# CHAPTER 10

# ADDITIONAL APPLICATIONS

## 10.1 Introduction

This chapter applies SPLC to a variety of stochastic models, in order to indicate the scope, applicability and flexibility of the methodology, and to suggest new applications. The chapter begins with the LC analysis of a replacement model, which is structured using renewal processes. In that model, we derive limiting pdf's of the excess life, age and total life of a renewal process, using LC. The chapter ends with the LC analysis of a classical renewal problem. The intervening sections analyze several models that suggest many additional potential applications of SPLC.

## 10.2 Renewal Processes

We shall derive steady-state pdf's of renewal processes in the context of a replacement model. This model is a variant of a GI/G/r(·) dam.

### 10.2.1 A Replacement Model

Consider a continuous-time stochastic process $\{X(t) \geq 0, t \geq 0\}$ having iid jumps of size $X_n > 0$ at $\tau_n$, where $0 = \tau_0 < \tau_1 \cdots < \tau_n < \cdots$. Thus $X(\tau_n) = X_n$, $n = 0, 1, 2, ...$, (Fig. 10.1). Consider a sample path of $\{X(t)\}$ ( we use $X(t)$ to denote the state variable and a sample path, for economy of notation). Assume $\frac{dX(t)}{dt} = -r(X(t))$, $t \in [\tau_n, \tau_{n+1})$, $n = 0, 1, ...$, where $r(x) > 0$, $x > 0$. Thus $X(t)$ is a piecewise deterministic


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function. Let the state space be $\boldsymbol{S} = [0, \infty)$. Assume that for all $v > 0$,

$$\lim_{u \downarrow 0} \int_{y=u}^{v} \frac{1}{r(y)} dy < \infty. \tag{10.1}$$

Condition (10.1) guarantees that a sample path $X(t), t \geq 0$, starting from any level $v > 0$, returns to level 0 in a finite time. The process $\{X(t)\}$ is a variant of the GI/G/r($\cdot$) dam such that inputs $\{X_n\}$ occur only at instants when the dam becomes empty. This mechanism can be thought of as that of a *replacement model*. New inputs replace the preceding inputs as soon as the latter become used up.

Denote the inter-replacement times by $\{Z_n\}$. The random variables $Z_n$ and $X_n$ are related by the equation

$$Z_n = \int_{y=0}^{X_n} \frac{1}{r(y)} dy, n = 0, 1, ... . \tag{10.2}$$

From (10.2), $Z_n$ is the *time* required for $\{X(t)\}$ to descend from level $X_n$ to level 0. The $\{Z_n\}$ are iid random variables.

## Renewal Processes $\{Z_n\}$ and $\{X_n\}$

The sequence $\{Z_n\}$ is a renewal process synchronized with the sequence $\{X_n\}$ and with the piecewise deterministic continuous efflux rate $r(X(t))$. Due to the structure of the model, the sequence $\{X_n\}$ is also a renewal process.

Let $X_n \underset{dist}{\equiv} X$ and $Z_n \underset{dist}{\equiv} Z$.

**Example 10.1** *Consider a newly-installed battery at $\tau_0$ with initial electrical charge $X_0 \underset{dist}{\equiv} X$. Assume that the charge declines at a rate that depends on the present charge. That is, $\frac{dX(t)}{dt} = -r(X(t)) < 0, t \in [\tau_0, \tau_1)$. Assume the battery operates continuously. Its charge dissipates non-uniformly and descends to 0 after a time $\tau_1 = Z_0 \underset{dist}{\equiv} Z$. The battery is immediately replaced by a new fully-charged one. This procedure is repeated as batteries wear out. Thus $Z_n \underset{dist}{\equiv} Z$, $X_n \underset{dist}{\equiv} X$, $n = 0, 1, 2, ....$ Then*

$$Z = \int_{y=0}^{X} \frac{1}{r(y)} dy, \tag{10.3}$$

*is the inter-replacement time. The dimension of $Z$ is $[Time]$. The dimension of $X$ is $[Coulombs]$. The function $r(X(t))$ has dimension $[Coulomb][Time]^{-1}$.*

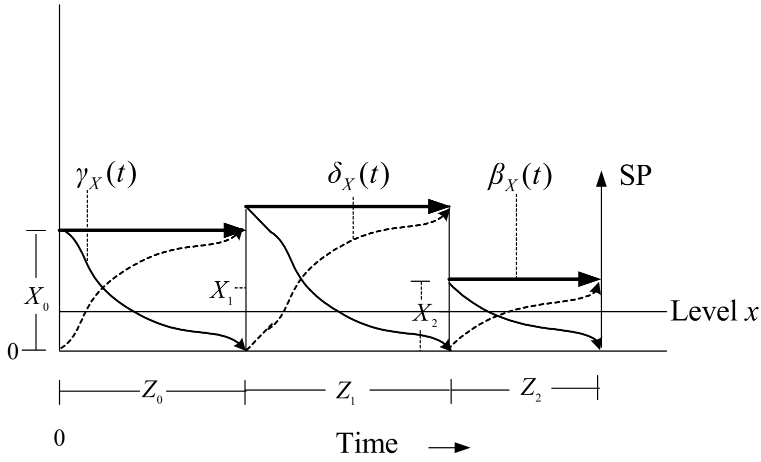


Figure 10.1: Sample path of excess life $\gamma_X(t)$, age $\delta_X(t)$, total life $\beta_X(t)$. Also shows a level $x$ in the state space.

### 10.2.2 Renewal Process $\{X_n\}$

**Excess Life, Age, Total Life**

Let $\gamma_X(t)$ $(\equiv X(t))$ denote the *excess life* of *content* at instant $t \geq 0$. Then $\frac{d\gamma_X(t)}{dt} = -r(\gamma_X(t))$. Let $\delta_X(t)$ denote the *age* of the *content*, i.e., amount of content used up at instant $t$, from the latest renewed amount prior to $t$. Then $\frac{d\delta_X(t)}{dt} = +r(\delta_X(t))$. Let $\beta_X(t)$ denote the *total life* (*span*) of the latest renewed amount of *content* at $t$ (Fig. 10.1). (In Example 10.1, $\gamma_X(t))$, $\delta_X(t)$, $\beta_X(t)$ are respectively to the remaining charge, the charge used up, and the total charge, of the battery in use at time $t$.)

In the sample paths of the processes $\{\gamma(t)\}$, $\{\delta(t)\}$, $\{\beta(t)\}$ all upward jumps start at level 0 and are $\underset{dist}{=} X$. All downward jumps start at a level $X$ and end at level 0.

**Limiting Distributions**

We now apply LC to derive the limiting pdf's $f_{\gamma_X}(x)$, $f_{\delta_X}(x)$, $f_{\beta_X}(x), x > 0$, of r.v.'s $\gamma_X(t)$, $\delta_X(t)$, $\beta_X(t)$, as $t \to \infty$, assuming the limits exist. Consider sample paths of $\{\gamma_X(t)\}$, $\{\delta_X(t)\}$, $\{\beta_X(t)\}$, $t \geq 0$ (Fig. 10.1).

Let $F_X(x)$, $f_X(x)$, $\mu_X$ be the cdf, pdf and expected value respectively of r.v. $X$. Let $\overline{F_X}(x) \equiv 1 - F_X(x)$.

## Limiting PDF of Excess Life

Consider a sample path of $\{\gamma(t)\}$. The long-run SP expected *downcrossing* rate of a *content* level $x > 0$, is

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t(x))}{t} = r(x) f_{\gamma_X}(x). \tag{10.4}$$

(as in Corollary 6.2).

The long-run SP expected *upcrossing* rate of level $x$ is

$$\lim_{t\to\infty} \frac{E(\mathcal{U}_t(x))}{t} = \frac{1}{E(Z)} \cdot \overline{F_X}(x), \tag{10.5}$$

since the expected time between upward jumps starting from level 0 is $E(Z)$ $(= E(\tau_{n+1} - \tau_n), n = 0, 1, ...)$; also $\overline{F_X}(x) = P($SP jump starting at level 0 is $> x)$. In (10.3), substituting from (10.2), conditioning on $X = x$ gives

$$\begin{aligned} E(Z) &= \int_{x=0}^{\infty} \left( \int_{y=0}^{x} \frac{1}{r(y)} dy \right) f_X(x) dx \\ &= \int_{y=0}^{\infty} \int_{x=y}^{x} \frac{1}{r(y)} f_X(x) dx dy = \int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)} dy. \end{aligned} \tag{10.6}$$

Equating (10.4) and (10.5) for rate balance across level $x$, and using (10.6), yields the equation

$$r(x) f_{\gamma_X}(x) = \frac{\overline{F_X}(x)}{E(Z)} = \frac{\overline{F_X}(x)}{\int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)} dy}, \tag{10.7}$$

$$f_{\gamma_X}(x) = \frac{\overline{F_X}(x)}{r(x) \int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)} dy}.. \tag{10.8}$$

The dimension of $f_{\gamma_X}(x)$ is $[content]^{-1}$ $([Coulomb]^{-1}$ in Example 10.1).

## Limiting PDF of Excess Life when $r(x) \equiv 1$

If the efflux rate $r(x) \equiv 1$, formula (10.8) reduces to

$$f_{\gamma_X}(x) = \frac{\overline{F_X}(x)}{\int_{y=0}^{\infty} \overline{F_X}(y) dy} = \frac{\overline{F_X}(x)}{\mu_X}, \tag{10.9}$$

since $\int_{y=0}^{\infty} \overline{F_X}(y) dy = E(X) = \mu_X$. (Note that $\gamma_X$ represents the limiting excess life of *content* having pdf $f_{\gamma_X}(x)$.) Formula (10.9) is exactly the same as the well known limiting pdf of the excess life in a "standard" renewal process. However, here the dimension of $f_{\gamma_X}(x)$ is $[content]^{-1}$ instead of $[Time]^{-1}$.

**Limiting PDF of Age**

For the process $\{\delta_X(t)\}$, the long-run SP expected upcrossing rate of a content level $x > 0$, is

$$\lim_{t\to\infty} \frac{E(\mathcal{U}_t(x))}{t} = +r(x)f_{\delta_X}(x), \tag{10.10}$$

(as in Corollary 6.2). The long-run SP (expected) downcrossing rate of level $x$ is

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t(x))}{t} = \frac{1}{E(Z)} \int_{y=x}^{\infty} f_X(y)dy = \frac{\overline{F_X}(x)}{E(Z)}, \tag{10.11}$$

since (1) downward jumps occur at rate $\frac{1}{E(Z)}$, (2) in order for the SP to downcross level $x$ by a jump at some $\tau_n^-$, the *upward* jump at $\tau_{n-1}$ from level 0 must have been such that $X_{n-1} > x$. Additionally, $X_{n-1}$ is equal to the downward jump size at $\tau_n^-$ (Fig. 10.1).

Equating (10.10) and (10.11) for rate balance across level $x$, gives

$$\begin{aligned} r(x)f_{\delta_X}(x) &= \frac{\overline{F_X}(x)}{E(Z)} = \frac{\overline{F_X}(x)}{\int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)}dy}; \\ f_{\delta_X}(x) &= \frac{\overline{F_X}(x)}{r(x)\int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)}dy}. \end{aligned} \tag{10.12}$$

Comparison of (10.8) with (10.12) shows that $f_{\delta_X}(x) \equiv f_{\gamma_X}(x)$. The dimension of $f_{\delta_X}(x)$ is $[content]^{-1}$.

**Limiting PDF of Age when $r(x) \equiv 1$**

If $r(x) \equiv 1$, we obtain similarly as in (10.9), the limiting pdf

$$f_{\delta_X}(x) = \frac{\overline{F_X}(x)}{\mu_X}. \tag{10.13}$$

The dimension of $f_{\delta_X}(x)$ is $[content]^{-1}$. It is well known that for a "standard" renewal process, the limiting distributions of the excess life and age are identical. In the variant of a GI/G/r(·) dam possessing the renewal structure here, these distributions are also identical with regard to the content, even when the efflux rate has a general form $r(x), x > 0$. That is, formulas (10.8) and (10.12) are identical.

## Limiting PDF of Total Life

For the process $\{\beta_X(t)\}$, the long-run SP expected downcrossing rate of a *content* level $x > 0$, is

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t(x))}{t} = \int_{y=x}^{\infty} \left( \frac{1}{\int_{u=0}^{y} \frac{1}{r(u)} du} \right) f_{\beta_X}(y)dy. \tag{10.14}$$

In (10.14), we have conditioned on $\beta_X(t) = y > x$. The SP *downward* jump rate across level $x$ starting at level $y$ is $\frac{1}{\left(\int_{u=0}^{y} \frac{1}{r(u)} du\right)}$, which is *the reciprocal of the expected sojourn time of* $\{\beta_X(t)\}$ *at level* $y$ (Fig. 10.1). At the end of a level-$y$ $(y > x)$ sojourn time, the SP jumps downward to level 0. It downcrosses every state-space level in $(0, y)$, including level $x$.

The SP long-run (expected) *upcrossing* rate of level $x$ is

$$\lim_{t\to\infty} \frac{E(\mathcal{U}_t(x))}{t} = \frac{1}{E(Z)} \int_{y=x}^{\infty} f_X(y)dy = \frac{\overline{F_X}(x)}{E(Z)}, \tag{10.15}$$

since the expected time between SP upward jumps out of level 0 is $E(Z)$, and the probability that such an SP jump exceeds level $x$ is $\overline{F_X}(x)$. Note that the SP *double jumps* in opposite directions at each renewal instant of the sequence $\{Z_n\}$. One jump is downward ending at level 0; the "opposite jump" is upward starting at level 0.

Equating (10.14) and (10.15) for rate balance across level $x$, results in the integral equation for $f_{\beta_X}(\cdot)$,

$$\int_{y=x}^{\infty} \frac{1}{\left(\int_{u=0}^{y} \frac{1}{r(u)} du\right)} f_{\beta_X}(y)dy = \frac{\overline{F_X}(x)}{E(Z)}. \tag{10.16}$$

In (10.16), we differentiate with respect to $x$ to yield

$$-\frac{1}{\left(\int_{u=0}^{x} \frac{1}{r(u)} du\right)} f_{\beta_X}(x) = -\frac{f_X(x)}{E(Z)}.$$

Hence

$$f_{\beta_X}(x) = \frac{\left(\int_{y=0}^{x} \frac{1}{r(y)} dy\right) f_X(x)}{E(Z)} = \frac{\left(\int_{y=0}^{x} \frac{1}{r(y)} dy\right) f_X(x)}{\int_{y=0}^{\infty} \frac{\overline{F}(y)}{r(y)} dy}. \qquad (10.17)$$

The dimension of $f_{\beta_X}(x)$ is $[content]^{-1}$.

**Limiting PDF of Total Life when $r(x) \equiv 1$**

Assume $r(x) \equiv 1$. Then $Z_n = X_n$ and $E(Z_n) = E(X_n) = \mu_X$ in value. However, the dimensions differ: thus $[X_n] = [content]$ and $[Z_n] = [Time]$. Formula (10.17) resembles the well known limiting pdf of total life (span) for a *standard* renewal process,

$$f_{\beta_X}(x) = \frac{x f_X(x)}{E(Z)} = \frac{x f_X(x)}{\mu_X}, \qquad (10.18)$$

except that the dimension of $f_{\beta_X}(x)$ is $[content]^{-1}$ instead of $[Time]^{-1}$. That is, in the variant of the GI/G/r(·) dam described, the "life" is measured in content dimensions.

**Remark 10.1** *This variant of GI/G/r(·) exhibits SP multiple jumps at the same instant (renewal instant). Recall that SP jumps in the state space* ***do not occur in Time****. (See Examples 2.2, 2.3 in Section 2.3, regarding SP multiple jumps.)*

**Example 10.2** *Suppose $r(x) = kx, x > 0$, where $k > 0$ is a constant. Then the inequality (10.1) does not hold. However, the SP returns to every level $x > 0$, however small. We may select a small $\varepsilon > 0$, such that when the content hits level $\varepsilon$ from above, a replenishment of new content is inserted (e.g., in Example 10.1, replace a battery with a new one when its charge decreases to $\varepsilon$ Coulombs).*

*Then for each positive $v > \varepsilon$,*

$$\int_{y=\varepsilon}^{v} \frac{1}{kx} dx = \frac{1}{k} \ln \frac{v}{\varepsilon} < \infty,$$

*so that the content returns to level $\varepsilon$ in a finite time.*

### 10.2.3 Renewal Process $\{Z_n\}$

**Excess Life, Age, Total Life of $\{Z_n\}$ Process**

Consider $\{Z_n\}$. Let $\gamma_{_Z}(t)$, $\delta_{_Z}(t)$, $\beta_{_Z}(t)$ denote the excess life, age, total life respectively, at instant $t > 0$. Denote the limiting r.v.'s by $\gamma_{_Z}$, $\delta_{_Z}$, $\beta_{_Z}$ respectively.

Define $\mathcal{G}(x) \equiv \int_{y=0}^{x} \frac{1}{r(y)} dy, x > 0$. Then $\mathcal{G}(x)$ is an increasing differentiable function of $x$ (since $\frac{d}{dx}\mathcal{G}(x) = \frac{1}{r(x)}$). This implies $\mathcal{G}^{-1}(x)$ (inverse of $\mathcal{G}(x)$) exists, and

$$\frac{d}{dx}\mathcal{G}^{-1}(x) = \frac{1}{\frac{d}{dx}\mathcal{G}(x)} = \frac{1}{\frac{1}{r(x)}} = r(x), x > 0.$$

Thus $\mathcal{G}^{-1}(x)$ is also an increasing (differentiable) function of $x$. The quantity $\mathcal{G}(x)$ is the time required for the SP to descend from level $x$ to level 0. The inverse $\mathcal{G}^{-1}(x)$ is the starting level of content, from which a descent to level 0 takes time $x$.

We may derive the pdf's of $\gamma_{_Z}$, $\delta_{_Z}$, $\beta_{_Z}$ from the the results for the pdf's of $\gamma_{_X}$, $\delta_{_X}$, $\beta_X$, respectively.

**Limiting PDF of Excess Life of $\{Z_n\}$**

The relation between $Z_n$ and $X(t)$ implies

$$\gamma_{_Z} \leq x \text{ iff } \gamma_X \leq \mathcal{G}^{-1}(x)).$$

Hence

$$F_{\gamma_{_Z}}(x) = F_{\gamma_{_X}}(\mathcal{G}^{-1}(x)). \tag{10.19}$$

(see Fig. 10.1).

Taking $\frac{d}{dx}$ on both sides of (10.19) and referring to (10.8) gives

$$\begin{aligned} f_{\gamma_{_Z}}(x) &= f_{\gamma_X}(\mathcal{G}^{-1}(x)) \cdot \frac{d}{dx}\mathcal{G}^{-1}(x) \\ &= f_{\gamma_X}(\mathcal{G}^{-1}(x)) \cdot r(x) = \frac{r(x) \cdot \overline{F_X}(\mathcal{G}^{-1}(x))}{r(\mathcal{G}^{-1}(x)) \int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)} dy}. \end{aligned} \tag{10.20}$$

The dimension of $f_{\gamma_{_Z}}(x)$ is $[Time]^{-1}$.

If $r(y) \equiv 1$ then $\mathcal{G}(x) = \mathcal{G}^{-1}(x) = x$. In that case $f_{\gamma_{_Z}}(x) = \frac{\overline{F_X}(x)}{\int_{y=0}^{\infty} \overline{F_X}(y) dy} = f_{\gamma_X}(x)$, but the dimension of $f_{\gamma_{_Z}}(x)$ is $[Time]^{-1}$, whereas the dimension of $f_{\gamma_X}(x)$ is $[content]^{-1}$.

**Limiting PDF of Age of $\{Z_n\}$**

In a similar manner as for the excess life, the age satisfies

$$\delta_Z \le x \ \text{ iff } \ \delta_X \le \mathcal{G}^{-1}(x).$$

Thus, $F_{\delta_Z}(x) = F_{\delta_X}(\mathcal{G}^{-1}(x))$. Taking $\frac{d}{dx}$ then yields

$$f_{\delta_Z}(x) = \frac{r(x)\overline{F_X}(\mathcal{G}^{-1}(x))}{r(\mathcal{G}^{-1}(x)) \int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)} dy}. \tag{10.21}$$

Thus $f_{\delta_Z}(x) \equiv f_{\gamma_Z}(x)$. The dimension of $f_{\delta_Z}(x)$ is $[Time]^{-1}$.

If $r(y) \equiv 1$ then $\mathcal{G}(x) = \mathcal{G}^{-1}(x) = x$. Then $f_{\delta_Z}(x) = \frac{\overline{F_X}(x)}{\int_{y=0}^{\infty} \overline{F_X}(y)dy} = f_{\delta_X}(x)$. The dimension of $f_{\delta_Z}(x)$ is $[Time]^{-1}$, whereas the dimension of $f_{\delta_X}(x)$ is $[content]^{-1}$.

**Limiting PDF of Total Life of $\{Z_n\}$**

Note that $\beta_Z \le x$ iff $\beta_X \le \mathcal{G}^{-1}(x)$. Hence, as for $f_{\delta_Z}(x)$, $f_{\gamma_X}(x)$ above, we obtain

$$\begin{aligned} f_{\beta_Z}(x) &= f_{\beta_X}(\mathcal{G}^{-1}(x)) \cdot \frac{d}{dx}\mathcal{G}^{-1}(x) \\ &= f_{\beta_X}(\mathcal{G}^{-1}(x)) \cdot r(x). \end{aligned}$$

From (10.17) we get

$$f_{\beta_Z}(x) = \frac{r(x) \cdot \left( \int_{y=0}^{x} \frac{1}{r(y)} dy \right) f_X(\mathcal{G}^{-1}(x))}{\int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)} dy}. \tag{10.22}$$

The dimension of $f_{\beta_Z}(x)$ is $[Time]^{-1}$ whereas the dimension of $f_{\beta_X}(x)$ is $[content]^{-1}$. When $r(x) \equiv 1$, $f_{\beta_Z}(x) = \frac{x f_X(x)}{\int_{y=0}^{\infty} \frac{\overline{F_X}(y)}{r(y)} dy}$, having dimension $[Time]^{-1}$.

### 10.2.4 Standard Renewal Process

We now obtain the steady-state pdf's for the standard renewal process as a special case of those for the replacement model. In the *standard* renewal process, we have $X_n = Z_n, n = 0, 1, 2...$, and $r(X(t)) \equiv 1$. The dimensions of $X_n$ and $Z_n$ are the same, usually $[Time]$. The pdf's $f_{\gamma_Z}(x)$, $f_{\delta_Z}(x)$, $f_{\beta_Z}(x), x > 0$ are the same as (10.9), (10.13), (10.18) respectively, and all have dimension $[Time]^{-1}$.

**Remark 10.2** *The LC derivations of the limiting pdf's of excess life, age and total life are* ***relatively*** *simple in the replacement model, and are much simpler for the standard renewal process. They are intuitive, and naturally suggest potential generalizations.*

**Remark 10.3** *The derivations in this section are based directly on my unpublished notes of June 18-July 26, 1992 [23]. These notes were motivated by a talk at the* $21^{st}$ *conference on Stochastic Processes and their Applications, York University, Toronto, June 15-19, 1992 by van Harn and Steutel (see Partial Bibliography).(Their generalization differs conceptually from LC.) Results using LC for* ***standard*** *renewal processes were published independently by Katayama (2002) (see Partial Bibliography).*

## 10.3 A Technique for Transient Distributions

In this section we outline a technique for deriving transient distributions of processes with a continuous or discrete state, and a continuous parameter. The technique is based on the general version of Theorem B (Theorem 4.1). We repeat formulas (4.1) and (4.2) of Theorem B here for reference. For fixed $t > 0$

$$E(\mathcal{I}_t(\boldsymbol{A})) = E(\mathcal{O}_t(\boldsymbol{A})) + P_t(\boldsymbol{A}) - P_0(\boldsymbol{A}),\ t \geq 0, \tag{10.23}$$

$$\frac{\partial}{\partial t}E(\mathcal{I}_t(\boldsymbol{A})) = \frac{\partial}{\partial t}E(\mathcal{O}_t(\boldsymbol{A})) + \frac{\partial}{\partial t}P_t(\boldsymbol{A}), t > 0, \tag{10.24}$$

where $\mathcal{I}_t(\boldsymbol{A})$ is the number of SP entrances and $\mathcal{O}_t(\boldsymbol{A})$ is the number of SP exits, of state-space set $\boldsymbol{A}$ during $[0,t]$. Let the parameter set be $\boldsymbol{T} = [0, \infty)$

**Remark 10.4** *If the* limiting *distribution of the state variable exists, it is obtained by taking the limit of the derived transient distribution as* $t \to \infty$.

### 10.3.1 State-space Set with Variable Boundary

**State Space $\boldsymbol{S} \subseteq \boldsymbol{R}$**

In formulas (10.23) and (10.24) assume set $\boldsymbol{A}$ depends on a continuous variable $x$ and define $\boldsymbol{A} \equiv \boldsymbol{A}_x, x \in \boldsymbol{S}$. Thus $x$ may be a state-space level, e.g., $\boldsymbol{T} \times \{x\}$ (a line in the $\boldsymbol{T}$-$\boldsymbol{S}$ coordinate system). For fixed $x$, replace formulas (10.23) and (10.24) by

$$E(\mathcal{I}_t(\boldsymbol{A}_x)) = E(\mathcal{O}_t(\boldsymbol{A}_x)) + P_t(\boldsymbol{A}_x) - P_0(\boldsymbol{A}_x) \tag{10.25}$$

$$\frac{\partial}{\partial t} E(\mathcal{I}_t(\boldsymbol{A}_x)) = \frac{\partial}{\partial t} E(\mathcal{O}_t(\boldsymbol{A}_x)) + \frac{\partial}{\partial t} P_t(\boldsymbol{A}_x). \tag{10.26}$$

Assume the following mixed partial derivatives exist and are equal, i.e.,

$$\frac{\partial^2}{\partial x \partial t} E(\mathcal{O}_t(\boldsymbol{A}_x)) = \frac{\partial^2}{\partial t \partial x} E(\mathcal{O}_t(\boldsymbol{A}_x)),$$
$$\frac{\partial^2}{\partial x \partial t} P_t(\boldsymbol{A}_x) = \frac{\partial^2}{\partial t \partial x} P_t(\boldsymbol{A}_x).$$

Taking $\frac{\partial}{\partial x}$ in (10.26) we obtain

$$\frac{\partial^2}{\partial x \partial t} E(\mathcal{I}_t(\boldsymbol{A}_x)) = \frac{\partial^2}{\partial t \partial x} E(\mathcal{O}_t(\boldsymbol{A}_x)) + \frac{\partial^2}{\partial t \partial x} P_t(\boldsymbol{A}_x). \tag{10.27}$$

## State Space $\boldsymbol{S} \subseteq \boldsymbol{R}^n$

Let $\{\boldsymbol{X}(t), t \geq 0\}$ denote a continuous-time, continuous-state stochastic process with $n$-dimensional state space $\boldsymbol{S} \subseteq \boldsymbol{R}^n$. The state space may be discrete or continuous. Let vector $\boldsymbol{x} = (x_1, ..., x_n)$, and state-space set $\boldsymbol{A}_{\boldsymbol{x}} = \cap_{i=1}^n (-\infty, x_i] \subseteq \boldsymbol{S}$. Then $P_t(\boldsymbol{A}_{\boldsymbol{x}}) = F_t(\boldsymbol{x}) = F_t(x_1, ..., x_n)$ is the joint cdf of the $n$ state variables at time $t \geq 0$.

From the general result (10.25) the joint cdf is given by

$$F_t(\boldsymbol{x}) = E(\mathcal{I}_t(\boldsymbol{A}_x)) - E(\mathcal{O}_t(\boldsymbol{A}_x)) + F_0(\boldsymbol{x})$$

where $F_0(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \ \boldsymbol{X}(0) \in \boldsymbol{A}_x, \\ 0 & \text{if } \ \boldsymbol{X}(0) \notin \boldsymbol{A}_x. \end{cases}$

Provided the derivatives exist, we obtain

$$\frac{\partial F_t(\boldsymbol{x})}{\partial x_i} = \frac{\partial}{\partial x_i} \left[ E(\mathcal{I}_t(\boldsymbol{A}_x)) - E(\mathcal{O}_t(\boldsymbol{A}_x)) \right], i = 1, ..., n,$$
$$\frac{\partial^n F_t(\boldsymbol{x})}{\partial x_1 \cdots \partial x_n} = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left[ E(\mathcal{I}_t(\boldsymbol{A}_x)) - E(\mathcal{O}_t(\boldsymbol{A}_x)) \right],$$
$$\frac{\partial F_t(\boldsymbol{x})}{\partial t} = \frac{\partial}{\partial t} \left[ E(\mathcal{I}_t(\boldsymbol{A}_x)) - E(\mathcal{O}_t(\boldsymbol{A}_x)) \right].$$

If $\frac{\partial E(\mathcal{I}_t(\boldsymbol{A}_x))}{\partial t}$, $\frac{\partial E(\mathcal{O}_t(\boldsymbol{A}_x))}{\partial t}$ can be expressed as functions of $F_t(\boldsymbol{x})$ or $f_t(\boldsymbol{x})$, then we may be able to derive an integro-differential equation for $F_t(\boldsymbol{x})$ or $f_t(\boldsymbol{x})$.

If $n = 1$ the state space is one-dimensional. We get $\boldsymbol{A}_x = (-\infty, x]$. Thus

$$f_t(x) = \frac{\partial}{\partial x}\left[E(\mathcal{I}_t((-\infty, x])) - E(\mathcal{O}_t((-\infty, x]))\right]$$

where $f_t(x)$ represents the transient pdf of $\boldsymbol{X}(t)$.

**LC Computation**

The expressions in this subsection can aid in estimating or computing the transient cdf and pdf of an $n$-dimensional continuous-parameter process using level crossing estimation or computation (LCE) for transient distributions. We will not expound on this transient LCE technique further in this monograph. Remarks 3.6 and 9.2 briefly discuss the technique.

## 10.4 Discrete-Parameter Processes

Let $\{X_n, n = 0, 1, 2, ...\}$ denote a discrete-parameter process taking values in a state space $\boldsymbol{S}$, which may be discrete or continuous. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be (measurable) subsets of $\boldsymbol{S}$. Let $P_n(\boldsymbol{A}) = P(X_n \in \boldsymbol{A})$ and $P_{m,n}(\boldsymbol{B}, \boldsymbol{C}) = P(X_m \in \boldsymbol{B}, \boldsymbol{X}_n \in \boldsymbol{C})$.

**Definition 10.1** *The SP* ***exits*** *set* $\mathbf{A}$ *at time* $n$ *if* $X_n \in \mathbf{A}$ *and* $X_{n+1} \notin \boldsymbol{A}$.
*The SP* ***enters*** *set* $\mathbf{A}$ *at time* $n$ *if* $X_{n-1} \notin \mathbf{A}$ *and* $X_n \in \mathbf{A}$.
$\mathcal{I}_n(\boldsymbol{A})) =$ ***number of*** *SP entrances into* $\mathbf{A}$ *during* $[0, n]$.
$\mathcal{O}_n(\boldsymbol{A}) =$ ***number of*** *SP exits out of* $\mathbf{A}$ *during* $[0, n]$.

We state a theorem for discrete-time processes which is analogous to Theorem B.

**Theorem 10.1** *Let* $\{X_n, n = 0, 1, 2, ...\}$ *be a discrete-time process with state space* $\boldsymbol{S}$. *Let* $\mathbf{A} \subseteq \boldsymbol{S}$.

$$E(\mathcal{I}_n(\boldsymbol{A})) = E(\mathcal{O}_n(\boldsymbol{A})) + P_n(\boldsymbol{A}) - P_0(\boldsymbol{A}). \tag{10.28}$$

*where* $P_0(\boldsymbol{A}) = \begin{cases} 1 \text{ if } X_0 \in A, \\ 0 \text{ if } X_0 \notin A. \end{cases}$

**Proof.** The proof is similar to that of Theorem 4.1 in Chapter 4, upon replacing $t$ by $n$. ■

### 10.4.1 Application to Markov Chains

Let $\{X_n, n = 0, 1, ...\}$ be a Markov chain with the discrete state space $\boldsymbol{S}$. For example, let $\boldsymbol{S} = \{0, \pm 1, \pm 2, ...\}$. Let the set $\boldsymbol{A} = j \in \boldsymbol{S}$. Then

$$E(\mathcal{I}_n(j)) = \sum_{i \neq j} \sum_{m=0}^{n-1} P_i^m P_{ij}, \quad \text{and} \quad E(\mathcal{O}_n(j)) = \sum_{i \neq j} \sum_{m=0}^{n} P_j^m P_{ji},$$

where $P_{ij}$ is the one-step transition probability from $i$ to $j$ and $P_j^m \equiv P_m(\boldsymbol{A}) = P_m(j)$. Substituting into (10.28) gives

$$P_j^n = \sum_{i \neq j} \sum_{m=0}^{n-1} P_i^m P_{ij} - \sum_{i \neq j} \sum_{m=0}^{n} P_j^m P_{ji} + P_j^0. \tag{10.29}$$

Assume the following limiting probabilities exist:

$$\lim_{n \to \infty} P_{ij}^n = \lim_{n \to \infty} P_{jj}^n = \lim_{n \to \infty} P_j^n \equiv \pi_j,$$

where $P_{ij}^n$ is the $n$-step transition probability from $i$ to $j$. That is, the chain is positive recurrent and aperiodic. Note that $\sum_{j \in \boldsymbol{S}} \pi_j = 1$. Dividing both sides of (10.29) by $n$ and letting $n \to \infty$ yields

$$\begin{aligned}
\lim_{n \to \infty} \frac{P_j^n}{n} &= \sum_{i \neq j} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} P_i^m \right) P_{ij} \\
&\quad - \sum_{i \neq j} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} P_j^m \right) P_{j,i} + \lim_{n \to \infty} \frac{P_j^0}{n}, \\
0 &= \sum_{i \neq j} \pi_i P_{ij} - \sum_{i \neq j} \pi_j P_{ji} + 0, \\
\sum_{i \neq j} \pi_j P_{ji} &= \sum_{i \neq j} \pi_i P_{ij}, \\
\pi_j (1 - P_{jj}) &= \sum_{i \neq j} \pi_i P_{ij}, \\
\pi_j &= \sum_{i \in \boldsymbol{S}} \pi_i P_{ij}, j \in \boldsymbol{S}.
\end{aligned}$$

Thus we have derived the classical equations for the limiting probabilities $\{\pi_j\}$ by using an LC method, namely

$$\begin{aligned}
&\pi_j = \textstyle\sum_{i \in \boldsymbol{S}} \pi_i P_{ij}, j \in \boldsymbol{S}, \\
&\textstyle\sum_{j \in \boldsymbol{S}} \pi_j = 1.
\end{aligned} \tag{10.30}$$

**Remark 10.5** *We have applied the discrete-time analog of Theorem B to a standard Markov chain in order to demonstrate its applicability to discrete-time discrete-state models. Note that Theorem B emphasizes the* ***system point aspect*** *of the SPLC method. SPLC utilizes SP entrance/exit rates of state-space sets. (SP level crossings are special cases of SP entrances and exits.)*

## 10.5 Semi-Markov Process

Consider a semi-Markov process (SMP) $\{X(t), t \geq 0\}$, with discrete state space $\boldsymbol{S}$ (also called a Markov renewal process ). Let the sojourn time in state $j \in \boldsymbol{S}$ have a general distribution with mean $\mu_j > 0$. The type of distribution of the sojourn time may differ from state to state; only the means are utilized in this analysis. At the end of a sojourn in state $i$, say instant $\tau^-$, assume $P\{X(t) = j|X(t^-) = i) = P_{ij}, j \neq i,\ j \in \boldsymbol{S}$. The matrix $\|P_{ij}\|$ is a Markov matrix. Assume the Markov chain with transition matrix $\|P_{ij}\|$ is positive recurrent and aperiodic so that the limiting probabilities $\pi_j, j \in \boldsymbol{S}$ exist.

Let $P_j(t) = P(X(t) = j), t \geq 0$; $P_j = \lim_{t\to\infty} P_j(t), j, j \in \boldsymbol{S}$. We shall derive the probabilities $P_j, j \in \boldsymbol{S}$, by using SPLC.

Consider a sample path of $\{X(t)\}$. Let $T_t(i)$ denote the total time spent by the SP in state $i$ during $(0, t)$. Then

$$E(T_t(i)) = \int_{s=0}^{t} P_i(s)ds. \tag{10.31}$$

The expected number of SP exits from state $i$ during $(0, t)$ is $\frac{E(T_t(i))}{\mu_i}$ since the mean of each sojourn time in $i$ is $\mu_i$. The expected number of SP $i \to j$ transitions during $(0, t)$ is $\frac{E(T_t(i))}{\mu_i} P_{ij}$. The expected total number of SP transitions into (*entrances* into) state $j$ during $(0, t)$ is

$$E(\mathcal{I}_t(j)) = \sum_{i \neq j} \frac{E(T_t(i))}{\mu_i} P_{ij}. \tag{10.32}$$

By a similar argument, the expected number of SP *exits* out of $j$ during $(0, t)$ is

$$E(\mathcal{O}_t(j)) = \frac{E(T_t(j))}{\mu_j}. \tag{10.33}$$

Substituting from (10.32) and (10.33) into Theorem B (10.23) gives

$$\sum_{i \neq j} \frac{E(T_t(i))}{\mu_i} P_{ij} = \frac{E(T_t(j))}{\mu_j} + P_j(t) - P_j(0). \tag{10.34}$$

(We assume the interchange of summation and the limit operation is valid. This applies if, e.g., $\boldsymbol{S}$ is finite.)

From (10.31), the proportion of time the SP is in state $i$ is

$$\lim_{t \to \infty} \frac{E(T_t(i))}{t} = P_i, i \in \boldsymbol{S}.$$

Also

$$\lim_{t \to \infty} \frac{P_j(t)}{t} = \lim_{t \to \infty} \frac{P_j(0)}{t} = 0,$$

since $0 \leq P_j(t) \leq 1, t \geq 0$. We divide (10.34) by $t > 0$ and let $t \to \infty$. This gives

$$\sum_{i \neq j} \frac{P_i}{\mu_i} P_{ij} = \frac{P_j}{\mu_j}, j \in \boldsymbol{S} \tag{10.35}$$

Suppose $\sum_{j \in S} \frac{1}{\mu_j} P_j = K > 0$. Then $\sum_{j \in S} \left( \frac{1}{K\mu_j} P_j \right) = 1$. Dividing (10.35) by $K$ and transposing terms gives the system of equations for $\{P_i\}$,

$$\begin{aligned} &\tfrac{1}{K\mu_j} P_j = \textstyle\sum_{i \neq j} \left( \tfrac{1}{K\mu_j} P_i \right) P_{ij}, j \in \boldsymbol{S} \\ &\textstyle\sum_{j \in S} \left( \tfrac{1}{K\mu_j} P_j \right) = 1. \end{aligned} \tag{10.36}$$

The system of equations (10.36) for $\left\{ \left( \frac{1}{K\mu_j} P_j \right) \right\}$ is identical to the system (10.30) for $\{\pi_j\}$. Thus

$$\begin{aligned} \frac{1}{K\mu_j} P_j &= \pi_j, j \in \boldsymbol{S}, \\ P_j &= (\pi_j \mu_j) K, j \in \boldsymbol{S}. \end{aligned} \tag{10.37}$$

We obtain $K$ from the normalizing condition

$$\sum_{j \in S} P_j = K \sum_{j \in S} \pi_j \mu_j = 1,$$

namely

$$K = \frac{1}{\sum_{j \in S} \pi_j \mu_j}. \tag{10.38}$$

Substituting from (10.37) into (10.38) gives the well known formula

$$P_j = \frac{\pi_j \mu_j}{\sum_{j \in \boldsymbol{S}} \pi_j \mu_j}, \ j \in \boldsymbol{S}. \tag{10.39}$$

The key steps in this SPLC derivation of (10.39) are: (1) obtain expressions for the expected SP entrance and exit rates of a state; (2) apply formula (10.23) of Theorem B; (3) divide by $t$ and take $\lim_{t\to\infty}$; (4) evaluate the constant $K$ by recognizing the role of the **linear Markov-chain equations** (10.30) for $\{\pi_j\}$.

## 10.6 Non-homogeneous Pure Birth Processes

Let $\{X(t), t \geq 0\}$ denote the number of births during $(0,t), t > 0$. Let $X(0) = i$, where $i$ is a non-negative integer. Consider the sequence of positive functions (birth rates) $\{\lambda_k(t), k = i, i+1, ...; i = 0, 1, ...\}$ such that

$$P(X(t+h) - X(t) = 1 | X(t) = k) = \lambda_t(k)h + o(h),$$
$$P(X(t+h) - X(t) = 0 | X(t) = k) = 1 - \lambda_t(k)h + o(h),$$

where $h > 0$.

Define $P_n(t) = P(X(t) = n)$. We shall compute $P_n(t), t > 0, n = 0, 1, 2, ...$; by utilizing Theorem B, i.e., (10.23) and (10.24).

The expected number of SP entrances into state $i$ during $(0,t)$ is $E(\mathcal{I}_t(i)) = 0$, since $X(0) = i$, and $X(\cdot)$ never returns to $i$, once it increases from $i$ to $i+1$. On the other hand the expected number of SP exits out of state $i$ during $(0,t)$ is $E(\mathcal{O}_t(i)) = \int_{s=0}^{t} \lambda_s(i) P_i(s) ds$, since an SP $i \to i+1$ exit can occur at any instant $s \in (0,t)$. Note that $P_i(0) = 1$. Substituting $E(\mathcal{I}_t(i))$, $E(\mathcal{O}_t(i))$, $P_i(0)$ into (10.23), we obtain

$$0 = \int_{s=0}^{t} \lambda_s(i) P_i(s) ds + P_i(t) - 1. \tag{10.40}$$

Differentiating (10.40) with respect to $t$ gives

$$\frac{d}{dt} P_i(t) + \lambda_t(i) P_i(t) = 0$$

having solution

$$P_i(t) = e^{-m_t(i)}, t \geq 0, \tag{10.41}$$

where $m_t(i) = \int_{s=0}^{t} \lambda_s(i) ds$, since $P_i(0) = 1$.

Next, consider an arbitrary state $j > i$. Then

$$E(\mathcal{I}_t(j)) = \int_{s=0}^{t} \lambda_{j-1}(s) P_{j-1}(s) ds, \tag{10.42}$$

$$E(\mathcal{O}_t(j)) = \int_{s=0}^{t} \lambda_j(s) P_j(s) ds. \tag{10.43}$$

Substituting from (10.42) and (10.43) into (10.23) gives

$$\int_{s=0}^{t} \lambda_s(j-1) P_{j-1}(s) ds = \int_{s=0}^{t} \lambda_s(j) P_j(s) ds + P_j(t) - 0. \tag{10.44}$$

Taking $\frac{d}{dt}$ in (10.44) yields

$$\frac{d}{dt} P_j(t) + \lambda_t(j) P_j(t) = \lambda_t(j-1) P_{j-1}(t),$$

with solution

$$P_j(t) = e^{-m_t(j)} \int_{s=0}^{t} e^{m_s(j)} \lambda_s(j-1) P_{j-1}(s) ds. \tag{10.45}$$

Formula (10.45) provides a recursive solution for $P_j(t), j = i, i+1, \ldots$ .

### 10.6.1 Non-homogeneous Poisson Process

The non-homogeneous Poisson process is a special case of the pure growth process. Assume $X(0) = 0$, $\lambda_t(j) \equiv \lambda_t$ independent of the state, so that $m(t) = \int_{s=0}^{t} \lambda_s ds$. Setting $i = 0$ gives $P_0(t) = e^{-m(t)}$. From (10.45) we obtain (by induction) the well known formula

$$P_n(t) = e^{-m(t)} \frac{(m(t))^n}{n!}, n = 0, 1, 2, \ldots . \tag{10.46}$$

Formula (10.46) is a Poisson distribution with mean $m(t)$. The $\{P_n(t)\}$ for the standard Poisson process are obtained by letting $\lambda_t \equiv \lambda$, so that $m(t) \equiv \lambda t$.

### 10.6.2 Yule Process

The Yule process is a special case of the pure growth process. Assume $X(0) = 1$ and $\lambda_t(i) = i\lambda, t \geq 0, i = 1, 2, \ldots$ . Thus the growth rate is directly proportional to the current population, but independent of $t$. Then $P_1(t) = e^{-\lambda t}$ (= probability of no births in $(0, t)$). Using (10.45) and mathematical induction, we obtain the well known geometric distribution for the Yule process

$$P_n(t) = (1 - e^{-\lambda t})^{n-1} e^{-\lambda t}, n = 1, 2, ... \ . \tag{10.47}$$

For completeness, we include the probability $P_{ik}(t)$ that $i$ *independent* Yule processes with the same parameter $\lambda$, yield a sum equal to $k \geq i$ at time $t > 0$ (total number of individuals $= k$ at time $t$). Assume each process starts in state 1 at time 0. Since $P_n(t)$ in (10.47) is a geometric distribution, we obtain a negative binomial distribution

$$P_{ik}(t) = \binom{k-1}{i-1} e^{-i\lambda t}(1 - e^{-\lambda t})^{k-i}, k = i, i+1, ... \ . \tag{10.48}$$

Formula (10.48) can be derived in several ways (e.g., [74], [91]). We shall outline a direct proof using LC.

We derive in a similar manner as for (10.45),

$$P_{ik}(t) = (k+1)\lambda e^{-k\lambda t} \int_{s=0}^{t} e^{k\lambda s} P_{i,k-1}(s) ds + C_k e^{-k\lambda t}, k \geq i, \tag{10.49}$$

where $C_k = \begin{cases} 1 \text{ if } k = i, \\ 0 \text{ if } k > i. \end{cases}$ Now, $P(\text{no births in } (0,t)) = P(\mathrm{E}_{i\lambda} > t)$ where $\mathrm{E}_{i\lambda}$ is an exponentially distributed r.v. with mean $\frac{1}{i\lambda}$. Hence

$$P_{ii} = e^{-i\lambda t}. \tag{10.50}$$

Thus (10.48) holds for $k = i$. From (10.50) and (10.49) with $k = i+1$, we obtain

$$P_{i,i+1}(t) = ie^{-i\lambda t}\left(1 - e^{-\lambda t}\right) = \binom{i+1-1}{i-1} e^{-i\lambda t}\left(1 - e^{-\lambda t}\right). \tag{10.51}$$

Therefore (10.48) holds for $k = i + 1$.

Assume (10.48) holds for an arbitrary integer $k > i$. We then show using (10.49) that it holds for $k+1$. Hence it holds for all $k = i, i+1, ...$, by mathematical induction.

## 10.7 Revisit of Transient M/G/1 Queue

We very briefly revisit the transient M/G/1 queue of Section 3.2. It is readily proved by a slight generalization of the proofs in Section 3.2, that the theory holds for models where the arrival rate $\lambda$ and cdf of service time $B(x)$ depend on time. Denote them by $\lambda_t$ and $B_t(x), x \geq 0$, respectively. We obtain

$$\begin{aligned} f_t(x) &= \tfrac{\partial}{\partial t}F_t(x) + \lambda_t\overline{B}_t(x)P_0(t) \\ &\qquad +\lambda_t \textstyle\int_{y=0}^{x} \overline{B}_t(x-y)f_t(y)dy, x > 0, \qquad (10.52) \\ f_t(0) &= \tfrac{\partial}{\partial t}P_0(t) + \lambda_t P_0(t). \end{aligned}$$

The solution of the differential equation for $P_0(t)$ in (10.52) is

$$P_0(t) = e^{-m(t)} \int_{s=0}^{t} e^{m(s)} f_s(0)ds + P_0(0)e^{-m(t)}, \qquad (10.53)$$

where $m(t) = \int_{s=0}^{t} \lambda_s ds$ and $P_0(0) = \begin{cases} 1 \text{ if } W(0) = 0, \\ 0 \text{ otherwise.} \end{cases}$

## 10.8 Pharmacokinetic Model

We outline an LC approach to pharmacokinetics with a brief discussion of a simplified one-compartment model. We assume bolus dosing, i.e., a full dose is absorbed into the blood stream immediately at a dosing instant. Also, inter-dose times are $\underset{dist}{=} \mathrm{E}_\alpha$. Thus doses occur in a Poisson process at rate $\lambda$. This assumption is valid outside of a controlled environment. Statistical tests have shown that many patients take certain medications non-uniformly over time in a Poisson process [33]. We first suppose the dose amounts are deterministic of size $D$.

We assume first-order kinetics. That is, the concentration of the drug in the blood stream decays at a rate which is proportional to the concentration. This is equivalent to a plot of the concentration over time having a negative exponential shape between doses (similar to Fig. 10.2).

This model is equivalent to an M/D/r($\cdot$) dam (Section 6.2). Let $W(t), t \geq 0$, denote the drug concentration at time $t$. Let the dose times be $\{\tau_n\}$. $\tau_n < \tau_{n+1}, n = 0, 1, 2, ...$ , where $\tau_0 \equiv 0$. The decay rate is

$$\frac{dW(t)}{dt} = -kW(t), \tau_n \leq t < \tau_{n+1}, n = 0, 1, 2, ..., \qquad (10.54)$$

where $k > 0$. The dimension of the concentration $W(t)$ is $[W(t)] = \left[\frac{Mass}{Volume}\right]$; $\left[\frac{dW(t)}{dt}\right] = \left[\frac{Mass}{Volume}\right] \cdot \left[Time^{-1}\right]$; $[k] = [Time]^{-1}$.

Let $f(x), x > 0$ denote the steady-state pdf of concentration. The steady-state probability that the concentration is zero, is equal to 0. This is because a sample path never declines to level 0 once dosing begins, due to the negative exponential shape of the decay. In theory, the concentration of the drug never vanishes. In practice, it goes to 0 or is negligible. (We are not discussing the treatment effects of dosing; only the concentration dynamics.)

### 10.8.1 Equation for PDF of Concentration

Consider a sample path of $\{W(t)\}$. Fix level $x > 0$ (Fig. 10.2). The SP downcrossing rate of level $x$ is $kxf(x)$. The SP upcrossing rate of $x$ is equal to $\lambda F(x) - \lambda F(x - D)$ (see Section 3.8). Rate balance across $x$ gives an equation for$f(x)$ and $F(x)$, namely

$$kxf(x) = \lambda F(x) - \lambda F(x - D), x > 0. \tag{10.55}$$

In integral equation (10.55) for the $F(\cdot)$, note that $F(x - D) = 0$ for $x \in (0, D)$. Also

$$\frac{f(x)}{F(x)} = \frac{d \ln F(x)}{dx} = \frac{\lambda}{kx},$$

with solution

$$F(x) = Ax^{\frac{\lambda}{\kappa}}, x \in (0, D), \tag{10.56}$$

where $A$ is a positive constant. The solution for $F(x)$ on the state-space intervals $[iD, (i + 1)D), i = 1, 2, ...$, can be obtained by an iteration procedure (not carried out here). We add that $F(x)$ is continuous for all $x > 0$. This continuity property helps to solve for $F(x)$ on successive state-space intervals $[iD, (i + 1)D), i = 1, 2, ...$ , in terms of $A$. The constant $A$ in (10.56) is then determined using the normalizing condition $F(\infty) = 1$. Once $F(x)$ is obtained, we can determine $f(x)$ by substituting into (10.55) (as in Section 3.8 ). Alternatively, we may solve for $f(x)$ using LC estimation, or a *hybrid* LC estimation procedure since we have a partial analytical solution in (10.56) (see Section 9.6).

### 10.8.2 Exponentially Distributed Doses

We may rationalize a model using exponentially distributed doses if the amount absorbed is affected by the dosing environment (e.g., acidity, presence of enzymes, interaction with other medications, etc.). Another

| M/G/r(·) Dam | Pharmacokinetic Model |
|---|---|
| Input instant | Bolus dose instant |
| Input amount (jump size) | Dose amount (jump size) |
| Content $W(t), t \geq 0$ | Concentration $W(t), t \geq 0$ |
| Sample-path slope $-r(x), x > 0$ | Sample-path slope $-r(x), x > 0$ |
| CDF/PDF of content | CDF/PDF of concentration |
| Mean content | Average drug concentration |
| Variance of content | Variance of concentration |

Table 10.1: M/G/r(·) Dam versus Pharmakokinetic model

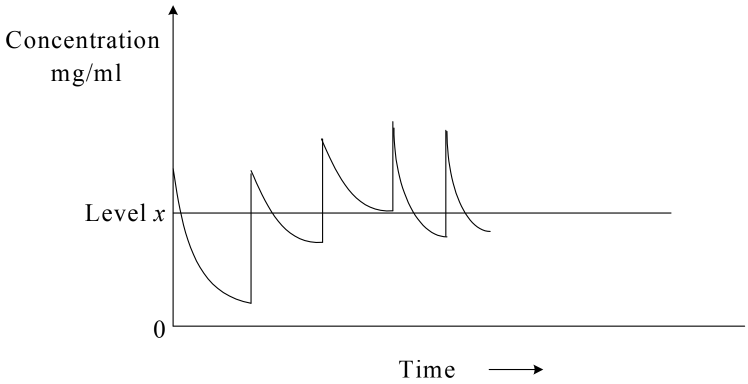


Figure 10.2: Sample path of drug concentration in one-compartment model with bolus dosing and first-order kinetics

instance could occur when eye drops are instilled by a patient, say approximately every six hours. The sizes of the individual drops may vary considerably, due to usually using a hand-squeezed container. The location on the cornea of the instillation may vary from dose to dose, thereby affecting absorption. This could create random increases in concentration with the successive doses during a dosing regime. Similar remarks apply to fast-acting sprays, such as nitrolingual pump sprays, or to nasal sprays. Also, for certain drugs it may be feasible to randomize dose sizes as an exponential random variable inherently in a prescription. Such randomization may tend to decrease variability in the long run concentration during the dosing regime.

Assume the bolus dose amounts are random, distributed as $E_\mu$. Then the equation for the pdf of concentration is

$$kxf(x) = \lambda \int_{y=0}^{x} e^{-\mu(x-y)} f(y)dy. \tag{10.57}$$

Equation (10.57) has the solution

$$f(x) = \frac{1}{\Gamma\left(\frac{\lambda}{k}\right)}(\mu x)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu x}\mu, x > 0. \tag{10.58}$$

where $\Gamma(\cdot)$ is the Gamma function (see Section 6.4). Let $W$ denote the steady-state concentration. The mean and second moment of $W$ are

$$E(W) = \frac{\lambda}{k\mu},\ E(W^2) = \frac{\lambda}{k\mu^2}\left(\frac{\lambda}{k}+1\right).$$

The variance of $W$ is

$$Var(X) = E(W^2) - (E(W))^2 = \frac{\lambda}{k\mu^2}.$$

We can find the probability that the steady-state concentration is between two threshold limits, say $\alpha < \beta$, using

$$P(\alpha < \text{concentration} < \beta) = \int_{x=\alpha}^{\beta} \frac{1}{\Gamma\left(\frac{\lambda}{k}\right)}\mu(\mu x)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu x}dx. \tag{10.59}$$

The information in (10.59) may be useful when dosing continues for a long time, e.g., when administering the blood thinner coumadin. If the concentration is $< \alpha$ coumadin is not effective for the intended treatment. If the concentration is $> \beta$ the blood becomes too thin.

The type of analysis outlined briefly here can be extended to various pharmacokinetic models of varying complexity.

**Remark 10.6** *We mention in passing that it is possible to apply Theorem B to compute the* ***time-dependent pdf and cdf of concentration*** *(see formulas (10.23) - (10.26)). Knowledge of transient distributions may be useful in dosing regimes where it important to estimate the concentration after a short dosing duration.*

**Remark 10.7** *Some related stochastic models have characteristics in common with the pharmacokinetic model. One group of models involves consumer response (CR) to non-uniform advertisements [30]. Such models can be analyzed along similar lines, using LC.*

## 10.9 Counter Models

We consider the transient total output of type-1 and type-2 counters. We first treat a type-2 counter.

### 10.9.1 Type-2 Counter

Consider a type-2 counter. Electrical pulses arrive in a Poisson process at rate $\lambda$. Each *arriving* pulse is followed immediately by a fixed *locked* period of length $D > 0$, during which new arrivals cannot be detected by the counter. If a new arrival occurs at a time $t$ when the counter is locked, then the locked period is extended to time $t + D$. Thus the locked time "telescopes". Assume the locked periods are $\underset{dist}{=} L$; note that $L \geq D$. Arrivals can be detected only when the counter is unlocked or *free*. Assume that the counter is free at time 0.

Let the amplitudes of the pulses be $\underset{dist}{\equiv} X$, having cdf $B(y), y > 0$. Let $\eta_i(t), t \geq \tau_i$, denote the output at time $t$ due to the *detected pulse* $X_i$ occurring at $\tau_i$. Assume that $\eta_i(t)$ dissipates at rate

$$\frac{d\eta_i(t)}{dt} = -k \cdot \eta_i(t), t > \tau_i, \tag{10.60}$$

where the constant $k > 0$ is the same for all $i = 1, 2, ...$ .

Let $\eta_t$ denote the *total output* at time $t$, due to all *registered* pulses that arrive during $(0, t)$ (see Fig. 10.3). Then

$$\eta_t = \sum_{i=1}^{n} \eta_i(t), \tau_n \leq t < t_{n+1}, n = 1, 2, ..., \tag{10.61}$$

$$\frac{d}{dt}\eta_t = -k \sum_{i=1}^{n} \eta_i(t) = -k\eta_t, \tau_n \leq t < t_{n+1}, n = 1, 2, .... \tag{10.62}$$

Denote the cdf and pdf of $\eta_t$ by $F_t(x)$ and $f_t(x) = \frac{d}{dt}F_t(x)$,$x > 0$, wherever the derivative exists.

### 10.9.2 Sample Path of Total Output

A sample path of the process $\{\eta_t, t \geq 0\}$ consists of segments that decay exponentially with decay constant $k$, between the $\tau_i$'s (Fig. 10.3). That is,

$$\eta_t = \sum_{i=1}^{n} X_i e^{-k(t-\tau_i)}, \tau_n \leq t < t_{n+1}, n = 1, 2, .... \tag{10.63}$$

Note that a sample path cannot descend to level 0 due to exponential decay.

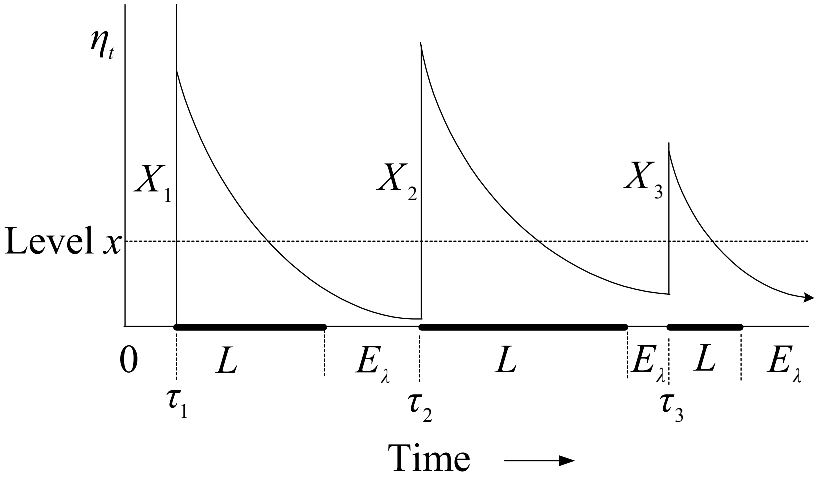


Figure 10.3: Sample path of total output $\eta_t$ for type-2 counter model. Locked periods are each $\underset{dist}{=} L \geq D$. Arrivals during $L$ are not detected, but extend the locked period. Arrival process of pulses is Poisson at rate $\lambda$.

**Probability that the Counter is Free at Time $t$**

Let $p(t) = P$(counter is free at time $t$). Then

$$p(t) = \begin{cases} e^{-\lambda t}, 0 < t < D, \\ e^{-\lambda D}, t \geq D. \end{cases} \tag{10.64}$$

The reason for (10.64) is that for $0 < t < D$, the counter is free at $t$ iff there is no arrival in $(0, t)$, which has probability $e^{-\lambda t}$. For $t \geq D$, the counter is free at time $t$ iff there has not been an arrival during the interval $(t - D, t)$. The probability of this event is $e^{-\lambda D}$, by the memoryless property of $\mathrm{E}_\lambda$ (see, e.g., [74]).

### 10.9.3 Integro-differential Equation for PDF of Output

Consider level $x > 0$ in the state space; and state-space set $A_x = (0, x]$. We can show as in Theorem 6.2.8, that for SP *entrances* into set $A_x$ (downcrossings of level $x$)

$$\frac{\partial}{\partial t} E(\mathcal{I}_t(A_x)) = \frac{\partial}{\partial t} E(\mathcal{D}_t(x)) = kx f_t(x), t > 0. \tag{10.65}$$

For SP *exits* out of $A_x$ (upcrossings of level $x$)

$$\begin{aligned}&\tfrac{\partial}{\partial t}E(\mathcal{U}_t(x)) = \tfrac{\partial}{\partial t}E(\mathcal{O}_t(A_x))\\ &= \begin{cases} \lambda e^{-\lambda t} \cdot \int_{y=0}^{x} \overline{B}(x-y) f_t(y) dy, x > 0, 0 < t < D, \\ \lambda e^{-\lambda D} \cdot \int_{y=0}^{x} \overline{B}(x-y) f_t(y) dy, x > 0, t \geq D. \end{cases}\end{aligned} \tag{10.66}$$

Substituting (10.65) and (10.66) into Theorem B (noting that $\frac{\partial}{\partial t}F_t(x) = -\frac{\partial}{\partial t}(1 - F_t(x))$, we get integro-differential equations for the pdf $f_t(x)$,

$$\begin{aligned} kxf_t(x) = \lambda e^{-\lambda t} \cdot \int_{y=0}^{x} \overline{B}(x-y) f_t(y) dy - \tfrac{\partial}{\partial t}(1 - F_t(x)),\\ x > 0, 0 < t < D, \end{aligned} \tag{10.67}$$

$$\begin{aligned} kxf_t(x) = \lambda e^{-\lambda D} \cdot \int_{y=0}^{x} \overline{B}(x-y) f_t(y) dy - \tfrac{\partial}{\partial t}(1 - F_t(x)),\\ x > 0, t \geq D, \end{aligned} \tag{10.68}$$

since the arrival rate is $\lambda$, and an arrival can be registered at time $t$ iff the counter is unlocked or free at time $t$.

### 10.9.4 Expected Value of Total Output

We obtain the expected value of $\eta_t$ by integrating both sides of (10.67) and (10.68) with respect to $x \in (0, \infty)$. (In (10.67) and (10.68), we assume that $\frac{\partial}{\partial t}F_t(x)$ is continuous with respect to $t > 0$. This condition is required to apply Fubini's Theorem on interchanging the operations $\int_{x=0}^{\infty}$ and $\frac{\partial}{\partial t}$.)

Upon integrating (10.67) we obtain

$$\begin{aligned} kE(\eta_t) &= \lambda e^{-\lambda t}E(X) - \frac{\partial}{\partial t}E(\eta_t),\\ \frac{\partial}{\partial t}e^{kt}E(\eta_t) &= \lambda e^{(k-\lambda)t}E(X),\\ E(\eta_t) &= \frac{\lambda e^{-\lambda t}E(X)}{k-\lambda} + Ae^{-kt}, 0 < t < D, (A \text{ constant}),\\ E(\eta_t) &= \frac{\lambda E(X)}{k-\lambda}\left(e^{-\lambda t} - e^{-kt}\right), 0 < t < D, \end{aligned} \tag{10.69}$$

since $E(\eta_0) = 0$.

Integrating (10.68), we obtain

$$\begin{aligned} kE(\eta_t) &= \lambda e^{-\lambda D}E(X) - \frac{\partial}{\partial t}E(\eta_t), \\ \frac{\partial}{\partial t}e^{kt}E(\eta_t) &= \lambda e^{-\lambda D}E(X)e^{kt}, \\ E(\eta_t) &= \frac{\lambda e^{-\lambda D}E(X)}{k} + Ae^{-kt}, t \geq D, \end{aligned} \tag{10.70}$$

where the constant $A$ is given by

$$A = \lambda E(X)\left(\frac{e^{-(\lambda-k)D}-1}{k-\lambda} - \frac{e^{-(\lambda-k)D}}{k}\right).$$

To obtain the value of $A$, we have used the fact that $\eta_{D^-} = \eta_D$ (see Fig. 10.3), which implies continuity of $E(\eta_t)$ at $t = D$ $(a.s.)$. Thus, from (10.69), $E(\eta_D) = \frac{\lambda E(X)}{k-\lambda}\left(e^{-\lambda D} - e^{-kD}\right)$.

If $t \to \infty$, then (10.70) reduces to

$$\lim_{t\to\infty} E(\eta_t) = \frac{\lambda e^{-\lambda D}E(X)}{k}.$$

If $D = 0$, then $A = -\frac{\lambda E(X)}{k}$. We then obtain $E(\eta_t) = \frac{\lambda E(X)}{k}\left(1 - e^{-kt}\right)$ and $\lim_{t\to\infty} E(\eta_t) = \frac{\lambda E(X)}{k}$, as in [74].

### 10.9.5 Type-1 Counter

A type-1 counter differs from a typre-2 counter (Subsection 10.9.1) only in the locking mechanism. In a type-1 counter, only *registered* arrivals when the counter is free, generate locked periods. Arrivals when the counter is locked, have no effect on the locked period. Thus every locked period has length $D > 0$. Aside from the locking mechanism, we generally use the same notation and assumptions for type-1 and type-2 counters.

Thus equations (10.60) - (10.63) hold for type-1 counters.

### 10.9.6 Sample Path of Total Output

A sample path of the process $\{\eta_t, t \geq 0\}$ consists of segments that decay exponentially with decay constant $k$, between the $\tau_i$'s (Fig. 10.4).

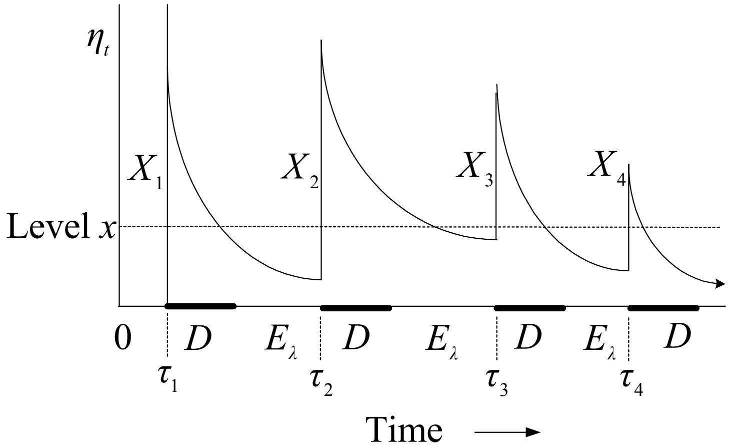


Figure 10.4: Sample path of total output $\eta_t$ for type-1 counter model. Locked periods are each $= D$ (arrivals not detected therein, and have no effect on locked period) Arrival process of pulses is Poisson at rate $\lambda$.

**Probability that the Counter is Free at Time $t$**

The probability that the counter is free to register a newly arriving pulse at time $t$ is given by the following recursion ([70]).

$$\begin{aligned} p_1(t) &= e^{-\lambda t}, 0 < t < D, \\ p_2(t) &= e^{-\lambda(t-D)} p_1(D) + \frac{(\lambda(t-D))\, e^{-\lambda(t-D)}}{1!}, D \le t < 2D, \\ &\cdots \\ p_n(t) &= \sum_{j=1}^{n-1} \frac{(\lambda(t-(n-1)D))^{j-1} \cdot e^{-\lambda(t-(n-1)D)}}{(j-1)!} p_{n-j}((n-j)D) \\ &\quad + \frac{(\lambda(t-(n-1)D))^{n-1}\, e^{-\lambda(t-(n-1)D)}}{(n-1)!}, \\ &\quad (n-1)\, D \le t < nD, n = 1, 2, \ldots\ , \end{aligned} \tag{10.71}$$

where $\sum_{j=1}^{0} \equiv 0$.

**Remark 10.8** *Let $p(t) = P$(the counter is free at time $t$),$t \ge 0$. Then $\lim_{t\to\infty} p(t) = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+D}$ (a known result for alternating renewal processes [49]). Hence we have proved using probability arguments that*

$$\lim_{n\to\infty} p_n(nD) = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+D},$$

*where $p_n(nD)$ is the series obtained by substituting $t = nD$ in (10.71). More strongly, for every $\alpha \in [0,1]$,*

$$\lim_{n\to\infty} p_n(\alpha(n-1)D + (1-\alpha)nD) = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+D}.$$

### 10.9.7 Integro-differential Equation for PDF of Output

Consider level $x > 0$ in the state space; and state-space set $A_x = (0, x]$. We can show as in Theorem 6.2.8, that for SP entrances into set $A_x$,

$$\frac{\partial}{\partial t}E(\mathcal{I}_t(A_x)) = \frac{\partial}{\partial t}E(\mathcal{D}_t(x)) = kxf_t(x), t > 0. \tag{10.72}$$

For SP exits out of $A_x$,

$$\begin{aligned}\frac{\partial}{\partial t}E(\mathcal{O}_t(A_x)) &= \frac{\partial}{\partial t}E(\mathcal{U}_t(x)) \\ &= \lambda p_n(t)\cdot\int_{y=0}^{x}\overline{B}(x-y)f_t(y)dy, \\ &\qquad (n-1)\,D \le t < nD, n = 1,2,\ldots\,. \end{aligned} \tag{10.73}$$

In (10.73), the factor $p_n(t)$ occurs because an arrival is registered iff it arrives when the counter is free.

Substituting (10.72) and (10.73) into Theorem B, we get an integro-differential equation for the pdf $f_t(x)$,

$$\begin{aligned} kxf_t(x) &= \lambda p_n(t)\cdot\int_{y=0}^{x}\overline{B}(x-y)f_t(y)dy + \frac{\partial}{\partial t}F_t(x), x > 0, \\ kxf_t(x) &= \lambda p_n(t)\cdot\int_{y=0}^{x}\overline{B}(x-y)f_t(y)dy, -\frac{\partial}{\partial t}(1-F_t(x)), x > 0, \\ &\qquad (n-1)\,D \le t < nD, n = 1,2,\ldots\,. \end{aligned} \tag{10.74}$$

### 10.9.8 Expected Value of Total Output

We obtain the expected value of $\eta_t$ by integrating both sides of the integral equations (10.74) with respect to $x \in (0,\infty)$. We obtain

$$E(\eta_t) = \frac{\lambda E(X)}{k-\lambda}\left(e^{-\lambda t} - e^{-kt}\right), 0 < t < D \tag{10.75}$$

in the same manner as (10.69). Similarly, we can obtain $E(\eta_t), nD \le t < (n+1)\,D, n = 1,2,\ldots$ . (We shall not carry out this computation here.)

**Remark 10.9** *If the locked period has value $D = 0$, then $p_n(t) = 1, n = 1, 2, \ldots$ . Then every arrival is registered. We then obtain the known result $E(\eta_t) = \frac{\lambda E(X)}{k}\left(1 - e^{-kt}\right), t > 0$ (e.g., [74]).*

*If $t \to \infty$, then (10.75) reduces to $\lim_{t\to\infty} E(\eta_t) = \frac{\lambda E(X)}{k}$.*

**Remark 10.10** *When there is no locked time ($D = 0$), the foregoing type-1 and type-2 counter models coincide with an M/G/r(·) dam with efflux rate proportional to content. Thus, results for a dam with $r(x) = kx, x > 0$, can be derived as a special case of either counter model.*

## 10.10 A Dam with Alternating Influx and Efflux

Consider a dam in which the content alternates between random periods of continuous influx and continuous efflux. We arbitrarily classify periods of emptiness as being parts of periods of efflux, for notational convenience. Periods of efflux are $\underset{dist}{=} \mathrm{E}_{\lambda_1}$ and periods of efflux are $\underset{dist}{=} \mathrm{E}_{\lambda_2}$. Let $W(t) \geq 0$ denote the content of the dam at time $t \geq 0$. Assume that during an influx period, the rate of *increase* of content is $\frac{dW(t)}{dt} = +q(W(t))$, where $q(x) > 0, x > 0$. Assume that during an efflux period, the rate of *decrease* of content is $\frac{dW(t)}{dt} = -r(W(t))$, where $r(x) > 0, x > 0$. When the dam is empty (i.e., $W(t) = 0$), $\frac{dW(t)}{dt} = 0$. By the memoryless property of $\mathrm{E}_{\lambda_2}$, sojourns at level 0 are also distributed as $\mathrm{E}_{\lambda_2}$ (Fig. 10.5). The empty period is analogous to an idle period in an M/G/1 queue or empty period in an M/G/r(·) dam. The efflux rate $r(x)$ is similar to that of the M/G/r(·) dam (Section 6.2).

Consider the stochastic process $\{W(t), M(t)\}$ where $W(t)$ denotes the content at instant $t$, and the configuration $M(t) \in \boldsymbol{M} = \{0, 1, 2\}$. The state space is $\boldsymbol{S} = [0, \infty) \times \boldsymbol{M}$. The meaning of $M(t)$ is given in the following table.(See Subsections 4.5 – 4.6 for discussions on system configuration.)

| $M(t)$ | Meaning |
|---|---|
| 0 | Empty period. |
| 1 | Influx phase; content increasing. |
| 2 | Efflux phase; content decreasing or at level 0. |

A sample path of $\{W(t), M(t)\}$ evolves on two sheets corresponding to configurations 1 and 2, and on one line corresponding to an empty period ($W(t) = 0$) (Fig. 10.6).

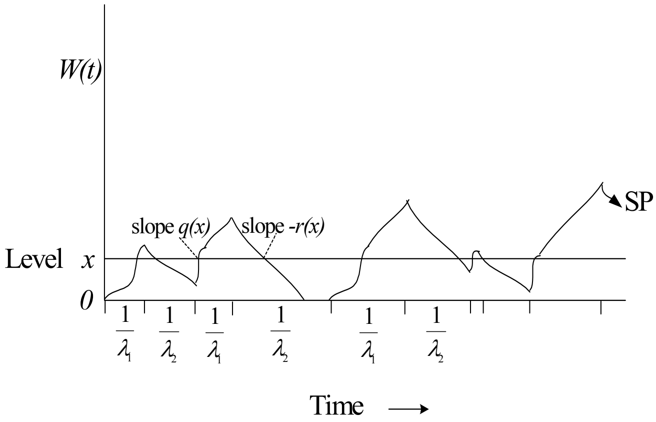


Figure 10.5: Sample path of dam with continuous influx and efflux. Slope at level $x$: during influx is $\frac{d}{dt}W(t) = q(x)$; during efflux is $-r(x)$. Slope at level 0 is $\frac{d}{dt}W(t) = 0$. Influx and efflux times are distributed as $\mathrm{E}_{\lambda_1}$, $\mathrm{E}_{\lambda_2}$, respectively.

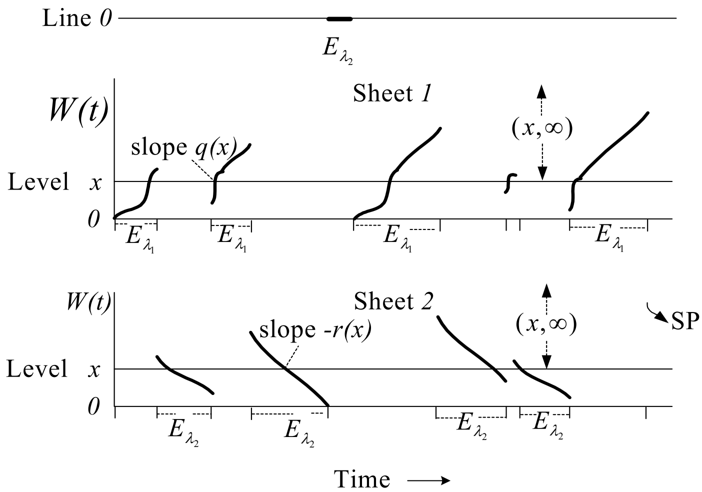


Figure 10.6: Sample path of dam with continuous influx and efflux, showing line and sheets (pages). Line 0 $\leftrightarrow W(t) = 0$, dam empty. Sheet 1 $\leftrightarrow M(t) = 1$, influx phase. Sheet 2 $\leftrightarrow M(t) = 2$, efflux phase. Also indicates composite states $\langle (x, \infty), i \rangle, i = 1, 2$. Slope at level $x > 0$: during influx is $\frac{d}{dt}W(t) = q(x)$; during efflux is $-r(x)$. Slope at level 0 is $\frac{d}{dt}W(t) = 0$. Influx and efflux durations are distributed as $\mathrm{E}_{\lambda_1}$, $\mathrm{E}_{\lambda_2}$, respectively. Empty duration is distributed as $\mathrm{E}_{\lambda_2}$.

### 10.10.1 Steady-state PDF of Content

Denote the "partial cdf's" of content by

$$F_i(x) = \lim_{t\to\infty} P(W(t) \le x, M(t) = i), x > 0, i = 1, 2.$$

Denote the steady-state "partial" pdf of content by

$$f_i(x) = \frac{d}{dx}F_i(x), i = 1, 2, x > 0,$$

wherever the derivative exists.

The *total* pdf of content (marginal pdf) is

$$f(x) = f_1(x) + f_2(x), x > 0. \tag{10.76}$$

Let $P_0 = \lim_{t\to\infty} P(W(t) = 0)$. We shall derive: $f_i(x), i = 1, 2$; $f(x)$; $P_0$; $F(x) = P_0 + \int_{y=0}^{x} f(y)dy$, in terms of the input parameters $\lambda_1$, $\lambda_2$, $q(x)$, $r(x)$. The steady-state probability that the dam is in the influx phase $(i = 1)$ or efflux phase $(i = 2)$ is $F_i(\infty) = \int_{x=0}^{\infty} f_i(x)dx, i = 1.2$.

### 10.10.2 Equations for PDF's

Consider composite state $((x,\infty), 1)\,, x > 0$, on sheet 1. The SP rate *out* of $((x,\infty), 1)$ is $\lambda_1 \int_{y=x}^{\infty} f_1(y)dy$, since the end of an influx period signals an instantaneous SP $1 \to 2$ transition from $((x,\infty), 1)$ to $((x,\infty), 2)$ *at the same level.*

The SP rate *into* $((x,\infty), 1)$ is

$$q(x)f_1(x) + \lambda_2 \int_{y=x}^{\infty} f_2(y)dy,$$

since: (1) the SP upcrosses level $x$ on sheet 1 at rate $q(x)f_1(x)$, (2) the SP enters $((x,\infty), 1)$ from $((x,\infty), 2)$ $(2 \to 1$ transition) at the same level (the rate at which efflux periods end when the SP is in $((x,\infty), 2)$ is $= \lambda_2$). Set balance, namely

$$\textbf{\textit{SP rate out of}}\ ((x,\infty), 1) = \textbf{\textit{SP rate into}}\ ((x,\infty), 1)\,,$$

gives an integral equation relating $f_1(x)$ and $f_2(x)$,

$$\lambda_1 \int_{y=x}^{\infty} f_1(y)dy = q(x)f_1(x) + \lambda_2 \int_{y=x}^{\infty} f_2(y)dy. \tag{10.77}$$

Similarly, balancing SP rates out of, and into $((x,\infty),2)$, $x>0$, on sheet 2 yields the integral equation

$$\lambda_2 \int_{y=x}^{\infty} f_2(y)dy + r(x)f_2(x) = \lambda_1 \int_{y=x}^{\infty} f_1(y)dy. \tag{10.78}$$

In (10.78), the left and right sides are the SP exit and entrance rates respectively, of $((x,\infty),2)$.

Addition of (10.77) and (10.78) yields

$$q(x)\cdot f_1(x) = r(x)\cdot f_2(x). \tag{10.79}$$

There is an easy alternative derivation 0f (10.79), which follows by viewing the sample-path via the "cover". That is, we *project* the segments of the sample path from sheets $1,2$ (pages) onto a single $t$-$W(t)$ coordinate system (Fig. 10.5). Then we apply SP rate balance across level $x$:

***total upcrossing rate*** $=$ ***total downcrossing rate***,

which translates to (10.79).

Using (10.79), we substitute $f_2(x) = \frac{q(x)}{r(x)} f_1(x)$ into (10.77), and take $\frac{d}{dx}$ in (10.77). Then we solve the resulting differential equation, and applying the initial condition

$$r(0^+)f_2(0) = \lambda_2 P_0 = q(0^+)f_1(0).$$

These operations result in the formula

$$f_1(x) = \frac{\lambda_2 P_0}{q(x)} \cdot e^{-\left(\lambda_1 \int_{y=0}^{x} \frac{1}{q(y)}dy - \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)}dy\right)}, x>0. \tag{10.80}$$

Since $f_2(x) = \frac{q(x)}{r(x)} f_1(x)$, we have

$$f_2(x) = \frac{\lambda_2 P_0}{r(x)} \cdot e^{-\left(\lambda_1 \int_{y=0}^{x} \frac{1}{q(y)}dy - \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)}dy\right)}, x>0. \tag{10.81}$$

The total pdf of content is $f(x) = f_1(x) + f_2(x)$. Adding (10.80) and (10.81) gives

$$\begin{aligned} f(x) &= \lambda_2 \left(\frac{1}{q(x)} + \frac{1}{r(x)}\right) P_0 \cdot e^{-\left(\lambda_1 \int_{y=0}^{x} \frac{1}{q(y)}dy - \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)}dy\right)}, x>0, \\ &= \lambda_2 \left(\frac{q(x)+r(x)}{q(x)r(x)}\right) P_0 \cdot e^{-\left(\lambda_1 \int_{y=0}^{x} \frac{1}{q(y)}dy - \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)}dy\right)}, x>0. \end{aligned} \tag{10.82}$$

The normalizing condition is

$$P_0 + \int_{x=0}^{\infty} f(x)dx = 1. \tag{10.83}$$

From (10.82) and (10.83)

$$P_0 = \frac{1}{1 + \lambda_2 \int_{x=0}^{\infty} \left( \left( \frac{q(x)+r(x)}{q(x)r(x)} \right) \cdot e^{-\left(\lambda_1 \int_{y=0}^{x} \frac{1}{q(y)} dy - \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)} dy\right)} \right) dx}. \tag{10.84}$$

**Remark 10.11** *Formulas (10.80)-(10.84) are asymmetric with respect to $\lambda_1$ and $\lambda_2$. This is because empty periods are distributed as $E_{\lambda_2}$ (classified as part of efflux phase).*

**Remark 10.12** *The model can be generalized in various ways. There may be several different important state-space levels at which there is no change in content (no influx or efflux), rather than only at level 0. Such levels may be due to a control policy or due to natural phenomena. There would then be **more than one atom** in the state space. Also, the influx and efflux periods may have general distributions. The content may be bounded above, resulting in an atom. Some of these variants are easy to analyze; others are more complex. We do not treat such variants here.*

**Stability Condition**

A necessary condition for the pdf to exist is $f(\infty) = 0$. Thus, the exponent $\left(\lambda_1 \int_{y=0}^{x} \frac{1}{q(y)} dy - \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)} dy\right)$ in (10.84) must be positive for all $\dot{x} > 0$. That is

$$\begin{aligned} \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)} dy &< \lambda_1 \int_{y=0}^{x} \frac{1}{q(y)} dy, \\ \lambda_1 \int_{y=0}^{x} \frac{1}{q(y)} dy - \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)} dy &> 0, \text{ for all } x > 0. \end{aligned} \tag{10.85}$$

### 10.10.3 Numerical Example

Let $\lambda_1 = 1$, $\lambda_2 = 2$, $q(x) = \sqrt{x}$, $r(x) = 3\sqrt{x}$. Substituting into (10.85) gives

$$\lambda_1 \int_{y=0}^{x} \frac{1}{q(y)} dy - \lambda_2 \int_{y=0}^{x} \frac{1}{r(y)} dy = 2\sqrt{x}\left(\lambda_1 - \frac{\lambda_2}{3}\right)$$
$$= 2\sqrt{x}\left(1 - \frac{2}{3}\right) > 0, x > 0,$$

implying stability. Thus the steady-state pdf $f(x)$ exists. From (10.82), we obtain

$$f(x) = \frac{8}{3\sqrt{x}} P_0 \cdot e^{-\frac{2}{3}\sqrt{x}}, x > 0. \tag{10.86}$$

From the normalizing condition (10.83),

$$P_0 = \frac{1}{1 + \int_{x=0}^{\infty} \frac{8}{3\sqrt{x}} e^{-\frac{2}{3}\sqrt{x}} dx} = \frac{1}{9} = 0.111111. \tag{10.87}$$

Thus

$$f(x) = \frac{8}{27\sqrt{x}} e^{-\frac{2}{3}\sqrt{x}}, x > 0. \tag{10.88}$$

From (10.87) and (10.88), the cdf is (see Figs. 10.7, 10.8),

$$F(x) = P_0 + \int_{y=0}^{x} f(y) dy = 1 - \frac{8}{9} e^{-\frac{2}{3}\sqrt{x}}. \tag{10.89}$$

## Proportion of Time in Influx and Efflux Phases

From ((10.79)) and (10.76) we obtain

$$f_1(x) = \frac{2}{9\sqrt{x}} e^{-\frac{2}{3}\sqrt{x}}, \ x > 0,$$
$$f_2(x) = \frac{2}{27\sqrt{x}} e^{-\frac{2}{3}\sqrt{x}}, \ x > 0.$$

Hence the proportion of time the dam is in the influx, efflux phase respectively is

$$F_1(\infty) = \int_{x=0}^{\infty} \frac{2}{9\sqrt{x}} e^{-\frac{2}{3}\sqrt{x}} dx = 0.666667,$$
$$F_2(\infty) = \int_{x=0}^{\infty} \frac{2}{27\sqrt{x}} e^{-\frac{2}{3}\sqrt{x}} dx = 0.222222.$$

These values are also the steady-state probabilities of the dam being in these phases at an arbitrary time point. A check on the normalizing condition is

$$P_0 + F_1(\infty) + F_2(\infty) = 0.111111 + 0.666667 + 0.222222 = 1.$$

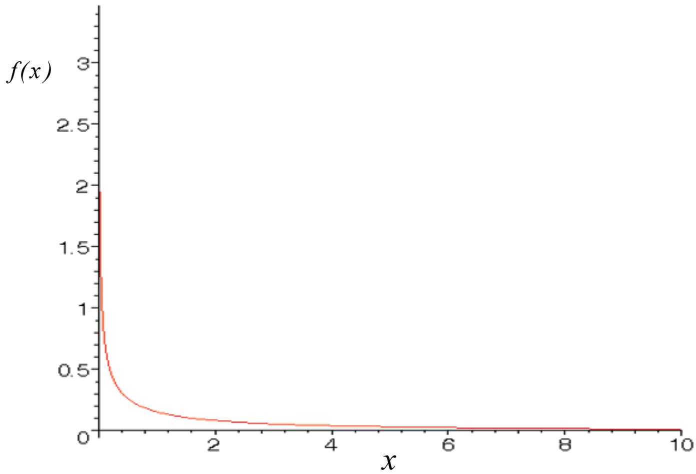


Figure 10.7: Steady-state pdf $f(x) = \frac{8}{27\sqrt{x}}e^{-\frac{2}{3}\sqrt{x}}, x > 0$, in continuous dam with alternating influx/efflux periods: $\lambda_1 = 1$, $\lambda_2 = 2$, $q(x) = \sqrt{x}$, $r(x) = 3\sqrt{x}$.

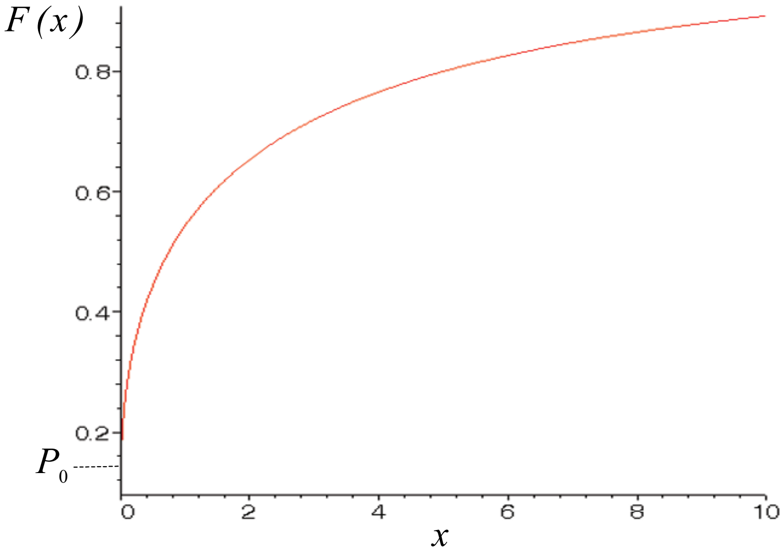


Figure 10.8: Steady-state cdf $F(x) = 1 - \frac{8}{9}e^{-\frac{2}{3}\sqrt{x}}$, $x > 0$, $P_0 = 0.1111$, in continuous dam with alternating influx/efflux periods: $\lambda_1 = 1$, $\lambda_2 = 2$, $q(x) = \sqrt{x}$, $r(x) = 3\sqrt{x}$.

## 10.11 Estimation of Laplace Transforms

We very briefly discuss a procedure for estimating the LST (Laplace-Stieltjes transform) of the state variable of a stochastic model. We shall use the virtual wait in a GI/G/1 queue as an example.

Suppose we want to estimate the LST of the steady-state pdf of the virtual wait in a GI/G/1 queue. Let the steady-state cdf of the virtual wait be $F(x), x \geq 0$, having pdf $f(x), x > 0$, and let $P_0 = F(0)$. The LST of the mixed pdf $\{P_0; f(x), x > 0\}$ is defined as

$$F^*(s) = \int_{x=0}^{\infty} e^{-sx} dF(x), s > 0. \tag{10.90}$$

### 10.11.1 Probabilistic Interpretation of LST

The probabilistic interpretation of the LST (10.90) is as follows ([78], and used in various papers, e.g., [31]). In (10.90), the right side is the probability that an independent "*catastrophe random variable*", distributed as $\mathrm{E}_s$, is greater than the virtual wait having cdf $F(x), x \geq 0$.

### 10.11.2 Estimation of LST

In order to estimate $F^*(s)$, we can simulate a sample path of the virtual wait $W(u), u \geq 0$, over a long period of simulated time $(0, t)$. Next, we generate a sample path of a renewal process $\{\mathcal{C}(u), u \geq 0\}$ with inter-renewal times equal to the catastrophe r.v., and overlay it on the *same time-state coordinate system* (see Fig 10.9). Fix $s > 0$. The SP jump sizes and inter-renewal times in the sample path of $\{\mathcal{C}(u)\}$, are iid r.v.'s distributed as $\mathrm{E}_s$. This is because the process $\mathcal{C}(u)$ represents the excess life $\gamma$ at time $u$ (see Subsection 10.2.4). The steady-state pdf of excess life is $f_\gamma(x) = s \cdot e^{-sx}, x > 0$.

Now we observe the sample paths of $\{W(u)\}$ and $\{\mathcal{C}(u)\}$ on the time interval $(0, t)$. We compute the **sum**, $T_s = \sum_i T_{si}$, of all time intervals such that $\mathcal{C}(u) > W(u), u \in (0, t)$ (Fig. 10.9). An estimate of $F^*(s)$ is then $\widehat{F^*}(s) = \frac{T_s}{t}$, which is the proportion of time that $\mathcal{C}(u)$ exceeds $W(u)$ during $(0, t)$. The probabilistic interpretation of the LST strongly suggests that $\frac{T_s}{t}$ is an appropriate estimate.

We repeat the procedure using different values of $s > 0$. For example, we may choose a partition of $N$ uniformly-spaced values for $s$, such as $\Delta, 2\Delta, 3\Delta, ..., N\Delta$, where $N$ is a large positive integer and $\Delta$ is a small positive number. (Different spacing for the partition may improve the

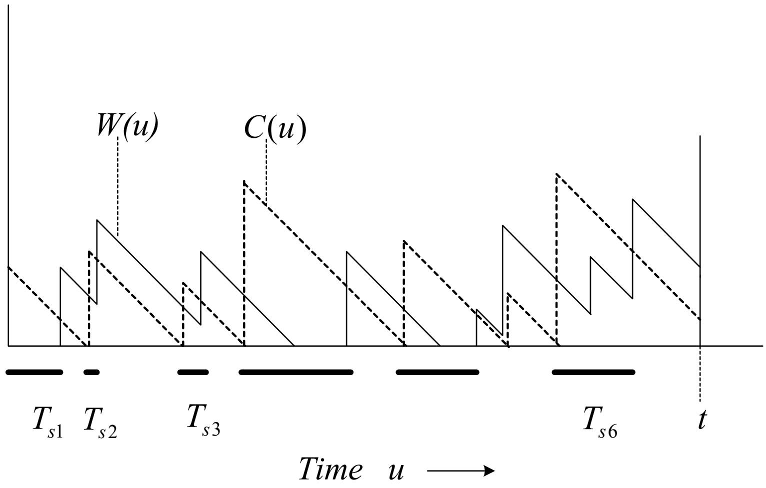


Figure 10.9: Sample paths of virtual wait $\{W(u), u \geq 0,\}$ and renewal process with inter-arrival time distributed as $E_s$, the catastrphe r.v., $\{\mathcal{C}(u), u \geq 0\}$ . $T_s = T_{s1} + T_{s2} + \cdots + T_{s6}$.

estimates, e.g., if $F(\cdot)$ is known to have certain properties such as a long tail.) This procedure results in a set of estimates $\widehat{F^*}(n\Delta) = \frac{T_{n\Delta}}{t}, n = 1, ..., N$. (From (10.90), $\widehat{F^*}(0) = 1.$, which is the normalizing condition.)

Finally, we can plot the points

$$\left(0, \widehat{F^*}(0)\right) = (0,1) \quad \text{and} \quad \left(n\Delta, \widehat{F^*}(n\Delta)\right), n = 1, ..., N,$$

on a two-dimensional $\left(s, \widehat{F^*}(s)\right)$ coordinate system. The $\{n\Delta\}$ grid is on the horizontal axis; the $\widehat{F^*}(n\Delta)$ terms are ordinates parallel to the vertical axis.

The plot will be a discrete estimate of the LST of the pdf of the virtual wait. It may be improved by smoothing techniques. In order to obtain an estimate of the pdf of the virtual wait from it, we can use numerical inversion of $\left\{\widehat{F^*}(n\Delta)\right\}$.

## 10.12 Simple Harmonic Motion

We analyze an elementary model of *deterministic* simple harmonic motion, using LC.

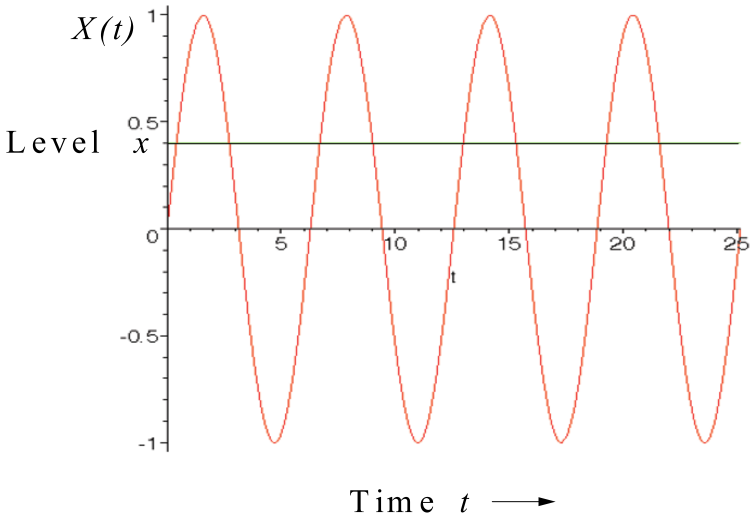


Figure 10.10: Sample path of simple harmonic motion $X(t) = \sin t$. State space is $\boldsymbol{S} = [-1, +1]$. Shows level $x$ in $\boldsymbol{S}$.

Consider a particle moving according to simple harmonic motion (SHM) (see, e.g., [6]). Let $X(t)$ denote the position of the particle at instant $t \geq 0$, and $X(0) = 0$. Let the state space be the interval $\boldsymbol{S} = [-1, +1]$. In this version of the standard SHM model there is only one sample path, namely,

$$X(t) = \sin(t), t \geq 0.$$

We wish to determine the stationary pdf $f(x)$ and cdf $F(x)$ of $X(t)$ when the particle is observed at an arbitrary time point, as $t \to \infty$.

Consider the sample path $X(t), t \geq 0$ (Fig. 10.10). The slope of the sample path at level $x$ is

$$r(x) == \frac{d}{dt} \sin t|_{t=\sin^{-1} x} = \cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}, x \in [-1, +1]. \tag{10.91}$$

Consider levels $x$, $x + h \in \boldsymbol{S}$, where $h > 0$ is small. The time required for the SP to ascend from level $x$ to level $x + h$ is

$$\int_{y=x}^{x+h} \frac{1}{r(y)} dy = \int_{y=x}^{x+h} \frac{1}{\sqrt{1 - y^2}} dy. \tag{10.92}$$

The symmetries of the sample path imply that the time required for the SP to descend from level $x + h$ to level $x$ is also given by (10.92).

Applying (10.92), we see that the long-run *proportion of time* the SP spends in state-space interval $(x, x+h)$ in a cycle of length $2\pi$ time units is

$$\frac{2}{2\pi}\int_{y=x}^{x+h}\frac{1}{\sqrt{1-y^2}}dy = F(x+h) - F(x). \tag{10.93}$$

Formula (10.93) leads to

$$\frac{1}{\pi}h\frac{1}{\sqrt{1-(x^*)^2}} = F(x+h) - F(x) \tag{10.94}$$

where $x^* \in (x, x+h)$, by the definition of $F(x)$ as the long-run proportion of time the process is in state-space interval $[-1, x]$. Dividing both sides of (10.94) by $h$ and letting $h \downarrow 0$, yields

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}, x \in [-1, +1]\,. \tag{10.95}$$

The stationary pdf $f(x)$ in (10.95) is interesting and suggests intuitive insights (Fig. 10.11). Note that $\lim_{x\downarrow(-1)} f(x) = \lim_{x\uparrow(+1)} f(x) = \infty$. Also, $\min_{x\in \boldsymbol{S}} f(x) = \frac{1}{\pi}$, at $x = 0$. The pdf $f(x)$ is symmetric about $x = 0$, and is convex.

From (10.95), the cdf is

$$\begin{aligned} F(x) &= \int_{y=-1}^{x} f(y)dy, \\ &= \frac{1}{\pi}\left(\sin^{-1}(x) - \sin^{-1}(-1)\right) \\ &= \frac{1}{\pi}\sin^{-1}(x) + \frac{1}{2}, x \in [-1, +1]. \end{aligned} \tag{10.96}$$

### 10.12.1 Inferences Based on PDF and CDF

From (10.91), the speed of the particle $r(x) = \sqrt{1-x^2} = 0$ at $x = \pm 1$. Hence, intuitively, it is much more likely to observe the particle close to the boundaries of $\boldsymbol{S}$ ($x = \pm 1$), at an arbitrary time point in the long run. This fact implies that the particle spends a much greater proportion of time near the boundaries $x = \pm 1$ than near the center $x = 0$. At the center, the speed is $r(0) = 1$. This is the maximum speed.

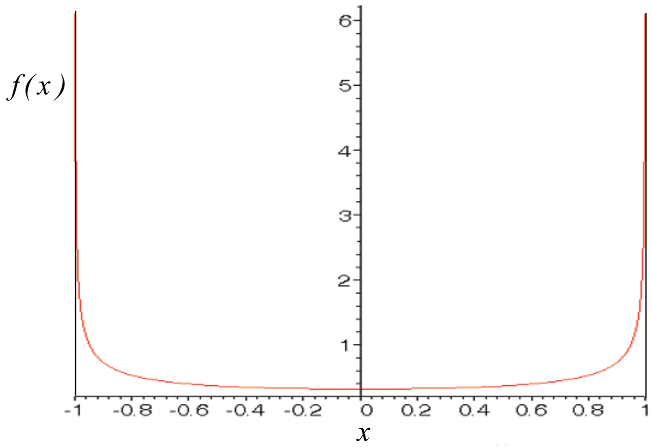


Figure 10.11: Stationary pdf $f(x) = \frac{1}{\pi\sqrt{1-x^2}}, x \in [-1, +1]$, for particle moving in simple harmonic motion, $X(t) = \sin t, t \geq 0$.

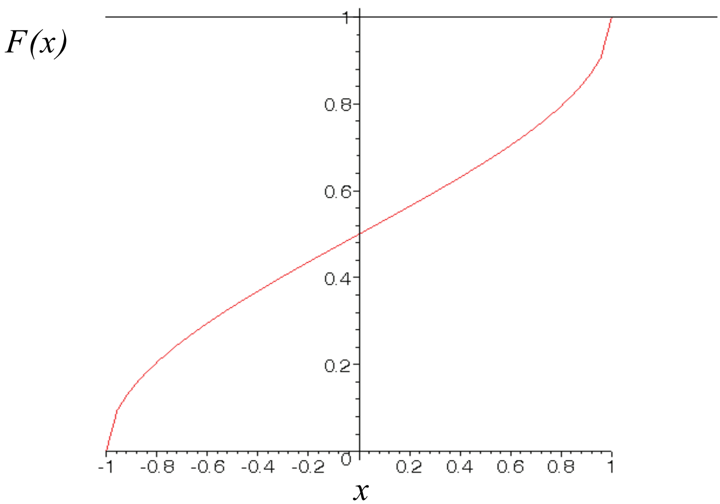


Figure 10.12: Stationary cdf $F(x) = \frac{1}{\pi}\sin^{-1}(x) + \frac{1}{2}, x \in [-1, +1]$, for particle moving in simple harmonic motion, $X(t) = \sin t, t \geq 0$.

From computations using (10.96), the proportion of time the SP (particle) spends in the central interval $[-.5, +.5]$ is equal to $F(.5) - F(-.5) = 0.333$. The proportion of time the particle spends in the outer regions $[-1.0, -.5] \cup [.5, 1.0]$, is equal to $2 \cdot (F(1.0) - F(.5)) = 0.667$. The "median" symmetric outer edges with respect to time spent by the particle, is $\boldsymbol{A}_{0.5} \equiv [-1.0, -.707] \cup [.707, 1.0]$. That is, $P(\text{particle} \in \boldsymbol{A}_{0.5}) = 0.5$. This indicates that it is equally likely to find the particle in two bands of equal width 0.293 touching the edges $\pm 1.0$ (total width .586), as it is to find it in a central interval of width 1.414 about 0. Arbitrary observations on operating pendulum clocks, readily corroborate these theoretical computations.

**Remark 10.13** *The type of LC analysis in this section, may be extendable to analyze random trigonometric functions (e.g., like* $A\sin(\theta t) + B\cos(\theta t), t \geq 0$*, where* $A$*,* $B$ *are random variables and* $\theta$ *is a constant). Extensions may also be applicable in some models of physics, and in the analysis of roots of equations.*

## 10.13 Renewal Problem with Barrier

Consider a renewal process $\{Z_n\}, n = 1, 2....$ . Assume $Z_n \underset{dist}{=} U_{(0,1)})$, a uniform random variable on $(0, 1)$ (Fig. 10.13). Let $N_K$ denote the number of renewals required to *first exceed* a barrier $K > 0$. In this section we derive the expected value $E(N_K), K = 1, 2, 3, ...$ , and related results. It is well known that $E(N_1) = e$, the base of natural logarithms. The general formula for $E(N_K)$ has not been reported previously in the literature or is not well known. It is usually shown that $E(N_1) = e$ by a standard renewal argument. That is, condition on the first renewal distance $s$ (Fig. 10.13). Derive a renewal equation, and solve it.

In this section we derive $E(N_1)$ by an alternative method, which also leads to the values of $E(N_K), K = 1, 2, ...$ . This alternative method facilitates finding the expected number of renewals required to exceed a barrier, in other (seemingly unrelated) models. The idea is to extend the one-dimensional renewal process to a two-dimensional *nested* renewal process. The new construct has applications in a variety of stochastic models.

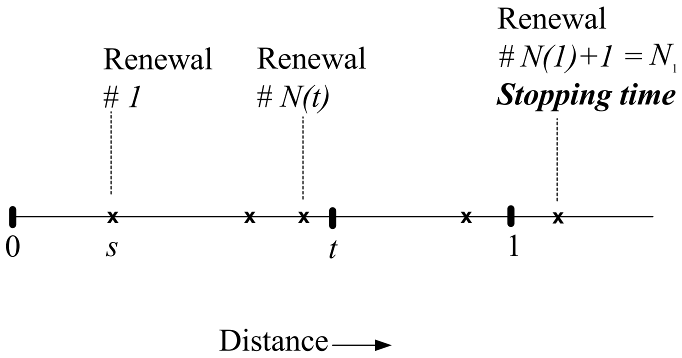


Figure 10.13: Renewal process $\{Z_n\}$ showing renewals. $N(t)$ is the number of renewals within $(0, t)$. $N_1 = N(1) + 1$ is number of renewals required to first exceed barrier $K = 1$. $N_1$ is a stopping time for the sequence $\{Z_n\}$ where $Z_n \underset{dist}{=} U_{(0,1)}$.

### 10.13.1 Alternative Solution Method

We construct a continuous-time continuous-state stochastic process

$$\{X(t), t \geq 0\}, X(0) = 0,$$

which is related to $\{Z_n\}$ (Fig. 10.14). A sample path of $\{X(t)\}$ is a nondecreasing step function. In sample paths of $\{X(t)\}$, SP *upward* jumps of size $\underset{dist}{=} U_{(0,1)}$, occur at an arbitrary Poisson rate $\lambda$. (We will select $\lambda = 1$ for convenience.) The upward jumps are denoted by

$$b_n \underset{dist}{\equiv} U_{(0,1)}, n = 1, 2, \ldots .$$

(Note that $Z_n \equiv b_n$. We replace symbol $Z_n$ by $b_n$ for generality beyond boundary $K = 1$, and because of applicability to other models.)

Let

$$N_K = \min\{n | \sum_{i=1}^{n} b_i > K\}, K = 1, 2, \ldots . \tag{10.97}$$

Random variable $N_K$ is a *stopping time* for the sequence $\{b_n\}$.

Let random variable $a \underset{dist}{=} \mathrm{E}_\lambda = \mathrm{E}_1$. Thus $E(a) = 1$.

Define random variable $c$ by

$$c = \sum_{i=1}^{N_K} a_i, \quad \text{where} \quad a_i \underset{dist}{\equiv} a. \tag{10.98}$$

Let $\{c_n\}$ be a renewal process where $c_n \underset{dist}{\equiv} c$. Then $\{c_n\}$ is a *nested renewal process* with components $\{c_n\}$ and sub-components $\{a_i\}$. Note that $N_K$ is also a stopping time for the sequence $\{a_i\}$. Taking the expected value in (10.98) yields

$$E(c) = E(N_K)E(\alpha) = E(N_K), \tag{10.99}$$

by Wald's equation (e.g., [91] or [101]).

At each instant when a sample path of $\{X(t)\}$ upcrosses level $K$, the SP jumps downward (rebounds) to level 0, and the process $\{X(t)\}$ starts over again at level 0. Our construction guarantees that the limiting distribution of $X(t)$ exists as $t \to \infty$. Random variable $N_K$ equals the number of SP jumps required for $\{X(t)\}$ to first exceed level $K$. R.v. $N_K$ is also equal to the number of subintervals which are $\underset{dist}{=} a$, that comprise a cycle $c$.

**Relation to $\langle s, S \rangle$ with No Decay**

It is notable that other stochastic models have a related structure. For example, the $\langle s, S \rangle$ inventory *with no decay* in Example 2.3 is the "flip" (like $\updownarrow$) of the $\{X(t)\}$ process, in which $K = S - s$, and the jump sizes are distributed as $\mathrm{E}_\mu$. In that $\langle s, S \rangle$ model $E(N_{S-s})$ is the expected number of orders in an ordering cycle.

### 10.13.2 Number of Renewals Required to Exceed 1

We first determine $E(N_1)$. Denote the limiting distribution of $\{X(t)\}$ as $t \to \infty$, by $\{\pi_0;\ f_0(x), 0 < x < 1\}$. Consider a sample path of $\{X(t)\}$. Fix level $x \in (0, 1)$ (Fig. 10.14). SP upcrossings of level $x$ are due to jumps starting at level 0 or at level $y, 0 < y < x$. Thus the SP upcrossing rate of level $x$ is

$$1 \cdot \pi_0 \cdot P(b > x) + 1 \cdot \int_{y=0}^{x} P(b > x - y) \cdot f(y) dy, \tag{10.100}$$

where r.v. $b \underset{dist}{\equiv} b_i$, and upward jumps occur at rate $\frac{1}{E(\alpha)} = \lambda = 1$.

The SP downcrossing rate of level $x$ is equal to the *upcrossing rate of level* 1 *for all* $x \in (0, 1)$. That is, the SP rebounds to level 0 at every instant it upcrosses level 1. (The SP makes a *double jump.* Compare with $\langle s, S \rangle$ inventory with no decay in Example 2.3.) The rate of SP downward jumps is also the rate of SP entrances into state $\{0\}$ from

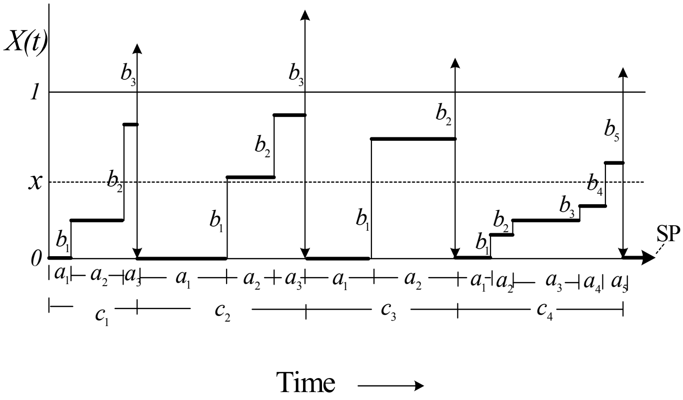


Figure 10.14: Sample path of $\{X(t), t \geq 0\}$, in renewal problem to determine $E(N_1)$ when renewal times $\underset{dist}{=} U_{(0,1)}$.

above. This rate is the same as the SP exit rate out of $\{0\}$, namely $\lambda\pi_0 = 1 \cdot \pi_0 = \pi_0$. Letting $x = 1$ in (10.100) we obtain

$$1 \cdot \pi_0 \cdot P(b > 1) + 1 \cdot \int_{y=0}^{1} P(b > 1 - y) \cdot f(y)dy = \pi_0. \tag{10.101}$$

Note that since $b \underset{dist}{=} U_{(0,1)}$,

$$P(b > x) = 1 - x, 0 < x < 1. \tag{10.102}$$

We substitute from (10.102) into (10.100). Then we apply rate balance across level $x$ to equate (10.100) to the right-hand side of (10.101), resulting in

$$\pi_0(1 - x) + \int_{y=0}^{x} (1 - x + y)f(y)dy = \pi_0, 0 < x < 1. \tag{10.103}$$

Taking $\frac{d}{dx}$ twice in (10.103), and solving the resulting ordinary differential equation gives

$$f(x) = \pi_0 e^x, 0 < x < 1. \tag{10.104}$$

We substitute from (10.104) into the normalizing condition $\pi_0 + \int_{x=0}^{1} f(x)dx = 1$. This gives

$$\pi_0 = \frac{1}{e}. \tag{10.105}$$

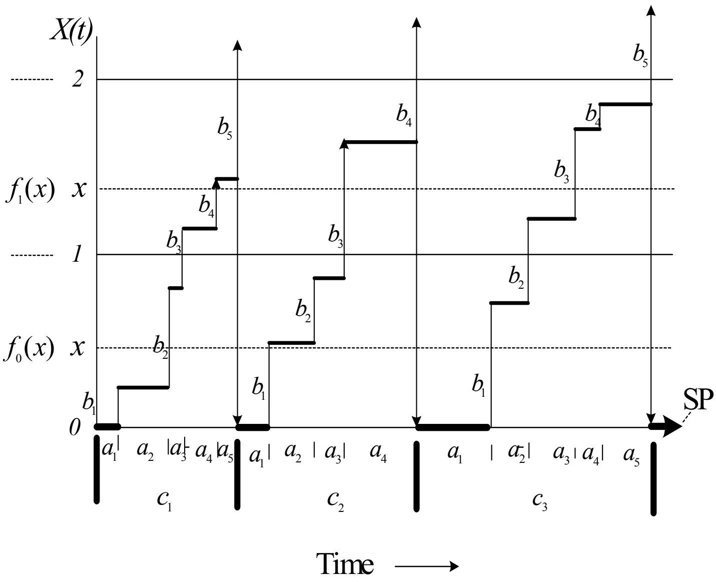


Figure 10.15: Sample path of $\{X(t)\}$ for renewal problem, with state space $\boldsymbol{S} = [0, 2)$. Facilitates solution for $E(N_2)$.

The renewal rate of $\{c_n\}$ is $\frac{1}{E(c)}$ = SP entrance rate into $\{0\} = \pi_0$. Thus $E(c) = \frac{1}{\pi_0}$. From (10.99) and (10.105),

$$E(N_1) = E(c) \cdot E(a) = \frac{1}{\pi_0} \cdot 1 = e = 2.71828. \qquad (10.106)$$

We have derived $E(N_1)$ in detail using the nested renewal process structure, to fix ideas. The following results are new (or not well known).

### 10.13.3 Number of Renewals Required to Exceed 2

Next we determine $E(N_2)$. Let the steady-state PDF of $\{X(t)\}$ be

$$\{\pi_0;\ f_0(x), 0 < x < 1\};\ \{\ f_1(x), 1 \le x < 2\}.$$

Consider a sample path of $\{X(t)\}$ (Fig. 10.15), where the state space is $\boldsymbol{S} = [0, 2)$. Balancing SP up- and downcrossing rates of $x \in (0, 1)$, as in the case $K = 1$, gives

$$\pi_0(1 - x) + \int_{y=0}^{x} (1 - x + y) f_0(y) dy = \pi_0, 0 < x < 1. \qquad (10.107)$$

Fix $x \in [1, 2)$. Balancing SP up- and downcrossing rates of $x$, gives

$$\int_{y=x-1}^{1} (1 - x + y) f_0(y) dy + \int_{y=1}^{x} (1 - x + y) f_1(y) dy = \pi_0. \quad (10.108)$$

The lower limit in the first integral of (10.108) is $y = x - 1$ because an SP jump upcrosses $x$ only if it starts in interval $(x - 1, x)$.

Taking $\frac{d}{dx}$ in (10.108) and solving in a similar manner as for $K = 1$, we obtain

$$\begin{aligned} f_0(x) &= \quad \pi_0 e^x, 0 < x < 1, \\ f_1(x) &= \quad \pi_0 (1 - e^{-1} x) e^x, 1 \leq x < 2. \end{aligned} \quad (10.109)$$

The normalizing condition is

$$\pi_0 + \int_{x=0}^{1} f_0(x) dx + \int_{x=1}^{2} f(x) dx = 1. \quad (10.110)$$

Substituting from (10.109) into (10.110) gives

$$\pi_0 = \frac{1}{-e + e^{2.}}. \quad (10.111)$$

From (10.99),

$$E(N_2) = E(c)E(a) = \frac{1}{\pi_0} = -e + e^2 = 4.67077. \quad (10.112)$$

### 10.13.4 Number of Renewals Required to Exceed 3

To explore further the pattern of $\{E(N_K)\}, K = 1, 2, ...$ we derive $E(N_3)$. The state space is $\boldsymbol{S} = [0, 3)$. Let the steady state PDF of $\{X(t)\}$ be

$$\{\pi_0;\ f_0(x), 0 < x < 1\};\ \{f_1(x), 1 \leq x < 2\};\ \ \{f_2(x), 2 \leq x < 3\}.$$

We now balance SP up- and downcrossing rates across arbitrary levels $x \in (0, 1)$; $x \in [1, 2)$; $x \in [2, 3)$. This gives respectively, integral equations

$$\pi_0(1 - x) + \int_{y=0}^{x} (1 - x + y) f_0(y) dy = \pi_0, \quad (10.113)$$

$$\int_{y=x-1}^{1} (1 - x + y) f_0(y) dy + \int_{y=1}^{x} (1 - x + y) f_1(y) dy = \pi_0, \quad (10.114)$$

$$\int_{y=x-1}^{2} (1 - x + y) f_1(y) dy + \int_{y=2}^{x} (1 - x + y) f_2(y) dy = \pi_0. \quad (10.115)$$

Solving integral equations (10.113), (10.114), (10.113) in a similar manner as for $K = 1, 2$ above, gives

$$\begin{aligned} f_0(x) = &\quad \pi_0 e^x, 0 < x < 1, \\ f_1(x) = &\quad \pi_0(1 - e^{-1}x)e^x, 1 \le x < 2, \\ f_2(x) = &\quad \tfrac{1}{2}\pi_0(-2xe^{-2} + e^{-2}x^2 - 2xe^{-1} + 2)e^x, 2 \le x < 3. \end{aligned} \tag{10.116}$$

The normalizing condition is

$$\pi_0 + \int_{x=0}^{1} f_0(x)dx + \int_{x=1}^{2} f_1(x)dx + \int_{x=2}^{3} f_2(x)dx = 1, \tag{10.117}$$

yielding

$$\pi_0 = \frac{1}{\frac{1}{2}e - 2e^2 + e^3}.$$

Substituting from (10.116) into (10.117) gives

$$E(N_3) = \frac{1}{\pi_0} = \frac{1}{2}e - 2e^2 + e^3 = 6.66656563. \tag{10.118}$$

### 10.13.5 Number of Renewals Required to Exceed $K$

After carrying out the procedure for several more steps, I hypothesized that the formula for general integer $K$ is $E(N_K) = \sum_{i=1}^{K} \frac{(-i)^{K-i}}{(K-i)!}e^i$. This formula can be verified by mathematical induction. Thus

$$E(N_K) = \sum_{i=1}^{K} \frac{(-i)^{K-i}}{(K-i)!}e^i, K = 1, 2, ... \,. \tag{10.119}$$

The induction is carried out by assuming that the formulas for $f_i(x), i = 0, ..., K-1$ are similar to those in (10.116). Then we obtain (10.119) in a similar manner as for the derivation of (10.118).

### 10.13.6 Asymptotic Formula for $E(N_K)$

We can show that $E(N_K)$ given in (10.119) is asymptotic to $2K + \frac{2}{3}$. That is

$$\lim_{K\to\infty} \frac{E(N_K)}{2K + \frac{2}{3}} = 1. \tag{10.120}$$

For example, using (10.120), an approximation to $E(N_{20})$ is $2(20) + \frac{2}{3} = 40.6667$. The analytical value using (10.119) is 40.6667. The accuracy of the computation depends on the number of digits carried, and on the computational algorithm.

Remarkably, from the analytical values of $E(N_2)$ and $E(N_3)$ given in (10.112) and (10.118), the approximation (10.120) is very accurate for $K = 2, 3, \ldots$ . Even for $K = 1$, we have $2K + \frac{2}{3} = 2.6666$, which is within 1.90% of $e = 2.71828$.

## Derivation of Asymptotic Formula

We give a renewal-theoretic derivation of formula (10.120).

Let $\gamma_x$ denote the excess life at a point $x \in \boldsymbol{S}$. The pdf of $\gamma_x$ as $x \to \infty$ is given by $f_\gamma(y) = \frac{1}{\mu}(1 - B(y)), y > 0$ where $B(y)$ is the common cdf of the renewal r.v. having mean $\mu$ (formula (10.9)). In the present context, the renewal r.v. $\underset{dist}{=} U_{(0,1)}$. Thus $B(y) = y, 0 < y < 1$ and $\mu = \frac{1}{2}$. Hence $\lim_{x\to\infty} E(\gamma_x)$ is given by

$$\begin{aligned}\lim_{x\to\infty} E(\gamma_x) &= \frac{1}{\mu}\int_{y=0}^{\infty} y f_\gamma(y)dy \\ &= 2\int_{y=0}^{1} y\,(1-y)\,dy = \frac{1}{3}.\end{aligned} \tag{10.121}$$

Let $\gamma_K$ denote the excess life at $K$; then $E(\gamma_K) \approx \frac{1}{3}$. Also,

$$K + \gamma_K = \sum_{j=1}^{N_K} Z_j, \tag{10.122}$$

where $\{Z_j\}$ are iid, $Z_j \underset{dist}{\equiv} U_{(0,1)}$, and $N_K$ is a stopping time for $\{Z_j\}$. Taking expected values in (10.122) yields $K + \frac{1}{3} \approx E(N_K)\frac{1}{2}$. If $K \to \infty$, we obtain (10.120). Moreover, if $\alpha > 0$ is a real number, then $E(N_\alpha) \approx 2\alpha + \frac{2}{3}$, where $N_\alpha$ is the number of renewals required to first exceed $\alpha$.

### 10.13.7 Number of Renewals Within an Interval

Let $N(a, b)$ denote the number of renewal instants occurring *within* interval $(a, b)$, during a single cycle of $\{c_n\}$. Without loss of generality, $X(0) = 0$, and we stop after $N_K$ renewals of $\{a_n\}$. Then

$$N(0, K) = N_K - 1, \quad \text{and} \quad E(N(0, K)) = E(N_K) - 1.$$

Thus the values of $E(N_1)$, $E(N_2)$, $E(N_3)$ lead to the expected number of renewal instants within intervals $(0,1)$, $(0,2)$, $(0,3)$, $(1,2)$, $(2,3)$, namely

$$\begin{aligned}
E(N(0,1)) &= E(N_1) - 1 = e - 1 = 1.7183, \\
E(N(0,2)) &= E(N_2) - 1 = -e + e^2 - 1 = 3.6708, \\
E(N(0,3)) &= E(N_3) - 1 = \frac{1}{2}e - 2e^2 + e^3 - 1 = 5.6666, \\
E(N(1,2)) &= E(N(0,2)) - E(N(0,1)) = E(N_2) - E(N_1) = 1.9525, \\
E(N(2,3)) &= E(N(0,3)) - E(N(0,2)) = E(N_3) - E(N_2) = 1.9958.
\end{aligned} \tag{10.123}$$

For large $K$,

$$\begin{aligned}
E(N(K, K+1) &= E(0, K+1) - E(0, K) \\
&= E(N_{K+1}) - E(N_K) \approx 2.
\end{aligned}$$

Note that in (10.123), the values of $E(N(1,2))$, $E(N(2,3))$ are already within 2.38% and 1.40% of the limiting value 2.0, respectively.

Suppose $0 < \alpha < \beta < 1$, where $\alpha, \beta$ are arbitrary real numbers. We obtain $E(N_\alpha) = e^\alpha$, and $E(N_\beta) = e^\beta$, analogously as for the solution for $E(N_1)$. Hence, $E(N(0,\alpha)) = e^\alpha - 1$, $E(N(0,\beta)) = e^\beta - 1$. Therefore, the expected number of renewals within $(\alpha, \beta)$ is

$$E(N(\alpha, \beta)) = E(N_\beta) - E(N_\alpha) = e^\beta - e^\alpha, 0 < \alpha < \beta < 1. \tag{10.124}$$

For example

$$\begin{aligned}
E\left(N\left(\tfrac{2}{3}, 1\right)\right) &= e - e^{\frac{2}{3}} = 0.77055, \\
E\left(N\left(\tfrac{1}{3}, \tfrac{2}{3}\right)\right) &= e^{\frac{2}{3}} - e^{\frac{1}{3}} = 0.55212, \\
E\left(N\left(0, \tfrac{1}{3}\right)\right) &= e^{\frac{1}{3}} - e^0 = 0.39561.
\end{aligned}$$

Thus approximately 44.84% of the renewals occur in the top third, 32.13% in the middle third and 23.02% in the bottom third, of interval $(0,1)$. Hence, renewal instants tend to accumulate in the top portion of $(0,1)$. For a possible intuitive explanation of this phenomenon, fix the length of a "sliding interval" $\boldsymbol{I}_h$ to be $|\boldsymbol{I}_h| = h, 0 < h < 1$. As $\boldsymbol{I}_h$ slides from position $(0, h)$ to position $(1-h, 1)$, the probability that $\boldsymbol{I}_h$ will contain $n$ renewals increases for every $n = 1, 2, ....$ .

We can extend the analysis to determine the expected number of renewals within an arbitrary interval $(\alpha, \beta), 0 \leq \alpha < \beta < \infty$.

### 10.13.8 Discussion

We can apply the nested renewal model of this section, to an arbitrary renewal process such that $\{b_n\}$ are non-lattice positive r.v.'s. The analysis can also be extended to models where $\{b_n\}$ are such that $-\infty < b_n < \infty$. In that case, $\{b_n\}$ is not a renewal process, but $\{c_n\}$ and $\{a_n\}$ are renewal processes, with $\{a_n\}$ nested in $\{c_n\}$.

Possible applications are to problems where it is required to determine the expected number of events until a stopping criterion is satisfied. Examples are the number of: customers served in a busy period of a queue; orders in an ordering cycle of an inventory; inputs until overflow of a dam; shocks until failure of a machine part; claims until ruin in an actuarial model; doses of a drug until an overdose; ads until a favorable consumer response to a product occurs.