Lagrange Principle of Wealth Distribution

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Summary. The Lagrange principle $L = f + \lambda g \rightarrow$ maximum! is used to maximize a function $f(x)$ under a constraint $g(x)$. Economists regard $f(x) = U$ as a rational production function, which has to be maximized under the constraint of prices $g(x)$. In physics $f(x) = lnP$ is regarded as entropy of a stochastic system, which has to be maximized under constraint of energy $g(x)$. In the discussion of wealth distribution it may be demonstrated that both aspects will apply. The stochastic aspect of physics leads to a Boltzmann distribution of wealth, which applies to the majority of the less affluent population. The rational approach of economics leads to a Pareto distribution, which applies to the minority of the super rich. The boundary corresponds to an economic phase transition similar to the liquid - gas transition in physical systems.

Key words: Lagrange principle, Cobb Douglas production function, entropy, Boltzmann distribution, Pareto distribution, econophysics.

1 Introduction

Since the work of Pareto [1] as long ago as 1897, it has been known that economic distributions strictly follow power law decays. These distributions have been observed across a wide variety of economic and financial data. More recently, Roegen [2], Foley [3], Weidlich,[4] Mimkes [5][6], Levy and Solomon [7][8], Solomon and Richmond [9], Mimkes and Willis [10], Yakovenko [11], Clementi and Gallegatti [12] and Nirei and Souma [13] have proposed statistical models for economic distributions. In this paper the Lagrange principle is applied to recent data of wealth in different countries.
The Lagrange equation

$$f - \lambda g \rightarrow maximum! \tag{1}$$

applies to all functions (f) that are to be maximized under constraints (g). The factor λ is called Lagrange parameter.

2 Calculation of the Boltzmann distribution

In stochastic systems the probability P is to be maximized under constraints of capital according to the Lagrange principle

$$lnP(x_j) - \lambda \Sigma_j w_j x_j \rightarrow maximum! \quad (2)$$

lnP is the logarithm of probability $P(x_j)$ or entropy that will be maximized under the constraints of the total capital in income $\Sigma w_j x_j$. The variable (x_j) is the relative number of people in the income class (w_j). The Lagrange factor $\lambda = 1/ < w >$ is equivalent to the mean income $< w >$ per person. Distributing N households to (w_j) property classes is a question of combinatorial statistics,

$$P = N!/\Pi(N_j!) \quad (3)$$

Using Sterlings formula $(lnN! = NlnN - N)$ and $x = N_j/N$ we may change to

$$-\Sigma_j x_j lnx_j - \lambda \Sigma_j w_j x_j \rightarrow maximum! \quad (4)$$

At equilibrium (maximum) the derivative of equation 4 with respect to x_j will be zero,

$$\partial lnP/\partial x_j = -(lnx_j + 1) = \lambda w_j \quad (5)$$

In this operation for x_j all other variables are kept constant and we may solve Eq.6 for $x = x_j$. The number $N(w)$ of people in income class (w) is given by

$$N(w) = A\, e^{-\frac{w}{<w>}} \quad (6)$$

In the physical model the relative number of people (x) in the income class (w) follows a Boltzmann distribution, Eq.6. The Lagrange parameter λ has been replaced by the mean income $< w >$. The constant A is determined by the total number of people N_1 with an income following a Boltzmann distribution and may be calculated by the integral from zero to infinity,

$$N_1 = \int N(w)dw = A < w > \quad (7)$$

The amount of capital K(w) in the property class (w) is

$$K(w) = A\, w\, e^{-\frac{w}{<w>}}. \quad (8)$$

The total amount K_1 is given by the integral from zero to infinity,

$$K_1 = A \int w\ x(w)\ dw = A < w >^2 \quad (9)$$

The ratio of total wealth, Eq.9 divided by the total number of households, Eq.8 leads to

$$K_1/N_1 =< w > \quad (10)$$

which is indeed the mean income $< w >$ per household. The Lorenz distribution $y = K_1(w)$ as a function of $x = N_1(w)$ in fig.1 may be calculated from the Boltzmann distribution, Eqs.8 and 6, and leads to

$$y = x + (1 - x)ln(1 - x) \tag{11}$$

This function will be applied to Lorenz distributions of wealth.

2.1 German wealth data 1993 and the Boltzmann distribution

Property data for Germany (1993) have been published [14] by the German Institute of Economics (DIW). The data show the number $N(w)$ of households and the amount of capital $K(w)$ in each property class (w). The data are generally presented by a Lorenz distribution. Fig.1 shows the Lorenz distribution of capital K vs. the number N of households in Germany 1993.

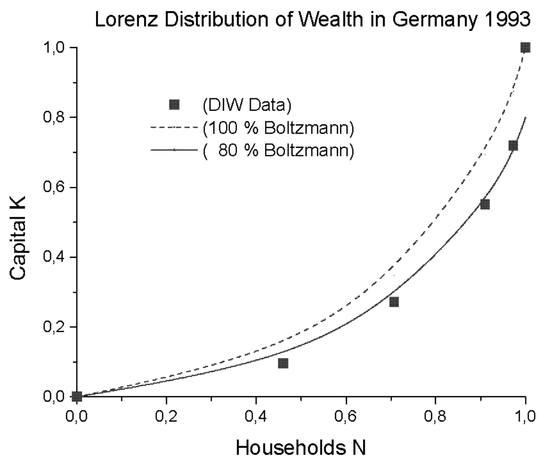

Fig. 1. Lorenz distribution, sum of capital K vs. sum of households N in Germany 1993, data from DIW [14], dotted line according to eq. 11. However, the Boltzmann approach only fits for 80 % of the total capital, solid line.

The data points are fitted by the dotted line for the Lorenz function of a Boltzmann distribution, eq. 11. The Boltzmann approach fits well only for 80% of the capital, solid line. A more detailed analysis of the super rich 20 % in Germany 1993 in fig.1 is not possible due to the few data points.

2.2 US wages data 1995 and the Boltzmann distribution

Wages like wealth may be expected to show a Boltzmann distribution according to eq.6. However, jobs below a wage minimum w_0 have the attrac-

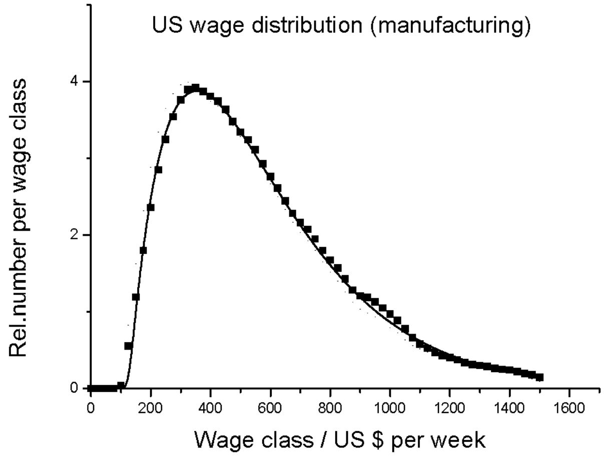

Fig. 2. Number of people in wage classes in manufacturing in the US [15]. The data have been fitted by the Boltzmann distribution, eq. 12 with $a(w, w_0,) = (w - w_0)$.

tiveness $a* = 0$. Accordingly, the distribution of income will be given by

$$N(w) = a(w, w_0)e^{-\frac{w}{<w>}} \tag{12}$$

The number of people earning a wage (w) will depend on the job attractiveness $a(w, w_0)$ and the Boltzmann function. Low wages will be very probable, high wages less probable. Fig. 2 shows the wage distribution for manufacturing in the US in 1995 [12]. Again the Boltzmann distribution seems to be a good fit. However, some authors prefer to fit similar data by a log normal distribution e.g. F. Clementi and M. Gallegatti [12]. Presently, both functions seem to apply equally well.

3 Calculation of the Pareto Distribution

Economic actions (f) are optimized under the constraints of capital, costs or prices (g). Economists are used to maximize the rational production function $U(x_j)$ under constraints of total income $\Sigma w_j dx_j$. The variable (x_j) is relative number of people in each income class (w_j). The Lagrange principle 27 is now given by

$$U(x_j) - \lambda \Sigma_j w_j x_j \to maximum! \tag{13}$$

At equilibrium (maximum) the derivative of equation 29 with respect to x_j will be zero,

$$\partial U/\partial x_j = \lambda w_j \tag{14}$$

In economic calculations a Cobb Douglas type Ansatz for the production function U is often applied,

$$U(x_j) = A\Pi x_j \alpha_j \tag{15}$$

A is a constant, the exponents α_j are the elasticity constants. Inserting Eq.15 into equation 14 we obtain

$$\partial U/\partial x_j = a_j A x_j^{\alpha_j - 1} = \lambda w_j \tag{16}$$

In this operation for x_j all other variables are kept constant and we may solve Eq.15 for $x = x_j$. The relative number $x(w)$ of people in income class (w) is now given by

$$N(w) = A(\lambda w/\alpha A)^{\frac{1}{(\alpha-1)}} = C(w_m/w)^{2+\delta} \tag{17}$$

According to this economic model the number of people $N(w)$ in the income class (w) follows a Pareto distribution! In eq. 17 the Lagrange parameter λ has been replaced by the minimum wealth class of the super rich, $w_m = 1/\lambda$, the constants have been combined to C. The Pareto exponent is given by $2 + \delta = 1/(1-\alpha)$. The relative number of people $x(w)$ decreases with rising income (w). The total number of rich people N_2 with an income following a Pareto distribution is given by the integral from a minimum wealth w_m to infinity,

$$N_2 = \int N(w)dw = Cw_m/(1+\delta) \tag{18}$$

The minimum wealth w_m is always larger than zero, $w_m > 0$. The amount of capital $K(w)$ in the property class (w) is

$$K(w) = N(w)w = Cw_m(w_m/w)^{1+\delta} \tag{19}$$

The total amount of capital K_2 of very rich people with an income following a Pareto distribution is given by the integral from a minimum wealth w m to infinity,

$$K_2 = \int N(w)wdw = Cw_m^2/\delta \tag{20}$$

For positive wealth the exponent δ needs to be positive, $\delta > 0$. For a high capital of the super rich the exponent δ is expected to be: $0 < \delta < 1$. The ratio of total wealth Eq.20 divided by the total number of super rich households Eq.19 leads to

$$w_m = (K_2/N_2)\delta/(1+\delta) \tag{21}$$

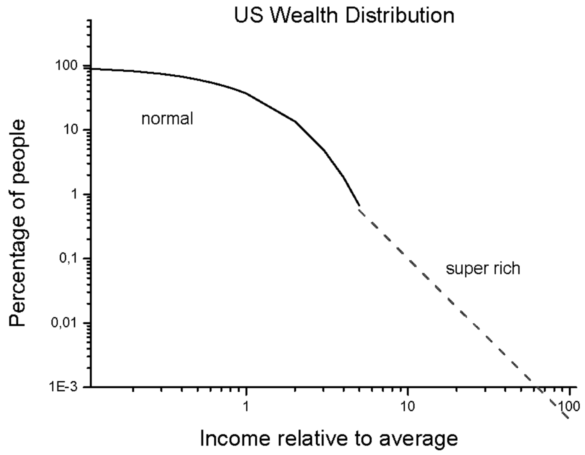

Fig. 3. shows the percentage of people in wage classes relative to average (USA 1983 – 2001) [Silva and Yakovenko] [11]: The distribution is clearly divided into two parts. The wealth of the majority of people (97%) follows a Boltzmann distribution, the wealth of the minority of super rich follows a Pareto law. The Pareto tail has a slope between 2 and 3.

3.1 Boltzmann and Pareto distribution in USA wealth data

Yakovenko [11] and others have presented data that follow a Boltzmann distribution for the majority of normal wages and a Pareto distribution for the income of the minority of very rich people, fig. 3.

A. Boltzmann distribution

A1. The wealth of the majority of the population in fig. 3 follows the Boltzmann distribution of Eq.6.

A2. The total number of normal rich people is $N_1 = 97\%$ of the total population,

$$N_1 = \int N(w)dw = A < w >= 0.97 \tag{22}$$

A3. The wealth of the normal population is given by

$$K_1 = \int N(w)wdw = N_1 < w >= 0.97 < w > \tag{23}$$

B. Pareto distribution

B1. The wealth of the super rich minority follows a Pareto law, Eq.17.

B2. The exponent of the Pareto tail in fig. 3 is between 2 and 3 or $0 < \delta < 1$, as required by Eq.20.

B3. The minimum wealth of the super rich according to fig. 3 is about eight

times the normal mean, $w_m = 8 < w >$.
B4. The total number of the super rich minority is $N_2 = 3\%$ of the total population,

$$N_2 = \int N(w)dw = Cw_m = 0,03 \tag{24}$$

B5. The total capital of the super rich minority (for $\delta = 0.5$) is

$$K_2 = N_2 w_m (1 + \delta)/\delta = 0.03 * 8 < w > *3 = 0.72 < w > \tag{25}$$

B6. The mean wealth of the super rich minority is

$$< w_2 >= (K_2/N_2) = w_m(1 + \delta)/\delta = 8 < w > *3 = 25 < w > \tag{26}$$

B7. In fig. 3 the super rich minority (3% of the population) owns $0.72 < w >=$ 40% of the national wealth and the normal majority (97% of the population) owns $0.97 < w >= 40\%$.

4 Boltzmann and Pareto phase transition

The normal rich majority and the super rich minority belong to two different states or phases. The majority is governed by the Boltzmann law, the minority by a Pareto law. This corresponds to two different phases, like liquid and gas in physical sciences, and may be calculated by the Lagrange principle,

$$L = f - \lambda g \rightarrow maximum! \tag{27}$$

$$L_1 = -xlnx - < w > \Sigma_j w_j x_j \rightarrow maximum! \tag{28}$$

$$L_2 = Ax^\alpha - < w > \Sigma_k w_k x_k \rightarrow maximum! \tag{29}$$

$$y = b - xm \tag{30}$$

Eq.27 is the general Lagrange equation, Eq.28 the Lagrange principle in stochastic systems and Eq.29 the Lagrange principle in rational systems. All may be considered linear equations of $< w >$. The $b-$ value in Eq.30 is given by the entropy or utility function, which is lower for the super rich population due to the small factor A, which is of the order of $A = 0.1$ in fig. 3. The slope "m" is given by the total wealth $\Sigma_j w_j x_j$, which is higher for the normal population (60%). The borderline between the normal and the super rich population is given by the intersection of the two lines at $< w >_c$ in 4. Below $< w >_c$ the solid line is higher (at maximum) and the normal phase dominates. Above $< w >_c$ the broken line is higher (at maximum) and the super rich phase dominates. The transition point $< w >_c$ is given by $L_1 = L_2$. However, the data are not yet sufficient to tell whether the transition "normal" - "super rich" really is of first order, as it is indicated by the sharp knee in fig.3 and the intersection in fig. 4. Other authors [13] find a smooth second

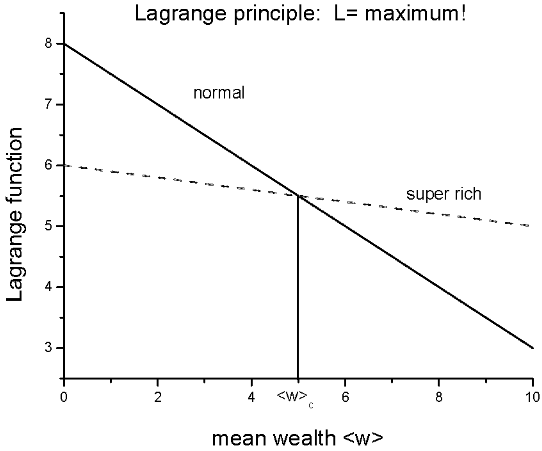

Fig. 4. For low mean income $< w >$ the Lagrange function L_1 (solid line) is at maximum, the normal rich phase dominates. For very large mean income $< w >$ the Lagrange function L_2 (broken line) is higher (at maximum), the super rich phase dominates.

order transition from the Boltzmann region of normal people to the Pareto region of the super rich. The point of transition is important for the full understanding of the system, but even more important is the mechanism that keeps normal and super rich people separated and drifting more and more apart. This topic will be discussed in a separate paper on the mechanism of economic growth.

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