CHAPTER 9

LEVEL CROSSING ESTIMATION

9.1 Introduction

This chapter describes a basic level crossing estimation method (LCE) for steady-state probability distributions in queues, storage processes and related stochastic models. LCE is also called: level crossing computation, system point estimation (or computation). LCE is related to non-parametric density estimation methods (e.g., [95]). In standard density estimation the data is assumed to be a random sample from an unknown pdf. The data is used to construct histograms, naive density estimators, kernel-density estimators, etc., for the unknown pdf, utilizing associated smoothing techniques.

In LCE we obtain the data from a simulated sample path of a stochastic process. We compute estimators of the pdf of the state variable from level-crossing time averages, or related averages. The estimators used in LCE can be combined with smoothing techniques to improve the estimates (e.g., [71], [72], [73]).

9.1.1 Main Steps of Level Crossing Estimation

The basic LCE procedure that we use here for steady-state distributions, has three main steps:

1. Simulate a single sample path of the process over a long simulated time period, say $[0, t]$.

P.H. Brill, *Level Crossing Methods in Stochastic Models*,
DOI: 10.1007/978-0-387-09421-2_9,

2. From the simulated sample path, compute *point estimators* of the pdf and cdf of the state variable, in terms of level-crossing time averages calculated on a state-space partition. Compute point estimators of moments and of expected values of measurable functions of the state variable.

3. Obtain confidence limits for the estimates of the pdf, cdf, moments and expected values of measurable functions.

Remark 9.1 *Step 2 may also include a sensitivity analysis of the estimates. Thus, we may vary the simulated total time t, and/or the state-space partition norm size (fixed bin size, defined below in Subsection 9.4.1), to ensure that estimates remain within preassigned tolerances.*

In addition to the three main steps, we also characterize the steady-state pdf and cdf according to continuity, boundedness, convexity, differentiability, etc., by utilizing sample-path properties for the model. For example, in $M_\lambda/G/1$ and in $G/M_\mu/1$ queues, the steady-state pdf's of wait are bounded by λ and μ respectively (Propositions 3.5, 5.9).

I have carried out numerous LCE computational experiments using the procedure described herein, as well as other LCE procedures (e.g., [13], [21], [22], [32]). These experiments have detected all pdf discontinuities and intervals of convexity or concavity in benchmark models, where the pdf properties are known. For example, an M/Discrete/1 queue may serve as a benchmark. Proposition 3.9 specifies continuity/discontinuity properties of the pdf of wait. We can also apply LCE to estimate the pdf of wait in *variants* of M/Discrete/1 with state dependencies, etc., in which analytical results are tedious to obtain, or are not available.

9.2 Theoretical Basis for LC Estimation

LCE is based on level crossing theorems. Consider M/G/1. Theorem 1.1 implies that virtual-wait sample-path level-crossing time averages converge to the steady-state pdf of wait (*a.s.*) as time $t \to \infty$ (Subsection 9.2.2 below). This implies that time averages computed from a simulated sample path over a long simulated time t, should approximate the pdf accurately for all state-space values up to the maximum state-space level attained during $[0, t]$, say χ_t. Thus, the state-space interval $[0, \chi_t]$ will contain an increasing measure of the total probability as t increases (Subsection 9.2.4). The measure will grow to 1 as $t \to \infty$.

Remark 9.2 *The LCE method described here is one of several LC estimation methods. I have developed a version of LCE based on Theorems 3.2, 3.3 and related theorems, for estimating* ***transient distributions of state variables*** *(e.g., Remark 3.6). (I have discussed this technique at several conferences, e.g., P. H. Brill (1982), "System Point Monte Carlo Simulation of Time Dependent Probability Distributions of Waiting Times in Queues", TIMS/ORSA National Meeting, Chicago, April.)*

9.2.1 Boundedness of Steady-state PDF

A bound on the steady-state pdf of the virtual wait in $M_\lambda/G/1$ queues is given in Proposition 3.5, and on the steady-state arrival-point pdf of wait in $G/M_\mu/1$ queues in Proposition 5.9. In $M_\lambda/G/1$, $f(x) < \lambda, x > 0$. In $G/M_\mu/1$, $f_\iota(x) < \mu, x > 0$. Recall that $\mathcal{D}_t(x)$, $\mathcal{U}_t(x)$ are the numbers of SP down- and upcrossings of level x during $(0,t]$ respectively. Boundedness implies that for a *typical* sample path in $M_\lambda/G/1$,

$$f(x) = \lim_{t\to\infty} \frac{\mathcal{D}_t(x)}{t} < \lambda, x \geq 0.$$

In $G/M_\mu/1$,

$$f_\iota(x) = \lim_{t\to\infty} \frac{\mathcal{U}_t(x)}{t} < \mu, x > 0.$$

Similarly, we can develop bounds on $f(x)$ for other models, e.g., for M/M/c, G/M/c, etc. In $M_\lambda/G/r(\cdot)$ dams, boundedness follows from integral equation (6.18) for the steady-state pdf of content $f(x)$. If the efflux rate satisfies $r(x) > m > 0$, $x > 0$, then $f(x) < \frac{\lambda}{m}, x > 0$.

9.2.2 Role of Level Crossing Theorems in LCE

Consider $M_\lambda/G/1$. A sample path of the virtual wait is diagrammed in figures 3.4 and 9.1. Let $F(x)$, $f(x)$ be the steady-state cdf and pdf of wait respectively. Let $P_0 = F(0)$. Theorem 1.1 asserts

$$\lim_{t\to\infty} \frac{\mathcal{D}_t(x)}{t} = f(x), x \geq 0, \ \lim_{t\to\infty} \frac{\mathcal{D}_t(0)}{t} = f(0) = \lambda P_0 \ (a.s.)$$

(recall that $f(0) \equiv f(0^+)$). Hence, given $\varepsilon > 0$, for each $x > 0 \ \exists \ t_{x\varepsilon}$ such that

$$t > t_{x\varepsilon} \implies \left| \frac{\mathcal{D}_t(x)}{t} - f(x) \right| < \varepsilon f(x) \ (a.s.), \qquad (9.1)$$

since $f(x)$ is bounded, i.e., $0 < f(x) < \lambda < \infty, x \geq 0$ (Subsection 9.2.1). Also $\exists\ t_{0\varepsilon}$ such that

$$t > t_{0\varepsilon} \implies \left| \frac{\mathcal{D}_t(0)}{\lambda t} - P_0 \right| < \varepsilon P_0. \tag{9.2}$$

Choose an arbitrary "small" $\delta, 0 < \delta << 1$. Let W_q denote the steady-state queue wait. Define $z_\delta > 0$ by $P(W_q > z_\delta) = \delta$. Then δ is the probability of the right tail of the distribution of W_q, i.e., on the interval (z_δ, ∞). Thus

$$1 - F(z_\delta) \equiv \int_{y=z_\delta}^{\infty} f(y)dy = \delta. \tag{9.3}$$

Suppose we could determine (finite) $t_\delta^* = \max_x\{t_{x\varepsilon}|x \in [0, z_\delta)\}$, where $t_{x\varepsilon}, x > 0$ is defined in (9.1) and $t_{0\varepsilon}$ is defined in (9.2). Then

$$\begin{aligned} & t > t_\delta^* \implies \left| \tfrac{\mathcal{D}_t(x)}{t} - f(x) \right| < \varepsilon f(x) \text{ for all } x \in (0, z_\delta)\ (a.s.), \\ & \left| \tfrac{\mathcal{D}_t(0)}{\lambda t} - P_0 \right| < \varepsilon P_0\ (a.s.). \end{aligned} \tag{9.4}$$

By the normalizing condition $P_0 + \int_{x=0}^{\infty} f(x)dx = 1$, we have

$$\begin{aligned} P_0 + \int_{x=0}^{z_\delta} f(x)dx &= 1 - \int_{x=z_\delta}^{\infty} f(x)dx \\ &= 1 - \delta > 0. \end{aligned} \tag{9.5}$$

Summing over all $x \in [0, \infty)$ in (9.4) and using (9.5), yields

$$\begin{aligned} t > t_\delta^* \implies & \left| \tfrac{\mathcal{D}_t(0)}{\lambda t} - P_0 \right| + \int_{x=0}^{z_\delta} \left| \tfrac{\mathcal{D}_t(x)}{t} - f(x) \right| dx < \varepsilon P_0 + \varepsilon \int_{x=0}^{z_\delta} f(x)dx \\ & = \varepsilon(1 - \delta) < \varepsilon\ (a.s.). \end{aligned} \tag{9.6}$$

Let $\left\{\widehat{P}_0; \widehat{f}(x)\right\}$ denote the estimate of $\{P_0; f(x)\}$. We assume that a sample path over a fixed simulated time interval $[0, t]$ is used to compute $\left\{\widehat{P}_0; \widehat{f}(x)\right\}$. (We omit subscript "$t$" in the symbols $\widehat{P}_0$ and $\widehat{f}(x)$, in order to distinguish $\widehat{P}_0$, $\widehat{f}(x)$ from estimators "$\widehat{P}_{0t}$, $\widehat{f}_t(x)$" for the *transient* pdf of wait, which we use outside this monograph.)

Assume we use the "natural" estimator based on the sample path, viz., $\widehat{P}_0 = \frac{\mathcal{D}_t(0)}{\lambda t}$, $\widehat{f}(x) = \frac{\mathcal{D}_t(x)}{t}, t > t_\delta^*$. Then (9.6) implies that the *total absolute error* of $\left\{\widehat{P}_0; \widehat{f}(x)\right\}$ in estimating $\{P_0; f(x), x \in (0, z_\delta)\}$ is less than ε.

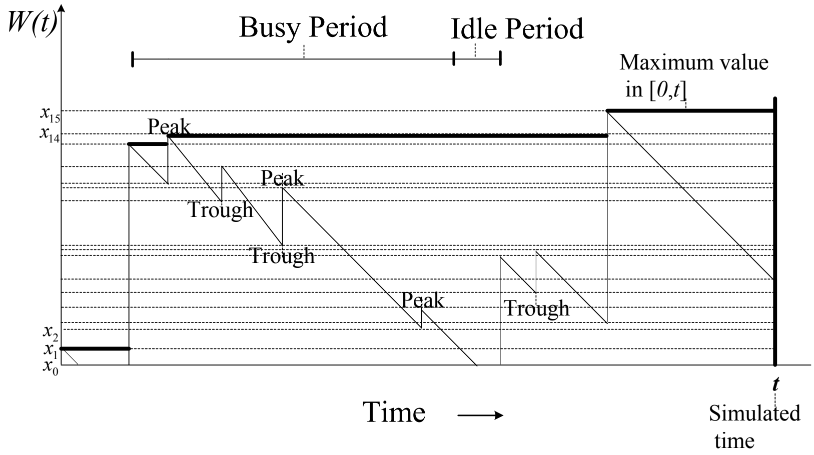

Figure 9.1: Sample path of virtual wait $\{W(t)\}$ in M/G/1. Shows peaks $\{W_n + S_n\}$, troughs $\{W_n\}$ and state-space partition $0 = x_0 < x_1 < x_2 \cdots < x_{15}$ in time interval $(0, t)$. Also shows maximum sample-path value attained in $[0, t]$.

Assume $\widehat{f}(x) = 0, x > z_\delta$. Then $t > t^*_\delta$ implies that the total absolute error in $\widehat{f}(x), x > z_\delta$ is equal to δ, i.e.,

$$t > t^*_\delta \implies \int_{x=z_\delta}^{\infty} \left|\widehat{f}(x) - f(x)\right| dx = \int_{x=z_\delta}^{\infty} f(x)dx = \delta, (a.s.). \quad (9.7)$$

Suppose we could simulate a sample path over a sufficiently large time interval $(0, t), t > t^*_\delta$. Statements (9.6) and (9.7) imply that the total absolute error would be

$$\begin{aligned} &\left|\widehat{P}_0 - P_0\right| + \int_{x=0}^{\infty} \left|\widehat{f}(x) - f(x)\right| dx \\ &\quad = \left|\tfrac{\mathcal{D}_t(0)}{\lambda t} - P_0\right| + \int_{x=0}^{\infty} \left|\tfrac{\mathcal{D}_t(x)}{t} - f(x)\right| dx < \varepsilon + \delta, (a.s.). \end{aligned} \quad (9.8)$$

In principle we can choose ε and δ arbitrarily small. Then we can simulate a sample path over a long simulated time $t > t^*_\delta$ and ensure that the total absolute error of $\left\{\widehat{P}_0; \widehat{f}(x)\right\}$ in estimating $\{P_0; f(x), x > 0\}$ is arbitrarily small. This procedure would amount to *computation* of the entire pdf $\{P_0; f(x), x > 0\}$ within a preassigned tolerance. The total error on $[0, z_\delta)$ is less than ε. The total error on (z_δ, ∞) is equal to δ.

9.2.3 Natural Partition of State Space

We illustrate a natural partition of the state space by means of an example.

Example 9.1 *Consider a* ***sample path of the virtual wait*** $\{W(t)\}$ ***in an M/G/1 queue*** *(Fig. 9.1). The state space is* $\boldsymbol{S} = [0, \infty)$*. For* ***fixed*** $x \in \boldsymbol{S}$, $\{\mathcal{D}_t(x)\}$ *is a counting process. For* ***fixed*** $t > 0$, $\mathcal{D}_t(x)$ *is a step function on* $\boldsymbol{S}$*. The jumps in the step function occur at the peaks* $\{W_n + S_n\}$ *and troughs* $\{W_n\}$*, where* W_n, $S_n, n = 1, 2, ...$ *are the customer waits and service times respectively. In Fig. 9.1 level* $W(0)$ *is a peak and level* $W(t)$ *is a trough. We* ***merge the peaks and troughs*** *to form a state-space partition*

$$\{x_i\} = W(0) \cup \{W_n\} \cup \{W_n + S_n\} \cup W(t),$$

arranged ***in ascending order of magnitude*** *in* $\boldsymbol{S}$,

$$0 = x_0 < x_1 < \cdots < x_{M(t)} < \infty.$$

The first partition point x_0 *corresponds to all troughs of* $W(0) \cup \{W_n\} \cup W(t)$ *such that the ordinate is* 0*. The second partition point is*

$$x_1 = \min_n \left\{ \begin{array}{c} W(0) \cup \{W_n\} \cup \{W_n + S_n\} \cup W(t) \\ \setminus \{troughs\ = 0\}. \end{array} \right\}$$

That is, $\min_n \{\cdot\}$ *excludes the troughs corresponding to* x_0 $(= 0)$*. The* j^{th} *partition point* x_j *is obtained similarly, excluding those troughs and/or peaks corresponding to* $\{x_0, x_1, ..., x_{j-1}\}$*. The number of subintervals of partition* $\{x_i\}$ *is* $M(t) \leq 2N_a(t)$*, where* $N_a(t)$ *is the number of arrivals during* $(0, t)$*. In Fig. 9.1,* $N_a(t) = 8$, $M(t) = 15$*.*

Note that t is fixed. Let

$$\mathcal{D}_t(x) = d_i, x \in [x_i, x_{i+1}), i = 0, 1, ..., M(t),$$

where $d_i \geq 0$ *is a constant. Then*

$$\frac{\mathcal{D}_t(x)}{t} = \frac{d_i}{t}, x \in [x_i, x_{i+1}), i = 0, 1, ..., M(t)$$

is a step function of $x \in \boldsymbol{S}$*. Suppose we can determine* t^*_δ *as in (9.4). Then, from (9.8)*

$$t > t^*_\delta \implies \left| \frac{d_0}{\lambda t} - P_0 \right| + \sum_{i=0}^{M(t)} \int_{x=x_i}^{x_{i+1}} \left| \frac{d_i}{t} - f(x) \right| dx < \varepsilon + \delta, (a.s.). \quad (9.9)$$

In Fig. 9.1

$$d_0 = 2, d_1 = 1, d_2 = 2, d_3 = 3, ..., d_{14} = 1, d_{15} = 0.$$

The recursion (9.11) below may simplify computation of $\{d_i\}$ using a computer program.

$$d_{i+1} = \begin{cases} d_i + 1 \text{ if } x_{i+1} \text{ is a trough,} \\ d_i - 1 \text{ if } x_{i+1} \text{ is a peak, } i = 0, ..., M(t) - 1, \end{cases} \quad (9.10)$$

$$d_{M(t)+1} = 0. \quad (9.11)$$

The sub-interval lengths of the partition $\{x_i\}$ are

$$\{x_{i+1} - x_i\}, i = 0, ..., M(t).$$

These lengths vary in a natural way (variable bin sizes).

9.2.4 Ladder Points and LCE Estimates

For the virtual wait, let χ_t denote the maximum sample-path level in $\boldsymbol{S}$ attained during $[0, t]$. For fixed t, $\chi_t = x_{M(t)}$, the greatest finite point of partition $\{x_i\}$. As t increases $\{\chi_t, t \geq 0\}$ is a non-decreasing step function with non-homogeneous inter-jump times. A sample path of $\{\chi_t\}$ is a non-decreasing right-continuous step function with upward jumps at embedded arrival instants $\tau_{ln}, n = 1, 2, ...$. The associated service-time jumps end strictly above $\chi_{\tau_{l(n-1)}} = \chi_{\tau_{ln}^-}$ (Fig. 9.1). Thus $\frac{d}{dt}\chi_t = 0, \tau_{l(n-1)} < t < \tau_{ln}, n = 0, 1, 2, ...,$ where $\tau_{l0} \equiv 0$. The increase in $\{\chi_t\}$ at arrival instant τ_{ln} is equal $\chi_{\tau_{ln}} - \chi_{\tau_{ln}^-}$ = excess service time above level $\chi_{\tau_{ln}^-}$. Random variables $\chi_{\tau_{ln}}, n = 1, 2, ...$ are ordinates of the "strict ascending ladder points" $\left\{\left(\tau_{ln}, \chi_{\tau_{ln}}\right)\right\}$ of the virtual wait process $\{W(s), s \geq 0\}$. The points $\left(\tau_{ln}, \chi_{\tau_{ln}}\right) \in \boldsymbol{T} \times \boldsymbol{S}, n = 1, 2, ...,$ are analogous to strict ascending ladder points for a random walk [56]. The LCE estimate of the pdf of wait $f(x), x \geq \chi_t$ is $\widehat{f}(x) = 0$. The number of strict ascending ladder points $\left(\tau_{ln}, \chi_{\tau_{ln}}\right)$ in time interval $[0, t]$ form a counting process as t increases. If the sample-path jump sizes are distributed as E_μ, then the n^{th} ascending ladder point is distributed as an Erlang-(n, μ) random variable. (We mention these ladder points because of their importance in the overall method. However, we shall not discuss them further in this introductory chapter on LCE.)

9.3 Computer Program for LCE

An LCE computer program can utilize different logical designs. Suppose we wish to estimate the steady-state pdf of wait. Assume that for fixed $t > 0$, we can simulate a sample path of the virtual wait over a simulated time interval $[0, t]$. We count the number of SP downcrossings of each state-space level $x \in \boldsymbol{S}$ during $[0, t]$. This is easier than it may seem at first glance, due to the step-function structure of $\mathcal{D}_t(x), x > 0$, for fixed $t > 0$.

9.3.1 Designs for Computer Program

We discuss two feasible designs for an LCE computer program.

State-space Partition with Variable Subintervals

One design is based directly on the discussion in Section 9.2, using partition $\{x_i\}$ having *variable* sub-interval lengths $\Delta_i = x_{i+1} - x_i$. The Δ_i's occur naturally in the simulated sample path (Fig. 9.1).

The embedded processes $\{W_n\}$ and $\{W_n + S_n\}$ are Markov processes. Thus, in a sample path the union $\{W_n\} \cup \{W_n + S_n\}$ of peaks and troughs, is everywhere dense in $\boldsymbol{S} = [0, \infty)$ as $t \to \infty$ (a.s.). That is, the entire state space will be covered *eventually* by the ordinates of the peaks and troughs.

An advantage of this design is that it takes every sample-path peak and trough during $[0, t]$ into consideration. In theory, any computed estimator will utilize all the information available in the sample path.

A possible disadvantage of this design is from a programming point of view. The points in $\{x_i\}$ become more dense as the sample path is generated over time. The Δ_i's in the region of higher probability, will become extremely small as simulated time t increases. The partition $\{x_i\}$ will contain on the order of $2N_a(t)$ distinct points, where $N_a(t)$ is the number of arrivals in time t (a peak and trough correspond to each arrival). If t is large, $N_a(t)$ will be large. Many Δ_i's will become less than a practical resolution size required for the estimation of the pdf of wait.

State-space Partition with Fixed Subintervals

A second design is to start with $x_0 = 0$ and a *fixed* partition norm size Δ. Thus $x_i = x_{i-1} + \Delta, i = 1, \ldots$. The program updates the count of SP downcrossings of each state-space level $x_i, i = 0, ..., M(t)$ as the sample path evolves over time interval $[0, t]$. We compute the maximum peak χ_s during $[0, s]$ as we generate a sample path over time. The state-space partition $\{x_i\}$ covers the state-space interval $[0, \chi_t]$. Generally the time intervals between successive ladder points of $\{W(s)\}$ increase. That is, $\tau_{l(n+1)} - \tau_{ln} > \tau_{ln} - \tau_{l(n-1)}$, after some $n \geq$ some integer $\in \{1, 2, ...\}$. Estimates of $\{P_0; f(x), x \geq 0\}$ that are computed using a fixed-Δ partition, very closely approximate estimates using a partition with variable Δ_i's, for most practical purposes. Moreover, the fixed-Δ design is easy to program.

9.4 LCE for M/G/1 Queue

This section describes LCE for the steady-state pdf of wait and related quantities for M/G/1 queues. A numerical example using this method is given in the next section. Let $\{W(t), t \geq 0\}$ denote the virtual wait. Without loss of generality assume $W(0) = 0$. The state space is $\boldsymbol{S} = [0, \infty)$. Let the arrival rate be λ. Let $S_n, n = 1, 2, ...$ denote the service times, which may be state dependent. Assume the parameters are such that the queue is stable, e.g., $\lambda E(S) < 1$. Assume $W(t) \underset{dist}{\rightarrow} W$ as $t \rightarrow \infty$ (weak convergence). Denote the cdf and pdf of W by $F(x), x \geq 0$, and $\{P_0; f(x), x > 0\}$ respectively. Here $P_0 = F(0) > 0$ and $f(x) = \frac{d}{dx}F(x)$ wherever the derivative exists. Denote the n^{th} moment of W by $m_n = \int_{x=0}^{\infty} x^n f(x) dx, n = 1, 2, ...$. Let $\psi(W)$ denote an arbitrary measurable function of W.

We use a computer program based on the fixed-norm size design of Subsection 9.3.1 to compute the estimators (fixed Δ). Definition 9.2 below incorporates minor modifications of the "basic" estimators, that retain theoretical consistency. The modified estimators are satisfactory for practical purposes.

9.4.1 Quantities Computed from a Sample Path

Fix finite time $t > 0$. Consider a simulated sample path of the virtual wait $\{W(s), 0 \leq s \leq t\}$. The SP is the leading point of a sample path when thought of as evolving over time (Section 2.3). In the fixed-Δ design, partition $\{x_i\}$ has a constant norm Δ. Define the following quantities.

Definition 9.1

$\mathcal{D}_t(x)$	number of SP downcrossings of level $x, x \geq 0$ during $[0,t]$,
χ_t	$\max\{W(s)\vert 0 \leq s \leq t\}$,
Δ	norm of preassigned uniform partition on $\boldsymbol{S}$,
ν	$\max\{n\vert n\Delta \leq \chi_t, n = 0,1,2,...\}$,
x_j	$x_j = j\Delta,\ j = 0,...,\nu+1;\ x_{\nu+2} \equiv \infty$,
$\{x_j\}$	preassigned uniform partition on $[0,(\nu+1)\Delta]$ with norm Δ,
$\boldsymbol{J}_j$	interval $\boldsymbol{J}_j = [x_j, x_{j+1}), j = 0,1,...,\nu$,
d_j	$\mathcal{D}_t(x_j), j = 0,...,\nu+1$,
A_t	$A_t = \frac{1}{t}\left(\frac{d_0}{\lambda} + \Delta\sum_{j=0}^{\nu} d_j\right) = \frac{1}{t}\left(\frac{\mathcal{D}_t(0)}{\lambda} + \Delta\sum_{j=0}^{\nu}\mathcal{D}_t(x_j)\right)$.

Remark 9.3 *Definition 9.1 retains the argument "t" for $\mathcal{D}_t(x)$, χ_t and A_t. Both ν and d_j also depend on t. We omit subscript t for ν to simplify notation since ν often appears as a subscript or index. We omit the subscript t for d_j for computer-programming purposes. The quantities Δ, x_j and $\boldsymbol{J}_j$ are defined in the state space, and are generally independent of t. (However, we may vary t and Δ jointly for a* ***sensitivity analysis*** *in order to increase accuracy.)*

Remark 9.4 *Note the inequality $x_\nu = \nu\Delta \leq \chi_t < (\nu+1)\Delta = x_{\nu+1}$. Also, for every $x \geq x_{\nu+1}$, $\mathcal{D}_t(x) = 0$, i.e., $d_{\nu+1} \equiv 0$.*

The term A_t is such that $A_t > 0, t > \tau_1$ (τ_1 = first arrival instant).

Proposition 9.1

$$\lim_{\substack{t\to\infty \\ \Delta\downarrow 0}} A_t = 1 \ (a.s.). \tag{9.12}$$

Proof. We *sketch* a proof of (9.12) in three steps.
(1) $P_0 + \int_{x=0}^{\infty} f(x)dx = 1$ (normalizing condition).
(2)For the first term of A_t we have

$$\lim_{t\to\infty}\frac{d_0}{t\lambda} = \lim_{t\to\infty}\frac{\mathcal{D}_t(0)}{t\lambda} = \frac{f(0)}{\lambda} = \frac{\lambda P_0}{\lambda} = P_0 \ (a.s.). \tag{9.13}$$

(3) First assume the virtual wait $W(t) \leq K, t > 0$ for some upper bound $K > 0$. Evidence for the existence of such queueing models is demonstrated in Section 3.14. Then $\chi_t \leq K$ for all $t > 0$. Also $\nu \leq \left[\frac{K}{\Delta}\right]$ where $[z]$ denotes the greatest integer $\leq z, z \in \boldsymbol{R}$. Thus ν is finite and positive

for all values of t. For the second term of A_t we have

$$\begin{aligned}\lim_{\substack{t\to\infty\\ \Delta\downarrow 0}}\left(\Delta\sum_{j=0}^{\nu}\frac{\mathcal{D}_t(x_j)}{t}\right) &= \lim_{\Delta\downarrow 0}\left(\lim_{t\to\infty}\left(\sum_{j=0}^{\nu}\frac{\mathcal{D}_t(x_j)}{t}\Delta\right)\right)\\ &= \lim_{\Delta\downarrow 0}\left(\sum_{j=0}^{\nu}\left(\lim_{t\to\infty}\frac{\mathcal{D}_t(x_j)}{t}\right)\Delta\right)\\ &= \lim_{\Delta\downarrow 0}\left(\sum_{j=0}^{\upsilon}f(x_j)\Delta\right)\\ &= \int_{x=0}^{K}f(x)dx\ (a.s.),\end{aligned} \tag{9.14}$$

since $\lim_{t\to\infty}\frac{\mathcal{D}_t(x_j)}{t} = ff(x_j,(a.s.)$ by Theorem 1.1. In the last equality of (9.14), the expression $\sum_{j=0}^{\upsilon}f(x_j)\Delta$ is a Riemann sum. It converges to the definite integral $\int_{x=0}^{K}f(x)dx$ as $\Delta\downarrow 0$, since $K-\Delta<x_\nu\le K$.

The result (9.14) holds for every $K>0$. If $K\to\infty$, then

$$\lim_{\substack{t\to\infty\\ \Delta\downarrow 0}}\left(\Delta\sum_{j=0}^{\nu}\frac{\mathcal{D}_t(x_j)}{t}\right) = \int_{x=0}^{\infty}f(x)dx\ (a.s.). \tag{9.15}$$

Equation (9.12) then follows from (9.13), (9.14) and the normalizing condition. ∎

9.4.2 Point Estimators

For fixed $t>0$ let

$$\widehat{f}(x), x>0,\ \widehat{F}(x), x\ge 0,\ \widehat{P}_0,\ \widehat{m}_n, n=1,2,...,\ \widehat{E}\left(\psi(W)\right),$$

denote *point estimators* of the corresponding quantities under the circumflexes. These point estimators are specified in Definition 9.2 below. Assume a "small" norm Δ is given ($\Delta =$ "bin size").

Definition 9.2 *For each fixed $t>0$, the* ***point estimators*** *are (see Definition 9.1):*

1. $\widehat{f}(x)\equiv\frac{d_j}{tA_t}=\frac{\mathcal{D}_t(x_j)}{tA_t}$, $x\in\boldsymbol{J}_j$, $j=0,...,\nu$,
2. $\widehat{P}_0=\frac{d_0}{\lambda tA_t}=\frac{\mathcal{D}_t(0)}{\lambda tA_t}$,
3. $\widehat{F}(x)=\widehat{P}_0+\Delta\sum_{i=0}^{j-1}\widehat{f}(x_i)+(x-x_j)\widehat{f}(x_j), x\in\boldsymbol{J}_j$, $j=0,...,\nu$,
4. $\widehat{m}_n=\Delta\sum_{i=0}^{\nu}x_i^n\widehat{f}(x_i)$,
5. $\widehat{E}(\psi(W))=\psi(0)\widehat{P}_0+\Delta\sum_{i=0}^{\nu}\psi(x_i)\widehat{f}(x_i)$.

Estimator of Laplace Stieltjes Transform

In Definition 9.2, set $\psi(W) = e^{-sW}, s > 0$. Then $E(\psi(W))$ is the Laplace-Stieltjes transform (LST) of W, namely

$$E(e^{-sW}) = \int_{x=0}^{\infty} e^{-sx} dF(x)dx.$$

The estimator of $E(e^{-sW})$ is

$$\widehat{E}(e^{-sW}) = \widehat{P}_0 + \Delta \sum_{i=0}^{\nu} e^{-sx_i} \widehat{f}(x_i), s > 0.$$

We may compute $\widehat{E}(e^{-sW})$ for $s = 0, h, 2h, ...$, where h is a small positive constant. Thus we can plot $\widehat{E}(e^{-sW})$ vs. s. Then we may substitute $\widehat{E}(e^{-sW})$ for the LST in formulas where it appears.

The value of Δ may be adjusted after a computer run, to increase accuracy or investigate an estimator's convergence rate with respect to Δ.

Remark 9.5 *In Definition 9.2 the quantities under the symbol "$\widehat{\ }$" omit the argument t, to distinguish them from* estimators of transient distributions. *(The latter estimators are not included in this monograph, but are discussed briefly in Remark 9.2 and remarks referred to therein.) The quantities also omit the argument Δ for notational simplicity.*

Remark 9.6 *For fixed $t > 0$, $\widehat{f}(x)$ is a step function of $x \in \cup_{j=0}^{\nu+1} \boldsymbol{J}_j$ having constant values on the intervals $\{\boldsymbol{J}_j\}$. The **term** A_t **is a normalizing constant** which guarantees that $\widehat{F}(x) = 1, x \geq x_{\nu+1}$, for any $t > 0$. Also, $\widehat{f}(x) = 0, x \in J_{\nu+1}$.*

Consistency of Estimators

An estimator $\widehat{\varphi}_t$ of quantity ϕ is *consistent* if $\lim_{t\to\infty} P(\widehat{\varphi}_t = \phi) = 1$. An estimator $\widehat{\varphi}_t$ of ϕ is *strongly consistent* if $P(\lim_{t\to\infty} \widehat{\varphi}_t = \phi) = 1$; equivalently $\lim_{t\to\infty} \widehat{\varphi}_t = \phi$ $(a.s.)$.

The estimators

$$\widehat{f}(x), x > 0, \ \widehat{F}(x), x \geq 0, \ \widehat{P}_0, \ \widehat{m}_n, n = 1, 2, ..., \ \widehat{E}(\psi(W))$$

in Definition 9.2 are strongly consistent. The gist of the proofs utilizes level crossing theorems discussed in Subsection 9.2.2.

Proposition 9.2

1. (a). For each x_j, $\widehat{f}(x_j)$ is strongly consistent.
   (b). For each fixed $x \neq x_j$ $\lim_{\Delta\downarrow 0} \widehat{f}(x)$ is strongly consistent.

2. (a). For each fixed $t > 0$, $0 \leq \widehat{P}_0 \leq 1$.
   (b). $\widehat{P}_0$ is strongly consistent.

3. (a). For each fixed $t > 0$, $0 \leq \widehat{F}(x) \leq 1, x \geq 0$, and $\widehat{F}(\infty) = 1$.
   (b). For each fixed $x \geq 0$, $\lim_{\Delta\downarrow 0} \widehat{F}(x)$ is strongly consistent.

4. $\lim_{\Delta\downarrow 0} m_n$ is strongly consistent, $n = 1, 2, ...$

5. $\lim_{\Delta\downarrow 0} \widehat{E}(\psi(W))$ is strongly consistent.

Proof. 1(a).

$$\lim_{t\to\infty} \widehat{f}(x_j) = \lim_{t\to\infty} \frac{d_j}{tA_t} = \lim_{t\to\infty} \frac{D_t(x_j)}{tA_t} \underset{a.s.}{=} \frac{f(x_j)}{\lim_{t\to\infty} A_t} = f(x_j),$$

since $\lim_{t\to\infty} A_t = 1$ by formula (9.12).
1(b). Fix $t > 0$. Fix $x \in \boldsymbol{S}$. Let $\delta > 0$ be given. We can make the fixed norm size Δ arbitrarily small. There exists $\Delta > 0$ and x_j in the fixed norm partition such that $0 < x - x_j < \Delta$. Also we have $x - x_j < \Delta \implies |f(x) - f(x_j)| < \delta$, since $f(\cdot)$ is defined to be right continuous. Note that $\widehat{f}(x) \equiv \widehat{f}(x_j)$. Now let $t > t_{x_j\varepsilon}$, such that $t > t_{x_j\varepsilon}$ $\implies \left|\widehat{f}(x_j) - f(x_j)\right| < \varepsilon$. (Such $t_{x_j\varepsilon}$ exists by 1(a).) Hence for Δ sufficiently small and $t > t_{x_j\varepsilon}$,

$$\begin{aligned}\left|f(x) - \widehat{f}(x)\right| &= \left|f(x) - \widehat{f}(x_j)\right| = \left|f(x) - f(x_j) + f(x_j) - \widehat{f}(x_j)\right| \\ &\leq |f(x) - f(x_j)| + \left|f(x_j) - \widehat{f}(x_j)\right| \\ &< \delta + \varepsilon.\end{aligned}$$

As $t \to \infty$, $\left|f(x_j) - \widehat{f}(x_j)\right| \downarrow 0$. Thus $\left|f(x) - \widehat{f}(x)\right| < \delta$, implying that $\lim_{t\to\infty} \left(\lim_{\Delta\downarrow 0} \widehat{f}(x)\right) = f(x)$ $(a.s.)$.
2 (a). For fixed t, $\mathcal{D}_t(0) \geq 0$. hence

$$0 \leq \frac{D_t(0)}{\lambda t A_t} = \widehat{P}_0 = \frac{\frac{D_t(0)}{\lambda t}}{\left(\frac{\mathcal{D}_t(0)}{\lambda t} + \Delta \sum_{j=0}^{\nu} \frac{\mathcal{D}_t(x_j)}{t}\right)} \leq 1.$$

2 (b). For a stable queue, state $\{0\}$ is positive recurrent. Hence

$$\lim_{t\to\infty} \widehat{P}_0 = \lim_{t\to\infty} \frac{D_t(0)}{\lambda t A_t} = \frac{f(0)}{\lambda \lim_{t\to\infty} A_t} = \frac{\lambda P_0}{\lambda \cdot 1} = P_0 \ (a.s.).$$

3(a). This follows because the denominators of $\widehat{P}_0$ and $\widehat{f}(x_j), j = 1, ..., \upsilon$ contain the normalizing factor $A_t = \widehat{P}_0 + \Delta \sum_{j=0}^{\nu} \widehat{f}(x_j)$, which exceeds or equals the value of the total numerator.

3(b). This follows because $\lim_{t\to\infty} \widehat{P}_0 = P_0$. Also,

$$\begin{aligned} &\lim_{t\to\infty}\left(\lim_{\Delta\downarrow 0}\left(\Delta\sum_{i=0}^{j-1}\widehat{f}(x_i) + (x - x_j)\widehat{f}(x_j)\right)\right) \\ &= \lim_{\Delta\downarrow 0}\left(\lim_{t\to\infty}\left(\Delta\sum_{i=0}^{j-1}\widehat{f}(x_i) + (x - x_j)\widehat{f}(x_j)\right)\right) \\ &= \lim_{\Delta\downarrow 0}\left(\Delta\sum_{i=0}^{j-1} f(x_i) + (x - x_j) f(x_j)\right), \end{aligned}$$

since for fixed Δ, the values of the partition points $\{x_j\}$ are fixed (thus interchange of limits permitted). Hence

$$\lim_{t\to\infty}\left(\lim_{\Delta\downarrow 0}\widehat{F}(x)\right) = P_0 + \int_{y=0}^{x} f(x)dx = F(x) \ (a.s.).$$

4, 5. These follow using similar reasoning as in the proof of 3(b). ■

Remark 9.7 *In the estimation procedure of this section, we must make two important preset choices: (1) the value of simulated time t; (2) the value of Δ. Since t is finite and $\Delta > 0$, the estimators in Proposition 9.2 are **approximately** consistent. We consider the partition norm Δ to be sufficiently "small" if the following holds. We repeat the estimation procedure with a smaller Δ, say $\frac{\Delta}{10}$ or $\frac{\Delta}{100}$, etc.; this leaves the estimates within a preassigned tolerance.*

Similarly, we consider t to be sufficiently "large" if repeating the procedure with a larger t, say $10t$ or $100t$, etc., leaves the estimates within a preassigned tolerance (compare with Cauchy condition for convergence of series). The joint choice of (t, Δ) poses an interesting exercise. Experimentation may be informative. A discussion is given in [20]. Computational experimentation has shown that the estimation procedure is robust over a wide range of (t, Δ) values. With the advent of fast computer processors, fast random access memories, fast storage drives, etc., *a sensitivity analysis can be carried out very efficiently. Computer speeds will increase in the future. Sensitivity analyses of the estimates with respect to* (t, Δ) *will become ever more efficient.*

9.4.3 Statistical Properties and Confidence Limits

For an arbitrary sample path $W(s), 0 \leq s \leq t$, define the following quantities.

d_x	time between successive SP downcrossings of level x,
$Var(d_x)$	variance of d_x,
$\sqrt{Var(d_x)}$	standard deviation of d_x,
b_x	time SP is in state-space interval $[0, x]$ during d_x = sojourn time at or below level x,
$\mathcal{A}((W(\cdot))^n)$	area under the sample path of $(W(s))^n$ during a busy cycle of $W(s), 0 \leq s \leq t$,
λP_0	long-run rate at which arrivals initiate busy periods.

Asymptotic Normality of Estimators

The following proposition describes the asymptotic normality of the estimators. Let $N(0,1)$ denote a standard normal random variate with mean 0 and variance 1. Let $Var(Z)$ denote the variance of a generic random variable Z.

Proposition 9.3

1. For every $x_j, j = 0, ..., \nu$

$$\frac{\widehat{f}(x_j) - f(x_j)}{Var(d_{x_j})\left((tA_t)^{-1}\left(f(x_j)\right)^3\right)^{\frac{1}{2}}} \to N(0,1) \text{ as } t \to \infty.$$

2.

$$\frac{\widehat{P}_0 - P_0}{Var(d_0)\left((tA_t)^{-1}\lambda\left(P_0\right)^3\right)^{\frac{1}{2}}} \to N(0,1) \text{ as } t \to \infty.$$

3. If Δ is small then for every $x \geq 0$ approximately

$$\frac{\widehat{F}(x) - F(x)}{\left((tA_t)^{-1} Var(b_x - b_0) f(x)\right)^{\frac{1}{2}}} \to N(0,1) \text{ as } t \to \infty.$$

4. If Δ is small then approximately

$$\frac{\widehat{m}_n - m_n}{(t^{-1}Var(\mathcal{A}((W(\cdot))^n))\lambda P_0)^{\frac{1}{2}}} \to N(0,1) \text{ as } t \to \infty.$$

Proof. The proofs of statements 1 - 4 follow from the asymptotic normality of renewal processes (see e.g., [91] or [49]). This proposition is also discussed in Section 6 of [20], based on the same references. ■

Confidence Intervals for Estimators

Assume t is large and define $z_{\frac{\alpha}{2}}$ by $P(N(0,1) > z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$. The following $100(1-\alpha)\%$ confidence limits apply.

1. $f(x_j)$: $\quad \widehat{f}(x_j) \pm z_{\frac{\alpha}{2}} \cdot \widehat{Var}(d_{x_j}) \cdot \left((tA_t)^{-1}\widehat{f}(x_j)^3\right)^{\frac{1}{2}},$

2. P_0: $\quad \widehat{P}_0 \pm z_{\frac{\alpha}{2}} \cdot \widehat{Var}(d_0) \cdot \left((tA_t)^{-1}\lambda\left(\widehat{P}_0\right)^3\right)^{\frac{1}{2}},$

3. $F(x)$: $\quad \widehat{F}(x) \pm z_{\frac{\alpha}{2}} \cdot \left((tA_t)^{-1}\widehat{Var}(b_x - b_0)\widehat{f}(x_j)\right)^{\frac{1}{2}},$

4. m_n: $\quad \widehat{m}_n \pm z_{\frac{\alpha}{2}} \cdot \left(t^{-1}\widehat{Var}(\mathcal{A}((W(\cdot))^n))\lambda\widehat{P}_0\right)^{\frac{1}{2}}.$

Proof. The profs are based on Proposition 9.3. ■

9.5 LCE Example: M/M/1 with Reneging

We consider an $M_\lambda/M_\mu/1$ queue in which customers may renege from the waiting line, or wait and balk at start of service (Section 3.11, Subsection 3.11.7 and equations (3.166), (3.167)). Alternatively customers may wait and stay for complete service. We compare LCE estimates of the steady-state pdf, cdf and mean wait of stayers with the analytical solutions for the same quantities.

We assume customers that wait less than 1 time unit stay ("reach" the server) and get complete service. Customers that are required to wait ≥ 1 time unit to reach the server, renege from the waiting line or wait the full time and then balk at service. In the notation of Section

3.11 the *staying function* $\overline{R}(x), x \geq 0$ has the same form as in Fig. 3.21, i.e.,

$$\overline{R}(x) = \begin{cases} 1, 0 \leq x < 1, \\ 0, x \geq 1. \end{cases} \tag{9.16}$$

The arrival rate λ and service rates μ may be arbitrary positive numbers since the queue is stable for all values of λ, μ (Theorem 3.8). We arbitrarily set $\lambda = 1$, $\mu = 5$.

Analytical Solution

We obtain the analytical solution for the pdf of the wait of stayers $\{P_0; f(x), x > 0\}$ from the model equations

$$f(x) = \begin{cases} \lambda P_0 e^{-\mu x} + \lambda \int_{y=0}^{x} e^{-\mu(x-y)} f(y)dy, 0 < x < 1, \\ \lambda P_0 e^{-\mu x} + \lambda \int_{y=0}^{1} e^{-\mu(x-y)} f(y)dy, x \geq 1. \end{cases} \tag{9.17}$$

The solution of (9.17) is

$$f(x) = \begin{cases} \lambda P_0 e^{-(\mu-\lambda))x}, \ 0 < x < 1, \\ \lambda P_0 e^{\lambda} e^{-\mu x}, \ 1 \leq x < \infty. \end{cases} \tag{9.18}$$

We substitute (9.18) into the normalizing condition $P_0 + \int_{x=0}^{\infty} f(x)dx = 1$, yielding

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu-\lambda}(1 - e^{-(\mu-\lambda)}) + \frac{\lambda}{\mu} e^{-(\mu-\lambda)}}. \tag{9.19}$$

Substituting $\lambda = 1$, $\mu = 5$ in (9.19) and (9.18) results in (Fig. 9.3)

$$P_0 = 0.8006, \tag{9.20}$$

$$f(x) = \begin{cases} 0.8006 \cdot e^{-4.0x}, 0 < x < 1, \\ 2.1763 \cdot e^{-5.0x}, 1 \leq x < \infty. \end{cases} \tag{9.21}$$

From (9.21) the derivative is

$$f'(x) = \begin{cases} -3.2024 \cdot e^{-4.0x}, 0 < x < 1, \\ -10.8815 \cdot e^{-5.0x}, 1 \leq x < \infty. \end{cases}$$

The pdf $f(x)$ is continuous at $x = 1$. The derivative $f'(x)$ is discontinuous at $x = 1$. Thus $f'(1^-) = -0.058654$, $f'(1) = -0.073319$. The pdf is bounded above by the arrival rate λ, i.e.,

$$\max_{x \geq 0} f(x) = f(0) = 0.8006 < 1 = \lambda.$$

LC Estimation using $t = 3000, \Delta = 0.1$				
Estimated values			Analytical Values	
$\widehat{P}_0 = 0.7995$			$P_0 = .800587$	
x	$\widehat{f}(x)$	$\widehat{F}(x)$	$f(x)$	$F(x)$
0.1	.7995	.7995	.8006	.8006
0.2	.5265	.8652	.5366	.8666
0.3	.2447	.9395	.2411	.9403
0.4	.1602	.9591	.1616	.9603
0.5	.1142	.9734	1083	.9736
0.6	.0729	.9828	.0726	.9826
0.7	.0484	.9809	.0487	.9886
0.8	.0317	.9929	.0326	.9926
0.9	.0208	.9955	.0219	.9953
1.0	.0147	.9973	.0147	.9971
1.1	.0092	.9984	.0089	.9982
1.2	.0058	.9992	.0054	.9989
1.3	.0031	.9996	.0033	.9993
1.4	.0010	.9998	.0020	.9996
1.5	.0007	.9999	.0012	.9998
1.6	.0003	1.000	.0007	.9999
1.7	.0000	1.000	.0004	.9999

Table 9.1: Comparison of LC estimation with steady-state analytic values for M/M/1 with reneging or balking at service

LCE Estimates of PDF and CDF of Wait of Stayers

We present the LCE estimates of $f(x)$, $F(x)$ and P_0 in Table 9.1, using $t = 3000$, $\Delta = 0.1$.

LCE Estimates of Mean of Wait of Stayers and P_0

From (9.21), $E(W_q) \equiv \int_{x=0}^{\infty} xf(x)dx \equiv m_1 = 0.049$, where W_q denotes the required wait of stayers before service. Simulation of 10 independent sample paths using $t = 3000$, $\Delta = 0.1$, resulted in the sample-average point estimate $\overline{\widehat{m}_1} = 0.0489$. A 95% confidence interval for m_1 is obtained using $t_{9,0.025} \cdot s_{\widehat{m}_1}$ where $t_{9,0.025}$ is the right 2.5% tail of the Student "t" distribution with 9 degrees of freedom (Student "t" because 10 is a small sample size) and $s_{\widehat{m}_1}$ is the sample standard deviation of $\widehat{m}_1$. The value of $t_{9,0.025} \cdot s_{\widehat{m}_1}$ turned out to be 0.0013. Thus a 95% confidence interval is $m_1 = \overline{\widehat{m}_1} \pm t_{9,0.025} \cdot s_{\widehat{m}_1}$ or $m_1 = 0.0489 \pm 0.0013$, which covers the true mean wait.

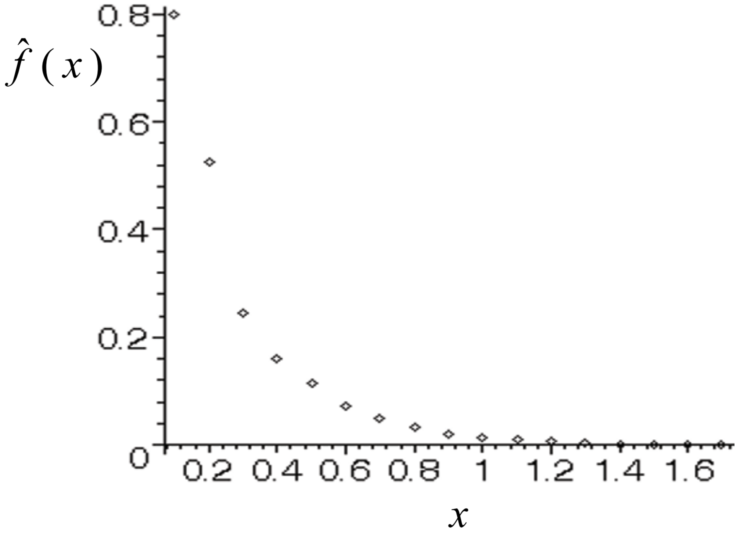

Figure 9.2: Point estimate $\widehat{f}(x)$ based on Table 9.1, for $f(x)$ in $M_\lambda/M_\mu/1$ queue with reneging or balking at service: $\lambda = 1$, $\mu = 5$. Compare with Fig. 9.3.

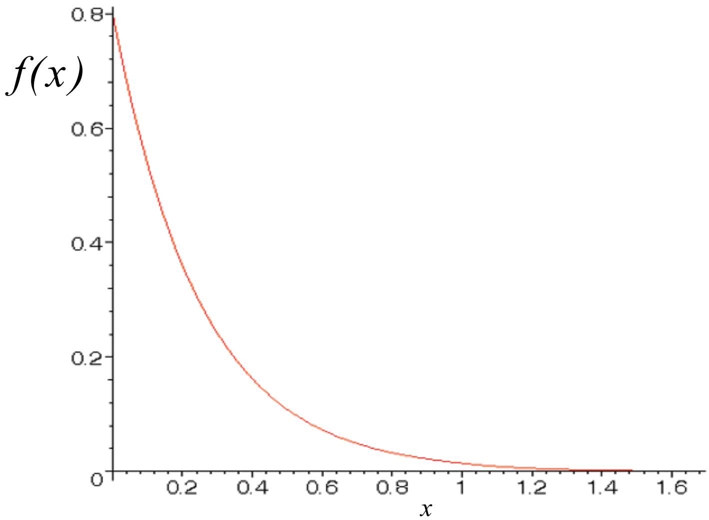

Figure 9.3: Analytical solution for $f(x)$ in $M_\lambda/M_\mu/1$ queue with reneging or balking at service: $\lambda = 1$, $\mu = 5$. See formulas (9.16), (9.18), (9.21). $f(x)$ is continuous at $x = 1$. $f'(x)$ is discontinuous at $x = 1$.

Similarly a 95% confidence interval for P_0 is $P_0 = \overline{\widehat{P}}_0 \pm t_{9,0.025} \cdot s_{\widehat{P_0}}$ or $P_0 = 0.7996 \pm 0.0025$, which covers the true value of P_0.

Discussion of Numerical Example

The probability that an arbitrary arrival stays and receives full service is

$$\begin{aligned} q_S &= P_0 + \int_{x=0}^{\infty} \overline{R}(x) f(x) dx \\ &= P_0 + \int_{x=0}^{1} f(x) dx \\ &= 0.8006 + \int_{x=0}^{1} 0.8006 e^{-4.0x} dx \\ &= 0.9971. \end{aligned}$$

For the particular choice of $(\lambda, \mu) = (1, 5)$ and $\overline{R}(\cdot)$ in the example, nearly all customers say, i.e., wait and get full service. Only $(1 - q_S) \cdot 100\% = 0.29\%$ either renege or balk at start of service. The reason is that the service rate is very fast relative to the arrival rate. The vast majority of arrivals (99.71%) are required to wait less than one time unit, and therefore stay for a full service.

The expected busy period is

$$E(\mathcal{B}) = \frac{1 - P_0}{\lambda P_0} = 0.24906.$$

The expected idle period is $E(\mathcal{I}) = \frac{1}{\lambda} = 1$. The proportion of time the server is idle is $\frac{E(\mathcal{I})}{E(\mathcal{I}) + E(\mathcal{B})} = 0.8006 = P_0$. Different values of (λ, μ) would, of course, give quite different results.

9.6 Discussion

LCE is useful for confirming theoretical results derived by various methods of analysis. LCE can be used to investigate the pdf of a state variable in a new model where the model equations are difficult to formulate, or‘ if formulated, are analytically intractable. It is an alternative approach for estimating pdf's, cdf's, moments, and expected values of functions of state variables (e.g., Laplace transforms) in stochastic models.

LCE for steady-state distributions has several advantages. It uses a single simulated sample path of the model. It requires the analyst to be sufficiently familiar with the model dynamics to construct a sample path using a computer program. It may help to uncover and explain subtleties about the pdf and cdf of the state variable, which enhance intuition about the model. It may help to discover unexpected properties about the pdf of the state variable.

LCE can be incorporated into a *hybrid* technique combining partially-known analytical solutions and statistical estimation. For example, in a single-server queue, the theoretical values of P_0 (probability of a zero wait) and $E(\mathcal{B})$ (expected busy period) may be known in terms of the model parameters. On the other hand, equations for the pdf of wait $f(x), x > 0$, may be analytically intractable. It may be possible to utilize the theoretical values of P_0 and $E(\mathcal{B})$ in the LCE computer program, to estimate $f(x), x > 0$.

LCE methods similar to that described here for M/G/1, have been applied to M/G/r($\cdot$) dams including cases where G is deterministic or discrete [22]; and to more complex models such as M/G^{a,b}/1 bulk-service queues [32]. The LCE technique is applicable in a vast array of other stochastic models as well.

We may classify the LCE method as an estimation method, or a *computational* method. With sensible values of the simulated time t and state-space partition norm size Δ, the technique gives almost-analytical values for the distribution of the state variable and related values, in many benchmark computational experiments already carried out.