Detailed Simulation Results for Some Wealth Distribution Models in Econophysics

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Summary. In this paper we present detailed simulation results on the wealth distribution model with quenched saving propensities. Unlike other wealth distribution models where the saving propensities are either zero or constant, this model is not found to be ergodic and self-averaging. The wealth distribution statistics with a single realization of quenched disorder is observed to be significantly different in nature from that of the statistics averaged over a large number of independent quenched configurations. The peculiarities in the single realization statistics refuses to vanish irrespective of whatever large sample size is used. This implies that previously observed Pareto law is essentially a convolution of the single member distributions.

In a society different members possess different amounts of wealth. Individual members often make economic transactions with other members of the society. Therefore in general the wealth of a member fluctuates with time and this is true for all other members of the society as well. Over a reasonably lengthy time interval of observation, which is small compared to the inherent time scales of the economic society this situation may be looked upon as a stationary state which implies that statistical properties like the individual wealth distribution, mean wealth, its fluctuation etc. are independent of time.

More than a century before, Pareto observed that the individual wealth (m) distribution in a society is characterized by a power-law tail like: $P(m) \sim m^{-(1+\nu)}$ and predicted a value for the constant $\nu \approx 1$, known as the Pareto exponent [1]. Very recently, i.e., over the last few years, the wealth distribution in a society has attracted renewed interests in the context of the study of *Econophysics* and various models have been proposed and studied. A number of analyses have also been done on the real-world wealth distribution data in different countries [2, 3, 4]. All these recent data indeed show that Pareto like power-law tails do exist in the wealth distributions in the large wealth regime but with different values of the Pareto exponent ranging from $\nu = 1$ to 3. It has also been observed that only a small fraction of very rich members

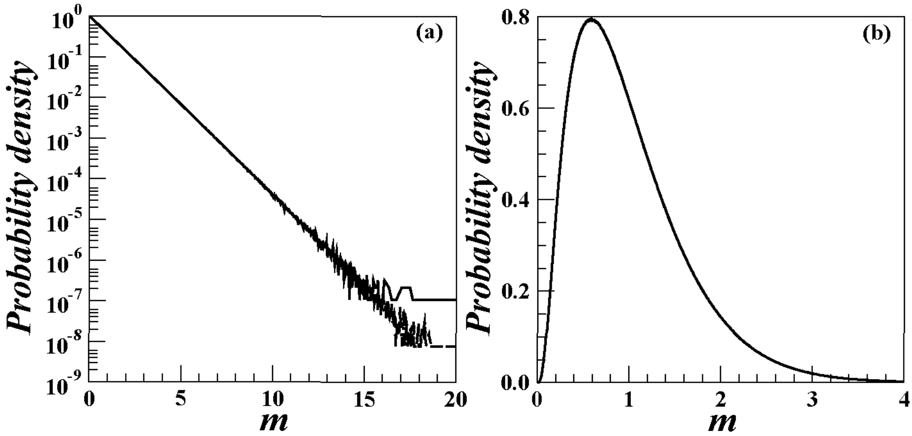

Fig. 1. The three probability densities of wealth distribution, namely $\text{Prob}_1(m)$ (solid line), $\text{Prob}_2(m)$ (dashed line) and $\text{Prob}(m)$ (dot-dashed line) are plotted with wealth m for $N = 256$ in (a) for the DY model and in (b) for the CC model for $\lambda = 0.35$. The excellent overlapping of all three curves indicate that both the DY and CC models are ergodic as well as self averaging.

actually contribute to the Pareto behavior whereas the middle and the low wealth individuals follow either exponential or log-normal distributions.

In this paper we report our detailed simulation results on the three recent models of wealth distribution. The three models are: (i) the model of Drăgulescu and Yakovenko (DY) [5] which gives an exponential decay of the wealth distribution, (ii) the model of Chakraborti and Chakrabarti (CC) [6] with a fixed saving propensity giving a Gamma function for the wealth distribution and (iii) the model of Chatterjee, Chakrabarti and Manna (CCM) [7] with a distribution of quenched individual saving propensities giving a Pareto law for the wealth distribution.

All these three models have some common features. The society consists of a group of N individuals, each has a wealth $m_i(t), i = 1, N$. The wealth distribution $\{m_i(t)\}$ dynamically evolves with time following the pairwise conservative money shuffling method of economic transactions. Randomly selected pairs of individuals make economic transactions one after another in a time sequence and thus the wealth distribution changes with time. For example, let two randomly selected individuals i and j, $(i \neq j)$ have wealths m_i and m_j. They make transactions by a random bi-partitioning of their total wealth $m_i + m_j$ and then receiving one part each randomly:

$$\begin{aligned} m_i(t+1) &= \epsilon(t)(m_i(t) + m_j(t)) \\ m_j(t+1) &= (1 - \epsilon(t))(m_i(t) + m_j(t)). \end{aligned} \tag{1}$$

Here time t is simply the number of transactions and $\epsilon(t)$ is the t-th random fraction with uniform distribution drawn for the t-th transaction.

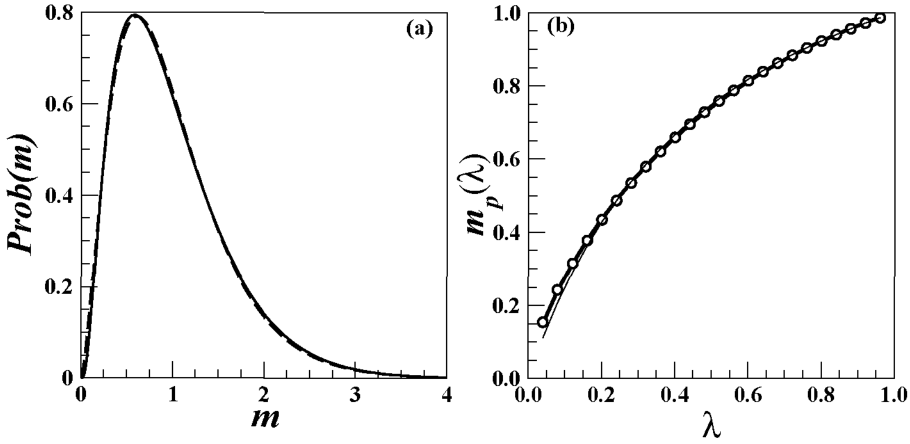

Fig. 2. For the CC model with $N = 256$ and $\lambda = 0.35$ these plots show the functional fits of the wealth distribution in (a) and the variation of the most probable wealth $m_p(\lambda)$ in (b). In (a) the simulation data of Prob(m) is shown by the solid black line where as the fitted Gamma function of Eqn. (5) is shown by the dashed line. In (b) the $m_p(\lambda)$ data for 24 different λ values denoted by circles is fitted to the Gamma function given in Eqn. (6) (solid line). The thin line is a comparison with the $m_p(\lambda)$ values obtained from the analytical expression of $a(\lambda)$ and $b(\lambda)$ in [10].

In all three models the system dynamically evolves to a stationary state which is characterized by a time independent probability distribution Prob(m) of wealths irrespective of the details of the initial distribution of wealths to start with. Typically in all our simulations a fixed amount of wealth is assigned to all members of the society, i.e. Prob$(m, t = 0) = \delta(m - \langle m \rangle)$. The model described so far is precisely the DY model in [5]. The stationary state wealth distribution for this model is [5, 8, 9]:

$$\text{Prob}(m) = \frac{1}{\langle m \rangle} \exp(-m/\langle m \rangle). \tag{2}$$

Typically $\langle m \rangle$ is chosen to be unity without any loss of generality.

A fixed saving propensity is introduced in the CC model [6]. During the economic transaction each member saves a constant λ fraction of his wealth. The total sum of the remaining wealths of both the traders is then randomly partitioned and obtained by the individual members randomly as follows:

$$\begin{aligned} m_i(t+1) &= \lambda m_i(t) + \epsilon(t)(1-\lambda)(m_i(t) + m_j(t)) \\ m_j(t+1) &= \lambda m_j(t) + (1-\epsilon(t))(1-\lambda)(m_i(t) + m_j(t)). \end{aligned} \tag{3}$$

The stationary state wealth distribution is an asymmetric distribution with a single peak. The distribution vanishes at $m = 0$ as well as for large m values. The most probable wealth $m_p(\lambda)$ increases monotonically with λ and the

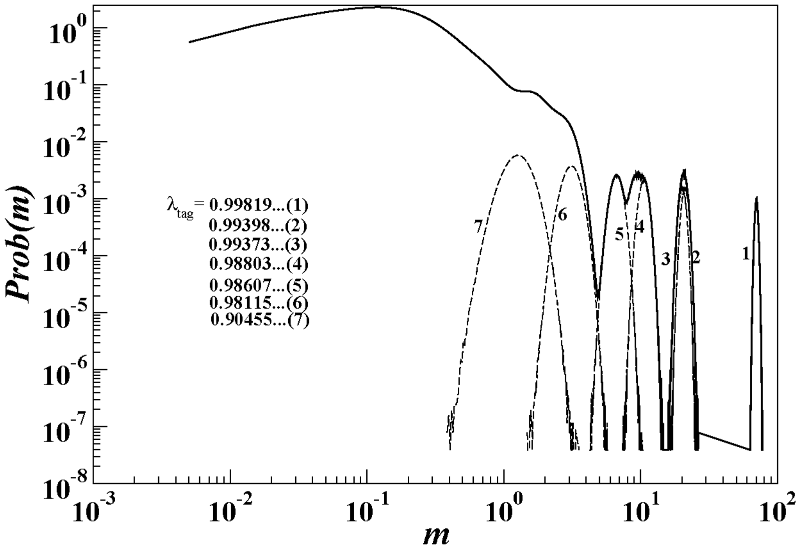

Fig. 3. The wealth distribution Prob(m) in the stationary state for the CCM model for a single initial configuration of saving propensities $\{\lambda_i\}$ with N=256 is shown by the solid line. Also the wealth distributions of the individual members with seven different tagged values of λ_{tag} are also plotted on the same curve with dashed lines. This shows that the averaged (over all members) distribution Prob(m) is the convolution of wealth distributions of all individual members.

distribution tends to the delta function again in the limit of $\lambda \to 1$ irrespective of the initial distribution of wealth.

In the third CCM model different members have their own fixed individual saving propensities and therefore the set of $\{\lambda_i, i = 1, N\}$ is a quenched variable. Economic transactions therefore take place following these equations:

$$
\begin{aligned}
m_i(t+1) &= \lambda_i m_i(t) + \epsilon(t)[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)] \\
m_j(t+1) &= \lambda_j m_j(t) + (1-\epsilon(t))[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)] \qquad (4)
\end{aligned}
$$

where λ_i and λ_j are the saving propensities of the members i and j. The stationary state wealth distribution shows a power law decay with a value of the Pareto exponent $\nu \approx 1$ [7].

In this paper we present the detailed numerical evidence to argue that while the first two models are ergodic and self-averaging, the third model is not. This makes the third model difficult to study numerically.

We simulated DY model with $N = 256, 512$ and 1024. Starting from an initial equal wealth distribution Prob(m) $= \delta(m-1)$ we skipped some transactions corresponding to a relaxation time $t_\times$ to reach the stationary state. Typically $t_\times \propto N$. In the stationary state we calculated the three different probability distributions, namely: (i) the wealth distribution $\mathrm{Prob}_1(m)$

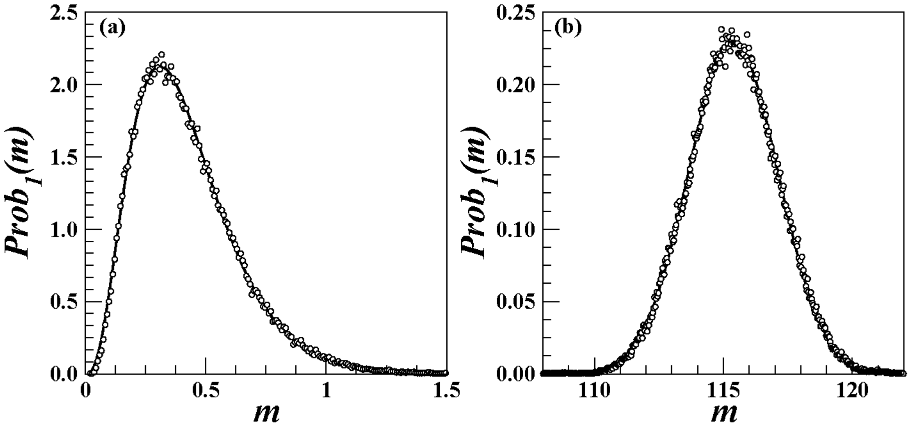

Fig. 4. The individual member's wealth distribution in the CCM model. A member is tagged with a fixed saving propensity λ_{tag}=0.05 in (a) and 0.999 in (b) for N=256. In the stationary state the distribution $\text{Prob}_1(m)$ is asymmetric in (a) and is fitted to a Gamma function. However for very large λ the distribution in (b) is symmetric and fits very nicely to a Gaussian distribution.

of an arbitrarily selected tagged member (ii) the overall wealth distribution $\text{Prob}_2(m)$ (averaged over all members of the society) on a long single run (single initial configuration, single sequence of random numbers) and (iii) the overall wealth distribution $\text{Prob}(m)$ averaged over many initial configurations. In Fig. 1(a) we show all three plots for $N = 256$ and observe that these three plots overlap excellent, i.e., these distributions are same. This implies that the DY model is ergodic as well as self-averaging.

Similar calculations are done for the CC model as well (Fig. 1(b)). We see a similar collapse of the data for the same three probability distributions. This lead us to conclude again that the CC model is also ergodic and self-averaging. Further we fit in Fig. 2(a) the CC model distribution $\text{Prob}(m)$ using a Gamma function as cited in [10] as:

$$\text{Prob}(m) \sim m^{a(\lambda)} \exp(-b(\lambda)m) \tag{5}$$

which gives excellent non-linear fits by *xmgrace* to all values of λ in the range between say 0.1 to 0.9. Once fitting is done the most-probable wealth is estimated by the relation: $m_p(\lambda) = a(\lambda)/b(\lambda)$ using the values of fitted parameters $a(\lambda)$ and $b(\lambda)$. Functional dependences of a and b on λ are also predicted in [10]. We plot $m_p(\lambda)$ so obtained with λ for 24 different values of λ in Fig. 2(b). We observe that these data points fit very well to another Gamma function as:

$$m_p(\lambda) = A\lambda^{\alpha} \exp(-\beta\lambda). \tag{6}$$

The values of $A \approx 1.46$, $\alpha \approx 0.703$ and $\beta \approx 0.377$ are estimated for $N = 256$, 512 and 1024 and we observe a concurrence of these values up to three decimal

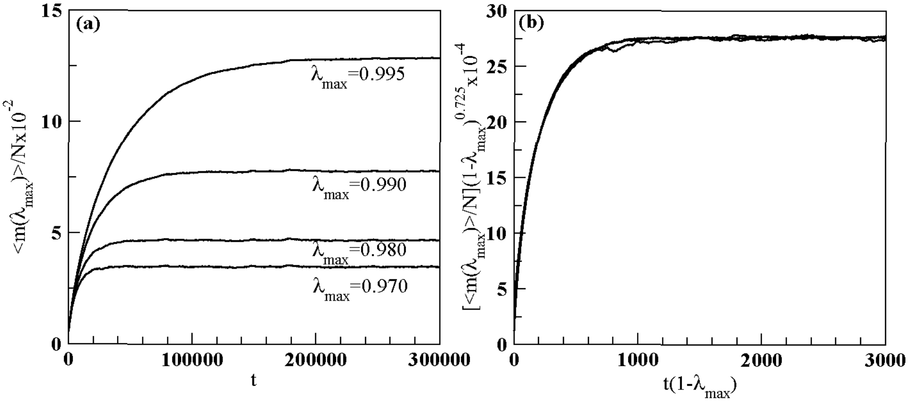

Fig. 5. (a) The mean wealth of a tagged member who has the maximal saving propensity is plotted as a function of time for four different values of λ_{max}. In (b) this data is scaled to obtain the data collapse.

places for the three different system sizes. While $m_p(0) = 0$ from Eqn. (6) is consistent, $m_p(1) = 1$ implies $A = \exp(\beta)$ is also consistent with estimated values of A and β. Following [10] we plotted $m_p(\lambda) = 3\lambda/(1+2\lambda)$ in Fig. 2(b) for the same values of λ and observe that these values deviate from our points for the small values of λ.

However, for the CCM model many inherent structures are observed. We argue that this model is neither ergodic nor self-averaging. For a society of $N = 256$ members a set of quenched individual saving propensities $\{0 \leq \lambda_i < 1, i = 1, N\}$ are assigned drawing these numbers from an independent and identical distribution of random numbers. The system then starts evolving with random pairwise conservative exchange rules cited in Eqn. (4). First we reproduced the Prob(m) vs. m curve given in [7] by averaging the wealth distribution over 500 uncorrelated initial configurations. The data looked very similar to that given in [7] and the Pareto exponent ν is found to be very close to 1.

Next we plot the same data for a single quenched configuration of saving propensities as shown in Fig. 3. It is observed that the wealth distribution plotted by the continuous solid line is far from being a nice power law as observed in [7] for the configuration averaged distribution. This curve in Fig. 3 has many humps, especially in the large wealth limit. To explain this we made further simulations by keeping track of the wealth distributions of the individual members. We see that the individual wealth distributions are significantly different from being power laws, they have single peaks as shown in Fig. 4. For small values of λ, the $\text{Prob}_1(m)$ distribution is asymmetric and has the form of a Gamma function similar to what is already observed for the CC model (Fig. 4(a)). On the other hand as $\lambda \to 1$ the variation becomes

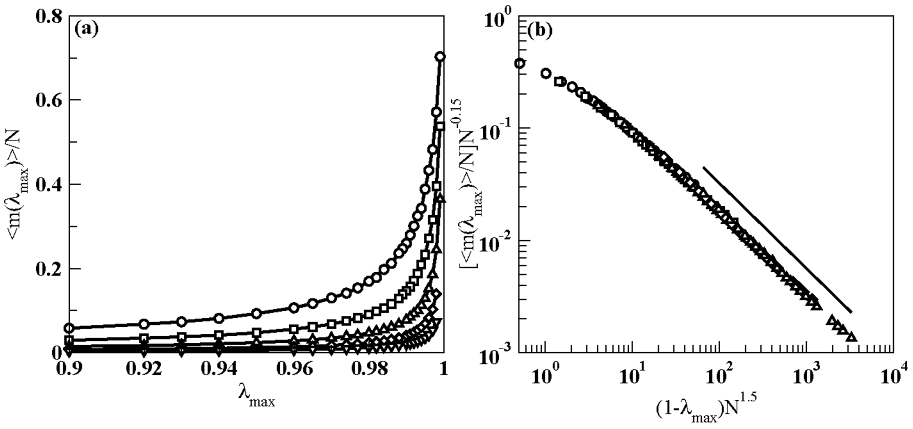

Fig. 6. In the stationary state the mean value of the wealth of the member with maximum saving propensity λ_{max} is plotted with λ_{max}. This value diverges as $\lambda_{max} \to 1$ for $N = 64$ (circle), 128 (square), 256 (triangle up), 512 (diamond) and 1024 (triangle down). (b) This data is scaled to obtain a data collapse of the three different sizes.

more and more symmetric which finally attains a simple Gaussian function (Fig. 4(b)). The reason is for small λ the individual wealth distribution does feel the presence of the infinite wall at $m = 0$ since no debt is allowed in this model, where as for $\lambda \to 1$ no such wall is present and consequently the distribution becomes symmetric. This implies that the wealth possessed by an individual varies within a limited region around an average value and certainly the corresponding phase trajectory does not explore the whole phase space. Therefore we conclude that the CCM model is not ergodic.

Seven individual wealth distributions have been plotted in Fig. 3. corresponding to six top most λ values and one with somewhat smaller value. We see that top parts of these $\text{Prob}_1(m)$ distributions almost overlap with the $\text{Prob}_2(m)$ distribution. This shows that $\text{Prob}_2(m)$ distribution is truly a superposition of N $\text{Prob}_1(m)$ distributions. In the limit of $\lambda \to 1$, large gaps are observed in the $\text{Prob}_2(m)$ distribution due to slight differences in the λ values of the corresponding individuals. These gaps remain there no matter whatever large sample size is used for the $\text{Prob}_2(m)$ distribution.

We further argue that even the configuration averaging may be difficult due to very slow relaxation modes present in the system. To demonstrate this point we consider the CCM model where the maximal saving propensity λ_{max} is continuously tuned. The N-th member is assigned λ_{max} and all other members are assigned values $\{0 \leq \lambda_i < \lambda_{max}, i = 1, N-1\}$. The average wealth $\langle m(\lambda_{max})\rangle/N$ of the N-th member is estimated at different times for $N = 256$ and they are plotted in Fig. 5(a) for four different values of λ_{max}. It is seen that as $\lambda_{max} \to 1$ it takes increasingly longer relaxation times to reach the stationary state and the saturation value of the mean wealth in the stationary state also increases very rapidly. In Fig. 5(b) we made a scaling of these plots like

$$[\langle m(\lambda_{max})\rangle/N](1-\lambda_{max})^{0.725} \sim \mathcal{G}[t(1-\lambda_{max})]. \quad (7)$$

This implies that the stationary state of the member with maximal saving propensity is reached after a relaxation time $t_\times$ given by

$$t_\times \propto (1-\lambda_{max})^{-1}. \quad (8)$$

Therefore we conclude that in CCM the maximal λ member takes the longest time to reach the stationary state where as rest of the members reach their individual stationary states earlier.

This observation poses a difficulty in the simulation of the CCM model. Since this is a problem of quenched disorder it is necessary that the observables should be averaged over many independent realizations of uncorrelated disorders. Starting from an arbitrary initial distribution of m_i values one generally skips the relaxation time $t_\times$ to reach the stationary state and then collect the data. In the CCM model the $0 \le \lambda_i < 1$ is used. Therefore if M different quenched disorders are used for averaging it means the maximal of all $M \times N$ λ values is around $1-1/(MN)$. From Eqn. (8) this implies that the slowest relaxation time grows proportional to MN. Therefore the main message is more accurate simulation one intends to do by increasing the number of quenched configurations, larger relaxation time $t_\times$ it has to skipp for each quenched configuration to ensure that it had really reached the stationary state.

Next, we calculate the variation of the mean wealth $\langle m(\lambda_{\rm max})\rangle/N$ of the maximally tagged member in the stationary state as a function of λ_{max} and for the different values of N. In Fig. 6(a) we plot this variation for $N = 64$, 128, 256, 512 and 1024 with different symbols. It is observed that larger the value of N the $\langle m(\lambda_{max})\rangle/N$ is closer to zero for all values of λ_{max} except for those which are very close to 1. For $\lambda_{max} \to 1$ the mean wealth increases very sharply to achieve the condensation limit of $\langle m(\lambda_{max} = 1)\rangle/N = 1$.

It is also observed that the divergence of the mean wealth near $\lambda_{max} = 1$ is associated with a critical exponent. In Fig. 6(b) we plot the same mean wealth with the deviation $(1-\lambda_{max})$ from 1 on a double logarithmic scale and observe power law variations. A scaling of these plots is done corresponding to a data collapse like:

$$[\langle m(\lambda_{max})\rangle/N]N^{-0.15} \sim \mathcal{F}[(1-\lambda_{max})N^{1.5}]. \quad (9)$$

Different symbols representing the data for the same five system sizes fall on the same curve which has a slope around 0.76. The scaling function $\mathcal{F}[x] \to x^{-\delta}$ as $x \to 0$ with $\delta \approx 0.76$. This means $\langle m(\lambda_{max})\rangle N^{-1.15} \sim (1-\lambda_{max})^{-0.76}N^{-1.14}$ or $\langle m(\lambda_{max})\rangle \sim (1-\lambda_{max})^{-0.76}N^{0.01}$. Since for a society of N traders $(1-\lambda_{max}) \sim 1/N$ this implies

$$\langle m(\lambda_{max})\rangle \sim N^{0.77}. \tag{10}$$

This result is therefore different from the claim that $\langle m(\lambda_{max})\rangle \sim N$ [7].

To summarize, we have revisited the three recent models of wealth distribution in Econophysics. Detailed numerical analysis yields that while the DY and CC models are ergodic and self-averaging, the CCM model with quenched saving propensities does not seem to be so. In CCM existence of slow modes proportional to the total sample size makes the numerical analysis difficult.

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