The Monomodal, Polymodal, Equilibrium and Nonequilibrium Distribution of Money

Juan C. Ferrero

Centro Laser de Ciencias Moleculares, INFIQC y Facultad de Ciencias Químicas, Universidad Nacional de Córdoba. 5000 Córdoba, Argentina
jferrero@mail.fcq.unc.edu.ar

Summary. The distribution of money for several countries is analyzed according to the Bolztmann-Gibbs distribution with explicit consideration of the degeneracy of states. At high values of money the experimental data are systematically larger than the values corresponding to the BG statistics. The use of Tsallis non extensive statistics results in a good fit in the whole range of income values, converging to Paretos law in the high money limit and indicating the fractal nature of the distribution. In some cases, the distribution has two or more components, which, according to model calculations, arise from the different degeneracy of each ensemble. Criteria to determine whether this situation corresponds to equilibrium are analysed.

1 Introduction

The shape and origin of the income distribution is of utmost importance in order to develop models to explain it and to analyse the causes of inequality [1].

The higher end of the distribution of money seems to follow a power law of universal character, as shown by Pareto more than a century ago [2]. Several attempts were made in order to explain this intriguing behaviour [3-7] as well as the low and medium region income [7-11]. Recently, a Boltzmann-Gibbs (BG) distribution has been proposed to account for the income distribution for several countries, that, notwithstanding, does not follow Paretos law in the high income limit [12-14]. Therefore, the behaviour of the distribution in the whole range of money requires the use of two functions, one for the high and one for the low and medium income region. As shown in the present communication, this discrepancy can be settled using the non extensive statistics proposed by Tsallis [15,16], thus indicating the fractal nature of the distribution. Tsallis statistics has also been used by other authors in connection with the distribution of money [17,18].

In addition to this difficulty, the income distribution of several countries shows the presence of more than one component in the intermediate region.

While this polymodal character of the distribution can be easily accounted for by a superposition of functions it should also be considered whether this behaviour corresponds to a real equilibrium or to an intermediate state in the temporal evolution of the system. The purpose of this work is to show that the Tsallis function can account for the distribution of money in the whole range of money values, to find criteria to establish if the distribution corresponds to equilibrium and to analyze bimodal cases to obtain information on the equilibrium and its social consequences.

2 The distribution of money

In a previous report the income distributions of the UK, Japan and New Zealand were shown to follow quite closely the BG function, when the degeneracy of states is considered [14]. Assuming that the degeneracy is proportional to money, m, the BG equation becomes the Gamma function, i.e.

$$P_i(m) = Nm^{(\alpha-1)} \exp(-m/\beta) \tag{1}$$

The data for several countries, exemplified by those of Japan, New Zealand and the UK in selected years, could be well reproduced by this function.

However, more detailed consideration, as evidenced in a log plot, shows that this distribution strongly deviates from the experimental data in the high income limit (Fig 1).

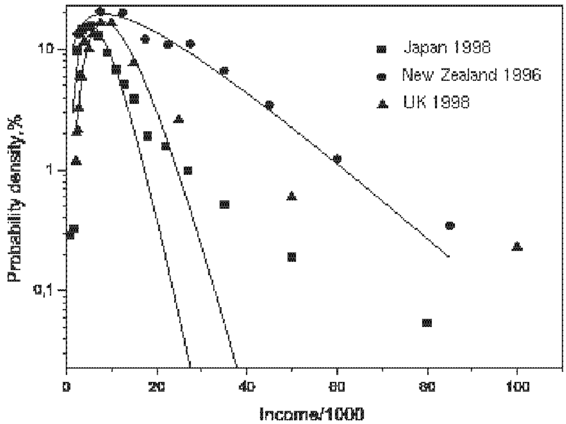

Fig. 1. Probability density vs. money for the income data of Japan and UK, 1998 and New Zealand 1996. The solid lines correspond to the Gamma function. The income axis has been scaled according to the following factors: New Zealand: 1; UK: 2; Japan: 500.

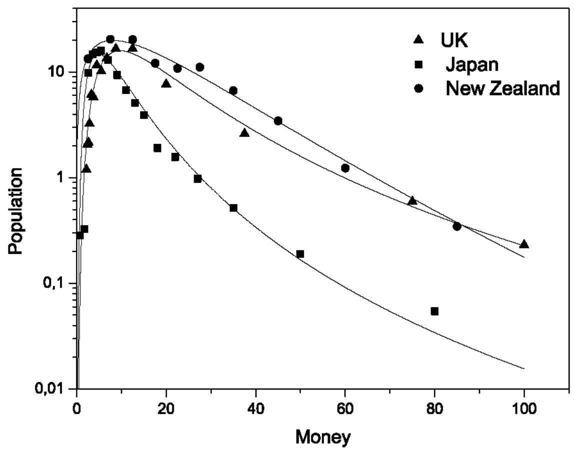

Fig. 2. Probability density vs. money for the income data of Japan and UK, 1998, and New Zealand 1996. The solid lines correspond to the Tsallis function.

However, the data could be fit to a Tsallis function in the whole money range, with a value of q close to 1.1 (Fig. 2). The lines shown in that figure corresponds to $q = 1.13$, 1.13 and 1.10, for Japan, UK and New Zealand, respectively.

Tsallis equation for the probability density is:

$$P_i = Ng_i[1 - (1 - q)Bx]^{1/1-q} \tag{2}$$

and reduces to the usual BG equation

$$P_i = Ng_i \exp(-x/\beta) \tag{3}$$

for $q = 1$. In Eq.2, g_i is the degeneracy of states, B is a constant and q is a parameter associated with the dimension of the system. This equation has been successfully applied to a variety of problems. For $x = m$, at high values of m Tsallis function becomes $P_i = Nm^n$, where $n = \alpha - 1 + 1/1 - q$, which, for $q > 1$ and $(\alpha - 1) < |1/(q - 1)|$ becomes Paretos law.

Therefore, non extensive statistics not only accounts for the distribution of money in the whole income range, when the degeneracy of states is properly considered but also shows its fractal nature. However, taking into account that Eq.2 is more difficult to use than Eq. 3 and that the BG statistics produces results in satisfactory agreement with the Tsallis function if the small fraction associated with the tail of the distribution is neglected, we will assume, for the present purposes, than Eq. 3 applies.

Another point to be considered is that the income distribution in many cases shows bimodal (or, in general, polymodal) behaviour. One example is obtained from the data of Japan for fiscal year 1998. These data are presented in Fig. 3, together with the fit to a double gamma function and the individual components of the distribution.

Since β is related to temperature [12, 13, 19], polymodal distributions for systems in equilibrium should be characterised by a unique value of β. This seems to be the case for the income distribution of Japan 1998, shown in Fig.3, where the value β for both components are the same, within the experimental error ($\beta_1 = 0.8 \pm .3$ and $\beta_2 = 1.1 \pm 2.1$).

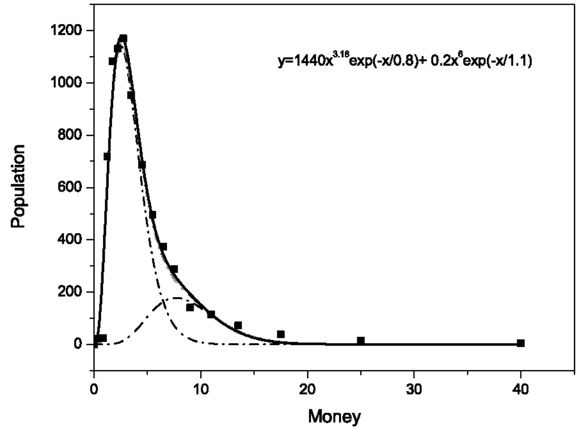

Fig. 3. Income distribution for Japan 1998, showing the two components of the distribution

A notable example of polymodal distributions is provided by the data from Argentina, during the economic crisis at the beginning of 2002 (Fig. 4) [20]. In this case, the components cannot be satisfactorily fit to a combination of BG functions, but they are well reproduced by the addition of Gaussian functions.

3 The evolution of the distribution

In an attempt to gain a further insight on the nature of multiple components in the income distribution, we made model calculations based on the

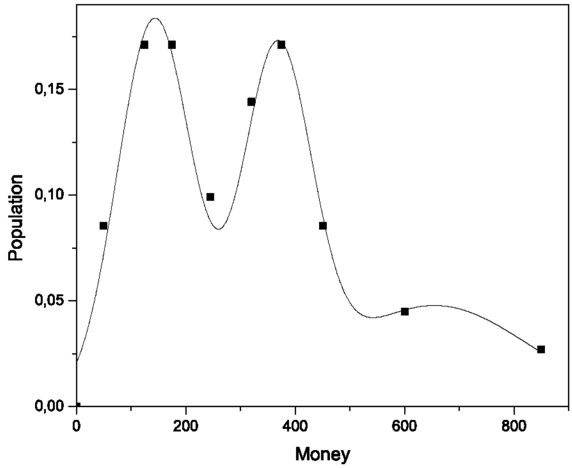

Fig. 4. Income distribution for Argentina, May 2002, showing the three components of the distribution

rate equations for the transference on money between pairs of agents in a society, following the same method than in our previous work [14]. However, we have now considered the case that an initial BG distribution, characterized by single values of α and β could change to two different ensembles, A and B, with the same value of β but different values of α, $alpha_A$ and α_B at constant total energy.

In these calculations we used $\beta = 40$ a.u. and an initial monomodal gamma distribution with a value of $\alpha = 3$, which yields an average money $< M >= \alpha\beta = 120$ a.u. The final statecharacterized by $\alpha_A = 2$ and $\alpha_B = 5$, that is, $< M_A >= 80$ a.u. and $< M_B >= 200$ a.u., that, in order to keep the total money constant requires a fractional final population for A, $P_A = 2/3$ and for B, $P_B = 1/3$.

In the absence of any flow of money in and out of the ensemble, the populations of A and B in money level i change in time according to the following master equations:

$$\frac{dn_i^A}{dt} = \omega_{AA} \sum_j P_{ij}^{AA} n_j^A + \omega_{AB} \sum_j P_{ij}^{AB} n_j^A - \omega_{BA} \sum_j P_{ij}^{BA} n_j^A - \omega_{AA} \sum_j P_{ij}^{AA} n_j^A - k_{AB} n_i^A + k_{BA} n_I^B \quad (4)$$

$$\frac{dn_i^B}{dt} = \omega_{BB} \sum_j P_{ij}^{BB} n_j^B + \omega_{AB} \sum_j P_{ij}^{AB} n_j^B - \omega_{BA} \sum_j P_{ij}^{BA} n_j^B - \omega_{BB} \sum_j P_{ij}^{BB} n_j^B - k_{BA} n_i^B + k_{AB} n_i^A \quad (5)$$

where P_{ij}^{XZ} is, in general, the probability of money transference from level j to i of X by interaction with Z, with interaction frequency ω_{XZ} and population n_i^X. The coefficients k_{zx} stand to account the rate for the rate of change between both ensembles, X and Z.

Integration of the set of equations 3 and 4 requires values for P_{ij}^{XZ}. The values of these elements for the gain of money are related to those for money loss by detailed balance, i.e.

$$\frac{P_{ji}}{P_{ij}} = \left(\frac{n_j}{n_i}\right)_{eq} = \left(\frac{g_j}{g_m}\right) \exp(-(M_j - M_i)/\beta) \quad (6)$$

This restriction, together with the condition of detailed balance for the back and forward rate of conversion of A into B, assures that the composition of equilibrium will be that of the BG distribution. A similar equation could also be imposed on the Tsallis distribution, if it were used instead of the BGs.

Therefore, the final state to be reached is determined by detailed balance, while the instantaneous value of the population distribution will depend on the values of the transition probabilities and the rate constants.

Several different calculations were made. In all of them, the elements P_{ij}^{XZ} were calculated using a normalized exponential model

$$P_{ij}^{XZ} = N \exp[-(M_i - M_j)/ < \Delta M >] \quad (7)$$

so that the transference of small amounts of money prevails.

In most of the calculations the rate coefficients were taken as constant, independent of the level of money and with the condition $k_{AB} = k_{BA}/2$. In a few cases, not presented here, these coefficients were assumed to increase with i. The average amount of money transferred per interaction, $< M >$, was set equal to 10.

A representative calculation is shown in Fig.5. The initial BG distribution separates into two different sets to finally reach the corresponding BG equilibrium composition. A similar calculation but starting from two well separated initial distributions should merge into the same final state, if the same parameters were used.

It should be noted that the curves presented in Fig. 5 for the intermediate states in the evolution to equilibrium can be very well reproduced by BG functions although with different values of β. Thus, the curves whose maxima are shifted to lower money values show a decrease of β from the initial

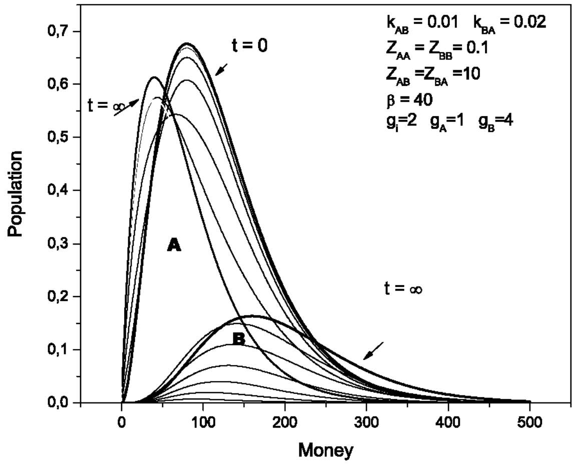

Fig. 5. Dissociation of a single initial distribution into two different ensembles, A and B.

value of 40 a.u. to around 20 a.u., followed by a relatively slower increase to the equilibrium value. In correlation with this, the value of β for the other ensemble initially increases and then decreases as money is equilibrated. This behaviour arises from the fast rate of change of A into B, as compared with money transference, so that an initial disequilibrium appears.

These results show that an equilibrium society could dissociate in two different groups, while still maintaining equilibrium, if a change if the properties of its components takes place. This change is evidenced by the value of α, that is, the degeneracy of states. A larger value of α increases the ability of the agents in that group to accommodate the money they have and the distribution moves to larger money values, while a decrease of α produces the opposite effect. The segregation results in a broader total distribution, given by the addition of the distributions of A and B, with more differences between the rich and the poor. Note however that both ensembles have poor and rich components, even thought in different proportions.

The same argument applies if the two groups were two different countries. The conclusion is that the key to a richer and egalitarian society (world) depends on the ability to increase $< m >$ by increasing the value of α. On the contrary, an increase in richness as a result of a larger value of β causes a broader distribution, with more differences between the poor and the rich.

An additional difficulty, not easy to overcome, is the lack of experimental data of the nonequilibrum distribution of income a various times. One recent example could be obtained from the evolution of the economy in Argentina around the end of the 20th century and the beginning of the present. The crisis attained its maximum intensity between the end of 2001 and the first quarter of 2002, which is clearly evidenced in the income distribution. According to the data available, the money distribution in Argentina was variable and showed certain bimodality. A few representative examples are shown in Figs.

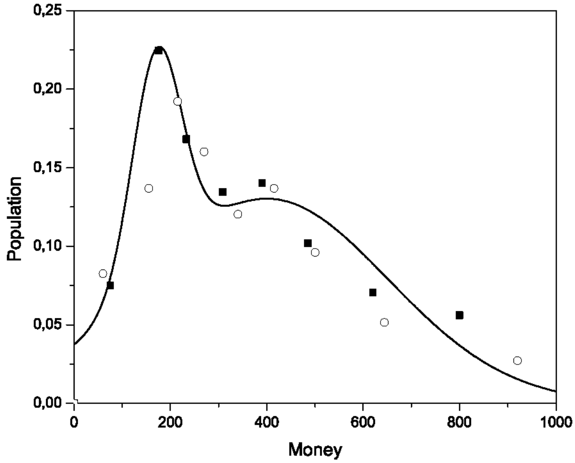

Fig. 6. Income distribution for Argentina, (open circles) October 2001, (filled squares) May 2004. The solid curve represents the fit to the data of October.

The data corresponding to May 2002 are fitted by three Gaussian functions, in agreement with the previous study that a system far from equilibrium evolves to the BG distribution through Gaussian distributions.

The income distribution in previous years could be fit to a single Gamma function, although bimodality was always present. However, as the crisis developed, the low and medium region of the data could only be fit to Gaussian functions. The distortion reached its maximum in May 2003 and seemed to tend to return to a more normal shape in 2004.

The appearance of a Gaussian shape in the distribution is expected according to model calculations presented before, for the evolution of a system far from equilibrium.

4 Conclusions

The main findings reported in this work are:

1. Monomodal distributions can be reproduced by the sole use of Tsallis non extensive statistics, with scaling factors close to 1.1. While others studies required the addition of two functions which separately fit the medium and

high income regions, where Paretos law is obeyed, Tsallis function fits both regions simultaneously.
2. Bimodal distributions *per se* do not indicate a deviation from equilibrium. Equilibrium is characterised by a single value of β for all the ensembles of the system. A rate equation analysis of the evolution of the populations indicate that a society with a monomodal BG distribution could dissociate in separate ensembles, to attain a new equilibrium.
3. The income distribution of Argentina during the economy crisis in the period 2001-2004 shows polymodal components, with Gaussian shapes, which is one of the characteristics of a system out of equilibrium. The other criterium is observing BG functions but with different values of β.

Acknowledgments: This work was supported by CONICET; SECYT and ACC.

References

1. Mantegna RN, Stanley HE (2000) An Introduction to Econophysics, Cambridge University Press, UK
2. Pareto V (1987) Le Cours d'Economie Politique, Macmillan, London.
3. Souma W (2001) Fractals 9: 463
4. Slanina F (2004) Phys. Rev. E 69: 46102-1-7
5. Fujiwara Y, Souma W, Hideaki A, Kaizoji T, Aoki M (2003) Physica A 231:598
6. Chakraborti A, Chakrabarti BK (2000) Eur Phys J B 17:167
7. Di Matteo T, Aste T, Hyde ST (2003) Il Nuovo Cimento, in press ; also available at arXiv (cond-mat/0310544).
8. Chaterjee A, Chakrabarti BK, Manna SS (2004) Physica A 335:155
9. Reed WJ (2003) Physica A 319:469
10. Patriarca M, Chacraborti A, Kaski K (2004) Physica A 340:334
11. Pianegonda S, Iglesas JR (2004) Physica A 342:193
12. Dragulescu AA, Yakovenko VM (2001) Eur Phys J B 20:585
13. Dragulescu AA, Yakovenko VM (2001) Eur Phys J B 17:723
14. Ferrero JC (2004) Physica A 341:575
15. Tsallis C (1998) J Stat Phys 52:479
16. Tsallis C, Mendes RS, Plastino AR (1998) Physica A 261:534
17. Anazawa A, Ishikawa A, Suzuki T, Tomoyose M (2004) Physica A 335:616
18. Richmond P, Sabatelli L (2004) Proc. XVIII Max Born Symposium Statistical Physics outside Physics 336:27
19. Silva AC, Yakovenko VM (2005) Europhysics Letters 69:304
20. http://www.indec.mecon.ar