CHAPTER 8

EMBEDDED LEVEL CROSSING METHOD

8.1 Dams and Queues

Consider a system modelled by $\{W(t), t \geq 0\}$, a continuous-parameter process with state space $\boldsymbol{S} = [0, \infty)$. (The state space can be extended to $\boldsymbol{S} \subseteq \boldsymbol{R^n}$ in more general models.) Let $\{\tau_n\}$ be an infinite set of embedded time points such that

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1} < \cdots .$$

Let $\{W_n, n = 1, 2, ...\}$ be the embedded discrete-parameter process, where $W(\tau_n^-) \equiv W_n$ and $W(\tau_n) \equiv W_n + S_n, n = 1, 2, ...$. Assume $W(t)$ is monotone in the interval $[\tau_n, \tau_{n+1})$.'Let

$$\frac{dW(t)}{dt} = -r(W(t)), t \in [\tau_n, \tau_{n+1}), n = 1, 2, ... ,$$

where $r(x) \geq 0$. Denote the cdf of $S_n, n = 1, 2, ...$, by $B(x), x \geq 0$, with $B(0) = 0$, and pdf $b(x) = \frac{d}{dx}B(x), x > 0$, wherever the derivative exists. Denote the cdf of W_n by $F_n(x)$ with pdf $\frac{dF_n(x)}{dx} = f_n(x)$, wherever it exists.

Definition 8.1 *An* ***embedded downcrossing*** *of state-space level x occurs during the closed interval $[\tau_n, \tau_{n+1}]$ if $W_n > x \geq W_{n+1}$.*

An ***embedded upcrossing*** *of level x occurs during $[\tau_n, \tau_{n+1}]$ if $W_n \leq x < W_{n+1}$.*

P.H. Brill, *Level Crossing Methods in Stochastic Models*,
DOI: 10.1007/978-0-387-09421-2_8,

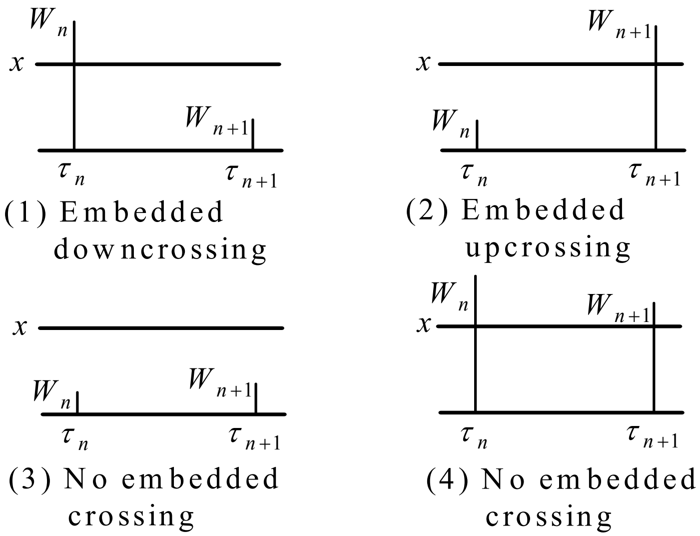

Figure 8.1: Embedded level crossings and non-crossings during time interval $[\tau_n, \tau_{n+1}]$.

Fix level $x \in \boldsymbol{S}$. Definition 8.1 classifies the set of intervals

$$\{[\tau_n, \tau_{n+1}], n = 1, 2, ...\}$$

into three mutually exclusive and exhaustive subsets with respect to level x (Fig. 8.1):

1. intervals that contain an embedded downcrossing,
2. intervals that contain an embedded upcrossing,
3. intervals that contain no embedded level crossing.

8.1.1 Rate Balance Across State-space Levels

Consider the time interval $[0, \tau_n], n \geq 2$ and a fixed level $x \in \boldsymbol{S}$. Let $\mathcal{D}_n(x)$, $\mathcal{U}_n(x)$ denote respectively the number of embedded down- and upcrossings of level x during $[0, \tau_n]$. Assume that the set of sample paths (sample functions) having an infinite number of embedded time points, has measure 1. The principle of rate balance across level x is

$$\begin{aligned} &\lim_{n\to\infty} \tfrac{\mathcal{D}_n(x)}{n} = \lim_{n\to\infty} \tfrac{\mathcal{U}_n(x)}{n} \ (a.s.), \\ &\lim_{n\to\infty} \tfrac{E(\mathcal{D}_n(x))}{n} = \lim_{n\to\infty} \tfrac{E(\mathcal{U}_n(x))}{n}. \end{aligned} \tag{8.1}$$

8.1.2 Method of Analysis

If the process is stable, the steady-state distribution of $W(t)$ as $t \to \infty$ and of W_n as $n \to \infty$, exist. Let $f(x) = \lim_{n\to\infty} f_n(x)$, $F(x) = \lim_{n\to\infty} F_n(x), x \in \boldsymbol{S}$. In the following sections, we shall derive an integral equation for $f(x)$ and $F(x)$ by using only:

1. the concept of embedded level crossings,
2. the principle of rate balance,
3. properties of the model,
4. knowledge of the efflux function $r(x), x \geq 0$.

8.2 GI/G/r(·) Dam

Assume that inputs to the dam occur in a renewal process with inter-input times having common cdf $A(\cdot)$. The model description is the same as for the M/G/r(·) dam in Subsection 6.2.1 except for the general renewal input stream.

The embedded process $\{W_n\}$ is a Markov chain, since

$$W_{n+1} = \max\{W_n + S_n - \Delta_n, 0\}$$

where S_n is the input amount at instant τ_n^- and Δ_n is the change in content during the time interval $[\tau_n, \tau_{n+1})$.

Define $\mathcal{G}(x)$ as the anti-derivative of $\frac{1}{r(x)}$ for $r(x) > 0$. Then $\mathcal{G}(x)$ is a continuous increasing function of x, since $\frac{d}{dx}\mathcal{G}(x) = \frac{1}{r(x)} > 0$. The time for the content to decline from state-space level v to level u, $v > u$, is

$$\int_u^v \frac{1}{r(x)} dx = \mathcal{G}(v) - \mathcal{G}(u).$$

A necessary and sufficient condition for the content of the dam to return to level 0 is: for every $v > 0$,

$$\begin{aligned} \lim_{u\downarrow 0} \int_{x=u}^{v} \frac{1}{r(x)} dx &= \lim_{u\downarrow 0}(\mathcal{G}(v) - \mathcal{G}(u)) \\ &= \mathcal{G}(v) - \lim_{u\downarrow 0} \mathcal{G}(u) < \infty. \end{aligned} \tag{8.2}$$

For example, in a pharmacokinetic model (Section 10.8 below) with "first order" kinetics, $r(x) = kx, x > 0$. In theory the drug concentration never returns to level 0. In practice, the drug may be entirely removed from the body after some finite time.

8.2.1 Embedded Downcrossing Rate

Proposition 8.1 *The probability of an embedded downcrossing of level x occurring in $[\tau_n, \tau_{n+1}]$ is*

$$\begin{aligned} d_n(x) &= \int_{y=0}^{\infty} \int_{\alpha=x}^{\gamma(x,y)} B(\gamma(x,y)-\alpha) dF_n(\alpha) dA(y) \\ &= \int_{\alpha=x}^{\infty} \int_{y=\eta(\alpha,x)}^{\infty} B(\gamma(x,y)-\alpha) dA(y) dF_n(\alpha), n=1,2,\dots , \quad (8.3) \end{aligned}$$

where $\gamma(x,y) = \mathcal{G}^{-1}(\mathcal{G}(x)+y)$, and $\eta(\alpha,x) = \mathcal{G}(\alpha) - \mathcal{G}(x)$.

Proof. An embedded downcrossing occurs in $[\tau_n, \tau_{n+1}] \iff W_n > x$ and the time for $W(t)$ to descend from level $W_n + S_n$ to level x is $\leq (\tau_{n+1} - \tau_n) \iff$

$$\int_{z=x}^{W_n+S_n} \frac{1}{r(z)} dz = \mathcal{G}(W_n+S_n) - \mathcal{G}(x) \leq \tau_{n+1} - \tau_n. \quad (8.4)$$

Conditioning on $\tau_n - \tau_{n+1} = y$, (8.4) is equivalent to

$$\begin{aligned} \mathcal{G}(W_n+S_n) - \mathcal{G}(x) &\leq y, \\ \mathcal{G}(W_n+S_n) &\leq \mathcal{G}(x) + y. \end{aligned} \quad (8.5)$$

Note that $\mathcal{G}(\cdot)$ and its inverse $\mathcal{G}^{-1}(\cdot)$ are both continuous and increasing functions. Taking the inverse $\mathcal{G}^{-1}$ on both sides of (8.5) gives

$$S_n \leq \mathcal{G}^{-1}(\mathcal{G}(x)+y) - W_n = \gamma(x,y) - W_n.$$

Conditioning on $W_n = \alpha$, gives

$$\begin{aligned} &P(\text{embedded downcrossing in } [\tau_n,\tau_{n+1}] | \tau_n - \tau_{n+1} = y) \\ &\qquad = \int_{\alpha=x}^{\gamma(x,y)} B(\gamma(x,y)-\alpha) dF_n(\alpha). \end{aligned}$$

We obtain the unconditional probability of an embedded downcrossing of x during $[\tau_n, \tau_{n+1}]$ by integrating with respect to the inter-input time y having distribution $A(y)$. This yields $d_n(x)$ given in (8.3). ∎

Let

$$\delta_n(x) = \begin{cases} 1 \text{ if there is an embedded downcrossing of } x \text{ in } [\tau_n, \tau_{n+1}], \\ 0 \text{ if there is no embedded downcrossing of } x \text{ in } [\tau_n, \tau_{n+1}]. \end{cases}$$

Then $E(\delta_n(x)) = d_n(x)$. The number of embedded downcrossings of level x in $[0, \tau_{n+1}]$ is

$$\mathcal{D}_n(x) = \sum_{i=1}^{n} \delta_i(x).$$

Thus

$$E(\mathcal{D}_n(x)) = \sum_{i=1}^{n} d_i(x).$$

The long-run expected embedded downcrossing *rate* of level x is

$$\lim_{n\to\infty} \frac{E(D_n(x))}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d_i(x).$$

From (8.3), since $\lim_{n\to\infty} F_n(x) \equiv F(x)$, then $\lim_{n\to\infty} d_n(x) = d(x)$ where

$$d(x) = \int_{\alpha=x}^{\infty} \int_{y=\eta(\alpha,x)}^{\infty} B(\gamma(x,y) - \alpha) dA(y) dF(\alpha).$$

Also,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d_i(x) = \lim_{n\to\infty} d_n(x) = d(x)$$

implies the expected embedded level downcrossing rate of level x is

$$\lim_{n\to\infty} \frac{E(\mathcal{D}_n(x))}{n} = \int_{\alpha=x}^{\infty} \int_{y=\eta(\alpha,x)}^{\infty} B(\gamma(x,y) - \alpha) dA(y) dF(\alpha). \tag{8.6}$$

8.2.2 Embedded Upcrossing Rate

Proposition 8.2 *The probability of an embedded upcrossing of level x occurring in $[\tau_n, \tau_{n+1}]$ is*

$$\begin{aligned} u_n(x) &= \int_{y=0}^{\infty} \int_{\alpha=0}^{x} \overline{B}(\gamma(x,y) - \alpha) dF_n(\alpha) dA(y) \\ &= \int_{\alpha=0}^{x} \int_{y=0}^{\infty} \overline{B}(\gamma(x,y) - \alpha) dA(y) dF_n(\alpha), n = 1, 2, \ldots . \end{aligned} \tag{8.7}$$

Proof. An embedded upcrossing of level x occurs in $[\tau_n, \tau_{n+1}] \iff W_n \le x$, $W_n + S_n > x$, and the time for $W(t)$ to descend from level $W_n + S_n$ to level x exceeds $\tau_{n+1} - \tau_n$

$$\iff \int_{z=x}^{W_n+S_n} \frac{1}{r(z)} dz = \mathcal{G}(W_n + S_n) - \mathcal{G}(x) > \tau_{n+1} - \tau_n$$

$$\iff S_n > \mathcal{G}^{-1}(\mathcal{G}(x) + y) - W_n = \gamma(x,y) - W_n,$$

where we have conditioned on $\tau_n - \tau_{n+1} = y$. Therefore

$$P(\text{embedded upcrossing in } [\tau_n,\tau_{n+1}] | \tau_n - \tau_{n+1} = y)$$
$$= \int_{\alpha=0}^{x} \overline{B}(\gamma(x,y) - \alpha) dF_n(\alpha),$$

where $\overline{B}(z) = 1 - B(z), z \geq 0$. The unconditional probability of an embedded upcrossing of x in $[\tau_n,\tau_{n+1}]$ is therefore given by (8.7). ■

As in the derivation of (8.4), it follows that the long-run expected embedded upcrossing rate of level x is

$$\lim_{n\to\infty} \frac{E(\mathcal{U}_n(x))}{n} = \int_{\alpha=0}^{x} \int_{y=0}^{\infty} \overline{B}(\gamma(x,y) - \alpha) dA(y) dF(\alpha). \tag{8.8}$$

8.2.3 Steady-state PDF of Content

We obtain an integral equation for the steady-state pdf of content. Applying rate balance (8.1) to formulas (8.6) and (8.8) gives an integral equation for $f(x)$ and $F(x)$, namely,

$$\int_{\alpha=x}^{\infty} \int_{y=\eta(\alpha,x)}^{\infty} B(\gamma(x,y) - \alpha) dA(y) dF(\alpha)$$
$$- \int_{\alpha=0}^{x} \int_{y=0}^{\infty} \overline{B}(\gamma(x,y) - \alpha) dA(y) dF(\alpha) = 0, x \geq 0. \tag{8.9}$$

CDF Form of Integral Equation

In the second term of (8.9) write $\overline{B}(\cdot) = 1 - B(\cdot)$ and apply $F(x) = \int_{\alpha=0}^{x} dF(\alpha)$. This yields a ***cdf form*** with $F(x)$ on the left side explicitly,

$$\begin{aligned} F(x) &= \textstyle\int_{\alpha=0}^{x} \int_{y=0}^{\infty} B(\gamma(x,y) - \alpha) dA(y) dF(\alpha) \\ &\quad + \textstyle\int_{\alpha=x}^{\infty} \int_{y=\eta(\alpha,x)}^{\infty} B(\gamma(x,y) - \alpha) dA(y) dF(\alpha), x \geq 0. \end{aligned} \tag{8.10}$$

PDF Form of Integral Equation

Differentiation of (8.10) with respect to $x > 0$, gives a **pdf form** with $f(x)$ explicitly on the left side,

$$\begin{aligned} f(x) &= \textstyle\int_{\alpha=0}^{x} \int_{y=0}^{\infty} \varrho(x,y) \cdot b(\gamma(x,y) - \alpha) dA(y) dF(\alpha) \\ &\quad + \textstyle\int_{\alpha=x}^{\infty} \int_{y=\eta(\alpha,x)}^{\infty} \varrho(x,y) \cdot b(\gamma(x,y) - \alpha) dA(y) dF(\alpha), x > 0, \end{aligned} \tag{8.11}$$

where $\varrho(x,y) = \frac{\partial}{\partial x}\gamma(x,y) = \frac{r(\gamma(x,y))}{r(x)}$.

Probability of Zero Content

Letting $x \downarrow 0$ in (8.10) gives

$$F(0) = \frac{\int_{\alpha=0^+}^{\infty} \int_{y=\eta(\alpha,0)}^{\infty} B(\gamma(0,y)-\alpha)dA(y)dF(\alpha)}{\int_{y=0}^{\infty} \overline{B}(\gamma(0,y))dA(y)}. \tag{8.12}$$

The normalizing condition is

$$F(0) + \int_{\alpha=0}^{\infty} f(\alpha)d\alpha = 1 \tag{8.13}$$

If condition (8.2) does not hold, then $F(0) = 0$ (recall that $f(0) \equiv f(0^+)$).

Solution Method

The solution method in the following sections will be to obtain the functional form of $f(x)$ and $F(x)$ using (8.10) or (8.11), and applying the boundary conditions (8.12) and (8.13) to specify $f(x)$, $F(x), x \geq 0$.

8.2.4 M/G/r(·) Dam

In this model, $A(y) = 1 - e^{-\lambda y}, y \geq 0$. Note that

$$\frac{\partial\left(\gamma(x,y)\right)}{\partial y} = \frac{\partial(\mathcal{G}^{-1}(\mathcal{G}(x)+y))}{\partial y} = r(\gamma(x,y)) = r(\mathcal{G}^{-1}(\mathcal{G}(x)+y)).$$

Integrating (8.11) by parts, using parts

$$\frac{\lambda e^{-\lambda y}}{r(y)} \quad \text{and } r(\gamma(x,y)) \cdot b(\gamma(x,y)-\alpha)dy,$$

simplifying and substituting from (8.10) results in

$$r(x)f(x) = \lambda \int_{\alpha=0}^{x} \overline{B}(x-\alpha)dF(\alpha), x > 0. \tag{8.14}$$

Equation (8.14) is identical to the integral equation (6.18) for the steady-state pdf of content in the M/G/r(·) dam (derived using "continuous" LC)

Remark 8.1 *In equation (8.14) $f(x) = \lim_{n\to\infty} f_n(x)$ since (8.14) has been derived using* ***embedded*** *LC. In Chapter 6, equation (6.18), $f(x) = \lim_{t\to\infty} f_t(x)$ is the* ***time-average*** *steady-state pdf of content. The fact*

GI/G/r(·) Dam	Gi/G/1 Queue
Input instant τ_n^-	Customer arrival instant τ_n^-
Input amount at τ_n^-	Service time (jump size) S_n
Content at τ_n^-	Customer wait W_n in queue at τ_n^-
Content at instant τ_n	Virtual wait $W(\tau_n) = W_n + S_n$
Content at time $t \geq 0$	Virtual wait $W(t)$ at time $t \geq 0$
$r(x) > 0, x > 0$; $r(0) = 0$	$r(x) = 1, x > 0$; $r(0) = 0$
Distribution of content	Distribution of waiting time

Table 8.1: GI/G/r(.) dam versus GI/G/1queue.

that $\lim_{n\to\infty} f_n(x)$ *and* $\lim_{t\to\infty} f_t(x)$ *satisfy the same integral equation, demonstrates that the content of an M/G/r(·) dam satisfies the PASTA principle that Poisson arrivals "see" time averages [102]. Here we have derived PASTA for the M/G/r(·) dam by using continuous and embedded LC concepts only.*

8.3 GI/G/1 Queue

The GI/G/1 queue is closely related to the Gi/G/r(·) dam (Table 8.1). For the virtual wait of the GI/G/1 queue $r(x) = \begin{cases} 1, x > 0, \\ 0, x = 0. \end{cases}$

The anti-derivative of $\frac{1}{r(x)}, x > 0$, is

$$\mathcal{G}(x) = \int \frac{1}{r(x)} dx = \int 1 \cdot dx = x.$$

Thus,

$$\begin{aligned} \gamma(x, y) &= \mathcal{G}^{-1}(\mathcal{G}(x) + y)) = \mathcal{G}^{-1}(x + y)) = x + y \\ \eta(\alpha, x) &= \mathcal{G}(\alpha) - \mathcal{G}(x) = \alpha - x, \\ \varrho(x, y) &= \frac{r(\gamma(x, y))}{r(x)} = \frac{r(x + y)}{1} = \frac{1}{1} = 1. \end{aligned}$$

For the GI/G/1 queue, equations (8.10), (8.11) and (8.13) reduce respectively to

$$\begin{aligned} F(x) = \textstyle\int_{\alpha=0}^{x} \int_{y=0}^{\infty} B(x + y - \alpha) dA(y) dF(\alpha) \\ + \textstyle\int_{\alpha=x}^{\infty} \int_{y=\alpha-x}^{\infty} B(x + y - \alpha) dA(y) dF(\alpha), x \geq 0, \end{aligned} \tag{8.15}$$

$$\begin{aligned} f(x) = \textstyle\int_{\alpha=0}^{x} \int_{y=0}^{\infty} b(x + y - \alpha) dA(y) dF(\alpha) \\ + \textstyle\int_{\alpha=x}^{\infty} \int_{y=\alpha-x}^{\infty} b(x + y - \alpha) dA(y) dF(\alpha), x > 0, \end{aligned} \tag{8.16}$$

$$F(0) = \frac{\int_{\alpha=0^+}^{\infty} \int_{y=\alpha}^{\infty} B(y-\alpha)dA(y)dF(\alpha)}{\int_{y=0}^{\infty} \overline{B}(y))dA(y)}. \tag{8.17}$$

The normalizing condition is

$$F(0) + \int_{\alpha=0}^{\infty} f(\alpha)d\alpha = 1. \tag{8.18}$$

Applications

Some single-server queueing models can be solved using embedded LC, by applying equations (8.15) - (8.18). Other models are solved by deriving integral equations for the pdf of the state variables from first principles using embedded LC. The next four subsections illustrate some applications.

8.3.1 M/G/1 Queue

The M/G/1 queue is a special case of the M/G/r(·) dam, with $r(x) = 1, x > 0$ and $A(y) = 1 - e^{-\lambda y}, y \geq 0$. Substituting directly into equation (8.14) or into (8.16) followed by some algebra yields

$$\begin{aligned} f(x) &= \lambda \int_{\alpha=0}^{x} \overline{B}(x-\alpha)dF(\alpha) \\ &= \lambda P_0 \overline{B}(x) + \lambda \int_{\alpha=0}^{x} \overline{B}(x-\alpha)f(\alpha)d\alpha, x > 0, \end{aligned} \tag{8.19}$$

which is identical to equation (3.29). Remark 8.1 applies also to this model.

8.3.2 GI/M/1 Queue

The GI/M/1 queue is a special case of the GI/G/1 queue with

$$B(x) = 1 - e^{-\mu x}, x \geq 0, \quad b(x) = \mu e^{-\mu x} = \mu - \mu B(x), x > 0.$$

Substituting $b(x) = \mu - \mu B(x)$ into (8.16), simplifying and combining with (8.15) gives the integral equation

$$f(x) = \mu \int_{y=x}^{\infty} \overline{A}(y-x)f(y)dy, \ x > 0, \tag{8.20}$$

which is identical to equation (5.6).

Equation (8.19) for M/G/1	Equation (8.20) for G/M/1
λ	μ
x is upper bound of integral	x is lower bound of integral
$\overline{B}(x-y)$	$\overline{A}(y-x)$
P_0 appears explicitly	P_0 does not appear explicitly

Table 8.2: Interchanged roles of terms in integral equations for M/G/1 and G/M/1.

Duality of M/G/1 and GI/M/1 Queues

Upon comparing integral equations (8.19) and (8.20) it is evident that they are duals, in the sense that the roles of certain terms are interchanged (see Table 8.2). The significance of this "duality" is that we analyze the M/G/1 queue via LC using the virtual wait process. On the other hand, we are led to analyzing the G/M/1 queue via LC using the extended "age" process (see Subsection 5.1.1 and [11]).

Remark 8.1 applies also to GI/M/1, provided we analyze the extended age process, for which departures from the system occur in a Poisson process at rate μ conditional on the server being occupied. This implies that in (8.20), $f(x)$ on the left side (equal to time-average pdf of virtual wait) is the same function as $f(y)$ in the integrand on the right side (pdf of system time at departure instants).

Solution for Steady-state PDF of Wait in GI/M/1

Assume the solution for the pdf of wait has the form $f(x) = Ke^{-\gamma x}, x > 0$. Substituting into (8.20) yields the equation for γ

$$\int_{z=0}^{\infty} \overline{A}(z)e^{-\gamma z}dz = \frac{1}{\mu},$$

or

$$\frac{1}{\gamma} - \frac{1}{\gamma}A^*(\gamma) = \frac{1}{\mu}. \tag{8.21}$$

In (8.21) $A^*(\cdot)$ is the Laplace-Stieltjes transform of $A(\cdot)$ defined by

$$A^*(s) = \int_{y=0}^{\infty} e^{-sy}a(y)dy, s \geq 0,$$

and $a(y) = \frac{d}{dy}A(y)$, assuming the inter-arrival times are continuous r.v.'s. We obtain an expression for $P_0 = F(0)$ upon substituting $B(y) = 1 -$

$e^{-\mu y}$, $f(\alpha) = Ke^{-\gamma\alpha}$ in (8.17), namely

$$F(0) = [A^*(\mu)]^{-1} \left\lfloor \frac{\gamma - \mu + \mu A^*(\gamma) - \gamma A^*(\mu)}{\gamma(\gamma - \mu)} \right\rfloor \cdot K. \tag{8.22}$$

From (8.21)

$$\mu - \mu A^*(\gamma) = \gamma,$$

which substituted into (8.22) leads directly to

$$F(0) = \frac{K}{\mu - \gamma}. \tag{8.23}$$

The normalizing condition (8.18) gives

$$\frac{K}{\mu - \gamma} + \frac{K}{\gamma} = 1.$$

Then (8.23) implies

$$F(0) = \frac{\gamma}{\mu}. \tag{8.24}$$

Formula (8.24) is important because $F(0) = P_{0\iota}$ in (5.23) which was derived using "continuous" or "time-average" LC. This provides further evidence of the overall logical correctness of the LC methodology.

Check with M/M/1 Queue

It is instructive to check the result for the M/M/1 queue. Consider M/M/1 with arrival rate λ and service rate μ. Then $A^*(s) = \frac{\lambda}{\lambda+s}$. From (8.21) $\gamma = \mu - \lambda$, which substituted into (8.22), gives $F(0) = P_0 = \frac{K}{\lambda}$. Applying the normalizing condition $F(0)+\int_{y=0}^{\infty} f(y)dy = 1$, gives

$$\frac{K}{\lambda} + K\int_{y=0}^{\infty} e^{-(\mu-\lambda)y}dy = 1,$$
$$K = \lambda(1 - \frac{\lambda}{\mu}).$$

Thus

$$P_0 = \frac{K}{\lambda} = 1 - \frac{\lambda}{\mu}, \checkmark$$
$$f(x) = \lambda P_0 e^{-(\mu-\lambda)x}, x > 0, \checkmark$$

which checks with the M/M/1 solution given in (3.86) and (3.87).

8.3.3 $E_k/M/1$ Queue

Assume the common pdf of the inter-arrival times $a(\cdot) \underset{dist}{=}$ Erlang-(k, λ). Thus for integer $k = 1, 2, \ldots$, $a(y) = e^{-\lambda y}\frac{(\lambda y)^{k-1}}{(k-1)!}\lambda, y > 0$. Let $A(\cdot)$ denote the cdf corresponding to $a(\cdot)$. Then the LST of $A(\cdot)$ is $A^*(\gamma) = \left(\frac{\lambda}{\lambda+\gamma}\right)^k$, which substituted into equation (8.21) gives an equation for γ,

$$\frac{1}{\gamma} - \frac{1}{\gamma}\left(\frac{\lambda}{\lambda+\gamma}\right)^k = \frac{1}{\mu}, k = 1, 2, \ldots . \tag{8.25}$$

We seek a unique positive solution of (8.25) for γ. Assume that $\lambda, \mu > 0$ and $\lambda < k\mu$ (stability condition for G/M/1 is $\boldsymbol{a} < \mu$, where $\boldsymbol{a} = \frac{k}{\lambda} =$ arrival rate). Then equation (8.25) has exactly one *real* positive root for γ (see [11]). If k is odd, all other roots are *complex*. If k is even, one other root is negative real and all other roots are complex. Thus the solution for γ is unique. Denote it by γ_k.

To solve for $K \equiv \eta_k$ we first substitute γ_k into (8.22) and use (8.25) to obtain

$$F(0) = \frac{\eta_k}{\mu - \gamma_k}.$$

(We use η_k instead of K_k in this subsection only, for notational contrast.) Then apply the normalizing condition (8.18) to obtain

$$\eta_k = \frac{\gamma_k(\mu - \gamma_k)}{\mu} = \gamma_k\left(1 - \frac{\gamma_k}{\mu}\right).$$

The steady-state pdf of wait is then given by

$$\begin{aligned} P_0 &= \frac{\eta_k}{\mu - \gamma_k} = \frac{\gamma_k}{\mu}, \\ f(x) &= \eta_k e^{-\gamma_k x} = \gamma_k(1 - \frac{\gamma_k}{\mu})e^{-\gamma_k x}, x > 0. \end{aligned}$$

Remark 8.2 *The solution of equation (8.25) can be readily obtained numerically for any specified values of λ, μ, k.*

8.3.4 D/M/1 Queue

Assume the common inter-arrival time is $D > 0$. Then $A^*(s) = e^{-sD}, s > 0$. Let the steady-state pdf of wait be $f(x) = Ke^{-\gamma x}, x > 0$. Substituting $A^*(\gamma) = e^{-\gamma D}$ into (8.21) gives the equation

$$\mu e^{-\gamma D} + \gamma - \mu = 0$$

for γ, whose solution we call γ_{D}. From (8.22)

$$F(0) = \frac{K}{\mu - \gamma_{D}}.$$

Let $K \equiv K_{D}$. Substituting into (8.18) gives

$$\frac{K_{D}}{\mu - \gamma_{D}} + \frac{K_{D}}{\gamma_{D}} = 1,$$
$$K_{D} = \gamma_{D}\left(1 - \frac{\gamma_{D}}{\mu}\right).$$

The steady-state pdf of wait is

$$\begin{aligned} P_0 &= \frac{K_{D}}{\mu - \gamma_{D}}, \\ f(x) &= K_{D} e^{-\gamma_{D} \cdot x}, x > 0. \end{aligned}$$

8.4 M/G/1 with Reneging

We apply the embedded LC method to an M/G/1 queue in which customers can either: (1) renege from the waiting line; (2) wait and balk at service; (3) wait and stay for a full service. Assume the *staying function* is $\overline{R}(y) = P(\text{arrival stays for service}|\text{required wait} = y)$. We verify that the pdf $f(\cdot)$ on the left side of (3.162) and the pdf $f(\cdot)$ on the right side of (3.162) are the same functions. In (3.162) the pdf on the left side is $\lim_{t\to\infty} f_t(x)$ (time-average pdf). The pdf on the right side is $\lim_{n\to\infty} f_n(x)$ (pdf at arrival instants, or arrival-point pdf). We now use embedded LC to derive an integral equation for $f(x) = \lim_{n\to\infty} f_n(x)$ and show that it is identical to equation (3.162).

8.4.1 Embedded Crossing Probabilities

The limiting probability of an SP *embedded upcrossing* of level x is

$$u = \int_{y=0^-}^{x} \int_{z=0}^{\infty} \overline{B}(x - y + z)\overline{R}(y) f(y) \lambda e^{-\lambda z} dz dy, \tag{8.26}$$

where the lower limit $y = 0^-$ means that the term $\overline{B}(x + z)P_0$ for the atom $\{0\}$ is included in the evaluation of u. The right side of (8.26) holds since an embedded upcrossing of x occurs iff $0 \le W_n = y < x$, the arrival at τ_n stays for service (probability $\overline{R}(y)$), and given that the time to the next arrival is z, the service time exceeds $x - y + z$.

The limiting probability of an SP *embedded downcrossing* of level x consists of two terms,

$$\begin{aligned} d = &\int_{y=x}^{\infty} \int_{z=y-x}^{\infty} B(x-y+z)\overline{R}(y)f(y)\lambda e^{-\lambda z} dz dy \\ &+ \int_{y=x}^{\infty} \int_{z=y-x}^{\infty} R(y)f(y)\lambda e^{-\lambda z} dz dy. \end{aligned} \tag{8.27}$$

The first term on the right of (8.27) is similar to (8.26), except that an SP jump starts at a level $y > x$ and the service time must be less than $x - y + z$ for an embedded downcrossing to occur. The second term is due to arrivals that *do not stay for service* (renege or balk at service); arrivals renege or balk at service with probability $R(y) = 1 - \overline{R}(y)$. We can assume that an SP "jump" is of size 0 (probability $R(y)$) when a reneger or service-balker arrives. Equivalently there is *no SP jump* when a reneger or service-balker arrives. In this case the SP makes an embedded downcrossing of level x provided the next inter-arrival time $z > y - x$. The second term in (8.27) simplifies to $\int_{y=x}^{\infty} R(y)f(y)e^{-\lambda(y-x)}dy$.

Since $\overline{B}(\cdot) \equiv 1 - B(\cdot)$, equation (8.26) can be written as

$$u = \int_{y=0^-}^{x} \overline{R}(y)f(y)dy - \int_{y=0^-}^{x} \int_{z=0}^{\infty} B(x-y+z)\overline{R}(y)f(y)\lambda e^{-\lambda z} dz dy \tag{8.28}$$

8.4.2 Steady-State PDF of Wait of Stayers

Applying *embedded* rate balance across level x, we set $u = d$. This yields from equations (8.27) and (8.28), the integral equation

$$\begin{aligned} \int_{y=0^-}^{x} \overline{R}(y)f(y)dy = &\int_{y=0^-}^{x} \int_{z=0}^{\infty} B(x-y+z)\overline{R}(y)f(y)\lambda e^{-\lambda z} dz dy \\ &+ \int_{y=x}^{\infty} \int_{z=y-x}^{\infty} B(x-y+z)\overline{R}(y)f(y)\lambda e^{-\lambda z} dz dy \\ &+ \int_{y=x}^{\infty} \int_{z=y-x}^{\infty} R(y)f(y)\lambda e^{-\lambda z} dz dy. \end{aligned} \tag{8.29}$$

We take $\frac{d}{dx}$ on both sides of (8.29). This involves differentiation under the integral sign. Some algebra including cancellation of terms and using

$R(y) + \overline{R}(y) = 1$ gives

$$\begin{aligned} f(x) = & \int_{y=0^-}^{x} \int_{z=0}^{\infty} b(x-y+z)\overline{R}(y)f(y)\lambda e^{-\lambda z}dzdy \\ & + \int_{y=x}^{\infty} \int_{z=y-x}^{\infty} b(x-y+z)\overline{R}(y)f(y)\lambda e^{-\lambda z}dzdy \\ & + \lambda \int_{y=x}^{\infty} \int_{z=y-x}^{\infty} R(y)f(y)\lambda e^{-\lambda z}dzdy. \end{aligned} \tag{8.30}$$

Integrating each of the inner integrals

$$\int_{z=0}^{\infty} b(x-y+z)\lambda e^{-\lambda z}dz \quad \text{and} \quad \int_{z=y-x}^{\infty} b(x-y+z)\lambda e^{-\lambda z}dz$$

in (8.30) by parts, using parts $\lambda e^{-\lambda z}$ and $b(x-y+z)$, leads to the integral equation (assuming $B(0) = 0$)

$$\begin{aligned} f(x) = \quad & -\lambda \int_{y=0^-}^{x} \overline{R}(y)f(y)B(x-y)dy \\ & +\lambda \int_{y=0^-}^{x} \int_{z=0}^{\infty} B(x-y+z)\overline{R}(y)f(y)\lambda e^{-\lambda z}dzdy \\ & +\lambda \int_{y=x}^{\infty} \int_{z=y-x}^{\infty} B(x-y+z)\overline{R}(y)f(y)\lambda e^{-\lambda z}dzdy \\ & +\lambda \int_{y=x}^{\infty} \int_{z=y-x}^{\infty} R(y)f(y)\lambda e^{-\lambda z}dzdy. \end{aligned} \tag{8.31}$$

From (8.29) the sum of the last three terms on the right of (8.31) is

$$\lambda \int_{y=0^-}^{x} \overline{R}(y)f(y)dy.$$

Hence

$$\begin{aligned} f(x) &= \lambda \int_{y=0^-}^{x} \overline{R}(y)f(y)dy - \lambda \int_{y=0^-}^{x} \overline{R}(y)f(y)B(x-y)dy, \\ f(x) &= \lambda \int_{y=0^-}^{x} \overline{B}(x-y)\overline{R}(y)f(y)dy. \end{aligned} \tag{8.32}$$

Equation (8.32) is *identical to* (3.162). Hence, in (3.162), the time-average pdf of stayers (left side) is equal to the arrival-point pdf of stayers (in integral on right side). The derivation of (3.162) using "continuous-time" LC is far simpler than that of (8.32). Nevertheless, the embedded LC method is very useful in this case, and elsewhere. It helps to confirm that "continuous" LC works in the reneging problem. The embedded LC method can often be applied to determine whether the time-average and arrival-point pdf's are equal. The embedded LC method is inherently very intuitive, and has additional applications as well.