Chapter 11
Time–Frequency Analysis and the Carleson–Hunt Theorem

In this chapter we discuss in detail the proof of the almost everywhere convergence of the partial Fourier integrals of L^p functions on the line. The proof of this theorem is based on techniques involving both spatial and frequency decompositions. These techniques are referred to as time–frequency analysis. The underlying goal is to decompose a given function at any scale as a sum of pieces perfectly localized in frequency and well localized in space. The action of an operator on each piece is carefully studied and the interaction between different parts of this action are analyzed. Ideas from combinatorics are employed to organize the different pieces of the decomposition.

11.1 Almost Everywhere Convergence of Fourier Integrals

In this section we study the proof of one of the most celebrated theorems in Fourier analysis, Carleson's theorem on the almost everywhere convergence of Fourier series of square integrable functions on the circle. The same result is also valid for functions f on the line if the partial sums of the Fourier series are replaced by the (partial) Fourier integrals

$$\int_{|\xi|\le N} \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi\,.$$

The equivalence of these assertions follows from the transference methods discussed in Chapter 3.

For square-integrable functions f on the line, define the *Carleson operator*

$$\mathscr{C}(f)(x) = \sup_{N>0} \big| \big(\widehat{f}\chi_{[-N,N]}\big)^{\vee} \big| = \sup_{N>0} \left| \int_{|\xi|\le N} \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi \right|. \tag{11.1.1}$$

We note that the operators $(\widehat{f}\chi_{[a,b]})^{\vee}$ are well defined when $-\infty \le a < b \le \infty$ for f in $L^2(\mathbf{R})$, and thus so is $\mathscr{C}(f)$. We have the following result concerning $\mathscr{C}$.

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Theorem 11.1.1. *There is a constant $C>0$ such that for all square-integrable functions f on the line the following estimate is valid:*

$$\big\|\mathscr{C}(f)\big\|_{L^{2,\infty}} \le C\big\|f\big\|_{L^2}\,.$$

It follows that for all f in $L^2(\mathbf{R})$ we have

$$\lim_{N\to\infty}\int_{|\xi|\le N}\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi = f(x) \tag{11.1.2}$$

for almost all $x\in\mathbf{R}$.

Proof. Because of the simple identity

$$\int_{|\xi|\le N}\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi = \int_{-\infty}^{N}\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi - \int_{-\infty}^{-N}\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi\,,$$

it suffices to obtain $L^2\to L^{2,\infty}$ bounds for the *one-sided maximal operators*

$$\mathscr{C}_1(f)(x) = \sup_{N>0}\left|\int_{-\infty}^{N}\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi\right|,$$
$$\mathscr{C}_2(f)(x) = \sup_{N>0}\left|\int_{-\infty}^{-N}\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi\right|.$$

Once these bounds are obtained, we can use the simple fact that (11.1.2) holds for Schwartz functions and Theorem 2.1.14 to obtain (11.1.2) for all square-integrable functions f on the line. Note that $\widetilde{\mathscr{C}_2(f)} = \mathscr{C}_1(\widetilde{f})$, where $\widetilde{f}(x) = f(-x)$ is the usual reflection operator. Therefore, it suffices to obtain bounds only for $\mathscr{C}_1$. Just as is the case with $\mathscr{C}$, the operators $\mathscr{C}_1$ and $\mathscr{C}_2$ are well defined on $L^2(\mathbf{R})$.

For $a>0$ and $y\in\mathbf{R}$ we define the translation operator τ^y, the modulation operator M^a, and the dilation operator D^a as follows:

$$\tau^y(f)(x) = f(x-y)\,,$$
$$D^a(f)(x) = a^{-\frac{1}{2}}f(a^{-1}x)\,,$$
$$M^y(f)(x) = f(x)e^{2\pi i y x}\,.$$

These operators are isometries on $L^2(\mathbf{R})$.

We break down the proof of Theorem 11.1.1 into several steps.

11.1.1 Preliminaries

We denote rectangles of area 1 in the (x,ξ) plane by s, t, u, etc. All rectangles considered in the sequel have sides parallel to the axes. We think of x as the time coordinate and of ξ as the frequency coordinate. For this reason we refer to the (x,ξ)

coordinate plane as the time–frequency plane. The projection of a rectangle s on the time axis is denoted by I_s, while its projection on the frequency axis is denoted by ω_s. Thus a rectangle s is just $s = I_s \times \omega_s$. Rectangles with sides parallel to the axes and area equal to one are called *tiles*.

The center of an interval I is denoted by $c(I)$. Also for $a > 0$, aI denotes an interval with the same center as I whose length is $a|I|$. Given a tile s, we denote by $s(1)$ its bottom half and by $s(2)$ its upper half defined by

$$s(1) = I_s \times \big(\omega_s \cap (-\infty, c(\omega_s))\big), \qquad s(2) = I_s \times \big(\omega_s \cap [c(\omega_s), +\infty)\big).$$

These sets are called *semitiles*. The projections of these sets on the frequency axes are denoted by $\omega_{s(1)}$ and $\omega_{s(2)}$, respectively.

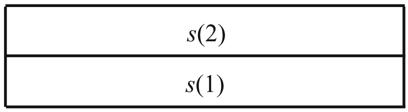

Fig. 11.1 The lower and the upper parts of a tile s.

A dyadic interval is an interval of the form $[m2^k, (m+1)2^k)$, where k and m are integers. We denote by $\mathbf{D}$ the set of all rectangles $I \times \omega$ with I, ω dyadic intervals and $|I|\,|\omega| = 1$. Such rectangles are called *dyadic tiles*. We denote by $\mathbf{D}$ the set of all dyadic tiles.

We fix a Schwartz function φ such that $\widehat{\varphi}$ is real, nonnegative, and supported in the interval $[-1/10, 1/10]$. For each tile s, we introduce a function φ_s as follows:

$$\varphi_s(x) = |I_s|^{-\frac{1}{2}}\, \varphi\left(\frac{x - c(I_s)}{|I_s|}\right) e^{2\pi i c(\omega_{s(1)})x}. \tag{11.1.3}$$

This function is localized in frequency near $c(\omega_{s(1)})$. Using the previous notation, we have

$$\varphi_s = M^{c(\omega_{s(1)})} \tau^{c(I_s)} D^{|I_s|}(\varphi).$$

Observe that

$$\widehat{\varphi_s}(\xi) = |\omega_s|^{-\frac{1}{2}}\, \widehat{\varphi}\left(\frac{\xi - c(\omega_{s(1)})}{|\omega_s|}\right) e^{2\pi i (c(\omega_{s(1)}) - \xi) c(I_s)}, \tag{11.1.4}$$

from which it follows that $\widehat{\varphi_s}$ is supported in $\frac{1}{5}\omega_{s(1)}$. Also observe that the functions φ_s have the same $L^2(\mathbf{R})$ norm.

Recall the complex inner product notation for $f, g \in L^2(\mathbf{R})$:

$$\langle f \,|\, g \rangle = \int_{\mathbf{R}} f(x)\overline{g(x)}\, dx. \tag{11.1.5}$$

Given a nonzero real number ξ, we introduce an operator

$$A_\xi(f) = \sum_{s\in\mathbf{D}} \chi_{\omega_{s(2)}}(\xi)\langle f\,|\,\varphi_s\rangle\,\varphi_s \tag{11.1.6}$$

initially defined for f in the Schwartz class. We show in the next subsection that the series in (11.1.6) converges absolutely for f in the Schwartz class and thus A_ξ is well defined on this class. Moreover, we show in Lemma 11.1.2 that A_ξ admits an extension that is L^2 bounded, and therefore it can thought of as well defined on $L^2(\mathbf{R})$.

For every integer m, let us denote by $\mathbf{D}_m$ the set of all tiles $s\in\mathbf{D}$ such that $|I_s| = 2^m$. We call these dyadic tiles *of scale m*. Then

$$A_\xi(f) = \sum_{m\in\mathbf{Z}} A_\xi^m(f)\,,$$

where

$$A_\xi^m(f) = \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(\xi)\,\langle f\,|\,\varphi_s\rangle\,\varphi_s\,, \tag{11.1.7}$$

and observe that for each scale m, the second sum above ranges over all dyadic rectangles of a fixed scale whose tops contain the line perpendicular to the frequency axis at height ξ. The operators A_ξ^m are discretized versions of the multiplier operator $f\mapsto \big(\widehat{f}\chi_{(-\infty,\xi]}\big)^\vee$. Indeed, the Fourier transform of $A_\xi^m(f)$ is supported in the frequency projection of the lower part $s(1)$ of the dyadic tiles s that appear in (11.1.7). But the sum in (11.1.7) is taken over all dyadic tiles s whose frequency projection of the upper part $s(2)$ contains ξ. So the Fourier transform of $A_\xi^m(f)$ is supported in $(-\infty,\xi]$. On the other hand, summing over all s in (11.1.7) yields essentially the identity operator; cf. Exercise 11.1.9. Therefore, A_ξ^m can be viewed as the "part" of the identity operator whose frequency multiplier consists of the function $\chi_{(-\infty,\xi]}$ instead of the function 1. As m becomes larger, we obtain a better and better approximation to this multiplier. This heuristic explanation motivates the introduction and study of the operators A_ξ^m and A_ξ.

Lemma 11.1.2. *For any fixed ξ, the operators A_ξ^m are bounded on $L^2(\mathbf{R})$ uniformly in m and ξ; moreover, the operator A_ξ is L^2 bounded uniformly in ξ.*

Proof. We make a few observations about the operators A_ξ^m. First recall that the adjoint of an operator T is uniquely defined by the identity

$$\langle T(f)\,|\,g\rangle = \langle f\,|\,T^*(g)\rangle$$

for all f and g. Observe that A_ξ^m are self-adjoint operators, meaning that $(A_\xi^m)^* = A_\xi^m$. Moreover, we claim that if $m\neq m'$, then

$$A_\xi^{m'}(A_\xi^m)^* = (A_\xi^{m'})^*A_\xi^m = 0\,.$$

Indeed, given f and g we have

$$\begin{aligned}\big\langle (A_\xi^{m'})^* A_\xi^m(f)\,|\,g\big\rangle &= \big\langle A_\xi^m(f)\,|\,A_\xi^{m'}(g)\big\rangle \qquad (11.1.8)\\ &= \sum_{s\in\mathbf{D}_m}\sum_{s'\in\mathbf{D}_{m'}} \langle f\,|\,\varphi_s\rangle\overline{\langle g\,|\,\varphi_{s'}\rangle}\langle\varphi_s\,|\,\varphi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)\,.\end{aligned}$$

Suppose that $\langle\varphi_s\,|\,\varphi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)$ is nonzero. Then $\langle\varphi_s\,|\,\varphi_{s'}\rangle$ is also nonzero, which implies that $\omega_{s(1)}$ and $\omega_{s'(1)}$ intersect. Also, the function $\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)$ is nonzero; hence $\omega_{s(2)}$ and $\omega_{s'(2)}$ must intersect. Thus the dyadic intervals ω_s and $\omega_{s'}$ are not disjoint, and one must contain the other. If ω_s were properly contained in $\omega_{s'}$, then it would follow that ω_s is contained in $\omega_{s'(1)}$ or in $\omega_{s'(2)}$. But then either $\omega_{s(1)}\cap\omega_{s'(1)}$ or $\omega_{s(2)}\cap\omega_{s'(2)}$ would have to be empty, which does not happen, as observed. It follows that if $\langle\varphi_s\,|\,\varphi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)$ is nonzero, then $\omega_s=\omega_{s'}$, which is impossible if $m\neq m'$. Thus the expression in (11.1.8) has to be zero.

We first discuss the boundedness of each operator A_ξ^m. We have

$$\begin{aligned}\big\|A_\xi^m(f)\big\|_{L^2}^2 &= \sum_{s\in\mathbf{D}_m}\sum_{s'\in\mathbf{D}_m} \langle f\,|\,\varphi_s\rangle\overline{\langle f\,|\,\varphi_{s'}\rangle}\langle\varphi_s\,|\,\varphi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)\\ &= \sum_{s\in\mathbf{D}_m}\sum_{\substack{s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}} \langle f\,|\,\varphi_s\rangle\overline{\langle f\,|\,\varphi_{s'}\rangle}\langle\varphi_s\,|\,\varphi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)\\ &\le \sum_{s\in\mathbf{D}_m}\sum_{\substack{s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}} \big|\langle f\,|\,\varphi_s\rangle\big|^2\chi_{\omega_{s(2)}}(\xi)\big|\langle\varphi_s\,|\,\varphi_{s'}\rangle\big|\\ &\le C_1\sum_{s\in\mathbf{D}_m}\big|\langle f\,|\,\varphi_s\rangle\big|^2\chi_{\omega_{s(2)}}(\xi)\,, \qquad (11.1.9)\end{aligned}$$

where we used an earlier observation about s and s', the Cauchy–Schwarz inequality, and the fact that

$$\sum_{\substack{s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}}\big|\langle\varphi_s\,|\,\varphi_{s'}\rangle\big|\le C\sum_{\substack{s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}}\left(1+\frac{\operatorname{dist}(I_s,I_{s'})}{2^m}\right)^{-10}\le C_1\,,$$

which follows from the result in Appendix K.1. To estimate (11.1.9), we use that

$$\begin{aligned}\big|\langle f\,|\,\varphi_s\rangle\big| &\le C_2\int_{\mathbf{R}}|f(y)|\,|I_s|^{-\frac{1}{2}}\left(1+\frac{|y-c(I_s)|}{|I_s|}\right)^{-10}dy\\ &= C_3\,|I_s|^{\frac{1}{2}}\int_{\mathbf{R}}|f(y)|\left(1+\frac{|y-z|}{|I_s|}\right)^{-10}\frac{dy}{|I_s|}\\ &\le C_4\,|I_s|^{\frac{1}{2}}M(f)(z),\end{aligned}$$

for all $z\in I_s$, in view of Theorem 2.1.10. Since the preceding estimate holds for all $z\in I_s$, it follows that

$$\big|\langle f\,|\,\varphi_s\rangle\big|^2 \le (C_3)^2|I_s|\inf_{z\in I_s} M(f)(z)^2 \le (C_3)^2\int_{I_s} M(f)(x)^2\,dx. \tag{11.1.10}$$

Next we observe that the rectangles $s \in \mathbf{D}_m$ with the property that $\xi \in \omega_{s(2)}$ are all disjoint. This implies that the corresponding time intervals I_s are also disjoint. Thus, summing (11.1.10) over all $s \in \mathbf{D}_m$ with $\xi \in \omega_{s(2)}$, we obtain that

$$\begin{aligned}\sum_{s\in\mathbf{D}_m}\big|\langle f\,|\,\varphi_s\rangle\big|^2\chi_{\omega_{s(2)}}(\xi) &\le (C_3)^2\sum_{s\in\mathbf{D}_m}\chi_{\omega_{s(2)}}(\xi)\int_{I_s} M(f)(x)^2\,dx\\ &\le (C_3)^2\int_{\mathbf{R}} M(f)(x)^2\,dx,\end{aligned}$$

which establishes the required claim using the boundedness of the Hardy–Littlewood maximal operator M on $L^2(\mathbf{R})$.

Finally, we discuss the boundedness of $A_\xi = \sum_{m\in\mathbf{Z}} A_\xi^m$. For every fixed $m \in \mathbf{Z}$, the dyadic tiles that appear in the sum defining $A_\xi^m(f)$ have the form

$$s = [k2^m,(k+1)2^m)\times[\ell 2^{-m},(\ell+1)2^{-m}),$$

where $(\ell+\frac{1}{2})2^{-m} \le \xi < (\ell+1)2^{-m}$. Thus $\ell = [2^m\xi]$, and since $\widehat{\varphi_s}$ is supported in the lower part of the dyadic tile s, we may replace f by f_m, where

$$\widehat{f_m} = \widehat{f}\chi_{[2^{-m}[2^m\xi],2^{-m}([2^m\xi]+\frac{1}{2}))}$$

As already observed, we have $\big\langle A_\xi^m(f)\,|\,A_\xi^{m'}(f)\big\rangle = 0$ whenever $m \neq m'$. Consequently,

$$\begin{aligned}\Big\|\sum_{m\in\mathbf{Z}} A_\xi^m(f)\Big\|_{L^2}^2 &= \sum_{m\in\mathbf{Z}}\big\|A_\xi^m(f)\big\|_{L^2}^2\\ &= \sum_{m\in\mathbf{Z}}\big\|A_\xi^m(f_m)\big\|_{L^2}^2\\ &\le C_4\sum_{m\in\mathbf{Z}}\big\|f_m\big\|_{L^2}^2\\ &= C_4\sum_{m\in\mathbf{Z}}\big\|\widehat{f_m}\big\|_{L^2}^2\\ &\le C_4\big\|f\big\|_{L^2}^2,\end{aligned}$$

since the supports of $\widehat{f_m}$ are disjoint for different values of $m \in \mathbf{Z}$. □

11.1.2 Discretization of the Carleson Operator

We let $h \in \mathscr{S}(\mathbf{R})$, $\xi \in \mathbf{R}\setminus\{0\}$, and for each $m \in \mathbf{Z}$, $y,\eta \in \mathbf{R}$, and $\lambda \in [0,1]$ we introduce the operators

$$B^m_{\xi,y,\eta,\lambda}(h) = \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\left\langle D^{2^\lambda}\tau^y M^\eta(h)\,|\,\varphi_s\right\rangle M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\varphi_s)\,.$$

It is not hard to see that for all $x\in\mathbf{R}$ and $\lambda\in[0,1]$ we have

$$B^m_{\xi,y,\eta,\lambda}(h)(x) = B^m_{\xi,y+2^{m-\lambda},\eta,\lambda}(h)(x) = B^m_{\xi,y,\eta+2^{-m+\lambda},\lambda}(h)(x)\,;$$

in other words, the function $(y,\eta)\mapsto B^m_{\xi,y,\eta,\lambda}(h)(x)$ is periodic in $\mathbf{R}^2$ with period $(2^{m-\lambda},2^{-m+\lambda})$. See Exercise 11.1.1.

Using Exercise 11.1.2, we obtain that for $|m|$ large (with respect to ξ) we have

$$\begin{aligned}
&\Big|\sum_{s\in\mathbf{D}_m}\chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\left\langle D^{2^\lambda}\tau^y M^\eta(h)\,|\,\varphi_s\right\rangle M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\varphi_s)(x)\Big|\\
&\quad\le C_h\min(2^m,1)\sum_{s\in\mathbf{D}_m}\chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))2^{-m/2}\Big|\varphi\Big(\frac{x+y-c(I_s)}{2^{m-\lambda}}\Big)\Big|\\
&\quad\le C_h\min(2^{m/2},2^{-m/2})\sum_{k\in\mathbf{Z}}\Big|\varphi\Big(\frac{x+y-k2^m}{2^{m-\lambda}}\Big)\Big|\\
&\quad\le C_h\min(2^{m/2},2^{-m/2})\,,
\end{aligned}$$

since the last sum is seen easily to converge to some quantity that remains bounded in x, y, η, and λ. It follows that for $h\in\mathscr{S}(\mathbf{R})$ we have

$$\sup_{x\in\mathbf{R}}\sup_{y\in\mathbf{R}}\sup_{\eta\in\mathbf{R}}\sup_{0\le\lambda\le1}\left|B^m_{\xi,y,\eta,\lambda}(h)(x)\right|\le C_h\min(2^{m/2},2^{-m/2})\,. \tag{11.1.11}$$

Using Exercise 11.1.3 and the periodicity of the functions $B^m_{\xi,y,\eta,\lambda}(h)$, we conclude that the averages

$$\frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_0^1 B^m_{\xi,y,\eta,\lambda}(h)\,d\lambda\,dy\,d\eta$$

converge pointwise to some $\Pi^m_\xi(h)$ as $K,L\to\infty$. Estimate (11.1.11) implies the uniform convergence for the series $\sum_{m\in\mathbf{Z}} B^m_{\xi,y,\eta,\lambda}(h)$ and therefore

$$\begin{aligned}
&\lim_{\substack{K\to\infty\\L\to\infty}}\frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_0^1 M^{-\eta}\tau^{-y}D^{2^{-\lambda}}A_{\frac{\xi+\eta}{2^\lambda}}D^{2^\lambda}\tau^y M^\eta(h)\,d\lambda\,dy\,d\eta \qquad (11.1.12)\\
&\quad=\lim_{\substack{K\to\infty\\L\to\infty}}\frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_0^1\sum_{m\in\mathbf{Z}}B^m_{\xi,y,\eta,\lambda}(h)\,d\lambda\,dy\,d\eta\\
&\quad=\sum_{m\in\mathbf{Z}}\lim_{\substack{K\to\infty\\L\to\infty}}\frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_0^1 B^m_{\xi,y,\eta,\lambda}(h)\,d\lambda\,dy\,d\eta\\
&\quad=\sum_{m\in\mathbf{Z}}\Pi^m_\xi(h)\,.
\end{aligned}$$

We now make a few observations about the operator Π_ξ defined on $\mathscr{S}(\mathbf{R})$ in terms of the expression in (11.1.12), that is:

$$\Pi_\xi(h) = \sum_{m\in\mathbf{Z}} \Pi_\xi^m(h).$$

First we observe that in view of Lemma 11.1.2 and Fatou's lemma, we have that Π_ξ is bounded on L^2 uniformly in ξ. Next we observe that Π_ξ commutes with all translations τ^z for $z\in\mathbf{R}$. To see this, we use the fact that $\tau^{-z}M^{-\eta} = e^{-2\pi i\eta z}M^{-\eta}\tau^{-z}$ to obtain

$$\begin{aligned}
&\sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}\big(2^{-\lambda}(\xi+\eta)\big)\big\langle D^{2^\lambda}\tau^y M^\eta \tau^z(h)\,|\,\varphi_s\big\rangle\, \tau^{-z}M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\varphi_s)\\
&\qquad = \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}\big(2^{-\lambda}(\xi+\eta)\big)\big\langle h\,|\,\tau^{-z}M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\varphi_s)\big\rangle\, \tau^{-z}M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\varphi_s)\\
&\qquad = \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}\big(2^{-\lambda}(\xi+\eta)\big)\big\langle h\,|\,M^{-\eta}\tau^{-y-z}D^{2^{-\lambda}}(\varphi_s)\big\rangle\, M^{-\eta}\tau^{-y-z}D^{2^{-\lambda}}(\varphi_s).
\end{aligned}$$

Recall that $\tau^{-z}\Pi_\xi^m\tau^z(h)$ is equal to the limit of the averages of the preceding expressions over all $(y,\eta,\lambda)\in[-K,K]\times[-L,L]\times[0,1]$. But in view of the previous identity, this is equal to the limit of the averages of the expressions

$$\sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}\big(2^{-\lambda}(\xi+\eta)\big)\big\langle D^{2^\lambda}\tau^{y'}M^\eta(h)\,|\,\varphi_s\big\rangle M^{-\eta}\tau^{-y'}D^{2^{-\lambda}}(\varphi_s) \tag{11.1.13}$$

over all $(y',\eta,\lambda)\in[-K+z,K+z]\times[-L,L]\times[0,1]$. Since (11.1.13) is periodic in (y',η), it follows that its average over the set $[-K+z,K+z]\times[-L,L]\times[0,1]$ is equal to its average over the set $[-K,K]\times[-L,L]\times[0,1]$. Taking limits as $K,L\to\infty$, we obtain the identity $\tau^{-z}\Pi_\xi^m\tau^z(h) = \Pi_\xi^m(h)$. Summing over all $m\in\mathbf{Z}$, it follows that $\tau^{-z}\Pi_\xi\tau^z(h) = \Pi_\xi(h)$.

A similar argument using averages over shifted rectangles of the form $[-K,K]\times[-L+\theta,L+\theta]$ yields the identity

$$M^{-\theta}\Pi_{\xi+\theta}M^\theta = \Pi_\xi \tag{11.1.14}$$

for all $\xi,\theta\in\mathbf{R}$. The details are left to the reader. Next, we claim that the operator $M^{-\xi}\Pi_\xi M^\xi$ commutes with dilations D^{2^a}, $a\in\mathbf{R}$. First we observe that for all integers k we have

$$A_\xi(h) = D^{2^{-k}}A_{2^{-k}\xi}D^{2^k}(h), \tag{11.1.15}$$

which is simply saying that A_ξ is well behaved under change of scale. This identity is left as an exercise to the reader. Identity (11.1.15) may not hold for noninteger k, and this is exactly why we have averaged over all dilations 2^λ, $0\le\lambda\le 1$, in (11.1.12).

Let us denote by $[a]$ the integer part of a real number a. Using the identities $D^bM^\eta = M^{\eta/b}D^b$ and $D^b\tau^z = \tau^{bz}D^b$, we obtain

$$
\begin{aligned}
&D^{2^{-a}} M^{-(\xi+\eta)} \tau^{-y} D^{2^{-\lambda}} A_{\frac{\xi+\eta}{2^\lambda}} D^{2^\lambda} \tau^y M^{\xi+\eta} D^{2^a} &(11.1.16)\\
&\quad = M^{-2^a(\xi+\eta)} \tau^{-2^{-a}y} D^{2^{-(a+\lambda)}} A_{\frac{\xi+\eta}{2^\lambda}} D^{2^{a+\lambda}} \tau^{2^{-a}y} M^{2^a(\xi+\eta)}\\
&\quad = M^{-2^a(\xi+\eta)} \tau^{-y'} D^{2^{-\lambda'}} D^{2^{-[a+\lambda]}} A_{\frac{2^a(\xi+\eta)}{2^{\lambda'}2^{[a+\lambda]}}} D^{2^{[a+\lambda]}} D^{2^{\lambda'}} \tau^{y'} M^{2^a(\xi+\eta)}\\
&\quad = M^{-2^a\xi} M^{-\eta'} \tau^{-y'} D^{2^{-\lambda'}} A_{\frac{2^a\xi+2^a\eta}{2^{\lambda'}}} D^{2^{\lambda'}} \tau^{y'} M^{\eta'} M^{2^a\xi}\\
&\quad = M^{-\xi} M^{-\theta} \big(M^{-\eta'} \tau^{-y'} D^{2^{-\lambda'}} A_{\frac{\xi+\theta+\eta'}{2^{\lambda'}}} D^{2^{\lambda'}} \tau^{y'} M^{\eta'}\big) M^\theta M^\xi, &(11.1.17)
\end{aligned}
$$

where we set $y' = 2^{-a}y$, $\eta' = 2^a\eta$, $\lambda' = a+\lambda-[a+\lambda]$, and $\theta = (2^a-1)\xi$. The average of (11.1.16) over all (y,η,λ) in $[-K,K]\times[-L,L]\times[0,1]$ converges to the operator $D^{2^{-a}} M^{-\xi} \Pi_\xi M^\xi D^{2^a}$ as $K,L\to\infty$. But this limit is equal to the limit of the averages of the expression in (11.1.17) over all (y',η',λ') in $[-2^{-a}K, 2^{-a}K]\times[-2^aL, 2^aL]\times[0,1]$, which is

$$M^{-\xi} M^{-\theta} \Pi_{\xi+\theta} M^\theta M^\xi .$$

Using the identity (11.1.14), we obtain that

$$D^{2^{-a}} M^{-\xi} \Pi_\xi M^\xi D^{2^a} = M^{-\xi} \Pi_\xi M^\xi ,$$

saying that the operator $M^{-\xi}\Pi_\xi M^\xi$ commutes with dilations.

Next we observe that if $\widehat{h}$ is supported in $[0,\infty)$, then $M^{-\xi}\Pi_\xi M^\xi(h) = 0$. This is a consequence of the fact that the inner products

$$\big\langle D^{2^\lambda} \tau^y M^\eta M^\xi(h) \,|\, \varphi_s \big\rangle = \big\langle M^\xi(h) \,|\, M^{-\eta} \tau^{-y} D^{2^{-\lambda}}(\varphi_s) \big\rangle$$

vanish, since the Fourier transform of $\tau^{-z} M^{-\eta} \tau^{-y} D^{2^{-\lambda}} \varphi_s$ is supported in the set $(-\infty, 2^\lambda c(\omega_{s(1)}) - \eta + \frac{2^\lambda}{10}|\omega_s|)$, which is disjoint from the interval $(\xi,+\infty)$ whenever $2^{-\lambda}(\xi+\eta)\in\omega_{s(2)}$. Finally, we observe that Π_ξ is a positive semidefinite operator, that is, it satisfies

$$\big\langle \Pi_\xi(h) \,|\, h \big\rangle \ge 0. \tag{11.1.18}$$

This follows easily from the fact that the inner product in (11.1.18) is equal to

$$\lim_{\substack{K\to\infty\\ L\to\infty}} \frac{1}{4KL} \int_{-L}^{L}\int_{-K}^{K}\int_0^1 \sum_{s\in\mathbf{D}} \chi_{\omega_{s(2)}}\big(\tfrac{\xi+\eta}{2^\lambda}\big) \big|\big\langle D^{2^\lambda} \tau^y M^\eta(h) \,|\, \varphi_s \big\rangle\big|^2 \, d\lambda\, dy\, d\eta. \tag{11.1.19}$$

This identity also implies that Π_ξ is not the zero operator; indeed, notice that

$$\sum_{s\in\mathbf{D}_0} \chi_{\omega_{s(2)}}\big(\tfrac{\xi+\eta}{2^\lambda}\big) \big|\big\langle D^{2^\lambda} \tau^y M^\eta(h) \,|\, \varphi_s \big\rangle\big|^2 = \big\langle h \,|\, B^0_{\xi,y,\eta,\lambda}(h) \big\rangle$$

is periodic with period $(2^{-\lambda}, 2^{\lambda})$ in (y, η), and consequently the limit in (11.1.19) is at least as big as

$$\int_0^{2^\lambda} \int_0^{2^{-\lambda}} \int_0^1 \sum_{s \in \mathbf{D}_0} \chi_{\omega_{s(2)}}\big(\tfrac{\xi+\eta}{2^\lambda}\big) \big|\big\langle D^{2^\lambda} \tau^y M^\eta(h) \,|\, \varphi_s \big\rangle\big|^2 \, d\lambda \, dy \, d\eta$$

(cf. Exercise 11.1.3). Since we can always find a Schwartz function h and a dyadic tile s such that $\big\langle D^{2^\lambda} \tau^y M^\eta(h) \,|\, \varphi_s \big\rangle$ is not zero for (y, η, λ) near $(0,0,0)$, it follows that the expression in (11.1.19) is strictly positive for some function h. The same is valid for the inner product in (11.1.18); hence the operators and $M^{-\xi}\Pi_\xi M^\xi$ are nonzero for every ξ.

Let us summarize what we have already proved: The operator $M^{-\xi}\Pi_\xi M^\xi$ is nonzero, is bounded on $L^2(\mathbf{R})$, commutes with translations and dilations, and vanishes when applied to functions whose Fourier transform is supported in the positive semiaxis $[0, \infty)$. In view of Exercise 4.1.11, it follows that for some constant $c_\xi \neq 0$ we have

$$M^{-\xi}\Pi_\xi M^\xi(h)(x) = c_\xi \int_{-\infty}^0 \widehat{h}(\eta) e^{2\pi i x \eta} \, d\eta \,,$$

which identifies Π_ξ with the convolution operator whose multiplier is the function $c_\xi \chi_{(-\infty,\xi]}$. Using the identity (11.1.14), we obtain

$$c_{\xi+\theta} = c_\xi$$

for all ξ and θ, saying that c_ξ does not depend on ξ. We have therefore proved that for all Schwartz functions h the following identity is valid:

$$\Pi_\xi(h) = c\,\big(\widehat{h}\chi_{(-\infty,\xi]}\big)^\vee \tag{11.1.20}$$

for some fixed nonzero constant c. This completely identifies the operator Π_ξ. By density it follows that

$$\mathscr{C}_1(f) = \frac{1}{|c|} \sup_{\xi > 0} |\Pi_\xi(f)| \tag{11.1.21}$$

for all $f \in \bigcup_{1 \le p < \infty} L^p(\mathbf{R})$.

11.1.3 Linearization of a Maximal Dyadic Sum

Our goal is to show that there exists a constant $C > 0$ such that for all $f \in L^2(\mathbf{R})$ we have

$$\big\| \sup_{\xi > 0} |A_\xi(f)| \big\|_{L^{2,\infty}(\mathbf{R})} \le C \big\| f \big\|_{L^2(\mathbf{R})} . \tag{11.1.22}$$

Once (11.1.22) is established, averaging yields the same conclusion for the operator $f \mapsto \sup_{\xi>0} |\Pi_\xi(f)|$, establishing the required bound for $\mathscr{C}_1$. Let us describe this averaging argument. Identity (11.1.12) gives

$$\Pi_\xi(f) = \lim_{\substack{K\to\infty\\ L\to\infty}} \frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_0^1 G_{\xi,y,\eta,\lambda}(f)\,d\lambda\,dy\,d\eta\,,$$

where

$$G_{\xi,y,\eta,\lambda}(f) = M^{-\eta}\tau^{-y}D^{2^{-\lambda}}A_{\frac{\xi+\eta}{2^\lambda}}D^{2^\lambda}\tau^y M^\eta(f)\,.$$

This, in turn, implies

$$\sup_{\xi\in\mathbf{R}}|\Pi_\xi(f)| \le \liminf_{\substack{K\to\infty\\ L\to\infty}} \frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_0^1 \sup_{\xi\in\mathbf{R}}|G_{\xi,y,\eta,\lambda}(f)|\,d\lambda\,dy\,d\eta\,. \tag{11.1.23}$$

We now take the $L^{2,\infty}$ quasinorms of both sides, and we use Fatou's lemma for weak L^2 [Exercise 1.1.12(d)]. We thus reduce the estimate for the operator $\sup_{\xi>0}|\Pi_\xi(f)|$ to the corresponding estimate for $\sup_{\xi>0}|A_\xi(f)|$. In this way we obtain the $L^{2,\infty}$ boundedness of $\sup_{\xi>0}|\Pi_\xi(f)|$ and therefore that of $\mathscr{C}_1$ in view of identity (11.1.21).

Matters are now reduced to the study of the discretized maximal operator $\sup_{\xi>0}|A_\xi(f)|$ and, in particular, to the proof of estimate (11.1.22). It will be convenient to study the maximal operator $\sup_{\xi>0}|A_\xi(f)|$ via a linearization. Here is how this is achieved. Given $f\in L^2(\mathbf{R})$, we select a measurable real-valued function[1] $N_f:\ \mathbf{R}\to\mathbf{R}^+$ such that for all $x\in\mathbf{R}$ we have

$$\sup_{\xi>0}|A_\xi(f)(x)| \le 2\,|A_{N_f(x)}(f)(x)|\,.$$

For a general measurable function $N:\ \mathbf{R}\to\mathbf{R}^+$, we define a linear operator $\mathfrak{D}_N$ by setting for $f\in L^2(\mathbf{R})$,

$$\mathfrak{D}_N(f)(x) = A_{N(x)}(f)(x) = \sum_{s\in\mathbf{D}}(\chi_{\omega_{s(2)}}\circ N)(x)\left\langle f\,|\,\varphi_s\right\rangle\varphi_s(x)\,, \tag{11.1.24}$$

where the sum on the right converges in $L^2(\mathbf{R})$ [and also uniformly for $f\in\mathscr{S}(\mathbf{R})$].

To prove (11.1.22), it suffices to show that there exists $C>0$ such that for all $f\in L^2(\mathbf{R})$ and all measurable functions $N:\ \mathbf{R}\to\mathbf{R}^+$ we have

$$\big\|\mathfrak{D}_N(f)\big\|_{L^{2,\infty}} \le C\big\|f\big\|_{L^2}\,. \tag{11.1.25}$$

Applying (11.1.25) to the measurable function N_f and using the estimate

$$\sup_{\xi>0}|A_\xi(f)| \le 2\,\mathfrak{D}_{N_f}(f)$$

yields the required conclusion for the maximal dyadic sum operator $\sup_{\xi>0}|A_\xi(f)|$ and thus for $\mathscr{C}_1(f)$.

---

[1] The range $\xi>0$ may be replaced by a finite subset of the positive rationals by density; in this case N_f could be taken to be the point ξ at which the supremum is attained.

To justify certain algebraic manipulations we fix a finite subset $\mathbf{P}$ of $\mathbf{D}$ and we define

$$\mathfrak{D}_{N,\mathbf{P}}(f)(x) = \sum_{s\in\mathbf{P}} (\chi_{\omega_{s(2)}} \circ N)(x) \left\langle f \,|\, \varphi_s \right\rangle \varphi_s(x)\,. \tag{11.1.26}$$

To prove (11.1.25) it suffices to show that there exists a $C > 0$ such that for all $f \in L^2(\mathbf{R})$, all finite subsets $\mathbf{P}$ of $\mathbf{D}$, and all real-valued measurable functions N on the line we have

$$\big\|\mathfrak{D}_{N,\mathbf{P}}(f)\big\|_{L^{2,\infty}} \le C\big\|f\big\|_{L^2}\,. \tag{11.1.27}$$

The important point is that the constant C in (11.1.27) is independent of f, $\mathbf{P}$, and the measurable function N.

To prove (11.1.27) we use duality. In view of the results of Exercises 1.4.12(c) and 1.4.7, it suffices to prove that for all measurable subsets E of the real line with finite measure we have

$$\left|\int_E \mathfrak{D}_{N,\mathbf{P}}(f)\,dx\right| = \left|\sum_{s\in\mathbf{P}} \big\langle (\chi_{\omega_{s(2)}} \circ N)\varphi_s, \chi_E \big\rangle \big\langle \varphi_s \,|\, f \big\rangle\right| \le C|E|^{\frac{1}{2}} \big\|f\big\|_{L^2}\,. \tag{11.1.28}$$

We obtain estimate (11.1.28) as a consequence of

$$\sum_{s\in\mathbf{P}} \big|\big\langle (\chi_{\omega_{s(2)}} \circ N)\varphi_s, \chi_E \big\rangle \big\langle f \,|\, \varphi_s \big\rangle\big| \le C|E|^{\frac{1}{2}} \big\|f\big\|_{L^2} \tag{11.1.29}$$

for all f in L^2, all measurable functions N, all measurable sets E of finite measure, and all finite subsets $\mathbf{P}$ of $\mathbf{D}$. It is estimate (11.1.29) that we shall concentrate on.

11.1.4 Iterative Selection of Sets of Tiles with Large Mass and Energy

We introduce a partial order in the set of dyadic tiles that provides a way to organize them. In this section, dyadic tiles are simply called tiles.

Definition 11.1.3. We define a *partial order* $<$ in the set of dyadic tiles $\mathbf{D}$ by setting

$$s < s' \iff I_s \subseteq I_{s'} \quad \text{and} \quad \omega_{s'} \subseteq \omega_s\,.$$

If two tiles $s, s' \in \mathbf{D}$ intersect, then we must have either $s < s'$ or $s' < s$. Indeed, both the time and frequency components of the tiles must intersect; then either $I_s \subseteq I_{s'}$ or $I_{s'} \subseteq I_s$. In the first case we must have $|\omega_s| \ge |\omega_{s'}|$, thus $\omega_{s'} \subseteq \omega_s$, which gives $s < s'$, while in the second case a similar argument gives $s' < s$. As a consequence of this observation, if $\mathbf{R}_0$ is a finite set of tiles, then all maximal elements of $\mathbf{R}_0$ under $<$ must be disjoint sets.

Definition 11.1.4. A finite set of tiles $\mathbf{P}$ is called a *tree* if there exists a tile $t \in \mathbf{P}$ such that all $s \in \mathbf{P}$ satisfy $s < t$. We call t the top of $\mathbf{P}$ and we denote it by $t = \mathrm{top}(\mathbf{P})$. Observe that the top of a tree is unique.

We denote trees by $\mathbf{T}$, $\mathbf{T}'$, $\mathbf{T}_1$, $\mathbf{T}_2$, and so on.

We observe that every finite set of tiles $\mathbf{P}$ can be written as a union of trees whose tops are maximal elements. Indeed, consider all maximal elements of $\mathbf{P}$ under the partial order $<$. Then every nonmaximal element s of $\mathbf{P}$ satisfies $s < t$ for some maximal element $t \in \mathbf{P}$, and thus it belongs to a tree with top t.

Tiles can be written as a union of two *semitiles* $I_s \times \omega_{s(1)}$ and $I_s \times \omega_{s(2)}$. Since tiles have area 1, semitiles have area $1/2$.

Definition 11.1.5. A tree $\mathbf{T}$ is called a 1-tree if

$$\omega_{\mathrm{top}(\mathbf{T})(1)} \subseteq \omega_{s(1)}$$

all $s \in \mathbf{T}$. A tree $\mathbf{T}'$ is called a 2-tree if for all $s \in \mathbf{T}'$ we have

$$\omega_{\mathrm{top}(\mathbf{T}')(2)} \subseteq \omega_{s(2)} \,.$$

We make a few observations about 1-trees and 2-trees. First note that every tree can be written as the union of a 1-tree and a 2-tree, and the intersection of these is exactly the top of the tree. Also, if $\mathbf{T}$ is a 1-tree, then the intervals $\omega_{\mathrm{top}(\mathbf{T})(2)}$ and $\omega_{s(2)}$ are disjoint for all $s \in \mathbf{T}$, and similarly for 2-trees. See Figure 11.2.

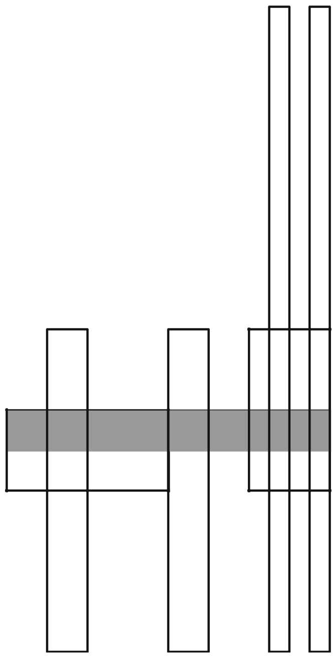

Fig. 11.2 A tree of seven tiles including the darkened top. The top together with the three tiles on the right forms a 1-tree, while the top together with the three tiles on the left forms a 2-tree.

Definition 11.1.6. Let $N: \mathbf{R} \to \mathbf{R}^+$ be a measurable function, let $s \in \mathbf{D}$, and let E be a set of finite measure. Then we introduce the quantity

$$\mathscr{M}(E;\{s\}) = \frac{1}{|E|} \sup_{\substack{u\in\mathbf{D}\\ s<u}} \int_{E\cap N^{-1}[\omega_u]} \frac{|I_u|^{-1}\,dx}{(1+\frac{|x-c(I_u)|}{|I_u|})^{10}}.$$

We call $\mathscr{M}(E;\{s\})$ the *mass* of E with respect to $\{s\}$. Given a subset $\mathbf{P}$ of $\mathbf{D}$, we define the mass of E with respect to $\mathbf{P}$ as

$$\mathscr{M}(E;\mathbf{P}) = \sup_{s\in\mathbf{P}} \mathscr{M}(E;\{s\}).$$

We observe that the mass of E with respect to any set of tiles is at most

$$\frac{1}{|E|}\int_{-\infty}^{+\infty} \frac{dx}{(1+|x|)^{10}} \le \frac{1}{|E|}.$$

Definition 11.1.7. Given a finite subset $\mathbf{P}$ of $\mathbf{D}$ and a function f in $L^2(\mathbf{R})$, we introduce the quantity

$$\mathscr{E}(f;\mathbf{P}) = \frac{1}{\big\|f\big\|_{L^2}} \sup_{\mathbf{T}} \bigg(\frac{1}{|I_{\mathrm{top}(\mathbf{T})}|}\sum_{s\in\mathbf{T}} \big|\langle f\,|\,\varphi_s\rangle\big|^2\bigg)^{\frac{1}{2}},$$

where the supremum is taken over all 2-trees $\mathbf{T}$ contained in $\mathbf{P}$. We call $\mathscr{E}(f;\mathbf{P})$ the *energy* of the function f with respect to the set of tiles $\mathbf{P}$.

We now state three important lemmas which we prove in the remaining three subsections, respectively.

Lemma 11.1.8. *There exists a constant C_1 such that for any measurable function $N:\ \mathbf{R}\to\mathbf{R}^+$, for any measurable subset E of the real line with finite measure, and for any finite set of tiles $\mathbf{P}$ there is a subset $\mathbf{P}'$ of $\mathbf{P}$ such that*

$$\mathscr{M}(E;\mathbf{P}\setminus\mathbf{P}') \le \frac{1}{4}\mathscr{M}(E;\mathbf{P})$$

and $\mathbf{P}'$ is a union of trees $\mathbf{T}_j$ satisfying

$$\sum_j |I_{\mathrm{top}(\mathbf{T}_j)}| \le \frac{C_1}{\mathscr{M}(E;\mathbf{P})}. \tag{11.1.30}$$

Lemma 11.1.9. *There exists a constant C_2 such that for any finite set of tiles $\mathbf{P}$ and for all functions f in $L^2(\mathbf{R})$ there is a subset $\mathbf{P}''$ of $\mathbf{P}$ such that*

$$\mathscr{E}(f;\mathbf{P}\setminus\mathbf{P}'') \le \frac{1}{2}\mathscr{E}(f;\mathbf{P})$$

and $\mathbf{P}''$ is a union of trees $\mathbf{T}_j$ satisfying

$$\sum_j |I_{\mathrm{top}(\mathbf{T}_j)}| \le \frac{C_2}{\mathscr{E}(f;\mathbf{P})^2}. \tag{11.1.31}$$

Lemma 11.1.10. *(The basic estimate) There is a finite constant C_3 such that for all trees $\mathbf{T}$, all functions f in $L^2(\mathbf{R})$, for any measurable function $N:\ \mathbf{R}\to\mathbf{R}^+$, and for all measurable sets E we have*

$$\sum_{s\in\mathbf{T}} \big|\big\langle f\,|\,\varphi_s\big\rangle\big\langle \chi_{E\cap N^{-1}[\omega_{s(2)}]}\,|\,\varphi_s\big\rangle\big| \\ \le C_3\,|I_{\mathrm{top}(\mathbf{T})}|\,\mathscr{E}(f;\mathbf{T})\mathscr{M}(E;\mathbf{T})\big\|f\big\|_{L^2}\,|E|\,. \tag{11.1.32}$$

In the rest of this subsection, we conclude the proof of Theorem 11.1.1 assuming Lemmas 11.1.8, 11.1.9, and 11.1.10.

Given a finite set of tiles $\mathbf{P}$, a measurable set E of finite measure, a measurable function $N:\mathbf{R}\to\mathbf{R}^+$, and a function f in $L^2(\mathbf{R})$, we find a very large integer n_0 such that

$$\mathscr{E}(f;\mathbf{P}) \ \le\ 2^{n_0}\,,$$
$$\mathscr{M}(E;\mathbf{P}) \ \le\ 2^{2n_0}\,.$$

We shall construct by decreasing induction a sequence of pairwise disjoint sets

$$\mathbf{P}_{n_0},\,\mathbf{P}_{n_0-1},\,\mathbf{P}_{n_0-2},\,\mathbf{P}_{n_0-3},\dots$$

such that

$$\bigcup_{j=-\infty}^{n_0}\mathbf{P}_j=\mathbf{P} \tag{11.1.33}$$

and such that the following properties are satisfied:

(1) $\mathscr{E}(f;\mathbf{P}_j)\le 2^{j+1}$ for all $j\le n_0$;

(2) $\mathscr{M}(E;\mathbf{P}_j)\le 2^{2j+2}$ for all $j\le n_0$;

(3) $\mathscr{E}\big(f;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_j)\big)\le 2^j$ for all $j\le n_0$;

(4) $\mathscr{M}\big(E;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_j)\big)\le 2^{2j}$ for all $j\le n_0$;

(5) $\mathbf{P}_j$ is a union of trees $\mathbf{T}_{jk}$ such that for all $j\le n_0$ we have

$$\sum_k |I_{\mathrm{top}(\mathbf{T}_{jk})}|\le C_0\,2^{-2j}\,,$$

where $C_0=C_1+C_2$ and C_1 and C_2 are the constants that appear in Lemmas 11.1.8 and 11.1.9, respectively.

Assume momentarily that we have constructed a sequence $\{\mathbf{P}_j\}_{j\le n_0}$ with the described properties. Then to obtain estimate (11.1.29) we use (1), (2), (5), the observation that the mass of any set of tiles is always bounded by $|E|^{-1}$, and Lemma 11.1.10 to obtain

$$\begin{aligned}
&\sum_{s\in\mathbf{P}} \big|\big\langle f \,|\, \varphi_s\big\rangle\big\langle \chi_{E\cap N^{-1}[\omega_{s(2)}]} \,|\, \varphi_s\big\rangle\big| \\
&\quad = \sum_j \sum_{s\in\mathbf{P}_j} \big|\big\langle f \,|\, \varphi_s\big\rangle\big\langle \chi_{E\cap N^{-1}[\omega_{s(2)}]} \,|\, \varphi_s\big\rangle\big| \\
&\quad \le \sum_j \sum_k \sum_{s\in\mathbf{T}_{jk}} \big|\big\langle f \,|\, \varphi_s\big\rangle\big\langle \chi_{E\cap N^{-1}[\omega_{s(2)}]} \,|\, \varphi_s\big\rangle\big| \\
&\quad \le C_3 \sum_j \sum_k |I_{\mathrm{top}(\mathbf{T}_{jk})}|\,\mathscr{E}(f;\mathbf{T}_{jk})\,\mathscr{M}(E;\mathbf{T}_{jk})\big\|f\big\|_{L^2}|E| \\
&\quad \le C_3 \sum_j \sum_k |I_{\mathrm{top}(\mathbf{T}_{jk})}|\,2^{j+1}\min(|E|^{-1},2^{2j+2})\big\|f\big\|_{L^2}|E| \\
&\quad \le C_3 \sum_j C_0 2^{-2j} 2^{j+1}\min(|E|^{-1},2^{2j+2})\big\|f\big\|_{L^2}|E| \\
&\quad \le 8C_0C_3 \sum_j \min(2^{-j}|E|^{-\frac{1}{2}},2^{j}|E|^{\frac{1}{2}})\big\|f\big\|_{L^2}|E|^{\frac{1}{2}} \\
&\quad \le C|E|^{\frac{1}{2}}\,\big\|f\big\|_{L^2}\,.
\end{aligned}$$

This proves estimate (11.1.29).

It remains to construct a sequence of disjoint sets $\mathbf{P}_j$ satisfying properties (1)–(5). The selection of these sets is based on decreasing induction. We start the induction at $j = n_0$ by setting $\mathbf{P}_{n_0} = \emptyset$. Then (1), (2), and (5) are clearly satisfied, while

$$\begin{aligned}
\mathscr{E}(f;\mathbf{P}\setminus\mathbf{P}_{n_0}) &= \mathscr{E}(f;\mathbf{P}) \le 2^{n_0}\,, \\
\mathscr{M}(E;\mathbf{P}\setminus\mathbf{P}_{n_0}) &= \mathscr{M}(E;\mathbf{P}) \le 2^{2n_0}\,;
\end{aligned}$$

hence (3) and (4) are also satisfied for $\mathbf{P}_{n_0}$.

Suppose we have selected pairwise disjoint sets $\mathbf{P}_{n_0}$, $\mathbf{P}_{n_0-1},\dots,\mathbf{P}_n$ for some $n < n_0$ such that (1)–(5) are satisfied for all $j \in \{n_0, n_0-1,\dots,n\}$. We construct a set of tiles $\mathbf{P}_{n-1}$ disjoint from all $\mathbf{P}_j$ with $j \ge n$ such that (1)–(5) are satisfied for $j = n-1$.

We define first an auxiliary set $\mathbf{P}'_{n-1}$. If $\mathscr{M}\big(E;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n)\big) \le 2^{2(n-1)}$ set $\mathbf{P}'_{n-1} = \emptyset$. If $\mathscr{M}\big(E;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n)\big) > 2^{2(n-1)}$ apply Lemma 11.1.8 to find a subset $\mathbf{P}'_{n-1}$ of $\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n)$ such that

$$\mathscr{M}\big(E;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n\cup\mathbf{P}'_{n-1})\big) \le \frac{1}{4}\mathscr{M}\big(E;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n)\big) \le \frac{2^{2n}}{4} = 2^{2(n-1)}$$

[by the induction hypothesis (4) with $j = n$] and $\mathbf{P}'_{n-1}$ is a union of trees $\mathbf{T}'_k$ satisfying

$$\sum_k |I_{\mathrm{top}(\mathbf{T}'_k)}| \le C_1\,\mathscr{M}\big(f;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n)\big)^{-1} \le C_1\,2^{-2(n-1)}. \tag{11.1.34}$$

Likewise, if $\mathscr{E}\big(f;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n)\big) \le 2^{n-1}$ set $\mathbf{P}''_{n-1} = \emptyset$; otherwise, apply Lemma 11.1.9 to find a subset $\mathbf{P}''_{n-1}$ of $\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n)$ such that

$$\mathscr{E}\big(f;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n\cup\mathbf{P}''_{n-1})\big) \le \frac{1}{2}\mathscr{E}\big(f;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\dots\cup\mathbf{P}_n)\big) \le \frac{1}{2}2^n = 2^{n-1}$$

[by the induction hypothesis (3) with $j = n$] and $\mathbf{P}''_{n-1}$ is a union of trees $\mathbf{T}''_k$ satisfying

$$\sum_k |I_{\mathrm{top}(\mathbf{T}''_k)}| \le C_2\, \mathscr{E}\big(f;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n)\big)^{-2} \le C_2\, 2^{-2(n-1)}. \tag{11.1.35}$$

Whether the sets $\mathbf{P}'_{n-1}$ and $\mathbf{P}''_{n-1}$ are empty or not, we note that

$$\mathscr{M}\big(E;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n\cup\mathbf{P}'_{n-1})\big) \le 2^{2(n-1)}, \tag{11.1.36}$$
$$\mathscr{E}\big(f;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n\cup\mathbf{P}''_{n-1})\big) \le 2^{n-1}. \tag{11.1.37}$$

We set $\mathbf{P}_{n-1} = \mathbf{P}'_{n-1}\bigcup\mathbf{P}''_{n-1}$, and we verify properties (1)–(5) for $j = n-1$. Since $\mathbf{P}_{n-1}$ is contained in $\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n)$ we have

$$\mathscr{E}(f;\mathbf{P}_{n-1}) \le \mathscr{E}(f;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n) \le 2^n = 2^{(n-1)+1},$$

where the last inequality is a consequence of the induction hypothesis (3) for $j = n$; thus (1) holds with $j = n-1$. Likewise,

$$\mathscr{M}(E;\mathbf{P}_{n-1}) \le \mathscr{M}(E;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n) \le 2^{2n} = 2^{2(n-1)+2}$$

in view of the induction hypothesis (4) for $j = n$; thus (2) holds with $j = n-1$.

To prove (3) with $j = n-1$ notice that $\mathbf{P}\setminus\big(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n\cup\mathbf{P}_{n-1}\big)$ is contained in $\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n\cup\mathbf{P}''_{n-1})$, and the latter has energy at most 2^{n-1} by (11.1.37). To prove (4) with $j = n-1$ note that $\mathbf{P}\setminus\big(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n\cup\mathbf{P}_{n-1}\big)$ is contained in $\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_n\cup\mathbf{P}'_{n-1})$ and the latter has mass at most $2^{2(n-1)}$ by (11.1.36). Finally, adding (11.1.34) and (11.1.35) yields (5) for $j = n-1$ with $C_0 = C_1 + C_2$.

Pick $j\in\mathbf{Z}$ with $0 < 2^{2j} < \min_{s\in\mathbf{P}}\mathscr{M}(E;\{s\})$. Then $\mathscr{M}\big(E;\mathbf{P}\setminus(\mathbf{P}_{n_0}\cup\cdots\cup\mathbf{P}_j)\big) = 0$, and since the only set of tiles with zero mass is the empty set, we conclude that (11.1.33) holds. It also follows that there exists an n_1 such that for all $n\le n_1$, $\mathbf{P}_n = \emptyset$. The construction of the $\mathbf{P}_j$'s is now complete.

11.1.5 Proof of the Mass Lemma 11.1.8

Proof. Given a finite set of tiles $\mathbf{P}$, we set $\mu = \mathscr{M}(E;\mathbf{P})$ to be the mass of $\mathbf{P}$. We define

$$\mathbf{P}' = \{s\in\mathbf{P}:\ \mathscr{M}(E;\{s\}) > \tfrac14\mu\}$$

and we observe that $\mathscr{M}(E;\mathbf{P}\setminus\mathbf{P}') \le \frac14\mu$. We now show that $\mathbf{P}'$ is a union of trees whose tops satisfy (11.1.30).

It follows from the definition of mass that for each $s\in\mathbf{P}'$, there is a tile $u(s)\in\mathbf{D}$ such that $u(s) > s$ and

$$\frac{1}{|E|}\int_{E\cap N^{-1}[\omega_{u(s)}]}\frac{|I_{u(s)}|^{-1}\,dx}{(1+\frac{|x-c(I_{u(s)})|}{|I_{u(s)}|})^{10}}>\frac{\mu}{4}. \tag{11.1.38}$$

Let $\mathbf{U}=\{u(s):\ s\in\mathbf{P}'\}$. Also, let $\mathbf{U}_{\max}$ be the subset of $\mathbf{U}$ containing all maximal elements of $\mathbf{U}$ under the partial order of tiles $<$. Likewise define $\mathbf{P}'_{\max}$ as the set of all maximal elements in $\mathbf{P}'$. Tiles in $\mathbf{P}'$ can be grouped in trees $\mathbf{T}_j=\{s\in\mathbf{P}':\ s<t_j\}$ with tops $t_j\in\mathbf{P}'_{\max}$. Observe that if $t_j<u$ and $t_{j'}<u$ for some $u\in\mathbf{U}_{\max}$, then ω_{t_j} and $\omega_{t_{j'}}$ intersect, and since t_j and $t_{j'}$ are disjoint sets, it follows that I_{t_j} and $I_{t_{j'}}$ are disjoint subsets of I_u. Consequently, we have

$$\sum_j|I_{t_j}|=\sum_{u\in\mathbf{U}_{\max}}\sum_{j:\,t_j<u}|I_{t_j}|\le\sum_{u\in\mathbf{U}_{\max}}|I_u|.$$

Therefore, estimate (11.1.30) will be a consequence of

$$\sum_{u\in\mathbf{U}_{\max}}|I_u|\le C\mu^{-1} \tag{11.1.39}$$

for some constant C. For $u\in\mathbf{U}_{\max}$ we rewrite (11.1.38) as

$$\frac{1}{|E|}\sum_{k=0}^{\infty}\int_{E\cap N^{-1}[\omega_u]\cap\left(2^kI_u\setminus 2^{k-1}I_u\right)}\frac{|I_u|^{-1}\,dx}{\left(1+\frac{|x-c(I_u)|}{|I_u|}\right)^{10}}>\frac{\mu}{8}\sum_{k=0}^{\infty}2^{-k}$$

with the interpretation that $2^{-1}I_u=\emptyset$. It follows that for all u in $\mathbf{U}_{\max}$ there exists an integer $k\ge 0$ such that

$$|E|\,\frac{\mu}{8}\,|I_u|2^{-k}<\int_{E\cap N^{-1}[\omega_u]\cap\left(2^kI_u\setminus 2^{k-1}I_u\right)}\frac{dx}{(1+\frac{|x-c(I_u)|}{|I_u|})^{10}}\le\frac{|E\cap N^{-1}[\omega_u]\cap 2^kI_u|}{(\frac{4}{5})^{10}(1+2^{k-2})^{10}}.$$

We therefore conclude that

$$\mathbf{U}_{\max}=\bigcup_{k=0}^{\infty}\mathbf{U}_k,$$

where

$$\mathbf{U}_k=\{u\in\mathbf{U}_{\max}:\ |I_u|\le 8\cdot 5^{10}2^{-9k}\mu^{-1}|E|^{-1}|E\cap N^{-1}[\omega_u]\cap 2^kI_u|\}.$$

The required estimate (11.1.39) will be a consequence of the sequence of estimates

$$\sum_{u\in\mathbf{U}_k}|I_u|\le C2^{-8k}\mu^{-1},\qquad k\ge 0. \tag{11.1.40}$$

We now fix a $k\ge 0$ and we concentrate on (11.1.40). Select an element $v_0\in\mathbf{U}_k$ such that $|I_{v_0}|$ is the largest possible among elements of $\mathbf{U}_k$. Then select an element $v_1\in\mathbf{U}_k\setminus\{v_0\}$ such that the enlarged rectangle $(2^kI_{v_1})\times\omega_{v_1}$ is disjoint from the enlarged rectangle $(2^k I_{v_0}) \times \omega_{v_0}$ and $|I_{v_1}|$ is the largest possible. Continue this process by induction. At the jth step select an element of $\mathbf{U}_k \setminus \{v_0, \dots, v_{j-1}\}$ such that the enlarged rectangle $(2^k I_{v_j}) \times \omega_{v_j}$ is disjoint from all the enlarged rectangles of the previously selected tiles and the length $|I_{v_j}|$ is the largest possible. This process will terminate after a finite number of steps. We denote by $\mathbf{V}_k$ the set of all selected tiles in $\mathbf{U}_k$.

We make a few observations. Recall that all elements of $\mathbf{U}_k$ are maximal rectangles in $\mathbf{U}$ and therefore disjoint. For any $u \in \mathbf{U}_k$ there exists a selected $v \in \mathbf{V}_k$ with $|I_u| \le |I_v|$ such that the enlarged rectangles corresponding to u and v intersect. Let us associate this u to the selected v. Observe that if u and u' are associated with the same selected v, they are disjoint, and since both ω_u and $\omega_{u'}$ contain ω_v, the intervals I_u and $I_{u'}$ must be disjoint. Thus, tiles $u \in \mathbf{U}_k$ associated with a fixed $v \in \mathbf{V}_k$ have disjoint I_u's and satisfy

$$I_u \subseteq 2^{k+2} I_v .$$

Consequently,

$$\sum_{\substack{u \in \mathbf{U}_k \\ u \text{ associated with } v}} |I_u| \le |2^{k+2} I_v| = 2^{k+2} |I_v| .$$

Putting these observations together, we obtain

$$\begin{aligned}
\sum_{u \in \mathbf{U}_k} |I_u| &\le \sum_{v \in \mathbf{V}_k} \sum_{\substack{u \in \mathbf{U}_k \\ u \text{ associated with } v}} |I_u| \\
&\le 2^{k+2} \sum_{v \in \mathbf{V}_k} |I_v| \\
&\le 2^{k+5} 5^{10} \mu^{-1} |E|^{-1} 2^{-9k} \sum_{v \in \mathbf{V}_k} |E \cap N^{-1}[\omega_v] \cap 2^k I_v| \\
&\le 32 \cdot 5^{10} \mu^{-1} \, 2^{-8k} ,
\end{aligned}$$

since the enlarged rectangles $2^k I_v \times \omega_v$ of the selected tiles v are disjoint and therefore so are the subsets $E \cap N^{-1}[\omega_v] \cap 2^k I_v$ of E. This concludes the proof of estimate (11.1.40) and therefore of Lemma 11.1.8. □

11.1.6 Proof of Energy Lemma 11.1.9

Proof. We work with a finite set of tiles $\mathbf{P}$. For a 2-tree $\mathbf{T}'$, let us denote by

$$\Delta(f; \mathbf{T}') = \frac{1}{\|f\|_{L^2}} \left\{ \frac{1}{|I_{\mathrm{top}(\mathbf{T}')}|} \sum_{s \in \mathbf{T}'} \big|\langle f \,|\, \varphi_s \rangle\big|^2 \right\}^{\frac{1}{2}}$$

the quantity associated with $\mathbf{T}'$ appearing in the definition of the energy. Consider the set of all 2-trees $\mathbf{T}'$ contained in $\mathbf{P}$ that satisfy

$$\Delta(f;\mathbf{T}') \geq \frac{1}{2}\mathscr{E}(f;\mathbf{P}) \tag{11.1.41}$$

and among them select a 2-tree $\mathbf{T}_1'$ with $c(\omega_{\mathrm{top}(\mathbf{T}_1')})$ as small as possible. We let $\mathbf{T}_1$ be the set of $s \in \mathbf{P}$ satisfying $s < \mathrm{top}(\mathbf{T}_1')$. Then $\mathbf{T}_1$ is the largest tree in $\mathbf{P}$ whose top is $\mathrm{top}(\mathbf{T}_1')$. We now repeat this procedure with the set $\mathbf{P}\setminus\mathbf{T}_1$. Among all 2-trees contained in $\mathbf{P}\setminus\mathbf{T}_1$ that satisfy (11.1.41) we pick a 2-tree $\mathbf{T}_2'$ with $c(\omega_{\mathrm{top}(\mathbf{T}_2')})$ as small as possible. Then we let $\mathbf{T}_2$ be the $s \in \mathbf{P}\setminus\mathbf{T}_1$ satisfying $s < \mathrm{top}(\mathbf{T}_2')$. Then $\mathbf{T}_2$ is the largest tree in $\mathbf{P}\setminus\mathbf{T}_1$ whose top is $\mathrm{top}(\mathbf{T}_2')$. We continue this procedure by induction until there is no 2-tree left in $\mathbf{P}$ that satisfies (11.1.41). We have therefore constructed a finite sequence of pairwise disjoint 2-trees $\mathbf{T}_1', \mathbf{T}_2', \mathbf{T}_3', \dots, \mathbf{T}_q'$, and a finite sequence of pairwise disjoint trees $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \dots, \mathbf{T}_q$, such that $\mathbf{T}_j' \subseteq \mathbf{T}_j$, $\mathrm{top}(\mathbf{T}_j) = \mathrm{top}(\mathbf{T}_j')$, and the $\mathbf{T}_j'$ satisfy (11.1.41). We now let

$$\mathbf{P}'' = \bigcup_j \mathbf{T}_j\,,$$

and observe that this selection of trees ensures that

$$\mathscr{E}(f;\mathbf{P}\setminus\mathbf{P}'') \leq \frac{1}{2}\mathscr{E}(f;\mathbf{P})\,.$$

It remains to prove (11.1.31). Using (11.1.41), we obtain that

$$\begin{aligned}
\frac{1}{4}\mathscr{E}(f;\mathbf{P})^2 \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}| &\leq \frac{1}{\|f\|_{L^2}^2} \sum_j \sum_{s\in\mathbf{T}_j'} |\langle f \,|\, \varphi_s\rangle|^2 \\
&= \frac{1}{\|f\|_{L^2}^2} \sum_j \sum_{s\in\mathbf{T}_j'} \langle f \,|\, \varphi_s\rangle \overline{\langle f \,|\, \varphi_s\rangle} \\
&= \frac{1}{\|f\|_{L^2}^2} \Big\langle f \,\Big|\, \sum_j \sum_{s\in\mathbf{T}_j'} \langle f \,|\, \varphi_s\rangle \varphi_s \Big\rangle \\
&\leq \frac{1}{\|f\|_{L^2}} \Big\| \sum_j \sum_{s\in\mathbf{T}_j'} \langle \varphi_s \,|\, f\rangle \varphi_s \Big\|_{L^2}\,,
\end{aligned} \tag{11.1.42}$$

and we use this estimate to obtain (11.1.31). We set $\mathbf{U} = \bigcup_j \mathbf{T}_j'$. We shall prove that

$$\frac{1}{\|f\|_{L^2}} \Big\| \sum_{s\in\mathbf{U}} \langle \varphi_s \,|\, f\rangle \varphi_s \Big\|_{L^2} \leq C\Big(\mathscr{E}(f;\mathbf{P})^2 \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}|\Big)^{\frac{1}{2}}. \tag{11.1.43}$$

Once this estimate is established, then (11.1.42) combined with (11.1.43) yields (11.1.31). (All involved quantities are finite, since $\mathbf{P}$ is a finite set of tiles.)

We estimate the square of the left-hand side in (11.1.43) by

$$\sum_{\substack{s,u\in\mathbf{U}\\ \omega_s=\omega_u}} \big|\langle \varphi_s \,|\, f\rangle \langle \varphi_u \,|\, f\rangle \langle \varphi_s \,|\, \varphi_u\rangle\big| + 2 \sum_{\substack{s,u\in\mathbf{U}\\ \omega_s \subsetneq \omega_u}} \big|\langle \varphi_s \,|\, f\rangle \langle \varphi_u \,|\, f\rangle \langle \varphi_s \,|\, \varphi_u\rangle\big|\,, \tag{11.1.44}$$

since $\langle \varphi_s \,|\, \varphi_u \rangle = 0$ unless ω_s contains ω_u or vice versa. We now estimate the quantities $\big|\langle \varphi_s \,|\, f \rangle\big|$ and $\big|\langle \varphi_u \,|\, f \rangle\big|$ by the larger one and we use Exercise 11.1.4 to obtain the following bound for the first term in (11.1.44):

$$\begin{aligned}
&\sum_{s\in\mathbf{U}} \big|\langle f \,|\, \varphi_s \rangle\big|^2 \sum_{\substack{u\in\mathbf{U}\\ \omega_u=\omega_s}} \big|\langle \varphi_s \,|\, \varphi_u \rangle\big| \\
&\qquad \le \sum_{s\in\mathbf{U}} \big|\langle f \,|\, \varphi_s \rangle\big|^2 \sum_{\substack{u\in\mathbf{U}\\ \omega_u=\omega_s}} C' \int_{I_u} \frac{1}{|I_s|}\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{-100} dx \\
&\qquad \le C'' \sum_{s\in\mathbf{U}} \big|\langle f \,|\, \varphi_s \rangle\big|^2 \\
&\qquad = C'' \sum_j \sum_{s\in\mathbf{T}'_j} \big|\langle f \,|\, \varphi_s \rangle\big|^2 \\
&\qquad \le C'' \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}|\,|I_{\mathrm{top}(\mathbf{T}_j)}|^{-1} \sum_{s\in\mathbf{T}'_j} \big|\langle f \,|\, \varphi_s \rangle\big|^2 \\
&\qquad \le C'' \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}|\,\mathscr{E}(f;\mathbf{P})^2 \big\|f\big\|_{L^2}^2 ,
\end{aligned} \tag{11.1.45}$$

where in the derivation of the second inequality we used the fact that for fixed $s \in \mathbf{U}$, the intervals I_u with $\omega_u = \omega_s$ are pairwise disjoint.

Our next goal is to obtain a similar estimate for the second term in (11.1.44). That is, we need to prove that

$$\sum_{\substack{s,u\in\mathbf{U}\\ \omega_s \subsetneq \omega_u}} \big|\langle f \,|\, \varphi_s \rangle \langle f \,|\, \varphi_u \rangle \langle \varphi_s \,|\, \varphi_u \rangle\big| \le C\,\mathscr{E}(f;\mathbf{P})^2 \big\|f\big\|_{L^2}^2 \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}| . \tag{11.1.46}$$

Then the required estimate (11.1.43) follows by combining (11.1.45) and (11.1.46). To prove (11.1.46), we argue as follows:

$$\begin{aligned}
&\sum_{\substack{s,u\in\mathbf{U}\\ \omega_s \subsetneq \omega_u}} \big|\langle f \,|\, \varphi_s \rangle \langle f \,|\, \varphi_u \rangle \langle \varphi_s \,|\, \varphi_u \rangle\big| \\
&\quad = \sum_j \sum_{s\in\mathbf{T}'_j} \big|\langle f \,|\, \varphi_s \rangle\big| \sum_{\substack{u\in\mathbf{U}\\ \omega_s \subsetneq \omega_u}} \big|\langle f \,|\, \varphi_u \rangle \langle \varphi_s \,|\, \varphi_u \rangle\big| \\
&\quad \le \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}|^{\frac12} \Delta(f;\mathbf{T}'_j) \big\|f\big\|_{L^2} \Bigg\{ \sum_{s\in\mathbf{T}'_j} \Bigg(\sum_{\substack{u\in\mathbf{U}\\ \omega_s \subsetneq \omega_u}} \big|\langle f \,|\, \varphi_u \rangle \langle \varphi_s \,|\, \varphi_u \rangle\big| \Bigg)^2 \Bigg\}^{\frac12} \\
&\quad \le \mathscr{E}(f;\mathbf{P}) \big\|f\big\|_{L^2} \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}|^{\frac12} \Bigg\{ \sum_{s\in\mathbf{T}'_j} \Bigg(\sum_{\substack{u\in\mathbf{U}\\ \omega_s \subseteq \omega_{u(1)}}} \big|\langle f \,|\, \varphi_u \rangle \langle \varphi_s \,|\, \varphi_u \rangle\big| \Bigg)^2 \Bigg\}^{\frac12} ,
\end{aligned}$$

where we used the Cauchy–Schwarz inequality and the fact that if $\omega_s \subsetneq \omega_u$ and $\langle \varphi_s \,|\, \varphi_u \rangle \neq 0$, then $\omega_s \subseteq \omega_{u(1)}$. The proof of (11.1.46) will be complete if we can show that the expression inside the curly brackets is at most a multiple of $\mathscr{E}(f;\mathbf{P})^2 \|f\|_{L^2}^2 |I_{\mathrm{top}(\mathbf{T}_j)}|$. Since any singleton $\{s\} \subseteq \mathbf{P}$ is a 2-tree, we have

$$\mathscr{E}(f;\{u\}) = \frac{1}{\|f\|_{L^2}} \left(\frac{|\langle f \,|\, \varphi_u \rangle|^2}{|I_u|} \right)^{\frac{1}{2}} = \frac{1}{\|f\|_{L^2}} \frac{|\langle f \,|\, \varphi_u \rangle|}{|I_u|^{\frac{1}{2}}} \leq \mathscr{E}(f;\mathbf{P});$$

hence

$$|\langle f \,|\, \varphi_u \rangle| \leq \|f\|_{L^2} |I_u|^{\frac{1}{2}} \mathscr{E}(f;\mathbf{P})$$

and it follows that

$$\sum_{s\in\mathbf{T}'_j} \Bigg[\sum_{\substack{u\in\mathbf{U} \\ \omega_s\subseteq\omega_{u(1)}}} |\langle f \,|\, \varphi_u \rangle \langle \varphi_s \,|\, \varphi_u \rangle| \Bigg]^2 \leq \mathscr{E}(f;\mathbf{P})^2 \|f\|_{L^2}^2 \sum_{s\in\mathbf{T}'_j} \Bigg[\sum_{\substack{u\in\mathbf{U} \\ \omega_s\subseteq\omega_{u(1)}}} |I_u|^{\frac{1}{2}} |\langle \varphi_s \,|\, \varphi_u \rangle| \Bigg]^2 .$$

Thus (11.1.46) will be proved if we can establish that

$$\sum_{s\in\mathbf{T}'_j} \Bigg(\sum_{\substack{u\in\mathbf{U} \\ \omega_s\subseteq\omega_{u(1)}}} |I_u|^{\frac{1}{2}} |\langle \varphi_s \,|\, \varphi_u \rangle| \Bigg)^2 \leq C |I_{\mathrm{top}(\mathbf{T}_j)}| . \tag{11.1.47}$$

We need the following crucial lemma.

Lemma 11.1.11. *Let $\mathbf{T}_j$, $\mathbf{T}'_j$ be as previously. Let $s \in \mathbf{T}'_j$ and $u \in \mathbf{T}'_k$. Then if $\omega_s \subseteq \omega_{u(1)}$, we have $I_u \cap I_{\mathrm{top}(\mathbf{T}_j)} = \emptyset$. Moreover, if $u \in \mathbf{T}'_k$ and $v \in \mathbf{T}'_l$ are different tiles and satisfy $\omega_s \subseteq \omega_{u(1)}$ and $\omega_s \subseteq \omega_{v(1)}$ for some fixed $s \in \mathbf{T}'_j$, then $I_u \cap I_v = \emptyset$.*

Proof. We observe that if $s \in \mathbf{T}'_j$, $u \in \mathbf{T}'_k$, and $\omega_s \subseteq \omega_{u(1)}$, then the 2-trees $\mathbf{T}'_j$ and $\mathbf{T}'_k$ have different tops and therefore they cannot be the same tree; thus $j \neq k$.

Next we observe that the center of $\omega_{\mathrm{top}(\mathbf{T}'_j)}$ is contained in ω_s, which is contained in $\omega_{u(1)}$. Therefore, the center of $\omega_{\mathrm{top}(\mathbf{T}'_j)}$ is contained in $\omega_{u(1)}$, and therefore it must be smaller than the center of $\omega_{\mathrm{top}(\mathbf{T}'_k)}$, since $\mathbf{T}'_k$ is a 2-tree. This means that the 2-tree $\mathbf{T}'_j$ was selected before $\mathbf{T}'_k$, that is, we must have $j < k$. If I_u had a nonempty intersection with $I_{\mathrm{top}(\mathbf{T}_j)} = I_{\mathrm{top}(\mathbf{T}'_j)}$, then since

$$|I_{\mathrm{top}(\mathbf{T}'_j)}| = \frac{1}{|\omega_{\mathrm{top}(\mathbf{T}'_j)}|} \geq \frac{1}{|\omega_s|} \geq \frac{1}{|\omega_{u(1)}|} = \frac{2}{|\omega_u|} = 2|I_u| ,$$

I_u would have to be contained in $I_{\mathrm{top}(\mathbf{T}'_j)}$. Since also $\omega_{\mathrm{top}(\mathbf{T}'_j)} \subseteq \omega_s \subseteq \omega_u$, it follows that $u < \mathrm{top}(\mathbf{T}'_j)$; thus u would belong to the tree $\mathbf{T}_j$ [which is the largest tree with top $\mathrm{top}(\mathbf{T}'_j)$], since this tree was selected first. But if u belonged to $\mathbf{T}_j$, then it could not belong to $\mathbf{T}'_k$, which is disjoint from $\mathbf{T}_j$; hence we get a contradiction. We conclude that $I_u \cap I_{\mathrm{top}(\mathbf{T}_j)} = \emptyset$.

Next assume that $u \in \mathbf{T}'_k$, $v \in \mathbf{T}'_l$, $u \neq v$, and that $\omega_s \subseteq \omega_{u(1)} \cap \omega_{v(1)}$ for some fixed $s \in \mathbf{T}'_j$. Since the left halves of two dyadic intervals ω_u and ω_v intersect, three things can happen: (a) $\omega_u \subseteq \omega_{v(1)}$, in which case I_v is disjoint from $I_{\mathrm{top}(\mathbf{T}'_k)}$ and thus from I_u; (b) $\omega_v \subseteq \omega_{u(1)}$, in which case I_u is disjoint from $I_{\mathrm{top}(\mathbf{T}'_l)}$ and thus from I_v; and (c) $\omega_u = \omega_v$, in which case $|I_u| = |I_v|$, and thus I_u and I_v are either disjoint or they coincide. Since $u \neq v$, it follows that I_u and I_v cannot coincide; thus $I_u \cap I_v = \emptyset$. This finishes the proof of the lemma. □

We now return to (11.1.47). In view of Lemma 11.1.11, different $u \in \mathbf{U}$ that appear in the interior sum in (11.1.47) have disjoint intervals I_u, and all of these are contained in $(I_{\mathrm{top}(\mathbf{T}_j)})^c$. Set $t_j = \mathrm{top}(\mathbf{T}_j)$. Using Exercise 11.1.4, we obtain

$$\begin{aligned}
&\sum_{s\in\mathbf{T}'_j}\Bigg(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subseteq\omega_{u(1)}}} |I_u|^{\frac12}\big|\big\langle \varphi_s \,|\, \varphi_u\big\rangle\big|\Bigg)^2\\
&\qquad\le C\sum_{s\in\mathbf{T}'_j}\Bigg(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subseteq\omega_{u(1)}}} |I_u|^{\frac12}\bigg(\frac{|I_s|}{|I_u|}\bigg)^{\frac12}\int_{I_u}\frac{|I_s|^{-1}\,dx}{\big(1+\frac{|x-c(I_s)|}{|I_s|}\big)^{20}}\Bigg)^2\\
&\qquad\le C\sum_{s\in\mathbf{T}'_j}|I_s|\Bigg(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subseteq\omega_{u(1)}}}\int_{I_u}\frac{|I_s|^{-1}\,dx}{\big(1+\frac{|x-c(I_s)|}{|I_s|}\big)^{20}}\Bigg)^2\\
&\qquad\le C\sum_{s\in\mathbf{T}'_j}|I_s|\Bigg(\int_{(I_{t_j})^c}\frac{|I_s|^{-1}\,dx}{\big(1+\frac{|x-c(I_s)|}{|I_s|}\big)^{20}}\Bigg)^2\\
&\qquad\le C\sum_{s\in\mathbf{T}'_j}|I_s|\int_{(I_{t_j})^c}\frac{|I_s|^{-1}\,dx}{\big(1+\frac{|x-c(I_s)|}{|I_s|}\big)^{20}}\,,
\end{aligned}$$

since $\int_{\mathbf{R}}(1+|x|)^{-20}\,dx \le 1$. For each scale $k \ge 0$ the sets I_s, $s \in \mathbf{T}'_j$, with $|I_s| = 2^{-k}|I_{t_j}|$ are pairwise disjoint and contained in I_t; therefore, we have

$$\begin{aligned}
\sum_{s\in\mathbf{T}'_j}|I_s|\int_{(I_{t_j})^c}\frac{|I_s|^{-1}\,dx}{\big(1+\frac{|x-c(I_s)|}{|I_s|}\big)^{20}} &\le \sum_{k=0}^{\infty}\frac{2^k}{|I_{t_j}|}\sum_{\substack{s\in\mathbf{T}'_j\\ |I_s|=2^{-k}|I_{t_j}|}}|I_s|\int_{(I_{t_j})^c}\frac{dx}{\big(1+\frac{|x-c(I_s)|}{|I_s|}\big)^{20}}\\
&\le C\sum_{k=0}^{\infty}\frac{2^k}{|I_{t_j}|}\sum_{\substack{s\in\mathbf{T}'_j\\ |I_s|=2^{-k}|I_{t_j}|}}\int_{I_s}\int_{(I_{t_j})^c}\frac{dx}{\big(1+\frac{|x-y|}{|I_s|}\big)^{20}}\,dy\\
&\le C\sum_{k=0}^{\infty}2^k|I_{t_j}|^{-1}\int_{I_{t_j}}\int_{(I_{t_j})^c}\frac{1}{\big(1+\frac{|x-y|}{2^{-k}|I_{t_j}|}\big)^{20}}\,dx\,dy\\
&\le C'\sum_{k=0}^{\infty}2^k|I_{t_j}|^{-1}(2^{-k}|I_{t_j}|)^2\\
&= C''|I_{t_j}|\,,
\end{aligned}$$

in view of Exercise 11.1.5. This completes the proof of (11.1.47) and thus of Lemma 11.1.9. □

11.1.7 Proof of the Basic Estimate Lemma 11.1.10

Proof. In the proof of the required estimate we may assume that $\|f\|_{L^2} = 1$, for we can always replace f by $f/\|f\|_{L^2}$. Throughout this subsection we fix a square-integrable function with L^2 norm 1, a tree $\mathbf{T}$ contained in $\mathbf{P}$, a measurable function $N: \mathbf{R} \to \mathbf{R}^+$, and a measurable set E with finite measure.

Let $\mathscr{J}'$ be the set of all dyadic intervals J such that $3J$ does not contain any I_s with $s \in \mathbf{T}$. It is not hard to see that any point in $\mathbf{R}$ belongs to a set in $\mathscr{J}'$. Let $\mathscr{J}$ be the set of all maximal (under inclusion) elements of $\mathscr{J}'$. Then $\mathscr{J}$ consists of disjoint sets that cover $\mathbf{R}$; thus it forms a partition of $\mathbf{R}$. This partition of $\mathbf{R}$ is shown in Figure 11.3 when the tree consists of two tiles.

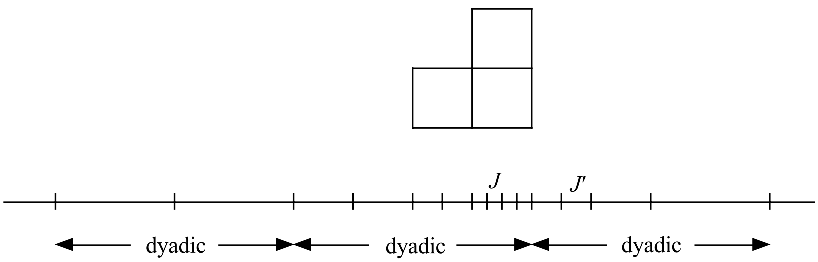

Fig. 11.3 A tree of two tiles and the partition $\mathscr{J}$ of $\mathbf{R}$ corresponding to it. The intervals J and J' are members of the partition $\mathscr{J}$.

For each $s \in \mathbf{T}$ pick an $\varepsilon_s \in \mathbf{C}$ with $|\varepsilon_s| = 1$ such that

$$\big|\big\langle f \,|\, \varphi_s\big\rangle\big\langle \chi_{E\cap N^{-1}[\omega_{s(2)}]} \,|\, \varphi_s\big\rangle\big| = \varepsilon_s\big\langle f \,|\, \varphi_s\big\rangle\big\langle \varphi_s \,|\, \chi_{E\cap N^{-1}[\omega_{s(2)}]}\big\rangle .$$

We can now write the left-hand side of (11.1.32) as

$$\begin{aligned}\sum_{s\in\mathbf{T}} \varepsilon_s\big\langle f \,|\, \varphi_s\big\rangle\big\langle \varphi_s \,|\, \chi_{E\cap N^{-1}[\omega_{s(2)}]}\big\rangle &\le \Big\|\sum_{s\in\mathbf{T}} \varepsilon_s\big\langle f \,|\, \varphi_s\big\rangle \chi_{E\cap N^{-1}[\omega_{s(2)}]}\varphi_s\Big\|_{L^1(\mathbf{R})} \\ &= \sum_{J\in\mathscr{J}} \Big\|\sum_{s\in\mathbf{T}} \varepsilon_s\big\langle f \,|\, \varphi_s\big\rangle \chi_{E\cap N^{-1}[\omega_{s(2)}]}\varphi_s\Big\|_{L^1(J)} \\ &\le \Sigma_1 + \Sigma_2 ,\end{aligned}$$

where

$$\Sigma_1 = \sum_{J\in\mathscr{J}} \Big\| \sum_{\substack{s\in\mathbf{T}\\ |I_s|\le 2|J|}} \varepsilon_s \langle f \,|\, \varphi_s\rangle \chi_{E\cap N^{-1}[\omega_{s(2)}]}\varphi_s \Big\|_{L^1(J)}, \tag{11.1.48}$$

$$\Sigma_2 = \sum_{J\in\mathscr{J}} \Big\| \sum_{\substack{s\in\mathbf{T}\\ |I_s|> 2|J|}} \varepsilon_s \langle f \,|\, \varphi_s\rangle \chi_{E\cap N^{-1}[\omega_{s(2)}]}\varphi_s \Big\|_{L^1(J)}. \tag{11.1.49}$$

We start with Σ_1. Observe that for every $s\in\mathbf{T}$, the singleton $\{s\}$ is a 2-tree contained in $\mathbf{T}$ and we therefore have the estimate

$$\big|\langle f \,|\, \varphi_s\rangle\big| \le |I_s|^{\frac{1}{2}}\mathscr{E}(f;\mathbf{T}). \tag{11.1.50}$$

Using this, we obtain

$$\begin{aligned}
\Sigma_1 &\le \sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbf{T}\\ |I_s|\le 2|J|}} \mathscr{E}(f;\mathbf{T}) \int_{J\cap E\cap N^{-1}[\omega_{s(2)}]} |I_s|^{\frac{1}{2}} |\varphi_s(x)|\,dx \\
&\le C\sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbf{T}\\ |I_s|\le 2|J|}} \mathscr{E}(f;\mathbf{T})|I_s| \int_{J\cap E\cap N^{-1}[\omega_{s(2)}]} \frac{|I_s|^{-1}}{\big(1+\frac{|x-c(I_s)|}{|I_s|}\big)^{20}}\,dx \\
&\le C\sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbf{T}\\ |I_s|\le 2|J|}} \mathscr{E}(f;\mathbf{T})\,|E|\mathscr{M}(E;\mathbf{T})|I_s| \sup_{x\in J} \frac{1}{\big(1+\frac{|x-c(I_s)|}{|I_s|}\big)^{10}} \\
&\le C\mathscr{E}(f;\mathbf{T})\,|E|\mathscr{M}(E;\mathbf{T}) \sum_{J\in\mathscr{J}} \sum_{k=-\infty}^{\log_2 2|J|} 2^k \sum_{\substack{s\in\mathbf{T}\\ |I_s|=2^k}} \frac{1}{\big(1+\frac{\operatorname{dist}(J,I_s)}{2^k}\big)^5} \frac{1}{\big(1+\frac{\operatorname{dist}(J,I_s)}{2^k}\big)^5}.
\end{aligned}$$

But note that all I_s with $s\in\mathbf{T}$ and $|I_s|=2^k$ are pairwise disjoint and contained in $I_{\mathrm{top}(\mathbf{T})}$. Therefore, $2^{-k}\operatorname{dist}(J,I_s)\ge |I_{\mathrm{top}(\mathbf{T})}|^{-1}\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})$, and we have the estimate

$$\left(1+\frac{\operatorname{dist}(J,I_s)}{2^k}\right)^{-5} \le \left(1+\frac{\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^{-5}.$$

Moreover, the sum

$$\sum_{\substack{s\in\mathbf{T}\\ |I_s|=2^k}} \frac{1}{\big(1+\frac{\operatorname{dist}(J,I_s)}{2^k}\big)^5} \tag{11.1.51}$$

is controlled by a finite constant, since for every nonnegative integer m there exist at most two tiles $s\in\mathbf{T}$ with $|I_s|=2^k$ such that I_s are not contained in $3J$ and $m2^k\le \operatorname{dist}(J,I_s)<(m+1)2^k$. Therefore, we obtain

$$\begin{aligned}
\Sigma_1 &\le C\mathscr{E}(f;\mathbf{T})\,|E|\mathscr{M}(E;\mathbf{T}) \sum_{J\in\mathscr{J}} \sum_{k=-\infty}^{\log_2 2|J|} \frac{2^k}{\left(1+\frac{\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^5} \\
&\le C\mathscr{E}(f;\mathbf{T})\,|E|\mathscr{M}(E;\mathbf{T}) \sum_{J\in\mathscr{J}} \frac{|J|}{\left(1+\frac{\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^5} \\
&\le C\mathscr{E}(f;\mathbf{T})\,|E|\mathscr{M}(E;\mathbf{T}) \sum_{J\in\mathscr{J}} \int_J \frac{1}{\left(1+\frac{|x-c(I_{\mathrm{top}(\mathbf{T})})|}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^5}\,dx \\
&\le C\,|I_{\mathrm{top}(\mathbf{T})}|\,\mathscr{E}(f;\mathbf{T})\,|E|\mathscr{M}(E;\mathbf{T}),
\end{aligned} \tag{11.1.52}$$

since $\mathscr{J}$ forms a partition of $\mathbf{R}$. We need to justify, however, the penultimate inequality in (11.1.52). Since J and $I_{\mathrm{top}(\mathbf{T})}$ are dyadic intervals, there are only two possibilities: (a) $J\cap I_{\mathrm{top}(\mathbf{T})}=\emptyset$ and (b) $J\subseteq I_{\mathrm{top}(\mathbf{T})}$. [The third possibility $I_{\mathrm{top}(\mathbf{T})}\subseteq J$ is excluded, since $3J$ does not contain $I_{\mathrm{top}(\mathbf{T})}$.] In case (a) we have $|J|\le \operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})$, since $3J$ does not contain $I_{\mathrm{top}(\mathbf{T})}$. In case (b) we have $|J|\le|I_{\mathrm{top}(\mathbf{T})}|$. Thus in both cases we have $|J|\le \operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})+|I_{\mathrm{top}(\mathbf{T})}|$. Consequently, for any $x\in J$ one has

$$\begin{aligned}
|x-c(I_{\mathrm{top}(\mathbf{T})})| &\le |J|+\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})+\frac{1}{2}|I_{\mathrm{top}(\mathbf{T})}| \\
&\le 2\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})+\frac{3}{2}|I_{\mathrm{top}(\mathbf{T})}|.
\end{aligned}$$

Therefore, it follows that

$$\int_J \frac{dx}{\left(1+\frac{|x-c(I_{\mathrm{top}(\mathbf{T})})|}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^5} \ge \frac{|J|}{\left(\frac{5}{2}+\frac{2\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^5} \ge \frac{\left(\frac{2}{5}\right)^5|J|}{\left(1+\frac{\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^5}.$$

In case (b) we have $J\subseteq I_{\mathrm{top}(\mathbf{T})}$, and therefore any point x in J lies in $I_{\mathrm{top}(\mathbf{T})}$; thus $|x-c(I_{\mathrm{top}(\mathbf{T})})|\le\frac{1}{2}|I_{\mathrm{top}(\mathbf{T})}|$. We conclude that

$$\int_J \frac{dx}{\left(1+\frac{|x-c(I_{\mathrm{top}(\mathbf{T})})|}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^5} \ge \frac{|J|}{(3/2)^5} = \left(\frac{2}{3}\right)^5 \frac{|J|}{\left(1+\frac{\operatorname{dist}(J,I_{\mathrm{top}(\mathbf{T})})}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^5}.$$

These observations justify the second-to-last inequality in (11.1.52) and complete the required estimate for Σ_1.

We now turn attention to Σ_2. We may assume that for all J appearing in the sum in (11.1.49), the set of s in $\mathbf{T}$ with $2|J|<|I_s|$ is nonempty. Thus, if J appears in the sum in (11.1.49), we have $2|J|<|I_{\mathrm{top}(\mathbf{T})}|$, and it is easy to see that J is contained in $3I_{\mathrm{top}(\mathbf{T})}$. [The intervals J in $\mathscr{J}$ that are not contained in $3I_{\mathrm{top}(\mathbf{T})}$ have size larger than $|I_{\mathrm{top}(\mathbf{T})}|$.]

We let $\mathbf{T}_2$ be the 2-tree of all s in $\mathbf{T}$ such that $\omega_{\mathrm{top}(\mathbf{T})(2)}\subseteq\omega_{s(2)}$, and we also let $\mathbf{T}_1=\mathbf{T}\setminus\mathbf{T}_2$. Then $\mathbf{T}_1$ is a 1-tree minus its top. We set

$$F_{1J} = \sum_{\substack{s\in\mathbf{T}_1\\ |I_s|>2|J|}} \varepsilon_s \left\langle f \,|\, \varphi_s \right\rangle \varphi_s \chi_{E\cap N^{-1}[\omega_{s(2)}]}\,,$$
$$F_{2J} = \sum_{\substack{s\in\mathbf{T}_2\\ |I_s|>2|J|}} \varepsilon_s \left\langle f \,|\, \varphi_s \right\rangle \varphi_s \chi_{E\cap N^{-1}[\omega_{s(2)}]}\,.$$

Clearly

$$\Sigma_2 \le \sum_{J\in\mathscr{J}} \big\|F_{1J}\big\|_{L^1(J)} + \sum_{J\in\mathscr{J}} \big\|F_{2J}\big\|_{L^1(J)} = \Sigma_{21} + \Sigma_{22}\,,$$

and we need to estimate both sums. We start by estimating F_{1J}. If the tiles s and s' that appear in the definition of F_{1J} have different scales, then the sets $\omega_{s(2)}$ and $\omega_{s'(2)}$ are disjoint and thus so are the sets $E\cap N^{-1}[\omega_{s(2)}]$ and $E\cap N^{-1}[\omega_{s'(2)}]$. Let us set

$$G_J = J \cap \bigcup_{\substack{s\in\mathbf{T}\\ |I_s|>2|J|}} E\cap N^{-1}[\omega_{s(2)}]\,.$$

Then F_{1J} is supported in the set G_J and we have

$$\begin{aligned}
\big\|F_{1J}\big\|_{L^1(J)} &\le \big\|F_{1J}\big\|_{L^\infty(J)}|G_J| \\
&= \bigg\| \sum_{k>\log_2 2|J|} \sum_{\substack{s\in\mathbf{T}_1\\ |I_s|=2^k}} \varepsilon_s \left\langle f \,|\, \varphi_s \right\rangle \varphi_s \chi_{E\cap N^{-1}[\omega_{s(2)}]} \bigg\|_{L^\infty(J)} |G_J| \\
&\le \sup_{k>\log_2 2|J|} \bigg\| \sum_{\substack{s\in\mathbf{T}_1\\ |I_s|=2^k}} \varepsilon_s \left\langle f \,|\, \varphi_s \right\rangle \varphi_s \chi_{E\cap N^{-1}[\omega_{s(2)}]} \bigg\|_{L^\infty(J)} |G_J| \\
&\le \sup_{k>\log_2 2|J|} \sup_{x\in J} \sum_{\substack{s\in\mathbf{T}_1\\ |I_s|=2^k}} \mathscr{E}(f;\mathbf{T}) 2^{k/2} \frac{2^{-k/2}}{\left(1+\frac{|x-c(I_s)|}{2^k}\right)^{10}} |G_J| \\
&\le C\mathscr{E}(f;\mathbf{T})|G_J|\,,
\end{aligned}$$

using (11.1.50) and the fact that all the I_s that appear in the sum are disjoint. We now claim that for all $J\in\mathscr{J}$ we have

$$|G_J| \le C\,|E|\mathscr{M}(E;\mathbf{T})|J|\,. \tag{11.1.53}$$

Once (11.1.53) is established, summing over all the intervals J that appear in the definition of F_{1J} and keeping in mind that all of these intervals are pairwise disjoint and contained in $3I_{\mathrm{top}(\mathbf{T})}$, we obtain the desired estimate for Σ_{21}.

To prove (11.1.53), we consider the unique dyadic interval $\widetilde{J}$ of length $2|J|$ that contains J. Then by the maximality of $\mathscr{J}$, $3\widetilde{J}$ contains the time interval I_{s_J} of a tile s_J in $\mathbf{T}$. We consider the following two cases: (a) If I_{s_J} is either $\big(\widetilde{J}-|\widetilde{J}|\big)\cup\widetilde{J}$ or $\widetilde{J}\cup\big(\widetilde{J}+|\widetilde{J}|\big)$, we let $u_J = s_J$; in this case $|I_{u_J}| = 2|\widetilde{J}|$. (This is the case for the interval J in Figure 11.3.) Otherwise, we have case (b), in which I_{s_J} is contained in one of the two dyadic intervals $\widetilde{J}-|\widetilde{J}|$, $\widetilde{J}+|\widetilde{J}|$. (This is the case for the interval J' in Figure 11.3.) Whichever of these two dyadic intervals contains I_{s_J} is also contained in $I_{\mathrm{top}(\mathbf{T})}$, since it intersects it and has smaller length than it. In case (b) there exists a tile $u_J \in \mathbf{D}$ with $|I_{u_J}| = |\widetilde{J}|$ such that $I_{s_J} \subseteq I_{u_J} \subseteq I_{\mathrm{top}(\mathbf{T})}$ and $\omega_{\mathrm{top}(\mathbf{T})} \subseteq \omega_{u_J} \subseteq \omega_{s_J}$. In both cases we have a tile u_J satisfying $s_J < u_J < \mathrm{top}(\mathbf{T})$ with $|\omega_{u_J}|$ being either $\frac{1}{4}|J|^{-1}$ or $\frac{1}{2}|J|^{-1}$.

Then for any $s \in \mathbf{T}$ with $|I_s| > 2|J|$ we have $|\omega_s| \le |\omega_{u_J}|$. But since both ω_s and ω_{u_J} contain $\omega_{\mathrm{top}(\mathbf{T})}$, they must intersect, and thus $\omega_s \subseteq \omega_{u_J}$. We conclude that any $s \in \mathbf{T}$ with $|I_s| > 2|J|$ must satisfy $N^{-1}[\omega_s] \subseteq N^{-1}[\omega_{u_J}]$. It follows that

$$G_J \subseteq J \cap E \cap N^{-1}[\omega_{u_J}] \tag{11.1.54}$$

and therefore we have

$$\begin{aligned} |E|\mathscr{M}(E;\mathbf{T}) &= \sup_{s\in\mathbf{T}} \sup_{\substack{u\in\mathbf{D}\\ s<u}} \int_{E\cap N^{-1}[\omega_u]} \frac{|I_u|^{-1}}{\left(1+\frac{|x-c(I_u)|}{|I_u|}\right)^{10}}\,dx \\ &\ge \int_{J\cap E\cap N^{-1}[\omega_{u_J}]} \frac{|I_{u_J}|^{-1}}{\left(1+\frac{|x-c(I_{u_J})|}{|I_{u_J}|}\right)^{10}}\,dx \\ &\ge c\,|I_{u_J}|^{-1}\,|J\cap E\cap N^{-1}[\omega_{u_J}]| \\ &\ge c\,|I_{u_J}|^{-1}\,|G_J|\,, \end{aligned}$$

using (11.1.54) and the fact that for $x \in J$ we have $|x-c(I_{u_J})| \le 4|J| = 2|I_{u_J}|$. It follows that

$$|G_J| \le \frac{1}{c}\,|E|\mathscr{M}(E;\mathbf{T})|I_{u_J}| = \frac{2}{c}\,|E|\mathscr{M}(E;\mathbf{T})|J|\,,$$

and this is exactly (11.1.53), which we wanted to prove.

We now turn to the estimate for $\Sigma_{22} = \sum_{J\in\mathscr{J}} \big\|F_{2J}\big\|_{L^1(J)}$. All the intervals $\omega_{s(2)}$ with $s \in \mathbf{T}_2$ are nested, since $\mathbf{T}_2$ is a 2-tree. Therefore, for each $x \in J$ for which $F_{2J}(x)$ is nonzero, there exists a largest dyadic interval ω_{u_x} and a smallest dyadic interval ω_{v_x} (for some $u_x, v_x \in \mathbf{T}_2 \cap \{s:\ |I_s| \ge 4|J|\}$) such that for $s \in \mathbf{T}_2 \cap \{s:\ |I_s| \ge 4|J|\}$ we have $N(x) \in \omega_{s(2)}$ if and only if $\omega_{v_x} \subseteq \omega_s \subseteq \omega_{u_x}$. Then we have

$$\begin{aligned} F_{2J}(x) &= \sum_{\substack{s\in\mathbf{T}_2\\ |I_s|\ge 4|J|}} \varepsilon_s \langle f\,|\,\varphi_s\rangle (\varphi_s \chi_{E\cap N^{-1}[\omega_{s(2)}]})(x) \\ &= \chi_E(x) \sum_{\substack{s\in\mathbf{T}_2\\ |\omega_{v_x}|\le|\omega_s|\le|\omega_{u_x}|}} \varepsilon_s \langle f\,|\,\varphi_s\rangle\, \varphi_s(x)\,. \end{aligned}$$

Pick a Schwartz function ψ whose Fourier transform $\widehat{\psi}(t)$ is supported in $|t| \le 1+\frac{1}{100}$ and that is equal to 1 on $|t| \le 1$. We can easily check that for all $z \in \mathbf{R}$, if $|\omega_{v_x}| \le |\omega_s| \le |\omega_{u_x}|$, then

$$\left(\varphi_s * \left\{\frac{M^{c(\omega_{u_x})}D^{|\omega_{u_x}|^{-1}}(\psi)}{|\omega_{u_x}|^{-\frac{1}{2}}} - \frac{M^{c(\omega_{v_x(2)})}D^{|\omega_{v_x(2)}|^{-1}}(\psi)}{|\omega_{v_x(2)}|^{-\frac{1}{2}}}\right\}\right)(z) = \varphi_s(z) \tag{11.1.55}$$

by a simple examination of the Fourier transforms. Basically, the Fourier transform (in z) of the function inside the curly brackets is equal to

$$\widehat{\psi}\Big(\tfrac{\xi - c(\omega_{u_x})}{|\omega_{u_x}|}\Big) - \widehat{\psi}\Big(\tfrac{\xi - c(\omega_{v_x(2)})}{|\omega_{v_x(2)}|}\Big),$$

which is equal to 1 on the support of $\widehat{\varphi_s}$ for all s in $\mathbf{T}_2$ that satisfy $|\omega_{v_x}| \le |\omega_s| \le |\omega_{u_x}|$ but vanishes on $\omega_{v_x(2)}$. Taking $z = x$ in (11.1.55) yields

$$\begin{aligned} F_{2J}(x) &= \sum_{\substack{s\in\mathbf{T}_2\\ |\omega_{v_x}|\le|\omega_s|\le|\omega_{u_x}|}} \varepsilon_s \langle f \,|\, \varphi_s\rangle\, \varphi_s(x)\chi_E(x) \\ &= \Big[\sum_{s\in\mathbf{T}_2} \varepsilon_s\langle f\,|\,\varphi_s\rangle\,\varphi_s\Big] * \left\{\frac{M^{c(\omega_{u_x})}D^{|\omega_{u_x}|^{-1}}(\psi)}{|\omega_{u_x}|^{-\frac{1}{2}}} - \frac{M^{c(\omega_{v_x(2)})}D^{|\omega_{v_x(2)}|^{-1}}(\psi)}{|\omega_{v_x(2)}|^{-\frac{1}{2}}}\right\}(x)\chi_E(x). \end{aligned}$$

Since all s that appear in the definition of F_{2J} satisfy $|\omega_s| \le (4|J|)^{-1}$, it follows that we have the estimate

$$\begin{aligned} |F_{2J}(x)| &\le 2\chi_E(x) \sup_{\delta > |\omega_{u_x}|^{-1}} \int_{\mathbf{R}} \Big|\sum_{s\in\mathbf{T}_2} \varepsilon_s\langle f\,|\,\varphi_s\rangle\varphi_s(z)\Big| \tfrac{1}{\delta}\Big|\psi\big(\tfrac{x-z}{\delta}\big)\Big|\,dz \\ &\le C \sup_{\delta>4|J|} \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} \Big|\sum_{s\in\mathbf{T}_2} \varepsilon_s\langle f\,|\,\varphi_s\rangle\varphi_s(z)\Big|\,dz. \end{aligned} \tag{11.1.56}$$

(The last inequality follows from Exercise 2.1.14.) Observe that the maximal function in (11.1.56) satisfies the property

$$\sup_{x\in J}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}|h(t)|\,dt \le 2\inf_{x\in J}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}|h(t)|\,dt.$$

Using this property, we obtain

$$\begin{aligned} \Sigma_{22} &\le \sum_{J\in\mathscr{J}} \big\|F_{2J}\big\|_{L^1(J)} \le \sum_{J\in\mathscr{J}} \big\|F_{2J}\big\|_{L^\infty(J)}|G_J| \\ &\le C \sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3I_{\mathrm{top}(\mathbf{T})}}} |E|\mathscr{M}(E;\mathbf{T})|J| \sup_{x\in J}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\Big|\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle f\,|\,\varphi_s\rangle\varphi_s(z)\Big|\,dz \\ &\le 2C|E|\mathscr{M}(E;\mathbf{T}) \sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3I_{\mathrm{top}(\mathbf{T})}}} \int_J \sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\Big|\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle f\,|\,\varphi_s\rangle\varphi_s(z)\Big|\,dz\,dx \\ &\le C|E|\mathscr{M}(E;\mathbf{T})\Big\|M\Big(\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle f\,|\,\varphi_s\rangle\varphi_s\Big)\Big\|_{L^1(3I_{\mathrm{top}(\mathbf{T})})}, \end{aligned}$$

where M is the Hardy–Littlewood maximal operator. Using the Cauchy–Schwarz inequality and the boundedness of M on $L^2(\mathbf{R})$, we obtain the following estimate:

$$\Sigma_{22} \le C|E|\mathscr{M}(E;\mathbf{T})\,|I_{\mathrm{top}(\mathbf{T})}|^{\frac{1}{2}}\Big\|\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle f\,|\,\varphi_s\rangle\varphi_s\Big\|_{L^2}.$$

Appealing to the result of Exercise 11.1.6(a), we deduce

$$\Big\|\sum_{s\in\mathbf{T}_2}\varepsilon_s\langle f\,|\,\varphi_s\rangle\varphi_s\Big\|_{L^2} \le C\Big(\sum_{s\in\mathbf{T}_2}\big|\varepsilon_s\langle f\,|\,\varphi_s\rangle\big|^2\Big)^{\frac{1}{2}} \le C'|I_{\mathrm{top}(\mathbf{T})}|^{\frac{1}{2}}\,\mathscr{E}(f;\mathbf{T}).$$

The first estimate was also shown in (11.1.43); the same argument applies here, and the presence of the ε_s's does not introduce any change. We conclude that

$$\Sigma_{22} \le C|E|\mathscr{M}(E;\mathbf{T})|I_{\mathrm{top}(\mathbf{T})}|\,\mathscr{E}(f;\mathbf{T}),$$

which is what we needed to prove. This completes the proof of Lemma 11.1.10. □

The proof of the theorem is now complete. □

Exercises

11.1.1. Show that for every f in the Schwartz class, $x,\xi\in\mathbf{R}$, and $\lambda\in[0,1]$, the function $(y,\eta)\mapsto B^m_{\xi,y,\eta,\lambda}(f)(x)$ is periodic in y with period $2^{m-\lambda}$ and periodic in η with period $2^{-m+\lambda}$.

11.1.2. Fix a function h in the Schwartz class, $\xi,y,\eta\in\mathbf{R}$, $s\in\mathbf{D}_m$, and $\lambda\in[0,1]$. Suppose that $2^{-\lambda}(\xi+\eta)\in\omega_{s(2)}$.
(a) Assume that $m\le 0$ and that $2^{-m}\ge 40|\xi|$. Show that for some C that does not depend on y, η, and λ we have

$$\begin{aligned}\big|\big\langle D^{2^\lambda}\tau^yM^\eta(h)\,|\,\varphi_s\big\rangle\big| &= \big|\big\langle h\,|\,M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\varphi_s)\big\rangle\big|\\ &\le C2^{\frac{m}{2}}\big\|\widehat{h}\big\|_{L^1((-\infty,-\frac{1}{40\cdot 2^m})\cup(\frac{1}{40\cdot 2^m},\infty))}.\end{aligned}$$

[*Hint:* Use Plancherel's theorem, noting that $\eta\ge 2^\lambda c(\omega_{s(1)})+\frac{9}{40}2^{-m}$.]
(b) Using the trivial fact that $\big|\big\langle D^{2^\lambda}\tau^yM^\eta(h)\,|\,\varphi_s\big\rangle\big|\le C\big\|h\big\|_{L^2}$, conclude that whenever $|m|$ is large with respect to ξ, we have

$$\chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\big|\big\langle D^{2^\lambda}\tau^yM^\eta(h)\,|\,\varphi_s\big\rangle\big|\le C_h\min(1,2^m),$$

where C_h may depend on h but is independent of y, η, and λ.

11.1.3. (a) Let g be a bounded periodic function on $\mathbf{R}$ with period κ. Show that

$$\lim_{K\to\infty}\frac{1}{2K}\int_{-K}^{K} g(t)\,dt \to \frac{1}{\kappa}\int_0^{\kappa} g(t)\,dt\,.$$

(b) Let g be a bounded periodic function on $\mathbf{R}^n$ that is periodic with period $(\kappa_1,\dots,\kappa_n)$. Show that

$$\lim_{K_1,\dots,K_n\to\infty}\frac{2^{-n}}{K_1\cdots K_n}\int_{-K_1}^{K_1}\cdots\int_{-K_n}^{K_n} g(t)\,dt = \frac{1}{\kappa_1\cdots\kappa_n}\int_0^{\kappa_1}\cdots\int_0^{\kappa_n} g(t)\,dt$$

11.1.4. Use the result in Appendix K.1 to obtain the size estimate

$$\big|\langle \varphi_s\,|\,\varphi_u\rangle\big| \le C_M \frac{\min\left(\frac{|I_s|}{|I_u|},\frac{|I_u|}{|I_s|}\right)^{\frac{1}{2}}}{\left(1+\frac{|c(I_s)-c(I_u)|}{\max(|I_s|,|I_u|)}\right)^M}$$

for every $M>0$. Conclude that if $|I_u|\le|I_s|$, then

$$\big|\langle \varphi_s\,|\,\varphi_u\rangle\big| \le C_M'\left(\frac{|I_s|}{|I_u|}\right)^{\frac{1}{2}}\int_{I_u}\frac{|I_s|^{-1}\,dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^M}\,.$$

[*Hint:* Use that

$$\left|\frac{|x-c(I_s)|}{|I_s|}-\frac{|c(I_u)-c(I_s)|}{|I_s|}\right|\le\frac{1}{2}$$

for all $x\in I_u$.]

11.1.5. Prove that there is a constant $C>0$ such that for any interval J and any $b>0$,

$$\int_J\int_{J^c}\frac{1}{\left(1+\frac{|x-y|}{b|J|}\right)^{20}}\,dx\,dy \le Cb^2|J|^2\,.$$

[*Hint:* Translate J to the interval $[-\frac{1}{2}|J|,\frac{1}{2}|J|]$ and change variables. The resulting integral can be computed explicitly.]

11.1.6. Let φ_s be as in (11.1.3). Let $\mathbf{T}_2$ be a 2-tree and $f\in L^2(\mathbf{R})$.
(a) Show that there is a constant C such that for all sequences of complex scalars $\{\lambda_s\}_{s\in\mathbf{T}_2}$ we have

$$\Big\|\sum_{s\in\mathbf{T}_2}\lambda_s\,\varphi_s\Big\|_{L^2(\mathbf{R})}\le C\Big(\sum_{s\in\mathbf{T}_2}|\lambda_s|^2\Big)^{\frac{1}{2}}.$$

(b) Use duality to conclude that

$$\sum_{s\in\mathbf{T}_2}\big|\langle f\,|\,\varphi_s\rangle\big|^2\le C^2\|f\|_{L^2}^2\,.$$

[*Hint:* To prove part (a) define $\mathscr{G}_m = \{s \in \mathbf{T}_2 : |I_s| = 2^m\}$. Then for $s \in \mathscr{G}_m$ and $s' \in \mathscr{G}_{m'}$, the functions φ_s and $\varphi_{s'}$ are orthogonal to each other, and it suffices to obtain the corresponding estimate when the summation is restricted to a given $\mathscr{G}_m$. But for s in $\mathscr{G}_m$, the intervals I_s are disjoint, and we may use the idea of the proof of Lemma 11.1.2. Use that $\sum_{u:\, \omega_u=\omega_s} \big|\big\langle \varphi_s \,|\, \varphi_u \big\rangle\big| \le C$ for every fixed s.]

11.1.7. Fix $A \ge 1$. Let $\mathbf{S}$ be a finite collection of dyadic tiles such that for all s_1, s_2 in $\mathbf{S}$ we have either $\omega_{s_1} \cap \omega_{s_2} = \emptyset$ or $AI_{s_1} \cap AI_{s_2} = \emptyset$. Let $N_{\mathbf{S}}$ be the *counting function* of $\mathbf{S}$, defined by

$$N_{\mathbf{S}} = \sup_{x\in\mathbf{R}} \#\{I_s : \ s \in \mathbf{S} \text{ and } x \in I_s\}.$$

(a) Show that for any $M > 0$ there exists a $C_M > 0$ such that for all $f \in L^2(\mathbf{R})$ we have

$$\sum_{s\in\mathbf{S}} \left| \left\langle f, |I_s|^{-\frac{1}{2}} \left(1 + \frac{\operatorname{dist}(\cdot, I_s)}{|I_s|}\right)^{-\frac{M}{2}} \right\rangle \right|^2 \le C_M N_{\mathbf{S}} \big\|f\big\|_{L^2}^2 .$$

(b) Let φ_s be as in (11.1.3). Show that for any $M > 0$ there exists a $C_M > 0$ such that for all finite sequences of scalars $\{a_s\}_{s\in\mathbf{S}}$ we have

$$\Big\| \sum_{s\in\mathbf{S}} a_s \varphi_s \Big\|_{L^2}^2 \le C_M (1 + A^{-M} N_{\mathbf{S}}) \sum_{s\in\mathbf{S}} |a_s|^2 .$$

(c) Conclude that for any $M > 0$ there exists a $C_M > 0$ such that for all $f \in L^2(\mathbf{R})$ we have

$$\sum_{s\in\mathbf{S}} \big|\big\langle f, \varphi_s \big\rangle\big|^2 \le C_M (1 + A^{-M} N_{\mathbf{S}}) \big\|f\big\|_{L^2}^2 .$$

[*Hint:* Use the idea of Lemma 11.1.2 to prove part (a) when $N_{\mathbf{S}} = 1$. Suppose now that $N_{\mathbf{S}} > 1$. Call an element $s \in \mathbf{S}$ h-maximal if the region in $\mathbf{R}^2$ that is directly horizontally above the tile s does not intersect any other tile $s' \in \mathbf{S}$. Let $\mathbf{S}_1$ be the set of all h-maximal tiles in $\mathbf{S}$. Then $N_{\mathbf{S}_1} = 1$; otherwise, some $x \in \mathbf{R}$ would belong to both I_s and $I_{s'}$ for $s \ne s' \in \mathbf{S}_1$, and thus the horizontal regions directly above s and s' would have to intersect, contradicting the h-maximality of $\mathbf{S}_1$. Now define $\mathbf{S}_2$ to be the set of all h-maximal tiles in $\mathbf{S} \setminus \mathbf{S}_1$. As before, we have $N_{\mathbf{S}_2} = 1$. Continue in this way and write $\mathbf{S}$ as a union of at most $N_{\mathbf{S}}$ families of tiles $\mathbf{S}_j$, each of which has the property $N_{\mathbf{S}_j} = 1$. Apply the result to each $\mathbf{S}_j$ and then sum over j. Part (b): observe that whenever $s_1, s_2 \in \mathbf{S}$ and $s_1 \ne s_2$ we must have either $\big\langle \varphi_{s_1}, \varphi_{s_2} \big\rangle = 0$ or $\operatorname{dist}(I_{s_1}, I_{s_2}) \ge (A-1)\max(|I_{s_1}|, |I_{s_2}|)$, which implies

$$\left(1 + \frac{\operatorname{dist}(I_{s_1}, I_{s_2})}{\max(|I_{s_1}|, |I_{s_2}|)}\right)^{-M} \le A^{-\frac{M}{2}} \left(1 + \frac{\operatorname{dist}(I_{s_1}, I_{s_2})}{\max(|I_{s_1}|, |I_{s_2}|)}\right)^{-\frac{M}{2}} .$$

Use this estimate to obtain

$$\Big\| \sum_{s\in\mathbf{S}} a_s \varphi_s \Big\|_{L^2}^2 \le \sum_{s\in\mathbf{S}} |a_s|^2 + \frac{C_M}{A^{\frac{M}{2}}} \left\| \sum_{s\in\mathbf{S}} \frac{|a_s|}{|I_s|^{\frac{1}{2}}} \left(1 + \frac{\operatorname{dist}(x, I_s)}{|I_s|}\right)^{-\frac{M}{2}} \right\|_{L^2}^2$$

by expanding the square on the left. The required estimate follows from the dual statement to part (a). Part (c) follows from part (b) by duality.]

11.1.8. Let φ_s be as in (11.1.3) and let $\mathbf{D}_m$ be the set of all dyadic tiles s with $|I_s| = 2^m$. Show that there is a constant C (independent of m) such that for square-integrable sequences of scalars $\{a_s\}_{s\in\mathbf{D}_m}$ we have

$$\Big\| \sum_{s\in\mathbf{D}_m} a_s \varphi_s \Big\|_{L^2}^2 \le C \sum_{s\in\mathbf{D}_m} |a_s|^2 .$$

Conclude from this that

$$\sum_{s\in\mathbf{D}_m} \big|\langle f, \varphi_s\rangle\big|^2 \le C\|f\|_{L^2}^2 .$$

11.1.9. Fix a Schwartz function φ whose Fourier transform is supported in the interval $[-\frac{3}{8}, \frac{3}{8}]$ and that satisfies

$$\sum_{l\in\mathbf{Z}} |\widehat{\varphi}(t + \tfrac{l}{2})|^2 = c_0$$

for all real numbers t. Define functions φ_s as follows. Fix an integer m and set

$$\varphi_s(x) = 2^{-\frac{m}{2}} \varphi(2^{-m}x - k) e^{2\pi i 2^{-m} x \frac{l}{2}}$$

whenever $s = [k2^m, (k+1)2^m) \times [l2^{-m}, (l+1)2^{-m})$ is a tile in $\mathbf{D}$. Prove that for all Schwartz functions f we have

$$\sum_{s\in\mathbf{D}_m} \langle f \,|\, \varphi_s\rangle \varphi_s = c_0 f .$$

Observe that m does not appear on the right of this identity.
[*Hint:* First prove that

$$\sum_{s\in\mathbf{D}_m} \varphi_s(x)\overline{\widehat{\varphi_s}(y)} = c_0\, e^{2\pi i x y}$$

using the Poisson summation formula.]

11.1.10. This is a continuous version of Exercise 11.1.9. Fix a Schwartz function φ on $\mathbf{R}^n$ and define a *continuous wave packet*

$$\varphi_{y,\xi}(x) = \varphi(x-y) e^{2\pi i \xi\cdot x} .$$

Prove that for all f Schwartz functions on $\mathbf{R}^n$, the following identity is valid:

$$\|\varphi\|_{L^2}^2 f(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \varphi_{y,\xi}(x)\langle f \,|\, \varphi_{y,\xi}\rangle \, dy\, d\xi .$$

[*Hint:* Prove first that $\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \varphi_{y,\xi}(x)\overline{\widehat{\varphi_{y,\xi}}(z)}\, dy\, d\xi = \|\varphi\|_{L^2}^2 e^{2\pi i x\cdot z}$.]

11.2 Distributional Estimates for the Carleson Operator

In this section we derive estimates for the distribution function of the Carleson operator acting on characteristic functions of measurable sets. These estimates imply, in particular, that the Carleson operator is bounded on $L^p(\mathbf{R})$ for $1 < p < \infty$. To achieve this we build on the time–frequency analysis approach developed in the previous section. Working with characteristic functions of measurable sets of finite measure is crucial in obtaining an improved energy estimate, which is the key to the proof. Later in this section we obtain weighted estimates for the Carleson operator $\mathscr{C}$. These estimates are reminiscent of the corresponding estimates for the maximal singular integrals we encountered in the previous chapter.

11.2.1 The Main Theorem and Preliminary Reductions

In the sequel we use the notation introduced in Section 11.1. The following is the main result of this section.

Theorem 11.2.1. *(a) There exist finite constants $C, \kappa > 0$ such that for any measurable subset F of the reals with finite measure we have*

$$\big|\{x\in\mathbf{R}: \mathscr{C}(\chi_F)(x) > \alpha\}\big| \le C|F| \begin{cases} \frac{1}{\alpha}\Big(1+\log\big(\frac{1}{\alpha}\big)\Big) & \text{when } 0<\alpha<1, \\ e^{-\kappa\alpha} & \text{when } \alpha \ge 1. \end{cases} \tag{11.2.1}$$

(b) For any $1<p<\infty$ there is a constant $C_p>0$ such that for all f in $L^p(\mathbf{R})$ we have the estimate

$$\big\|\mathscr{C}(f)\big\|_{L^p(\mathbf{R})} \le C_p \big\|f\big\|_{L^p(\mathbf{R})}. \tag{11.2.2}$$

Proof. Assuming statement (a), we obtain

$$\big\|\mathscr{C}(\chi_F)\big\|_{L^p}^p = p\int_0^\infty \big|\{\mathscr{C}(\chi_F)>\alpha\}\big|\lambda^{p-1}\,d\alpha \le pC^p|F|\int_0^\infty \varphi(\alpha)\alpha^{p-1}\,d\alpha,$$

where $\varphi(\alpha)=\alpha^{-1}(1+\log(\alpha)^{-1})$ for $\alpha<1$ and $\varphi(\alpha)=e^{-\kappa\alpha}$ for $\alpha\ge 1$. The last integral is convergent, and consequently one obtains a restricted strong type (p,p) estimate

$$\big\|\mathscr{C}(\chi_F)\big\|_{L^p(\mathbf{R})} \le C_p'|F|^{\frac{1}{p}}$$

for the Carleson operator. The required strong type (p,p) estimate follows by applying Theorem 1.4.19. Thus (a) implies (b).

It remains to prove (a). This follows from the corresponding estimate for $\mathscr{C}_1$ and requires a considerable amount of work. The proof of (a) is based on a modification of the proof of Theorem 11.1.1. Recall that in (11.1.21) we identified the one-sided Carleson operator $\mathscr{C}_1(f)$ with

$$\mathscr{C}_1(f)(x) = \sup_{N>0}\left|\int_{-\infty}^{N} \widehat{f}(\eta)e^{2\pi i x\cdot\eta}\,d\eta\right| = \frac{1}{|c|}\sup_{\xi>0}|\Pi_\xi(f)|, \tag{11.2.3}$$

where $c \neq 0$ and Π_ξ, $\xi \in \mathbf{R}$ is given by

$$\Pi_\xi(f) = \lim_{\substack{K\to\infty\\L\to\infty}} \frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_{0}^{1} G_{\xi,y,\eta,\lambda}(f)\,d\lambda\,dy\,d\eta\,. \tag{11.2.4}$$

Also recall that $G_{\xi,y,\eta,\lambda}(f)$ is

$$G_{\xi,y,\eta,\lambda}(f) = M^{-\eta}\tau^{-y}D^{2^{-\lambda}}A_{\frac{\xi+\eta}{2^\lambda}}D^{2^\lambda}\tau^{y}M^{\eta}(f), \tag{11.2.5}$$

where A_ξ is defined in (11.1.6). Note that

$$\begin{aligned} G_{\xi,y,\eta,\lambda}(f)(x) &= \sum_{\substack{s\in\mathbf{D}\\ \xi\in\omega_{u(2)}}} \left\langle f\,|\,M^{-\eta}\tau^{-y}D^{2^{-\lambda}}\varphi_u\right\rangle M^{-\eta}\tau^{-y}D^{2^{-\lambda}}\varphi_u(x)\\ &= \sum_{\substack{s\in\mathbf{D}_{y,\eta,\lambda}\\ \xi\in\omega_{s(2)}}} \left\langle f\,|\,\varphi_s\right\rangle \varphi_s(x), \end{aligned}$$

where $\mathbf{D}_{y,\eta,\lambda}$ is the set of all rectangles of the form $(2^\lambda\otimes I_u - y)\times(2^{-\lambda}\otimes\omega_u-\eta)$, where u ranges over $\mathbf{D}$. Here $a\otimes I$ denotes the set $\{ax:\ x\in I\}$. For such s, φ_s is defined in (11.1.3). The rectangles in $\mathbf{D}_{y,\eta,\lambda}$ are formed by dilating the dyadic tiles in $\mathbf{D}$ by the amount 2^λ in the time coordinate axis and by $2^{-\lambda}$ in the frequency coordinate axis and then translating them by the amounts y and η, respectively.

In view of identity (11.1.12), for a Schwartz function f we have

$$|\Pi_\xi(f)(x)| = \left|\lim_{\substack{K\to\infty\\L\to\infty}}\frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_{0}^{1}\sum_{\substack{s\in\mathbf{D}_{y,\eta,\lambda}\\ \xi\in\omega_{s(2)}}}\left\langle f\,|\,\varphi_s\right\rangle\varphi_s(x)\,d\lambda\,dy\,d\eta\right|.$$

Since both terms of this identity are well defined L^2-bounded operators, (11.2.1) is also valid for L^2 functions f. For such functions f, for a measurable function $N:\ \mathbf{R}\to\mathbf{R}^+$, $y,\eta\in\mathbf{R}$, and $\lambda\in[0,1]$ we define operators

$$\mathfrak{D}_{N,y,\eta,\lambda}(f) = \sum_{s\in\mathbf{D}_{y,\eta,\lambda}}\left\langle f\,|\,\varphi_s\right\rangle(\chi_{\omega_{s(2)}}\circ N)\,\varphi_s$$

and

$$\mathfrak{D}_N(f) = \lim_{\substack{K\to\infty\\L\to\infty}}\frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_{0}^{1}\sum_{s\in\mathbf{D}_{y,\eta,\lambda}}\left\langle f\,|\,\varphi_s\right\rangle(\chi_{\omega_{s(2)}}\circ N)\,\varphi_s\,d\lambda\,dy\,d\eta\,.$$

For every square-integrable function f and $x \in \mathbf{R}$ we pick, in a measurable way, a positive real number $\xi = N_f(x)$ such that

$$\sup_{\xi>0} |\Pi_\xi(f)(x)| \le 2\,|\Pi_{N_f(x)}(f)(x)| \le 2\,\mathfrak{D}_{N_f}(f)(x)\,.$$

Then

$$\mathscr{C}_1(f) \le \frac{2}{|c|}\,|\mathfrak{D}_{N_f}(f)|\,. \tag{11.2.6}$$

We work with functions $f = \chi_F$, where F is a measurable set of finite measure; certainly such functions are square-integrable. We show the validity of statement (a) of Theorem 11.2.1 for $\mathfrak{D}_N$, where $N:\ \mathbf{R} \to \mathbf{R}^+$ is measurable with bounds independent of N. Then (11.2.6) implies the same statement for $\mathscr{C}_1$.

We claim that the following estimate is valid for $\mathfrak{D}_N$. There is a constant C' such that for any pair of measurable subsets (E,F) of the real line with nonzero finite measure there is a subset E' of E with $|E'| \ge \frac{1}{2}|E|$ such that for any measurable function $N:\ \mathbf{R} \to \mathbf{R}^+$ we have

$$\left| \int_{E'} \mathfrak{D}_N(\chi_F)(x)\,dx \right| \le 2C' \min(|E|,|F|) \left(1 + \left|\log \frac{|E|}{|F|}\right|\right). \tag{11.2.7}$$

This is a fundamental estimate that implies (11.2.1). We derive this estimate from an analogous estimate for the operators $\mathfrak{D}_{N,y,\eta,\lambda}$ by picking a set E' that is independent of y, η, and λ.

We introduce a set

$$\Omega_{E,F} = \left\{ M(\chi_F) > 8 \min\left(1, \tfrac{|F|}{|E|}\right) \right\}.$$

It follows that $|\Omega_{E,F}| \le \frac{1}{2}|E|$, since the Hardy–Littlewood maximal operator is of weak type $(1,1)$ with norm 2. We conclude that the set

$$E' = E \setminus \Omega_{E,F}$$

satisfies $|E'| \ge \frac{1}{2}|E|$. (Notice that in the case $|F| \ge |E|$ the set $\Omega_{E,F}$ is empty.)

Let $\mathbf{P}$ be a finite subset of $\mathbf{D}_{y,\eta,\lambda}$. The required inequality (11.2.7) will be a consequence of the following two estimates:

$$\left| \int_{E'} \sum_{\substack{s\in\mathbf{P} \\ I_s \subseteq \Omega_{E,F}}} \langle \chi_F \,|\, \varphi_s \rangle (\chi_{\omega_{s(2)}} \circ N)\, \varphi_s \,dx \right| \le C' \min(|E|,|F|) \tag{11.2.8}$$

and

$$\left| \int_{E'} \sum_{\substack{s\in\mathbf{P} \\ I_s \nsubseteq \Omega_{E,F}}} \langle \chi_F \,|\, \varphi_s \rangle (\chi_{\omega_{s(2)}} \circ N)\varphi_s \,dx \right| \le C' \min(|E|,|F|) \left(1 + \left|\log \frac{|E|}{|F|}\right|\right), \tag{11.2.9}$$

where the constant C' is independent of the sets E,F, of the measurable function N, and of the finite subset $\mathbf{P}$ of $\mathbf{D}_{y,\eta,\lambda}$. Estimates (11.2.8) and (11.2.9) are proved in the next three subsections.

In the rest of this subsection we show that (11.2.7) implies statement (a) of Theorem 11.2.1. Given $\alpha > 0$ we define sets

$$\begin{aligned} E_\alpha^1 &= \{\operatorname{Re}\mathfrak{D}_N(\chi_F) > \alpha\}, \qquad & E_\alpha^2 &= \{\operatorname{Re}\mathfrak{D}_N(\chi_F) < -\alpha\}, \\ E_\alpha^3 &= \{\operatorname{Im}\mathfrak{D}_N(\chi_F) > \alpha\}, \qquad & E_\alpha^4 &= \{\operatorname{Im}\mathfrak{D}_N(\chi_F) < -\alpha\}. \end{aligned}$$

We apply (11.2.7) to the pair (E_α^j, F) for any $j = 1,2,3,4$. We find a subset $(E_\alpha^j)'$ of E_α^j of at least half its measure so that (11.2.7) holds for this pair. Then we have

$$\begin{aligned} \frac{\alpha}{2}|E_\alpha^j| \le \alpha|(E_\alpha^j)'| &\le \left|\int_{(E_\alpha^j)'} \mathfrak{D}_N(\chi_F)(x)\,dx\right| \\ &\le 2C' \min(|E_\alpha^j|, |F|)\left(1 + \left|\log\frac{|E_\alpha^j|}{|F|}\right|\right). \end{aligned} \tag{11.2.10}$$

If $|E_\alpha^j| \le |F|$, this estimate implies that

$$|E_\alpha^j| \le |F| e\, e^{-\frac{1}{4C'}\alpha}, \tag{11.2.11}$$

while if $|E_\alpha^j| > |F|$, it implies that

$$\alpha \le 4C' \frac{|F|}{|E_\alpha^j|}\left(1 + \log\frac{|E_\alpha^j|}{|F|}\right). \tag{11.2.12}$$

Case 1: $\alpha > 4C'$. If $|E_\alpha^j| > |F|$, setting $t = |E_\alpha^j|/|F| > 1$ and using the fact that $\sup_{1<t<\infty} \frac{1}{t}(1+\log t) = 1$, we obtain that (11.2.12) fails. In this case we must therefore have that $|E_\alpha^j| \le |F|$. Applying (11.2.11) four times, we deduce

$$|\{\mathfrak{D}_N(\chi_F) > 4\alpha\}| \le 4e\,|F|\,e^{-\frac{1}{4C'}\alpha}. \tag{11.2.13}$$

Case 2: $\alpha \le 4C'$. If $|E_\alpha^j| > |F|$, we use the elementary fact that if $t > 1$ satisfies $t(1+\log t)^{-1} < \frac{B}{\alpha}$, then $t < \frac{2B}{\alpha}(1+\log\frac{2B}{\alpha})$; to prove this fact one may use the inequalities $t < \frac{2B}{\alpha}(1+\log\sqrt{t})$ and $\log\sqrt{t} \le \log t - \log(1+\log\sqrt{t}) \le \log\frac{2B}{\alpha}$ for $t > 1$. Taking t = $|E_\alpha^j|/|F|$ and $B = 4C'$ in (11.2.12) yields

$$\frac{|E_\alpha^j|}{|F|} \le \frac{8C'}{\alpha}\left(1 + \log\frac{8C'}{\alpha}\right). \tag{11.2.14}$$

If $|E_\alpha^j| \le |F|$, then we use (11.2.11), but we note that for some constant $c' > 1$ we have

$$e\,e^{-\frac{1}{4C'}\alpha} \le c'\frac{8C'}{\alpha}\left(1 + \log\frac{8C'}{\alpha}\right)$$

whenever $\alpha \le 4C'$. Thus, when $\alpha \le 4C'$ we always have

$$|\{\mathfrak{D}_N(\chi_F) > 4\alpha\}| \le c' \frac{32C'}{\alpha} |F| \left(1 + \log \frac{8C'}{\alpha}\right). \tag{11.2.15}$$

Combining (11.2.13) and (11.2.15), we obtain estimate (11.2.7) for $\mathfrak{D}_N$. The same estimate holds for $\mathscr{C}_1$ in view of (11.2.6). Since $\widetilde{\mathscr{C}_2(f)} = \mathscr{C}_1(\widetilde{f})$, where $\widetilde{f}(x) = f(-x)$, the same estimate holds for $\mathscr{C}_2$ and hence estimate (11.2.7) is valid for $\mathscr{C}$. □

11.2.2 The Proof of Estimate (11.2.8)

In proving (11.2.8), we may assume that $|F| \le |E|$; otherwise, the set $\Omega_{E,F}$ is empty and there is nothing to prove.

Let $\mathbf{P}$ be a finite subset of $\mathbf{D}_{y,\eta,\lambda}$. We denote by $\mathscr{I}(\mathbf{P})$ the grid that consists of all the time projections I_s of tiles s in $\mathbf{P}$. For a fixed interval J in $\mathscr{I}(\mathbf{P})$ we define

$$\mathbf{P}(J) = \{s \in \mathbf{P} : \ I_s = J\}$$

and a function

$$\psi_J(x) = |J|^{-\frac{1}{2}} \left(1 + \frac{|x - c(J)|}{|J|}\right)^{-M},$$

where M is a large integer to be chosen momentarily. We note that for each $s \in \mathbf{P}(J)$ we have $|\varphi_s(x)| \le C_M\, \psi_J(x)$.

For each $k = 0, 1, 2, \dots$ we introduce families

$$\mathscr{F}_k = \big\{J \in \mathscr{I}(\mathbf{P}) : \ 2^k J \subseteq \Omega_{E,F},\ 2^{k+1} J \nsubseteq \Omega_{E,F}\big\}.$$

We begin by writing the left-hand side of (11.2.8) as

$$\begin{aligned}
&\sum_{\substack{J \in \mathscr{I}(\mathbf{P}) \\ J \subseteq \Omega_{E,F}}} \bigg| \sum_{s \in \mathbf{P}(J)} \int_{E'} \langle \chi_F \,|\, \varphi_s \rangle\, \chi_{\omega_{s(2)}}(N(x))\, \varphi_s(x)\, dx \bigg| \\
&\quad = \sum_{k=0}^{\infty} \sum_{\substack{J \in \mathscr{I}(\mathbf{P}) \\ J \in \mathscr{F}_k}} \bigg| \int_{E'} \sum_{s \in \mathbf{P}(J)} \langle \chi_F \,|\, \varphi_s \rangle\, \chi_{\omega_{s(2)}}(N(x))\, \varphi_s(x)\, dx \bigg| .
\end{aligned} \tag{11.2.16}$$

Using Exercise 9.2.8(b) we obtain the existence of a constant $C_0 < \infty$ such that for each $k = 0, 1, \dots$ and $J \in \mathscr{F}_k$ we have

$$\begin{aligned}\langle \chi_F, \psi_J \rangle &\le |J|^{\frac{1}{2}} \inf_J M(\chi_F) \\ &\le |J|^{\frac{1}{2}} C_0^k \inf_{2^{k+1}J} M(\chi_F) \\ &\le 4C_0^k |J|^{\frac{1}{2}} \frac{|F|}{|E|},\end{aligned} \tag{11.2.17}$$

since $2^{k+1}J$ meets the complement of $\Omega_{E,F}$.

For $J \in \mathscr{F}_k$ we also have that $E' \cap 2^k J = \emptyset$ and hence

$$\int_{E'} \psi_J(y)\,dy \le \int_{(2^kJ)^c} \psi_J(y)\,dy \le |J|^{\frac{1}{2}} C_M 2^{-kM}. \tag{11.2.18}$$

Next we note that for each $J \in \mathscr{I}(\mathbf{P})$ and $x \in \mathbf{R}$ there is at most one $s = s_x \in \mathbf{P}(J)$ such that $N(x) \in \omega_{s_x(2)}$. Using this observation along with (11.2.17) and (11.2.18), we can therefore estimate the expression on the right in (11.2.16) as follows:

$$\begin{aligned}&\sum_{k=0}^{\infty} \sum_{\substack{J\in\mathscr{I}(\mathbf{P})\\ J\in\mathscr{F}_k}} \left| \int_{E'} \langle \chi_F \mid \varphi_{s_x} \rangle \chi_{\omega_{s_x(2)}}(N(x)) \varphi_{s_x}(x)\,dx \right| \\ &\qquad \le C_M^2 \sum_{k=0}^{\infty} \sum_{\substack{J\in\mathscr{I}(\mathbf{P})\\ J\in\mathscr{F}_k}} \int_{E'} \langle \chi_F, \psi_J \rangle \psi_J(x)\,dx \\ &\qquad \le C_M^2\, 4\frac{|F|}{|E|} \sum_{k=0}^{\infty} C_0^k \sum_{J\in\mathscr{F}_k} |J|^{\frac{1}{2}} \int_{E'} \psi_J(x)\,dx \\ &\qquad \le 4C_M^3 \frac{|F|}{|E|} \sum_{k=0}^{\infty} (C_0 2^{-M})^k \sum_{J\in\mathscr{F}_k} |J|,\end{aligned} \tag{11.2.19}$$

and we pick $M > \log C_0 / \log 2$. It remains to control

$$\sum_{J\in\mathscr{F}_k} |J|$$

for each nonnegative integer k. In doing this we let $\mathscr{F}_k^*$ be all elements of $\mathscr{F}_k$ that are maximal under inclusion. Then we observe that if $J \in \mathscr{F}_k^*$ and $J' \in \mathscr{F}_k$ satisfy $J' \subseteq J$ then

$$\operatorname{dist}(J', J^c) = 0,$$

otherwise $2J'$ would be contained in J and thus

$$2^{k+1}J' \subseteq 2^k J \subseteq \Omega_{E,F}.$$

Therefore, for any J in $\mathscr{F}_k^*$ and any scale m there are at most two intervals J' from $\mathscr{F}_k$ contained in J with $|J'| = 2^m$. Summing over all possible scales, we obtain a bound of at most four times the length of J. We conclude that

$$\sum_{J\in\mathscr{F}_k}|J| = \sum_{J\in\mathscr{F}_k^*}\sum_{\substack{J'\in\mathscr{F}_k\\ J'\subseteq J}}|J'| \le \sum_{J\in\mathscr{F}_k^*}4\,|J| \le 4\,|\Omega_{E,F}|\,,$$

since elements of $\mathscr{F}_k^*$ are disjoint and contained in $\Omega_{E,F}$. Inserting this estimate in (11.2.19), we obtain the required bound

$$C'_M\frac{|F|}{|E|}\,|\Omega_{E,F}| \le C''_M\,|F| = C''_M\,\min(|E|,|F|)$$

for the expression on the right in (11.2.16). This concludes the proof of (11.2.8).

11.2.3 The Proof of Estimate (11.2.9)

For fixed y,η,λ we define a partial order in the set of tiles in $\mathbf{D}_{y,\eta,\lambda}$ just as in Definition 11.1.3. All properties of dyadic tiles obtained in the previous section also hold for the tiles in $\mathbf{D}_{y,\eta,\lambda}$. Throughout this section, $\mathbf{P}$ is a finite subset of $\mathbf{D}_{y,\eta,\lambda}$.

To simplify notation, in the sequel we set

$$\mathbf{P}_{E,F} = \left\{s\in\mathbf{P}:\ I_s \nsubseteq \Omega_{E,F}\right\}.$$

Setting $N^{-1}[A] = \{x:\ N(x)\in A\}$ for a set $A\subseteq\mathbf{R}$, we note that (11.2.9) is a consequence of

$$\sum_{s\in\mathbf{P}_{E,F}}\left|\left\langle\chi_F,\varphi_s\right\rangle\left\langle\chi_{E'\cap N^{-1}[\omega_{s(2)}]},\varphi_s\right\rangle\right| \le C\min(|E|,|F|)\left(1+\left|\log\frac{|E|}{|F|}\right|\right). \tag{11.2.20}$$

The following lemma is the main ingredient of the proof and is proved in the next section.

Lemma 11.2.2. *There is a constant C such that for all measurable sets E and F of finite measure we have*

$$\mathscr{E}\left(\chi_F;\mathbf{P}_{E,F}\right) \le C\,|F|^{-\frac{1}{2}}\min\left(\frac{|F|}{|E|},1\right). \tag{11.2.21}$$

Assuming Lemma 11.2.2, we argue as follows to prove (11.2.9). Given the finite set of tiles $\mathbf{P}_{E,F}$, we write it as the union

$$\mathbf{P}_{E,F} = \bigcup_{j=-\infty}^{n_0}\mathbf{P}_j\,,$$

where the sets $\mathbf{P}_j$ satisfy properties (1)–(5) of page 437.

Given the sequence of sets $\mathbf{P}_j$, we use properties (1), (2), (5) on page 437, the observation that the mass is always bounded by $|E|^{-1}$, and Lemmas 11.2.2 and

11.1.10 to obtain the following bound for the expression on the left in (11.2.9):

$$\begin{aligned}
&\sum_{s\in\mathbf{P}_{E,F}} \left|\left\langle \chi_F \,|\, \varphi_s\right\rangle\right| \left|\left\langle \chi_{E'\cap N^{-1}[\omega_{s(2)}]}, \varphi_s\right\rangle\right| \\
&\quad = \sum_{j\in\mathbf{Z}} \sum_{s\in\mathbf{P}_j} \left|\left\langle \chi_F \,|\, \varphi_s\right\rangle\right| \left|\left\langle \chi_{E'\cap N^{-1}[\omega_{s(2)}]}, \varphi_s\right\rangle\right| \\
&\quad \le \sum_{j\in\mathbf{Z}} \sum_{k} \sum_{s\in\mathbf{T}_{jk}} \left|\left\langle \chi_F \,|\, \varphi_s\right\rangle\right| \left|\left\langle \chi_{E'\cap N^{-1}[\omega_{s(2)}]}, \varphi_s\right\rangle\right| \\
&\quad \le C_3 \sum_{j} \sum_{k} |I_{\mathrm{top}(\mathbf{T}_{jk})}|\, \mathscr{E}(f;\mathbf{T}_{jk})\, \mathscr{M}(E',\mathbf{T}_{jk})\, |E'|\, |F|^{\frac{1}{2}} \\
&\quad \le C_3 \sum_{j\in\mathbf{Z}} \sum_{k} |I_{\mathrm{top}(\mathbf{T}_{jk})}| \min\Big(2^{j+1}, C\frac{|F|^{\frac{1}{2}}}{|E|}, C|F|^{-\frac{1}{2}}\Big) \min(|E'|^{-1}, 2^{2j+2})\, |E|\, |F|^{\frac{1}{2}} \\
&\quad \le C_4 \sum_{j\in\mathbf{Z}} 2^{-2j} \min\big(2^j, |F|^{\frac{1}{2}}|E|^{-1}, |F|^{-\frac{1}{2}}\big) \min(|E|^{-1}, 2^{2j})\, |E|\, |F|^{\frac{1}{2}} \\
&\quad \le C_5 \sum_{j\in\mathbf{Z}} \min\left(2^j|E|^{\frac{1}{2}}, \min\Big(\frac{|F|}{|E|}, \frac{|E|}{|F|}\Big)^{\frac{1}{2}}\right) \min\big((2^j|E|^{\frac{1}{2}})^{-2}, 1\big)\, |E|^{\frac{1}{2}}\, |F|^{\frac{1}{2}} \\
&\quad \le C_6 \sum_{j\in\mathbf{Z}} \min\left(2^j, \min\Big(\frac{|F|}{|E|}, \frac{|E|}{|F|}\Big)^{\frac{1}{2}}\right) \min(2^{-2j}, 1)\, |E|^{\frac{1}{2}}\, |F|^{\frac{1}{2}} \\
&\quad \le C_7 \min(|E|, |F|)\left(1 + \left|\log\frac{|E|}{|F|}\right|\right).
\end{aligned}$$

The last estimate follows by a simple calculation considering the three cases $1 < 2^j$, $\min\left(\frac{|F|}{|E|}, \frac{|E|}{|F|}\right)^{\frac{1}{2}} \le 2^j \le 1$, and $2^j < \min\left(\frac{|F|}{|E|}, \frac{|E|}{|F|}\right)^{\frac{1}{2}}$.

11.2.4 The Proof of Lemma 11.2.2

It remains to prove Lemma 11.2.2.

Fix a 2-tree $\mathbf{T}$ contained in $\mathbf{P}_{E,F}$ and let $t = \mathrm{top}(\mathbf{T})$ denote its top. We show that

$$\frac{1}{|I_t|} \sum_{s\in\mathbf{T}} \left|\left\langle \chi_F \,|\, \varphi_s\right\rangle\right|^2 \le C \min\left(\frac{|F|}{|E|}, 1\right)^2 \tag{11.2.22}$$

for some constant C independent of F, E, and $\mathbf{T}$. Then (11.2.21) follows from (11.2.22) by taking the supremum over all 2-trees $\mathbf{T}$ contained in $\mathbf{P}_{E,F}$.

We decompose the function χ_F as follows:

$$\chi_F = \chi_{F\cap 3I_t} + \chi_{F\cap(3I_t)^c}\,.$$

We begin by observing that for s in $\mathbf{P}_{E,F}$ we have

$$\begin{aligned}
\big|\big\langle \chi_{F\cap(3I_t)^c}\,|\,\varphi_s\big\rangle\big| &\le \frac{C_M|I_s|^{\frac{1}{2}}\inf\limits_{I_s} M(\chi_F)}{\Big(1+\dfrac{\operatorname{dist}((3I_t)^c,c(I_s))}{|I_s|}\Big)^M} \\
&\le 8\,C_M|I_s|^{\frac{1}{2}}\min\Big(\frac{|F|}{|E|},1\Big)\Big(\frac{|I_s|}{|I_t|}\Big)^M,
\end{aligned}$$

since I_s meets the complement of $\Omega_{E,F}$ for every $s\in\mathbf{P}_F$. Square this inequality and sum over all s in $\mathbf{T}$ to obtain

$$\sum_{s\in\mathbf{T}}|\langle\chi_{F\cap(3I_t)^c}\,|\,\varphi_s\rangle|^2\le C\,|I_t|\,\min\Big(\frac{|F|}{|E|},1\Big)^2,$$

using Exercise 11.2.1.

We now turn to the corresponding estimate for the function $\chi_{F\cap 3I_t}$. At this point it is convenient to distinguish the simple case $|F|>|E|$ from the difficult case $|F|\le|E|$. In the first case the set $\Omega_{E,F}$ is empty and Exercise 11.1.6(b) yields

$$\begin{aligned}
\sum_{s\in\mathbf{T}}\big|\big\langle\chi_{F\cap 3I_t}\,|\,\varphi_s\big\rangle\big|^2 &\le C\big\|\chi_{F\cap 3I_t}\big\|_{L^2}^2 \\
&\le C\,|I_t| \\
&= C\,|I_t|\,\min\Big(\frac{|F|}{|E|},1\Big)^2,
\end{aligned}$$

since $|F|>|E|$.

We may therefore concentrate on the case $|F|\le|E|$. In proving (11.2.21) we may assume that there exists a point $x_0\in I_t$ such that

$$M(\chi_F)(x_0)\le 8\,\frac{|F|}{|E|};$$

otherwise there is nothing to prove.

We write the set $\Omega_{E,F}=\big\{M(\chi_F)>8\,\frac{|F|}{|E|}\big\}$ as a disjoint union of dyadic intervals J'_ℓ such that the dyadic parent $\widetilde{J'_\ell}$ of J'_ℓ is not contained in $\Omega_{E,F}$ and therefore

$$|F\cap J'_\ell|\le|F\cap\widetilde{J'_\ell}|\le 16\,\frac{|F|}{|E|}\,|J'_\ell|.$$

Now some of these dyadic intervals may have size larger than or equal to $|I_t|$. Let J'_ℓ be such an interval. Then we split J'_ℓ into $\frac{|J'_\ell|}{|I_t|}$ intervals $J'_{\ell,m}$ each of size exactly $|I_t|$. Since there is an $x_0\in I_t$ with

$$M(\chi_F)(x_0)\le 8\,\frac{|F|}{|E|},$$

if K is the smallest interval that contains x_0 and $J'_{\ell,m}$, then

$$\frac{1}{|K|}\int_K \chi_F\,dx \le 8\,\frac{|F|}{|E|} \implies |F\cap J'_{\ell,m}| \le 8\,\frac{|F|}{|E|}\,|I_t|\,\frac{|K|}{|I_t|}.$$

We conclude that

$$|F\cap J'_{\ell,m}| \le c\,\frac{|F|}{|E|}\,|I_t|\left(1+\frac{\operatorname{dist}(I_t,J'_{\ell,m})}{|I_t|}\right). \tag{11.2.23}$$

We now have a new collection of dyadic intervals $\{J_k\}_k$ contained in $\Omega_{E,F}$ consisting of all the previous J'_ℓ when $|J'_\ell|<|I_t|$ and the $J'_{\ell,m}$'s when $|J'_\ell|\ge|I_t|$. In view of the construction we have

$$|F\cap J_k| \le \begin{cases} 2c\,\dfrac{|F|}{|E|}\,|J_k| & \text{when } |J_k|<|I_t|,\\[2ex] 2c\,\dfrac{|F|}{|E|}\,|J_k|\left(1+\dfrac{\operatorname{dist}(I_t,J_k)}{|I_t|}\right) & \text{when } |J_k|=|I_t|,\end{cases} \tag{11.2.24}$$

for all k. We now define the "bad functions"

$$b_k(x) = \left(e^{-2\pi i c(\omega_t)x}\chi_{F\cap 3I_t}(x) - \frac{1}{|J_k|}\int_{J_k} e^{-2\pi i c(\omega_t)y}\chi_{F\cap 3I_t}(y)\,dy\right)\chi_{J_k}(x),$$

which are supported in J_k, have mean value zero, and satisfy

$$\big\|b_k\big\|_{L^1} \le 2c\,|F|\,|J_k|\left(1+\frac{\operatorname{dist}(I_t,J_k)}{|I_t|}\right).$$

We also set

$$g(x) = e^{-2\pi i c(\omega_t)x}\chi_{F\cap 3I_t}(x) - \sum_k b_k(x),$$

the "good function" of this Calderón–Zygmund-type decomposition. We have therefore decomposed the function $\chi_{F\cap 3I_t}$ as follows:

$$\chi_{F\cap 3I_t}(x) = g(x)e^{2\pi i c(\omega_t)x} + \sum_k b_k(x)e^{2\pi i c(\omega_t)x}. \tag{11.2.25}$$

We show that $\big\|g\big\|_{L^\infty} \le C\frac{|F|}{|E|}$. Indeed, for x in J_k we have

$$g(x) = \frac{1}{|J_k|}\int_{J_k} e^{-2\pi i c(\omega_t)y}\chi_{F\cap 3I_t}(y)\,dy,$$

which implies

$$|g(x)| \le \frac{|F\cap 3I_t \cap J_k|}{|J_k|} \le \begin{cases} \dfrac{|F\cap J_k|}{|J_k|} & \text{when } |J_k|<|I_t|, \\[2ex] \dfrac{|F\cap 3I_t|}{|I_t|} & \text{when } |J_k|=|I_t|, \end{cases}$$

and both of the preceding are at most a multiple of $\frac{|F|}{|E|}$; the latter is because there is an $x_0 \in I_t$ with $M(\chi_F)(x_0) \le 8\,\frac{|F|}{|E|}$. Also, for $x \in (\bigcup_k J_k)^c = (\Omega_{E,F})^c$ we have

$$|g(x)| = \chi_{F\cap 3I_t}(x) \le M(\chi_F)(x) \le 8\,\frac{|F|}{|E|}\,.$$

We conclude that $\big\|g\big\|_{L^\infty} \le C\,\frac{|F|}{|E|}$. Moreover,

$$\big\|g\big\|_{L^1} \le \sum_k \int_{J_k} \frac{|F\cap 3I_t\cap J_k|}{|J_k|}\,dx + \big\|\chi_{F\cap 3I_t}\big\|_{L^1} \le C\,|F\cap 3I_t| \le C\,\frac{|F|}{|E|}\,|I_t|\,,$$

since the J_k are disjoint. It follows that

$$\big\|g\big\|_{L^2} \le C\Big(\frac{|F|}{|E|}\Big)^{\frac{1}{2}}\Big(\frac{|F|}{|E|}\Big)^{\frac{1}{2}}\,|I_t|^{\frac{1}{2}} = C\frac{|F|}{|E|}\,|I_t|^{\frac{1}{2}}\,.$$

Using Exercise 11.1.6, we have

$$\sum_{s\in\mathbf{T}} \big|\big\langle g e^{2\pi i c(\omega_t)(\cdot)} \,\big|\, \varphi_s\big\rangle\big|^2 \le C\,\big\|g\big\|_{L^2}^2\,,$$

from which we obtain the required conclusion for the first function in the decomposition (11.2.25).

Next we turn to the corresponding estimate for the second function,

$$\sum_k b_k e^{2\pi i c(\omega_t)(\cdot)}\,,$$

in the decomposition (11.2.25), which requires some further analysis. We have the following two estimates for all s and k:

$$\big|\big\langle b_k e^{2\pi i c(\omega_t)(\cdot)} \,\big|\, \varphi_s\big\rangle\big| \le \frac{C_M\,|F|\,|E|^{-1}\,|J_k|^2\,|I_s|^{-\frac{3}{2}}}{\big(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|}\big)^M}\,, \tag{11.2.26}$$

$$\big|\big\langle b_k e^{2\pi i c(\omega_t)(\cdot)} \,\big|\, \varphi_s\big\rangle\big| \le \frac{C_M\,|F|\,|E|^{-1}\,|I_s|^{\frac{1}{2}}}{\big(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|}\big)^M}\,, \tag{11.2.27}$$

for all $M>0$, where C_M depends only on M.

To prove (11.2.26) we use the mean value theorem together with the fact that b_k has vanishing integral to write for some ξ_y,

$$
\begin{aligned}
&\big|\big\langle b_k\, e^{2\pi i c(\omega_t)(\cdot)} \,|\, \varphi_s \big\rangle\big| \\
&\quad = \Big| \int_{J_k} b_k(y) e^{2\pi i c(\omega_t) y} \overline{\varphi_s(y)}\, dy \Big| \\
&\quad = \Big| \int_{J_k} b_k(y) \big(e^{2\pi i c(\omega_t) y} \overline{\varphi_s(y)} - e^{2\pi i c(\omega_t) c(J_k)} \overline{\varphi_s(c(J_k))}\, dy \Big| \\
&\quad \le |J_k| \int_{J_k} |b_k(y)| \Big[2\pi \tfrac{|c(\omega_s)-c(\omega_t)|}{|I_s|^{\frac{1}{2}}} \big|\varphi\big(\tfrac{\xi_y - c(I_s)}{|I_s|}\big)\big| + |I_s|^{-\frac{3}{2}} \big|\varphi'\big(\tfrac{\xi_y - c(I_s)}{|I_s|}\big)\big| \Big]\, dy \\
&\quad \le \big\|b_k\big\|_{L^1} |J_k| \sup_{\xi\in J_k} \frac{C_M |I_s|^{-\frac{3}{2}}}{(1+\frac{|\xi - c(I_s)|}{|I_s|})^{M+1}} \\
&\quad \le C_M \frac{|F|}{|E|} |J_k| \Big(1 + \frac{\operatorname{dist}(J_k, I_t)}{|I_t|}\Big) \frac{|J_k|\, |I_s|^{-\frac{3}{2}}}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^{M+1}} \\
&\quad \le \frac{C_M |F|\, |E|^{-1} |J_k|^2 |I_s|^{-\frac{3}{2}}}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^{M}},
\end{aligned}
$$

where we used the fact that $1+\frac{\operatorname{dist}(J_k,I_t)}{|I_t|} \le 1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|}$. To prove estimate (11.2.27) we note that

$$
\big|\big\langle b_k e^{2\pi i c(\omega_t)(\cdot)} \,|\, \varphi_s \big\rangle\big| \le \frac{C_M |I_s|^{\frac{1}{2}} \inf_{I_s} M(b_k)}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^{M}}
$$

and that

$$
M(b_k) \le M(\chi_F) + \frac{|F\cap 3I_t \cap J_k|}{|J_k|} M(\chi_{J_k}),
$$

and since $I_s \not\subseteq \Omega_{E,F}$, we have $\inf_{I_s} M(\chi_F) \le 8\,\frac{|F|}{|E|}$, while the second term in the sum was observed earlier to be at most $C\,\frac{|F|}{|E|}$.

Finally, we have the estimate

$$
\big|\big\langle b_k e^{2\pi i c(\omega_t)(\cdot)} \,|\, \varphi_s \big\rangle\big| \le \frac{C_M |F|\, |E|^{-1} |J_k|\, |I_s|^{-\frac{1}{2}}}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^{M}}, \tag{11.2.28}
$$

which follows by taking the geometric mean of (11.2.26) and (11.2.27).

Now for a fixed $s \in \mathbf{P}_{E,F}$ we may have either $J_k \subseteq I_s$ or $J_k \cap I_s = \emptyset$ (since I_s is not contained in $\Omega_{E,F}$). Therefore, for fixed $s \in \mathbf{P}_{E,F}$ there are only three possibilities for J_k:

(a) $J_k \subseteq 3I_s$;

(b) $J_k \cap 3I_s = \emptyset$;

(c) $J_k \cap I_s = \emptyset$, $J_k \cap 3I_s \neq \emptyset$, and $J_k \not\subseteq 3I_s$.

Observe that case (c) is equivalent to the following statement:

(c) $J_k \cap I_s = \emptyset$, $\operatorname{dist}(J_k, I_s) = 0$, and $|J_k| \geq 2|I_s|$.

Note that in case (c), for each I_s there exists exactly one $J_k = J_{k(s)}$ with the previous properties; but for a given J_k there may be a sequence of I_s's that lie on the left of J_k such that $|J_k| \geq 2|I_s|$ and $\operatorname{dist}(J_k, I_s) = 0$ and another sequence with similar properties on the right of J_k. The I_s's that lie on either side of J_k must be nested, and their lengths must add up to $|I^L_{s_k}| + |I^R_{s_k}|$, where $I^L_{s_k}$ is the largest one among them on the left of J_k and $I^R_{s_k}$ is the largest one among them on the right of J_k. Using (11.2.27), we obtain

$$\begin{aligned}
\sum_{s\in\mathbf{T}} \Big| \sum_{\substack{k:\, J_k\cap I_s=\emptyset \\ \operatorname{dist}(J_k,I_s)=0 \\ |J_k|\geq 2|I_s|}} \big\langle b_k e^{2\pi i c(\omega_t)(\cdot)} \,\big|\, \varphi_s \big\rangle \Big|^2 &= \sum_{s\in\mathbf{T}} \Big| \big\langle b_{k(s)} e^{2\pi i c(\omega_t)(\cdot)} \,\big|\, \varphi_s \big\rangle \Big|^2 \\
&\leq C \Big(\frac{|F|}{|E|}\Big)^2 \sum_{\substack{s\in\mathbf{T}:\, J_k\cap I_s=\emptyset \\ \operatorname{dist}(J_k,I_s)=0 \\ |J_k|\geq 2|I_s|}} |I_s| \\
&\leq C \Big(\frac{|F|}{|E|}\Big)^2 \sum_k \big(|I^L_{s_k}| + |I^R_{s_k}|\big)\,.
\end{aligned}$$

But note that $I^L_{s_k} \subseteq 2J_k$, and since $I^L_{s_k} \cap J_k = \emptyset$, we must have $I^L_{s_k} \subseteq 2J_k \setminus J_k$ (and likewise for $I^R_{s_k}$). We define sets

$$\begin{aligned}
I^{L+}_{s_k} &= I^L_{s_k} + \frac{1}{2}|J_k|\,, \\
I^{R-}_{s_k} &= I^R_{s_k} - \frac{1}{2}|J_k|\,.
\end{aligned}$$

We have $I^{L+}_{s_k} \cup I^{R-}_{s_k} \subseteq J_k$, and hence the sets $I^{L+}_{s_k}$ are pairwise disjoint for different k, and the same is true for the $I^{R-}_{s_k}$. Moreover, since $\frac{1}{2}|J_k| \leq \frac{1}{2}|I_t|$ for all k, all the shifted sets $I^{L+}_{s_k}$, $I^{R-}_{s_k}$ are contained in $3I_t$. We conclude that

$$\begin{aligned}
\sum_k |I^L_{s_k}| + \sum_k |I^R_{s_k}| &= \sum_k \big(|I^{L+}_{s_k}| + |I^{R-}_{s_k}|\big) \\
&\leq \Big|\bigcup_k I^{L+}_{s_k}\Big| + \Big|\bigcup_k I^{R-}_{s_k}\Big| \\
&\leq 2\,|3I_t|\,,
\end{aligned}$$

which combined with the previously obtained estimate yields the required result in case (c).

We now consider case (a). Using (11.2.26), we can write

$$\Big(\sum_{s\in\mathbf{T}} \Big| \sum_{k: J_k\subseteq 3I_s} \big\langle b_k e^{2\pi i c(\omega_t)(\cdot)} \,\big|\, \varphi_s \big\rangle \Big|^2 \Big)^{\frac{1}{2}} \leq C_M \Big(\frac{|F|}{|E|}\Big)^2 \Big(\sum_{s\in T} \Big| \sum_{k:\, J_k\subseteq 3I_s} |J_k|^{\frac{1}{2}} \frac{|J_k|^{\frac{3}{2}}}{|I_s|^{\frac{3}{2}}} \Big|^2 \Big)^{\frac{1}{2}},$$

and we control the second expression by

$$C_M \frac{|F|}{|E|}\bigg\{\sum_{s\in\mathbf{T}}\bigg(\sum_{k:\,J_k\subseteq 3I_s}|J_k|\bigg)\bigg(\sum_{k:\,J_k\subseteq 3I_s}\frac{|J_k|^3}{|I_s|^3}\bigg)\bigg\}^{\frac12}$$

$$\le C_M \frac{|F|}{|E|}\bigg\{\sum_{k:\,J_k\subseteq 3I_t}|J_k|^3\sum_{\substack{s\in\mathbf{T}\\ J_k\subseteq 3I_s}}\frac{1}{|I_s|^2}\bigg\}^{\frac12},$$

where we used that the dyadic intervals J_k are disjoint and the Cauchy–Schwarz inequality. We note that the last sum is equal to at most $C|J_k|^{-2}$, since for every dyadic interval J_k there exist at most three dyadic intervals of a given length whose triples contain it. The required estimate $C|F|\,|E|^{-1}\,|I_t|^{\frac12}$ now follows in case (a).

Finally, we deal with case (b), which is the most difficult case. We split the set of k into two subsets, those for which $J_k\subseteq 3I_t$ and those for which $J_k\nsubseteq 3I_t$ (recall that $|J_k|\le|I_t|$). Whenever $J_k\nsubseteq 3I_t$, we have

$$\operatorname{dist}(J_k,I_s)\approx\operatorname{dist}(J_k,I_t)\,.$$

In this case we use Minkowski's inequality and estimate (11.2.28) to deduce

$$\bigg(\sum_{s\in\mathbf{T}}\Big|\sum_{k:\,J_k\nsubseteq 3I_t}\big\langle b_k e^{2\pi i c(\omega_t)(\cdot)}\,|\,\varphi_s\big\rangle\Big|^2\bigg)^{\frac12}$$

$$\le\sum_{k:\,J_k\nsubseteq 3I_t}\bigg(\sum_{s\in\mathbf{T}}\big|\big\langle b_k e^{2\pi i c(\omega_t)(\cdot)}\,|\,\varphi_s\big\rangle\big|^2\bigg)^{\frac12}$$

$$\le C_M\frac{|F|}{|E|}\sum_{k:\,J_k\nsubseteq 3I_t}|J_k|\bigg(\sum_{s\in\mathbf{T}}\frac{|I_s|^{2M-1}}{\operatorname{dist}(J_k,I_s)^{2M}}\bigg)^{\frac12}$$

$$\le C_M\frac{|F|}{|E|}\sum_{k:\,J_k\nsubseteq 3I_t}\frac{|J_k|}{\operatorname{dist}(J_k,I_t)^M}\bigg(\sum_{s\in\mathbf{T}}|I_s|^{2M-1}\bigg)^{\frac12}$$

$$\le C_M\frac{|F|}{|E|}|I_t|^{M-\frac12}\sum_{k:\,J_k\nsubseteq 3I_t}\frac{|J_k|}{\operatorname{dist}(J_k,I_t)^M}$$

$$\le C_M\frac{|F|}{|E|}|I_t|^{M-\frac12}\sum_{l=1}^{\infty}\sum_{\substack{k:\\ \operatorname{dist}(J_k,I_t)\approx 2^l|I_t|}}\frac{|J_k|}{(2^l|I_t|)^M},$$

where $\operatorname{dist}(J_k,I_t)\approx 2^l|I_t|$ means that $\operatorname{dist}(J_k,I_t)\in[2^l|I_t|,2^{l+1}|I_t|]$. But note that all the J_k with $\operatorname{dist}(J_k,I_t)\approx 2^l|I_t|$ are contained in $2^{l+2}I_t$, and since they are disjoint, we estimate the last sum by $C2^l|I_t|(2^l|I_t|)^{-M}$. The required estimate $C_M|F|\,|E|^{-1}|I_t|^{\frac12}$ follows.

Next we consider the case $J_k \subseteq 3I_t$, $J_k \cap 3I_s = \emptyset$, and $|J_k| \le |I_s|$, in which we use estimate (11.2.26). We have

$$\begin{aligned}
&\bigg(\sum_{s\in\mathbf{T}}\bigg|\sum_{\substack{k:\,J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\langle b_k e^{2\pi i c(\omega_t)(\cdot)}\,|\,\varphi_s\rangle\bigg|^2\bigg)^{\frac12}\\
&\le C_M\frac{|F|}{|E|}\bigg(\sum_{s\in\mathbf{T}}\bigg|\sum_{\substack{k:\,J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}|J_k|^2|I_s|^{-\frac32}\frac{|I_s|^M}{\operatorname{dist}(J_k,I_s)^M}\bigg|^2\bigg)^{\frac12}\\
&\le C_M\frac{|F|}{|E|}\bigg\{\sum_{s\in\mathbf{T}}\bigg[\sum_{\substack{k:\,J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|^3}{|I_s|^2}\Big(\frac{|I_s|}{\operatorname{dist}(J_k,I_s)}\Big)^M\bigg]\\
&\qquad\qquad\times\bigg[\sum_{\substack{k:\,J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|}{|I_s|}\Big(\frac{\operatorname{dist}(J_k,I_s)}{|I_s|}\Big)^{-M}\bigg]\bigg\}^{\frac12}\\
&\le C_M\frac{|F|}{|E|}\bigg\{\sum_{s\in\mathbf{T}}\bigg[\sum_{\substack{k:\,J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|^3}{|I_s|^2}\Big(\frac{|I_s|}{\operatorname{dist}(J_k,I_s)}\Big)^M\bigg]\\
&\qquad\qquad\times\bigg[\sum_{\substack{k:\,J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\int_{J_k}\Big(\frac{|x-c(I_s)|}{|I_s|}\Big)^{-M}\frac{dx}{|I_s|}\bigg]\bigg\}^{\frac12}\\
&\le C_M\frac{|F|}{|E|}\bigg\{\sum_{s\in\mathbf{T}}\bigg[\sum_{\substack{k:\,J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|^3}{|I_s|^2}\Big(\frac{|I_s|}{\operatorname{dist}(J_k,I_s)}\Big)^M\bigg]\\
&\qquad\qquad\times\bigg[\int_{(3I_s)^c}\Big(\frac{|x-c(I_s)|}{|I_s|}\Big)^{-M}\frac{dx}{|I_s|}\bigg]\bigg\}^{\frac12}\\
&\le C_M\frac{|F|}{|E|}\bigg\{\sum_{s\in\mathbf{T}}\sum_{\substack{k:\,J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}|J_k|^3|I_s|^{-2}\Big(\frac{|I_s|}{\operatorname{dist}(J_k,I_s)}\Big)^M\bigg\}^{\frac12}.
\end{aligned}$$

But since the last integral contributes at most a constant factor, we can estimate the last displayed expression by

$$C_M \frac{|F|}{|E|} \Bigg\{ \sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| \le |I_s|}} |J_k|^3 \sum_{m \ge \log |J_k|} 2^{-2m} \sum_{\substack{s \in \mathbf{T} \\ |I_s| = 2^m}} \Big(\frac{\operatorname{dist}(J_k, I_s)}{2^m} \Big)^{-M} \Bigg\}^{\frac{1}{2}}$$

$$\le C_M \frac{|F|}{|E|} \Bigg\{ \sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| \le |I_s|}} |J_k|^3 \sum_{m \ge \log |J_k|} 2^{-2m} \Bigg\}^{\frac{1}{2}}$$

$$\le C_M \frac{|F|}{|E|} \Bigg\{ \sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| \le |I_s|}} |J_k|^3 |J_k|^{-2} \Bigg\}^{\frac{1}{2}}$$

$$\le C_M \frac{|F|}{|E|} |I_t|^{\frac{1}{2}} .$$

There is also the subcase of case (b) in which $|J_k| > |I_s|$. Here we have the two special subcases $I_s \cap 3J_k = \emptyset$ and $I_s \subseteq 3J_k$. We begin with the first of these special subcases, in which we use estimate (11.2.27). We have

$$\Bigg(\sum_{s \in \mathbf{T}} \Bigg| \sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \cap 3J_k = \emptyset}} \langle b_k e^{2\pi i c(\omega_t)(\cdot)} \,|\, \varphi_s \rangle \Bigg|^2 \Bigg)^{\frac{1}{2}}$$

$$\le C_M \frac{|F|}{|E|} \Bigg(\sum_{s \in T} \Bigg| \sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \cap 3J_k = \emptyset}} |I_s|^{\frac{1}{2}} \frac{|I_s|^M}{\operatorname{dist}(J_k, I_s)^M} \Bigg|^2 \Bigg)^{\frac{1}{2}}$$

$$\le C_M \frac{|F|}{|E|} \Bigg\{ \sum_{s \in \mathbf{T}} \Bigg[\sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \cap 3J_k = \emptyset}} \frac{|I_s|^2}{|J_k|} \frac{|I_s|^M}{\operatorname{dist}(J_k, I_s)^M} \Bigg] \Bigg[\sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \cap 3J_k = \emptyset}} \frac{|J_k|}{|I_s|} \frac{|I_s|^M}{\operatorname{dist}(J_k, I_s)^M} \Bigg] \Bigg\}^{\frac{1}{2}} .$$

Since $I_s \cap 3J_k = \emptyset$, we have that $\operatorname{dist}(J_k, I_s) \approx |x - c(I_s)|$ for every $x \in J_k$, and therefore the second term inside square brackets satisfies

$$\sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \cap 3J_k = \emptyset}} \frac{|J_k|}{|I_s|} \frac{|I_s|^M}{\operatorname{dist}(J_k, I_s)^M} \le \sum_k \int_{J_k} \Big(\frac{|x - c(I_s)|}{|I_s|} \Big)^{-M} \frac{dx}{|I_s|} \le C_M .$$

Using this estimate, we obtain

$$C_M \frac{|F|}{|E|} \left\{ \sum_{s\in\mathbf{T}} \Bigg[\sum_{\substack{k:\, J_k\subseteq 3I_t \\ J_k\cap 3I_s=\emptyset \\ |J_k|>|I_s| \\ I_s\cap 3J_k=\emptyset}} \frac{|I_s|^2}{|J_k|} \frac{|I_s|^M}{\operatorname{dist}(J_k,I_s)^M} \Bigg] \Bigg[\sum_{\substack{k:\, J_k\subseteq 3I_t \\ J_k\cap 3I_s=\emptyset \\ |J_k|>|I_s| \\ I_s\cap 3J_k=\emptyset}} \frac{|J_k|}{|I_s|} \frac{|I_s|^M}{\operatorname{dist}(J_k,I_s)^M} \Bigg] \right\}^{\frac{1}{2}}$$

$$\leq C_M \frac{|F|}{|E|} \left\{ \sum_{s\in\mathbf{T}} \Bigg[\sum_{\substack{k:\, J_k\subseteq 3I_t \\ J_k\cap 3I_s=\emptyset \\ |J_k|>|I_s| \\ I_s\cap 3J_k=\emptyset}} \frac{|I_s|^2}{|J_k|} \frac{|I_s|^M}{\operatorname{dist}(J_k,I_s)^M} \Bigg] \right\}^{\frac{1}{2}}$$

$$= C_M \frac{|F|}{|E|} \left\{ \sum_{k:\, J_k\subseteq 3I_t} \frac{1}{|J_k|} \sum_{\substack{s\in\mathbf{T} \\ J_k\cap 3I_s=\emptyset \\ |J_k|>|I_s| \\ I_s\cap 3J_k=\emptyset}} |I_s|^2 \frac{|I_s|^M}{\operatorname{dist}(J_k,I_s)^M} \right\}^{\frac{1}{2}}$$

$$\leq C_M \frac{|F|}{|E|} \left\{ \sum_{k:\, J_k\subseteq 3I_t} \frac{1}{|J_k|} \sum_{m=-\infty}^{\log_2|J_k|} 2^{2m} \sum_{\substack{s\in\mathbf{T}:\, |I_s|=2^m \\ J_k\cap 3I_s=\emptyset \\ |J_k|>|I_s| \\ I_s\cap 3J_k=\emptyset}} \frac{|I_s|^M}{\operatorname{dist}(J_k,I_s)^M} \right\}^{\frac{1}{2}}$$

$$\leq C_M \frac{|F|}{|E|} \left\{ \sum_{k:\, J_k\subseteq 3I_t} \frac{1}{|J_k|} \sum_{m=-\infty}^{\log_2|J_k|} 2^{2m} \right\}^{\frac{1}{2}}$$

$$\leq C_M \frac{|F|}{|E|} \left\{ \sum_{k:\, J_k\subseteq 3I_t} \frac{1}{|J_k|} |J_k|^2 \right\}^{\frac{1}{2}}$$

$$\leq C_M \frac{|F|}{|E|} |I_t|^{\frac{1}{2}} .$$

Finally, there is the subcase of case (b) in which $|J_k| \geq |I_s|$ and $I_s \subseteq 3J_k$. Here again we use estimate (11.2.27). We have

$$\begin{aligned} &\left\{ \sum_{s\in\mathbf{T}} \Bigg| \sum_{\substack{k:\, J_k\subseteq 3I_t \\ J_k\cap 3I_s=\emptyset \\ |J_k|>|I_s| \\ I_s\subseteq 3J_k}} \langle b_k e^{2\pi i c(\omega_t)(\cdot)} \,|\, \varphi_s \rangle \Bigg|^2 \right\}^{\frac{1}{2}} \\ &\qquad \leq\; C_M \frac{|F|}{|E|} \left\{ \sum_{s\in\mathbf{T}} |I_s| \Bigg| \sum_{\substack{k:\, J_k\subseteq 3I_t \\ J_k\cap 3I_s=\emptyset \\ |J_k|>|I_s| \\ I_s\subseteq 3J_k}} \frac{|I_s|^M}{\operatorname{dist}(J_k,I_s)^M} \Bigg|^2 \right\}^{\frac{1}{2}} . \end{aligned} \tag{11.2.29}$$

Let us make some observations. For a fixed s there exist at most finitely many J_k's contained in $3I_t$ with size at least $|I_s|$. Let $J_L^1(s)$ be the interval that lies to the left of I_s and is closest to I_s among all J_k that satisfy the conditions in the preceding sum. Then $|J_L^1(s)| > |I_s|$ and

$$\operatorname{dist}(J_L^1(s), I_s) \geq |I_s|\,.$$

Let $J_L^2(s)$ be the interval to the left of $J_L^1(s)$ that is closest to $J_L^1(s)$ and that satisfies the conditions of the sum. Since $3J_L^2(s)$ contains I_s, it follows that $|J_L^2(s)| > 2|I_s|$ and

$$\operatorname{dist}(J_L^2(s), I_s) \geq 2|I_s|\,.$$

Continuing in this way, we can find a finite number of intervals $J_L^r(s)$ that lie to the left of I_s and inside $3I_t$, satisfy $|J_L^r(s)| > 2^r|I_s|$ and $\operatorname{dist}(J_L^r(s), I_s) \geq 2^r|I_s|$, and whose triples contain I_s. Likewise we find a finite collection of intervals $J_R^1(s), J_R^2(s), \dots$ that lie to the right of I_s and satisfy similar conditions. Then, using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\Bigg|\sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \subseteq 3J_k}} \frac{|I_s|^M}{\operatorname{dist}(J_k, I_s)^M}\Bigg|^2 \\
&\qquad \leq 2\Bigg|\sum_{r=1}^{\infty} \frac{|I_s|^{\frac{M}{2}}}{\operatorname{dist}(J_L^r(s), I_s)^{\frac{M}{2}}} \frac{1}{2^{\frac{rM}{2}}}\Bigg|^2 + 2\Bigg|\sum_{r=1}^{\infty} \frac{|I_s|^{\frac{M}{2}}}{\operatorname{dist}(J_R^r(s), I_s)^{\frac{M}{2}}} \frac{1}{2^{\frac{rM}{2}}}\Bigg|^2 \\
&\qquad \leq C_M \sum_{r=1}^{\infty} \frac{|I_s|^M}{\operatorname{dist}(J_L^r(s), I_s)^M} + C_M \sum_{r=1}^{\infty} \frac{|I_s|^M}{\operatorname{dist}(J_R^r(s), I_s)^M} \\
&\qquad \leq C_M \sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \subseteq 3J_k}} \frac{|I_s|^M}{\operatorname{dist}(J_k, I_s)^M}\,.
\end{aligned}$$

We use this estimate to control the expression on the left in (11.2.29) by

$$\begin{aligned}
&C_M \frac{|F|}{|E|}\Bigg\{\sum_{s \in \mathbf{T}} |I_s| \sum_{\substack{k:\, J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \subseteq 3J_k}} \frac{|I_s|^M}{\operatorname{dist}(J_k, I_s)^M}\Bigg\}^{\frac{1}{2}} \\
&\qquad\qquad \leq C_M \frac{|F|}{|E|}\Bigg\{\sum_{k: J_k \subseteq 3I_t} |J_k| \sum_{m=0}^{\infty} 2^{-m} \sum_{\substack{s:\, I_s \subseteq 3J_k \\ J_k \cap 3I_s = \emptyset \\ |I_s| = 2^{-m}|J_k|}} \frac{|I_s|^M}{\operatorname{dist}(J_k, I_s)^M}\Bigg\}^{\frac{1}{2}}.
\end{aligned}$$

Since the last sum is at most a constant, it follows that the term on the left in (11.2.29) also satisfies the estimate $C_M \frac{|F|}{|E|}|I_t|^{\frac{1}{2}}$. This concludes the proof of Lemma 11.2.2.

Exercises

11.2.1. Let $\mathbf{T}$ be a 2-tree with top I_t and let $M > 1$ and L be such that $2^L < |I_t|$. Show that there exists a constant $C_M > 0$ such that

$$\sum_{s\in\mathbf{T}} |I_s|^M \le C_M |I_t|^M ,$$

$$\sum_{\substack{s\in\mathbf{T}\\ |I_s|\ge 2^L}} |I_s|^{-M} \le C_M \frac{|I_t|}{(2^L)^{M+1}} ,$$

$$\sum_{\substack{s\in\mathbf{T}\\ |I_s|\le 2^L}} |I_s|^M \le C_M |I_t| (2^L)^{M-1} .$$

$\big[$*Hint:* Group the s that appear in each sum in families $\mathscr{G}_m$ such that $|I_s| = 2^{-m}|I_t|$ for each $s \in \mathscr{G}_m$.$\big]$

11.2.2. Show that the operator

$$g \mapsto \sup_{-\infty<a<b<\infty} \big|(\widehat{g}\chi_{[a,b]})^\vee\big|$$

defined on the line is L^p bounded for all $1 < p < \infty$.

11.2.3. On $\mathbf{R}^n$ fix a unit vector b and consider the maximal operator

$$T(g)(x) = \sup_{N>0} \bigg| \int_{|b\cdot\xi|\le N} \widehat{g}(\xi) e^{2\pi i x\cdot\xi}\, d\xi \bigg| .$$

Show that T maps $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.
$\big[$*Hint:* Apply a rotation.$\big]$

11.2.4. Define the *directional Carleson operators* by

$$\mathscr{C}^\theta(f)(x) = \sup_{a\in\mathbf{R}} \bigg| \lim_{\varepsilon\to 0} \int_{\varepsilon<|t|<\varepsilon^{-1}} e^{2\pi i a t} f(x-t\theta)\, \frac{dt}{t} \bigg| ,$$

for functions f on $\mathbf{R}^n$. Here θ is a vector in $\mathbf{S}^{n-1}$.
(a) Show that $\mathscr{C}^\theta$ is bounded on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.
(b) Let Ω be an odd integrable function on $\mathbf{S}^{n-1}$. Define an operator

$$\mathscr{C}^\Omega(f)(x) = \sup_{\xi\in\mathbf{R}^n} \bigg| \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<\varepsilon^{-1}} e^{2\pi i \xi\cdot y} f(x-y)\, \frac{\Omega\big(\frac{y}{|y|}\big)}{|y|^n}\, dy \bigg| .$$

Show that $\mathscr{C}^\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.
$\big[$*Hint:* Part (a): Reduce to the case $\theta = e_1 = (1,0,\dots,0)$ via a rotation and use Theorem 11.2.1(b). Part (b): Use the method of rotations and part (a).$\big]$

11.3 The Maximal Carleson Operator and Weighted Estimates

Recall the one-sided Carleson operator $\mathscr{C}_1$ defined in the previous section:

$$\mathscr{C}_1(f)(x) = \sup_{N>0} \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} \, d\xi \right| .$$

Recall also the modulation operator $M^a(g)(x) = g(x)e^{2\pi i a x}$. We begin by observing that the following identity is valid:

$$\big(\widehat{f}\chi_{(-\infty,b]}\big)^{\vee} = M^b \frac{I - iH}{2} M^{-b}(f) = \frac{1}{2} f - \frac{i}{2} M^b H M^{-b}(f) , \tag{11.3.1}$$

where H is the Hilbert transform. It follows from (11.3.1) that

$$\mathscr{C}_1(f) \le \frac{1}{2}|f| + \frac{1}{2} \sup_{\xi \in \mathbf{R}} |H(M^{\xi}(f))|$$

and that

$$\sup_{\xi \in \mathbf{R}} |H(M^{\xi}(f))| \le |f| + 2\mathscr{C}_1(f) .$$

We conclude that the L^p boundedness of the sublinear operator $f \mapsto \mathscr{C}_1(f)$ is equivalent to that of the sublinear operator

$$f \mapsto \sup_{\xi \in \mathbf{R}} |H(M^{\xi}(f))| .$$

Definition 11.3.1. The *maximal Carleson operator* is defined by

$$\begin{aligned} \mathscr{C}_*(f)(x) &= \sup_{\varepsilon>0} \sup_{\xi \in \mathbf{R}} \left| \int_{|x-y|>\varepsilon} f(y) e^{2\pi i \xi y} \frac{dy}{x-y} \right| \\ &= \sup_{\xi \in \mathbf{R}} \left| H^{(*)}(M^{\xi}(f))(x) \right| , \end{aligned} \tag{11.3.2}$$

where $H^{(*)}$ is the maximal Hilbert transform. Observe that $\mathscr{C}_*(f)$ is well defined for all f in $\bigcup_{1 \le p < \infty} L^p(\mathbf{R})$ and that $\mathscr{C}_*(f)$ controls the Carleson operator $\mathscr{C}(f)$ pointwise.

We begin with the following pointwise estimate, which reduces the boundedness of $\mathscr{C}_*$ to that of $\mathscr{C}$:

Lemma 11.3.2. *There is a positive constant $c > 0$ such that for all functions f in $\bigcup_{1 \le p < \infty} L^p(\mathbf{R})$ we have*

$$\mathscr{C}_*(f) \le c M(f) + M(\mathscr{C}(f)) , \tag{11.3.3}$$

where M is the Hardy–Littlewood maximal function.

Proof. The proof of (11.3.3) is based on the classical inequality

$$H^{(*)}(g) \le cM(g) + M(H(g))$$

obtained in (4.1.32). Applying this to the functions $M^{\xi}(f)$ and taking the supremum over $\xi \in \mathbf{R}$, we obtain

$$\mathscr{C}_*(f) \le cM(f) + \sup_{\xi\in\mathbf{R}} M\big(H(M^{\xi}(f))\big),$$

from which (11.3.3) easily follows by passing the supremum inside the maximal function. □

It is convenient to work with a variant of the Hardy–Littlewood maximal operator. For $0 < r < \infty$ define

$$M_r(f) = M(|f|^r)^{\frac{1}{r}}$$

for f such that $|f|^r$ is locally integrable over the real line. Note that $M(f) \le M_r(f)$ for any $r \in (1,\infty)$. Our next goal is to obtain the boundedness of the Carleson operator on weighted L^p spaces.

Theorem 11.3.3. *For every $p \in (1,\infty)$ and $w \in A_p$ there is a constant $C(p,[w]_{A_p})$ such that for all $f \in L^p(\mathbf{R})$ we have*

$$\big\|\mathscr{C}(f)\big\|_{L^p(w)} \le C(p,[w]_{A_p})\big\|f\big\|_{L^p(w)}, \tag{11.3.4}$$

$$\big\|\mathscr{C}_*(f)\big\|_{L^p(w)} \le C(p,[w]_{A_p})\big\|f\big\|_{L^p(w)}. \tag{11.3.5}$$

Proof. Fix a $1 < p < \infty$ and pick an $r \in (1,p)$ such that $w \in A_r$. We show that for all $f \in L^p(w)$ we have the estimate

$$\int_{\mathbf{R}} \mathscr{C}(f)(x)^p\, w(x)\,dx \le C_p([w]_{A_p}) \int_{\mathbf{R}} M_r(f)(x)^p w(x)\,dx. \tag{11.3.6}$$

Then the boundedness of $\mathscr{C}$ on $L^p(w)$ is a consequence of the boundedness of the Hardy–Littlewood maximal operator on $L^{\frac{p}{r}}(w)$.

If we show that for any $w \in A_p$ there is a constant $C_p([w]_{A_p})$ such that

$$\int_{\mathbf{R}} M(\mathscr{C}(f))^p\, w\,dx \le C_p([w]_{A_p}) \int_{\mathbf{R}} M_r(f)^p\, w\,dx, \tag{11.3.7}$$

then the trivial fact $\mathscr{C}(f) \le M(\mathscr{C}(f))$, inserted in (11.3.7), yields (11.3.6).

Estimate (11.3.7) will be a consequence of the following two important observations:

$$M^{\#}(\mathscr{C}(f)) \le C_r M_r(f) \qquad \text{a.e.} \tag{11.3.8}$$

and

$$\big\|M(\mathscr{C}(f))\big\|_{L^p(w)} \le c_p([w]_{A_p})\big\|M^{\#}(\mathscr{C}(f))\big\|_{L^p(w)}, \tag{11.3.9}$$

where $c_p([w]_{A_p})$ depends on $[w]_{A_p}$ and C_r depends only on r.

We begin with estimate (11.3.8), which was obtained in Theorem 7.4.9 for singular integral operators. Here this estimate is extended to maximally modulated singular integrals. To prove (11.3.8) we use the result in Proposition 7.4.2 (2). We fix $x \in \mathbf{R}$ and we pick an interval I that contains x. We write $f = f_0 + f_\infty$, where $f_0 = f\chi_{3I}$ and $f_\infty = f\chi_{(3I)^c}$. We set $a_I = \mathscr{C}(f_\infty)(c_I)$, where c_I is the center of I. Then we have

$$\begin{aligned}\frac{1}{|I|}\int_I |\mathscr{C}(f)(y) - a_I|\,dx &\le \frac{1}{|I|}\int_I \sup_{\xi\in\mathbf{R}} \big|H(M^\xi(f))(y) - H(M^\xi(f_\infty))(c_I)\big|\,dy \\ &\le B_1 + B_2\,,\end{aligned}$$

where

$$\begin{aligned}B_1 &= \frac{1}{|I|}\int_I \sup_{\xi\in\mathbf{R}} \big|H(M^\xi(f_0))(y)\big|\,dy\,,\\ B_2 &= \frac{1}{|I|}\int_I \sup_{\xi\in\mathbf{R}} \big|H(M^\xi(f_\infty))(y) - H(M^\xi(f_\infty))(c_I)\big|\,dy\,.\end{aligned}$$

But

$$\begin{aligned}B_1 &\le \frac{1}{|I|}\int_I \mathscr{C}(f_0)(y)\,dy\\ &\le \frac{1}{|I|}\big\|\mathscr{C}(f_0)\big\|_{L^r}\big\|\chi_I\big\|_{L^{r'}}\\ &\le \frac{\big\|\mathscr{C}\big\|_{L^r\to L^r}}{|I|}\big\|f_0\big\|_{L^r}|I|^{\frac{1}{r'}}\\ &\le C_r M_r(f)(x)\,,\end{aligned}$$

where we used the boundedness of the Carleson operator $\mathscr{C}$ from L^r to L^r and Theorem 1.4.17 (v).

We turn to the corresponding estimate for B_2. We have

$$\begin{aligned}B_2 &\le \frac{1}{|I|}\int_I\int_{\mathbf{R}^n} |f_\infty(z)|\left|\frac{1}{y-z} - \frac{1}{c_I - z}\right| dz\,dy\\ &= \frac{1}{|I|}\int_I\int_{(3I)^c} |f(z)|\left|\frac{y-c_I}{(y-z)(c_I-z)}\right| dz\,dy\\ &\le \int_I\left(\int_{(3I)^c} |f(z)|\,\frac{C}{(|c_I - z| + |I|)^2}\,dz\right)dy\\ &\le \int_I \frac{C}{|I|}M(f)(x)\,dy\\ &\le CM(f)(x)\\ &\le CM_r(f)(x)\,.\end{aligned}$$

This completes the proof of estimate (11.3.8), and we now turn to the proof of estimate (11.3.9). We derive (11.3.9) as a consequence of Exercise 9.4.9, provided we have that

$$\big\|M(\mathscr{C}(f))\big\|_{L^r(w)} < \infty. \tag{11.3.10}$$

Unfortunately, the finiteness estimate (11.3.10) for general functions f in $L^p(w)$ cannot be deduced easily without knowledge of the sought estimate (11.3.4) for $p = r$. However, we can show the validity of (11.3.10) for functions f with compact support and weights $w \in A_p$ that are bounded. This argument requires a few technicalities, which we now present. For a fixed constant B we introduce a truncated Carleson operator

$$\mathscr{C}^B(f) = \sup_{|\xi|\le B} |H(M^\xi(f))|.$$

Next we work with a weight w in A_p that is bounded. In fact, we work with $w_k = \min(w,k)$, which satisfies

$$[w_k]_{A_p} \le (1+2^{p-2})\big(1+[w]_{A_p}\big)$$

for all $k \ge 1$ (see Exercise 9.1.9). Finally, we take $f = h$ to be a smooth function with support contained in an interval $[-R,R]$. Then for $|\xi| \le B$ we have

$$|H(M^\xi(h))(x)| \le 2R\big\|(M^\xi(h))'\big\|_{L^\infty}\chi_{|x|\le 2R} + \frac{\big\|h\big\|_{L^1}}{|x|+R}\chi_{|x|>2R} \le \frac{BC_h R}{|x|+R},$$

where C_h is a constant that depends on h. This implies that the last estimate also holds for $\mathscr{C}^B(h)$. Using Example 2.1.8, we now obtain

$$M(\mathscr{C}^B(h))(x) \le BC_h \frac{\log\big(1+\frac{|x|}{R}\big)}{1+\frac{|x|}{R}}.$$

It follows that $M(\mathscr{C}^B(h))$ lies in $L^r(w_k)$, since $r > 1$ and $w_k \le k$. Therefore,

$$\big\|M(\mathscr{C}^B(f))\big\|_{L^r(w_k)} < \infty,$$

and thus (11.3.10) holds in this setting. Applying the previous argument to $\mathscr{C}^B(h)$ and the weight w_k [in lieu of $\mathscr{C}(f)$ and w], we obtain (11.3.7) and thus (11.3.4) for $M(\mathscr{C}^B(h))$ and the weight w_k. This establishes the estimate

$$\big\|\mathscr{C}^B(h)\big\|_{L^p(w_k)} \le C(p,[w]_{A_p})\big\|h\big\|_{L^p(w_k)} \tag{11.3.11}$$

for some constant $C(p,[w]_{A_p})$ that is independent of B and k, for functions h that are smooth and compactly supported. Letting $k \to \infty$ in (11.3.11) and applying Fatou's lemma, we obtain (11.3.4) for smooth functions h with compact support. From this we deduce the validity of (11.3.4) for general functions f in $L^p(w)$ by density (cf. Exercise 4.3.11).

Finally, to obtain (11.3.5) for general $f \in L^p(w)$, we raise (11.3.3) to the power p, use the inequality $(a+b)^p \le 2^p(a^p+b^p)$, and integrate over $\mathbf{R}$ with respect to the measure $w\,dx$ to obtain

$$\int_{\mathbf{R}} \mathscr{C}_*(f)^p w\,dx \le 2^p c \int_{\mathbf{R}} M(f)^p w\,dx + 2^p \int_{\mathbf{R}} M(\mathscr{C}(f))^p w\,dx. \tag{11.3.12}$$

Then we use estimate (11.3.4) and the boundedness of the Hardy–Littlewood maximal operator on $L^p(w)$ to obtain the required conclusion. □

Exercises

11.3.1. (a) Let $\theta \in \mathbf{S}^{n-1}$. Define the *maximal directional Carleson operator*

$$\mathscr{C}_*^\theta(f)(x) = \sup_{a\in\mathbf{R}} \sup_{\varepsilon>0} \left| \int_{\varepsilon<|t|<\varepsilon^{-1}} e^{2\pi i a t} f(x-t\theta)\,\frac{dt}{t} \right|$$

for functions f on $\mathbf{R}^n$. Prove that $\mathscr{C}_*^\theta$ is bounded on $L^p(\mathbf{R}^n, w)$ for any weight $w \in A_p$ and $1<p<\infty$.
(b) Let Ω be an odd integrable function on $\mathbf{S}^{n-1}$. Obtain the same conclusion for the maximal operator

$$\mathscr{C}_*^\Omega(f)(x) = \sup_{\xi\in\mathbf{R}^n} \sup_{\varepsilon>0} \left| \int_{\varepsilon<|y|<\varepsilon^{-1}} e^{2\pi i \xi\cdot y} f(x-y)\,\frac{\Omega\big(\frac{y}{|y|}\big)}{|y|^n}\,dy \right|.$$

[*Hint:* Part (a): Reduce to the case $\theta = e_1 = (1,0,\dots,0)$ via a rotation and use Theorem 11.3.3 with $w=1$. Part (b): Use the method of rotations and part (a).]

11.3.2. For a fixed $\lambda > 0$ write

$$\big\{x\in\mathbf{R} :\ \mathscr{C}_*(f)(x) > \lambda\big\} = \bigcup_j I_j\,,$$

where $I_j = (\alpha_j, \alpha_j+\delta_j)$ are open disjoint intervals. Let $1<r<\infty$. Show that there exists a $\gamma_0 > 0$ such that for every $0<\gamma<\gamma_0$ there exists a constant $C_\gamma > 0$ such that $\lim_{\gamma\to 0} C_\gamma = 0$ and

$$\big|\{x\in I_j : \mathscr{C}_*(f)(x) > 3\lambda,\ M_r(f)(x) \le \gamma\lambda\}\big| \le C_\gamma \big|I_j\big|$$

for all f for which $C_*(f)$ is defined.
[*Hint:* Note that we must have $\mathscr{C}_*(f)(\alpha_j) \le \lambda$ and $\mathscr{C}_*(f)(\alpha_j+\delta_j) \le \lambda$ for all j. Set $I_j^* = (\alpha_j - 5\delta_j, \alpha_j + 6\delta_j)$, $f_1(x) = f(x)$ for $x \in I_j^*$, $f_1(x) = 0$ for $x \notin I_j^*$, and $f_2(x) = f(x) - f_1(x)$. We may assume that for all j there exists a z_j in I_j such that $M_r(f)(z_j) \le \gamma\lambda$. For fixed $x \in I_j$ estimate $|H^{(\varepsilon)}(f_2)(x) - H^{(\varepsilon)}(f_2)(\alpha_j)|$ by the threefold sum

$$\begin{aligned}\bigg|\int_{|\alpha_j-t|>\varepsilon} f_2(t)e^{2\pi i\xi t}\bigg(\frac{2}{\alpha_j-t}-\frac{2}{x-t}\bigg)\,dt\bigg| \\ +\bigg|\int_{|x-t|>\varepsilon\geq|\alpha_j-t|} f_2(t)e^{2\pi i\xi t}\frac{1}{x-t}\,dt\bigg| \\ +\bigg|\int_{|\alpha_j-t|>\varepsilon\geq|x-t|} f_2(t)e^{2\pi i\xi t}\frac{1}{\alpha_j-t}\,dt\bigg|,\end{aligned}$$

which is easily shown to be controlled by $c_0 M(f)(z_j)$ for some constant c_0. Thus $\mathscr{C}_*(f_2)(x)\leq\mathscr{C}_*(f_2)(\alpha_j)+c_0M(f)(z_j)\leq\lambda+c_0\gamma\lambda$. Select γ_0 such that $c_0\gamma_0<\frac{1}{2}$. Then $\lambda+c_0\gamma\lambda<\frac{3}{2}\lambda$ for $\gamma<\gamma_0$; hence we have $\mathscr{C}_*(f)(x)\leq\mathscr{C}_*(f_1)(x)+\frac{3}{2}\lambda$ for $x\in I_j$ and thus $I_j\cap\{\mathscr{C}_*(f)>3\lambda\}\subseteq\{\mathscr{C}_*(f_1)>\lambda\}$. Using the boundedness of $\mathscr{C}_*$ on L^r and the fact that $M_r(f)(z_j)\leq\gamma\lambda$, we obtain that the last set has measure at most a constant multiple of $\gamma^r|I_j|$.]

11.3.3. (*Hunt and Young [173]*) Show that for every w in A_∞ there is a finite constant $\gamma_0>0$ such that for all $0<\gamma<\gamma_0$ and all $1<r<\infty$ there is a constant B_γ such that

$$w\big(\{\mathscr{C}_*(f)>3\lambda\}\cap\{M_r(f)\leq\gamma\lambda\}\big)\leq B_\gamma w\big(\{\mathscr{C}_*(f)>\lambda\}\big)$$

for all f for which $C_*(f)$ is finite. Moreover, the constants B_γ satisfy $B_\gamma\to 0$ as $\gamma\to 0$.

[*Hint:* Start with positive constants C_0 and δ such that for all intervals I and any measurable set E we have $|E\cap I|\leq\varepsilon|I| \implies w(E\cap I)\leq C_0\,\varepsilon^\delta w(I)$. Use the estimate of Exercise 11.3.3 with $I=I_j$ and sum over j to obtain the required estimate with $B_\gamma=C_0\,(C\gamma)^\delta$.]

11.3.4. Prove the following vector-valued version of Theorem 11.2.1:

$$\bigg\|\Big(\sum_j|\mathscr{C}(f_j)|^r\Big)^{\frac{1}{r}}\bigg\|_{L^p(w)}\leq C_{p,r}(w)\bigg\|\Big(\sum_j|f_j|^r\Big)^{\frac{1}{r}}\bigg\|_{L^p(w)}$$

for all $1<p,r<\infty$, all weights $w\in A_p$, and all sequences of functions f_j in $L^p(w)$.

[*Hint:* You may want to use Corollary 9.5.7.]

HISTORICAL NOTES

A version of Theorem 11.1.1 concerning the maximal partial sum operator of Fourier series of square-integrable functions on the circle was first proved by Carleson [55]. An alternative proof of Carleson's theorem was provided by Fefferman [126], pioneering a set of ideas called time–frequency analysis. Lacey and Thiele [205] provided the first independent proof on the line of the boundedness of the maximal Fourier integral operator (11.1.1). The proof of Theorem 11.1.1 given in this text follows closely the one given in Lacey and Thiele [205], which improves in some ways that of Fefferman's [126], by which it was inspired. One may also consult the expository article of Thiele [312]. The proof of Lacey and Thiele was a byproduct of their work [203], [204] on the boundedness of the bilinear Hilbert transforms $H_\alpha(f_1,f_2)(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbf{R}} f_1(x-t)f_2(x-\alpha t)\,\frac{dt}{t}$. This family of operators arose in early attempts of A. Calderón to show that the first commutator (Example 8.3.8, $m=1$) is bounded on L^2 when A' is in L^∞, an approach completed only using the uniform boundedness of H_α obtained by Thiele [311], Grafakos and Li [150], and Li [212].

A version of Theorem 11.2.1 concerning the L^p boundedness, $1<p<\infty$, of the maximal partial sum operator on the circle was obtained by Hunt [170]. Sjölin [283] extended this result to $L(\log^+ L)(\log^+\log^+ L)$ and Antonov [5] to $L(\log^+ L)(\log^+\log^+\log^+ L)$. Counterexamples of Kolmogorov [191], [192], Körner [197], and Konyagin [193] indicate that the everywhere convergence of partial Fourier sums (or integrals) may fail for functions in L^1 and in spaces near L^1. The exponential decay estimate for $\alpha \ge 1$ in (11.2.1) and the restricted weak type (p,p) estimate with constant $Cp^2(p-1)^{-1}$ for the maximal partial sum operator on the circle are contained in Hunt's article [170]. The estimate for $\alpha < 1$ in (11.2.1) appears in the article of Grafakos, Tao, and Terwilleger [153]; the proof of Theorem 11.2.1 is based on this article. This article also investigates higher-dimensional analogues of the theory that were initiated in Pramanik and Terwilleger [266]. Theorem 11.3.3 was first obtained by Hunt and Young [173] using a good lambda inequality for the Carleson operator. An improved good lambda inequality for the Carleson operator is contained in of Grafakos, Martell, and Soria [152]. The particular proof of Theorem 11.3.3 given in the text is based on the approach of Rubio de Francia, Ruiz, and Torrea [276]. The books of Jørsboe and Melbro [179], Mozzochi [236], and Arias de Reyna [6] contain detailed presentations of the Carleson–Hunt theorem on the circle.

The subject of Fourier analysis is currently enjoying a surge of activity. Emerging connections with analytic number theory, combinatorics, geometric measure theory, partial differential equations, and multilinear analysis introduce new dynamics and present promising developments. These connections are also creating new research directions that extend beyond the scope of this book.

Glossary

$A \subseteq B$	A is a subset of B (not necessarily a proper subset)
$A \subsetneqq B$	A is a proper subset of B
A^c	the complement of a set A
χ_E	the characteristic function of the set E
d_f	the distribution function of a function f
f^*	the decreasing rearrangement of a function f
$f_n \uparrow f$	f_n increases monotonically to a function f
$\mathbf{Z}$	the set of all integers
$\mathbf{Z}^+$	the set of all positive integers $\{1,2,3,\dots\}$
$\mathbf{Z}^n$	the n-fold product of the integers
$\mathbf{R}$	the set of real numbers
$\mathbf{R}^+$	the set of positive real numbers
$\mathbf{R}^n$	the Euclidean n-space
$\mathbf{Q}$	the set of rationals
$\mathbf{Q}^n$	the set of n-tuples with rational coordinates
$\mathbf{C}$	the set of complex numbers
$\mathbf{C}^n$	the n-fold product of complex numbers
$\mathbf{T}$	the unit circle identified with the interval $[0,1]$
$\mathbf{T}^n$	the n-dimensional torus $[0,1]^n$,
$\lvert x\rvert$	$\sqrt{\lvert x_1\rvert^2+\cdots+\lvert x_n\rvert^2}$ when $x=(x_1,\dots,x_n)\in\mathbf{R}^n$
$\mathbf{S}^{n-1}$	the unit sphere $\{x\in\mathbf{R}^n:\ \lvert x\rvert=1\}$

e_j	the vector $(0,\dots,0,1,0,\dots,0)$ with 1 in the jth entry and 0 elsewhere
$\log t$	the logarithm to base e of $t>0$
$\log_a t$	the logarithm to base a of $t>0$ $(1\neq a>0)$
$\log^+ t$	$\max(0,\log t)$ for $t>0$
$[t]$	the integer part of the real number t
$x\cdot y$	the quantity $\sum_{j=1}^n x_j y_j$ when $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$
$B(x,R)$	the ball of radius R centered at x in $\mathbf{R}^n$
ω_{n-1}	the surface area of the unit sphere $\mathbf{S}^{n-1}$
v_n	the volume of the unit ball $\{x\in\mathbf{R}^n:\ \lvert x\rvert<1\}$
$\lvert A\rvert$	the Lebesgue measure of the set $A\subseteq\mathbf{R}^n$
dx	Lebesgue measure
$\mathrm{Avg}_B f$	the average $\frac{1}{\lvert B\rvert}\int_B f(x)\,dx$ of f over the set B
$\langle f,g\rangle$	the real inner product $\int_{\mathbf{R}^n} f(x)g(x)\,dx$
$\langle f\,\vert\,g\rangle$	the complex inner product $\int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx$
$\langle u,f\rangle$	the action of a distribution u on a function f
p'	the number $p/(p-1)$, whenever $0<p\neq 1<\infty$
$1'$	the number ∞
∞'	the number 1
$f=O(g)$	means $\lvert f(x)\rvert\leq M\lvert g(x)\rvert$ for some M for x near x_0
$f=o(g)$	means $\lvert f(x)\rvert\,\lvert g(x)\rvert^{-1}\to 0$ as $x\to x_0$
A^t	the transpose of the matrix A
A^*	the conjugate transpose of a complex matrix A
A^{-1}	the inverse of the matrix A
$O(n)$	the space of real matrices satisfying $A^{-1}=A^t$
$\lVert T\rVert_{X\to Y}$	the norm of the (bounded) operator $T:X\to Y$
$A\approx B$	means that there exists a $c>0$ such that $c^{-1}\leq\frac{B}{A}\leq c$
$\lvert\alpha\rvert$	indicates the size $\lvert\alpha_1\rvert+\cdots+\lvert\alpha_n\rvert$ of a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$
$\partial_j^m f$	the mth partial derivative of $f(x_1,\dots,x_n)$ with respect to x_j
$\partial^\alpha f$	$\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f$
$\mathscr{C}^k$	the space of functions f with $\partial^\alpha f$ continuous for all $\lvert\alpha\rvert\leq k$

$\mathscr{C}_0$	space of continuous functions with compact support
$\mathscr{C}_{00}$	the space of continuous functions that vanish at infinity
$\mathscr{C}_0^\infty$	the space of smooth functions with compact support
$\mathscr{D}$	the space of smooth functions with compact support
$\mathscr{S}$	the space of Schwartz functions
$\mathscr{C}^\infty$	the space of smooth functions $\bigcap_{k=1}^\infty \mathscr{C}^k$
$\mathscr{D}'(\mathbf{R}^n)$	the space of distributions on $\mathbf{R}^n$
$\mathscr{S}'(\mathbf{R}^n)$	the space of tempered distributions on $\mathbf{R}^n$
$\mathscr{E}'(\mathbf{R}^n)$	the space of distributions with compact support on $\mathbf{R}^n$
$\mathscr{P}$	the set of all complex-valued polynomials of n real variables
$\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}$	the space of tempered distributions on $\mathbf{R}^n$ modulo polynomials
$\ell(Q)$	the side length of a cube Q in $\mathbf{R}^n$
∂Q	the boundary of a cube Q in $\mathbf{R}^n$
$L^p(X,\mu)$	the Lebesgue space over the measure space (X,μ)
$L^p(\mathbf{R}^n)$	the space $L^p(\mathbf{R}^n, \lvert\cdot\rvert)$
$L^{p,q}(X,\mu)$	the Lorentz space over the measure space (X,μ)
$L^p_{\mathrm{loc}}(\mathbf{R}^n)$	the space of functions that lie in $L^p(K)$ for any compact set K in $\mathbf{R}^n$
$\lvert d\mu\rvert$	the total variation of a finite Borel measure μ on $\mathbf{R}^n$
$\mathscr{M}(\mathbf{R}^n)$	the space of all finite Borel measures on $\mathbf{R}^n$
$\mathscr{M}_p(\mathbf{R}^n)$	the space of L^p Fourier multipliers, $1 \le p \le \infty$
$\mathscr{M}^{p,q}(\mathbf{R}^n)$	the space of translation-invariant operators that map $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$
$\lVert\mu\rVert_{\mathscr{M}}$	$\int_{\mathbf{R}^n} \lvert d\mu\rvert$ the norm of a finite Borel measure μ on $\mathbf{R}^n$
$\mathcal{M}$	the centered Hardy–Littlewood maximal operator with respect to balls
M	the uncentered Hardy–Littlewood maximal operator with respect to balls
$\mathcal{M}_c$	the centered Hardy–Littlewood maximal operator with respect to cubes
M_c	the uncentered Hardy–Littlewood maximal operator with respect to cubes
$\mathcal{M}_\mu$	the centered maximal operator with respect to a measure μ
M_μ	the uncentered maximal operator with respect to a measure μ
M_s	the strong maximal operator
M_d	the dyadic maximal operator
$M^\#$	the sharp maximal operator

$\mathscr{M}$	the grand maximal operator
$L_s^p(\mathbf{R}^n)$	the inhomogeneous L^p Sobolev space
$\dot{L}_s^p(\mathbf{R}^n)$	the homogeneous L^p Sobolev space
$\Lambda_\alpha(\mathbf{R}^n)$	the inhomogeneous Lipschitz space
$\dot{\Lambda}_\alpha(\mathbf{R}^n)$	the homogeneous Lipschitz space
$H^p(\mathbf{R}^n)$	the real Hardy space on $\mathbf{R}^n$
$B_{s,q}^p(\mathbf{R}^n)$	the inhomogeneous Besov space on $\mathbf{R}^n$
$\dot{B}_{s,q}^p(\mathbf{R}^n)$	the homogeneous Besov space on $\mathbf{R}^n$
$\dot{B}_{s,q}^p(\mathbf{R}^n)$	the homogeneous Besov space on $\mathbf{R}^n$
$F_{s,q}^p(\mathbf{R}^n)$	the inhomogeneous Triebel–Lizorkin space on $\mathbf{R}^n$
$\dot{F}_{s,q}^p(\mathbf{R}^n)$	the homogeneous Triebel–Lizorkin space on $\mathbf{R}^n$
$BMO(\mathbf{R}^n)$	the space of functions of bounded mean oscillation on $\mathbf{R}^n$