Chapter 10
Boundedness and Convergence of Fourier Integrals

In this chapter we return to fundamental questions in Fourier analysis related to convergence of Fourier series and Fourier integrals. Our main goal is to understand in what sense the inversion property of the Fourier transform

$$f(x) = \int_{\mathbf{R}^n} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$$

holds when f is a function on $\mathbf{R}^n$. This question is equivalent to the corresponding question for the Fourier series

$$f(x) = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m) e^{2\pi i x\cdot m}$$

when f is a function on $\mathbf{T}^n$. The main problem is that the function (or sequence) $\widehat{f}$ may not be integrable and the convergence of the preceding integral (or series) needs to be suitably interpreted. To address this issue, a summability method is employed. This is achieved by the introduction of a localizing factor $\Phi(\xi/R)$, leading to the study of the convergence of the expressions

$$\int_{\mathbf{R}^n} \Phi(\xi/R)\widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$$

as $R\to\infty$. Here Φ is a function on $\mathbf{R}^n$ that decays sufficiently rapidly at infinity and satisfies $\Phi(0) = 1$. For instance, we may take $\Phi = \chi_{B(0,1)}$, where $B(0,1)$ is the unit ball in $\mathbf{R}^n$. Analogous summability methods arise in the torus.

An interesting case arises when $\Phi(\xi) = (1-|\xi|^2)_+^\lambda$, $\lambda \ge 0$, in which we obtain the Bochner–Riesz means introduced by Riesz when $n = 1$ and $\lambda = 0$ and Bochner for $n \ge 2$ and general $\lambda > 0$. The question is whether the Bochner–Riesz means

$$\sum_{m_1^2+\cdots+m_n^2\le R^2} \left(1-\frac{m_1^2+\cdots+m_n^2}{R^2}\right)^\lambda \widehat{f}(m_1,\dots,m_n)\, e^{2\pi i(m_1x_1+\cdots+m_nx_n)}$$

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converge in L^p. This question is equivalent to whether the function $(1-|\xi|^2)_+^\lambda$ is an L^p multiplier on $\mathbf{R}^n$ and is investigated in this chapter. Analogous questions concerning the almost everywhere convergence of these families are also studied.

10.1 The Multiplier Problem for the Ball

In this section we show that the characteristic function of the unit disk in $\mathbf{R}^2$ is not an L^p multiplier when $p \neq 2$. This implies the same conclusion in dimensions $n \geq 3$, since sections of higher-dimensional balls are disks and by Theorem 2.5.16 we have that if $\chi_{B(0,r)} \notin \mathscr{M}_p(\mathbf{R}^2)$ for all $r > 0$, then $\chi_{B(0,1)} \notin \mathscr{M}_p(\mathbf{R}^n)$ for any $n \geq 3$.

10.1.1 Sprouting of Triangles

We begin with a certain geometric construction that at first sight has no apparent relationship to the multiplier problem for the ball in $\mathbf{R}^n$. Given a triangle ABC with base $b = AB$ and height h_0 we let M be the midpoint of AB. We construct two other triangles AMF and BME from ABC as follows. We fix a height $h_1 > h_0$ and we extend the sides AC and BC in the direction away from its base until they reach a certain height h_1. We let E be the unique point on the line passing through the points B and C such that the triangle EMB has height h_1. Similarly, F is uniquely chosen on the line through A and C so that the triangle AMF has height h_1.

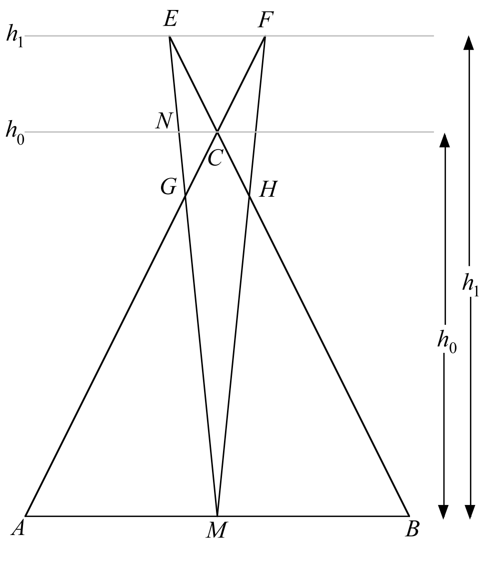

Fig. 10.1 The sprouting of the triangle *ABC*.

The triangle ABC now gives rise to two triangles AMF and BME called the *sprouts* of ABC. The union of the two sprouts AMF and BME is called the *sprouted figure* obtained from ABC and is denoted by $\mathrm{Spr}(ABC)$. Clearly $\mathrm{Spr}(ABC)$ contains ABC. We call the difference

$$\mathrm{Spr}(ABC) \setminus ABC$$

the *arms* of the sprouted figure. The sprouted figure $\mathrm{Spr}(ABC)$ has two arms of equal area, the triangles EGC and FCH as shown in Figure 10.1, and we can precisely compute the area of each arm. One may easily check (see Exercise 10.1.1) that

$$\text{Area (each arm of Spr}(ABC)) = \frac{b}{2}\frac{(h_1-h_0)^2}{2h_1-h_0}, \tag{10.1.1}$$

where $b = AB$.

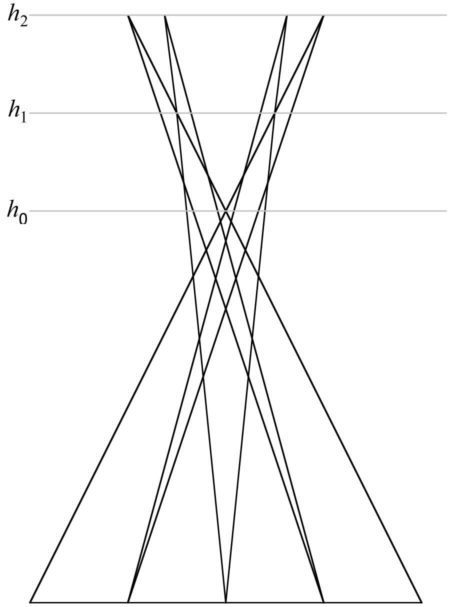

Fig. 10.2 The second step of the construction.

We start with an isosceles triangle $\Lambda = ABC$ in $\mathbf{R}^2$ with base AB of length $b_0 = \varepsilon$ and height $MC = h_0 = \varepsilon$, where M is the midpoint of AB. We define the heights

$$\begin{aligned}
h_1 &= \Big(1+\frac{1}{2}\Big)\varepsilon\,,\\
h_2 &= \Big(1+\frac{1}{2}+\frac{1}{3}\Big)\varepsilon\,,\\
&\cdots\\
h_j &= \Big(1+\frac{1}{2}+\cdots+\frac{1}{j+1}\Big)\varepsilon\,.
\end{aligned}$$

We apply the previously described sprouting procedure to Λ to obtain two sprouts $\Lambda_1 = AMF$ and $\Lambda_2 = EMB$, as in Figure 10.1, each with height h_1 and base length $b_0/2$. We now apply the same procedure to the triangles Λ_1 and Λ_2. We then obtain two sprouts Λ_{11} and Λ_{12} from Λ_1 and two sprouts Λ_{21} and Λ_{22} from Λ_2, a total of four sprouts with height h_2. See Figure 10.2. We continue this process, obtaining at the jth step 2^j sprouts $\Lambda_{r_1 \dots r_j}$, $r_1, \dots, r_j \in \{1,2\}$ each with base length $b_j = 2^{-j} b_0$ and height h_j. We stop this process when the kth step is completed.

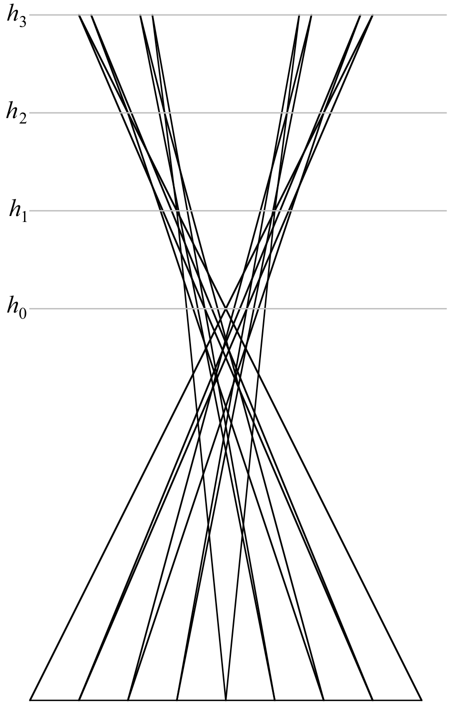

Fig. 10.3 The third step of the construction.

We let $E(\varepsilon, k)$ be the union of the triangles $\Lambda_{r_1 \dots r_k}$ over all sequences r_j of 1's and 2's. We obtain an estimate for the area of $E(\varepsilon, k)$ by adding to the area of Λ the areas of the arms of all the sprouted figures obtained during the construction. By (10.1.1) we have that each of the 2^j arms obtained at the jth step has area

$$\frac{b_{j-1}}{2} \frac{(h_j - h_{j-1})^2}{2h_j - h_{j-1}}.$$

Summing over all these areas and adding the area of the original triangle, we obtain the estimate

$$
\begin{aligned}
|E(\varepsilon,k)| &= \frac{1}{2}\varepsilon^2 + \sum_{j=1}^{k} 2^j \frac{b_{j-1}}{2} \frac{(h_j - h_{j-1})^2}{2h_j - h_{j-1}} \\
&\le \frac{1}{2}\varepsilon^2 + \sum_{j=1}^{k} 2^j \frac{2^{-(j-1)}b_0}{2} \frac{\varepsilon^2}{(j+1)^2\varepsilon} \\
&\le \frac{1}{2}\varepsilon^2 + \sum_{j=2}^{\infty} \frac{\varepsilon^2}{j^2} \le \Big(\frac{1}{2} + \frac{\pi^2}{6} - 1\Big)\varepsilon^2 \\
&\le \frac{3}{2}\varepsilon^2,
\end{aligned}
$$

where we used the fact that $2h_j - h_{j-1} \ge \varepsilon$ for all $j \ge 1$.

Having completed the construction of the set $E(\varepsilon,k)$, we are now in a position to indicate some of the ideas that appear in the solution of the Kakeya problem. We first observe that no matter what k is, the measure of the set $E(\varepsilon,k)$ can be made as small as we wish if we take ε small enough. Our purpose is to make a needle of infinitesimal width and unit length move continuously from one side of this angle to the other utilizing each sprouted triangle in succession. To achieve this, we need to apply a similar construction to any of the 2^k triangles that make up the set $E(\varepsilon,k)$ and repeat the sprouting procedure a large enough number of times. We refer to [99] for details. An elaborate construction of this sort yields a set within which the needle can be turned only through a fixed angle. But adjoining a few such sets together allows us to rotate a needle through a half-turn within a set that still has arbitrarily small area. This is the idea used to solve the aforementioned needle problem.

10.1.2 The counterexample

We now return to the multiplier problem for the ball, which has an interesting connection with the Kakeya needle problem.

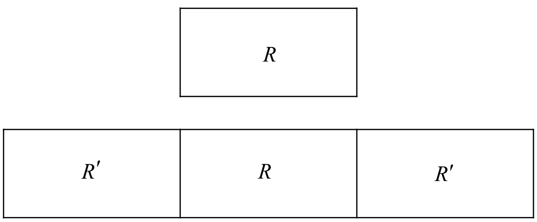

Fig. 10.4 A rectangle R and its adjacent rectangles R'.

In the discussion that follows we employ the following notation. Given a rectangle R in $\mathbf{R}^2$, we let R' be two copies of R adjacent to R along its shortest side so that $R \cup R'$ has the same width as R but three times its length. See Figure 10.4.

We need the following lemma.

Lemma 10.1.1. *Let $\delta > 0$ be a given number. Then there exists a measurable subset E of $\mathbf{R}^2$ and a finite collection of rectangles R_j in $\mathbf{R}^2$ such that*

(1) The R_j's are pairwise disjoint.
(2) We have $1/2 \le |E| \le 3/2$.
(3) We have $|E| \le \delta \sum_j |R_j|$.
(4) For all j we have $|R'_j \cap E| \ge \frac{1}{12}|R_j|$.

Proof. We start with an isosceles triangle ABC in the plane with height 1 and base AB, where $A = (0,0)$ and $B = (1,0)$. Given $\delta > 0$, we find a positive integer k such that $k+2 > e^{1/\delta}$. For this k we set $E = E(1,k)$, the set constructed earlier with $\varepsilon = 1$. We then have $1/2 \le |E| \le 3/2$; thus (2) is satisfied.

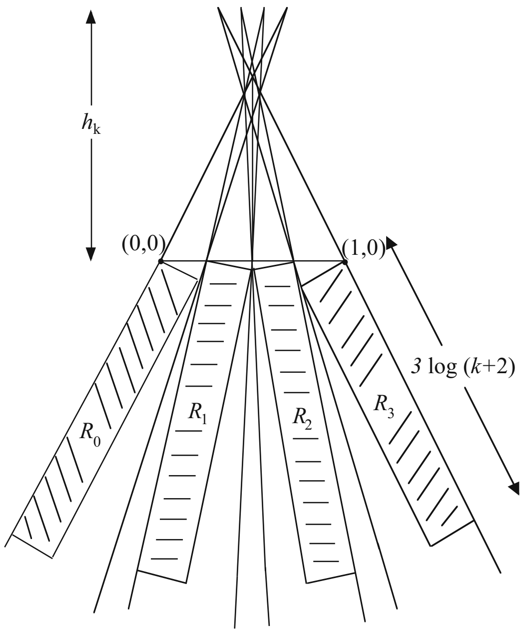

Fig. 10.5 The rectangles R_j.

Recall that each dyadic interval $[j2^{-k},(j+1)2^{-k}]$ in $[0,1]$ is the base of exactly one sprouted triangle $A_jB_jC_j$, where $j \in \{0,1,\dots,2^k-1\}$. Here we set $A_j = (j2^{-k},0)$, $B_j = ((j+1)2^{-k},0)$, and C_j the other vertex of the sprouted triangle. We define a rectangle R_j inside the angle $\angle A_jC_jB_j$ as in Figure 10.6. The rectangle R_j is defined so that one of its vertices is either A_j or B_j and the length of its longest side is $3\log(k+2)$.

We now make some calculations. First we observe that the longest possible length that either A_jC_j or B_jC_j can achieve is $\sqrt{5}h_k/2$. By symmetry we may assume that the length of A_jC_j is larger than that of B_jC_j as in Figure 10.6. We now have that

$$\frac{\sqrt{5}}{2}h_k < \frac{3}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{k+1}\right) < \frac{3}{2}\big(1+\log(k+1)\big) < 3\log(k+2),$$

since $k \geq 1$ and $e < 3$. Hence R'_j contains the triangle $A_jB_jC_j$. We also have that

$$h_k = 1+\frac{1}{2}+\cdots+\frac{1}{k+1} > \log(k+2).$$

Using these two facts, we obtain

$$|R'_j \cap E| \geq \text{Area}(A_jB_jC_j) = \frac{1}{2}2^{-k}h_k > 2^{-k-1}\log(k+2). \tag{10.1.2}$$

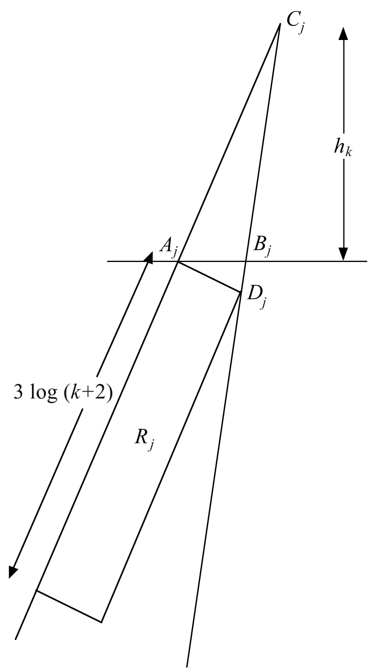

Fig. 10.6 A closer look at R_j.

Denote by $|XY|$ the length of the line segment through the points X and Y. The law of sines applied to the triangle $A_jB_jD_j$ gives

$$|A_jD_j| = 2^{-k}\frac{\sin(\angle A_jB_jD_j)}{\sin(\angle A_jD_jB_j)} \leq \frac{2^{-k}}{\cos(\angle A_jC_jB_j)}. \tag{10.1.3}$$

But the law of cosines applied to the triangle $A_jB_jC_j$ combined with the estimates $h_k \leq |A_jC_j|, |B_jC_j| \leq \sqrt{5}h_k/2$ give that

$$\cos(\angle A_jC_jB_j) \geq \frac{h_k^2 + h_k^2 - (2^{-k})^2}{2\,\frac{5}{4}\,h_k^2} \geq \frac{4}{5} - \frac{2}{5}\cdot\frac{1}{4} \geq \frac{1}{2}\,. \tag{10.1.4}$$

Combining (10.1.3) and (10.1.4), we obtain

$$|A_jD_j| \leq 2^{-k+1} = 2|A_jB_j|\,.$$

Using this fact and (10.1.2), we deduce

$$|R_j' \cap E| \geq 2^{-k-1}\log(k+2) = \frac{1}{12}2^{-k+1}3\log(k+2) \geq \frac{1}{12}|R_j|\,,$$

which proves the required conclusion (4).

Conclusion (1) in Lemma 10.1.1 follows from the fact that the regions inside the angles $\angle A_jC_jB_j$ and under the triangles $A_jC_jB_j$ are pairwise disjoint. This is shown in Figure 10.5. This can be proved rigorously by a careful examination of the construction of the sprouted triangles $A_jC_jB_j$, but the details are omitted.

It remains to prove (3). To achieve this we first estimate the length of the line segment A_jD_j from below. The law of sines gives

$$\frac{|A_jD_j|}{\sin(\angle A_jB_jD_j)} = \frac{2^{-k}}{\sin(\angle A_jD_jB_j)}\,,$$

from which we obtain that

$$|A_jD_j| \geq 2^{-k}\sin(\angle A_jB_jD_j) \geq 2^{-k-1}\angle A_jB_jD_j \geq 2^{-k-1}\angle B_jA_jC_j\,.$$

(All angles are measured in radians.) But the smallest possible value of the angle $\angle B_jA_jC_j$ is attained when $j=0$, in which case $\angle B_0A_0C_0 = \arctan 2 > 1$. This gives that

$$|A_jD_j| \geq 2^{-k-1}\,.$$

It follows that each R_j has area at least $2^{-k-1}3\log(k+2)$. Therefore,

$$\left|\bigcup_{j=0}^{2^k-1} R_j\right| = \sum_{j=0}^{2^k-1}|R_j| \geq 2^k 2^{-k-1}3\log(k+2) \geq |E|\log(k+2) \geq \frac{|E|}{\delta}\,,$$

since $|E| \leq 3/2$ and k was chosen so that $k+2 > e^{1/\delta}$. □

Next we have a calculation involving the Fourier transforms of characteristic functions of rectangles.

Proposition 10.1.2. *Let R be a rectangle whose center is the origin in $\mathbf{R}^2$ and let v be a unit vector parallel to its longest side. Consider the half-plane*

$$\mathscr{H} = \{x \in \mathbf{R}^2 :\ x\cdot v \geq 0\}$$

and the multiplier operator

$$S_{\mathscr{H}}(f) = (\widehat{f}\chi_{\mathscr{H}})^{\vee}.$$

Then we have $|S_{\mathscr{H}}(\chi_R)| \ge \frac{1}{10}\chi_{R'}$.

Remark 10.1.3. Applying a translation, we see that the same conclusion is valid for any rectangle in $\mathbf{R}^2$ whose longest side is parallel to v.

Proof. Applying a rotation, we reduce the problem to the case $R = [-a,a] \times [-b,b]$, where $0 < a \le b < \infty$, and $v = e_2 = (0,1)$. Since the Fourier transform acts in each variable independently, we have the identity

$$\begin{aligned} S_{\mathscr{H}}(\chi_R)(x_1,x_2) &= \chi_{[-a,a]}(x_1)\big(\widehat{\chi_{[-b,b]}}\chi_{[0,\infty)}\big)^{\vee}(x_2) \\ &= \chi_{[-a,a]}(x_1)\,\frac{I+iH}{2}(\chi_{[-b,b]})(x_2). \end{aligned}$$

It follows that

$$\begin{aligned} |S_{\mathscr{H}}(\chi_R)(x_1,x_2)| &\ge \frac{1}{2}\chi_{[-a,a]}(x_1)|H(\chi_{[-b,b]})(x_2)| \\ &= \frac{1}{2\pi}\chi_{[-a,a]}(x_1)\left|\log\left|\frac{x_2+b}{x_2-b}\right|\right|. \end{aligned}$$

But for $(x_1,x_2) \in R'$ we have $\chi_{[-a,a]}(x_1) = 1$ and $b < |x_2| < 3b$. So we have two cases, $b < x_2 < 3b$ and $-3b < x_2 < -b$. When $b < x_2 < 3b$ we see that

$$\left|\frac{x_2+b}{x_2-b}\right| = \frac{x_2+b}{x_2-b} > 2,$$

and similarly, when $-3b < x_2 < -b$ we have

$$\left|\frac{x_2-b}{x_2+b}\right| = \frac{b-x_2}{-b-x_2} > 2.$$

It follows that for $(x_1,x_2) \in R'$ the lower estimate is valid:

$$|S_{\mathscr{H}}(\chi_R)(x_1,x_2)| \ge \frac{\log 2}{2\pi} \ge \frac{1}{10}.$$

□

Next we have a lemma regarding vector-valued inequalities of half-plane multipliers.

Lemma 10.1.4. *Let* $v_1, v_2, \dots, v_j, \dots$ *be a sequence of unit vectors in* $\mathbf{R}^2$. *Define the half-planes*

$$\mathscr{H}_j = \{x \in \mathbf{R}^2 : \; x \cdot v_j \ge 0\} \tag{10.1.5}$$

and linear operators

$$S_{\mathscr{H}_j}(f) = (\widehat{f}\,\chi_{\mathscr{H}_j})^{\vee}.$$

Assume that the disk multiplier operator

$$T(f) = (\widehat{f}\,\chi_{B(0,1)})^{\vee}$$

maps $L^p(\mathbf{R}^2)$ *to itself with norm* $B_p < \infty$*. Then we have the inequality*

$$\Big\|\Big(\sum_j |S_{\mathscr{H}_j}(f_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le B_p \Big\|\big(\sum_j |f_j|^2\big)^{\frac{1}{2}}\Big\|_{L^p} \tag{10.1.6}$$

for all bounded and compactly supported functions f_j*.*

Proof. We prove the lemma for Schwartz functions f_j and we obtain the general case by a simple limiting argument. We define disks $D_{j,R} = \{x \in \mathbf{R}^2 :\ |x - Rv_j| \le R\}$ and we let

$$T_{j,R}(f) = (\widehat{f}\,\chi_{D_{j,R}})^{\vee}$$

be the multiplier operator associated with the disk $D_{j,R}$. We observe that $\chi_{D_{j,R}} \to \chi_{\mathscr{H}_j}$ pointwise as $R \to \infty$, as shown in Figure 10.7.

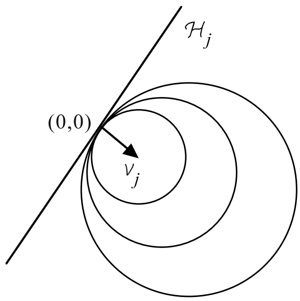

Fig. 10.7 A sequence of disks converging to a half-plane.

For $f \in \mathscr{S}(\mathbf{R}^2)$ and every $x \in \mathbf{R}^2$ we have

$$\lim_{R\to\infty} T_{j,R}(f)(x) = S_{\mathscr{H}_j}(f)(x)$$

by passing the limit inside the convergent integral. Fatou's lemma now yields

$$\Big\|\big(\sum_j |S_{\mathscr{H}_j}(f_j)|^2\big)^{\frac{1}{2}}\Big\|_{L^p} \le \liminf_{R\to\infty} \Big\|\big(\sum_j |T_{j,R}(f_j)|^2\big)^{\frac{1}{2}}\Big\|_{L^p}. \tag{10.1.7}$$

Next we observe that the following identity is valid:

$$T_{j,R}(f)(x) = e^{2\pi i R v_j \cdot x} T_R(e^{-2\pi i R v_j \cdot (\cdot)} f)(x), \tag{10.1.8}$$

where T_R is the multiplier operator $T_R(f) = (\widehat{f}\chi_{B(0,R)})^{\vee}$. Setting $g_j = e^{-2\pi i R v_j \cdot (\cdot)} f_j$ and using (10.1.7) and (10.1.8), we deduce

$$\Big\|\Big(\sum_j |S_{\mathscr{H}_j}(f_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le \liminf_{R\to\infty} \Big\|\Big(\sum_j |T_R(g_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \tag{10.1.9}$$

Observe that the operator T_R is L^p bounded with the same norm B_p as T in view of identity (2.5.15). Applying Theorem 4.5.1, we obtain that the last term in (10.1.9) is bounded by

$$\liminf_{R\to\infty} \|T_R\|_{L^p\to L^p} \Big\|\Big(\sum_j |g_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} = B_p \Big\|\Big(\sum_j |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}.$$

Combining this inequality with (10.1.9), we obtain (10.1.6). □

We have now completed all the preliminary material we need to prove that the characteristic function of the unit disk in $\mathbf{R}^2$ is not an L^p multiplier if $p \neq 2$.

Theorem 10.1.5. *The characteristic function of the unit ball in $\mathbf{R}^n$ is not an L^p multiplier when $1 < p \neq 2 < \infty$.*

Proof. As mentioned earlier, in view of Theorem 2.5.16, it suffices to prove the result in dimension $n = 2$. By duality it suffices to prove the result when $p > 2$. Suppose that $\chi_{B(0,1)} \in \mathscr{M}_p(\mathbf{R}^2)$ for some $p > 2$, say with norm $B_p < \infty$.

Suppose that $\delta > 0$ is given. Let E and R_j be as in Lemma 10.1.1. We let $f_j = \chi_{R_j}$. Let v_j be the unit vector parallel to the long side of R_j and let H_j be the half-plane defined as in (10.1.5). Using Proposition 10.1.2, we obtain

$$\begin{aligned}
\int_E \sum_j |S_{\mathscr{H}_j}(f_j)(x)|^2\,dx &= \sum_j \int_E |S_{\mathscr{H}_j}(f_j)(x)|^2\,dx \\
&\ge \sum_j \int_E \frac{1}{10^2}\chi_{R_j'}(x)\,dx \\
&= \frac{1}{100}\sum_j |E\cap R_j'| \\
&\ge \frac{1}{1200}\sum_j |R_j|,
\end{aligned} \tag{10.1.10}$$

where we used condition (4) of Lemma 10.1.1 in the last inequality. Hölder's inequality with exponents $p/2$ and $(p/2)' = p/(p-2)$ gives

$$\begin{aligned}
\int_E \sum_j |S_{\mathscr{H}_j}(f_j)(x)|^2\,dx &\le |E|^{\frac{p-2}{p}} \Big\|\Big(\sum_j |S_{\mathscr{H}_j}(f_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}^2 \\
&\le B_p^2 |E|^{\frac{p-2}{p}} \Big\|\Big(\sum_j |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}^2 \\
&= B_p^2 |E|^{\frac{p-2}{p}} \Big(\sum_j |R_j|\Big)^{\frac{2}{p}} \\
&\le B_p^2 \delta^{\frac{p-2}{p}} \sum_j |R_j|,
\end{aligned} \tag{10.1.11}$$

where we used Lemma 10.1.4, the disjointness of the R_j's, and condition (3) of Lemma 10.1.1 successively. Combining (10.1.10) with (10.1.11), we obtain the inequality

$$\sum_j |R_j| \le 1200 B_p \delta^{\frac{p-2}{p}} \sum_j |R_j| ,$$

which provides a contradiction when δ is very small. □

Exercises

10.1.1. Prove identity (10.1.1).
$\big[$*Hint:* With the notation of Figure 10.1, first prove

$$\frac{h_1 - h_0}{h_1} = \frac{NC}{b/2}, \qquad \frac{\text{height}\,(NGC)}{h_0} = \frac{NC}{NC + b/2}$$

using similar triangles.$\big]$

10.1.2. Given a rectangle R, let R'' denote either of the two parts that make up R'. Prove that for any $k \in \mathbf{Z}^+$ and any $\delta > 0$, there exist rectangles S_j in $\mathbf{R}^2$, $0 \le j < 2^k$, with dimensions proportionate to $2^{-k} \times \log(k+1)$,

$$\Big| \bigcup_{j=0}^{2^k-1} S_j \Big| < \delta ,$$

such that for some choice of S_j'', the S_j'''s are disjoint.
$\big[$*Hint:* Consider the 2^k triangles that make up the set $E(\varepsilon, k)$ and choose each rectangle S_j inside a corresponding triangle. Then the parts of the S_j''s that point downward are disjoint. Choose ε depending on δ.$\big]$

10.1.3. Is the characteristic function of the cylinder

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \; \xi_1^2 + \xi_2^2 < 1\}$$

a Fourier multiplier on $L^p(\mathbf{R}^3)$ for $1 < p < \infty$ and $p \neq 2$?

10.1.4. Modify the ideas of the proof of Lemma 10.1.4 to show that the characteristic function of the set

$$\{(\xi_1, \xi_2) \in \mathbf{R}^2 : \; \xi_2 > \xi_1^2\}$$

is not in $\mathscr{M}_p(\mathbf{R}^2)$ when $p \neq 2$.
$\big[$*Hint:* Let $\mathscr{H}_j = \big\{(\xi_1, \xi_2) \in \mathbf{R}^2 : \; \xi_2 > s_j \xi_1^2\big\}$ for some $s_j > 0$. The parabolic regions $\big\{(\xi_1, \xi_2) \in \mathbf{R}^2 : \; \xi_2 + R\frac{s_j^2}{4} > \frac{1}{R}\big(\xi_1 + R\frac{s_j}{2}\big)^2\big\}$ are contained in $\mathscr{H}_j$, are translates of the region $\big\{(\xi_1, \xi_2) \in \mathbf{R}^2 : \; \xi_2 > \frac{1}{R}\xi_1^2\big\}$, and tend to $\mathscr{H}_j$ as $R \to \infty$.$\big]$

10.1.5. Let $a_1,\dots,a_n > 0$. Show that the characteristic function of the ellipsoid

$$\Big\{(\xi,\dots,\xi_n)\in\mathbf{R}^n:\ \frac{\xi_1^2}{a_1^2}+\cdots+\frac{\xi_n^2}{a_n^2}<1\Big\}$$

is not in $\mathscr{M}_p(\mathbf{R}^n)$ when $p\neq 2$.
[*Hint:* Think about dilations.]

10.2 Bochner–Riesz Means and the Carleson–Sjölin Theorem

We now address the problem of norm convergence for the Bochner–Riesz means. In this section we provide a satisfactory answer in dimension $n=2$, although a key ingredient required in the proof is left for the next section.

Definition 10.2.1. For a function f on $\mathbf{R}^n$ we define its *Bochner–Riesz means* of complex order λ with $\operatorname{Re}\lambda>0$ to be the family of operators

$$B_R^\lambda(f)(x)=\int_{\mathbf{R}^n}(1-|\xi/R|^2)_+^\lambda\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi\,,\quad R>0.$$

We are interested in the convergence of the family $B_R^\lambda(f)$ as $R\to\infty$. Observe that when $R\to\infty$ and f is a Schwartz function, the sequence $B_R^\lambda(f)$ converges pointwise to f. Does it also converge in norm? Using Exercise 10.2.1, this question is equivalent to whether the function $(1-|\xi|^2)_+^\lambda$ is an L^p multiplier [it lies in $\mathscr{M}_p(\mathbf{R}^n)$], that is, whether the linear operator

$$B^\lambda(f)(x)=\int_{\mathbf{R}^n}(1-|\xi|^2)_+^\lambda\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi$$

maps $L^p(\mathbf{R}^n)$ to itself. The question that arises is given λ with $\operatorname{Re}\lambda>0$ find the range of p's for which $(1-|\xi|^2)_+^\lambda$ is an $L^p(\mathbf{R}^n)$ Fourier multiplier; this question is investigated in this section when $n=2$.

The analogous question for the operators B_R^λ on the n-torus introduced in Definition 3.4.1 is also equivalent to the fact that the function $(1-|\xi|^2)_+^\lambda$ is a Fourier multiplier in $\mathscr{M}_p(\mathbf{R}^n)$. This was shown in Corollary 3.6.10. Therefore the Bochner–Riesz problem for the torus $\mathbf{T}^n$ and the Euclidean space $\mathbf{R}^n$ are equivalent. Here we focus attention on the Euclidean case, and we start our investigation by studying the kernel of the operator B^λ.

10.2.1 The Bochner–Riesz Kernel and Simple Estimates

In view of the last identity in Appendix B.5, B^λ is a convolution operator with kernel

$$K_\lambda(x) = \frac{\Gamma(\lambda+1)}{\pi^\lambda} \frac{J_{\frac{n}{2}+\lambda}(2\pi|x|)}{|x|^{\frac{n}{2}+\lambda}}. \tag{10.2.1}$$

Following Appendix B.6, we have for $|x| \le 1$,

$$|K_\lambda(x)| = \frac{|\Gamma(\lambda+1)|}{|\pi^\lambda|} \frac{|J_{\frac{n}{2}+\lambda}(2\pi|x|)|}{|x|^{\frac{n}{2}+\operatorname{Re}\lambda}} \le \frac{\Gamma(\operatorname{Re}\lambda+1)}{\pi^{\operatorname{Re}\lambda}} C_0\, e^{\pi^2|\operatorname{Im}\lambda|^2},$$

where C_0 is a constant that depends only on $n/2+\operatorname{Re}\lambda$. Consequently, $K_\lambda(x)$ is bounded by a constant (that grows at most exponentially in $|\operatorname{Im}\lambda|^2$) in the unit ball of $\mathbf{R}^n$.

For $|x| \ge 1$, following Appendix B.7, we have

$$|K_\lambda(x)| = \frac{|\Gamma(\lambda+1)|}{|\pi^\lambda|} \frac{|J_{\frac{n}{2}+\lambda}(2\pi|x|)|}{|x|^{\frac{n}{2}+\operatorname{Re}\lambda}} \le C_0 \frac{e^{\pi|\operatorname{Im}\lambda|+\pi^2|\operatorname{Im}\lambda|^2}}{\pi^{\operatorname{Re}\lambda}(2\pi|x|)^{\frac{1}{2}}} \frac{\Gamma(\operatorname{Re}\lambda+1)}{|x|^{\frac{n}{2}+\operatorname{Re}\lambda}},$$

where C_0 depends only on $n/2+\operatorname{Re}\lambda$. Thus $K_\lambda(x)$ is pointwise bounded by a constant (that grows at most exponentially in $|\operatorname{Im}\lambda|$) times $|x|^{-\frac{n+1}{2}-\operatorname{Re}\lambda}$ for $|x| \ge 1$.

Combining these two observations, we obtain that for $\operatorname{Re}\lambda > \frac{n-1}{2}$, K_λ is a smooth integrable function on $\mathbf{R}^n$. Hence B^λ is a bounded operator on L^p for $1 \le p \le \infty$.

Proposition 10.2.2. *For all $1 \le p \le \infty$ and $\lambda > \frac{n-1}{2}$, B^λ is a bounded operator on $L^p(\mathbf{R}^n)$ with norm at most $C_1 e^{c_1|\operatorname{Im}\lambda|^2}$, where C_1, c_1 depend only on $n, \operatorname{Re}\lambda$.*

Proof. The ingredients of the proof have already been discussed. □

We refer to Exercise 10.2.8 for an analogous result for the maximal Bochner–Riesz operator.

According to the asymptotics for Bessel functions in Appendix B.8, K_λ is a smooth function equal to

$$\frac{\Gamma(\lambda+1)}{\pi^{\lambda+1}} \frac{\cos(2\pi|x| - \frac{\pi(n+1)}{4} - \frac{\pi\lambda}{2})}{|x|^{\frac{n+1}{2}+\lambda}} + O(|x|^{-\frac{n+3}{2}-\lambda}) \tag{10.2.2}$$

for $|x| \ge 1$. It is natural to examine whether the operators B^λ are bounded on certain L^p spaces by testing them on specific functions. This may provide some indication as to the range of p's for which these operators may be bounded on L^p.

Proposition 10.2.3. *When $\lambda > 0$ and $p \le \frac{2n}{n+1+2\lambda}$ or $p \ge \frac{2n}{n-1-2\lambda}$, the operators B^λ are not bounded on $L^p(\mathbf{R}^n)$.*

Proof. Let h be a Schwartz function whose Fourier transform is equal to 1 on the ball $B(0,2)$ and vanishes off the ball $B(0,3)$. Then

$$B^\lambda(h)(x) = \int_{|\xi|\le 1} (1-|\xi|^2)^\lambda e^{2\pi i \xi\cdot x}\, dx = K_\lambda(x),$$

and it suffices to show that K_λ is not in $L^p(\mathbf{R}^n)$ for the claimed range of p's. Notice that

$$\sqrt{2}/2 \le \cos(2\pi|x| - \tfrac{\pi(n+1)}{4} - \tfrac{\pi\lambda}{2}) \le 1 \tag{10.2.3}$$

for all x lying in the annuli

$$A_k = \Big\{x \in \mathbf{R}^n : \, k + \frac{n+2\lambda}{8} \le |x| \le k + \frac{n+2\lambda}{8} + \frac{1}{4}\Big\}, \qquad k \in \mathbf{Z}^+.$$

Since in this range, the argument of the cosine in (10.2.2) lies in $[2\pi k, 2\pi k + \frac{\pi}{4}]$.

Consider the range of p's that satisfy

$$\frac{2n}{n+1+2\lambda} \ge p > \frac{2n}{n+3+2\lambda}. \tag{10.2.4}$$

If we can show that B^λ is unbounded in this range, it will also have to be unbounded in the bigger range $\frac{2n}{n+1+2\lambda} \ge p$. This follows by interpolation between the values $p = \frac{2n}{n+3+2\lambda} - \delta$ and $p = \frac{2n}{n+1+2\lambda} + \delta$, $\delta > 0$, for λ fixed.

In view of (10.2.2) and (10.2.3), we have that

$$\big\|K_\lambda\big\|_{L^p}^p \ge C' \sum_{k=n+2\lambda}^{\infty} \int_{A_k} |x|^{-p\frac{n+1}{2} - p\lambda} dx - C'' - C''' \int_{|x|\ge 1} |x|^{-p\frac{n+3}{2} - p\lambda} dx, \tag{10.2.5}$$

where C'' is the integral of K_λ in the unit ball. It is easy to see that for p in the range (10.2.4), the integral outside the unit ball converges, while the series diverges in (10.2.5).

The unboundedness of B^λ on $L^p(\mathbf{R}^n)$ in the range of $p \ge \frac{2n}{n-1-2\lambda}$ follows by duality. □

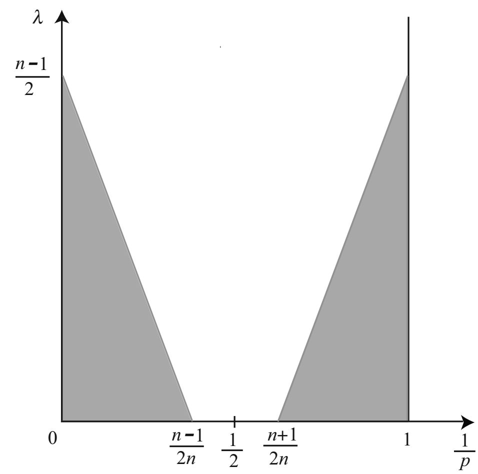

Fig. 10.8 The operator B^λ is unbounded on $L^p(\mathbf{R}^n)$ when $(1/p, \lambda)$ lies in the shaded region.

In Figure 10.8 the shaded region is the set of all pairs $(\frac{1}{p}, \lambda)$ for which the operators B^λ are known to be unbounded on $L^p(\mathbf{R}^n)$.

10.2.2 The Carleson–Sjölin Theorem

We now pass to the main result in this section. We prove the boundedness of the operators B^λ, $\lambda > 0$, in the range of p's not excluded by the previous proposition in dimension $n = 2$.

Theorem 10.2.4. *Suppose that* $0 < \operatorname{Re}\lambda \le 1/2$. *Then the Bochner–Riesz operator* B^λ *maps* $L^p(\mathbf{R}^2)$ *to itself when* $\frac{4}{3+2\operatorname{Re}\lambda} < p < \frac{4}{1-2\operatorname{Re}\lambda}$. *Moreover, for this range of* p*'s and for all* $f \in L^p(\mathbf{R}^2)$ *we have that*

$$B_R^\lambda(f) \to f$$

in $L^p(\mathbf{R}^2)$ *as* $R \to \infty$.

Proof. Once the first assertion of the theorem is established, the second assertion will be a direct consequence of it and of the fact that the means $B_R^\lambda(h)$ converge to h in L^p for h in a dense subclass of L^p. Such a dense class is, for instance, the class of all Schwartz functions h whose Fourier transforms are compactly supported (Exercise 5.2.9). For a function h in this class, we see easily that $B_R^\lambda(h) \to h$ pointwise. But if $\widehat{h}$ is supported in $|\xi| \le c$, then for $R \ge 2c$, integration by parts gives that the functions $B_R^\lambda(h)(x)$ are pointwise controlled by the function $(1+|x|)^{-N}$ with N large; then the Lebesgue dominated convergence theorem gives that the $B_R^\lambda(h)$ converge to h in L^p. Finally, a standard $\varepsilon/3$ argument, using that

$$\sup_{R>0} \big\|B_R^\lambda\big\|_{L^p \to L^p} = \big\|(1-|\xi|^2)_+^\lambda\big\|_{\mathscr{M}_p} < \infty,$$

yields $B_R^\lambda(f) \to f$ in L^p for general L^p functions f.

It suffices to focus our attention on the first part of the theorem. We therefore fix a complex number λ with positive real part and we keep track of the growth of all involved constants in $\operatorname{Im}\lambda$.

We start by picking a smooth function φ supported in $[-\frac{1}{2}, \frac{1}{2}]$ and a smooth function ψ supported in $[\frac{1}{8}, \frac{5}{8}]$ that satisfy

$$\varphi(t) + \sum_{k=0}^{\infty} \psi\Big(\frac{1-t}{2^{-k}}\Big) = 1$$

for all $t \in [0,1)$. We now decompose the multiplier $(1-|\xi|^2)_+^\lambda$ as

$$(1-|\xi|^2)_+^\lambda = m_{00}(\xi) + \sum_{k=0}^{\infty} 2^{-k\lambda} m_k(\xi), \tag{10.2.6}$$

where $m_{00}(\xi) = \varphi(|\xi|)(1-|\xi|^2)^\lambda$ and for $k \ge 0$, m_k is defined by

$$m_k(\xi) = \Big(\frac{1-|\xi|}{2^{-k}}\Big)^\lambda \psi\Big(\frac{1-|\xi|}{2^{-k}}\Big)(1+|\xi|)^\lambda .$$

Note that m_{00} is a smooth function with compact support; hence the multiplier m_{00} lies in $\mathscr{M}_p(\mathbf{R}^2)$ for all $1 \le p \le \infty$. Each function m_k is also smooth, radial, and supported in the small annulus

$$1-\tfrac{5}{8}2^{-k} \le |\xi| \le 1-\tfrac{1}{8}2^{-k}$$

and therefore also lies in $\mathscr{M}_p$; nevertheless the $\mathscr{M}_p$ norms of the m_k's grow as k increases, and it is crucial to determine how this growth depends on k so that we can sum the series in (10.2.6).

Next we show that the Fourier multiplier norm of each m_k on $L^4(\mathbf{R}^2)$ is at most $C(1+|k|)^{1/2}(1+|\mathrm{Im}\,\lambda|)^3$. Summing on k implies that B^λ maps $L^4(\mathbf{R}^2)$ to itself with norm at most a multiple of $(1+|\mathrm{Im}\,\lambda|)^3$ when $\mathrm{Re}\,\lambda > 0$. Given this bound, we conclude the first (and main) statement of the theorem via Theorem 1.3.7 (precisely Exercise 1.3.4), which permits interpolation for the analytic family of operators $\lambda \mapsto B^\lambda$ between the estimates

$$\big\|B^\lambda\big\|_{L^4(\mathbf{R}^2)\to L^4(\mathbf{R}^2)} \le C(1+|\mathrm{Im}\,\lambda|)^3 \qquad \text{when } \mathrm{Re}\,\lambda > 0,$$
$$\big\|B^\lambda\big\|_{L^1(\mathbf{R}^2)\to L^1(\mathbf{R}^2)} \le C_1\, e^{c_1|\mathrm{Im}\,\lambda|^2} \qquad \text{when } \mathrm{Re}\,\lambda > \tfrac{1}{2},$$

where C, C_1, c_1 depend only on $\mathrm{Re}\,\lambda$. The second estimate above is proved in Proposition 10.2.2 while the set of points $(1/p, \lambda)$ obtained by interpolation can be seen in Figure 10.8.

To estimate the norm of each m_k in $\mathscr{M}_4(\mathbf{R}^2)$, we need an additional decomposition of the operator m_k that takes into account the radial nature of m_k. For each $k \ge 0$ we define the sectorial arcs (parts of a sector between two arcs)

$$\Gamma_{k,\ell} = \big\{re^{2\pi i\theta} \in \mathbf{R}^2 :\ |\theta - \ell 2^{-\frac{k}{2}}| < 2^{-\frac{k}{2}}, \quad 1-\tfrac{5}{8}2^{-k} \le r \le 1-\tfrac{1}{8}2^{-k}\big\}$$

for all $\ell \in \{0,1,2,\dots,[2^{k/2}]-1\}$. We now introduce a smooth function ω supported in $[-1,1]$ and equal to 1 on $[-1/4,1/4]$ such that for all $x \in \mathbf{R}$ we have

$$\sum_{\ell\in\mathbf{Z}} \omega(x-\ell) = 1 .$$

Then we define $m_{k,\ell}(re^{2\pi i\theta}) = m_k(re^{2\pi i\theta})\omega(2^{k/2}\theta - \ell)$ for integers ℓ in the set $\{0,1,2,\dots,[2^{k/2}]-1\}$. If k is an even integer, it follows from the construction that

$$m_k(\xi) = \sum_{\ell=0}^{[2^{k/2}]-1} m_{k,\ell}(\xi) \tag{10.2.7}$$

for all ξ in $\mathbf{R}^2$. If k is odd we replace the function $\theta \mapsto \omega(2^{k/2}\theta - ([2^{k/2}]-1))$ by a function $\omega_k(\theta)$ supported in the bigger interval $\big[([2^{k/2}]-2)2^{-k/2}, 1\big]$ that satisfies $\omega_k(\theta) + \omega(2^{k/2}(\theta-1)) = 1$ on the interval $\big[([2^{k/2}]-1)2^{-k/2}, 1\big]$. This leads to a new definition of the function $m_{k,[2^{k/2}]-1}$ so that (10.2.7) is satisfied.

This provides the circular (angular) decomposition of m_k. Observe that for all positive integers α and β there exist constants $C_{\alpha,\beta}$ such that

$$|\partial_r^\alpha \partial_\theta^\beta m_{k,\ell}(re^{2\pi i\theta})| \le C_{\alpha,\beta}(1+|\lambda|)^{\alpha+\beta} 2^{k\alpha} 2^{\frac{k}{2}\beta}$$

and such that each $m_{k,\ell}$ is a smooth function supported in the sectorial arcs $\Gamma_{k,\ell}$.

We fix $k \ge 0$ and we group the set of all $\{m_{k,\ell}\}_\ell$ into five subsets: (a) those whose supports are contained in $Q = \{(x,y) \in \mathbf{R}^2 : \, x > 0, \; |y| < |x|\}$; (b) those $m_{k,\ell}$ whose supports are contained in the sector $Q' = \{(x,y) \in \mathbf{R}^2 : \, x < 0, \; |y| < |x|\}$; (c) those whose supports are contained in $Q'' = \{(x,y) \in \mathbf{R}^2 : \, y > 0, \; |y| > |x|\}$; (d) the $m_{k,\ell}$ with supports contained in $Q''' = \{(x,y) \in \mathbf{R}^2 : \, y < 0, \; |y| > |x|\}$; and finally (e) those $m_{k,\ell}$ whose supports intersect the lines $|y| = |x|$.

There are only at most eight $m_{k,\ell}$ in case (e), and their sum is easily shown to be an L^4 Fourier multilpier with a constant that grows like $(1+|\lambda|)^3$, as shown below. The remaining cases are symmetric, and we focus attention on case (a).

Let I be the set of all indices ℓ in the set $\{0,1,2,\dots,[2^{k/2}]-1\}$ corresponding to case (a), i.e., the sectorial arcs $\Gamma_{k,\ell}$ are contained in the quarter-plane Q. Let $T_{k,\ell}$ be the operator given on the Fourier transform by multiplication by the function $m_{k,\ell}$. We have

$$\begin{aligned}\Big\| \sum_{\ell\in I} T_{k,\ell}(f) \Big\|_{L^4}^4 &= \int_{\mathbf{R}^2} \Big| \sum_{\ell\in I} T_{k,\ell}(f) \Big|^4 dx \\ &= \int_{\mathbf{R}^2} \Big| \sum_{\ell\in I}\sum_{\ell'\in I} T_{k,\ell}(f) T_{k,\ell'}(f) \Big|^2 dx \\ &= \int_{\mathbf{R}^2} \Big| \sum_{\ell\in I}\sum_{\ell'\in I} \widehat{T_{k,\ell}(f)} * \widehat{T_{k,\ell'}(f)} \Big|^2 d\xi \,, \end{aligned} \tag{10.2.8}$$

where we used Plancherel's identity in the last equality. Each function $\widehat{T_{k,\ell}(f)}$ is supported in the sectorial arc $\Gamma_{k,\ell}$. Therefore, the function $\widehat{T_{k,\ell}(f)} * \widehat{T_{k,\ell'}(f)}$ is supported in $\Gamma_{k,\ell} + \Gamma_{k,\ell'}$ and we write the last integral as

$$\int_{\mathbf{R}^2} \Big| \sum_{\ell\in I}\sum_{\ell'\in I} \big(\widehat{T_{k,\ell}(f)} * \widehat{T_{k,\ell'}(f)}\big) \chi_{\Gamma_{k,\ell}+\Gamma_{k,\ell'}} \Big|^2 d\xi \,.$$

In view of the Cauchy–Schwarz inequality, the last expression is controlled by

$$\int_{\mathbf{R}^2} \Big(\sum_{\ell\in I}\sum_{\ell'\in I} \big|\widehat{T_{k,\ell}(f)} * \widehat{T_{k,\ell'}(f)}\big|^2 \Big) \Big(\sum_{\ell\in I}\sum_{\ell'\in I} \big|\chi_{\Gamma_{k,\ell}+\Gamma_{k,\ell'}}\big|^2 \Big) d\xi \,. \tag{10.2.9}$$

At this point we make use of the following lemma, in which the curvature of the circle is manifested.

Lemma 10.2.5. *There exists a constant C_0 such that for all $k \geq 0$ the following estimate holds:*

$$\sum_{\ell \in I} \sum_{\ell' \in I} \chi_{\Gamma_{k,\ell}+\Gamma_{k,\ell'}} \leq C_0 \,.$$

We postpone the proof of this lemma until the end of this section. Using Lemma 10.2.5, we control the expression in (10.2.9) by

$$C_0 \int_{\mathbf{R}^2} \sum_{\ell \in I} \sum_{\ell' \in I} \big|\widehat{T_{k,\ell}(f)} * \widehat{T_{k,\ell'}(f)}\big|^2 \, d\xi \, = C_0 \Big\| \Big(\sum_{\ell \in I} |T_{k,\ell}(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^4}^4 . \qquad (10.2.10)$$

We examiine each $T_{k,\ell}$ a bit more carefully. We have that $m_{k,0}$ is supported in a rectangle with sides parallel to the axes and dimensions 2^{-k} (along the ξ_1-axis) and $2^{-\frac{k}{2}+1}$ (along the ξ_2-axis). Moreover, in that rectangle, $\partial_{\xi_1} \approx \partial_r$ and $\partial_{\xi_2} \approx \partial_\theta$, and it follows that the smooth function $m_{k,0}$ satisfies

$$|\partial_{\xi_1}^{\alpha} \partial_{\xi_2}^{\beta} m_{k,0}(\xi_1, \xi_2)| \leq C_{\alpha,\beta}(1+|\lambda|)^{\alpha+\beta} 2^{k\alpha} 2^{\frac{k}{2}\beta}$$

for all positive integers α and β. This estimate can also be written as

$$\big|\partial_{\xi_1}^{\alpha} \partial_{\xi_2}^{\beta} \big[m_{k,0}(2^{-k}\xi_1, 2^{-\frac{k}{2}}\xi_2) \big] \big| \leq C_{\alpha,\beta}(1+|\lambda|)^{\alpha+\beta} ,$$

which easily implies that

$$2^{\frac{3}{2}k} |m_{k,0}^{\vee}(2^k x_1, 2^{\frac{k}{2}} x_2)| \leq C_{\alpha,\beta}(1+|\lambda|)^3 (1+|x_1|+|x_2|)^{-3} .$$

Let V_ℓ be the unit vector representing the point $e^{2\pi i \ell 2^{-k/2}}$ and $V_\ell^{\perp}$ the unit vector representing the point $ie^{2\pi i \ell 2^{-k/2}}$. Applying a rotation, we obtain that the functions $m_{k,\ell}^{\vee}$ satisfy

$$\big|m_{k,\ell}^{\vee}(x_1,x_2)\big| \leq C(1+|\lambda|)^3 3^{-\frac{3k}{2}} (1+2^{-k}|x \cdot V_\ell| + 2^{-\frac{k}{2}}|x \cdot V_\ell^{\perp}|)^{-3} \qquad (10.2.11)$$

and hence

$$\sup_{k \geq 0} \sup_{\ell \in I} \big\| m_{k,\ell}^{\vee} \big\|_{L^1} \leq C(1+|\lambda|)^3 . \qquad (10.2.12)$$

The crucial fact is that the constant C in (10.2.12) is independent of ℓ and k.

At this point, for each fixed $k \geq 0$ and $\ell \in I$ we let $J_{k,\ell}$ be the ξ_2-projection of the support of $m_{k,\ell}$. Based on the earlier definition of $m_{k,\ell}$, we easily see that when $\ell > 0$,

$$J_{k,\ell} = \big[(1-\tfrac{5}{8}2^{-k}) \sin(2\pi 2^{-\frac{k}{2}}(\ell-1)), (1-\tfrac{1}{8}2^{-k}) \sin(2\pi 2^{-\frac{k}{2}}(\ell+1)) \big] .$$

A similar formula holds for $\ell < 0$ in I. The crucial observation is that for any fixed $k \geq 0$ the sets $J_{k,\ell}$ are "almost disjoint" for different $\ell \in I$. Indeed, the sets $J_{k,\ell}$ are contained in the intervals

$$\widetilde{J_{k,\ell}} = \big[(1-\tfrac{3}{8}2^{-k})\sin(2\pi 2^{-\frac{k}{2}}\ell) - 10\cdot 2^{-\frac{k}{2}}, (1-\tfrac{3}{8}2^{-k})\sin(2\pi 2^{-\frac{k}{2}}\ell) + 10\cdot 2^{-\frac{k}{2}}\big],$$

which have length $20\cdot 2^{-\frac{k}{2}}$ and are centered at the points $(1-\frac{3}{8}2^{-k})\sin(2\pi 2^{-\frac{k}{2}}\ell)$. For $\sigma \in \mathbf{Z}$ and $\tau \in \{0,1,\dots,39\}$ we define the strips

$$S_{k,\sigma,\tau} = \big\{(\xi_1,\xi_2):\ \xi_2 \in [40\sigma 2^{-\frac{k}{2}} + \tau 2^{-\frac{k}{2}}, 40(\sigma+1)2^{-\frac{k}{2}} + \tau 2^{-\frac{k}{2}})\big\}.$$

These strips have length $40\cdot 2^{-\frac{k}{2}}$ and have the property that each $\widetilde{J_{k,\ell}}$ is contained in one of them; say $\widetilde{J_{k,\ell}}$ is contained in some $S_{k,\sigma_\ell,\tau_\ell}$, which we call $B_{k,\ell}$. Then we have

$$T_{k,\ell}(f) = T_{k,\ell}(f_{k,\ell}),$$

where we set

$$f_{k,\ell} = \big(\chi_{B_{k,\ell}}\widehat{f}\,\big)^\vee = \chi_{B_{k,\ell}}^\vee * f.$$

As a consequence of the Cauchy–Schwarz inequality (with respect to the measure $|m_{k,\ell}^\vee|\,dx$), we obtain

$$\begin{aligned}|T_{k,\ell}(f_{k,\ell})|^2 &\leq \big\|m_{k,\ell}^\vee\big\|_{L^1}\big(|m_{k,\ell}^\vee| * |f_{k,\ell}|^2\big)\\ &\leq C(1+|\lambda|)^3\big(|m_{k,\ell}^\vee| * |f_{k,\ell}|^2\big)\end{aligned}$$

in view of (10.2.12). We now return to (10.2.10), which controls (10.2.9) and hence (10.2.8). Using this estimate, we bound the term in (10.2.10) by

$$\begin{aligned}\Big\|\Big(\sum_{\ell\in I}|T_{k,\ell}(f)|^2\Big)^{\frac12}\Big\|_{L^4}^4 &= \Big\|\sum_{\ell\in I}|T_{k,\ell}(f_{k,\ell})|^2\Big\|_{L^2}^2\\ &\leq C^2(1+|\lambda|)^6\Big\|\sum_{\ell\in I}|m_{k,\ell}^\vee| * |f_{k,\ell}|^2\Big\|_{L^2}^2\\ &= C^2(1+|\lambda|)^6\Big(\int_{\mathbf{R}^2}\sum_{\ell\in I}(|m_{k,\ell}^\vee| * |f_{k,\ell}|^2)\,g\,dx\Big)^2\\ &= C^2(1+|\lambda|)^6\Big(\sum_{\ell\in I}\int_{\mathbf{R}^2}(|\widehat{m_{k,\ell}}| * g)\,|f_{k,\ell}|^2\,dx\Big)^2\\ &\leq C^2(1+|\lambda|)^6\Big(\int_{\mathbf{R}^2}\sup_{\ell\in I}\big(|\widehat{m_{k,\ell}}| * g|\big)\sum_{\ell\in I}|f_{k,\ell}|^2\,dx\Big)^2\\ &\leq C^2(1+|\lambda|)^6\Big\|\sup_{\ell\in I}\big(|\widehat{m_{k,\ell}}| * g\big)\Big\|_{L^2}^2\Big\|\Big(\sum_{\ell\in I}|f_{k,\ell}|^2\Big)^{\frac12}\Big\|_{L^4}^4,\end{aligned}$$

where g is an appropriate nonnegative function in $L^2(\mathbf{R}^2)$ of norm 1.

If we knew the validity of the estimates

$$\Big\| \sup_{\ell \in I} \big(|\widehat{m_{k,\ell}}| * g \big) \Big\|_{L^2} \le C(1+|\lambda|)^3(1+k) \big\| g \big\|_{L^2} \tag{10.2.13}$$

and

$$\Big\| \Big(\sum_{\ell \in I} |f_{k,\ell}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^4} \le C \big\| f \big\|_{L^4}, \tag{10.2.14}$$

then we would be able to conclude that

$$\big\| m_k \big\|_{\mathscr{M}_p} \le C(1+|\lambda|)^3(1+k)^{\frac{1}{2}} \tag{10.2.15}$$

and hence we could sum the series in (10.2.6).

Estimates (10.2.13) and (10.2.14) are discussed in the next two subsections. □

10.2.3 The Kakeya Maximal Function

We showed in the previous subsection that $m_{k,0}^{\vee}$ is integrable over $\mathbf{R}^2$ and satisfies the estimate

$$2^{\frac{3}{2}k} |m_{k,0}^{\vee}(2^k x_1, 2^{\frac{k}{2}} x_2)| \le \frac{C(1+|\lambda|)^3}{(1+|x|)^3}.$$

Since

$$\frac{1}{(1+|x|)^3} \le C \sum_{s=0}^{\infty} \frac{2^{-s}}{2^{2s}} \chi_{[-2^s,2^s]\times[-2^s,2^s]}(x),$$

it follows that

$$|\widehat{m_{k,0}}(x)| \le C'(1+|\lambda|)^3 \sum_{s=0}^{\infty} 2^{-s} \frac{1}{|R_s|} \chi_{R_s}(x),$$

where $R_s = [-2^s 2^k, 2^s 2^k] \times [-2^s 2^{\frac{k}{2}}, 2^s 2^{\frac{k}{2}}]$. Since a general $\widehat{m_{k,\ell}}$ is obtained from $\widehat{m_{k,0}}$ via a rotation, a similar estimate holds for it. Precisely, we have

$$|\widehat{m_{k,\ell}}(x)| \le C'\,(1+|\lambda|)^3 \sum_{s=0}^{\infty} 2^{-s} \frac{1}{|R_{s,\ell}|} \chi_{R_{s,\ell}}(x), \tag{10.2.16}$$

where $R_{s,\ell}$ is a rectangle with principal axes along the directions V_ℓ and $V_\ell^{\perp}$ and side lengths $2^s 2^k$ and $2^s 2^{\frac{k}{2}}$, respectively. Using (10.2.16), we obtain the following pointwise estimate for the maximal function in (10.2.13):

$$\sup_{\ell \in I} \big(|\widehat{m_{k,\ell}}| * g \big) \;\le\; C' \sum_{s=0}^{\infty} 2^{-s} \sup_{\ell \in I} \frac{1}{|R_{s,\ell}|} \int_{R_{s,\ell}} g(x-y)\,dy, \tag{10.2.17}$$

where $R_{s,\ell}$ are rectangles with dimensions $2^s 2^k$ and $2^s 2^{\frac{k}{2}}$.

Motivated by (10.2.17), for fixed $N \geq 10$ and $a > 0$, we introduce the *Kakeya maximal operator without dilations*

$$\mathscr{K}_N^a(g)(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |g(y)|\, dy, \tag{10.2.18}$$

acting on functions $g \in L^1_{\mathrm{loc}}$, where the supremum is taken over all rectangles R in $\mathbf{R}^2$ of dimensions a and aN and arbitrary orientation. What makes this maximal operator interesting is that the rectangles R that appear in the supremum in (10.2.19) are allowed to have arbitrary orientation. We also define the *Kakeya maximal operator* $\mathscr{K}_N$ by

$$\mathscr{K}_N(w)(x) = \sup_{a>0} \mathscr{K}_N^a(w), \tag{10.2.19}$$

for w locally integrable. The maximal function $\mathscr{K}_N(w)(x)$ is therefore obtained as the supremum of the averages of a function w over all rectangles in $\mathbf{R}^2$ that contain the point x and have arbitrary orientation but fixed eccentricity equal to N. (The eccentricity of a rectangle is the ratio of its longer side to its shorter side.)

We see that $\mathscr{K}_N(f)$ is pointwise controlled by a $cNM(f)$, where M is the Hardy–Littlewood maximal operator M. This implies that $\mathscr{K}_N$ is of weak type $(1,1)$ with bound at most a multiple of N. Since $\mathscr{K}_N$ is bounded on L^∞ with norm 1, it follows that $\mathscr{K}_N$ maps $L^p(\mathbf{R}^2)$ to itself with norm at most a multiple of $N^{1/p}$. However, we show in the next section that this estimate is very rough and can be improved significantly. In fact, we obtain an L^p estimate for $\mathscr{K}_N$ with norm that grows logarithmically in N (when $p \geq 2$), and this is very crucial, since $N = 2^{k/2}$ in the following application.

Using this new terminology, we write the estimate in (10.2.17) as

$$\sup_{\ell \in I} \big(|\widehat{m_{k,\ell}}| * g \big) \ \leq \ C'(1+|\lambda|)^3 \sum_{s=0}^{\infty} 2^{-s} \mathscr{K}_{2^{k/2}}^{2^{s+k/2}}(g). \tag{10.2.20}$$

The required estimate (10.2.13) is a consequence of (10.2.20) and of the following theorem, whose proof is discussed in the next section.

Theorem 10.2.6. *There exists a constant C such that for all* $N \geq 10$ *and all* f *in* $L^2(\mathbf{R}^2)$ *the following norm inequality is valid:*

$$\sup_{a>0} \big\| \mathscr{K}_N^a(f) \big\|_{L^2(\mathbf{R}^2)} \leq C(\log N) \, \|f\|_{L^2(\mathbf{R}^2)}.$$

Theorem 10.2.6 is a consequence of Theorem 10.3.5, in which the preceding estimate is proved for a more general maximal operator $\mathfrak{M}_{\Sigma_N}$, which in particular controls $\mathscr{K}_N$ and hence $\mathscr{K}_N^a$ for all $a > 0$. This maximal operator is introduced in the next section.

10.2.4 Boundedness of a Square Function

We now turn to the proof of estimate (10.2.14). This is a consequence of the following result, which is a version of the Littlewood–Paley theorem for intervals of equal length.

Theorem 10.2.7. *For $j \in \mathbf{Z}$, let I_j be intervals of equal length with disjoint interior whose union is $\mathbf{R}$. We define operators P_j with multipliers χ_{I_j}. Then for $2 \le p < \infty$, there is a constant C_p such that for all $f \in L^p(\mathbf{R})$ we have*

$$\Big\| \Big(\sum_j |P_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R})} \le C_p \big\| f \big\|_{L^p(\mathbf{R})} . \tag{10.2.21}$$

In particular, the same estimate holds if the intervals I_j have disjoint interiors and equal length but do not necessarily cover $\mathbf{R}$.

Proof. Multiplying the function f by a suitable exponential, we may assume that the intervals I_j have the form $\big((j-\frac{1}{2})a, (j+\frac{1}{2})a\big)$ for some $a > 0$. Applying a dilation to f reduces matters to the case $a = 1$. We conclude that the constant C_p is independent of the common size of the intervals I_j and it suffices to obtain estimate (10.2.21) in the case $a = 1$.

We assume therefore that $I_j = (j-\frac{1}{2}, j+\frac{1}{2})$ for all $j \in \mathbf{Z}$. Next, our goal is to replace the operators P_j by smoother analogues of them. To achieve this we introduce a smooth function ψ with compact support that is identically equal to 1 on the interval $[-\frac{1}{2}, \frac{1}{2}]$ and vanishes off the interval $[-\frac{3}{4}, \frac{3}{4}]$. We introduce operators S_j by setting

$$\widehat{S_j(f)}(\xi) = \widehat{f}(\xi)\psi(\xi - j)$$

and we note that the identity

$$P_j = P_j S_j \tag{10.2.22}$$

is valid for all $j \in \mathbf{Z}$. For $t \in \mathbf{R}$ we define multipliers m_t as

$$m_t(\xi) = \sum_{j \in \mathbf{Z}} e^{-2\pi i j t} \psi(\xi - j),$$

and we set $k_t = m_t^{\vee}$. With $I_0 = (-1/2, 1/2)$, we have

$$\begin{aligned} \int_{I_0} |(k_t * f)(x)|^2 \, dt &= \int_{I_0} \Big| \sum_{j \in \mathbf{Z}} e^{-2\pi i j t} S_j(f)(x) \Big|^2 dt \\ &= \sum_{j \in \mathbf{Z}} |S_j(f)(x)|^2 , \end{aligned} \tag{10.2.23}$$

where the last equality is just Plancherel's identity on $I_0 = [-\frac{1}{2}, \frac{1}{2}]$. In view of the last identity, it suffices to analyze the operator given by convolution with the family of kernels k_t. By the Poisson summation formula (Theorem 3.1.17) applied to the function $x \mapsto \psi(x)e^{2\pi i x t}$, we obtain

$$
\begin{aligned}
m_t(\xi) &= e^{-2\pi i\xi t}\sum_{j\in\mathbf{Z}}\psi(\xi-j)e^{2\pi i(\xi-j)t}\\
&= \sum_{j\in\mathbf{Z}}\big(\psi(\cdot)e^{2\pi i(\cdot)t}\big)\widehat{\;}(j)\,e^{2\pi ij\xi}\,e^{-2\pi i\xi t}\\
&= \sum_{j\in\mathbf{Z}}e^{2\pi i(j-t)\xi}\,\widehat{\psi}(j-t)\,.
\end{aligned}
$$

Taking inverse Fourier transforms, we obtain

$$
k_t = \sum_{j\in\mathbf{Z}}\widehat{\psi}(j-t)\delta_{-j+t}\,,
$$

where δ_b denotes Dirac mass at the point b. Therefore, k_t is a sum of Dirac masses with rapidly decaying coefficients. Since each Dirac mass has Borel norm at most 1, we conclude that

$$
\big\|k_t\big\|_{\mathscr{M}} \le \sum_{j\in\mathbf{Z}}|\widehat{\psi}(j-t)| \le \sum_{j\in\mathbf{Z}}(1+|j-t|)^{-10}\le 10\,, \tag{10.2.24}
$$

which is independent of t. This says that the measures k_t have uniformly bounded norms. Take now $f\in L^p(\mathbf{R})$ and $p\ge 2$. Using identity (10.2.22), we obtain

$$
\begin{aligned}
\int_{\mathbf{R}}\Big(\sum_{j\in\mathbf{Z}}|P_j(f)(x)|^2\Big)^{\frac{p}{2}}dx &= \int_{\mathbf{R}}\Big(\sum_{j\in\mathbf{Z}}|P_jS_j(f)(x)|^2\Big)^{\frac{p}{2}}dx\\
&\le c_p\int_{\mathbf{R}}\Big(\sum_{j\in\mathbf{Z}}|S_j(f)(x)|^2\Big)^{\frac{p}{2}}dx\,,
\end{aligned}
$$

and the last inequality follows from Exercise 4.6.1(a). The constant c_p depends only on p. Recalling identity (10.2.23), we write

$$
\begin{aligned}
c_p\int_{\mathbf{R}}\Big(\sum_{j\in\mathbf{Z}}|S_j(f)(x)|^2\Big)^{\frac{p}{2}}dx &\le c_p\int_{\mathbf{R}}\bigg(\int_{I_0}|(k_t*f)(x)|^2\,dt\bigg)^{\frac{p}{2}}dx\\
&\le c_p\int_{\mathbf{R}}\bigg(\int_{I_0}|(k_t*f)(x)|^p dt\bigg)^{\frac{p}{p}}dx\\
&= c_p\int_{I_0}\int_{\mathbf{R}}|(k_t*f)(x)|^p\,dx\,dt\\
&\le 10\,c_p\int_{I_0}\int_{\mathbf{R}}|f(x)|^p\,dx\,dt\\
&= 10\,c_p\big\|f\big\|_{L^p}^p\,,
\end{aligned}
$$

where we used Hölder's inequality on the interval I_0 (together with the fact that $p\ge 2$) and (10.2.24). The proof of the theorem is complete with constant $C_p = (10c_p)^{1/p}$. □

We now return to estimate (10.2.14). First recall the strips

$$S_{k,\sigma,\tau} = \left\{(\xi_1,\xi_2) : \xi_2 \in [40\sigma 2^{-\frac{k}{2}} + \tau, 40(\sigma+1)2^{-\frac{k}{2}} + \tau)\right\}$$

defined for $\sigma \in \mathbf{Z}$ and $\tau \in \{0,1,\dots,39\}$. These strips have length $40 \cdot 2^{-\frac{k}{2}}$, and each $\widetilde{J}_{k,\ell}$ is contained in one of them, which we called $S_{k,\sigma_\ell,\tau_\ell} = B_{k,\ell}$.

The family $\{B_{k,\ell}\}_{\ell\in I}$ does not consist of disjoint sets, but we split it into 40 subfamilies by placing $B_{k,\ell}$ in different subfamilies if the indices τ_ℓ and $\tau_{\ell'}$ are different. We now write the set I as

$$I = I^1 \cup I^2 \cup \cdots \cup I^{40},$$

where for each $\ell,\ell' \in I^j$ the sets $B_{k,\ell}$ and $B_{k,\ell'}$ are disjoint.

We now use Theorem 10.2.7 to obtain the required quadratic estimate (10.2.14). Things now are relatively simple. We observe that the multiplier operators $f \mapsto (\chi_{B_{k,\ell}}\widehat{f})^\vee$ on $\mathbf{R}^2$ obey the estimates (10.2.21), in which $L^p(\mathbf{R})$ is replaced by $L^p(\mathbf{R}^2)$, since they are the identity operators in the ξ_1-variable.

We conclude that

$$\Big\|\Big(\sum_{\ell\in I^j} |T_{k,\ell}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^2)} \le C_p \big\|f\big\|_{L^p(\mathbf{R}^2)} \tag{10.2.25}$$

holds for all $p \ge 2$ and, in particular, for $p = 4$. This proves (10.2.14) for a single I^j, and the same conclusion follows for I with a constant 40 times as big.

10.2.5 The Proof of Lemma 10.2.5

We finally discuss the proof of Lemma 10.2.5.

Proof. If $k = 0,1,\dots,k_0$ up to a fixed integer k_0, then there exist only finitely many pairs of sets $\Gamma_\ell + \Gamma_{\ell'}$ depending on k_0, and the lemma is trivially true. We may therefore assume that k is a large integer; in particular we may take $\delta = 2^{-k} \le 2400^{-2}$. In the sequel, for simplicity we replace 2^{-k} by δ and we denote the set $\Gamma_{k,\ell}$ by Γ_ℓ. In the proof that follows we are working with a fixed $\delta \in [0, 2400^{-2}]$. Elements of the set $\Gamma_\ell + \Gamma_{\ell'}$ have the form

$$re^{2\pi i(\ell+\alpha)\delta^{1/2}} + r'e^{2\pi i(\ell'+\alpha')\delta^{1/2}}, \tag{10.2.26}$$

where α, α' range in the interval $[-1,1]$ and r, r' range in $[1-\frac{5}{8}\delta, 1-\frac{1}{8}\delta]$. We set

$$w(\ell,\ell') = e^{2\pi i\ell\delta^{1/2}} + e^{2\pi i\ell'\delta^{1/2}} = 2\cos(\pi|\ell-\ell'|\delta^{\frac{1}{2}})e^{\pi i(\ell+\ell')\delta^{1/2}}, \tag{10.2.27}$$

where the last equality is a consequence of a trigonometric identity that can be found in Appendix E. Using similar identities (see Appendix E) and performing algebraic manipulations, one may verify that the general element (10.2.26) of the set $\Gamma_\ell + \Gamma_{\ell'}$ can be written as

$$
\begin{aligned}
w(\ell,\ell') &+ \Big\{ r\,\frac{\cos(2\pi\alpha\delta^{\frac12})+\cos(2\pi\alpha'\delta^{\frac12})-2}{2}\Big\}\, w(\ell,\ell') \\
&+ \Big\{ r\,\frac{\sin(2\pi\alpha\delta^{\frac12})+\sin(2\pi\alpha'\delta^{\frac12})}{2}\Big\}\, i w(\ell,\ell') \\
&+ \mathrm{E}(r,\ell,\ell',\alpha,\alpha',\delta)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{E}(r,\ell,\ell',\alpha,\alpha',\delta) &= (r-1)\Big(e^{2\pi i\ell\delta^{1/2}}+e^{2\pi i\ell'\delta^{1/2}}\Big) \\
&+ (r'-r)e^{2\pi i(\ell'+\alpha')\delta^{1/2}} \\
&+ r\Big(e^{2\pi i\ell\delta^{1/2}}-e^{2\pi i\ell'\delta^{1/2}}\Big)\Big(\frac{\cos(2\pi\alpha\delta^{\frac12})-\cos(2\pi\alpha'\delta^{\frac12})}{2}\Big) \\
&+ r\Big(e^{2\pi i\ell\delta^{1/2}}-e^{2\pi i\ell'\delta^{1/2}}\Big)\Big(\frac{\sin(2\pi\alpha\delta^{\frac12})-\sin(2\pi\alpha'\delta^{\frac12})}{2}\Big).
\end{aligned}
$$

The coefficients in the curly brackets are real, and $\mathrm{E}(r,\ell,\ell',\alpha,\alpha',\delta)$ is an error of magnitude at most $2\delta+8\pi^2|\ell-\ell'|\delta$. These observations and the facts $|\sin x|\le|x|$ and $|1-\cos x|\le|x|^2/2$ (see Appendix E) imply that the set $\Gamma_\ell+\Gamma_{\ell'}$ is contained in the rectangle $R(\ell,\ell')$ centered at the point $w(\ell,\ell')$ with half-width

$$
2\pi^2\delta+(2\delta+8\pi^2|\ell-\ell'|\delta)\le 80\,(1+|\ell-\ell'|)\delta
$$

in the direction along $w(\ell,\ell')$ and half-length

$$
2\pi\delta^{\frac12}+(2\delta+8\pi^2|\ell-\ell'|\delta)\le 30\,\delta^{\frac12}
$$

in the direction along $iw(\ell,\ell')$ [which is perpendicular to that along $w(\ell,\ell')$]. Since $2\pi|\ell-\ell'|\delta^{\frac12}<\frac{\pi}{2}$, this rectangle is contained in a disk of radius $105\,\delta^{\frac12}$ centered at the point $w(\ell,\ell')$.

We immediately deduce that if $|w(\ell,\ell')-w(m,m')|$ is bigger than $210\,\delta^{\frac12}$, then the sets $\Gamma_\ell+\Gamma_{\ell'}$ and $\Gamma_m+\Gamma_{m'}$ do not intersect. Therefore, if these sets intersect, we should have

$$
|w(\ell,\ell')-w(m,m')|\le 210\,\delta^{\frac12}\,.
$$

In view of Exercise 10.2.2, the left-hand side of the last expression is at least

$$
2\,\tfrac{2}{\pi}\cos(\tfrac{\pi}{4})|\pi(\ell+\ell')-\pi(m+m')|\delta^{\frac12}
$$

(here we use the hypothesis that $|2\pi\ell\delta^{\frac12}|<\frac{\pi}{4}$ twice). We conclude that if the sets $\Gamma_\ell+\Gamma_{\ell'}$ and $\Gamma_m+\Gamma_{m'}$ intersect, then

$$
|(\ell+\ell')-(m+m')|\le 210/2\sqrt{2}\le 150\,. \tag{10.2.28}
$$

In this case the angle between the vectors $w(\ell,\ell')$ and $w(m,m')$ is

$$\varphi_{\ell,\ell',m,m'} = \pi|(\ell+\ell')-(m+m')|\delta^{\frac{1}{2}},$$

which is smaller than $\pi/16$, provided (10.2.28) holds and $\delta < 2400^{-2}$. This says that in this case, the rectangles $R(\ell,\ell')$ and $R(m,m')$ are essentially parallel to each other (the angle between them is smaller than $\pi/16$).

Let us fix a rectangle $R(\ell,\ell')$, and for another rectangle $R(m,m')$ we denote by $\widetilde{R}(m,m')$ the smallest rectangle containing $R(m,m')$ with sides parallel to the corresponding sides of $R(\ell,\ell')$. An easy trigonometric argument shows that $\widetilde{R}(m,m')$ has the same center as $R(m,m')$ and has half-sides at most

$$30\delta^{\frac{1}{2}}\cos(\varphi_{\ell,\ell',m,m'})+80(1+|\ell-\ell'|)\delta\sin(\varphi_{\ell,\ell',m,m'}),$$
$$80(1+|\ell-\ell'|)\delta\cos(\varphi_{\ell,\ell',m,m'})+30\delta^{\frac{1}{2}}\sin(\varphi_{\ell,\ell',m,m'}),$$

in view of Exercise 10.2.3. Then $\widetilde{R}(m,m')$ has half-sides at most $60000\delta^{\frac{1}{2}}$ and $18000(1+|\ell-\ell'|)\delta$ and is therefore contained in a fixed multiple of $R(m,m')$. If $\Gamma_\ell+\Gamma_{\ell'}$ and $\Gamma_m+\Gamma_{m'}$ intersect, then so do $\widetilde{R}(m,m')$ and $R(\ell,\ell')$, and both of these rectangles have sides parallel to the vectors $w(\ell,\ell')$ and $iw(\ell,\ell')$. But in the direction of $w(\ell,\ell')$, these rectangles have sides with half-lengths at most $80(1+|\ell-\ell'|)\delta$ and $16000(1+|m-m'|)\delta$. Note that the distance of the lines parallel to the direction $iw(\ell,\ell')$ and passing through the centers of the rectangles $\widetilde{R}(m,m')$ and $R(\ell,\ell')$ is

$$2\big|\cos(\pi|\ell-\ell'|\delta^{\frac{1}{2}})-\cos(\pi|m-m'|\delta^{\frac{1}{2}})\big|,$$

as we easily see using (10.2.27). If these rectangles intersect, we must have

$$2\big|\cos(\pi|\ell-\ell'|\delta^{\frac{1}{2}})-\cos(\pi|m-m'|\delta^{\frac{1}{2}})\big| \le 16080\,(2+|\ell-\ell'|+|m-m'|)\delta.$$

We conclude that if the sets $R(m,m')$ and $R(\ell,\ell')$ intersect and $(\ell,\ell')\neq(m,m')$, then

$$\big|\cos(\pi|\ell-\ell'|\delta^{\frac{1}{2}})-\cos(\pi|m-m'|\delta^{\frac{1}{2}})\big| \le 50000\,(|\ell-\ell'|+|m-m'|)\delta.$$

But the expression on the left is equal to

$$2\big|\sin(\pi\tfrac{|\ell-\ell'|-|m-m'|}{2}\delta^{\frac{1}{2}})\sin(\pi\tfrac{|\ell-\ell'|+|m-m'|}{2}\delta^{\frac{1}{2}})\big|,$$

which is at least

$$2\big||\ell-\ell'|-|m-m'|\big|\,\big(|\ell-\ell'|+|m-m'|\big)\delta$$

in view of the simple estimate $|\sin t| \ge \frac{2}{\pi}|t|$ for $|t|<\frac{\pi}{2}$. We conclude that if the sets $R(m,m')$ and $R(\ell,\ell')$ intersect and $(\ell,\ell')\neq(m,m')$, then

$$\big||\ell-\ell'|-|m-m'|\big| \le 25000. \tag{10.2.29}$$

Combining (10.2.28) with (10.2.29), it follows that if $\Gamma_m + \Gamma_{m'}$ and $\Gamma_\ell + \Gamma_{\ell'}$ intersect, then

$$\max\Big(\big|\min(m,m') - \min(\ell,\ell')\big|, \big|\max(m,m') - \max(\ell,\ell')\big|\Big) \le \frac{25150}{2}.$$

We conclude that the set $\Gamma_m + \Gamma_{m'}$ intersects the fixed set $\Gamma_\ell + \Gamma_{\ell'}$ for at most $(25151)^2$ pairs (m,m'). This finishes the proof of the lemma. $\square$

Exercises

10.2.1. For $\lambda \ge 0$ show that for all $f \in L^p(\mathbf{R}^n)$ the Bochner–Riesz operators

$$B_R^\lambda(f)(x) = \int_{\mathbf{R}^n} (1 - |\xi/R|^2)_+^\lambda \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$$

converge to f in $L^p(\mathbf{R}^n)$ if and only if the function $(1 - |\xi|^2)_+^\lambda$ lies in $\mathscr{M}_p(\mathbf{R}^n)$.
$\big[$*Hint:* In the beginning of the proof of Theorem 10.2.4 it was shown that if $(1 - |\xi|^2)_+^\lambda$ lies in $\mathscr{M}_p(\mathbf{R}^n)$, then the $B_R^\lambda(f)$ converge to f in $L^p(\mathbf{R}^n)$. Conversely, if for all $f \in L^p(\mathbf{R}^n)$ the $B_R^\lambda(f)$ converge to f in L^p as $R \to \infty$, then for every f in $L^p(\mathbf{R}^n)$ there is a constant C_f such that $\sup_{R>0} \big\|B_R^\lambda(f)\big\|_{L^p} \le C_f < \infty$. It follows that $\sup_{R>0} \big\|B_R^\lambda\big\|_{L^p\to L^p} < \infty$ by the uniform boundedness principle; hence $\big\|B^\lambda\big\|_{L^p\to L^p} < \infty$.$\big]$

10.2.2. Let $|\theta_1|, |\theta_2| < \frac{\pi}{4}$ be two angles. Show geometrically that

$$|r_1 e^{i\theta_1} - r_2 e^{i\theta_2}| \ge \min(r_1, r_2) \sin|\theta_1 - \theta_2|$$

and use the estimate $|\sin t| \ge \frac{2|t|}{\pi}$ for $|t| < \frac{\pi}{2}$ to obtain a lower bound for the second expression in terms of $|\theta_1 - \theta_2|$.

10.2.3. Let R be a rectangle in $\mathbf{R}^2$ having length $b > 0$ along a direction $\vec{v} = (\xi_1, \xi_2)$ and length $a > 0$ along the perpendicular direction $\vec{v}^\perp = (-\xi_2, \xi_1)$. Let $\vec{w}$ be another vector that forms an angle $\varphi < \frac{\pi}{2}$ with $\vec{v}$. Show that the smallest rectangle R' that contains R and has sides parallel to $\vec{w}$ and $\vec{w}^\perp$ has side lengths $a\sin(\varphi) + b\cos(\varphi)$ along the direction $\vec{w}$ and $a\cos(\varphi) + b\sin(\varphi)$ along the direction $\vec{w}^\perp$.

10.2.4. Prove that Theorem 10.2.7 does not hold when $p < 2$.
$\big[$*Hint:* Try the intervals $I_j = [j, j+1]$ and $\widehat{f} = \chi_{[0,N]}$ as $N \to \infty$.$\big]$

10.2.5. Let $\{I_k\}_k$ be a family of intervals in the real line with $|I_k| = |I_{k'}|$ and $I_k \cap I_{k'} = \emptyset$ for all $k \ne k'$. Define the sets

$$S_k = \big\{(\xi_1, \dots, \xi_n) \in \mathbf{R}^n : \ \xi_1 \in I_k\big\}.$$

Prove that for all $p \ge 2$ and all $f \in L^p(\mathbf{R}^n)$ we have

$$\Big\|\Big(\sum_k |(\widehat{f}\chi_{S_k})^\vee|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n)} \le C_p \big\|f\big\|_{L^p(\mathbf{R}^n)},$$

where C_p is the constant of Theorem 10.2.7.

10.2.6. (a) Let $\{I_k\}_k$, $\{J_\ell\}_\ell$ be two families of intervals in the real line with $|I_k| = |I_{k'}|$, $I_k \cap I_{k'} = \emptyset$ for all $k \neq k'$, and $|J_\ell| = |J_{\ell'}|$, $J_\ell \cap J_{\ell'} = \emptyset$ for all ℓ, ℓ'. Prove that for all $p \ge 2$ there is a constant C_p such that

$$\Big\|\Big(\sum_k \sum_\ell |(\widehat{f}\chi_{I_k\times J_\ell})^\vee|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^2)} \le C_p^2 \big\|f\big\|_{L^p(\mathbf{R}^2)},$$

for all $f \in L^p(\mathbf{R}^2)$.
(b) State and prove an analogous result on $\mathbf{R}^n$.
[*Hint:* Use the Rademacher functions and apply Theorem 10.2.7 twice.]

10.2.7. (*Rubio de Francia [273]*) On $\mathbf{R}^n$ consider the points $x_\ell = \ell\sqrt{\delta}$, $\ell \in \mathbf{Z}^n$. Fix a Schwartz function h whose Fourier transform is supported in the unit ball in $\mathbf{R}^n$. Given a function f on $\mathbf{R}^n$, define $\widehat{f_\ell}(\xi) = \widehat{f}(\xi)\widehat{h}(\delta^{-\frac{1}{2}}(\xi - x_\ell))$. Prove that for some constant C (which depends only on h and n) the estimate

$$\Big(\sum_{\ell\in\mathbf{Z}^n} |f_\ell|^2\Big)^{\frac{1}{2}} \le CM(|f|^2)^{\frac{1}{2}}$$

holds for all functions f. Deduce the $L^p(\mathbf{R}^n)$ boundedness of the preceding square function for all $p > 2$.
[*Hint:* For a sequence λ_ℓ with $\sum_\ell |\lambda_\ell|^2 = 1$, set

$$G(f)(x) = \sum_{\ell\in\mathbf{Z}^n} \lambda_\ell f_\ell(x) = \int_{\mathbf{R}^n}\Big[\sum_{\ell\in\mathbf{Z}^n} \lambda_\ell\, e^{2\pi i \frac{x_\ell\cdot y}{\sqrt{\delta}}}\Big] f\big(x - \tfrac{y}{\sqrt{\delta}}\big)h(y)\,dy.$$

Split $\mathbf{R}^n$ as the union of $Q_0 = [-\frac{1}{2},\frac{1}{2}]^n$ and $2^{j+1}Q_0 \setminus 2^jQ_0$ for $j \ge 0$ and control the integral on each such set using the decay of h and the $L^2(2^{j+1}Q_0)$ norms of the other two terms. Finally, exploit the orthogonality of the functions $e^{2\pi i\ell\cdot y}$ to estimate the $L^2(2^{j+1}Q_0)$ norm of the expression inside the square brackets by $C2^{nj/2}$. Sum over $j \ge 0$ to obtain the required conclusion.]

10.2.8. For $\lambda > 0$ define the *maximal Bochner–Riesz operator*

$$B_*^\lambda(f)(x) = \sup_{R>0}\bigg|\int_{\mathbf{R}^n}(1 - |\xi/R|^2)_+^\lambda \widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi\bigg|.$$

Prove that B_*^λ maps $L^p(\mathbf{R}^n)$ to itself when $\lambda > \frac{n-1}{2}$ for $1 \le p \le \infty$.
[*Hint:* Use Corollary 2.1.12.]

10.3 Kakeya Maximal Operators

We recall the Hardy–Littlewood maximal operator with respect to cubes on $\mathbf{R}^n$ defined as

$$M_c(f)(x) = \sup_{\substack{Q\in\mathscr{F}\\ Q\ni x}} \frac{1}{|Q|}\int_Q |f(y)|\,dy\,, \tag{10.3.1}$$

where $\mathscr{F}$ is the set of all closed cubes in $\mathbf{R}^n$ (with sides not necessarily parallel to the axes). The operator M_c is equivalent (bounded above and below by constants) to the corresponding maximal operator M_c' in which the family $\mathscr{F}$ is replaced by the more restrictive family $\mathscr{F}'$ of cubes in $\mathbf{R}^n$ with sides parallel to the coordinate axes.

It is interesting to observe that if the family of all cubes $\mathscr{F}$ in (10.3.1) is replaced by the family of all rectangles (or parallelepipeds) $\mathscr{R}$ in $\mathbf{R}^n$, then we obtain an operator M_0 that is unbounded on $L^p(\mathbf{R}^n)$; see also Exercise 2.1.9. If we substitute the family of all parallelepipeds $\mathscr{R}$, however, with the more restrictive family $\mathscr{R}'$ of all parallelepipeds with sides parallel to the coordinate axes, then we obtain the so-called strong maximal function

$$M_s(f)(x) = \sup_{\substack{R\in\mathscr{R}'\\ R\ni x}} \frac{1}{|R|}\int_R |f(y)|\,dy\,, \tag{10.3.2}$$

which was introduced in Exercise 2.1.6. The operator M_s is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$ but it is not of weak type $(1,1)$. See Exercise 10.3.1.

These examples indicate that averaging over long and skinny rectangles is quite different than averaging over squares. In general, the direction and the dimensions of the averaging rectangles play a significant role in the boundedness properties of the maximal functions. In this section we investigate aspects of this topic.

10.3.1 Maximal Functions Associated with a Set of Directions

Definition 10.3.1. Let Σ be a set of unit vectors in $\mathbf{R}^2$, i.e., a subset of the unit circle $\mathbf{S}^1$. Associated with Σ, we define $\mathscr{R}_\Sigma$ to be the set of all closed rectangles in $\mathbf{R}^2$ whose longest side is parallel to some vector in Σ. We also define a maximal operator $\mathfrak{M}_\Sigma$ associated with Σ as follows:

$$\mathfrak{M}_\Sigma(f)(x) = \sup_{\substack{R\in\mathscr{R}_\Sigma\\ R\ni x}} \frac{1}{|R|}\int_R |f(y)|\,dy\,,$$

where f is a locally integrable function on $\mathbf{R}^2$.

We also recall the definition given in (10.2.19) of the *Kakeya maximal operator*

$$\mathscr{K}_N(w)(x) = \sup_{R\ni x} \frac{1}{|R|}\int_R |w(y)|\,dy\,, \tag{10.3.3}$$

where the supremum is taken over all rectangles R in $\mathbf{R}^2$ of dimensions a and aN where $a > 0$ is arbitrary. Here N is a fixed real number that is at least 10.

Example 10.3.2. Let $\Sigma = \{v\}$ consist of only one vector $v = (a,b)$. Then

$$\mathfrak{M}_\Sigma(f)(x) = \sup_{0<r\le 1}\sup_{N>0} \frac{1}{rN^2}\int_{-N}^{N}\int_{-rN}^{rN} |f(x - t(a,b) - s(-b,a))|\,ds\,dt\,.$$

If $\Sigma = \{(1,0),(0,1)\}$ consists of the two unit vectors along the axes, then

$$\mathfrak{M}_\Sigma = M_s\,,$$

where M_s is the strong maximal function defined in (10.3.2).

It is obvious that for each $\Sigma \subseteq \mathbf{S}^1$, the maximal function $\mathfrak{M}_\Sigma$ maps $L^\infty(\mathbf{R}^2)$ to itself with constant 1. But $\mathfrak{M}_\Sigma$ may not always be of weak type $(1,1)$, as the example M_s indicates; see Exercise 10.3.1. The boundedness of $\mathfrak{M}_\Sigma$ on $L^p(\mathbf{R}^2)$ in general depends on the set Σ.

An interesting case arises in the following example as well.

Example 10.3.3. For $N \in \mathbf{Z}^+$, let

$$\Sigma = \Sigma_N = \left\{\left(\cos(\tfrac{2\pi j}{N}), \sin(\tfrac{2\pi j}{N})\right) :\ j = 0,1,2,\dots,N-1\right\}$$

be the set of N uniformly spread directions on the circle. Then we expect $\mathfrak{M}_{\Sigma_N}$ to be L^p bounded with constant depending on N. There is a connection between the operator $\mathfrak{M}_{\Sigma_N}$ previously defined and the Kakeya maximal operator $\mathscr{K}_N$ defined in (10.2.19). In fact, Exercise 10.3.3 says that

$$\mathscr{K}_N(f) \le 20\,\mathfrak{M}_{\Sigma_N}(f) \tag{10.3.4}$$

for all locally integrable functions f on $\mathbf{R}^2$.

We now indicate why the norms of $\mathscr{K}_N$ and $\mathfrak{M}_{\Sigma_N}$ on $L^2(\mathbf{R}^2)$ grow as $N \to \infty$. We refer to Exercises 10.3.4 and 10.3.7 for the corresponding result for $p \neq 2$.

Proposition 10.3.4. *There is a constant c such that for any* $N \ge 10$ *we have*

$$\big\|\mathscr{K}_N\big\|_{L^2(\mathbf{R}^2)\to L^2(\mathbf{R}^2)} \ge c\,(\log N) \tag{10.3.5}$$

and

$$\big\|\mathscr{K}_N\big\|_{L^2(\mathbf{R}^2)\to L^{2,\infty}(\mathbf{R}^2)} \ge c\,(\log N)^{\frac{1}{2}}\,. \tag{10.3.6}$$

Therefore, a similar conclusion follows for $\mathfrak{M}_{\Sigma_N}$.

Proof. We consider the family of functions $f_N(x) = \frac{1}{|x|}\chi_{3\le|x|\le N}$ defined on $\mathbf{R}^2$ for $N \ge 10$. Then we have

$$\big\|f_N\big\|_{L^2(\mathbf{R}^2)} \le c_1(\log N)^{\frac{1}{2}}\,. \tag{10.3.7}$$

On the other hand, for every x in the annulus $6 < |x| < N$, we consider the rectangle R_x of dimensions $|x| - 3$ and $\frac{|x|-3}{N}$, one of whose shorter sides touches the circle $|y| = 3$ and the other has midpoint x. Then

$$\mathscr{K}_N(f_N)(x) \geq \frac{1}{|R_x|}\int_{R_x} |f_N(y)|\,dy \geq \frac{c_2 N}{(|x|-3)^2} \iint\limits_{\substack{6 \leq y_1 \leq |x| \\ |y_2| \leq \frac{|x|-3}{2N}}} \frac{dy_1 dy_2}{y_1} \geq c_3 \frac{\log|x|}{|x|}.$$

It follows that

$$\big\|\mathscr{K}_N(f_N)\big\|_{L^2(\mathbf{R}^2)} \geq c_3 \bigg(\int\limits_{6 \leq |x| \leq N} \Big(\frac{\log|x|}{|x|}\Big)^2 dx\bigg)^{\frac{1}{2}} \geq c_4 (\log N)^{\frac{3}{2}}. \tag{10.3.8}$$

Combining (10.3.7) with (10.3.8) we obtain (10.3.5) with $c = c_4/c_1$.

We now turn to estimate (10.3.6). Since for all $6 < |x| < N$ we have

$$\mathscr{K}_N(f_N)(x) \geq c_3 \frac{\log|x|}{|x|} > c_3 \frac{\log N}{N},$$

it follows that $\big|\big\{\mathscr{K}_N(f_N) > c_3 \frac{\log N}{N}\big\}\big| \geq \pi(N^2 - 6^2) \geq c_5 N^2$ and hence

$$\begin{aligned}
\frac{\big\|\mathscr{K}_N(f_N)\big\|_{L^{2,\infty}}}{\big\|f_N\big\|_{L^2}} &\geq \frac{\sup\limits_{\lambda>0} \lambda \big|\big\{\mathscr{K}_N(f_N) > \lambda\big\}\big|^{\frac{1}{2}}}{c_1 (\log N)^{\frac{1}{2}}} \\
&\geq c_3 \frac{\log N}{N} \frac{\big|\big\{\mathscr{K}_N(f_N) > c_3 \frac{\log N}{N}\big\}\big|^{\frac{1}{2}}}{c_1 (\log N)^{\frac{1}{2}}} \\
&\geq \frac{c_3 \sqrt{c_5}}{c_1} (\log N)^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof. □

10.3.2 The Boundedness of $\mathfrak{M}_{\Sigma_N}$ on $L^p(\mathbf{R}^2)$

It is rather remarkable that both estimates of Proposition 10.3.4 are sharp in terms of their behavior as $N \to \infty$, as the following result indicates.

Theorem 10.3.5. *There exist constants $0 < B, C < \infty$ such that for every $N \geq 10$ and all $f \in L^2(\mathbf{R}^2)$ we have*

$$\big\|\mathfrak{M}_{\Sigma_N}(f)\big\|_{L^{2,\infty}(\mathbf{R}^2)} \leq B (\log N)^{\frac{1}{2}} \big\|f\big\|_{L^2(\mathbf{R}^2)} \tag{10.3.9}$$

and

$$\big\|\mathfrak{M}_{\Sigma_N}(f)\big\|_{L^2(\mathbf{R}^2)} \le C(\log N)\big\|f\big\|_{L^2(\mathbf{R}^2)}. \tag{10.3.10}$$

In view of (10.3.4), similar estimates also hold for $\mathscr{K}_N$.

Proof. We deduce (10.3.10) from the weak type estimate (10.3.9), which we rewrite as

$$\big|\big\{x\in\mathbf{R}^2:\ \mathfrak{M}_{\Sigma_N}(f)(x)>\lambda\big\}\big| \le B^2(\log N)\,\frac{\big\|f\big\|_{L^2}^2}{\lambda^2}. \tag{10.3.11}$$

We prove this estimate for some constant $B>0$ independent of N. But prior to doing this we indicate why (10.3.11) implies (10.3.10).

Using Exercise 10.3.2, we have that $\mathfrak{M}_{\Sigma_N}$ maps $L^p(\mathbf{R}^2)$ to $L^p(\mathbf{R}^2)$ (and hence into $L^{p,\infty}$) with constant at most a multiple of $N^{1/p}$ for all $1<p<\infty$. Using this with $p=3/2$, we have

$$\big\|\mathfrak{M}_{\Sigma_N}\big\|_{L^{\frac32}\to L^{\frac32,\infty}} \le \big\|\mathfrak{M}_{\Sigma_N}\big\|_{L^{\frac32}\to L^{\frac32}} \le A\,N^{\frac23} \tag{10.3.12}$$

for some constant $A>0$. Now split f as the sum $f=f_1+f_2+f_3$, where

$$\begin{aligned} f_1 &= f\,\chi_{|f|\le\frac14\lambda}\,,\\ f_2 &= f\,\chi_{\frac14\lambda<|f|\le N^2\lambda}\,,\\ f_3 &= f\,\chi_{N^2\lambda<|f|}\,. \end{aligned}$$

It follows that

$$\big|\big\{\mathfrak{M}_{\Sigma_N}(f)>\lambda\big\}\big| \le \big|\big\{\mathfrak{M}_{\Sigma_N}(f_2)>\tfrac{\lambda}{3}\big\}\big| + \big|\big\{\mathfrak{M}_{\Sigma_N}(f_3)>\tfrac{\lambda}{3}\big\}\big|, \tag{10.3.13}$$

since the set $\big\{\mathfrak{M}_{\Sigma_N}(f_1)>\frac{\lambda}{3}\big\}$ is empty. To obtain the required result we use the $L^{2,\infty}$ estimate (10.3.11) for f_2 and the $L^{\frac32,\infty}$ estimate (10.3.12) for f_3. We have

$$\begin{aligned}
&\big\|\mathfrak{M}_{\Sigma_N}(f)\big\|_{L^2}^2\\
&= 2\int_0^\infty \lambda\,\big|\big\{\mathfrak{M}_{\Sigma_N}(f)>\lambda\big\}\big|\,d\lambda\\
&\le \int_0^\infty 2\lambda\,\big|\big\{\mathfrak{M}_{\Sigma_N}(f_2)>\tfrac{\lambda}{3}\big\}\big|\,d\lambda + \int_0^\infty 2\lambda\,\big|\big\{\mathfrak{M}_{\Sigma_N}(f_3)>\tfrac{\lambda}{3}\big\}\big|\,d\lambda\\
&\le \int_0^\infty \frac{2\lambda B^2(\log N)}{\lambda^2}\int\limits_{\frac14\lambda<|f|\le N^2\lambda} |f|^2\,dx\,d\lambda + \int_0^\infty \frac{2\lambda A^{\frac32}N}{\lambda^{\frac32}}\int\limits_{|f|>N^2\lambda} |f|^{\frac32}\,dx\,d\lambda\\
&\le 2B^2(\log N)\int_{\mathbf{R}^2}|f(x)|^2\int_{\frac{|f(x)|}{N^2}}^{4|f(x)|}\frac{d\lambda}{\lambda}\,dx + 2A^{\frac32}N\int_{\mathbf{R}^2}|f(x)|^{\frac32}\int_0^{\frac{|f(x)|}{N^2}}\frac{d\lambda}{\lambda^{\frac12}}\,dx\\
&= \big(4B^2(\log 2N)(\log N)+4A^{\frac32}\big)\big\|f\big\|_{L^2}^2\\
&\le C(\log N)^2\big\|f\big\|_{L^2}^2
\end{aligned}$$

using Fubini's theorem for integrals. This proves (10.3.10).

To avoid problems with antipodal points, it is convenient to split Σ_N as the union of eight sets, in each of which the angle between any two vectors does not exceed $2\pi/8$. It suffices therefore to obtain (10.3.11) for each such subset of Σ_N. Let us fix one such subset of Σ_N, which we call Σ_N^1. To prove (10.3.11), we fix a $\lambda > 0$ and we start with a compact subset K of the set $\{x \in \mathbf{R}^2 : \mathfrak{M}_{\Sigma_N^1}(f)(x) > \lambda\}$. Then for every $x \in K$, there exists an open rectangle R_x that contains x and whose longest side is parallel to a vector in Σ_N^1. By compactness of K, there exists a finite subfamily $\{R_\alpha\}_{\alpha\in\mathscr{A}}$ of the family $\{R_x\}_{x\in K}$ such that

$$\int_{R_\alpha} |f(y)|\,dy > \lambda\,|R_\alpha|$$

for all $\alpha \in \mathscr{A}$ and such that the union of the R_α's covers K.

We claim that there is a constant C such that for any finite family $\{R_\alpha\}_{\alpha\in\mathscr{A}}$ of rectangles whose longest side is parallel to a vector in Σ_N^1 there is a subset $\mathscr{B}$ of $\mathscr{A}$ such that

$$\int_{\mathbf{R}^2} \Big(\sum_{\beta\in\mathscr{B}} \chi_{R_\beta}(x)\Big)^2 dx \le C\Big|\bigcup_{\beta\in\mathscr{B}} R_\beta\Big| \tag{10.3.14}$$

and that

$$\Big|\bigcup_{\alpha\in\mathscr{A}} R_\alpha\Big| \le C(\log N)\Big|\bigcup_{\beta\in\mathscr{B}} R_\beta\Big|. \tag{10.3.15}$$

Assuming (10.3.14) and (10.3.15), we easily deduce (10.3.11). Indeed,

$$\begin{aligned}
\Big|\bigcup_{\beta\in\mathscr{B}} R_\beta\Big| &\le \sum_{\beta\in\mathscr{B}} |R_\beta| \\
&< \frac{1}{\lambda}\sum_{\beta\in\mathscr{B}}\int_{R_\beta} |f(y)|\,dy \\
&= \frac{1}{\lambda}\int_{\mathbf{R}^2}\Big(\sum_{\beta\in\mathscr{B}}\chi_{R_\beta}\Big)|f(y)|\,dy \\
&\le \frac{1}{\lambda}\bigg(\int_{\mathbf{R}^2}\Big(\sum_{\beta\in\mathscr{B}}\chi_{R_\beta}\Big)^2 dx\bigg)^{\frac{1}{2}}\|f\|_{L^2} \\
&\le \frac{C^{\frac{1}{2}}}{\lambda}\Big|\bigcup_{\beta\in\mathscr{B}} R_\beta\Big|^{\frac{1}{2}}\|f\|_{L^2},
\end{aligned}$$

from which it follows that

$$\Big|\bigcup_{\beta\in\mathscr{B}} R_\beta\Big| \le \frac{C}{\lambda^2}\|f\|_{L^2}^2.$$

Then, using (10.3.15), we obtain

$$|K| \le \Big|\bigcup_{\alpha\in\mathscr{A}} R_\alpha\Big| \le C(\log N)\Big|\bigcup_{\beta\in\mathscr{B}} R_\beta\Big| \le \frac{C^2}{\lambda^2}(\log N)\big\|f\big\|_{L^2}^2,$$

and since K was an arbitrary compact subset of $\{x:\ \mathfrak{M}_{\Sigma_N^1}(f)(x)>\lambda\}$, the same estimate is valid for the latter set.

We now turn to the selection of the subfamily $\{R_\beta\}_{\beta\in\mathscr{B}}$ and the proof of (10.3.14) and (10.3.15).

Let R_{β_1} be the rectangle in $\{R_\alpha\}_{\alpha\in\mathscr{A}}$ with the longest side. Suppose we have chosen $R_{\beta_1}, R_{\beta_2},\dots,R_{\beta_{j-1}}$ for some $j\ge 2$. Then among all rectangles R_α that satisfy

$$\sum_{k=1}^{j-1}|R_{\beta_k}\cap R_\alpha| \le \frac{1}{2}|R_\alpha|, \tag{10.3.16}$$

we choose a rectangle R_{β_j} such that its longer side is as large as possible. Since the collection $\{R_\alpha\}_{\alpha\in\mathscr{A}}$ is finite, this selection stops after m steps. Define

$$\mathscr{B}=\{\beta_1,\beta_2,\dots,\beta_m\}.$$

Using (10.3.16), we obtain

$$\begin{aligned}
\int_{\mathbf{R}^2}\Big(\sum_{\beta\in\mathscr{B}}\chi_{R_\beta}\Big)^2dx &\le 2\sum_{j=1}^m\sum_{k=1}^j|R_{\beta_k}\cap R_{\beta_j}|\\
&= 2\sum_{j=1}^m\Big[\Big(\sum_{k=1}^{j-1}|R_{\beta_k}\cap R_{\beta_j}|\Big)+|R_{\beta_j}|\Big]\\
&\le 2\sum_{j=1}^m\Big[\frac{1}{2}|R_{\beta_j}|+|R_{\beta_j}|\Big]\\
&= 3\sum_{j=1}^m|R_{\beta_j}|.
\end{aligned}\tag{10.3.17}$$

A consequence of this fact is that

$$\begin{aligned}
\sum_{j=1}^m|R_{\beta_j}| &= \int_{\bigcup_{j=1}^m R_{\beta_j}}\Big(\sum_{j=1}^m\chi_{R_{\beta_j}}\Big)dx\\
&\le \Big|\bigcup_{j=1}^m R_{\beta_j}\Big|^{\frac12}\Big(\int_{\mathbf{R}^n}\Big(\sum_{\beta\in\mathscr{B}}\chi_{R_\beta}\Big)^2dx\Big)^{\frac12}\\
&\le \Big|\bigcup_{j=1}^m R_{\beta_j}\Big|^{\frac12}\sqrt{3}\Big(\sum_{j=1}^m|R_{\beta_j}|\Big)^{\frac12},
\end{aligned}$$

which implies that

$$\sum_{j=1}^{m} |R_{\beta_j}| \leq 3 \Big| \bigcup_{j=1}^{m} R_{\beta_j} \Big|. \tag{10.3.18}$$

Using (10.3.18) in conjunction with the last estimate in (10.3.17), we deduce the desired inequality (10.3.14) with $C = 9$.

We now turn to the proof of (10.3.15). Let M_c be the usual Hardy–Littlewood maximal operator with respect to cubes in $\mathbf{R}^n$ (or squares in $\mathbf{R}^2$; recall $n = 2$). Since M_c is of weak type $(1,1)$, (10.3.15) is a consequence of the estimate

$$\bigcup_{\alpha \in \mathscr{A} \setminus \mathscr{B}} R_\alpha \subseteq \Big\{ x \in \mathbf{R}^2 : \, M_c \Big(\sum_{\beta \in \mathscr{B}} \chi_{(R_\beta)^*} \Big)(x) > c\,(\log N)^{-1} \Big\} \tag{10.3.19}$$

for some absolute constant c, where $(R_\beta)^*$ is the rectangle R_β expanded 5 times in both directions. Indeed, if (10.3.19) holds, then

$$\begin{aligned} \Big| \bigcup_{\alpha \in \mathscr{A}} R_\alpha \Big| &\leq \Big| \bigcup_{\beta \in \mathscr{B}} R_\beta \Big| + \Big| \bigcup_{\alpha \in \mathscr{A} \setminus \mathscr{B}} R_\alpha \Big| \\ &\leq \Big| \bigcup_{\beta \in \mathscr{B}} R_\beta \Big| + \frac{10}{c} (\log N) \sum_{\beta \in \mathscr{B}} |(R_\beta)^*| \\ &\leq \Big| \bigcup_{\beta \in \mathscr{B}} R_\beta \Big| + \frac{250}{c} (\log N) \sum_{\beta \in \mathscr{B}} |R_\beta| \\ &\leq C\,(\log N) \Big| \bigcup_{\beta \in \mathscr{B}} R_\beta \Big| , \end{aligned}$$

where we just used (10.3.18) and the fact that N is large.

It remains to prove (10.3.19). At this point we need the following lemma. In the sequel we denote by θ_α the angle between the x axis and the vector pointing in the longer direction of R_α for any $\alpha \in \mathscr{A}$. We also denote by l_α the shorter side of R_α and by L_α the longer side of R_α for any $\alpha \in \mathscr{A}$. Finally, we set

$$\omega_k = \frac{2\pi 2^k}{N}$$

for $k \in \mathbf{Z}^+$ and $\omega_0 = 0$.

Lemma 10.3.6. *Let R_α be a rectangle in the family $\{R_\alpha\}_{\alpha \in \mathscr{A}}$ and let $0 \leq k < \left[\frac{\log(N/8)}{\log 2}\right]$. Suppose that $\beta \in \mathscr{B}$ is such that*

$$\omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}$$

and such that $L_\beta \geq L_\alpha$. Let $s_\alpha = 8\max(l_\alpha, \omega_k L_\alpha)$. For an arbitrary $x \in R_\alpha$, let Q be a square centered at x with sides of length s_α parallel to the sides of R_α. Then we have

$$\frac{|R_\beta \cap R_\alpha|}{|R_\alpha|} \leq 32\, \frac{|(R_\beta)^* \cap Q|}{|Q|} \, . \tag{10.3.20}$$

Assuming Lemma 10.3.6, we conclude the proof of (10.3.19). Fix $\alpha \in \mathscr{A} \setminus \mathscr{B}$. Then the rectangle R_α was not selected in the selection procedure. This means that for all $l \in \{2,\dots,m+1\}$ we have exactly one of the following: either

$$\sum_{j=1}^{l-1} |R_{\beta_j} \cap R_\alpha| > \frac{1}{2}|R_\alpha| \tag{10.3.21}$$

or

$$\sum_{j=1}^{l-1} |R_{\beta_j} \cap R_\alpha| \le \frac{1}{2}|R_\alpha| \qquad \text{and} \qquad L_\alpha \le L_{\beta_l}. \tag{10.3.22}$$

If (10.3.22) holds for $l = 2$, we let $\mu \le m$ be the largest integer such that (10.3.22) holds for all $l \le \mu$. Then (10.3.22) fails for $l = \mu + 1$; hence (10.3.21) holds for $l = \mu + 1$; thus

$$\frac{1}{2}|R_\alpha| < \sum_{j=1}^{\mu} |R_{\beta_j} \cap R_\alpha| \le \sum_{\substack{\beta \in \mathscr{B} \\ L_\beta \ge L_\alpha}} |R_\beta \cap R_\alpha|. \tag{10.3.23}$$

If (10.3.22) fails for $l = 2$, then (10.3.21) holds for $l = 2$, and this implies that

$$\frac{1}{2}|R_\alpha| < |R_{\beta_1} \cap R_\alpha| \le \sum_{\substack{\beta \in \mathscr{B} \\ L_\beta \ge L_\alpha}} |R_\beta \cap R_\alpha|.$$

In either case we have

$$\frac{1}{2}|R_\alpha| < \sum_{\substack{\beta \in \mathscr{B} \\ L_\beta \ge L_\alpha}} |R_\beta \cap R_\alpha|,$$

and from this it follows that there exists a k with $0 \le k < \left[\frac{\log(N/8)}{\log 2}\right]$ such that

$$\frac{\log 2}{2\log(N/8)}\, |R_\alpha| \; < \sum_{\substack{\beta \in \mathscr{B} \\ L_\beta \ge L_\alpha \\ \omega_k \le |\theta_\beta - \theta_\alpha| < \omega_{k+1}}} |R_\beta \cap R_\alpha|. \tag{10.3.24}$$

By Lemma 10.3.6, for any $x \in R_\alpha$ there is a square Q such that (10.3.20) holds for any R_β with $\beta \in \mathscr{B}$ satisfying $L_\beta \ge L_\alpha$ and $\omega_k \le |\theta_\beta - \theta_\alpha| < \omega_{k+1}$. It follows that

$$\frac{\log 2}{2\log(N/8)} < 2 \sum_{\substack{\beta \in \mathscr{B} \\ L_\beta \ge L_\alpha \\ \omega_k \le |\theta_\beta - \theta_\alpha| < \omega_{k+1}}} \frac{|(R_\beta)^* \cap Q|}{|Q|},$$

which implies

$$\frac{c}{\log N} < \frac{\log 2}{4\log(N/8)} < \frac{1}{|Q|}\int_Q \sum_{\beta\in\mathscr{B}} \chi_{(R_\beta)^*}\,dx\,.$$

This proves (10.3.19), since for $\alpha \in \mathscr{A}\setminus\mathscr{B}$, any $x \in R_\alpha$ must be an element of the set $\big\{x\in\mathbf{R}^2:\ M_c\big(\sum_{\beta\in\mathscr{B}}\chi_{(R_\beta)^*}\big)(x) > c\,(\log N)^{-1}\big\}$. □

It remains to prove Lemma 10.3.6.

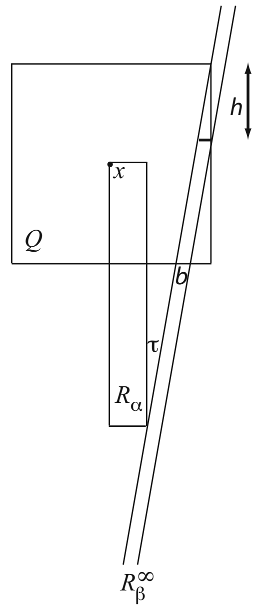

Fig. 10.9 For angles τ less than that displayed, the strip R_β^∞ meets the upper side of Q. The length of the intersection of R_β^∞ with the lower side of Q is denoted by b.

Proof. We fix R_α and R_β so that $L_\beta \ge L_\alpha$ and we assume that $\overline{R_\beta}$ intersects $\overline{R_\alpha}$; otherwise, (10.3.20) is obvious. Let τ be the angle between the directions of the rectangles R_α and R_β, that is,

$$\tau = |\theta_\alpha - \theta_\beta|\,.$$

By assumption we have $\tau < \omega_{k+1} \le \frac{\pi}{4}$, since $k+1 \le [\frac{\log(N/8)}{\log 2}] \le \frac{\log(N/8)}{\log 2}$.

Let R_β^∞ denote the smallest closed infinite strip in the direction of the longer side of R_β that contains it. We make the following observation: if

$$\tan\tau \le \frac{\frac{1}{2}s_\alpha - l_\alpha}{\frac{1}{2}s_\alpha + L_\alpha}\,, \tag{10.3.25}$$

then the strip R_β^∞ intersects the upper side (according to Figure 10.9) of the square Q. Indeed, the worst possible case is drawn in Figure 10.9, in which equality holds in (10.3.25). For $\tau \le \pi/4$ we have $\tan\tau < 3\tau/2$, and since $\tau < 2\omega_k$, it follows that $\tan\tau < 3\omega_k$. Our choice of s_α implies

$$s_\alpha \ge 6\,\omega_k L_\alpha + 2\,l_\alpha \implies 3\omega_k \le \frac{\frac{1}{2}s_\alpha - l_\alpha}{\frac{1}{2}s_\alpha + L_\alpha};$$

hence (10.3.25) holds.

We have now proved that R_β^∞ meets the upper side of Q. We examine the size of the intersection $R_\beta^\infty \cap Q$. According to the picture in Figure 10.9, this intersection contains a parallelogram of base $b = l_\beta / \cos\tau$ and height $s_\alpha - h$ and a right triangle with base b and height h (with $0 \le h \le s_\alpha$). Then we have

$$\frac{|R_\beta^\infty \cap Q|}{|Q|} \ge \frac{1}{s_\alpha^2}\,\frac{l_\beta}{\cos\tau}\Big(s_\alpha - h + \frac{1}{2}h\Big) \ge \frac{1}{s_\alpha^2}\,\frac{l_\beta}{\cos\tau}\Big(\frac{1}{2}s_\alpha\Big) \ge \frac{1}{2}\,\frac{l_\beta}{s_\alpha}.$$

Since $(R_\beta)^*$ has length $5L_\beta$ and R_β meets R_α, we have that $R_\beta^\infty \cap Q \subseteq (R_\beta)^* \cap Q$ and therefore

$$\frac{|(R_\beta)^* \cap Q|}{|Q|} \ge \frac{1}{2}\,\frac{l_\beta}{s_\alpha}. \tag{10.3.26}$$

On the other hand, let $R_{\alpha,\beta}$ be the smallest parallelogram two of whose opposite sides are parallel to the shorter sides of R_α and whose remaining two sides are contained in the boundary lines of R_β^∞. Then

$$|R_\alpha \cap R_\beta| \le |R_{\alpha,\beta}| \le \frac{l_\beta}{\cos\tau} L_\alpha \le 2\,l_\beta L_\alpha.$$

Another geometric argument shows that

$$|R_\alpha \cap R_\beta| \le l_\beta\,\frac{l_\alpha}{\sin(\tau)} \le l_\alpha\, l_\beta\,\frac{\pi}{2\tau} \le l_\alpha\, l_\beta\,\frac{\pi}{2\omega_k} \le 2\,\frac{l_\alpha\, l_\beta}{\omega_k}.$$

Combining these estimates, we deduce

$$\frac{|R_\alpha \cap R_\beta|}{|R_\alpha|} \le 2\min\Big(\frac{l_\beta}{l_\alpha}, \frac{l_\beta}{\omega_k L_\alpha}\Big) \le 16\,\frac{l_\beta}{s_\alpha}. \tag{10.3.27}$$

Finally, (10.3.26) and (10.3.27) yield (10.3.20). □

We end this subsection with an immediate corollary of the theorem just proved.

Corollary 10.3.7. *For every $1 < p < \infty$ there exists a constant c_p such that*

$$\big\|\mathscr{K}_N\big\|_{L^p(\mathbf{R}^2)\to L^p(\mathbf{R}^2)} \le c_p \begin{cases} N^{\frac{2}{p}-1}(\log N)^{\frac{1}{p'}} & \text{when } 1 < p < 2, \\ (\log N)^{\frac{1}{p}} & \text{when } 2 < p < \infty. \end{cases} \tag{10.3.28}$$

Proof. We see that

$$\big\|\mathscr{K}_N\big\|_{L^1(\mathbf{R}^2)\to L^{1,\infty}(\mathbf{R}^2)} \le CN \tag{10.3.29}$$

by replacing a rectangle of dimensions $a \times aN$ by the smallest square of side length aN that contains it. Interpolating between (10.3.9) and (10.3.29), we obtain the first statement in (10.3.28). The second statement in (10.3.28) follows by interpolation between (10.3.9) and the trivial $L^\infty \to L^\infty$ estimate. (In both cases we use Theorem 1.3.2.) □

10.3.3 The Higher-Dimensional Kakeya Maximal Operator

The Kakeya maximal operator without dilations $\mathscr{K}_N^a$ on $L^2(\mathbf{R}^2)$ was crucial in the study of the boundedness of the Bochner–Riesz operator B^λ on $L^4(\mathbf{R}^2)$. An analogous maximal operator could be introduced on $\mathbf{R}^n$.

Definition 10.3.8. Given fixed $a > 0$ and $N \geq 10$, we introduce the *Kakeya maximal operator without dilations* on $\mathbf{R}^n$ as

$$\mathscr{K}_N^a(f)(x) = \sup_R \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where the supremum is taken over all rectangular parallelepipeds (boxes) of arbitrary orientation in $\mathbf{R}^n$ that contain the point x and have dimensions

$$\underbrace{a \times a \times \cdots \times a}_{n-1 \text{ times}} \times aN.$$

We also define the centered version $\mathfrak{K}_N^a$ of $\mathscr{K}_N^a$ as follows:

$$\mathfrak{K}_N^a(f)(x) = \sup_R \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where the supremum is restricted to those rectangles among the previous ones that are centered at x. These two maximal operators are comparable, and we have

$$\mathfrak{K}_N^a \leq \mathscr{K}_N^a \leq 2^n \mathfrak{K}_N^a$$

by a simple geometric argument.

We also define the higher-dimensional analogue of the Kakeya maximal operator $\mathscr{K}_N$ introduced in (10.3.3).

Definition 10.3.9. Let $N \geq 10$. We denote by $\mathscr{R}(N)$ the set of all rectangular parallelepipeds (boxes) in $\mathbf{R}^n$ with arbitrary orientation and dimensions

$$\underbrace{a \times a \times \cdots \times a}_{n-1 \text{ times}} \times aN$$

with arbitrary $a > 0$. Given a locally integrable function f on $\mathbf{R}^n$, we define

$$\mathscr{K}_N(f)(x) = \sup_{\substack{R \in \mathscr{R}(N) \\ R \ni x}} \frac{1}{|R|} \int_R |f(y)|\,dy$$

and

$$\mathfrak{K}_N(f)(x) = \sup_{\substack{R \in \mathscr{R}(N) \\ R \text{ has center } x}} \frac{1}{|R|} \int_R |f(y)|\,dy;$$

$\mathfrak{K}_N$ and $\mathscr{K}_N$ are called the centered and uncentered nth-*dimensional Kakeya maximal operators*, respectively.

For convenience we call rectangular parallelepipeds, i.e., elements of $\mathscr{R}(N)$, higher-dimensional rectangles, or simply rectangles. We clearly have

$$\sup_{a>0} \mathscr{K}_N^a = \mathscr{K}_N \qquad \text{and} \qquad \sup_{a>0} \mathfrak{K}_N^a = \mathfrak{K}_N\,;$$

hence the boundedness of $\mathscr{K}_N^a$ can be deduced from that of $\mathscr{K}_N$; however, this deduction can essentially be reversed with only logarithmic loss in N (see the references at the end of this chapter). In the sequel we restrict attention to the operator $\mathscr{K}_N^a$, whose study already presents all the essential difficulties and requires a novel set of ideas in its analysis. We consider a specific value of a, since a simple dilation argument yields that the norms of $\mathscr{K}_N^a$ and $\mathscr{K}_N^b$ on a fixed $L^p(\mathbf{R}^n)$ are equal for all $a, b > 0$.

Concerning $\mathscr{K}_N^1$, we know that

$$\big\|\mathscr{K}_N^1\big\|_{L^1(\mathbf{R}^n)\to L^{1,\infty}(\mathbf{R}^n)} \le c_n N^{n-1}\,. \tag{10.3.30}$$

This estimate follows by replacing a rectangle of dimensions $\overbrace{1 \times 1 \times \cdots \times 1}^{n-1 \text{ times}} \times N$ by the smallest cube of side length N that contains it. This estimate is sharp; see Exercise 10.3.7.

It would be desirable to know the following estimate for $\mathscr{K}_N^1$:

$$\big\|\mathscr{K}_N^1\big\|_{L^n(\mathbf{R}^n)\to L^{n,\infty}(\mathbf{R}^n)} \le c_n'(\log N)^{\frac{n-1}{n}} \tag{10.3.31}$$

for some dimensional constant c_n'. It would then follow that

$$\big\|\mathscr{K}_N^1\big\|_{L^n(\mathbf{R}^n)\to L^n(\mathbf{R}^n)} \le c_n'' \log N \tag{10.3.32}$$

for some other dimensional constant c_n''; see Exercise 10.3.8(b). Moreover, if estimate (10.3.31) were true, then interpolating between (10.3.30) and (10.3.31) would yield the bound

$$\big\|\mathscr{K}_N^1\big\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} \le c_{n,p} N^{\frac{n}{p}-1} (\log N)^{\frac{1}{p'}}\,, \qquad 1 < p < n\,. \tag{10.3.33}$$

It is estimate (10.3.33) that we would like to concentrate on. We have the following result for a certain range of p's in the interval $(1,n)$.

Theorem 10.3.10. *Let* $p_n = \frac{n+1}{2}$ *and* $N \geq 10$. *Then there exists a constant* C_n *such that*

$$\big\|\mathscr{K}_N^1\big\|_{L^{p_n,1}(\mathbf{R}^n)\to L^{p_n,\infty}(\mathbf{R}^n)} \leq C_n N^{\frac{n}{p_n}-1}, \tag{10.3.34}$$

$$\big\|\mathscr{K}_N^1\big\|_{L^{p_n}(\mathbf{R}^n)\to L^{p_n,\infty}(\mathbf{R}^n)} \leq C_n N^{\frac{n}{p_n}-1}(\log N)^{\frac{1}{p_n'}}, \tag{10.3.35}$$

$$\big\|\mathscr{K}_N^1\big\|_{L^{p_n}(\mathbf{R}^n)\to L^{p_n}(\mathbf{R}^n)} \leq C_n N^{\frac{n}{p_n}-1}(\log N). \tag{10.3.36}$$

Moreover, for every $1 < p < p_n$ *there exists a constant* $C_{n,p}$ *such that*

$$\big\|\mathscr{K}_N^1\big\|_{L^{p}(\mathbf{R}^n)\to L^{p}(\mathbf{R}^n)} \leq C_{n,p} N^{\frac{n}{p}-1}(\log N)^{\frac{1}{p'}}. \tag{10.3.37}$$

Proof. We begin by observing that (10.3.37) is a consequence of (10.3.30) and (10.3.35) using Theorem 1.3.2. We also observe that (10.3.36) is a consequence of (10.3.35), while (10.3.35) is a consequence of (10.3.34) (see Exercise 10.3.8). We therefore concentrate on estimate (10.3.34).

We choose to work with the centered version $\mathfrak{K}_N^1$ of $\mathscr{K}_N^1$, which is comparable to it. To make the geometric idea of the proof a bit more transparent, we pick $\delta < 1/10$, we set $N = 1/\delta$, and we work with the equivalent operator $\mathfrak{K}_{1/\delta}^{\delta}$, whose norm is the same as that of $\mathscr{K}_N^1$. Since the operators in question are positive, we work with nonnegative functions.

The proof is based on a linearization of the operator $\mathscr{K}_{1/\delta}^{\delta}$. Let us call a rectangle of dimensions $\delta \times \delta \times \cdots \times \delta \times 1$ a δ-*tube*. We call the line segment parallel to the longest edges that joins the centers of its two smallest faces, a δ-tube's *axis of symmetry*.

For every x in $\mathbf{R}^n$ we select (in some measurable way) a δ-tube $\tau(x)$ that contains x such that

$$\frac{1}{2}\mathscr{K}_{1/\delta}^{\delta}(f)(x) \leq \frac{1}{|\tau(x)|}\int_{\tau(x)} f(y)\,dy.$$

Suppose we have a grid of cubes in $\mathbf{R}^n$ each of side length $\delta' = \delta/(2\sqrt{n})$, and let Q_j be a cube in that grid with center c_{Q_j}. Then any δ-tube centered at a point $z \in Q_j$ must contain the entire Q_j, and it follows that

$$\mathfrak{K}_{1/\delta}^{\delta}(f)(z) \leq \mathscr{K}_{1/\delta}^{\delta}(f)(c_{Q_j}) \leq \frac{2}{|\tau(c_{Q_j})|}\int_{\tau(c_{Q_j})} f(y)\,dy. \tag{10.3.38}$$

This observation motivates the introduction of a grid of width $\delta' = \delta/(2\sqrt{n})$ in $\mathbf{R}^n$ so that for every cube Q_j in the grid there is an associated δ-tube τ_j satisfying

$$\tau_j \cap Q_j \neq \emptyset.$$

Then we define a linear operator

$$L^{\delta}(f) = \sum_{j} \left(\frac{1}{|\tau_j|} \int_{\tau_j} f(y)\, dy \right) \chi_{Q_j} ,$$

which certainly satisfies

$$L^{\delta}(f) \le 2^n \mathscr{K}^{2\delta}_{1/\delta}(f) \le 4^n \mathfrak{K}^{2\delta}_{1/\delta}(f) ,$$

and in view of (10.3.38), it also satisfies

$$\mathfrak{K}^{\delta}_{1/\delta}(f) \le 2L^{\delta}(f) .$$

It suffices to show that L^{δ} is bounded from $L^{p_n,1}$ to $L^{p_n,\infty}$ with constant $C_n(\delta^{-1})^{\frac{n}{p_n}-1}$, which is independent of the choice of δ-tubes τ_j.

Our next reduction is to take f to be the characteristic function of a set. The space $L^{p_n,\infty}$ is normable [i.e., it has an equivalent norm under which it is a Banach space (Exercise 1.1.12)]; hence by Exercise 1.4.7, the boundedness of L^{δ} from $L^{p_n,1}$ to $L^{p_n,\infty}$ is a consequence of the restricted weak type estimate

$$\sup_{\lambda>0} \lambda \big| \{ L^{\delta}(\chi_A) > \lambda \} \big|^{\frac{1}{p_n}} \le C_n' (\delta^{-1})^{\frac{n}{p_n}-1} |A|^{\frac{1}{p_n}} , \tag{10.3.39}$$

for some dimensional constant C_n and all sets A of finite measure. This estimate can be written as

$$\lambda^{\frac{n+1}{2}} \delta^{\frac{n-1}{2}} |E_\lambda| \le C_n |A| , \tag{10.3.40}$$

where

$$E_\lambda = \{ x \in \mathbf{R}^n : \, L^{\delta}(\chi_A)(x) > \lambda \} = \{ L^{\delta}(\chi_A) > \lambda \} .$$

Our final reduction stems from the observation that the operator L^{δ} is "local." This means that if f is supported in a cube Q, say of side length one, then $L^{\delta}(f)$ is supported in a fixed multiple of Q. Indeed, it is simple to verify that if $x \notin 10Q$ and f is supported in Q, then $L^{\delta}(f)(x) = 0$, since no δ-tube containing x can reach Q. For "local" operators, it suffices to prove their boundedness for functions supported in cubes of side length one; see Exercise 10.3.9. We may therefore work with a measurable set A contained in a cube in $\mathbf{R}^n$ of side length one. This assumption has as a consequence that E_λ is contained in a fixed multiple of Q, such as $10Q$.

Having completed all the required reductions, we proceed by proving the restricted weak type estimate (10.3.40) for sets A supported in a cube of side length one. In proving (10.3.40) we may take $\lambda \le 1$; otherwise, the set E_λ is empty. We consider the cases $c_0(n)\,\delta \le \lambda$ and $c_0(n)\,\delta > \lambda$, for some large constant $c_0(n)$ to be determined later. If $c_0(n)\,\delta > \lambda$, then

$$|E_\lambda| \le C_n^1 \, (1/\delta)^{n-1} \frac{|A|}{\lambda} \tag{10.3.41}$$

by the weak type $(1,1)$ boundedness of L^{δ} with constant $C_n^1 \delta^{1-n}$. It follows from (10.3.41) that

$$C_n^1|A| \geq |E_\lambda|\delta^{n-1}\lambda > c_0(n)^{-\frac{n-1}{2}}\,|E_\lambda|\lambda^{\frac{n+1}{2}}\delta^{\frac{n-1}{2}},$$

which proves (10.3.40) in this case.

We now assume $c_0(n)\,\delta \leq \lambda \leq 1$. Since $L^\delta(\chi_A)$ is constant on each Q_j, we have that each Q_j is either entirely contained in the set E_λ or disjoint from it. Consequently, setting

$$\mathscr{E} = \{j:\ Q_j \subseteq E_\lambda\},$$

we have

$$E_\lambda = \bigcup_{j\in\mathscr{E}} Q_j\,.$$

Hence

$$|\mathscr{E}| = \#\{j:\ j\in\mathscr{E}\} = |E_\lambda|(\delta')^{-n},$$

and for all $j\in\mathscr{E}$ we have

$$|\tau_j\cap A| > \lambda|\tau_j| = \lambda\,\delta^{n-1}\,.$$

It follows that

$$\begin{aligned}
|A|\,\sup_x\Big[\sum_{j\in\mathscr{E}}\chi_{\tau_j}(x)\Big] &\geq \int_A\sum_{j\in\mathscr{E}}\chi_{\tau_j}\,dx\\
&= \sum_{j\in\mathscr{E}}|\tau_j\cap A|\\
&> \lambda\,\delta^{n-1}|\mathscr{E}|\\
&= \lambda\,\delta^{n-1}\frac{|E_\lambda|}{(\delta')^n}\\
&= (2\sqrt{n})^n\,\frac{\lambda\,|E_\lambda|}{\delta}\,.
\end{aligned}$$

Therefore, there exists an x_0 in A such that

$$\#\{j\in\mathscr{E}:\ x_0\in\tau_j\} > (2\sqrt{n})^n\,\frac{\lambda\,|E_\lambda|}{\delta|A|}\,.$$

Let $S(x_0,\frac{1}{2})$ be a sphere of radius $\frac{1}{2}$ centered at the point x_0. We find on this sphere a finite set of points $\Theta = \{\theta_k\}_k$ that is maximal with respect to the property that the balls $B(\theta_k,\delta)$ are at distance at least $10\sqrt{n}\,\delta$ from each other. Define spherical caps

$$S_k = \mathbf{S}^{n-1}\cap B(\theta_k,\delta)\,.$$

Since the S_k's are disjoint and have surface measure a constant multiple of δ^{n-1}, it follows that there are about δ^{1-n} such points θ_k.

We count the number of δ-tubes that contain x_0 and intersect a fixed cap S_k. All these δ-tubes are contained in a cylinder of length 3 and diameter $c_1(n)\delta$ whose axis of symmetry contains x_0 and the center of the cap S_k. This cylinder has volume

$3\omega_{n-1}c_1(n)^{n-1}\delta^{n-1}$, and thus it intersects at most $c_2(n)\delta^{-1}$ cubes of the family Q_j, since the Q_j's are disjoint and all have volume equal to $(\delta')^n$. We deduce then that given such a cap S_k, there exist at most $c_3(n)\delta^{-1}$ δ-tubes (from the initial family) that contain the point x_0 and intersect S_k.

Let us call a set of δ-tubes ε-*separated* if for every τ and τ' in the set with $\tau \neq \tau'$ we have that the angle between the axis of symmetry of τ and τ' is at least $\varepsilon > 0$. Since we have at least $\frac{(2\sqrt{n})^n\lambda|E_\lambda|}{\delta|A|}$ δ-tubes that contain the given point x_0, and each cap S_k is intersected by at most $c_3(n)\delta^{-1}$ δ-tubes that contain x_0, it follows that at least $c_4(n)\frac{\lambda|E_\lambda|}{|A|}$ of these δ-tubes have to intersect different caps S_k. But δ-tubes that intersect different caps S_k and contain x_0 are δ-separated. We have therefore shown that there exist at least $c_4(n)\frac{\lambda|E_\lambda|}{|A|}$ δ-separated tubes from the original family that contain the point x_0. Call $\mathscr{T}$ the family of these δ-tubes.

We find a maximal subset Θ' of the θ_k's such that the balls $B(\theta_k,\delta)$, $\theta_k \in \Theta'$, have distance at least $\frac{30\sqrt{n}\delta}{\lambda}$ from each other. This is possible if $\lambda/\delta \geq c_0(n)$ for some large constant $c_0(n)$ [such as $c_0(n) = 1000\sqrt{n}$]. We "thin out" the family $\mathscr{T}$ by removing all the δ-tubes that intersect the caps S_k with $\theta_k \in \Theta \setminus \Theta'$. In other words, we essentially keep in $\mathscr{T}$ one out of every $1/\lambda^{n-1}$ δ-tubes. In this way we extract at least $c_5(n)\frac{\lambda^n|E_\lambda|}{|A|}$ δ-tubes from $\mathscr{T}$ that are $\frac{60\sqrt{n}\delta}{\lambda}$-separated and contain the point x_0. We denote these tubes by $\{\tau_j : j \in \mathscr{F}\}$.

We have therefore found a subset $\mathscr{F}$ of $\mathscr{E}$ such that

$$x_0 \in \tau_j \quad \text{for all} \quad j \in \mathscr{F}, \tag{10.3.42}$$

$$\tau_k, \tau_j \quad \text{are} \quad 60\sqrt{n}\,\frac{\delta}{\lambda}\text{ - separated} \quad \text{when} \quad j,k \in \mathscr{F},\ j \neq k, \tag{10.3.43}$$

$$|\mathscr{F}| \geq c_5(n)\frac{|E_\lambda|\lambda^n}{|A|}. \tag{10.3.44}$$

Notice that

$$\left|A \cap \tau_j \cap B(x_0, \tfrac{\lambda}{3})\right| \leq \left|\tau_j \cap B(x_0, \tfrac{\lambda}{3})\right| \leq \frac{2}{3}\lambda\delta^{n-1},$$

and since for any $j \in \mathscr{E}$ (and thus for $j \in \mathscr{F}$) we have $|A \cap \tau_j| > \lambda\delta^{n-1}$, it must be the case that

$$\left|A \cap \tau_j \cap B(x_0, \tfrac{\lambda}{3})^c\right| > \frac{1}{3}\lambda\delta^{n-1}. \tag{10.3.45}$$

Moreover, it is crucial to note that the sets

$$A \cap \tau_j \cap B(x_0, \tfrac{\lambda}{3})^c, \qquad\qquad j \in \mathscr{F}, \tag{10.3.46}$$

are pairwise disjoint. In fact, if x_j and x_k are points on the axes of symmetry of two $60\sqrt{n}\frac{\delta}{\lambda}$-separated δ-tubes τ_j and τ_k in $\mathscr{F}$ such that $|x_j - x_0| = |x_k - x_0| = \frac{\lambda}{3}$, then the distance from x_k to x_j must be at least $10\sqrt{n}\,\delta$. This implies that the distance between $\tau_j \cap B(x_0, \frac{\lambda}{3})^c$ and $\tau_k \cap B(x_0, \frac{\lambda}{3})^c$ is at least $6\sqrt{n}\,\delta > 0$. We now conclude the proof of the theorem as follows:

$$
\begin{aligned}
|A| &\geq \Big|A \cap \bigcup_{j \in \mathscr{F}} \big(\tau_j \cap B(x_0, \tfrac{\lambda}{3})^c\big)\Big| \\
&= \sum_{j \in \mathscr{F}} \big|A \cap \tau_j \cap B(x_0, \tfrac{\lambda}{3})^c\big| \\
&\geq \sum_{j \in \mathscr{F}} \frac{\lambda\, \delta^{n-1}}{3} \\
&= |\mathscr{F}|\, \frac{\lambda\, \delta^{n-1}}{3} \\
&\geq c_5(n) \frac{|E_\lambda|\lambda^n}{|A|} \frac{\lambda\, \delta^{n-1}}{3},
\end{aligned}
$$

using that the sets in (10.3.46) are disjoint, (10.3.45), and (10.3.44). We conclude that

$$
|A|^2 \geq \frac{1}{3} c_5(n)\, \lambda^{n+1} \delta^{n-1} |E_\lambda| \geq c_6(n) \lambda^{n+1} \delta^{n-1} |E_\lambda|^2,
$$

since, as observed earlier, the set E_λ is contained in a cube of side length 10. Taking square roots, we obtain (10.3.40). This proves (10.3.39) and hence (10.3.36). □

Exercises

10.3.1. Let h be the characteristic function of the square $[0,1]^2$ in $\mathbf{R}^2$. Prove that for any $0 < \lambda < 1$ we have

$$
\big|\{x \in \mathbf{R}^2 :\ M_s(h)(x) > \lambda\}\big| \geq \frac{1}{\lambda} \log \frac{1}{\lambda}.
$$

Use this to show that M_s is not of weak type $(1,1)$. Compare this result with that of Exercise 2.1.6.

10.3.2. (a) Given a unit vector v in $\mathbf{R}^2$ define the *directional maximal function along* $\vec{v}$ by

$$
M_{\vec{v}}(f)(x) = \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} |f(x - t\vec{v})|\, dt
$$

wherever f is locally integrable over $\mathbf{R}^2$. Prove that for such f, $M_{\vec{v}}(f)(x)$ is well defined for almost all x contained in any line not parallel to $\vec{v}$.
(b) For $1 < p < \infty$, use the method of rotations to show that $M_{\vec{v}}$ maps $L^p(\mathbf{R}^2)$ to itself with norm the same as that of the centered Hardy–Littlewood maximal operator M on $L^p(\mathbf{R})$.
(c) Let Σ be a finite set of directions. Prove that for all $1 \leq p \leq \infty$, there is a constant $C_p > 0$ such that

$$
\big\|\mathfrak{M}_\Sigma(f)\big\|_{L^p(\mathbf{R}^2)} \leq C_p\, |\Sigma|^{\frac{1}{p}} \big\|f\big\|_{L^p(\mathbf{R}^2)}
$$

for all f in $L^p(\mathbf{R}^2)$.
$\big[$*Hint:* Use the inequality $\mathfrak{M}_\Sigma(f)^p \le \sum_{\vec{v}\in\Sigma} [M_{\vec{v}} M_{\vec{v}^\perp}(f)]^p$.$\big]$

10.3.3. Show that

$$\mathscr{K}_N \le 20\,\mathfrak{M}_{\Sigma_N}\,,$$

where Σ_N is a set of N uniformly distributed vectors in $\mathbf{S}^1$.
$\big[$*Hint:* Use Exercise 10.2.3.$\big]$

10.3.4. This exercise indicates a connection between the Besicovitch construction in Section 10.1 and the Kakeya maximal function. Recall the set E of Lemma 10.1.1, which satisfies $\frac{1}{2} \le |E| \le \frac{3}{2}$.
(a) Show that there is a positive constant c such that for all $N \ge 10$ we have

$$\big|\big\{x \in \mathbf{R}^2 :\ \mathscr{K}_N(\chi_E)(x) > \tfrac{1}{144}\big\}\big| \ge c \log\log N\,.$$

(b) Conclude that for all $2 < p < \infty$ there is a constant c_p such that

$$\big\|\mathscr{K}_N\big\|_{L^p(\mathbf{R}^2)\to L^p(\mathbf{R}^2)} \ge c_p (\log\log N)^{\frac{1}{p}}\,.$$

$\big[$*Hint:* Using the notation of Lemma 10.1.1, first show that

$$\big|\big\{x \in \mathbf{R}^2 :\ \mathscr{K}_{3\cdot 2^k \log(k+2)}(\chi_E)(x) > \tfrac{1}{36}\big\}\big| \ge \log(k+2)\,,$$

by showing that the previous set contains all the disjoint rectangles R_j for $j = 1,2,\dots,2^k$; here k is a large positive integer. To show this, for x in $\bigcup_{j=1}^{2^k} R_j$ consider the unique rectangle R_{j_x} that contains x union $(R_{j_x})'$ and set $R_x = R_{j_x} \cup (R_{j_x})'$. Then $|R_x| = 3|R_{j_x}| = 3\cdot 2^{-k}\log(k+2)$, and we have

$$\frac{1}{|R_x|}\int_{R_x} |\chi_E(y)|\,dy = \frac{|E\cap R_x|}{|R_x|} \ge \frac{|E\cap (R_{j_x})'|}{3|R_{j_x}|} \ge \frac{1}{36}$$

in view of conclusion (4) in Lemma 10.1.1. Part (b): Express the L^p norm of $\mathscr{K}_N(\chi_E)$ in terms of its distribution function.$\big]$

10.3.5. Show that $\mathfrak{M}_{\mathbf{S}^1}$ is unbounded on $L^p(\mathbf{R}^2)$ for any $p < \infty$.
$\big[$*Hint:* You may use Proposition 10.3.4 when $p \le 2$. When $p > 2$ one may need Exercise 10.3.4.$\big]$

10.3.6. Consider the n-dimensional Kakeya maximal operator $\mathscr{K}_N$. Show that there exist dimensional constants c_n and c_n' such that for N sufficiently large we have

$$\begin{aligned}
\big\|\mathscr{K}_N\big\|_{L^n(\mathbf{R}^n)\to L^n(\mathbf{R}^n)} &\ge c_n\,(\log N)\,,\\
\big\|\mathscr{K}_N\big\|_{L^n(\mathbf{R}^n)\to L^{n,\infty}(\mathbf{R}^n)} &\ge c_n'\,(\log N)^{\frac{n-1}{n}}\,.
\end{aligned}$$

$\big[$*Hint:* Consider the functions $f_N(x) = \frac{1}{|x|}\chi_{3\le|x|\le N}$ and adapt the argument in Proposition 10.3.4 to an n-dimensional setting.$\big]$

10.3.7. For all $1 \le p < n$ show that there exist constants $c_{n,p}$ such that the n-dimensional Kakeya maximal operator $\mathscr{K}_N$ satisfies

$$\big\|\mathscr{K}_N\big\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} \ge \big\|\mathscr{K}_N\big\|_{L^p(\mathbf{R}^n)\to L^{p,\infty}(\mathbf{R}^n)} \ge c_{n,p} N^{\frac{n}{p}-1}.$$

$\big[$*Hint:* Consider the functions $h_N(x) = |x|^{-\frac{n+1}{p}} \chi_{3\le|x|\le N}$ and show that $\mathscr{K}_N(h_N)(x) > c/|x|$ for all x in the annulus $6 < |x| < N$.$\big]$

10.3.8. (*Carbery, Hernández, and Soria [51]*) Let T be a sublinear operator defined on $L^1(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$ and taking values in a set of measurable functions. Let $10 \le N < \infty$, $1 < p < \infty$, and $0 < a, M < \infty$.
(a) Suppose that

$$\begin{aligned} \big\|T\big\|_{L^1\to L^{1,\infty}} &\le C_1 N^a, \\ \big\|T\big\|_{L^{p,1}\to L^{p,\infty}} &\le M, \\ \big\|T\big\|_{L^\infty\to L^\infty} &\le 1. \end{aligned}$$

Show that

$$\big\|T\big\|_{L^p\to L^{p,\infty}} \le C(a,p,C_1)\, M\, (\log N)^{\frac{1}{p'}}.$$

(b) Suppose that

$$\begin{aligned} \big\|T\big\|_{L^1\to L^{1,\infty}} &\le C_1 N^a, \\ \big\|T\big\|_{L^p\to L^{p,\infty}} &\le M, \\ \big\|T\big\|_{L^\infty\to L^\infty} &\le 1. \end{aligned}$$

Show that

$$\big\|T\big\|_{L^p\to L^p} \le C'(a,p,C_1)\, M (\log N)^{\frac{1}{p}}.$$

$\big[$*Hint:* Part (a): Split $f = f_1 + f_2 + f_3$, where $f_3 = f\chi_{|f|\le\frac{\lambda}{4}}$, $f_2 = f\chi_{\frac{\lambda}{4}<|f|\le L\lambda}$, and $f_1 = f\chi_{|f|>L\lambda}$, where $L^{p-1} = N^a$. Use the weak type $(1,1)$ estimate for f_1 and the restricted weak type (p,p) estimate for f_2 and note that the measure of the set $\{|T(f_3)| > \lambda/3\}$ is zero. One needs the auxiliary result

$$\big\|f\chi_{a\le|f|\le b}\big\|_{L^{p,1}} \le C(p)\,(1+\log\tfrac{b}{a})^{\frac{1}{p'}}\,\big\|f\big\|_{L^p},$$

which can be proved as follows. First use the identity of Proposition 1.4.9. Then note that the distribution function $d_{f\chi_{a\le|f|\le b}}(s)$ is equal to $d_f(a)$ for $s < a$, to $d_f(s)$ for $a \le s < b$, and vanishes for $s \ge b$. It follows that

$$\big\|f\chi_{a\le|f|\le b}\big\|_{L^{p,1}} \le a\, d_f(a)^{\frac{1}{p}} + \int_a^b d_f(t)^{\frac{1}{p}}\,dt \le 2\int_{\frac{a}{2}}^a d_f(t)^{\frac{1}{p}}\,dt + \int_a^b d_f(t)^{\frac{1}{p}}\,dt,$$

from which the claimed estimate follows by Hölder's inequality and Proposition 1.1.4. Part (b): Use the same splitting and the method employed in the proof of Theorem 10.3.5.]

10.3.9. Suppose that T is a linear operator defined on a subspace of measurable functions on $\mathbf{R}^n$ with the property that whenever f is supported in a cube Q of side length s, then $T(f)$ is supported in aQ for some $a > 1$. Prove the following:
(a) If T is defined on $L^p(\mathbf{R}^n)$ for some $0 < p < \infty$ and

$$\big\|T(f)\big\|_{L^p} \le B\big\|f\big\|_{L^p}$$

for all f supported in a cube of side length s, then the same estimate holds (with a larger constant) for all functions in $L^p(\mathbf{R}^n)$.
(b) If T satisfies for some $0 < p < \infty$,

$$\big\|T(\chi_A)\big\|_{L^{p,\infty}} \le B|A|^{\frac{1}{p}}$$

for all measurable sets A contained in a cube of side length s, then the same estimate holds (with a larger constant) for all measurable sets A in $\mathbf{R}^n$.

10.4 Fourier Transform Restriction and Bochner–Riesz Means

If g is a continuous function on $\mathbf{R}^n$, its restriction to a hypersurface $S \subseteq \mathbf{R}^n$ is a well defined function. By a hypersurface we mean a submanifold of $\mathbf{R}^n$ of dimension $n-1$. So, if f is an integrable function on $\mathbf{R}^n$, its Fourier transform $\widehat{f}$ is continuous and hence its restriction $\widehat{f}\big|_S$ on S is well defined.

Definition 10.4.1. Let $1 \le p,q \le \infty$. We say that a compact hypersurface S in $\mathbf{R}^n$ satisfies a (p,q) *restriction theorem* if the restriction operator

$$f \to \widehat{f}\big|_S,$$

which is initially defined on $L^1(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$, has an extension that maps $L^p(\mathbf{R}^n)$ boundedly into $L^q(S)$. The norm of this extension may depend on p,q,n, and S. If S satisfies a (p,q) restriction theorem, we write that property $R_{p\to q}(S)$ holds. We say that property $R_{p\to q}(S)$ holds with constant C if for all $f \in L^1(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$ we have

$$\big\|\widehat{f}\big\|_{L^q(S)} \le C\big\|f\big\|_{L^p(\mathbf{R}^n)}.$$

Example 10.4.2. Property $R_{1\to\infty}(S)$ holds for any compact hypersurface S.

We denote by $\mathscr{R}(f) = \widehat{f}\big|_{\mathbf{S}^{n-1}}$ the restriction of the Fourier transform on a hypersurface S. Let $d\sigma$ be the canonically induced surface measure on S. Then for a function φ defined on S we have

$$\int_{\mathbf{S}^{n-1}} \widehat{f}\,\varphi\,d\sigma = \int_{\mathbf{R}^n} \widehat{f}\,(\widehat{\varphi\,d\sigma})^\vee\,d\xi = \int_{\mathbf{R}^n} f\,\widehat{\varphi\,d\sigma}\,dx,$$

which says that the transpose of the linear operator $\mathscr{R}$ is the linear operator

$$\mathscr{R}^t(\varphi) = \widehat{\varphi\,d\sigma}. \tag{10.4.1}$$

By duality, we easily see that a (p,q) restriction theorem for a compact hypersurface S is equivalent to the following (q',p') *extension theorem* for S:

$$\mathscr{R}^t : L^{q'}(S) \to L^{p'}(\mathbf{R}^n).$$

Our objective is to determine all pairs of indices (p,q) for which the sphere $\mathbf{S}^{n-1}$ satisfies a (p,q) restriction theorem. It becomes apparent in this section that this problem is relevant in the understanding of the norm convergence of the Bochner–Riesz means.

10.4.1 Necessary Conditions for $R_{p\to q}(\mathbf{S}^{n-1})$ to Hold

We look at basic examples that impose restrictions on the indices p,q in order for $R_{p\to q}(\mathbf{S}^{n-1})$ to hold. We first make an observation. If $R_{p\to q}(\mathbf{S}^{n-1})$ holds, then $R_{p\to s}(\mathbf{S}^{n-1})$ for any $s \le q$.

Example 10.4.3. Let $d\sigma$ be surface measure on the unit sphere $\mathbf{S}^{n-1}$. In view of the identity in Appendix B.4, we have

$$\widehat{d\sigma}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|).$$

Using the asymptotics in Appendix B.8, the last expression is equal to

$$\frac{2\sqrt{2\pi}}{|\xi|^{\frac{n-1}{2}}}\cos(2\pi|\xi| - \tfrac{\pi(n-1)}{4}) + O(|\xi|^{-\frac{n+1}{2}})$$

as $|\xi| \to \infty$. It follows that $\mathscr{R}^t(1)(\xi) = \widehat{d\sigma}(\xi)$ does not lie in $L^{p'}(\mathbf{R}^n)$ if $\frac{n-1}{2}p' \le n$ and $\frac{n+1}{2}p' > n$. Thus $R_{p\to q}(\mathbf{S}^{n-1})$ fails when $\frac{2n}{n+1} \le p < \frac{2n}{n-1}$. Since $R_{1\to q}(\mathbf{S}^{n-1})$ holds for all $q \in [1,\infty]$, by interpolation we deduce that $R_{p\to q}(\mathbf{S}^{n-1})$ fails when $p \ge \frac{2n}{n+1}$. We conclude that a necessary condition for $R_{p\to q}(\mathbf{S}^{n-1})$ to hold is that

$$1 \le p < \frac{2n}{n+1}. \tag{10.4.2}$$

In addition to this condition, there is another necessary condition for $R_{p\to q}(\mathbf{S}^{n-1})$ to hold. This is a consequence of the following revealing example.

Example 10.4.4. Let φ be a Schwartz function on $\mathbf{R}^n$ such that $\widehat{\varphi} \ge 0$ and $\widehat{\varphi}(\xi) \ge 1$ for all ξ in the closed ball $|\xi| \le 2$. For $N \ge 1$ define functions

$$f_N(x_1, x_2, \dots, x_{n-1}, x_n) = \varphi\Big(\frac{x_1}{N}, \frac{x_2}{N}, \dots, \frac{x_{n-1}}{N}, \frac{x_n}{N^2}\Big).$$

To test property $R_{p\to q}(\mathbf{S}^{n-1})$, instead of working with $\mathbf{S}^{n-1}$, we may work with the translated sphere $S = \mathbf{S}^{n-1} + (0,0,\dots,0,1)$ in $\mathbf{R}^n$ (cf. Exercise 10.4.2(a)). We have

$$\widehat{f_N}(\xi) = N^{n+1}\widehat{\varphi}(N\xi_1, N\xi_2, \dots, N\xi_{n-1}, N^2\xi_n).$$

We note that for all $\xi = (\xi_1, \dots, \xi_n)$ in the spherical cap

$$S' = S \cap \{\xi \in \mathbf{R}^n : \ \xi_1^2 + \cdots + \xi_{n-1}^2 \le N^{-2} \quad \text{and} \quad \xi_n < 1\}, \tag{10.4.3}$$

we have $\xi_n \le 1 - (1 - \frac{1}{N^2})^{\frac{1}{2}} \le \frac{1}{N^2}$ and therefore

$$|(N\xi_1, N\xi_2, \dots, N\xi_{n-1}, N^2\xi_n)| \le 2.$$

This implies that for all ξ in S' we have $\widehat{f_N}(\xi) \ge N^{n+1}$. But the spherical cap S' in (10.4.3) has surface measure $c(N^{-1})^{n-1}$. We obtain

$$\big\|\widehat{f_N}\big\|_{L^q(S)} \ge \big\|\widehat{f_N}\big\|_{L^q(S')} \ge c^{\frac{1}{q}} N^{n+1} N^{\frac{1-n}{q}}.$$

On the other hand, $\big\|f_N\big\|_{L^p(\mathbf{R}^n)} = \big\|\varphi\big\|_{L^p(\mathbf{R}^n)} N^{\frac{n+1}{p}}$. Therefore, if $R_{p\to q}(\mathbf{S}^{n-1})$ holds, we must have

$$\big\|\varphi\big\|_{L^p(\mathbf{R}^n)} N^{\frac{n+1}{p}} \ge C c^{\frac{1}{q}} N^{n+1} N^{\frac{1-n}{q}},$$

and letting $N \to \infty$, we obtain the following necessary condition on p and q for $R_{p\to q}(\mathbf{S}^{n-1})$ to hold:

$$\frac{1}{q} \ge \frac{n+1}{n-1}\frac{1}{p'}. \tag{10.4.4}$$

We have seen that the restriction property $R_{p\to q}(\mathbf{S}^{n-1})$ fails in the shaded region of Figure 10.10 but obviously holds on the closed line segment CD. It remains to investigate the validity of property $R_{p\to q}(\mathbf{S}^{n-1})$ for $(\frac{1}{p}, \frac{1}{q})$ in the unshaded region of Figure 10.10.

It is a natural question to ask whether the restriction property $R_{p\to q}(\mathbf{S}^{n-1})$ holds on the line segment BD minus the point B in Figure 10.10, i.e., the set

$$\Big\{(p,q): \ \frac{1}{q} = \frac{n+1}{n-1}\frac{1}{p'} \qquad 1 \le p < \frac{2n}{n+1}\Big\}. \tag{10.4.5}$$

If property $R_{p\to q}(\mathbf{S}^{n-1})$ holds for all points in this set, then it will also hold in the closure of the quadrilateral $ABDC$ minus the closed segment AB.

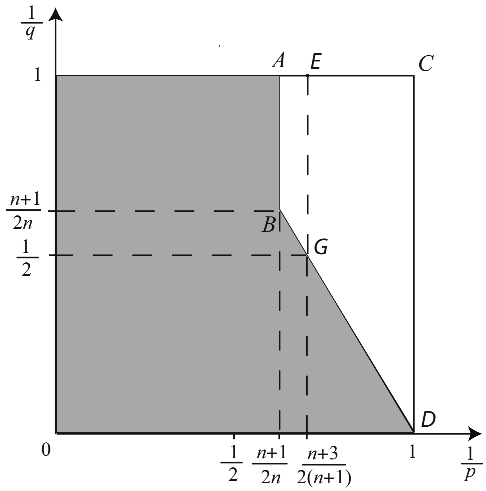

Fig. 10.10 The restriction property $R_{p\to q}(\mathbf{S}^{n-1})$ fails in the shaded region and on the closed line segment AB but holds on the closed line segment CD and could hold on the open line segment BD and inside the unshaded region.

10.4.2 A Restriction Theorem for the Fourier Transform

In this subsection we establish the following restriction theorem for the Fourier transform.

Theorem 10.4.5. *Property $R_{p\to q}(\mathbf{S}^{n-1})$ holds for the set*

$$\left\{(p,q):\ \frac{1}{q}=\frac{n+1}{n-1}\,\frac{1}{p'},\qquad 1\le p\le \frac{2(n+1)}{n+3}\right\} \tag{10.4.6}$$

and therefore for the closure of the quadrilateral with vertices E, G, D, and C in Figure 10.10.

Proof. The case $p=1$ and $q=\infty$ is trivial. Therefore, we need to establish only the case $p=\frac{2(n+1)}{n+3}$ and $q=2$, since the remaining cases follow by interpolation.

Using Plancherel's identity and Hölder's inequality, we obtain

$$\begin{aligned}\big\|\widehat{f}\big\|^2_{L^2(\mathbf{S}^{n-1})} &= \int_{\mathbf{S}^{n-1}} \overline{\widehat{f}(\xi)}\,\widehat{f}(\xi)\,d\sigma(\xi)\\ &= \int_{\mathbf{R}^n} \overline{f(x)}\,(f*d\sigma^\vee)(x)\,dx\\ &\le \big\|f\big\|_{L^p(\mathbf{R}^n)}\big\|f*d\sigma^\vee\big\|_{L^{p'}(\mathbf{R}^n)}\,.\end{aligned}$$

To establish the required conclusion it is enough to show that

$$\big\|f*d\sigma^\vee\big\|_{L^{p'}(\mathbf{R}^n)}\le C_n\big\|f\big\|_{L^p(\mathbf{R}^n)}\qquad \text{when}\quad p=\frac{2(n+1)}{n+3}\,. \tag{10.4.7}$$

To obtain this estimate we need to split the sphere into pieces. Each hyperplane $\xi_k = 0$ cuts the sphere $\mathbf{S}^{n-1}$ into two hemispheres, which we denote by H_k^1 and H_k^2. We introduce a partition of unity $\{\varphi_j\}_j$ of $\mathbf{R}^n$ with the property that for any j there exist $k \in \{1,2,\dots,\}$ and $l \in \{1,2\}$ such that

$$(\text{support}\, \varphi_j) \cap \mathbf{S}^{n-1} \subsetneqq H_k^l\,;$$

that is, the support of each φ_j intersected with the sphere $\mathbf{S}^{n-1}$ is properly contained in some hemisphere H_k^l. Then the family of all φ_j whose support meets $\mathbf{S}^{n-1}$ forms a finite partition of unity of the sphere when restricted to it. We therefore write

$$d\sigma = \sum_{j\in F} \varphi_j\, d\sigma\,,$$

where F is a finite set. If we obtain (10.4.7) for each measure $\varphi_j d\sigma$ instead of $d\sigma$, then (10.4.7) follows by summing on j. We fix such a measure $\varphi_j\, d\sigma$, which, without loss of generality, we assume is supported in $\{\xi \in \mathbf{S}^{n-1} :\ \xi_n > c\} \subsetneqq H_n^1$ for some $c \in (0,1)$. In the sequel we write elements $x \in \mathbf{R}^n$ as $x = (x',t)$, where $x' \in \mathbf{R}^{n-1}$ and $t \in \mathbf{R}$. Then for $x \in \mathbf{R}^n$ we have

$$(\varphi_j\, d\sigma)^\vee(x) = \int_{\mathbf{S}^{n-1}} \varphi_j(\xi) e^{2\pi i x\cdot\xi}\, d\sigma(\xi) = \int\limits_{\substack{\xi'\in\mathbf{R}^{n-1}\\ |\xi'|^2\le 1-c^2}} e^{2\pi i x\cdot \xi}\, \frac{\varphi_j\big(\xi', \sqrt{1-|\xi'|^2}\big)\, d\xi'}{\sqrt{1-|\xi'|^2}}\,,$$

where $\xi = (\xi',\xi_n)$; for the last identity we refer to Appendix D.5. Writing $x = (x',t) \in \mathbf{R}^{n-1}\times\mathbf{R}$, we have

$$\begin{aligned}(\varphi_j\, d\sigma)^\vee(x',t) \ &= \int\limits_{\substack{\xi'\in\mathbf{R}^{n-1}\\ |\xi'|^2\le 1-c^2}} e^{2\pi i x'\cdot\xi'}\, e^{2\pi i t\sqrt{1-|\xi'|^2}}\, \frac{\varphi_j\big(\xi', \sqrt{1-|\xi'|^2}\big)}{\sqrt{1-|\xi'|^2}}\, d\xi' \\ &= \left(e^{2\pi i t\sqrt{1-|\xi'|^2}}\, \frac{\varphi_j\big(\xi', \sqrt{1-|\xi'|^2}\big)}{\sqrt{1-|\xi'|^2}} \right)^{\triangledown} (x')\,,\end{aligned} \tag{10.4.8}$$

where $^\triangledown$ indicates the inverse Fourier transform in the ξ' variable. For each $t \in \mathbf{R}$ we introduce a function on $\mathbf{R}^{n-1}$ by setting

$$K_t(x') = (\varphi_j\, d\sigma)^\vee(x',t)\,.$$

We observe that identity (10.4.8) and the fact that $1-|\xi'|^2 \ge c^2 > 0$ on the support of φ_j imply that

$$\sup_{t\in\mathbf{R}}\ \sup_{\xi'\in\mathbf{R}^{n-1}} |(K_t)^\triangle(\xi')| \le C_n < \infty\,, \tag{10.4.9}$$

where $^\triangle$ indicates the Fourier transform on $\mathbf{R}^{n-1}$. We also have that

$$K_t(x') = (\varphi_j\, d\sigma)^\vee(x',t) = \big(\varphi_j^\vee * d\sigma^\vee\big)(x',t)\,.$$

Since $\varphi_j^\vee$ is a Schwartz function on $\mathbf{R}^n$ and the function $|d\sigma^\vee(x',t)|$ is bounded by $(1+|(x',t)|)^{-\frac{n-1}{2}}$ (see Appendices B.4, B.6, and B.7), it follows from Exercise 2.2.4 that

$$|K_t(x')| \le C(1+|(x',t)|)^{-\frac{n-1}{2}} \le C(1+|t|)^{-\frac{n-1}{2}} \tag{10.4.10}$$

for all $x' \in \mathbf{R}^{n-1}$. Estimate (10.4.9) says that the operator given by convolution with K_t maps $L^2(\mathbf{R}^{n-1})$ to itself with norm at most a constant, while (10.4.10) says that the same operator maps $L^1(\mathbf{R}^{n-1})$ to $L^\infty(\mathbf{R}^{n-1})$ with norm at most a constant multiple of $(1+|t|)^{-\frac{n-1}{2}}$. Interpolating between these two estimates yields

$$\big\|K_t \star g\big\|_{L^{p'}(\mathbf{R}^{n-1})} \le C_{p,n}|t|^{-(n-1)(\frac{1}{p}-\frac{1}{2})}\big\|g\big\|_{L^p(\mathbf{R}^{n-1})}$$

for all $1 \le p \le 2$, where $\star$ denotes convolution on $\mathbf{R}^{n-1}$ (and $*$ convolution on $\mathbf{R}^n$).

We now return to the proof of the required estimate (10.4.7) in which $d\sigma^\vee$ is replaced by $(\varphi_j\, d\sigma)^\vee$. Let $f(x) = f(x',t)$ be a function on $\mathbf{R}^n$. We have

$$\begin{aligned}
\big\|f*(\varphi_j\, d\sigma)^\vee\big\|_{L^{p'}(\mathbf{R}^n)} &= \bigg\|\Big\|\int_{\mathbf{R}} f(\,\cdot\,,\tau)\star K_{t-\tau}\, d\tau\Big\|_{L^{p'}(\mathbf{R}^{n-1})}\bigg\|_{L^{p'}(\mathbf{R})} \\
&\le \bigg\|\int_{\mathbf{R}}\big\|f(\,\cdot\,,\tau)\star K_{t-\tau}\big\|_{L^{p'}(\mathbf{R}^{n-1})}\, d\tau\bigg\|_{L^{p'}(\mathbf{R})} \\
&\le C_{p,n}\bigg\|\int_{\mathbf{R}}\frac{\big\|f(\,\cdot\,,\tau)\big\|_{L^p(\mathbf{R}^{n-1})}}{|t-\tau|^{(n-1)(\frac{1}{p}-\frac{1}{2})}}\, d\tau\bigg\|_{L^{p'}(\mathbf{R})} \\
&= C_{p,n}\Big\|I_\beta\Big(\big\|f(\,\cdot\,,t)\big\|_{L^p(\mathbf{R}^{n-1})}\Big)\Big\|_{L^{p'}(\mathbf{R},dt)},
\end{aligned}$$

where $\beta = 1-(n-1)(\frac{1}{p}-\frac{1}{2})$ and I_β is the Riesz potential (or fractional integral) given in Definition 6.1.1. Using Theorem 6.1.3 with $s=\beta$, $n=1$, and $q=p'$, we obtain that the last displayed equation is bounded by a constant multiple of

$$\Big\|\big\|f(\,\cdot\,,t)\big\|_{L^p(\mathbf{R}^{n-1})}\Big\|_{L^p(\mathbf{R},dt)} = \big\|f\big\|_{L^p(\mathbf{R}^n)}.$$

The condition $\frac{1}{p}-\frac{1}{q}=\frac{s}{n}$ on the indices p,q,s,n assumed in Theorem 6.1.3 translates exactly to

$$\frac{1}{p}-\frac{1}{p'}=\frac{\beta}{1}=1-\frac{n-1}{p}-\frac{n-1}{2},$$

which is equivalent to $p=\frac{2(n+1)}{n+3}$. This concludes the proof of estimate (10.4.7) in which the measure $\sigma^\vee$ is replaced by $(\varphi_j\, d\sigma)^\vee$. Estimates for the remaining $(\varphi_j\, d\sigma)^\vee$ follow by a similar argument in which the role of the last coordinate is played by some other coordinate. The final estimate (10.4.7) follows by summing j over the finite set F. The proof of the theorem is now complete. □

10.4.3 Applications to Bochner–Riesz Multipliers

We now apply the restriction theorem obtained in the previous subsection to the Bochner–Riesz problem. In this subsection we prove the following result.

Theorem 10.4.6. *For* $\mathrm{Re}\,\lambda > \frac{n-1}{2(n+1)}$, *the Bochner–Riesz operator* B^λ *is bounded on* $L^p(\mathbf{R}^n)$ *for* p *in the optimal range*

$$\frac{2n}{n+1+2\mathrm{Re}\,\lambda} < p < \frac{2n}{n-1-2\mathrm{Re}\,\lambda}\,.$$

Proof. The proof is based on the following two estimates:

$$\big\|B^\lambda\big\|_{L^1(\mathbf{R}^n)\to L^1(\mathbf{R}^n)} \le C_1(\mathrm{Re}\,\lambda)\, e^{c_0|\mathrm{Im}\,\lambda|^2} \qquad \text{when } \mathrm{Re}\,\lambda > \tfrac{n-1}{2}, \tag{10.4.11}$$

$$\big\|B^\lambda\big\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} \le C_2(\mathrm{Re}\,\lambda)\, e^{c_0|\mathrm{Im}\,\lambda|^2} \qquad \text{when } \mathrm{Re}\,\lambda > \tfrac{n-1}{2(n+1)}, \tag{10.4.12}$$

where $p = \frac{2(n+1)}{n+3}$ and C_1, C_2 are constants that depend on n and $\mathrm{Re}\,\lambda$, while c_0 is an absolute constant. Once (10.4.11) and (10.4.12) are known, the required conclusion is a consequence of Theorem 1.3.7. Recall that B^λ is given by convolution with the kernel K_λ defined in (10.2.1). This kernel satisfies

$$|K_\lambda(x)| \le C_3(\mathrm{Re}\,\lambda)\, e^{c_0|\mathrm{Im}\,\lambda|^2}(1+|x|)^{-\frac{n+1}{2}-\mathrm{Re}\,\lambda} \tag{10.4.13}$$

in view of the estimates in Appendices B.6 and B.7. Then (10.4.11) follows easily from (10.4.13) and we focus our attention on (10.4.12).

The key ingredient in the proof of (10.4.12) is a decomposition of the kernel. But first we isolate the smooth part of the multiplier near the origin and we focus attention on the part of it near the boundary of the unit disk. Precisely, we start with a Schwartz function $0 \le \eta \le 1$ supported in the ball $B(0,\frac{3}{4})$ that is equal to 1 on the smaller ball $B(0,\frac{1}{2})$. Then we write

$$m_\lambda(\xi) = (1-|\xi|^2)_+^\lambda = (1-|\xi|^2)_+^\lambda\eta(\xi) + (1-|\xi|^2)_+^\lambda(1-\eta(\xi))\,.$$

Since the function $(1-|\xi|^2)_+^\lambda\eta(\xi)$ is smooth and compactly supported, it is an L^p Fourier multiplier for all $1 \le p \le \infty$, with norm that is easily seen to grow polynomially in $|\lambda|$. We therefore need to concentrate on the nonsmooth piece of the multiplier $(1-|\xi|^2)_+^\lambda(1-\eta(\xi))$, which is supported in $B(0,\frac{1}{2})^c$. Let

$$K^\lambda(x) = \Big((1-|\xi|^2)_+^\lambda(1-\eta(\xi))\Big)^\vee(x)$$

be the kernel of the nonsmooth piece of the multiplier.

We pick a smooth *radial* function φ with support inside the ball $B(0,2)$ that is equal to 1 on the closed unit ball $\overline{B(0,1)}$. For $j = 1,2,\dots$ we introduce functions

$$\psi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$$

supported in the annuli $2^{j-1} \le |x| \le 2^{j+1}$. Then we write

$$K^\lambda * f = T_0^\lambda(f) + \sum_{j=1}^{\infty} T_j^\lambda(f), \tag{10.4.14}$$

where T_0^λ is given by convolution with φK^λ and each T_j^λ is given by convolution with $\psi_j K^\lambda$.

We begin by examining the kernel φK^λ. Introducing a compactly supported function ζ that is equal to 1 on $B(0,\frac{3}{2})$, we write

$$\begin{aligned} K^\lambda &= \Big((1-|\cdot|^2)_+^\lambda(1-\eta)\zeta\Big)^\vee \\ &= \big((1-|\cdot|^2)_+^\lambda\big)^\vee * \big((1-\eta)\zeta\big)^\vee \\ &= K_\lambda * \big((1-\eta)\zeta\big)^\vee . \end{aligned}$$

Using this and (10.4.13) implies that K^λ is a bounded function, and thus φK^λ is bounded and compactly supported. Thus the operator T_0^λ is bounded on all the L^p spaces, $1 \le p \le \infty$, with a bound that grows at most exponentially in $|\mathrm{Im}\,\lambda|^2$.

Next we study the boundedness of the operators T_j^λ; here the dependence on the index j plays a role. Fix $p < 2$ as in the statement of the theorem. Our goal is to show that there exist positive constants C, δ (depending only on n and $\mathrm{Re}\,\lambda$) such that for all functions f in $L^p(\mathbf{R}^n)$ we have

$$\big\|T_j^\lambda(f)\big\|_{L^p(\mathbf{R}^n)} \le C e^{c_0|\mathrm{Im}\,\lambda|^2} 2^{-j\delta} \big\|f\big\|_{L^p(\mathbf{R}^n)} . \tag{10.4.15}$$

Once (10.4.15) is established, the L^p boundedness of the operator $f \mapsto K^\lambda * f$ follows by summing the series in (10.4.14).

As a consequence of (10.4.13) we obtain that

$$\begin{aligned} |K_j^\lambda(x)| &\le C_3(\mathrm{Re}\,\lambda)\, e^{c_0|\mathrm{Im}\,\lambda|^2}\,(1+|x|)^{-\frac{n+1}{2}-\mathrm{Re}\,\lambda}|\psi_j(x)| \\ &\le C' 2^{-(\frac{n+1}{2}+\mathrm{Re}\,\lambda)j}, \end{aligned} \tag{10.4.16}$$

since $\psi_j(x) = \psi(2^{-j}x)$ and ψ is supported in the annulus $\frac{1}{2} \le |x| \le 2$. From this point on, the constants containing a prime are assumed to grow at most exponentially in $|\mathrm{Im}\,\lambda|^2$. Since K_j^λ is supported in a ball of radius 2^{j+1} and satisfies (10.4.16), we deduce the estimate

$$\big\|\widehat{K_j^\lambda}\big\|_{L^2}^2 = \big\|K_j^\lambda\big\|_{L^2}^2 \le C'' 2^{-(n+1+2\mathrm{Re}\,\lambda)j} 2^{nj} = C'' 2^{-(1+2\mathrm{Re}\,\lambda)j}. \tag{10.4.17}$$

We need another estimate for $\widehat{K_j^\lambda}$. We claim that for all $M \ge n+1$ there is a constant C_M such that

$$\int_{|\xi|\le\frac{1}{8}} |\widehat{K_j^\lambda}(\xi)|^2 |\xi|^{-\beta}\, d\xi \le C_{M,n,\beta}\, 2^{-2j(M-n)}, \quad \beta < n. \tag{10.4.18}$$

Indeed, since $\widehat{K^\lambda}(\xi)$ is supported in $|\xi| \ge \frac{1}{2}$ [recall that the function η was chosen equal to 1 on $B(0,\frac{1}{2})$], we have

$$|\widehat{K_j^\lambda}(\xi)| = |(\widehat{K^\lambda} * \widehat{\psi_j})(\xi)| \le 2^{jn} \int_{\frac{1}{2} \le |\xi-\omega| \le 1} (1-|\xi-\omega|^2)_+^{\operatorname{Re}\lambda} |\widehat{\psi}(2^j\omega)|\, d\omega .$$

Suppose that $|\xi| \le \frac{1}{8}$. Since $|\xi-\omega| \ge \frac{1}{2}$, we must have $|\omega| \ge \frac{3}{8}$. Then

$$|\widehat{\psi}(2^j\omega)| \le C_M (2^j|\omega|)^{-M} \le (8/3)^M C_M 2^{-jM},$$

from which it follows easily that

$$\sup_{|\xi|\le \frac{1}{8}} |\widehat{K_j^\lambda}(\xi)| \le C_M' 2^{-j(M-n)} . \tag{10.4.19}$$

Then (10.4.18) is a consequence of (10.4.19) and of the fact that the function $|\xi|^{-\beta}$ is integrable near the origin.

We now return to estimate (10.4.15). A localization argument (Exercise 10.4.4) allows us to reduce estimate (10.4.15) to functions f that are supported in a cube of side length 2^j. Let us therefore assume that f is supported in some cube Q of side length 2^j. Then $T_j^\lambda(f)$ is supported in $5Q$ and we have for $1 \le p < 2$ by Hölder's inequality

$$\begin{aligned} \big\|T_j^\lambda(f)\big\|_{L^p(5Q)}^2 &\le |5Q|^{2(\frac{1}{p}-\frac{1}{2})} \big\|T_j^\lambda(f)\big\|_{L^2(5Q)}^2 \\ &\le C_n 2^{(\frac{1}{p}-\frac{1}{2})2nj} \big\|\widehat{K_j^\lambda}\widehat{f}\big\|_{L^2}^2 . \end{aligned} \tag{10.4.20}$$

Having returned to L^2, we are able to use the $L^p \to L^2$ restriction theorem obtained in the previous subsection. To this end we use polar coordinates and the fact that K_j^λ is a radial function to write

$$\big\|\widehat{K_j^\lambda}\widehat{f}\big\|_{L^2}^2 = \int_0^\infty |\widehat{K_j^\lambda}(re_1)|^2 \bigg(\int_{\mathbf{S}^{n-1}} |\widehat{f}(r\theta)|^2\, d\theta \bigg) r^{n-1} dr, \tag{10.4.21}$$

where $e_1 = (1,0,\dots,0) \in \mathbf{S}^{n-1}$. Since the restriction of the function $x \mapsto r^{-n} f(x/r)$ on the sphere $\mathbf{S}^{n-1}$ is $\widehat{f}(r\theta)$, we have

$$\int_{\mathbf{S}^{n-1}} |\widehat{f}(r\theta)|^2\, d\theta \le C_{p,n}^2 \bigg[\int_{\mathbf{R}^n} r^{-np} |f(x/r)|^p\, dx \bigg]^{\frac{2}{p}} = C_{p,n}^2 r^{-\frac{2n}{p'}} \|f\|_{L^p}^2 , \tag{10.4.22}$$

where $C_{p,n}$ is the constant in Theorem 10.4.5 that holds whenever $p \le \frac{2(n+1)}{n+3}$. So assuming $p \le \frac{2(n+1)}{n+3}$ and inserting estimate (10.4.22) in (10.4.21) yields

$$\begin{aligned}\big\|\widehat{K_j^\lambda}\widehat{f}\big\|_{L^2}^2 &\le C_{p,n}^2\|f\|_{L^p}^2\int_0^\infty|\widehat{K_j^\lambda}(re_1)|^2 r^{n-1-\frac{2n}{p'}}\,dr\\ &\le \frac{C_{p,n}^2}{\omega_{n-1}}\|f\|_{L^p}^2\int_{\mathbf{R}^n}|\widehat{K_j^\lambda}(\xi)|^2|\xi|^{-\frac{2n}{p'}}\,d\xi\,,\end{aligned}\tag{10.4.23}$$

where $\omega_{n-1}=|\mathbf{S}^{n-1}|$. Appealing to estimate (10.4.18) for $|\xi|\le\frac{1}{8}$ with $\beta=\frac{2n}{p'}<n$ (since $p<2$) and to estimate (10.4.17) for $|\xi|\ge\frac{1}{8}$, we obtain

$$\big\|\widehat{K_j^\lambda}\widehat{f}\big\|_{L^2}^2\le C'''\,2^{-(1+2\mathrm{Re}\,\lambda)j}\|f\|_{L^p}^2\,.$$

Combining this inequality with the one previously obtained in (10.4.20) yields (10.4.15) with

$$\delta=\frac{n+1}{2}+\mathrm{Re}\,\lambda-\frac{n}{p}\,.$$

This number is positive exactly when $\frac{2n}{n+1+2\mathrm{Re}\,\lambda}<p$. This was the condition assumed by the theorem when $p<2$. The other condition $\mathrm{Re}\,\lambda>\frac{n-1}{2(n+1)}$ is naturally imposed by the restriction $p\le\frac{2(n+1)}{n+3}$. Finally, the analogous result in the range $p>2$ follows by duality. □

10.4.4 The Full Restriction Theorem on $\mathbf{R}^2$

In this section we prove the validity of the restriction condition $R_{p\to q}(\mathbf{S}^1)$ in dimension $n=2$, for the full range of exponents suggested by Figure 10.10.

To achieve this goal, we "fatten" the circle by a small amount 2δ. Then we obtain a restriction theorem for the "fattened circle" and then obtain the required estimate by taking the limit as $\delta\to0$. Precisely, we use the fact

$$\int_{\mathbf{S}^1}|\widehat{f}(\omega)|^q\,d\omega=\lim_{\delta\to0}\frac{1}{2\delta}\int_{1-\delta}^{1+\delta}\int_{\mathbf{S}^1}|\widehat{f}(r\theta)|^q\,d\theta\,r\,dr\tag{10.4.24}$$

to recover the restriction theorem for the circle from a restriction theorem for annuli of width 2δ.

Throughout this subsection, δ is a number satisfying $0<\delta<\frac{1}{1000}$, and for simplicity we use the notation

$$\chi^\delta(\xi)=\chi_{(1-\delta,1+\delta)}(|\xi|)\,,\qquad\qquad \xi\in\mathbf{R}^2\,.$$

We note that in view of identity (10.4.24), the restriction property $R_{p\to q}(\mathbf{S}^1)$ is a trivial consequence of the estimate

$$\frac{1}{2\delta}\int_0^\infty\int_{\mathbf{S}^1}|\chi^\delta(r\theta)\widehat{f}(r\theta)|^q\,d\theta\,r\,dr\le C^q\|f\|_{L^p}^q\,,\tag{10.4.25}$$

or, equivalently, of

$$\big\|\chi^\delta \widehat{f}\big\|_{L^q(\mathbf{R}^2)} \le (2\delta)^{\frac{1}{q}} C \big\|f\big\|_{L^p(\mathbf{R}^2)}. \tag{10.4.26}$$

We have the following result.

Theorem 10.4.7. *(a) Given $1 \le p < \frac{4}{3}$, set $q = \frac{p'}{3}$. Then there is a constant C_p such that for all L^p functions f on $\mathbf{R}^2$ and all small positive δ we have*

$$\big\|\chi^\delta \widehat{f}\big\|_{L^q(\mathbf{R}^2)} \le C_p \delta^{\frac{1}{q}} \big\|f\big\|_{L^p(\mathbf{R}^2)}. \tag{10.4.27}$$

(b) When $p = q = 4/3$, there is a constant C such that for all $L^{4/3}$ functions f on $\mathbf{R}^2$ and all small $\delta > 0$ we have

$$\big\|\chi^\delta \widehat{f}\big\|_{L^{\frac{4}{3}}(\mathbf{R}^2)} \le C\delta^{\frac{3}{4}} (\log \tfrac{1}{\delta})^{\frac{1}{4}} \big\|f\big\|_{L^{\frac{4}{3}}(\mathbf{R}^2)}. \tag{10.4.28}$$

Proof. To prove this theorem, we work with the *extension operator*

$$E^\delta(g) = \widehat{\chi^\delta g} = \widehat{\chi^\delta} * \widehat{g},$$

which is dual (i.e., transpose) to $f \mapsto \chi^\delta \widehat{f}$, and we need to show that

$$\big\|E^\delta(f)\big\|_{L^{p'}(\mathbf{R}^2)} \le C\delta^{\frac{1}{q}} (\log \tfrac{1}{\delta})^{\beta} \big\|f\big\|_{L^{q'}(\mathbf{R}^2)}, \tag{10.4.29}$$

where $\beta = \frac{1}{4}$ when $p = \frac{4}{3}$ and $\beta = 0$ when $p < \frac{4}{3}$.

We employ a splitting similar to that used in Theorem 10.2.4, with the only difference that the present partition of unity is nonsmooth and hence simpler. We define functions

$$\chi_\ell^\delta(\xi) = \chi^\delta(\xi)\chi_{2\pi\ell\delta^{1/2} \le \operatorname{Arg}\, \xi < 2\pi(\ell+1)\delta^{1/2}}$$

for $\ell \in \{0, 1, \dots, [\delta^{-1/2}]\}$. We suitably adjust the support of the function $\chi^\delta_{[\delta^{-1/2}]}$ so that the sum of all these functions equals χ^δ. We now split the indices that appear in the set $\{0, 1, \dots, [\delta^{-1/2}]\}$ into nine different subsets so that the supports of the functions indexed by them are properly contained in some sector centered at the origin of amplitude $\pi/4$. We therefore write E^δ as a sum of nine pieces, each properly supported in a sector of amplitude $\pi/4$. Let I be the set of indices that correspond to one of these nine sectors and let

$$E_I^\delta(f) = \sum_{\ell \in I} \widehat{\chi_\ell^\delta f}.$$

It suffices therefore to obtain (10.4.29) for each E_I^δ in lieu of E^δ. Let us fix such an index set I and without loss of generality we assume that

$$I = \{0, 1, \dots, [\tfrac{1}{8}\delta^{-1/2}]\}.$$

Since the theorem is trivial when $p=1$, to prove part (a) we fix a number p with $1<p<\frac{4}{3}$. We set

$$r=(p'/2)'$$

and we observe that this r satisfies $\frac{1}{r}=\frac{1}{p'}+\frac{1}{q'}$. We note that $1<r<2$ and we apply the Hausdorff–Young inequality $\big\|h\big\|_{L^{r'}}\le\big\|h^\vee\big\|_{L^r}$. We have

$$\begin{aligned}\big\|E_I^\delta(f)\big\|_{L^{p'}(\mathbf{R}^2)}^{p'} &= \int_{\mathbf{R}^2}|E_I^\delta(f)^2|^{r'}\,dx\\ &\le \bigg(\int_{\mathbf{R}^2}|(E_I^\delta(f)^2)^\vee|^r\,dx\bigg)^{\frac{r'}{r}}\\ &= \bigg(\int_{\mathbf{R}^2}\Big|\sum_{\ell\in I}\sum_{\ell'\in I}(\chi_\ell^\delta f)*(\chi_{\ell'}^\delta f)\Big|^r\,dx\bigg)^{\frac{r'}{r}}.\end{aligned}\tag{10.4.30}$$

We obtain the estimate

$$\bigg(\int_{\mathbf{R}^2}\Big|\sum_{\ell\in I}\sum_{\ell'\in I}(\chi_\ell^\delta f)*(\chi_{\ell'}^\delta f)\Big|^r\,dx\bigg)^{\frac{r'}{r}}\le C\delta^{\frac{p'}{q}}\big\|f\big\|_{L^{q'}(\mathbf{R}^2)}^{p'},\tag{10.4.31}$$

which suffices to prove the theorem.

Denote by $S_{\delta,\ell,\ell'}$ the support of $\chi_\ell^\delta+\chi_{\ell'}^\delta$. Then we write the left-hand side of (10.4.31) as

$$\bigg(\int_{\mathbf{R}^2}\Big|\sum_{\ell\in I}\sum_{\ell'\in I}\big((\chi_\ell^\delta f)*(\chi_{\ell'}^\delta f)\big)\chi_{S_{\delta,\ell,\ell'}}\Big|^r\,dx\bigg)^{\frac{r'}{r}},\tag{10.4.32}$$

which, via Hölder's inequality, is controlled by

$$\bigg(\int_{\mathbf{R}^2}\Big(\sum_{\ell\in I}\sum_{\ell'\in I}\big|(\chi_\ell^\delta f)*(\chi_{\ell'}^\delta f)\big|^r\Big)^{\frac{r}{r}}\Big(\sum_{\ell\in I}\sum_{\ell'\in I}\big|\chi_{S_{\delta,\ell,\ell'}}\big|^{r'}\Big)^{\frac{r}{r'}}\,dx\bigg)^{\frac{r'}{r}}.\tag{10.4.33}$$

We now recall Lemma 10.2.5, in which the curvature of the circle was crucial. In view of that lemma, the second factor of the integrand in (10.4.33) is bounded by a constant independent of δ. We have therefore obtained the estimate

$$\big\|E_I^\delta(f)\big\|_{L^{p'}}^{p'}\le C\bigg(\sum_{\ell\in I}\sum_{\ell'\in I}\int_{\mathbf{R}^2}\big|(\chi_\ell^\delta f)*(\chi_{\ell'}^\delta f)\big|^r\,dx\bigg)^{\frac{r'}{r}}.\tag{10.4.34}$$

We prove at the end of this section the following auxiliary result.

Lemma 10.4.8. *With the same notation as in the proof of Theorem 10.4.7, for any $1<r<\infty$, there is a constant C (independent of δ and f) such that*

$$\big\|(\chi_\ell^\delta f)*(\chi_{\ell'}^\delta f)\big\|_{L^r}\le C\bigg(\frac{\delta^{\frac{3}{2}}}{|\ell-\ell'|+1}\bigg)^{\frac{1}{r'}}\big\|\chi_\ell^\delta f\big\|_{L^r}\big\|\chi_{\ell'}^\delta f\big\|_{L^r}\tag{10.4.35}$$

for all $\ell,\ell'\in I=\{0,1,\dots,[\frac{1}{8}\delta^{-1/2}]\}$.

Assuming Lemma 10.4.8 and using (10.4.34), we write

$$\begin{aligned}\big\|E_I^\delta(f)\big\|_{L^{p'}}^{p'} &\le C\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^r}^{r}\Bigg(\sum_{\ell'\in I}\frac{\big\|\chi_{\ell'}^\delta f\big\|_{L^r}^{r}}{(|\ell-\ell'|+1)^{\frac{r}{r'}}}\Bigg)\Bigg]^{\frac{r'}{r}}\\ &\le C\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^r}^{rs}\Bigg]^{\frac{r'}{rs}}\Bigg[\sum_{\ell\in I}\Bigg(\sum_{\ell'\in I}\frac{\big\|\chi_{\ell'}^\delta f\big\|_{L^r}^{r}}{(|\ell-\ell'|+1)^{\frac{r}{r'}}}\Bigg)^{s'}\Bigg]^{\frac{r'}{rs'}},\end{aligned}\tag{10.4.36}$$

where we used Hölder's inequality for some $1<s<\infty$. We now recall the discrete fractional integral operator

$$\{a_j\}_j\mapsto\Big\{\sum_{j'}\frac{a_{j'}}{(|j-j'|+1)^{1-\alpha}}\Big\}_j,$$

which maps $\ell^s(\mathbf{Z})$ to $\ell^{s'}(\mathbf{Z})$ (see Exercise 6.1.10) when

$$\frac{1}{s}-\frac{1}{s'}=\alpha,\qquad 0<\alpha<1.\tag{10.4.37}$$

When $1<p<\frac{4}{3}$, we have $1<r<2$, and choosing $\alpha=2-r=1-\frac{r}{r'}$, we obtain from (10.4.36) that

$$\begin{aligned}\big\|E_I^\delta(f)\big\|_{L^{p'}}^{p'} &\le C'\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^r}^{rs}\Bigg]^{\frac{r'}{rs}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^r}^{rs}\Bigg]^{\frac{r'}{rs}}\\ &= C'\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^r}^{rs}\Bigg]^{\frac{2r'}{rs}}.\end{aligned}\tag{10.4.38}$$

The unique s that solves equation (10.4.37) is seen easily to be $s=q'/r$. Moreover, since $q=p'/3$, we have $1<s<2$. We use again Hölder's inequality to pass from $\big\|\chi_\ell^\delta f\big\|_{L^r}$ to $\big\|\chi_\ell^\delta f\big\|_{L^{q'}}$. Indeed, recalling that the support of χ_ℓ^δ has measure $\approx\delta^{\frac{3}{2}}$, we have

$$\big\|\chi_\ell^\delta f\big\|_{L^r}\le C(\delta^{\frac{3}{2}})^{\frac{1}{r}-\frac{1}{q'}}\big\|\chi_\ell^\delta f\big\|_{L^{q'}}.$$

Inserting this in (10.4.38) yields

$$\begin{aligned}\big\|E_I^\delta(f)\big\|_{L^{p'}}^{p'} &\le C\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\Big(C(\delta^{\frac{3}{2}})^{\frac{1}{r}-\frac{1}{q'}}\big\|\chi_\ell^\delta f\big\|_{L^{q'}}\Big)^{rs}\Bigg]^{\frac{2r'}{rs}}\\ &= C'\delta^{\frac{3}{2}}(\delta^{\frac{3}{2}})^{2r'(\frac{1}{r}-\frac{1}{q'})}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^{q'}}^{q'}\Bigg]^{\frac{2r'}{q'}}\\ &\le C\delta^{3}\big\|f\big\|_{L^{q'}}^{p'}\\ &= C\delta^{\frac{p'}{q}}\big\|f\big\|_{L^{q'}}^{p'},\end{aligned}$$

which is the required estimate since $\frac{1}{r}=\frac{1}{p'}+\frac{1}{q'}$ and $p'=2r'$. In the last inequality we used the fact that the supports of the functions χ_ℓ^δ are disjoint and that these add up to a function that is at most 1.

To prove part (b) of the theorem, we need to adjust the previous argument to obtain the case $p=\frac{4}{3}$. Here we repeat part of the preceding argument taking $r=r'=s=s'=2$.

Using (10.4.34) with $p=\frac{4}{3}$ (which forces r to be equal to 2) and Lemma 10.4.8 with $r=2$ we write

$$
\begin{aligned}
\big\|E_I(f)\big\|_{L^4(\mathbf{R}^2)}^4 &\le C\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^2}^2\Bigg(\sum_{\ell'\in I}\frac{\|\chi_{\ell'}^\delta f\|_{L^2}^2}{|\ell-\ell'|+1}\Bigg)\Bigg] \\
&\le C\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^2}^4\Bigg]^{\frac{1}{2}}\Bigg[\sum_{\ell\in I}\Bigg(\sum_{\ell'\in I}\frac{\|\chi_{\ell'}^\delta f\|_{L^2}^2}{|\ell-\ell'|+1}\Bigg)^2\Bigg]^{\frac{1}{2}} \\
&\le C\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^2}^4\Bigg]^{\frac{1}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^2}^4\Bigg]^{\frac{1}{2}}\Bigg[\sum_{\ell\in I}\frac{1}{|\ell|+1}\Bigg] \\
&\le C\delta^{\frac{3}{2}}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^2}^4\Bigg]\log(\delta^{-\frac{1}{2}}) \\
&\le C\delta^{\frac{3}{2}}(\delta^{\frac{3}{2}})^{(\frac{1}{2}-\frac{1}{4})4}\Bigg[\sum_{\ell\in I}\big\|\chi_\ell^\delta f\big\|_{L^4}^4\Bigg]\log\tfrac{1}{\delta} \\
&\le C\delta^3\big(\log\tfrac{1}{\delta}\big)\big\|f\big\|_{L^4}^4 .
\end{aligned}
$$

□

We now prove Lemma 10.4.8, which we had left open.

Proof. The proof is based on interpolation. For fixed $\ell,\ell'\in I$ we define the bilinear operator

$$T_{\ell,\ell'}(g,h)=(g\chi_\ell^\delta)*(h\chi_{\ell'}^\delta).$$

As we have previously observed, it is a simple geometric fact that the support of χ_ℓ^δ is contained in a rectangle of side length $\approx\delta$ in the direction $e^{2\pi i\delta^{1/2}\ell}$ and of side length $\approx\delta^{\frac{1}{2}}$ in the direction $ie^{2\pi i\delta^{1/2}\ell}$. Any two rectangles with these dimensions in the aforementioned directions have an intersection that depends on the angle between them. Indeed, if $\ell\neq\ell'$ this intersection is contained in a parallelogram of sides δ and $\delta/\sin(2\pi\delta^{\frac{1}{2}}|\ell-\ell'|)$, and hence the measure of the intersection is seen easily to be at most a constant multiple of

$$\delta\cdot\frac{\delta}{\sin(2\pi\delta^{\frac{1}{2}}|\ell-\ell'|)}.$$

As for ℓ,ℓ' in the index set I we have $2\pi\delta^{\frac{1}{2}}|\ell-\ell'|<\pi/4$, the sine is comparable to its argument, and we conclude that the measure of the intersection is at most

$$C\delta^{\frac{3}{2}}(1+|\ell-\ell'|)^{-1}.$$

It follows that

$$\|\chi_\ell^\delta * \chi_{\ell'}^\delta\|_{L^\infty} = \sup_{z\in\mathbf{R}^2} |(z-\text{supp}\,(\chi_\ell^\delta))\cap\text{supp}\,(\chi_{\ell'}^\delta)| \le \frac{C\delta^{\frac{3}{2}}}{1+|\ell-\ell'|},$$

which implies the estimate

$$\begin{aligned}\|T_{\ell,\ell'}(g,h)\|_{L^\infty} &\le \|\chi_\ell^\delta * \chi_{\ell'}^\delta\|_{L^\infty}\|g\|_{L^\infty}\|h\|_{L^\infty} \\ &\le \frac{C\delta^{\frac{3}{2}}}{1+|\ell-\ell'|}\|g\|_{L^\infty}\|h\|_{L^\infty}.\end{aligned} \tag{10.4.39}$$

Also, the estimate

$$\begin{aligned}\|T_{\ell,\ell'}(g,h)\|_{L^1} &\le \|g\chi_\ell^\delta\|_{L^1}\|h\chi_{\ell'}^\delta\|_{L^1} \\ &\le \|g\|_{L^1}\|h\|_{L^1}\end{aligned} \tag{10.4.40}$$

holds trivially. Interpolating between (10.4.39) and (10.4.40) yields the required estimate (10.4.35). Here we used bilinear interpolation (Exercise 1.4.17). □

Example 10.4.9. The presence of the logarithmic factor in estimate (10.4.28) is necessary. In fact, this estimate is sharp. We prove this by showing that the corresponding estimate for the "dual" extension operator E^δ is sharp. Let I be the set of indices we worked with in Theorem 10.4.7 (i.e., $I=\{0,1,\dots,[\frac{1}{8}\delta^{-1/2}]\}$.) Let

$$f^\delta = \sum_{\ell\in I}\chi_\ell^\delta.$$

Then

$$\|f^\delta\|_{L^4} \approx \delta^{\frac{1}{4}}.$$

However,

$$E^\delta(f^\delta) = \sum_{\ell\in I}\widehat{\chi_\ell^\delta},$$

and we have

$$\begin{aligned}\|E^\delta(f^\delta)\|_{L^4} &= \left(\int_{\mathbf{R}^2}\Big|\sum_{\ell\in I}\sum_{\ell'\in I}\widehat{\chi_\ell^\delta}\widehat{\chi_{\ell'}^\delta}\Big|^2 d\xi\right)^{\frac{1}{4}} \\ &= \left(\int_{\mathbf{R}^2}\Big|\sum_{\ell\in I}\sum_{\ell'\in I}\chi_\ell^\delta * \chi_{\ell'}^\delta\Big|^2 dx\right)^{\frac{1}{4}} \\ &\ge \left(\sum_{\ell\in I}\sum_{\ell'\in I}\int_{\mathbf{R}^2}|\chi_\ell^\delta * \chi_{\ell'}^\delta|^2 dx\right)^{\frac{1}{4}}.\end{aligned}$$

At this point observe that the function $\chi_\ell^\delta * \chi_{\ell'}^\delta$ is at least a constant multiple of $\delta^{\frac{3}{2}}(|\ell-\ell'|+1)^{-1}$ on a set of measure $c\delta^{\frac{3}{2}}(|\ell-\ell'|+1)$. (See Exercise 10.4.5.) Using this fact and the previous estimates, we deduce easily that

$$\big\|E^\delta(f^\delta)\big\|_{L^4} \ge c\Big(\sum_{\ell\in I}\sum_{\ell'\in I}\frac{\delta^3}{(|\ell-\ell'|+1)^2}\delta^{\frac{3}{2}}(|\ell-\ell'|+1)\Big)^{\frac{1}{4}} \approx \delta(\log\tfrac{1}{\delta})^{\frac{1}{4}},$$

since $|I|\approx\delta^{-\frac{1}{2}}$. It follows that

$$\frac{\big\|E^\delta(f^\delta)\big\|_{L^4}}{\big\|f^\delta\big\|_{L^4}} \ge c\delta^{\frac{3}{4}}(\log\tfrac{1}{\delta})^{\frac{1}{4}},$$

which justifies the sharpness of estimate (10.4.28).

Exercises

10.4.1. Let S be a compact hypersurface in $\mathbf{R}^n$ and let $d\sigma$ be surface measure on it. Suppose that for some $0<b<n$ we have

$$|\widehat{d\sigma}(\xi)| \le C(1+|\xi|)^{-b}$$

for all $\xi\in\mathbf{R}^n$. Prove that $R_{p\to q}(S)$ does not hold for any $1\le q\le\infty$ when $p\ge\frac{n}{n-b}$.

10.4.2. Let S be a compact hypersurface and let $1\le p,q\le\infty$.
(a) Suppose that $R_{p\to q}(S)$ holds for S. Show that $R_{p\to q}(\tau+S)$ holds for the translated hypersurface $\tau+S$.
(b) Suppose that the hypersurface S is compact and its interior contains the origin. For $r>0$ let $rS=\{r\xi:\ \xi\in S\}$. Suppose that $R_{p\to q}(\mathbf{S}^{n-1})$ holds with constant C_{pqn}. Show that $R_{p\to q}(r\mathbf{S}^{n-1})$ holds with constant $C_{pqn}r^{\frac{n-1}{q}-\frac{n}{p'}}$.

10.4.3. Obtain a different proof of estimate (10.4.7) (and hence of Theorem 10.4.5) by following the sequence of steps outlined here:
(a) Consider the analytic family of functions

$$(K_z)^\vee(\xi) = 2\pi^{1-z}\frac{J_{\frac{n-2}{2}+z}(2\pi|\xi|)}{|\xi|^{\frac{n-2}{2}+z}}$$

and observe that in view of the identity in Appendix B.4, $(K_z)^\vee(\xi)$ reduces to $d\sigma^\vee(\xi)$ when $z=0$, where $d\sigma$ is surface measure on $\mathbf{S}^{n-1}$.
(b) Use for free that the Bessel function $J_{-\frac{1}{2}+i\theta}$, $\theta\in\mathbf{R}$, satisfies

$$|J_{-\frac{1}{2}+i\theta}(x)| \le C_\theta|x|^{-\frac{1}{2}},$$

where C_θ grows at most exponentially in $|\theta|$, to obtain that the family of operators given by convolution with $(K_z)^\vee$ map $L^1(\mathbf{R}^n)$ to $L^\infty(\mathbf{R}^n)$ when $z = -\frac{n-1}{2} + i\theta$.
(c) Appeal to the result in Appendix B.5 to obtain that for z not equal to $0, -1, -2, \dots$ we have

$$K_z(x) = \frac{2}{\Gamma(z)}(1-|x|^2)^{z-1}.$$

Use this identity to deduce that for $z = 1 + i\theta$ the family of operators given by convolution with $(K_z)^\vee$ map $L^2(\mathbf{R}^n)$ to itself with constants that grow at most exponentially in $|\theta|$. (Appendix A.6 contains a useful lower estimate for $|\Gamma(1+i\theta)|$.)
(d) Use Exercise 1.3.4 to obtain that for $z = 0$ the operator given by convolution with $d\sigma^\vee$ maps $L^p(\mathbf{R}^n)$ to $L^{p'}(\mathbf{R}^n)$ when $p = \frac{2(n+1)}{n+3}$.

10.4.4. Suppose that T is a linear operator given by convolution with a kernel K that is supported in the ball $B(0,2R)$. Assume that there is a constant C such that for all functions f supported in a cube of side length R we have

$$\big\|T(f)\big\|_{L^p} \le B\big\|f\big\|_{L^p}$$

for some $1 \le p < \infty$. Show that this estimate also holds for all L^p functions f with constant $5^n B$.
$\big[$*Hint:* Write $f = \sum_j f\chi_{Q_j}$, where each cube Q_j has side length R.$\big]$

10.4.5. Using the notation of Theorem 10.4.7, show that there exist constants c, c' such that the function $\chi_\ell^\delta * \chi_{\ell'}^\delta$ is at least $c'\delta^{\frac{3}{2}}(|\ell-\ell'|+1)^{-1}$ on a set of measure $c\delta^{\frac{3}{2}}(|\ell-\ell'|+1)$.
$\big[$*Hint:* Prove the required conclusion for characteristic functions of rectangles with the same orientation and comparable dimensions. Then use that the support of each χ_ℓ^δ contains such a rectangle.$\big]$

10.5 Almost Everywhere Convergence of Bochner–Riesz Means

We recall the Bochner–Riesz means B_R^λ of complex order λ given in Definition 10.2.1. In this section we study the problem of almost everywhere convergence of $B_R^\lambda(f) \to f$ as $R \to \infty$. There is an intimate relationship between the almost everywhere convergence of a family of operators and boundedness properties of the associated maximal family (cf. Theorem 2.1.14).[1]

For $f \in L^p(\mathbf{R}^n)$, the *maximal Bochner–Riesz operator* or order λ is defined by

$$B_*^\lambda(f) = \sup_{R>0}\big|B_R^\lambda(f)\big|.$$

[1] In certain cases, Theorem 2.1.14 can essentially be reversed. Given a $1 \le p \le 2$ and a family of distributions u_j with the mild continuity property that $u_j * f_k \to u_j * f$ in measure whenever $f_k \to f$ in $L^p(\mathbf{R}^n)$ such that the maximal operator $\mathscr{M}(f) = \sup_j |f * u_j| < \infty$ whenever $f \in L^p(\mathbf{R}^n)$, then $\mathscr{M}$ maps $L^p(\mathbf{R}^n)$ to $L^{p,\infty}(K)$ for any compact subset K of $\mathbf{R}^n$. See Stein [289], [292].

10.5.1 A Counterexample for the Maximal Bochner–Riesz Operator

We have the following result.

Theorem 10.5.1. *Let $n \geq 2$, $\lambda > 0$, and let $1 < p < 2$ be such that*

$$\lambda < \frac{2n-1}{2p} - \frac{n}{2}.$$

Then B_*^λ does not map $L^p(\mathbf{R}^n)$ to weak $L^p(\mathbf{R}^n)$.

Proof. Figure 10.11 shows the region in which B_*^λ is known to be unbounded; this region contains the set of points $(1/p, \lambda)$ strictly below the line that joins the points $(1, (n-1)/2)$ and $(n/(2n-1), 0)$.

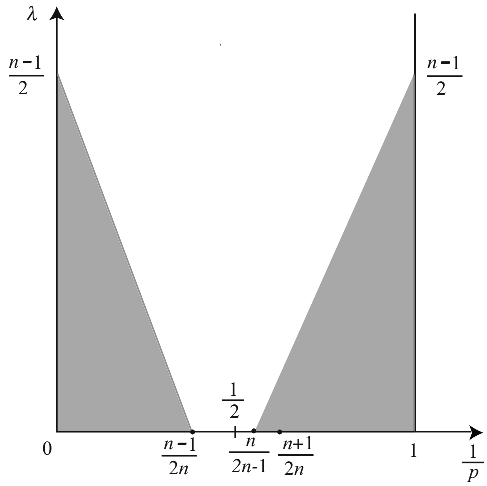

Fig. 10.11 The operators B_*^λ are unbounded on $L^p(\mathbf{R}^n)$ when $(1/p, \lambda)$ lies in the interior of the shaded region.

We denote points x in $\mathbf{R}^n$ by $x = (x', x_n)$, where $x' \in \mathbf{R}^{n-1}$, and we fix $M \geq 100$ and $\varepsilon < 1/100$. We let $\psi(y) = \chi_{|y'| \leq 1}(y')\,\zeta(y_n)$, where ζ is a smooth bump supported in the interval $[-1, 1]$ that is equal to 1 on $[-1/2, 1/2]$ and satisfies $0 \leq \zeta \leq 1$. We define

$$\psi_{\varepsilon,M}(y) = \psi(\varepsilon^{-1}y', \varepsilon^{-1}M^{-\frac{1}{2}}y_n) = \chi_{|y'| \leq \varepsilon}(y')\,\zeta(\varepsilon^{-1}M^{-\frac{1}{2}}y_n)$$

and we note that $\psi_{\varepsilon,M}(y)$ is supported in the set of y's that satisfy $|y'| \leq \varepsilon$ and $|y_n| \leq \varepsilon M^{\frac{1}{2}}$. We also define

$$f_M(y) = e^{2\pi i y_n}\psi_{\varepsilon,M}(y)$$

and

$$S_M = \{(x', x_n) : \; M \leq |x'| \leq 2M, \; M \leq |x_n| \leq 2M\}.$$

Then

$$\big\|f_M\big\|_{L^p} \approx M^{\frac{1}{2p}}\varepsilon^{\frac{n}{p}} \qquad \text{and} \qquad |S_M| \approx M^n\,. \tag{10.5.1}$$

Every point $x \in S_M$ must satisfy $M \le |x| \le 3M$. We fix $x \in S_M$ and we estimate $B_*^\lambda(f_M)(x) = \sup_{R>0}|B_R^\lambda(f_M)(x)|$ from below by picking $R = R_x = |x|/x_n$. Then $1/2 \le R_x \le 3$ and we have

$$B_*^\lambda(f_M)(x) \ge \frac{\Gamma(\lambda+1)}{\pi^\lambda}\bigg|\int_{\mathbf{R}^n}\frac{J_{\frac{n}{2}+\lambda}(2\pi R_x|x-y|)}{(R_x|x-y|)^{\frac{n}{2}+\lambda}}e^{2\pi i y_n}\psi_{\varepsilon,M}(y)\,dy\bigg|\,.$$

We make some observations. First $|x'-y'| \ge \frac{1}{2}|x'|$, since $|x'| \ge M$ and $|y'| \le \varepsilon$. Second, $|x_n - y_n| \ge |x_n| - |y_n| \ge \frac{1}{2}|x_n|$, since $|x_n| \ge M$ and $|y_n| \le \varepsilon M^{1/2}$. These facts imply that $|x-y| \ge \frac{1}{2}|x|$; thus $|x-y|$ is comparable to $|x|$, which is of the order of M. Since $2\pi R_x|x-y|$ is large, we use the asymptotics for the Bessel function $J_{\frac{n}{2}+\lambda}$ in Appendix B.8 to write

$$\frac{J_{\frac{n}{2}+\lambda}(2\pi R_x|x-y|)}{(R_x|x-y|)^{\frac{n}{2}+\lambda}} = \frac{C_\lambda\, e^{2\pi i R_x|x-y|}e^{i\varphi}}{(R_x|x-y|)^{\frac{n+1}{2}+\lambda}} + \frac{C_\lambda\, e^{-2\pi i R_x|x-y|}e^{-i\varphi}}{(R_x|x-y|)^{\frac{n+1}{2}+\lambda}} + V_{n,\lambda}(R_x|x-y|)\,,$$

where $\varphi = -\frac{\pi}{2}(\frac{n}{2}+\lambda) - \frac{\pi}{4}$ and

$$|V_{n,\lambda}(R_x|x-y|)| \le \frac{C_{n,\lambda}}{(R_x|x-y|)^{\frac{n+3}{2}+\lambda}} \le \frac{C'_{n,\lambda}}{M^{\frac{n+3}{2}+\lambda}}\,, \tag{10.5.2}$$

since $R_x = \frac{|x|}{x_n} \approx 1$ and $|x-y| \ge \frac{1}{2}M$. Using the preceding expression for the Bessel function, we write

$$\begin{aligned}
B_*^\lambda(f_M)(x) \ge C'_\lambda &\bigg|\int_{\mathbf{R}^n}\frac{e^{2\pi i R_x|x|}e^{i\varphi}}{(R_x|x-y|)^{\frac{n+1}{2}+\lambda}}\,\psi_{\varepsilon,M}(y)\,dy\bigg| \\
&- C'_\lambda\bigg|\int_{\mathbf{R}^n}\frac{(e^{2\pi i(R_x|x-y|+y_n)} - e^{2\pi i R_x|x|})e^{i\varphi}}{(R_x|x-y|)^{\frac{n+1}{2}+\lambda}}\,\psi_{\varepsilon,M}(y)\,dy\bigg| \\
&- C'_\lambda\bigg|\int_{\mathbf{R}^n}\frac{e^{2\pi i(-R_x|x-y|+y_n)}e^{-i\varphi}}{(R_x|x-y|)^{\frac{n+1}{2}+\lambda}}\,\psi_{\varepsilon,M}(y)\,dy\bigg| \\
&- \bigg|\int_{\mathbf{R}^n}V_{n,\lambda}(R_x|x-y|)e^{2\pi i y_n}\psi_{\varepsilon,M}(y)\,dy\bigg|\,.
\end{aligned}$$

The positive term is the main term and is bounded from below by

$$C'_\lambda\,(6M)^{-\frac{n+1}{2}-\lambda}\int_{\mathbf{R}^n}\psi_{\varepsilon,M}(y)\,dy = \frac{c_1\,\varepsilon^n M^{\frac{1}{2}}}{M^{\frac{n+1}{2}+\lambda}}\,. \tag{10.5.3}$$

The three terms with the minus signs are errors and are bounded in absolute value by smaller expressions. We notice that

$$\big|R_x|x-y|+y_n-R_x|x|\big| = \frac{|x|}{x_n}\Big||x-y|+\frac{x_n y_n}{|x|}-|x|\Big| = \frac{|x|}{x_n}\big|F_x(y)-F_x(0)\big|\,,$$

where $F_x(y)=|x-y|+|x|^{-1}x_n y_n$. Taylor's expansion yields

$$F_x(y)-F_x(0)=\nabla_y F_x(0)\cdot y + O\big(|y|^2 \sup_{j,k}|\partial_j\partial_k F_x(y)|\big)\,,$$

and a calculation gives $\nabla_y F_x(0)=(-|x|^{-1}x',0)$, while $|\partial_j\partial_k F_x(y)|\le C\,|x-y|^{-1}$. It follows that

$$\frac{|x|}{x_n}\big|F_x(y)-F_x(0)\big| \le 3\left[\frac{|x'\cdot y'|}{|x|}+C'\frac{|y|^2}{|x-y|}\right] \le C''\left[\varepsilon+\frac{(\varepsilon M^{1/2})^2}{M}\right] \le 2C''\varepsilon\,.$$

Using this fact and the support properties of ψ, we obtain

$$C'_\lambda\left|\int_{\mathbf{R}^n}\frac{(e^{2\pi i(R_x|x-y|+y_n)}-e^{2\pi i R_x|x|})e^{i\varphi}}{(R_x|x-y|)^{\frac{n+1}{2}+\lambda}}\,\psi_{\varepsilon,M}(y)\,dy\right| \le \frac{c_2\,\varepsilon(\varepsilon^n M^{\frac12})}{M^{\frac{n+1}{2}+\lambda}}\,. \qquad (10.5.4)$$

Next we examine the phase $R_x|x-y|+y_n$ as a function of y_n. Its derivative with respect to y_n is

$$\frac{\partial}{\partial y_n}\big(R_x|x-y|+y_n\big)=R_x\frac{x_n-y_n}{|x-y|}+1\ge 1\,,$$

since $x_n\ge M$ and $|y_n|\le \varepsilon M^{1/2}$, which implies that $x_n-y_n>0$. Also note that

$$\left|\frac{\partial}{\partial y_n}\left(R_x\frac{x_n-y_n}{|x-y|}+1\right)^{-1}\right|\le\frac{C'''}{M}$$

and that

$$\left|\frac{\partial}{\partial y_n}\frac{1}{|x-y|^{\frac{n+1}{2}+\lambda}}\right|\le\frac{C'''}{M^{\frac{n+3}{2}+\lambda}}\,,$$

while the derivative of $\zeta(\varepsilon^{-1}M^{-\frac12}y_n)$ with respect to y_n gives only a factor of $\varepsilon^{-1}M^{-\frac12}$. We integrate by parts one time with respect to y_n in the integral

$$\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}}\frac{e^{2\pi i(-R_x|x-y|+y_n)}e^{-i\varphi}}{(R_x|x-y|)^{\frac{n+1}{2}+\lambda}}\,\psi_{\varepsilon,M}(y)\,dy_n\,dy'$$

to obtain an additional factor of $\varepsilon^{-1}M^{-\frac12}$. Thus

$$\left|\int_{\mathbf{R}^n}\frac{e^{2\pi i(-R_x|x-y|+y_n)}e^{-i\varphi}}{(R_x|x-y|)^{\frac{n+1}{2}+\lambda}}\,\psi_{\varepsilon,M}(y)\,dy\right|\le\frac{c_3\,\varepsilon^n M^{\frac12}(\varepsilon^{-1}M^{-\frac12})}{M^{\frac{n+1}{2}+\lambda}}\,. \qquad (10.5.5)$$

Finally, using (10.5.2) we obtain that

$$\left| \int_{\mathbf{R}^n} V_{n,\lambda}(R_x|x-y|)e^{2\pi i y_n} \psi_{\varepsilon,M}(y)\, dy \right| \le \frac{c_4\, \varepsilon^n M^{\frac{1}{2}}}{M^{\frac{n+3}{2}+\lambda}}. \tag{10.5.6}$$

We combine (10.5.3), (10.5.4), (10.5.5), and (10.5.6) to deduce for $x \in S_M$,

$$B_*^\lambda(f_M)(x) \ge \frac{c_1\,\varepsilon^n}{M^{\frac{n}{2}+\lambda}} - \frac{c_2\,\varepsilon^{n+1}}{M^{\frac{n}{2}+\lambda}} - \frac{c_3\,\varepsilon^{n-1}}{M^{\frac{n+1}{2}+\lambda}} - \frac{c_4\,\varepsilon^n}{M^{\frac{n+2}{2}+\lambda}}.$$

We pick ε sufficiently small, say $\varepsilon \le c_1/(2c_2)$, and M_0 sufficiently large (depending on the constants c_1, c_2, c_3, c_4) that

$$x \in S_M \implies B_*^\lambda(f_M)(x) > c_0\, \frac{1}{M^{\frac{n}{2}+\lambda}}$$

whenever $M \ge M_0$. This fact together with (10.5.1) gives

$$\frac{\big\|B_*^\lambda(f_M)\big\|_{L^{p,\infty}}}{\big\|f_M\big\|_{L^p}} \ge \frac{c_0 M^{-\frac{n}{2}-\lambda}|S_M|^{\frac{1}{p}}}{c' M^{\frac{1}{2p}}} = c M^{\frac{2n-1}{2p}-\frac{n}{2}-\lambda},$$

and the required conclusion follows by letting $M \to \infty$. □

10.5.2 Almost Everywhere Summability of the Bochner–Riesz Means

We now focus attention on the case $p \ge 2$ and we investigate whether the Bochner–Riesz means converge almost everywhere outside the range in which they are known to be unbounded on L^p. Our goal is to prove the following result.

Theorem 10.5.2. *Let $\lambda > 0$ and $n \ge 2$. Then for all f in $L^p(\mathbf{R}^n)$ with $2 \le p < \frac{2n}{n-1-2\lambda}$ we have*

$$\lim_{R\to\infty} B_R^\lambda(f)(x) = f(x)$$

for almost all $x \in \mathbf{R}^n$.

Since the almost everywhere convergence is obvious for functions in the Schwartz class, to be able to use Theorem 2.1.14 to derive almost everywhere convergence for general L^p functions, it suffices to know a weak type (p,p) estimate for B_*^λ. However, instead of proving a weak type (p,p) estimate, we prove an L^2 and a weighted L^2 estimate for B_*^λ. Precisely, we prove the following result.

Proposition 10.5.3. *Let $\lambda > 0$ and $0 \le \alpha < 1 + 2\lambda \le n$. Then there is a constant $C = C(\alpha, \lambda, n)$ such that*

$$\int_{\mathbf{R}^n} |B_*^\lambda(f)(x)|^2 |x|^{-\alpha}\, dx \le C \int_{\mathbf{R}^n} |f(x)|^2 |x|^{-\alpha}\, dx$$

for all functions $f \in L^2(\mathbf{R}^n, |x|^{-\alpha}dx)$.

Assuming the result of Proposition 10.5.3, given p such that

$$2 \le p < p_\lambda = \frac{2n}{n-1-2\lambda},$$

choose α satisfying

$$0 \le n\left(1 - \frac{2}{p}\right) < \alpha < 1 + 2\lambda = n\left(1 - \frac{2}{p_\lambda}\right).$$

Then the maximal operator B_*^λ is bounded on L^2 and also on $L^2(|x|^{-\alpha}dx)$. Hence the almost everywhere convergence of the family $\{B_R^\lambda(f)\}_R$ holds on L^2 and also on $L^2(|x|^{-\alpha}dx)$. Since $0 \le \alpha < n$, we have

$$L^p \subseteqq L^2 + L^2(|x|^{-\alpha}),$$

and thus $B_R^\lambda(f)$ converges almost everywhere for functions $f \in L^p(\mathbf{R}^n)$. See Exercise 10.5.1 for this inclusion.

To prove Proposition 10.5.3, we decompose the multiplier $(1-|\xi|^2)_+^\lambda$ as an infinite sum of smooth bumps supported in small concentric annuli in the interior of the sphere $|\xi| = 1$ as we did in the proof of Theorem 10.2.4.

We pick a smooth function φ supported in $[-\frac{1}{2}, \frac{1}{2}]$ and a smooth function ψ supported in $[\frac{1}{8}, \frac{5}{8}]$ and with values in $[0,1]$ that satisfy

$$\varphi(t) + \sum_{k=0}^{\infty} \psi\Big(\frac{1-t}{2^{-k}}\Big) = 1$$

for all $t \in [0,1)$. We decompose the multiplier $(1-|\xi|^2)_+^\lambda$ as

$$(1-|\xi|^2)_+^\lambda = m_{00}(\xi) + \sum_{k=0}^{\infty} 2^{-k\lambda} m_k(\xi), \tag{10.5.7}$$

where $m_{00}(\xi) = \varphi(|\xi|)(1-|\xi|^2)^\lambda$, and for $k \ge 1$, m_k is defined by

$$m_k(\xi) = \Big(\frac{1-|\xi|}{2^{-k}}\Big)^\lambda \psi\Big(\frac{1-|\xi|}{2^{-k}}\Big)(1+|\xi|)^\lambda.$$

Then we define maximal operators associated with the multipliers m_{00} and m_k,

$$S_*^{m_k}(f)(x) = \sup_{R>0} \big|\big(\widehat{f}(\xi) m_k(\xi/R)\big)^\vee(x)\big|,$$

for $k \ge 0$, and analogously we define $S_*^{m_{00}}$. Using (10.5.7) we have

$$B_*^\lambda(f) \le S_*^{m_{00}}(f) + \sum_{k=0}^{\infty} 2^{-k\lambda} S_*^{m_k}(f). \tag{10.5.8}$$

Since $S_*^{m_{00}}$, $S_*^{m_0}$, $S_*^{m_1}$ and any finite number of them are pointwise controlled by the Hardy–Littlewood maximal operator, which is bounded on $L^2(|x|^\alpha)$ whenever $-n < \alpha < n$ (cf. Theorem 9.1.9 and Example 9.1.7), we focus attention on the remaining terms.

We make a small change of notation. Thinking of 2^{-k} as roughly being δ (precisely $\delta = 2^{-k-3}$), for $\delta < 1/10$ we let $m^\delta(t)$ be a smooth function supported in the interval $[1-5\delta, 1-\delta]$ and taking values in the interval $[0,1]$ that satisfies

$$\sup_{1\le t\le 2}\left|\frac{d^\ell}{dt^\ell}m^\delta(t)\right| \le C_\ell \delta^{-\ell} \tag{10.5.9}$$

for all $\ell \in \mathbf{Z}^+ \cup \{0\}$. We define a related function

$$\widetilde{m}^\delta(t) = \delta t\,\frac{d}{dt}m^\delta(t),$$

which obviously satisfies estimates (10.5.9) with another constant $\widetilde{C}_\ell$ in place of C_ℓ.

Next we introduce the multiplier operators

$$S_t^\delta(f)(x) = \big(\widehat{f}(\xi)m^\delta(t|\xi|)\big)^\vee(x), \qquad \widetilde{S}_t^\delta(f)(x) = \big(\widehat{f}(\xi)\widetilde{m}^\delta(t|\xi|)\big)^\vee(x),$$

and the $L^2(|x|^{-\alpha})$-bounded maximal multiplier operator

$$S_*^\delta(f) = \sup_{t>0}|S_t^\delta(f)|,$$

as well as the continuous square functions

$$G^\delta(f)(x) = \left(\int_0^\infty |S_t^\delta(f)(x)|^2\,\frac{dt}{t}\right)^{\frac{1}{2}}, \quad G^\delta(f)(x) = \left(\int_0^\infty |\widetilde{S}_t^\delta(f)(x)|^2\,\frac{dt}{t}\right)^{\frac{1}{2}}.$$

The operators S_t^δ and $\widetilde{S}_t^\delta$ are related. For $f \in L^2(|x|^{-\alpha})$ and $t > 0$ we have

$$\frac{d}{dt}S_t^\delta(f) = \frac{1}{\delta t}\,\widetilde{S}_t^\delta(f).$$

Indeed, this operator identity is obvious for Schwartz functions f by the Lebesgue dominated convergence theorem, and thus it holds for $f \in L^2(|x|^{-\alpha})$ by density.

The quadratic operators G^δ and $\widetilde{G}^\delta$ make their appearance in the application of the fundamental theorem of calculus in the following context:

$$|S_t^\delta(f)(x)|^2 = 2\,\mathrm{Re}\int_0^t \overline{S_u^\delta(f)(x)}\,\frac{d}{du}S_u^\delta(f)(x)\,du = \frac{2}{\delta}\,\mathrm{Re}\int_0^t \overline{S_u^\delta(f)(x)}\,\widetilde{S}_u^\delta(f)(x)\,\frac{du}{u},$$

which is valid for all functions f in $L^2(|x|^{-\alpha})$ and almost all $x \in \mathbf{R}^n$. This identity uses the fact that for almost all $x \in \mathbf{R}^n$ we have

$$\lim_{t\to 0} S_t^\delta(f)(x) = 0 \tag{10.5.10}$$

when $f \in L^2(|x|^{-\alpha})$. To see this, we observe that for Schwartz functions, (10.5.10) is trivial by the Lebesgue dominated convergence theorem, while for general f in $L^2(|x|^{-\alpha})$ it is a consequence of Theorem 2.1.14, since $S_*^\delta(f) \le C_\delta M(f)$, where M is the Hardy–Littlewood maximal operator. Consequently,

$$|S_t^\delta(f)(x)|^2 \le \frac{2}{\delta}\int_0^t |S_u^\delta(f)(x)|\,|\widetilde{S}_u^\delta(f)(x)|\,\frac{du}{u} \le \frac{2}{\delta}\,|G^\delta(f)(x)|\,|\widetilde{G}^\delta(f)(x)|$$

for all $t>0$, for $f \in L^2(|x|^{-\alpha})$ and for almost all $x \in \mathbf{R}^n$. It follows that

$$\big\|S_*^\delta(f)\big\|^2_{L^2(|x|^{-a})} \le \frac{2}{\delta}\,\big\|G^\delta(f)\big\|_{L^2(|x|^{-a})}\big\|\widetilde{G}^\delta(f)\big\|_{L^2(|x|^{-a})}, \tag{10.5.11}$$

and the asserted boundedness of S_*^δ reduces to that of the continuous square functions G^δ and $\widetilde{G}^\delta$ on weighted L^2 spaces with suitable constants depending on δ.

The boundedness of G^δ on $L^2(|x|^{-\alpha})$ is a consequence of the following lemma.

Lemma 10.5.4. *For $0<\delta<1/10$ and $0\le\alpha<n$ we have*

$$\int_{\mathbf{R}^n}\int_1^2 |S_t^\delta(f)(x)|^2\frac{dt}{t}\,\frac{dx}{|x|^\alpha} \le C_{n,\alpha}A_\alpha(\delta)\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{|x|^\alpha} \tag{10.5.12}$$

for all functions f in $L^2(|x|^{-\alpha})$, where for $\varepsilon>0$, $A_\alpha(\varepsilon)$ is defined by

$$A_\alpha(\varepsilon) = \begin{cases} \varepsilon^{2-\alpha} & \text{when } 1<\alpha<n,\\ \varepsilon\,(|\log\varepsilon|+1) & \text{when } \alpha=1,\\ \varepsilon & \text{when } 0\le\alpha<1. \end{cases} \tag{10.5.13}$$

Assuming the statement of the lemma, we conclude the proof of Proposition 10.5.3 as follows. We take a Schwartz function ψ such that $\widehat{\psi}$ vanishes in a neighborhood of the origin with $\widehat{\psi}(\xi)=1$ whenever $1/2\le|\xi|\le 2$ and we let $\psi_{2^k}(x)=2^{-kn}\psi(2^{-k}x)$. We make the observation that if $1-5\delta\le t|\xi|\le 1-\delta$ and $2^{k-1}\le t\le 2^k$, then $1/2\le 2^k|\xi|\le 2$, since $\delta<1/10$. This implies that $\widehat{\psi}(2^k\xi)=1$ on the support of the function $\xi\mapsto m^\delta(t|\xi|)$. Hence

$$S_t^\delta(f) = S_t^\delta(\psi_{2^k}*f)$$

whenever $2^{k-1}\le t\le 2^k$, and Lemma 10.5.4 (in conjunction with Exercise 10.5.2) yields

$$\int_{\mathbf{R}^n}\int_{2^{k-1}}^{2^k}|S_t^\delta(f)(x)|^2\,\frac{dt}{t}\,\frac{dx}{|x|^\alpha}\le C_{n,\alpha}A_\alpha(\delta)\int_{\mathbf{R}^n}|\psi_{2^k}*f(x)|^2\,\frac{dx}{|x|^\alpha}.$$

Summing over $k\in\mathbf{Z}$ we obtain

$$\big\|G^{\delta}(f)\big\|^2_{L^2(|x|^{-\alpha})} \le C_{n,\alpha}A_\alpha(\delta)\Big\|\Big(\sum_{k\in\mathbf{Z}}|\psi_{2^k}*f|^2\Big)^{\frac{1}{2}}\Big\|^2_{L^2(|x|^{-\alpha})}.$$

A randomization argument relates the weighted L^2 norm of the square function to the L^2 norm of a linear expression involving the Rademacher functions as in

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}|\psi_{2^k}*f|^2\Big)^{\frac{1}{2}}\Big\|^2_{L^2(|x|^{-\alpha})} = \int_0^1\Big\|\sum_{k\in\mathbf{Z}}r_k(t)(\psi_{2^k}*f)\Big\|^2_{L^2(|x|^{-\alpha})}\,dt\,,$$

where r_k denotes a renumbering of the Rademacher functions (Appendix C.1) indexed by the entire set of integers. For each $t\in[0,1]$ the operator

$$M_t(f) = \sum_{k\in\mathbf{Z}} r_k(t)(\psi_{2^k}*f)$$

is associated with a multiplier that satisfies Mihlin's condition (5.2.10) uniformly in t. It follows that M_t is a singular integral operator bounded on all the L^p spaces for $1<p<\infty$, and in view of Corollary 9.4.7, it is also bounded on $L^2(w)$ whenever $w\in A_2$. Since the weight $|x|^{-\alpha}$ is in A_2 whenever $-n<\alpha<n$, it follows that M_t is bounded on $L^2(|x|^{-\alpha})$ with a bound independent of $t>0$. We deduce that

$$\big\|G^{\delta}(f)\big\|_{L^2(|x|^{-\alpha})}+\big\|\widetilde{G}^{\delta}(f)\big\|_{L^2(|x|^{-\alpha})} \le C'_{n,\alpha}\big(A_\alpha(\delta)\big)^{\frac{1}{2}}\big\|f\big\|_{L^2(|x|^{-\alpha})}.$$

We now recall estimate (10.5.11) to obtain

$$\big\|S^{\delta}_{*}(f)\big\|_{L^2(|x|^{-\alpha})} \le C'(n,\alpha)\big(\delta^{-1}A_\alpha(\delta)\big)^{1/2}\big\|f\big\|_{L^2(|x|^{-\alpha})}.$$

Taking $\delta=2^{-k-3}$, recalling the value of $A_\alpha(\delta)$ from Lemma 10.5.4, and inserting this estimate in (10.5.8), we deduce Proposition 10.5.3. We note that the condition $\alpha<1+2\lambda$ is needed to make the series in (10.5.8) converge when $1<\alpha<n$.

10.5.3 Estimates for Radial Multipliers

It remains to prove Lemma 10.5.4. Since all subsequent estimates concern linear operators on weighted L^2 spaces, in the sequel we will be working with functions in the Schwartz class, unless it is otherwise specified.

We reduce estimate (10.5.12) to an estimate for a single t with the bound $A_\alpha(\delta)/\delta$, which is worse than $A_\alpha(\delta)$. The reduction to a single t is achieved via duality. Estimate (10.5.12) says that the operator $f\mapsto\{S^{\delta}_t(f)\}_{1\le t\le 2}$ is bounded from $L^2(\mathbf{R}^n,|x|^{-\alpha}dx)$ to $L^2(L^2(\frac{dt}{t}),|x|^{-\alpha}dx)$. The dual statement of this fact is that the operator

$$\{g_t\}_{1\le t\le 2}\mapsto\int_1^2 S^{\delta}_t(g_t)\,\frac{dt}{t}$$

maps $L^2(L^2(\frac{dt}{t}),|x|^\alpha dx)$ to $L^2(\mathbf{R}^n,|x|^\alpha dx)$. Here we use the fact that the operators S_t are self-transpose and self-adjoint, since they have real and radial multipliers. Thus estimate (10.5.12) is equivalent to

$$\int_{\mathbf{R}^n}\left|\int_1^2 S_t^\delta(g_t)(x)\,\frac{dt}{t}\right|^2|x|^\alpha\,dx\le C_{n,\alpha}A_\alpha(\delta)\int_{\mathbf{R}^n}\int_1^2\big|g_t(x)\big|^2\frac{dt}{t}\,|x|^\alpha\,dx\,,\quad(10.5.14)$$

which by Plancherel's theorem is also equivalent to

$$\int_{\mathbf{R}^n}\left|\mathscr{D}^{\frac{\alpha}{2}}\Big(\int_1^2 m^\delta(t|\cdot|)\widehat{g_t}(\cdot)\,\frac{dt}{t}\Big)(\xi)\right|^2 d\xi\le C_{n,\alpha}A_\alpha(\delta)\int_{\mathbf{R}^n}\int_1^2\big|\mathscr{D}^{\frac{\alpha}{2}}(\widehat{g_t})(\xi)\big|^2\frac{dt}{t}\,d\xi\,.$$

Here

$$\mathscr{D}^\beta(h)(x)=\left[\int_{\mathbf{R}^n}\frac{|D_y^{[\beta]+1}(h)(x)|^2}{|y|^{n+2\beta}}\,dy\right]^{\frac12},$$

where $D_y(f)(x)=f(x+y)-f(x)$ is the difference operator encountered in Section 6.3 and $D_y^k=D_y\circ\cdots\circ D_y$ (k times). The operator $\mathscr{D}^\beta$ obeys the identity (see Exercise 6.3.9)

$$\big\|\mathscr{D}^\beta(\widehat{h})\big\|_{L^2}^2=c_0(n,\beta)\int_{\mathbf{R}^n}|h(x)|^2\,|x|^{2\beta}\,dx\,.$$

Using the definition of $\mathscr{D}^{\alpha/2}$ we write

$$\left|\mathscr{D}^{\frac{\alpha}{2}}\Big(\int_1^2 m^\delta(t|\cdot|)\widehat{g_t}(\cdot)\,\frac{dt}{t}\Big)(\xi)\right|^2=\int_{\mathbf{R}^n}\left|\int_1^2 D_\eta^{[\frac{\alpha}{2}]+1}\big(m^\delta(t|\cdot|)\widehat{g_t}(\cdot)\big)(\xi)\,\frac{dt}{t}\right|^2\frac{d\eta}{|\eta|^{n+\alpha}}\,.$$

If the inner integrand on the right is nonzero, expressing D_y^{k+1} as in (6.3.2) and using the support properties of m^δ, we obtain that $1-5\delta\le t|\xi+s\eta|\le 1-\delta$ for some $s\in\{0,1,\dots,[\alpha/2]+1\}$; thus for each such s, t belongs to an interval of length $4\delta|\xi+s\eta|^{-1}\le 4\delta\,t(1-5\delta)^{-1}$. Since $t\le 2$ and $\delta\le 1/10$, it follows that t lies in a set of measure at most $2([\alpha/2]+2)\delta$. The Cauchy–Schwarz inequality then yields

$$\begin{aligned}&\left|\mathscr{D}^{\frac{\alpha}{2}}\Big(\int_1^2 m^\delta(t|\cdot|)\widehat{g_t}(\cdot)\,\frac{dt}{t}\Big)(\xi)\right|^2\\&\qquad\le c_\alpha\,\delta\int_{\mathbf{R}^n}\int_1^2\left|D_\eta^{[\frac{\alpha}{2}]+1}\big(m^\delta(t|\cdot|)\widehat{g_t}(\cdot)\big)(\xi)\right|^2\frac{dt}{t}\,\frac{d\eta}{|\eta|^{n+\alpha}}\,.\end{aligned}$$

In view of the preceding reduction, we deduce that (10.5.14) is a consequence of

$$\begin{aligned}&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_1^2\left|D_\eta^{[\frac{\alpha}{2}]+1}\big(m^\delta(t|\cdot|)\widehat{g_t}(\cdot)\big)(\xi)\right|^2\frac{dt}{t}\,\frac{d\eta}{|\eta|^{n+\alpha}}\,d\xi\\&\qquad\le C_{n,\alpha}\frac{A_\alpha(\delta)}{c_\alpha\,\delta}\int_{\mathbf{R}^n}\int_1^2\big|\mathscr{D}^{\frac{\alpha}{2}}(\widehat{g_t})(\xi)\big|^2\frac{dt}{t}\,d\xi\end{aligned}$$

which can also be written as

$$\int_{\mathbf{R}^n}\int_1^2 \left|\mathscr{D}^{\frac{\alpha}{2}}\big(m^\delta(t|\cdot|)\widehat{g_t}(\cdot)\big)(\xi)\right|^2 \frac{dt}{t}\,d\xi \le \frac{C_{n,\alpha}}{c_\alpha}\frac{A_\alpha(\delta)}{\delta}\int_{\mathbf{R}^n}\int_1^2 \big|\mathscr{D}^{\frac{\alpha}{2}}(\widehat{g_t})(\xi)\big|^2\frac{dt}{t}\,d\xi\,.$$

This estimate is a consequence of

$$\int_{\mathbf{R}^n} \left|\mathscr{D}^{\frac{\alpha}{2}}\big(m^\delta(t|\cdot|)\widehat{g_t}(\cdot)\big)(\xi)\right|^2 d\xi \le \frac{C_{n,\alpha}}{c_\alpha}\frac{A_\alpha(\delta)}{\delta}\int_{\mathbf{R}^n} \big|\mathscr{D}^{\frac{\alpha}{2}}(\widehat{g_t})(\xi)\big|^2 d\xi \tag{10.5.15}$$

for all $t\in[1,2]$. A simple dilation argument reduces (10.5.15) to the single estimate

$$\int_{\mathbf{R}^n} \left|\mathscr{D}^{\frac{\alpha}{2}}\big(m^\delta(|\cdot|)\widehat{g}(\cdot)\big)(\xi)\right|^2 d\xi \le \frac{C_{n,\alpha}}{c_\alpha}\frac{A_\alpha(\delta)}{\delta}\int_{\mathbf{R}^n} \big|\mathscr{D}^{\frac{\alpha}{2}}(\widehat{g})(\xi)\big|^2 d\xi\,, \tag{10.5.16}$$

which is equivalent to

$$\int_{\mathbf{R}^n} \big|S_1^\delta(g)(x)\big|^2\,|x|^\alpha\,dx \le \frac{C_{n,\alpha}}{c_\alpha}\frac{A_\alpha(\delta)}{\delta}\int_{\mathbf{R}^n}\big|g(x)\big|^2\,|x|^\alpha\,dx$$

and also equivalent to

$$\int_{\mathbf{R}^n} \big|S_1^\delta(f)(x)\big|^2\,\frac{dx}{|x|^\alpha} \le \frac{C_{n,\alpha}}{c_\alpha}\frac{A_\alpha(\delta)}{\delta}\int_{\mathbf{R}^n}\big|f(x)\big|^2\,\frac{dx}{|x|^\alpha} \tag{10.5.17}$$

by duality. We have now reduced estimate (10.5.12) to (10.5.17).

We denote by $K^\delta(x)$ the kernel of the operator S_1^δ, i.e., the inverse Fourier transform of the multiplier $m^\delta(|\xi|)$. Certainly K^δ is a radial kernel on $\mathbf{R}^n$, and it is convenient to decompose it radially as

$$K^\delta = K_0^\delta + \sum_{j=1}^\infty K_j^\delta\,,$$

where $K_0^\delta(x) = K^\delta(x)\phi(\delta x)$ and $K_j^\delta(x) = K^\delta(x)\big(\phi(2^{-j}\delta x)-\phi(2^{1-j}\delta x)\big)$, for some radial smooth function ϕ supported in the ball $B(0,2)$ and equal to one on $B(0,1)$.

To prove estimate (10.5.17) we make use of the subsequent lemmas.

Lemma 10.5.5. *For all $M\ge 2n$ there is a constant $C_M = C_M(n,\phi)$ such that for all $j=0,1,2,\dots$ we have*

$$\sup_{\xi\in\mathbf{R}^n}|\widehat{K_j^\delta}(\xi)| \le C_M 2^{-jM} \tag{10.5.18}$$

and also

$$|\widehat{K_j^\delta}(\xi)| \le C_M 2^{-(j+k)M} \tag{10.5.19}$$

whenever $||\xi|-1|\ge 2^k\delta$ and $k\ge 4$. Also

$$|\widehat{K_j^\delta}(\xi)| \le C_M 2^{-jM}\,\delta^M(1+|\xi|)^{-M} \tag{10.5.20}$$

whenever $|\xi|\le 1/8$ or $|\xi|\ge 15/8$.

Lemma 10.5.6. *Let $0 \le \alpha < n$. Then there is a constant $C(n,\alpha)$ such that for all Schwartz functions f and all $\varepsilon > 0$ we have*

$$\int_{||\xi|-1|\le\varepsilon} |\widehat{f}(\xi)|^2 \, d\xi \le C(n,\alpha)\, \varepsilon^{\alpha-1} A_\alpha(\varepsilon) \int_{\mathbf{R}^n} |f(x)|^2 \, |x|^\alpha dx \tag{10.5.21}$$

and also for $M \ge 2n$ there is a constant $C_M(n,\alpha)$ such that

$$\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2 \frac{1}{(1+|\xi|)^M} \, d\xi \le C_M(n,\alpha) \int_{\mathbf{R}^n} |f(x)|^2 \, |x|^\alpha dx. \tag{10.5.22}$$

Assuming Lemmas 10.5.5 and 10.5.6 we prove estimate (10.5.17) as follows. Using Plancherel's theorem we write

$$\int_{\mathbf{R}^n} |(K_j^\delta * f)(x)|^2 \, dx = \int_{\mathbf{R}^n} |\widehat{K_j^\delta}(\xi)|^2 |\widehat{f}(\xi)|^2 \, d\xi \le I + II + III,$$

where

$$\begin{aligned}
I &= \int_{|\xi|\le\frac{1}{8},|\xi|\ge\frac{15}{8}} |\widehat{K_j^\delta}(\xi)|^2 |\widehat{f}(\xi)|^2 \, d\xi\,, \\
II &= \sum_{k=4}^{[\log_2 \frac{7}{16}\delta^{-1}]+1} \int_{2^k\delta\le||\xi|-1|\le 2^{k+1}\delta} |\widehat{K_j^\delta}(\xi)|^2 |\widehat{f}(\xi)|^2 \, d\xi\,, \\
III &= \int_{||\xi|-1|\le 16\delta} |\widehat{K_j^\delta}(\xi)|^2 |\widehat{f}(\xi)|^2 \, d\xi\,.
\end{aligned}$$

Using (10.5.20) and (10.5.22) we obtain that

$$I \le C_M'(n,\alpha) 2^{-jM} \delta^M \int_{\mathbf{R}^n} |f(x)|^2 \, |x|^\alpha dx.$$

In view of (10.5.19) and (10.5.21) we have

$$\begin{aligned}
II &\le \sum_{k=4}^{[\log_2 \delta^{-1}]+1} C(n,\alpha)(2^{k+1}\delta)^{\alpha-1} A_\alpha(2^{k+1}\delta) 2^{-jM} 2^{-kM} \int_{\mathbf{R}^n} |f(x)|^2 \, |x|^\alpha dx \\
&\le C_M'(n,\alpha) 2^{-jM} \delta^{\alpha-1} A_\alpha(\delta) \int_{\mathbf{R}^n} |f(x)|^2 \, |x|^\alpha dx.
\end{aligned}$$

Finally, (10.5.18) and (10.5.21) yield

$$III \le C_M'(n,\alpha) 2^{-jM} \delta^{\alpha-1} A_\alpha(\delta) \int_{\mathbf{R}^n} |f(x)|^2 \, |x|^\alpha dx.$$

Summing the estimates for I, II, and III we deduce

$$\int_{\mathbf{R}^n} |(K_j^\delta * f)(x)|^2 \, dx \le C_M(n,\alpha) 2^{-jM} \delta^{\alpha-1} A_\alpha(\delta) \int_{\mathbf{R}^n} |f(x)|^2 \, |x|^\alpha dx.$$

By duality, this estimate can be written as

$$\int_{\mathbf{R}^n} |(K_j^\delta * f)(x)|^2 \frac{dx}{|x|^\alpha} \le C_M(n,\alpha) 2^{-jM} \delta^{\alpha-1} A_\alpha(\delta) \int_{\mathbf{R}^n} |f(x)|^2 \, dx. \tag{10.5.23}$$

Given a Schwartz function f, we write $f_0 = f\chi_{Q_0}$, where Q_0 is a cube centered at the origin of side length $C2^j/\delta$ for some C to be chosen. Then for $x \in Q_0$ we have $|x| \le C\sqrt{n}\,2^j/\delta$, hence

$$\begin{aligned}\int_{\mathbf{R}^n} |(K_j^\delta * f_0)(x)|^2 \frac{dx}{|x|^\alpha} &\le \frac{C_M'(n,\alpha)\,\delta^{\alpha-1} A_\alpha(\delta)}{2^{jM}} \Big(C\sqrt{n}\,\frac{2^j}{\delta}\Big)^\alpha \int_{Q_0} |f_0(x)|^2 \frac{dx}{|x|^\alpha} \\ &= C_M''(n,\alpha) 2^{j(\alpha-M)} \frac{A_\alpha(\delta)}{\delta} \int_{Q_0} |f_0(x)|^2 \frac{dx}{|x|^\alpha}. \end{aligned} \tag{10.5.24}$$

Now write $\mathbf{R}^n \setminus Q_0$ as a mesh of cubes Q_i, indexed by $i \in \mathbf{Z} \setminus \{0\}$, of side lengths $2^{j+2}/\delta$ and centers c_{Q_i}. Since K_j^δ is supported in a ball of radius $2^{j+1}/\delta$, if f_i is supported in Q_i, then $f_i * K_j^\delta$ is supported in the cube $2\sqrt{n}Q_i$. If the constant C is large enough, say $C \ge 1000n$, then for $x \in Q_i$ and $x' \in 2\sqrt{n}Q_i$ we have

$$|x| \approx |c_{Q_i}| \approx |x'|,$$

which says that the moduli of x and x' are comparable in the following inequality:

$$\int_{2\sqrt{n}Q_i} |(K_j^\delta * f_i)(x')|^2 \frac{dx'}{|x'|^\alpha} \le C_M' 2^{-jM} \int_{Q_i} |f_i(x)|^2 \frac{dx}{|x|^\alpha}. \tag{10.5.25}$$

Thus (10.5.25) is a consequence of

$$\int_{2\sqrt{n}Q_i} |(K_j^\delta * f_i)(x')|^2 \, dx' \le C_M 2^{-jM} \int_{Q_i} |f_i(x)|^2 \, dx, \tag{10.5.26}$$

which is certainly satisfied, as seen by applying Plancherel's theorem and using (10.5.18). Since for $\delta < 1/10$ we have $A_\alpha(\delta)/\delta \ge 1$, it follows that

$$\int_{\mathbf{R}^n} |(K_j^\delta * f_i)(x)|^2 \frac{dx}{|x|^\alpha} \le C_M 2^{-jM} \frac{A_\alpha(\delta)}{\delta} \int_{\mathbf{R}^n} |f_i(x)|^2 \frac{dx}{|x|^\alpha} \tag{10.5.27}$$

whenever f_i is supported in Q_i. We now pick $M = 2n$ and we recall that $\alpha < n$. We have now proved that

$$\int_{\mathbf{R}^n} |(K_j^\delta * f_i)(x)|^2 \frac{dx}{|x|^\alpha} \le C''(n,\alpha) 2^{-jn} \frac{A_\alpha(\delta)}{\delta} \int_{Q_i} |f_i(x)|^2 \frac{dx}{|x|^\alpha}$$

for functions f_i supported in Q_i.

Given a general f in the Schwartz class, write

$$f = \sum_{i \in \mathbf{Z}} f_i, \qquad \text{where} \qquad f_i = f\chi_{Q_i}.$$

Then

$$\begin{aligned}\big\|K_j^\delta * f\big\|^2_{L^2(|x|^{-\alpha})} &\le 2\big\|K_j^\delta * f_0\big\|^2_{L^2(|x|^{-\alpha})} + 2\big\|\sum_{i\neq 0} K_j^\delta * f_i\big\|^2_{L^2(|x|^{-\alpha})}\\
&\le 2\big\|K_j^\delta * f_0\big\|^2_{L^2(|x|^{-\alpha})} + 2C_n\sum_{i\neq 0}\big\|K_j^\delta * f_i\big\|^2_{L^2(|x|^{-\alpha})}\\
&\le C'''(n,\alpha)2^{-jn}\frac{A_\alpha(\delta)}{\delta}\Big[\big\|f_0\big\|^2_{L^2(|x|^{-\alpha})} + \sum_{i\neq 0}\big\|f_i\big\|^2_{L^2(|x|^{-\alpha})}\Big]\\
&= C'''(n,\alpha)2^{-jn}\frac{A_\alpha(\delta)}{\delta}\big\|f\big\|^2_{L^2(|x|^{-\alpha})},\end{aligned}$$

where we used the bounded overlap of the family $\{K_j * f_i\}_{i\neq 0}$ in the second displayed inequality (cf. Exercise 10.4.4). Taking square roots and summing over $j = 0,1,2,\dots$, we deduce (10.5.17).

We now address the proof of Lemma 10.5.5, which was left open.

Proof. For the purposes of this proof we set $\psi(x) = \phi(x) - \phi(2x)$. Then the inverse Fourier transform of the function $x \mapsto \psi(2^{-j}\delta x)$ is $\xi \mapsto 2^{jn}\delta^{-n}\widehat{\psi}(2^j\xi/\delta)$. Convolving the latter with the function $\xi \mapsto m^\delta(|\xi|)$, we obtain $\widehat{K_j^\delta}(\xi)$. We may therefore write for $j \ge 1$,

$$\widehat{K_j^\delta}(\xi) = \int_{\mathbf{R}^n} m^\delta(|\xi - 2^{-j}\delta\eta|)\widehat{\psi}(\eta)\,d\eta\,, \tag{10.5.28}$$

while for $j = 0$ an analogous formula holds with ϕ in place of ψ. Since $|m^\delta| \le 1$, (10.5.18) follows easily when $j = 0$. For $j \ge 1$ we expand the function $\xi \mapsto m^\delta(|\xi - 2^{-j}\delta\eta|)$ in a Taylor series and we make use of the fact that $\widehat{\psi}$ has vanishing moments of all orders to obtain

$$\begin{aligned}|\widehat{K_j^\delta}(\xi)| &\le \int_{\mathbf{R}^n}\sum_{|\gamma|=M}\frac{1}{\gamma!}\big\|\partial^\gamma m^\delta(|\cdot|)\big\|_{L^\infty}|2^{-j}\delta\eta|^M|\widehat{\psi}(\eta)|\,d\eta\\
&\le C(M)\delta^{-M}\delta^M 2^{-jM}\int_{\mathbf{R}^n}|\eta|^M|\widehat{\psi}(\eta)|\,d\eta\,.\end{aligned}$$

This proves (10.5.18).

We turn now to the proof of (10.5.19). Suppose that $\big||\xi|-1\big| \ge 2^k\delta$ and $k \ge 4$. Then for $|\xi| \le 1$, recalling that m^δ is supported in $[1-5\delta, 1-\delta]$, we write

$$|2^{-j}\delta\eta| \ge |\xi - 2^{-j}\delta\eta| - |\xi| \ge (1-5\delta) - (1-2^k\delta) \ge 2^{k-1}\delta\,,$$

since $k \ge 4$. For $|\xi| \ge 1$ we have

$$|2^{-j}\delta\eta| \ge |\xi| - |\xi - 2^{-j}\delta\eta| \ge (1+2^k\delta) - (1-\delta) \ge 2^k\delta\,.$$

In either case we conclude that $|\eta| \ge 2^{k+j-1}$, and using (10.5.28) we deduce

$$|\widehat{K_j^\delta}(\xi)| \le \int_{|\eta|\ge 2^{k+j-1}}|\widehat{\psi}(\eta)|\,d\eta \le C_M 2^{-(j+k)M}\,.$$

The proof of (10.5.20) is similar. Since $|\xi - 2^{-j}\delta\eta| \ge 1-5\delta \ge 1/2$, if $|\xi| \le 1/8$, it follows that $|2^{-j}\delta\eta| \ge 1/4$. Likewise, if $|\xi| \ge 15/8$, then $|2^{-j}\delta\eta| \ge |\xi| - 1 \ge |\xi|/4$. These estimates imply

$$|2^{-j}\delta\eta| \ge \frac{1}{8}(1+|\xi|) \implies |\eta| \ge 2^j \frac{1}{8\delta}(1+|\xi|)$$

in the support of the integral in (10.5.28). It follows that

$$|\widehat{K_j^\delta}(\xi)| \le \int_{|\eta|\ge 2^{j-3}(1+|\xi|)/\delta} |\widehat{\psi}(\eta)|\,d\eta \le C_M 2^{-jM}\delta^M(1+|\xi|)^{-M}$$

whenever $|\xi| \le 1/8$ or $|\xi| \ge 15/8$. □

We finish with the proof of Lemma 10.5.6, which had been left open.

Proof. We reduce estimate (10.5.21) by duality to

$$\int_{\mathbf{R}^n} |\widehat{g}(\xi)|^2 \frac{d\xi}{|\xi|^\alpha} \le C(n,\alpha)\varepsilon^{\alpha-1}A_\alpha(\varepsilon)\int_{||x|-1|\le\varepsilon} |g(x)|^2\,dx$$

for functions g supported in the annulus $||x|-1| \le \varepsilon$. Using that $(|\xi|^{-\alpha})^\wedge(x) = c_{n,\alpha}|x|^{\alpha-n}$ (cf. Theorem 2.4.6), we write

$$\begin{aligned}
\int_{\mathbf{R}^n} |\widehat{g}(\xi)|^2 \frac{d\xi}{|\xi|^\alpha} &= \int_{\mathbf{R}^n} \widehat{g}(\xi)\overline{\widehat{g}(\xi)} \frac{1}{|\xi|^\alpha}\,d\xi \\
&= \int_{\mathbf{R}^n} \big(\widehat{g}\overline{\widehat{g}}\big)^\vee(x) \frac{c_{n,\alpha}}{|x|^{n-\alpha}}\,dx \\
&= \int_{\mathbf{R}^n} (g*\overline{\widetilde{g}})(x) \frac{dx}{|x|^{n-\alpha}} \\
&= \int_{||y|-1|\le\varepsilon}\int_{||x|-1|\le\varepsilon} g(x)\overline{\widetilde{g}}(y) \frac{c_{n,\alpha}}{|x-y|^{n-\alpha}}\,dx\,dy \\
&\le B(n,\alpha)\,\|g\|_{L^2}^2,
\end{aligned}$$

where $\widetilde{g}(x) = g(-x)$ and

$$B(n,\alpha) = \sup_{||x|-1|\le\varepsilon} \int_{||y|-1|\le\varepsilon} \frac{c_{n,\alpha}}{|y-x|^{n-\alpha}}\,dy.$$

The last inequality is proved by interpolating between the $L^1 \to L^1$ and $L^\infty \to L^\infty$ estimates with bound $B(n,\alpha)$ for the linear operator

$$L(g)(x) = \int_{\mathbf{R}^n} g(y) \frac{c_{n,\alpha}}{|x-y|^{n-\alpha}}\,dy.$$

It remains to establish that

$$B(n,\alpha) \le C(n,\alpha)\varepsilon^{\alpha-1}A_\alpha(\varepsilon).$$

Applying a rotation and a change of variables, matters reduce to proving that

$$\sup_{||x|-1|\le\varepsilon}\int_{||y-|x|e_1|-1|\le\varepsilon}\frac{c_{n,\alpha}}{|y|^{n-\alpha}}\,dy\le C(n,\alpha)\varepsilon^{\alpha-1}A_\alpha(\varepsilon)\,,$$

where $e_1=(1,0,\dots,0)$. This, in turn, is a consequence of

$$\int_{||y-e_1|-1|\le 2\varepsilon}\frac{c_{n,\alpha}}{|y|^{n-\alpha}}\,dy\le C(n,\alpha)\varepsilon^{\alpha-1}A_\alpha(\varepsilon)\,, \tag{10.5.29}$$

since $||y-e_1|x||-1|\le\varepsilon$ and $||x|-1|\le\varepsilon$ imply $||y-e_1|-1|\le 2\varepsilon$. In proving (10.5.29), it suffices to assume that $\varepsilon<1/100$; otherwise, the left-hand side of (10.5.29) is bounded from above by a constant, and the right-hand side of (10.5.29) is bounded from below by another constant. The region of integration in (10.5.29) is a ring centered at e_1 and width 4ε. We estimate the integral in (10.5.29) by the sum of the integrals of the function $c_{n,\alpha}|y|^{\alpha-n}$ over the sets

$$\begin{aligned}
S_0&=\{y\in\mathbf{R}^n:\ |y|\le\varepsilon,\quad ||y-e_1|-1|\le 2\varepsilon\}\,,\\
S_\ell&=\{y\in\mathbf{R}^n:\ \ell\varepsilon\le|y|\le(\ell+1)\varepsilon,\quad ||y-e_1|-1|\le 2\varepsilon\}\,,\\
S_*&=\{y\in\mathbf{R}^n:\ |y|\ge 1,\quad ||y-e_1|-1|\le 2\varepsilon\}\,,
\end{aligned}$$

where $\ell=1,\dots,[\frac{1}{\varepsilon}]+1$. The volume of each S_ℓ is comparable to

$$\varepsilon\left[((\ell+1)\varepsilon)^{n-1}-(\ell\varepsilon)^{n-1}\right]\approx\varepsilon^n\ell^{n-2}\,.$$

Consequently,

$$\int_{S_0}\frac{dy}{|y|^{n-\alpha}}\le\omega_{n-1}\int_0^\varepsilon\frac{r^{n-1}}{r^{n-\alpha}}\,dr=\frac{\omega_{n-1}}{\alpha}\,\varepsilon^\alpha\,,$$

whereas

$$\sum_{\ell=1}^{[\frac{1}{\varepsilon}]+1}\int_{S_\ell}\frac{dy}{|y|^{n-\alpha}}\le C'_{n,\alpha}\sum_{\ell=1}^{2/\varepsilon}\frac{\varepsilon^n\ell^{n-2}}{(\ell\varepsilon)^{n-\alpha}}\le C'_{n,\alpha}\,\varepsilon^\alpha\sum_{\ell=1}^{2/\varepsilon}\frac{1}{\ell^{2-\alpha}}\,.$$

Finally, the volume of S_∞ is about ε; hence

$$\int_{S_\infty}\frac{dy}{|y|^{n-\alpha}}\le|S_\infty|\le C''_{n,\alpha}\,\varepsilon\,.$$

Combining these estimates, we obtain

$$\int_{||y-e_1|-1|\le 2\varepsilon}\frac{c_{n,\alpha}}{|y|^{n-\alpha}}\,dy\le C_{n,\alpha}\left[\varepsilon^\alpha+\varepsilon^\alpha\sum_{\ell=1}^{2/\varepsilon}\frac{1}{\ell^{2-\alpha}}+\varepsilon\right],$$

and it is an easy matter to check that the expression inside the square brackets is at most a constant multiple of $\varepsilon^{\alpha-1}A_\alpha(\varepsilon)$.

We now turn attention to (10.5.22). Switching the roles of f and $\widehat{f}$, we rewrite (10.5.22) as

$$\int_{\mathbf{R}^n} \frac{|f(x)|^2}{(1+|x|)^M}\,dx \le C_M'(n,\alpha)\int_{\mathbf{R}^n} |\widehat{(-\Delta)^{\frac{\alpha}{4}}(f)}(\xi)|^2\,d\xi$$
$$= C_M'(n,\alpha)\int_{\mathbf{R}^n} |(-\Delta)^{\frac{\alpha}{4}}(f)(x)|^2\,dx,$$

recalling the Laplacian introduced in (6.1.1). This estimate can also be restated in terms of the Riesz potential operator $I_{\alpha/2} = (-\Delta)^{-\alpha/4}$ as follows:

$$\int_{\mathbf{R}^n} \frac{|I_{\alpha/2}(g)(x)|^2}{(1+|x|)^M}\,dx \le C_M'(n,\alpha)\int_{\mathbf{R}^n} |g(x)|^2\,dx. \tag{10.5.30}$$

To show this, we use Hölder's inequality with exponents $q/2$ and n/α, where $q>2$ satisfies

$$\frac{1}{2}-\frac{1}{q}=\frac{\alpha}{2n}.$$

Then we have

$$\int_{\mathbf{R}^n} \frac{|I_{\alpha/2}(g)(x)|^2}{(1+|x|)^M}\,dx \le \bigg(\int_{\mathbf{R}^n} \frac{dx}{(1+|x|)^{Mn/\alpha}}\bigg)^{\frac{n}{\alpha}} \big\|I_{\alpha/2}(g)\big\|_{L^q(\mathbf{R}^n)}^2$$
$$\le C_M'(n,\alpha)\big\|g\big\|_{L^2(\mathbf{R}^n)}^2$$

in view of Theorem 6.1.3 and since $M>n$ and $\alpha<n$. This finishes the proof of the lemma. □

Exercises

10.5.1. Let $0<r<p<\infty$ and $n(1-\frac{r}{p})<\beta<n$. Show that $L^p(\mathbf{R}^n)$ is contained in $L^r(\mathbf{R}^n)+L^r(\mathbf{R}^n,|x|^{-\beta})$.
[*Hint:* Write $f=f_1+f_2$, where $f_1=f\chi_{|f|>1}$ and $f_2=f\chi_{|f|\le 1}$.]

10.5.2. (a) With the notation of Lemma 10.5.4, use dilations to show that the estimate

$$\int_{\mathbf{R}^n}\int_1^2 |S_t^\delta(f)(x)|^2\frac{dt}{t}\,\frac{dx}{|x|^\alpha} \le C_0\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{|x|^\alpha}$$

implies

$$\int_{\mathbf{R}^n}\int_a^{2a} |S_t^\delta(f)(x)|^2\frac{dt}{t}\,\frac{dx}{|x|^\alpha} \le C_0\int_{\mathbf{R}^n}|f(x)|^2\,\frac{dx}{|x|^\alpha}$$

for any $a>0$ and f in the Schwartz class.
(b) Using dilations also show that (10.5.16) implies (10.5.15).

10.5.3. Let h be a Schwartz function on $\mathbf{R}^n$. Prove that

$$\frac{1}{\varepsilon}\int_{|\,|x|-1|\le\varepsilon} h(x)\,dx \to 2|\mathbf{S}^{n-1}|\int_{\mathbf{S}^{n-1}} h(\theta)\,d\theta$$

as $\varepsilon \to 0$. Use Lemma 10.5.6 to show that for $1 < \alpha < n$ we have

$$\int_{\mathbf{S}^{n-1}} |\widehat{f}(\theta)|\,d\theta \le C(n,\alpha)\int_{\mathbf{R}^n} |f(x)|^2|x|^\alpha\,dx.$$

10.5.4. Let $w \in A_2$. Assume that the ball multiplier operator $B^0(f) = (\widehat{f}\chi_{B(0,1)})^\vee$ satisfies

$$\int_{\mathbf{R}^n} |B^0(f)(x)|^2\,w(x)\,dx \le C_{n,\alpha}\int_{\mathbf{R}^n} |f(x)|^2\,w(x)\,dx$$

for all $f \in L^2(w)$. Prove the same estimate for $\mathscr{B}(f) = \sup_{k\in\mathbf{Z}} |B^0_{2^k}(f)|$.
$\big[$*Hint:* Argue as in the proof of Theorem 5.3.1. Pick a smooth function with compact support $\widehat{\Phi}$ equal to one on $B(0,1)$ and vanishing in $B(0,2)$ and define $\widehat{\Psi}(\xi) = \widehat{\Phi}(\xi) - \widehat{\Phi}(2\xi)$. Then $\chi_{B(0,1)}\big(\widehat{\Phi}(\xi) - \widehat{\Phi}(2\xi)\big) = \chi_{B(0,1)} - \widehat{\Phi}(2\xi)$; hence

$$\begin{aligned}\mathscr{B}(f) &\le \sup_k |\Phi_{2^{-k}} * f| + \Big(\sum_{k\in\mathbf{Z}} |B^0_{2^k}(f) - \Phi_{2^{-(k-1)}} * f|^2\Big)^{\frac{1}{2}}\\ &\le C_\Phi M(f) + \Big(\sum_{k\in\mathbf{Z}} |B^0_{2^k}(f * \Psi_{2^{-k}})|^2\Big)^{\frac{1}{2}}\end{aligned}$$

and show that each term is bounded on $L^2(w)$.$\big]$

10.5.5. Show that the Bochner–Riesz operator B^λ does not map $L^p(\mathbf{R}^n)$ to $L^{p,\infty}(\mathbf{R}^n)$ when $\lambda = \frac{n-1}{2} - \frac{n}{p}$ and $2 < p < \infty$. Derive the same conclusion for B^λ_*.
$\big[$*Hint:* Suppose the contrary. Then by duality it would follow that B^λ maps $L^{p,1}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ when $1 < p < 2$ and $\lambda = \frac{n}{p} - \frac{n+1}{2}$. To contradict this statement test the operator on a Schwartz function whose Fourier transform is equal to 1 on the unit ball and argue as in Proposition 10.2.3.$\big]$

HISTORICAL NOTES

The geometric construction in Section 10.1 is based on ideas of Besicovitch, who used a similar construction to answer the following question posed in 1917 by the Japanese mathematician S. Kakeya: What is the smallest possible area of the trace of ink left on a piece of paper by an ink-covered needle of unit length when the positions of its two ends are reversed? This problem puzzled mathematicians for several decades until Besicovitch [22] showed that for any $\varepsilon > 0$ there is a way to move the needle so that the total area of the blot of ink left on the paper is smaller than ε. Fefferman [125] borrowed ideas from the construction of Besicovitch to provide the negative answer to the multiplier problem to the ball for $p \neq 2$ (Theorem 10.1.5). Prior to Fefferman's work, the fact that the characteristic function of the unit ball is not a multiplier on $L^p(\mathbf{R}^n)$ for $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2n}$ was pointed out by Herz [163], who also showed that this limitation is not necessary when this operator is restricted to radial L^p functions. The crucial Lemma 10.1.4 in Fefferman's proof is due to Y. Meyer.

The study of Bochner–Riesz means originated in the article of Bochner [27], who obtained their L^p boundedness for $\lambda > \frac{n-1}{2}$. Stein [287] improved this result to $\lambda > \frac{n-1}{2}|\frac{1}{p} - \frac{1}{2}|$ using interpolation for analytic families of operators. Theorem 10.2.4 was first proved by Carleson and Sjölin [58]. A second proof of this theorem was given by Fefferman [127]. A third proof was given by Hörmander [167]. The proof of Theorem 10.2.4 given in the text is due Córdoba [90]. This proof elaborated the use of the Kakeya maximal function in the study of spherical summation multipliers, which was implicitly pioneered in Fefferman [127]. The boundedness of the Kakeya maximal function $\mathscr{K}_N$ on $L^2(\mathbf{R}^2)$ with norm $C(\log N)^2$ was first obtained by Córdoba [89]. The sharp estimate $C\log N$ was later obtained by Strömberg [296]. The proof of Theorem 10.3.5 is taken from this article of Strömberg. Another proof of the boundedness of the Kakeya maximal function without dilations on $L^2(\mathbf{R}^2)$ was obtained by Müller [240]. Barrionuevo [17] showed that for any subset Σ of $\mathbf{S}^1$ with N elements the maximal operator $\mathfrak{M}_\Sigma$ maps $L^2(\mathbf{R}^2)$ to itself with norm $CN^{2(\log N)^{-1/2}}$ for some absolute constant C. Note that this bound is $O(N^\varepsilon)$ for any $\varepsilon > 0$. Katz [183] improved this bound to $C\log N$ for some absolute constant C; see also Katz [184]. The latter is a sharp bound, as indicated in Proposition 10.3.4. Katz [182] also showed that the maximal operator $\mathfrak{M}_K$ associated with a set of unit vectors pointing along a Cantor set K of directions is unbounded on $L^2(\mathbf{R}^2)$. If Σ is an infinite set of vectors in $\mathbf{S}^1$ pointing in lacunary directions, then $\mathfrak{M}_\Sigma$ was studied by Strömberg [295], Córdoba and Fefferman [93], and Nagel, Stein, and Wainger [244]. The last authors obtained its L^p boundedness for all $1 < p < \infty$. Theorem 10.2.7 was first proved by Carleson [56]. For a short account on extensions of this theorem, the reader may consult the historical notes at the end of Chapter 5.

The idea of restriction theorems for the Fourier transform originated in the work of E. M. Stein around 1967. Stein's original restriction result was published in the article of Fefferman [123], which was the first to point out connections between restriction theorems and boundedness of the Bochner–Riesz means. The full restriction theorem for the circle (Theorem 10.4.7 for $p < \frac{4}{3}$) is due to Fefferman and Stein and was published in the aforementioned article of Fefferman [123]. See also the related article of Zygmund [340]. The present proof of Theorem 10.4.7 is based in that of Córdoba [91]. This proof was further elaborated by Tomas [314], who pointed out the logarithmic blowup when $p = \frac{4}{3}$ for the corresponding restriction problem for annuli. The result in Example 10.4.4 is also due to Fefferman and Stein and was initially proved using arguments from spherical harmonics. The simple proof presented here was observed by A. W. Knapp. The restriction property in Theorem 10.4.5 for $p < \frac{2(n+1)}{n+3}$ is due to Tomas [313], while the case $p = \frac{2(n+1)}{n+3}$ is due to Stein [291]. Theorem 10.4.6 was first proved by Fefferman [123] for the smaller range of $\lambda > \frac{n-1}{4}$ using the restriction property $R_{p\to 2}(\mathbf{S}^{n-1})$ for $p < \frac{4n}{3n+1}$. The fact that the $R_{p\to 2}(\mathbf{S}^{n-1})$ restriction property (for $p < 2$) implies the boundedness of the Bochner–Riesz operator B^λ on $L^p(\mathbf{R}^n)$ is contained in the work of Fefferman [123]. A simpler proof of this fact, obtained later by E. M. Stein, appeared in the subsequent article of Fefferman [127]. This proof is given in Theorem 10.4.6, incorporating the Tomas–Stein restriction property $R_{p\to 2}(\mathbf{S}^{n-1})$ for $p \le \frac{2(n+1)}{n+3}$. It should be noted that the case $n = 3$ of this theorem was first obtained in unpublished work of Sjölin. For a short exposition and history of this material consult the book of Davis and Chang [106]. Much of the material in Sections 10.2, 10.3, and 10.4 is based on the notes of Vargas [322].

There is an extensive literature on restriction theorems for submanifolds of $\mathbf{R}^n$. It is noteworthy to mention (in chronological order) the results of Strichartz [294], Prestini [267], Greenleaf [155], Christ [62], Drury [112], Barceló [15], [16], Drury and Marshall [114], [115], Beckner, Carbery, Semmes, and Soria [18], Drury and Guo [113], De Carli and Iosevich [107], [108], Sjölin and Soria [284], Oberlin [250], Wolff [337], and Tao [306].

The boundedness of the Bochner–Riesz operators on the range excluded by Proposition 10.2.3 implies that the restriction property $R_{p\to q}(\mathbf{S}^{n-1})$ is valid when $\frac{1}{q} = \frac{n+1}{n-1}\frac{1}{p'}$ and $1 \le p < \frac{2n}{n+1}$, as shown by Tao [305]; in this article a hierarchy of conjectures in harmonic analysis and interrelations among them is discussed. In particular, the aforementioned restriction property would imply estimate (10.3.33) for the Kakeya maximal operator $\mathscr{K}_N$ on $\mathbf{R}^n$, which would in turn imply that Besicovitch sets have Minkowski dimension n. (A Besicovitch set is defined as a subset of $\mathbf{R}^n$ that contains a unit line segment in every direction.) Katz, Laba, and Tao [185] have obtained good estimates on the Minkowski dimension of such sets in $\mathbf{R}^3$.

A general sieve argument obtained by Córdoba [89] reduces the boundedness of the Kakeya maximal operator $\mathscr{K}_N$ to the one without dilations $\mathscr{K}_N^a$. For applications to the Bochner–Riesz multiplier problem, only the latter is needed. Carbery, Hernández, and Soria [51] have proved estimate (10.3.31) for radial functions in all dimensions. Igari [175] proved estimate (10.3.32) for products of one-variable functions of each coordinate. The norm estimates in Corollary 10.3.7 can be reversed, as shown by Keich [187] for $p > 2$. The corresponding estimate for $1 < p < 2$ in the same corollary can be improved to $N^{\frac{2}{p}-1}$. Córdoba [90] proved the partial case $p \le 2$ of Theorem 10.3.10 on $\mathbf{R}^n$. This range was extended by Drury [111] to $p \le \frac{n+1}{n-1}$ using estimates for the x-ray transform. Theorem 10.3.10 (i.e., the further extension to $p \le \frac{n+1}{2}$) is due to Christ, Duoandikoetxea, and Rubio de Francia [68], and its original proof also used estimates for the x-ray transform; the proof of Theorem 10.3.10 given in the text is derived from that in Bourgain [29]. This article brought a breakthrough in many of the previous topics. In particular, Bourgain [29] showed that the Kakeya maximal operator $\mathscr{K}_N$ maps $L^p(\mathbf{R}^n)$ to itself with bound $C_\varepsilon N^{\frac{n}{p}-1+\varepsilon}$ for all $\varepsilon > 0$ and some $p_n > \frac{n+1}{2}$. He also showed that the range of p's in Theorem 10.4.5 is not sharp, since there exist indices $p = p(n) > \frac{2(n+1)}{n+3}$ for which property $R_{p \to q}(\mathbf{S}^{n-1})$ holds, and that Theorem 10.4.6 is not sharp, since there exist indices $\lambda_n < \frac{n-1}{2(n+1)}$ for which the Bochner–Riesz operators are bounded on $L^p(\mathbf{R}^n)$ in the optimal range of p's when $\lambda \ge \lambda_n$. Improvements on these indices were subsequently obtained by Bourgain [30], [31]. Some of Bourgain's results in $\mathbf{R}^3$ were re-proved by Schlag [279] using different geometric methods. Wolff [335] showed that the Kakeya maximal operator $\mathscr{K}_N$ maps $L^p(\mathbf{R}^n)$ to itself with bound $C_\varepsilon N^{\frac{n}{p}-1+\varepsilon}$ for any $\varepsilon > 0$ whenever $p \le \frac{n+2}{2}$. In higher dimensions, this range of p's was later extended by Bourgain [32] to $p \le (1+\varepsilon)\frac{n}{2}$ for some dimension-free positive constant ε. When $n = 3$, further improvements on the restriction and the Kakeya conjectures were obtained by Tao, Vargas, and Vega [308]. For further historical advances in the subject the reader is referred to the survey articles of Wolff [336] and Katz and Tao [186].

Regarding the almost everywhere convergence of the Bochner–Riesz means, Carbery [50] has shown that the maximal operator $B_*^\lambda(f) = \sup_{R>0} |B_R^\lambda(f)|$ is bounded on $L^p(\mathbf{R}^2)$ when $\lambda > 0$ and $2 \le p < \frac{4}{1-2\lambda}$, obtaining the convergence $B_R^\lambda(f) \to f$ almost everywhere for $f \in L^p(\mathbf{R}^2)$. For $n \ge 3$, $2 \le p < \frac{2n}{n-1-2\lambda}$, and $\lambda \ge \frac{n-1}{2(n+1)}$ the same result was obtained by Christ [63]. Theorem 10.5.2 is due to Carbery, Rubio de Francia, and Vega [52]. Theorem 10.5.1 is contained in Tao [304]. Tao [307] also obtained boundedness for the maximal Bochner–Riesz operators B_*^λ on $L^p(\mathbf{R}^2)$ whenever $1 < p < 2$ for an open range of pairs $(\frac{1}{p}, \lambda)$ that lie below the line $\lambda = \frac{1}{2}(\frac{1}{p} - \frac{1}{2})$.

On the critical line $\lambda = \frac{n}{p} - \frac{n+1}{2}$, boundedness into weak L^p for the Bochner–Riesz operators is possible in the range $1 \le p \le \frac{2n}{n+1}$. Christ [65], [64] first obtained such results for $1 \le p < \frac{2(n+1)}{n+3}$ in all dimensions. The point $p = \frac{2(n+1)}{n+3}$ was later included by Tao [303]. In two dimensions, weak boundedness for the full range of indices was shown by Seeger [280]; in all dimensions the same conclusion was obtained by Colzani, Travaglini, and Vignati [87] for radial functions. Tao [304] has obtained a general argument that yields weak endpoint bounds for B^λ whenever strong type bounds are known above the critical line.