_i, and then we have $v_i = \sum_{i,j} \beta_{ij}^K \psi_{ij}^K$. The second equation in (6.83) implies that we can take $\beta_{1j}^K = \int_{e_j} v_1 \cdot n ds$ and $\beta_{ij}^K = 0$ for $i \neq 1$. Consequently,

$$\sum_{i,j} \beta_{ij}^K = \sum_j \int_{e_j} v_1 \cdot n ds = \int_K \text{div}(v_1) dx = 0,$$

which is the first equation in (6.83). One can obtain similar results for other v_i $(i = 2, ..., N)$.

Following our assumption, let

$$X = \{u | u = \sum_{i=1}^N a_i(x) v_i\}$$

be a subspace of $H(\text{div}, \Omega)$. For our analysis, we require that the integrals $\int_{e_j} a_i(x)v_i \cdot n ds$ are well defined. This is also needed in our computations because $\int_{e_j} a_i(x)v_i \cdot n ds$ determines the fluxes along the edges in two-phase flow simulations. One way to achieve this is to assume, as we did earlier, that $a_i(x) \in W^{1,\xi}(\Omega)$, $v_i \in L^\eta(\Omega)$, $\frac{1}{2} = 1/\xi + 1/\eta$. Because $a_i(x) \in W^{1,\xi}(\Omega)$ and $v_i \in L^\eta(\Omega)$ $(\frac{1}{2} = \frac{1}{\xi} + \frac{1}{\eta})$, Hölder inequality implies that $(\nabla a_i)v_i \in L^2(\Omega)$. Noticing that $\text{div}(v_i) = 0$, we have $\text{div}(a_i(x)v_i) \in L^2(\Omega)$ immediately. Invoking the Sobolev embedding theorem (see [18]), we get $a_i v_i \in L^\eta(\Omega)$ because $W^{1,\xi}(\Omega) \hookrightarrow L^\infty(\Omega)$. The integrals $\int_{e_j} a_i(x)v_i \cdot n ds$ are well defined by the fact that $a_i v_i \in L^\rho(\Omega)$ $(\rho > 2)$ and $\text{div}(a_i(x)v_i) \in L^2(\Omega)$ (see page 125 of [57]). We define an interpolation operator $\Pi_h : X \longrightarrow \mathcal{V}_h$ such that in each element K, for any $v = \sum_i a_i(x)v_i \in X$

$$\Pi_h|_K(\sum_i a_i(x)v_i) = \sum_{i,j} a_{ij}^K \psi_{ij}^K,$$

where $a_{ij}^K = \int_{e_j} a_i(x)v_i \cdot n ds$.

The proof of the following lemma can be found in [8].

Lemma 6.35. *Let Π_h be defined as above. Then $\forall v = \sum_{i=1}^N a_i v_i \in X, \quad q_h \in Q_h$,*
(1) $\int_\Omega \text{div}(v - \Pi_h v) q_h dx = 0$;
(2) $\|\Pi_h v\|_{H(\text{div},\Omega)} \le C\|v\|_{X,\Omega}$, if $\beta_1 \ge 2\beta_2$,
where $\|v\|_{X,\Omega} := \|\text{div}(v)\|_{0,\Omega} + \sum_{i=1}^N \|a_i\|_{1,\Omega}$ and C only depends on N, the constants in Assumption A2 (see (6.82)) and the pre-computed global fields v_i.

Remark 6.36. If $v_i \in L^\infty(\Omega)$, then $\beta_1 = 1$, $\beta_2 = 0$, and the proof of Lemma 6.35 implies that $\|\Pi_h v\|_{H(\text{div},\Omega)} \le C(\max_i \|v_i\|_{L^\infty(\Omega)}) \sum_i \|a_i\|_{1,\Omega}$.

Remark 6.37. For $v = \sum_{i=1}^N a_i v_i$, where $a_i \in W^{1,\xi}(\Omega)$ and $v_i \in L^\eta(\Omega)$ $(1/2 = 1/\xi + 1/\eta)$, one can also show that

$$\|\Pi_h v\|_{H(\text{div},\Omega)} \le C \sum_i \|a_i\|_{1,\xi,\Omega},$$

if $\alpha + \beta_1 - \beta_2 - 1 \ge 0$, where C only depends on N, the constants in Assumption A2 (see (6.82)), and the pre-computed global fields v_i.

Remark 6.38. We note that $\|v\|_{X,\Omega}$ may not be a norm in general because $v = \sum_i a_i v_i = 0$ may not imply that a_i are zero (this does not affect the derivation of the discrete inf-sup condition). In the problem setting considered here, one can assume that $\|v\|_{X,\Omega}$ is a norm. Indeed, a_i are coarse-scale functions, and v_i are fine-scale functions. Thus, in each coarse-grid block, the linear combination $\sum_i a_i v_i$ zero will imply that a_i are zero unless v_i are also coarse-scale functions. In the latter case, one can use standard mixed finite element basis functions. If $N = d$ (d being the dimension of the space), $\|v\|_{X,\Omega}$ is a norm when v_i are linearly independent. In the discrete setting, a_i are vectors defined on the coarse grid, whereas v_i are defined on the fine grid. If $\sum_i a_i v_i$ is zero, this implies that the vectors v_i are linearly dependent, and thus, the basis functions are linearly dependent.

Lemma 6.35 and the continuous inf-sup condition imply the discrete inf-sup condition (see page 58 of [57]). We assume that the continuous inf-sup condition holds (see [8] for more details). Assuming a continuous inf-sup condition, we have that for any $q_h \in Q_h$, there exists a constant C such that

$$\sup_{v_h \in \mathcal{V}_h} \frac{\int_\Omega \operatorname{div}(v_h) q_h dx}{\|v_h\|_{H(\operatorname{div},\Omega)}} \geq C \|q_h\|_{0,\Omega}. \tag{6.84}$$

Because of the inf-sup condition (6.84), we have the following optimal approximation (see [57, 71]).

Lemma 6.39. *Let* $\{v, p\}$ *and* $\{v_h, p_h\}$ *be the solution of* (4.4) *and* (4.7) *respectively. Then*

$$\|v - v_h\|_{H(\operatorname{div},\Omega)} + \|p - p_h\|_{0,\Omega} \leq C \inf_{w_h \in \mathcal{V}_h, w_h - g_{0,h} \in \mathcal{V}_h^0} \|v - w_h\|_{H(\operatorname{div},\Omega)} + C \inf_{q_h \in Q_h} \|p - q_h\|_{0,\Omega}. \tag{6.85}$$

Next, we formulate our main result.

Theorem 6.40. *Let* $\{v, p\}$ *and* $\{v_h, p_h\}$ *be the solution of* (4.4) *and* (4.7), *respectively. If* $\alpha + \beta_1 - \beta_2 - 1 > 0$, *we have*

$$\|v - v_h\|_{H(\operatorname{div},\Omega)} + \|p - p_h\|_{0,\Omega} \leq C h^{\alpha + \beta_1 - \beta_2 - 1},$$

where $\alpha = 1 - 2/\xi$, ξ *and* A_i *are defined in Assumption A1, and* β_i $(i = 1, 2)$ *are defined in Assumption A2. Here* C *is independent of* h *and depends on* N, *the constants in Assumption A2,* $\|A_i\|_{1,\xi,\Omega}$ $(i = 1, .., N)$ *and* $\|f\|_{1,\Omega}$.

Proof. For the proof, we need to choose a proper u_h and a proper q_h such that the right-hand side of (6.85) is small.

The second term on the right hand in (6.85) can be easily estimated. In fact, with the choice $q_h|_K = \langle p \rangle_K$ (i.e., the average of p in K), we have

$$\inf_{q_h \in Q_h} \|p - p_h\|_{0,\Omega} \leq C h |p|_{1,\Omega}.$$

Next, we try to find a $u_h \in \mathcal{V}_h$, say $u_h|_K = \sum_{i,j} c_{ij}^K \psi_{ij}^K$, and estimate the first term on the right-hand side in (6.85). Invoking Lemma 6.34 and its proof, it follows that in each K,

$$\begin{aligned} v - u_h &= \sum_i A_i(x) v_i - \sum_{i,j} c_{ij}^K \psi_{ij} \\ &= \sum_i (A_i(x) \sum_j \beta_{ij}^K \psi_{ij}^K) - \sum_{i,j} c_{ij}^K \psi_{ij}^K \\ &= \sum_{i,j} (A_i(x)\beta_{ij}^K - c_{ij}^K)\psi_{ij}^K, \end{aligned} \tag{6.86}$$

where $\beta_{ij}^K = \int_{e_j} v_i \cdot n ds$. Set $c_{ij}^K = A_{ij}^K = \int_{e_j} A_i(x) v_i \cdot n ds.$

Because $\int_K \sum_i \mathrm{div}(A_i(x)v_i)dx = f$, we get by the divergence theorem

$$\int_{\partial K} \sum_i A_i(x) v_i \cdot n ds = f.$$

This gives rise to

$$\begin{aligned} &\|\mathrm{div}(v - \sum_{i,j} c_{ij}^K \psi_{ij}^K)\|_{0,K} = \|f - \sum_{i,j} c_{ij}^K \frac{1}{|K|}\|_{0,K} \\ &= \|f - \sum_{i,j} \int_{e_j} A_i(x) v_i \cdot n ds \frac{1}{|K|}\|_{0,K} = \|f - \langle f \rangle_K\|_{0,K} \le Ch|f|_{1,K}. \end{aligned} \tag{6.87}$$

After summation over all K for (6.87), we have

$$\|\mathrm{div}(v - u_h)\|_{0,\Omega} \le Ch|f|_{1,\Omega}. \tag{6.88}$$

Next we estimate $\|v - \sum_{i,j} c_{ij}^K \psi_{ij}^K\|_{0,K}$. Because $A_i(x) \in W^{1,\xi}(\Omega)$, by using the Sobolev embedding theorem and Taylor expansion (or definition of C^α) we have

$$|A_i(x)|_{e_j} - \bar{A}_i^j| \le Ch^\alpha \|A_i\|_{C^\alpha(\Omega)},$$

where $\bar{A}_i^j$ is the average $A_i(x)$ along e_j and $\alpha = 1 - 2/\xi$. So

$$\begin{aligned} |A_{ij}^K - \bar{A}_i^j \beta_{ij}^K| &= |\int_{e_j} A_i v_i \cdot n ds - \bar{A}_i^j \int_{e_j} v_i \cdot n ds| \\ &= |\int_{e_j} (A_i - \bar{A}_i^j)(v_i \cdot n) ds| \le Ch^{\alpha+\beta_1} \|A_i\|_{C^\alpha(\Omega)}, \end{aligned} \tag{6.89}$$

where we have used the Assumption A2 (see (6.82)).

Next, we present an estimate for $\|\psi_{ij}^K\|_{0,K}$. For this reason, we introduce the lowest Raviart–Thomas basis functions R_j^K for velocity. We know that $\mathrm{div}(R_j^K) = 1/|K|$ and $R_j^K \cdot n = \delta_{jl}/|e_j|$ (e.g., [57]). We multiply (4.6) by a test function w; we have

$$\begin{aligned}\int_K k\nabla\phi_{ij}^K \nabla w dx &= -\int_K w\,\mathrm{div}(k\nabla\phi_{ij}^K)dx + \int_{\partial K}(k\nabla\phi_{ij}^K\cdot n)wds\\ &= -\int_K w\,\mathrm{div}R_j^K dx + \int_{\partial K}(k\nabla\phi_{ij}^K\cdot n)wds\\ &= \int_K(\nabla w)R_j^K dx + \int_{\partial K}(k\nabla\phi_{ij}^K\cdot n - R_j^K\cdot n)wds\\ &= \int_K(\nabla w)R_j^K dx + \int_{\partial K}\delta_{jl}\Big(\frac{v_i\cdot n}{\int_{e_l} v_i\cdot nds} - \langle\frac{v_i\cdot n}{\int_{e_l} v_i\cdot nds}\rangle_{e_l}\Big)wds,\end{aligned} \tag{6.90}$$

where we have used that $\langle\frac{v_i\cdot n}{\int_{e_j} v_i\cdot nds}\rangle_{e_j} = R_j^K\cdot n_{e_j} = \frac{1}{|e_j|}$.

If we set $w=\phi_{ij}^K$, then it follows that

$$\begin{aligned}C\|\nabla\phi_{ij}^K\|_{0,K}^2 &\le \|\nabla\phi_{ij}^K\|_{0,K}\|R_j^K\|_{0,K}\\ &\quad + \|\frac{v_i\cdot n}{\int_{e_j} v_i\cdot nds} - \langle\frac{v_i\cdot n}{\int_{e_j} v_i\cdot nds}\rangle_{e_j}\|_{L^r(e_j)}\|\phi_{ij}^K\|_{L^{r'}(\partial K)}\\ &\le C\|\nabla\phi_{ij}^K\|_{0,K} + Ch^{-\beta_2+1/r-1}\|\phi_{ij}^K\|_{L^{r'}(\partial K)}\\ &\le C\|\nabla\phi_{ij}^K\|_{0,K} + Ch^{-\beta_2+1/r-1}(h^{-1+1/r'}\|\phi_{ij}^K\|_{0,K} + h^{\frac{1}{r'}}\|\nabla\phi_{ij}^K\|_{0,K})\\ &\le C\|\nabla\phi_{ij}^K\|_{0,K} + Ch^{-\beta_2+1/r-1}h^{1/r'}\|\nabla\phi_{ij}^K\|_{0,K}\\ &\le C\|\nabla\phi_{ij}^K\|_{0,K} + Ch^{-\beta_2}\|\nabla\phi_{ij}^K\|_{0,K},\end{aligned}$$

where r' satisfies $1/r+1/r'=1$ (r is defined in Assumption A2), and we have used Assumption A2 (see (6.82)) and $\|R_j^K\|_{0,K}\le C$ (e.g., [57]) in the second step, the trace inequality (by rescaling) in the third step, and $\langle\phi_{ij}^K\rangle_K = 0$ along with the Poincaré–Friedrichs inequality (by rescaling) in the fourth step. Consequently, we have

$$\|\psi_{ij}^K\|_{0,K} \le C(1+h^{-\beta_2}), \tag{6.91}$$

where C only depends on $Assumption A2$ and the constants in trace inequality and Poincaré inequality in a fixed reference domain. Combining (6.89) and (6.91), it follows immediately

$$\begin{aligned}\|v-u_h\|_{0,K} &= \|\sum_{i,j}(A_i(x)\beta_{ij}^K - A_{ij}^K)\psi_{ij}^K\|_{0,K}\\ &\le \|\sum_{i,j}(A_i(x)-\bar{A}_i^j)\beta_{ij}^K\psi_{ij}^K\|_{0,K} + \|\sum_{i,j}(\bar{A}_i^j\beta_{ij}^K - A_{ij}^K)\psi_{ij}^K\|_{0,K}\\ &\le \|\sum_{i,j}|A_i(x)-\bar{A}_i^j|\beta_{ij}^K\psi_{ij}^K\|_{0,K} + \|\sum_{i,j}|\bar{A}_i^j\beta_{ij}^K - A_{ij}^K|\psi_{ij}^K\|_{0,K}\\ &\le Ch^{\alpha+\beta_1}(\sum_i\|A_i\|_{C^\alpha(\Omega)})\sum_{i,j}\|\psi_{ij}^K\|_{0,K}\\ &\le Ch^{\alpha+\beta_1-\beta_2}(\sum_i\|A_i\|_{C^\alpha(\Omega)}),\end{aligned} \tag{6.92}$$

where we have used Assumption A2 (see (6.82)) and C depends on N and the constants in Assumption A2. After summation over all K for (6.92) we have

$$\begin{aligned}\|v - u_h\|^2_{0,\Omega} &= \sum_K \|u - u_h\|^2_{0,K} \\ &\leq C(\sum_i \|A_i\|_{C^\alpha(\Omega)})^2 \sum_K h^{2(\alpha+\beta_1-\beta_2)} \\ &\leq C(\sum_i \|A_i\|_{C^\alpha(\Omega)})^2 \frac{1}{h^2} h^{2(\alpha+\beta_1-\beta_2)} \\ &= C(\sum_i \|A_i\|_{C^\alpha(\Omega)})^2 h^{2(\alpha+\beta_1-\beta_2-1)}.\end{aligned}$$

Consequently,

$$\|v - v_h\|_{0,\Omega} \leq C(\sum_i \|A_i\|_{C^\alpha(\Omega)}) h^{\alpha+\beta_1-\beta_2-1}. \qquad (6.93)$$

According to (6.85), for those K, $\partial K \cap \partial\Omega$, we adjust proper c^K_{ij} such that $\sum_{i,j} c^K_{ij}\psi^K_{i,j} - g_{0,h} \in \mathcal{V}^0_h$, but this does not affect our convergence rate. Therefore, invoking Lemma 6.39, (6.88), (6.93), and the Sobolev embedding theorem from $W^{1,\xi}$ into C^α, Theorem 6.40 follows.

From the proof of Theorem 6.40, one can easily get the following result. Let v and v_h be the velocity in (4.4) and (4.7), respectively; then we have

$$\|v - v_h\|_{0,\Omega} \leq C(\sum_i \|A_i\|_{C^\alpha(\Omega)}) h^{\alpha+\beta_1-\beta_2-1}.$$

Remark 6.41. If $A_i(x) \in C^1(\Omega)$ in Assumption A1 and v_i are defined such that $\beta_1 = 1$ and $\beta_2 = 0$ (e.g., v_i are bounded), then Theorem 6.40 implies that

$$\|v - v_h\|_{H(\text{div},\Omega)} + \|p - p_h\|_{0,\Omega} \leq Ch.$$

Remark 6.42. We note that the local mixed MsFEMs suffer from a resonance error and a typical convergence rate for periodic coefficients is

$$\|v_\epsilon - v_h\|_{H(\text{div},\Omega)} + \|p_\epsilon - p_h\|_{0,\Omega} \leq C(h + \left(\frac{\epsilon}{h}\right)^\gamma),$$

where $\gamma = 1/2$ for the mixed multiscale method introduced in [71]. In our global mixed MsFEM, the boundary condition for the velocity basis is heterogeneous and Theorem 6.40 implies that stability is independent of the small scale and the resonance error is removed.

Remark 6.43. One can relax the main assumption used here and assume that

$$\|v(x) - \sum_i A_i(x) v_i(x)\|_{H(\text{div},\Omega)} \leq C\delta.$$

In this case, we can expect the convergence as

$$\|v - v_h\|_{H(\text{div},\Omega)} + \|p - p_h\|_{0,\Omega} \leq C(h^{\alpha+\beta_1-\beta_2-1} + \delta).$$

Parabolic equations

Next, we extend the analysis to parabolic equations. We use the following assumption for the parabolic equation.

Assumption A1p. There exist functions $v_1, ..., v_N$ *and sufficiently smooth* $A_1(t,x), ..., A_N(t,x)$ *such that*

$$v(t,x) = \sum_{i=1}^{N} A_i(t,x) v_i,$$

where $v_i = k\nabla p_i$ *and* p_i *solves* $\text{div}(k(x)\nabla p_i) = 0$ *in* Ω *with appropriate boundary conditions.*

For our analysis, we assume, as before, $A_i(t,x) \in L^2(0,T; W^{1,\xi}(\Omega))(\xi > 2)$ and $v_i = k(x)\nabla p_i \in L^\eta(\Omega)$ $(1/2 = 1/\xi + 1/\eta)$, $i = 1, ..., N$.

Remark 6.44. Let $v_i = k(x)\nabla p_i$ $(i = 1, 2)$ be defined in (6.81), then Owhadi and Zhang in [217] show that $p(t,x) = p(t, p_1, p_2) \in L^2(0,T; W^{2,s})$ $(s > 2)$. Consequently, $v(t,x) = k(x)\nabla p = \sum_i (\partial p/\partial p_i) k\nabla p_i := \sum_i A_i(t,x) v_i$, where $A_i(t,x) = \partial p/\partial p_i \in L^2(0,T; W^{1,s})$.

We define

$$\|u\|^2_{L^2_k(\Omega)} = \int_\Omega u \cdot k^{-1}(x) u dx$$

and

$$\|u\|^2_{L^2(0,T;L^2_k(\Omega))} = \int_0^T \int_\Omega u \cdot k^{-1}(x) u dx ds.$$

Let $\Pi_h : H(\text{div}) \longrightarrow \mathcal{V}_h$ be the interpolation operator defined as in Section 6.3.1 and $P_{Q_h} : L^2(\Omega) \longrightarrow Q_h$ be the L^2 projection onto Q_h.

From (4.9) and (4.10), we have

$$\begin{aligned} \int_\Omega \frac{\partial}{\partial t}(p - p_h) q_h dx + \int_\Omega \text{div}(v - v_h) q_h dx = 0, \quad \forall q_h \in Q_h \\ \int_\Omega k^{-1}(v - v_h) \cdot w_h dx - \int_\Omega \text{div}(w_h)(p - p_h) dx = 0, \quad \forall w_h \in \mathcal{V}_h. \end{aligned} \tag{6.94}$$

Taking $w_h = \Pi_h v - v_h$ and $q_h = P_{Q_h} p - p_h$, we have

$$\begin{aligned} \int_\Omega \frac{\partial}{\partial t}(p - p_h)(P_{Q_h} p - p_h) dx + \int_\Omega \text{div}(v - v_h)(P_{Q_h} p - p_h) dx = 0 \\ \int_\Omega k^{-1}(v - v_h) \cdot (\Pi_h v - v_h) dx - \int_\Omega \text{div}(\Pi_h v - v_h)(p - p_h dx) = 0. \end{aligned} \tag{6.95}$$

Rewriting $p - p_h = p - P_{Q_h} p + P_{Q_h} p - p_h$ and $v - v_h = v - \Pi_h v + \Pi_h v - v_h$ in (6.95) and summation of the two equalities, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t}(P_{Q_h}p - p_h)(P_{Q_h}p - p_h)dx + \int_\Omega k^{-1}(\Pi_h v - v_h)\cdot(\Pi_h v - v_h)dx \\
&= -\int_\Omega \frac{\partial}{\partial t}(p - P_{Q_h}p)(P_{Q_h}p - p_h)dx - \int_\Omega k^{-1}(v - \Pi_h v)\cdot(\Pi_h v - v_h)dx \\
&+ \int_\Omega [\mathrm{div}(\Pi_h v - v_h)(p - P_{Q_h}p) - \mathrm{div}(v - \Pi_h v)(P_{Q_h}p - p_h)]dx.
\end{aligned}
\tag{6.96}
$$

Because P_{Q_h} is the $L^2(\Omega)$ projection onto Q_h, P_{Q_h} commutes with the time derivative operator $\partial/\partial t$. Consequently, the first and third terms of the right-hand side in (6.96) vanish. By Lemma 6.35, the fourth term of the right-hand side in (6.96) also vanishes. Consequently, (6.96) becomes

$$
\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t}(P_{Q_h}p - p_h)(P_{Q_h}p - p_h)dx + \int_\Omega k^{-1}(\Pi_h v - v_h)\cdot(\Pi_h v - v_h)dx \\
&= -\int_\Omega k^{-1}(v - \Pi_h v)\cdot(\Pi_h v - v_h)dx.
\end{aligned}
$$

The Schwarz inequality and Young's inequality give rise to

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|P_{Q_h}p - p_h\|_{0,\Omega}^2 + 2\|\Pi_h v - v_h\|_{L_k^2(\Omega)}^2 \\
&\leq \lambda\|\Pi_h v - v_h\|_{L_k^2(\Omega)}^2 + \frac{1}{4\lambda}\|v - \Pi_h v\|_{L_k^2(\Omega)}^2.
\end{aligned}
$$

Integrating with respect to time and applying Gronwall's inequality and after choosing the proper value for λ, we have

$$
\begin{aligned}
&\|P_{Q_h}p - p_h\|_{C^0(0,T;L^2(\Omega))}^2 + \|\Pi_h v - v_h\|_{L^2(0,T;L_k^2(\Omega))}^2 \\
&\quad \leq C(\|P_{Q_h}p(0) - p_{0,h}\|_{0,\Omega}^2 + \|v - \Pi_h v\|_{L^2(0,T;L_k^2(\Omega))}^2).
\end{aligned}
$$

Invoking the triangle inequality, we have

$$
\begin{aligned}
&\|p - p_h\|_{C^0(0,T;L^2(\Omega))}^2 + \|v - v_h\|_{L^2(0,T;L_k^2(\Omega))}^2 \\
&\quad \leq C(\|P_{Q_h}p(0) - p_{0,h}\|_{0,\Omega}^2 + \|v - \Pi_h v\|_{L^2(0,T;L_k^2(\Omega))}^2) \\
&\qquad\qquad + \|p - P_{Q_h}p\|_{C^0(0,T;L^2(\Omega))}^2.
\end{aligned}
\tag{6.97}
$$

Hence, we obtain the following lemma.

Lemma 6.45. *Let* $\{v, p\}$ *and* $\{v_h, p_h\}$ *be the solution of* (4.9) *and* (4.10), *respectively. Under Assumption A1p and the definition of* $\mathcal{V}_h$ *in Section* 6.3.1, *the estimate* (6.97) *holds.*

Utilizing Lemma 6.45 and the proof of Theorem 6.40, we can derive the convergence result.

Theorem 6.46. *Let* $\{v, p\}$ *and* $\{v_h, p_h\}$ *be the solution of* (4.9) *and* (4.10), *respectively. If* $\alpha + \beta_1 - \beta_2 - 1 > 0$ *then*

$$\|p - p_h\|_{C^0(0,T;L^2(\Omega))} + \|v - v_h\|_{L^2(0,T;L^2_k(\Omega))} \le Ch^{\alpha+\beta_1-\beta_2-1},$$

where $\alpha = 1-2/\xi$ *and* ξ *is from Assumption A1p, and* β_i $(i = 1, 2)$ *are defined in Assumption A2.*

Proof. Owing to the fact that P_{Q_h} is the $L^2(\Omega)$ projection onto Q_h,

$$\|p - P_{Q_h} p\|_{C^0(0,T;L^2(\Omega))} \le Ch|p|_{C^0(0,T;H^1(\Omega))}, \tag{6.98}$$

we estimate the first and the third term of right-hand side in (6.97). Next we estimate the term $\|v - \Pi_h v\|^2_{L^2(0,T;L^2_k(\Omega))}$. Define

$$A^K_{ij}(t) = \int_{e_j} A_i(t, s)(v_i \cdot n)ds$$

in each element K. Because $k^{-1}(x)$ is bounded, we have in each element K,

$$\begin{aligned}
&\|v - \Pi_h v\|^2_{L^2(0,T;L^2_k(K))} \\
&= \int_0^T \int_K \sum_{i,j} (A_i(t, x)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij} \cdot k^{-1} \sum_{i,j} (A_i(t, x)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij} dxdt \\
&\le C \int_0^T \int_K (\sum_{i,j} (A_i(t, x)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij})^2 dxdt \\
&= C\| \sum_{i,j} (A_i(t, x)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij} \|^2_{L^2(0,T;L^2(K))} \\
&\le C\| \sum_{i,j} (A_i(t, x) - \bar{A}^j_i(t))\beta^K_{ij}\psi^K_{ij} \|^2_{L^2(0,T;L^2(K))} \\
&+ C\| \sum_{i,j} (\bar{A}^j_i(t)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij} \|^2_{L^2(0,T;L^2(K))} \\
&\le Ch^{2(\alpha+\beta_1)} \sum \|\psi^K_{ij}\|^2_{0,K}.
\end{aligned} \tag{6.99}$$

In the last step, we used that facts that $A_i \in L^2(0, T; W^{1,\xi})$, Assumption A2 (see (6.82)) and proof of Theorem 6.40 (see (6.92)). After summation over all K for (6.99), we have

$$\|v - \Pi_h v\|_{L^2(0,T;L^2_k(\Omega))} \le Ch^{(\alpha+\beta_1-\beta_2-1)}. \tag{6.100}$$

Now, the proof can be completed taking into account (6.98) and (6.100).

6.3.2 Galerkin finite element methods with limited global information

We have proposed some analysis for modified MsFEMs in [103] and [3]. The main idea is to show that the pressure evolution in two-phase flow simulations is strongly influenced by the initial pressure. To demonstrate this, we consider a channelized permeability field, where the value of the permeability in the channel is large. We assume the permeability has the form kI, where I is an identity matrix. In a channelized medium, the dominant flow is within the channels. Our analysis assumes a single channel and is restricted to 2D. Here, we briefly mention the main findings. Denote the initial stream function and pressure by $\eta = \psi(x, t = 0)$ and $\zeta = p(x, t = 0)$ (ζ is also denoted by p^{sp} previously). The stream function is defined as

$$\partial\psi/\partial x_1 = -v_2, \quad \partial\psi/\partial x_2 = v_1. \tag{6.101}$$

Then the equation for the pressure can be written as

$$\frac{\partial}{\partial\eta}\left(|k|^2\lambda(S)\frac{\partial p}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\lambda(S)\frac{\partial p}{\partial\zeta}\right) = 0. \tag{6.102}$$

For simplicity, $S = 0$ at time zero is assumed. We consider a typical boundary condition that gives high flow within the channel, such that the high flow channel will be mapped into a large slab in (η, ζ) coordinate system. If the heterogeneities within the channel in the η direction are not strong (e.g., a narrow channel in Cartesian coordinates), the saturation within the channel will depend on ζ. In this case, the leading-order pressure will depend only on ζ, and it can be shown that

$$p(\eta, \zeta, t) = p_0(\zeta, t) + \text{high-order terms}, \tag{6.103}$$

where $p_0(\zeta, t)$ is the dominant pressure. Note that this result is shown when λ is smooth. This asymptotic expansion shows that the time-varying pressure strongly depends on the initial pressure (i.e., the leading-order term in the asymptotic expansion is a function of initial pressure and time only). We note that (6.103) does not hold when λ has discontinuities. In this case, our results hold away from the sharp interfaces and one can localize the interface by updating some basis functions. Our numerical results show that this update does not improve the results substantially. We believe this is because the discontinuities in λ are small compared to heterogeneities in porous media, the effects of which we capture using limited global information. In our analysis, we assume that $|p(x,t) - \hat{p}(p^{sp}, t)|_{H^1}$ is small.

Because the analysis of the multiscale finite element methods is carried out only for the pressure equation, we assume t (time) is fixed. We recall the assumption.

Assumption G. There exists a sufficiently smooth scalar-valued function $G(\eta)$ ($G \in W^{3,2s/(s-4)}$, $s > 4$), such that

$$|p - G(p^{sp})|_{1,\Omega} \leq C\delta, \tag{6.104}$$

where p^{sp} is single-phase flow pressure and δ is sufficiently small.

We note G is $p_0(\zeta, t)$ at fixed t in (6.103). Moreover, one does not need to know the function G for computing the multiscale approximation of the solution. It is only necessary that G have certain smoothness properties, however, it is important that the basis functions span p^{sp} in each coarse block.

Theorem 6.47. *Under Assumption G and $p^{sp} \in W^{1,s}(\Omega)$ $(s > 4)$, the MsFEM converges with the rate given by*

$$|p - p_h|_{1,\Omega} \leq C\delta + Ch^{1-2/s}. \tag{6.105}$$

The proof of this theorem is given in [3]. Note that Theorem 6.47 shows that MsFEM converges for problems without any scale separation and the proof of this theorem does not use homogenization techniques. Next, we present the proof.

Proof. Following standard practice of finite element estimation, we seek $p_I = c_i\phi_i$, where ϕ_i are single-phase flow-based multiscale finite element basis functions. In the proof, we assume that $|\phi_i^K|_{1,K} \leq C$. Then from Cea's lemma, we have

$$|p - p_h|_{1,\Omega} \leq |p - G(p^{sp})|_{1,\Omega} + |G(p^{sp}) - c_i\phi_i|_{1,\Omega}. \tag{6.106}$$

Next, we present an estimate for the second term. We choose $c_i = G(p^{sp}(x_i))$, where x_i are vertices of K. Furthermore, using a Taylor expansion of G around $\overline{p}_K$, which is the average of p^{sp} over K,

$$\begin{aligned} G(p^{sp}(x_i)) =& G(\overline{p}_K) + G'(\overline{p}_K)(p^{sp}(x_i) - \overline{p}_K) \\ &+ (p^{sp}(x_i) - \overline{p}_K)^2 \int_0^1 sG''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds. \end{aligned} \tag{6.107}$$

We have in each K

$$\begin{aligned} c_i\phi_i =& G(\overline{p}_K) \sum_i \phi_i + G'(\overline{p}_K)(p^{sp}(x_i) - \overline{p}_K)\phi_i \\ &+ (p^{sp}(x_i) - \overline{p}_K)^2 \phi_i \int_0^1 sG''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds \\ &= G(\overline{p}_K) + G'(\overline{p}_K)(p^{sp}(x_i)\phi_i - \overline{p}_K) \\ &+ (p^{sp}(x_i) - \overline{p}_K)^2 \phi_i \int_0^1 sG''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds. \end{aligned} \tag{6.108}$$

In the last step, we have used $\sum_i \phi_i = 1$. Similarly, in each K,

$$\begin{aligned} G(p^{sp}(x)) =& G(\overline{p}_K) + G'(\overline{p}_K)(p^{sp}(x) - \overline{p}_K) \\ &+ (p^{sp}(x) - \overline{p}_K)^2 \int_0^1 sG''(p^{sp}(x) + s(\overline{p}_K - p^{sp}(x)))ds. \end{aligned} \tag{6.109}$$

Using (6.108) and (6.109), we get

$$
\begin{aligned}
&|G(p^{sp}) - c_i\phi_i|_{1,K} \le |G'(\overline{p}_K)(p^{sp}(x) - p^{sp}(x_i)\phi_i)|_{1,K} \\
&+ |(p^{sp}(x_i) - \overline{p}_K)^2\phi_i \int_0^1 sG''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds|_{1,K} \\
&+ |(p^{sp}(x) - \overline{p}_K)^2 \int_0^1 sG''(p^{sp}(x) + s(\overline{p}_K - p^{sp}(x)))ds|_{1,K}.
\end{aligned}
\tag{6.110}
$$

Because $|p^{sp}(x) - p^{sp}(x_i)\phi_i|_{1,K} \le Ch\|f\|_{0,K}$, the estimate of the first term is the following,

$$
|G'(\overline{p}_K)(p^{sp}(x) - p^{sp}(x_i)\phi_i)|_{1,K} \le Ch\|f\|_{0,K}.
$$

For the second term on the right-hand side of (6.110), assuming $p^{sp}(x) \in W^{1,s}(\Omega)$ and $s > 4$, we have

$$
\begin{aligned}
&|(p^{sp}(x_i) - \overline{p}_K)^2\phi_i^K \int_0^1 sG''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds|_{1,K} \\
&\le Ch|p^{sp}|_{1,4,K}^2 |\phi_i^K|_{1,K} \\
&\le Ch|p^{sp}|_{1,4,K}^2,
\end{aligned}
$$

where we have used the assumption $|\phi_i^K|_{1,K} \le C$ and $W^{1,s} \subset W^{1,4}$ $(s \ge 4)$. Here, we have used the inequality (e.g., [18])

$$
|u(x) - u(y)| \le C|x-y|^{1-2/s}|u|_{1,s,K}.
$$

For the third term, a straightforward calculation gives

$$
\begin{aligned}
&|(p^{sp}(x) - \overline{p}_K)^2 \int_0^1 sG''(p^{sp}(x) + s(\overline{p}_K - p^{sp}(x)))ds|_{1,K} \\
\le& \|(p^{sp}(x) - \overline{p}_K)^2 \nabla p^{sp}(x) \int_0^1 (1-s)sG'''(p^{sp}(x) + s(\overline{p}_K - p^{sp}(x)))ds\|_{0,K} \\
&+ \|2(p^{sp}(x) - \overline{p}_K)\nabla p^{sp}(x) \int_0^1 sG''(p^{sp}(x) + s(\overline{p}_K - p^{sp}(x)))ds\|_{0,K} \\
&\le Ch^{2-2/s}\|\nabla p^{sp}\|_{L^s(K)}^3 \|G'''\|_{L^{2s/(s-4)}(K)} + Ch^{1-2/s}|p^{sp}|_{1,s,K}|p^{sp}|_{1,K} \\
&\le Ch^{2-2/s}\|\nabla p^{sp}\|_{L^s(K)}^3 + Ch^{1-2/s}|p^{sp}|_{1,K}
\end{aligned}
$$

where we used the Hölder inequality in the second step.

Combining the above estimates, we have for $s > 4$

$$
\begin{aligned}
|G(p^{sp}) - c_i\phi_i^K|_{1,K} \le& Ch|p^{sp}|_{1,4,K}^2 \\
&+ Ch^{2-2/s} + Ch^{1-2/s}|p^{sp}|_{1,K} + Ch\|f\|_{0,K}.
\end{aligned}
\tag{6.111}
$$

Summing (6.111) over all K and taking into account Assumption G, we have

$$
\begin{aligned}
|p - p_h|_{1,\Omega} &\leq C(\delta + h^{1-2/s}) + Ch|p^{sp}|^2_{1,4,\Omega} + Ch^{1-2/s}|p^{sp}|_{1,\Omega} + Ch\|f\|_{0,\Omega} \\
&\leq C(\delta + h^{1-2/s}) + Ch|p^{sp}|^2_{1,s,\Omega} + Ch^{1-2/s}|p^{sp}|_{1,s,\Omega} + Ch\|f\|_{0,\Omega}.
\end{aligned}
$$

Consequently, if $s > 4$ (see e.g., [28]), the single-phase flow-based MsFEM converges.

Remark 6.48. We can relax the assumption on G. In particular, it is sufficient to assume $G \in W^{2,m}$ $(m \geq 1)$. In this case, the proof can be carried out using Taylor polynomials in Sobolev spaces. Also, if $\nabla p^{sp} \in L^{\infty}(\Omega)$, then the convergence rate in (6.105) is $C\delta + Ch$.

Remark 6.49. One can similarly analyze Galerkin MsFEMs using multiple global fields (see [3]). This analysis can be extended to parabolic equations (see [163]).