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Econophysics of Wealth Distributions

Springer

New Economic Windows

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Econophysics of Wealth Distributions

Econophys-Kolkata I

Springer

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Preface

We all know the hard fact: neither wealth nor income is ever uniform for us all. Justified or not, they are unevenly distributed; few are rich and many are poor!

Investigations for more than hundred years and the recent availability of the income distribution data in the internet (made available by the finance ministries of various countries; from the tax return data of the income tax departments) have revealed some remarkable features. Irrespective of many differences in culture, history, language and, to some extent, the economic policies followed in different countries, the income distribution is seen to follow a particular universal pattern. So does the wealth distribution. Barring an initial rise in population with income (or wealth; for the destitutes), the population decreases either exponentially or in a log-normal way for the majority of 'middle income' group, and it eventually decreases following a power law (Pareto law, following Vilfredo Pareto's observation in 1896) for the richest 5-10 % of the population! This seems to be an universal feature – valid for most of the countries and civilizations; may be in ancient Egypt as well! Econophysicists tried to view this as a natural law for a statistical many-body-dynamical market system, analogous to gases, liquids or solids: classical or quantum.

Considerable developments have taken place recently, when econophysicists tried to model such a 'market' as a thermodynamic system and identified the income distribution there to be similar to the energy distribution in thermodynamic systems like gases etc. This workshop on 'Econophysics of Wealth Distributions', first in the '**Econophys-Kolkata**' series, was held in Kolkata under the auspices of the 'Centre for Applied Mathematics and Computational Science', Saha Institute of Nuclear Physics, during 15 - 19 March 2005. We plan to hold the next meeting in this series on 'Econophysics of Stock Markets' early next year. We hope to meet again along with the readers there.

We are extremely happy that almost all the leading economists and physicists, engaged in these studies on wealth distributions could come and participate in the workshop. In this "first ever conference on Econophysics of Wealth Distributions" (New Scientist, 12 March 2005), physicists and economists discussed about their latest researches in the field. There were some agreements and more disagreements; but many interdisciplinary collaborations also got started.

We believe, this Proceedings Volume will be able to convey the elctrifying atmosphere of the workshop, and will inspire further research in this important field.

Kolkata,
May 2005

Arnab Chatterjee
Sudhakar Yarlagadda
Bikas K Chakrabarti

Contents

Part I

Data and analysis

Pareto's Law of Income Distribution: Evidence for Germany, the United Kingdom, and the United States

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Summary. We analyze three sets of income data: the US Panel Study of Income Dynamics (PSID), the British Household Panel Survey (BHPS), and the German Socio-Economic Panel (GSOEP). It is shown that the empirical income distribution is consistent with a two-parameter lognormal function for the low-middle income group (97%–99% of the population), and with a Pareto or power law function for the high income group (1%–3% of the population). This mixture of two qualitatively different analytical distributions seems stable over the years covered by our data sets, although their parameters significantly change in time. It is also found that the probability density of income growth rates almost has the form of an exponential function.

Key words: Personal income; Lognormal distribution; Pareto's law; Income growth rate

1 Introduction

More than a century ago, the economist Vilfredo Pareto stated in his *Cours d'Économie Politique* that there is a simple law which governs the distribution of income in all countries and at all times. Briefly, if N represents all the number of income-receiving units cumulated from the top above a certain income limit x, and A and α are constants, then:

$$N = \frac{A}{x^{\alpha}} \tag{1}$$

and, therefore, $log\,(N) = log\,(A) - \alpha log\,(x)$. In other words, if the logarithms of the number of persons in receipt of incomes above definite amounts are plotted against the logarithms of the amount of these incomes, the points so obtained will be on a straight line whose slope with the axis on which the values of $log(x)$ are given will be α. Pareto examined the statistics of incomes in some countries and concluded that the inclination of the line with the $log(x)$ axis differed but little from 1.5.

Very recently, considerable investigations with modern data in capitalist economies have revealed that the upper tail of the income distribution (generally less than 5% of the individuals) indeed follows the above mentioned behaviour, and the variation of the slopes both from time to time and from country to country is large enough not to be negligible. Hence, characterization and understanding of income distribution is still an open problem. The interesting problem that remains to be answered is the functional form more adequate for the majority of population not belonging to the power law part of the income distribution. Using data coming from several parts of the world, a number of recent studies debate whether the low-middle income range of the income distribution may be fitted by an exponential [1–8] or lognormal [9–13] decreasing function.[4]

In this paper we have analyzed three data sets relating to a pool of major industrialized countries for several years in order to add some empirical investigations to the ongoing debate on income distribution. When fits are performed, a two-parameter lognormal distribution is used for the low-middle part of the distribution (97%–99% of the population), while the upper high-end tail (1%–3% of the population) is found to be consistent with a power law type distribution. Our results show that the parameters of income distribution change in time; furthermore, we find that the probability density of income growth rates almost scales as an exponential function.

The structure of the paper is as follows. Section 2 describes the data used in our study. Section 3 presents and analyzes the shape of the income distribution (Sect. 3.1) and its time development over the years covered by our data sets (Sect. 3.2). Section 4 concludes the paper.

2 The Data

We have used income data from the US Panel Study of Income Dynamics (PSID), the British Household Panel Survey (BHPS), and the German Socio-Economic Panel (GSOEP) as released in a cross-nationally comparable format in the Cross-National Equivalent File (CNEF). The CNEF brings together multiple waves of longitudinal data from the surveys above, and therefore

[4] Recently, a distribution proposed by [14,15] has the form of a deformed exponential function:

$$P_{\kappa}(x) = \left(\sqrt{1+\kappa^2 x^2} - \kappa x\right)^{\frac{1}{\kappa}}$$

which seems to capture well the behaviour of the income distribution at the low-middle range as well as the power law tail.

provides relatively long panels of information. The current release of the CNEF includes data from 1980 to 2001 for the PSID, from 1991 to 2001 for the BHPS, and from 1984 to 2002 for the GSOEP. Our data refer to the period 1980–2001 for the United States, and to the period 1991–2001 for the United Kingdom. As the eastern states of Germany were reunited with the western states of the Federal Republic of Germany in November 1990, the sample of families in the East Germany was merged with the existing data only at the beginning of the 1990s. Therefore, in order to perform analyses that represent the population of reunited Germany, we chose to refer to the subperiod 1990–2002 for the GSOEP.

A key advantage of the CNEF is that it provides reliable estimates of annual income variables defined in a similar manner for all the countries that are not directly available in the original data sets.[5] It includes pre- and post-government household income, estimates of annual labour income, assets, private and public transfers, and taxes paid at household level. In this paper, the household post-government income variable (equal to the sum of total family income from labour earnings, asset flows, private transfers, private pensions, public transfers, and social security pensions minus total household taxes) serves as the basis for all income calculations. Following a generally accepted methodology, the concept of *equivalent income* will serve as a substitute for personal income, which is unobservable. Equivalent income x is calculated as follows. In a first step, household income h is adjusted for by household type θ using an equivalence scale $e(\theta)$.[6] This adjusted household income $x = h/e(\theta)$ is then attributed to every member of the given household, which implies that income is distributed equally within households.

In the most recent release, the average sample size varies from about 7,300 households containing approximately 20,200 respondent individuals for the PSID-CNEF to 6,500 households with approximately 16,000 respondent individuals for the BHPS-CNEF; for the GSOEP-CNEF data from 1990 to 2002, we have about 7,800 households containing approximately 20,400 respondent individuals.

All the variables are in current year currency; therefore, we use the consumer price indices to convert into constant figures for all the countries. The base year is 1995.

[5] Reference [16] offers a detailed description of the CNEF. See also the CNEF web site for details: http://www.human.cornell.edu/pam/gsoep/equivfil.cfm.

[6] We use the so-called "modified OECD" equivalence scale, which is defined for each household as equal to $1 + 0.5 \times (\#\text{adults} - 1) + 0.3 \times (\#\text{children})$.

3 Empirical Findings

3.1 The Shape of the Distribution

The main panel of the pictures illustrated in Fig. 1 presents the empirical cumulative distribution of the equivalent income from our data sets for some randomly selected years in the log-log scale.[7] As shown in the lower insets, the upper income tail (about 1%–3% of the population) follows the Pareto's law:

$$1 - F(x) = P(X \geq x) = C_\alpha x^{-\alpha} \tag{2}$$

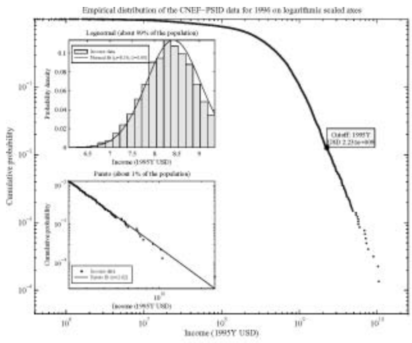
where $C_\alpha = k^\alpha$, $k, \alpha > 0$, and $k \leq x < \infty$. Since the values of x above some value x_R can not be observed due to tail truncation, to fit the (logarithm of the) data for the majority of the population (until the 97th–99th percentiles of the income distribution) we use a right-truncated normal probability density function:

$$f(y) = \begin{cases} \frac{f(y)}{\int_{-\infty}^{y_R} f(y)dy} \,, & -\infty < y \leq y_R \\ 0 \,, & y_R \leq y < \infty \end{cases} \tag{3}$$

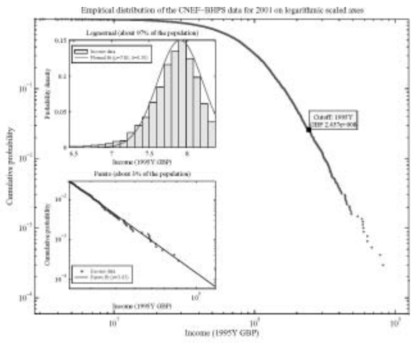
where $y = log(x)$, and $y_R = log(x_R)$. The fit to (3) is shown by the top insets of the pictures.

To select a suitable threshold or cutoff value x_R separating the lognormal part from the Pareto power law tail of the empirical income distribution, we use visually oriented statistical techniques such as the *quantile-quantile* (Q-Q) and *mean excess* plots. Figure 2 gives an example of these graphical tools for the countries at hand. The top pictures in the figure are the plots of the quantile function for the standard exponential distribution (i.e., a distribution with a medium-sized tail) against its empirical counterpart. If the sample comes from the hypothesized distribution, or a linear transformation of it, the Q-Q plot is linear. The concave presence in the plots is an indication of a fat-tailed distribution. Since a log-transformed Pareto random variable is exponentially distributed, we conduct experimental analysis on the log-transformed data by excluding some of the lower sample points to investigate the concave departure region on the plots and obtain a fit closer to the straight line. The results are shown by the insets of the top pictures in the figure. The lower pictures plot the empirical average of the data that are larger than or equal to x_R, $E(X|X \geq x_R)$, against x_R. If the plot is a linear curve, then it may be either a power type or an exponential type distribution. If the slope of the linear curve is greater than zero, then it suggests a power type (as in the main panels); otherwise, if the slope is equal to zero, it suggests an exponential type (as in the insets for the log-transformed data).

[7] To treat each wave of the surveys at hand as a cross-section, and to obtain population-based statistics, all calculations used sample weights which compensate for unequal probabilities of selection and sample attrition. Furthermore, to eliminate the influence of outliers, the data were trimmed. We also dropped observations with zero and negative incomes from all samples.



(a) US (1996)



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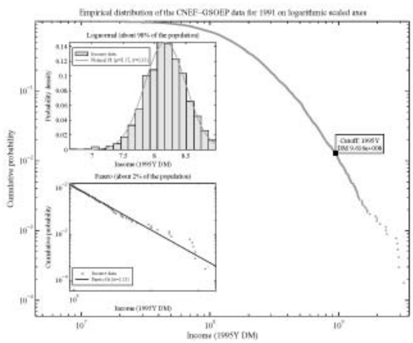
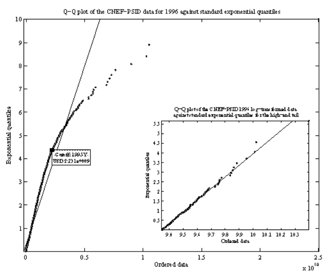


(c) Germany (1991)

Fig. 1. The cumulative probability distribution of the equivalent income in the log-log scale along with the lognormal (top insets) and Pareto (lower insets) fits for some randomly selected years

3.2 Temporal Change of the Distribution

The two-part structure of the empirical income distribution seems to hold all over the time span covered by our data sets. The distribution for all the years and countries are shown in Fig. 3. As one can easily recognize, the distribution shifts over the years covered by our data sets. It is conceivable to assume that the origin of this shift consists in the growth of the countries. To confirm this assumption, we study the fluctuations in the output and equivalent income growth rate, and try to show that the evolution of both these quantities is governed by similar mechanisms, pointing in this way to the existence of a correlation between them as one would expect. We calculate the



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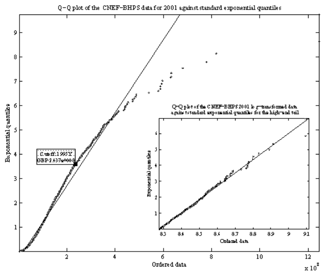

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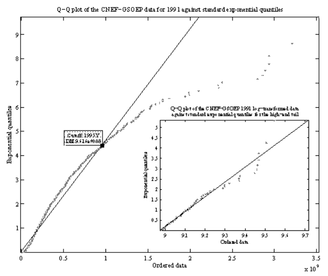

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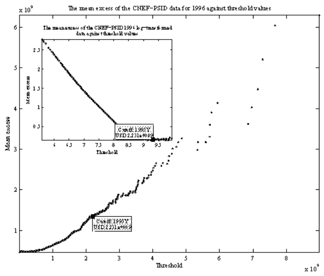
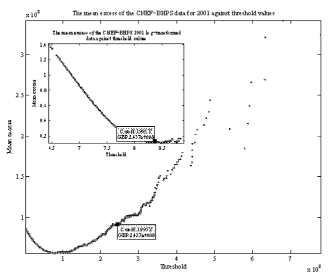
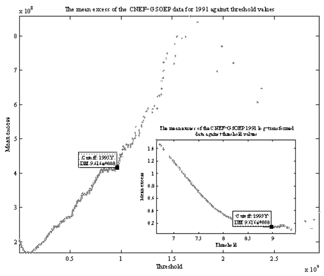


(d) US (1996)

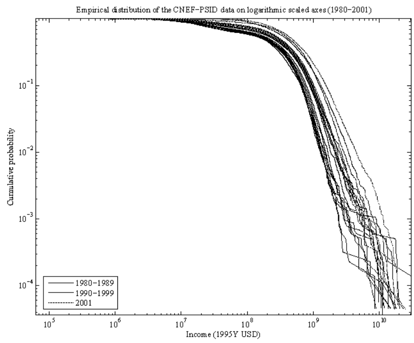


(e) UK (2001)

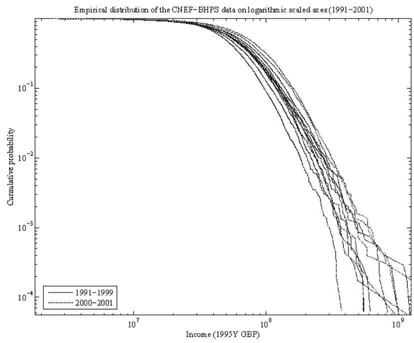


(f) Germany (1991)

Fig. 2. Q-Q plots (top pictures) against standard exponential quantiles and mean excess plots (lower pictures) against threshold values for some randomly selected years. A concave departure from the straight line in the Q-Q plot (as in the top main panels) or an upward sloping mean excess function (as in the lower main panels) indicate a heavy tail in the sample distribution. The insets in the pictures apply the same graphical tools to the log-transformed data



(a) US (1980–2001)



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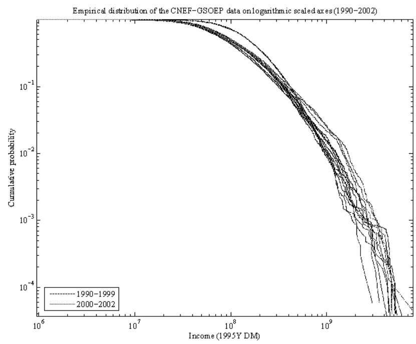


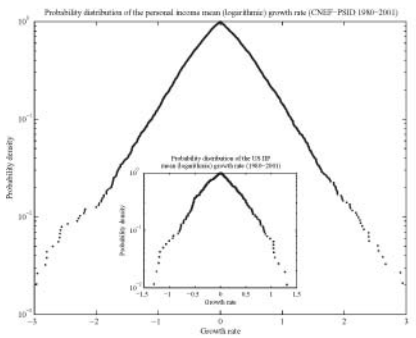
(c) Germany (1990–2002)

Fig. 3. Time development of the income distribution for all the countries and years

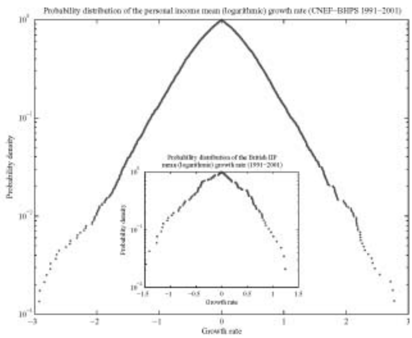
growth rates using the monthly series of the Index of Industrial Production (IIP) from [17] for output and connecting individual respondents' incomes over time for the equivalent income,[8] and express them in terms of their logarithm.[9] To account for the fact that the variance of the growth rates varies, we scale each growth rate by dividing by the corresponding estimated standard deviation. In Fig. 4 we graph the empirical probability density function for these scaled growth rates, where the data points for the equivalent income in the main panels are the average over the entire period covered by the

[8] To properly weight the sample of individuals represented in all the years of the CNEF surveys, we use the individual's longitudinal sample weights.

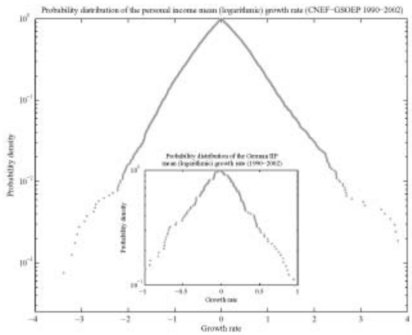
[9] All the data have been adjusted to 1995 prices and detrended by the average growth rate, so values for different years are comparable.



(a) US (1980–2001)



(b) UK (1991–2001)



(c) Germany (1990–2002)

Fig. 4. The probability distribution of equivalent income (main panels) and IIP (insets) growth rate for all the countries and years

CNEF surveys. As one can easily recognize, after scaling the resulting empirical probability density functions appear identical for observations drawn from different populations. Remarkably, both curves display a simple "tent-shaped" form; hence, the probability density functions are consistent with an exponential decay [18]:

$$f(r) = \frac{1}{\sigma\sqrt{2}} exp\left(-\frac{|r-\bar{r}|}{\sigma}\right) \tag{4}$$

where $-\infty < r < \infty$, $-\infty < \bar{r} < \infty$, and $\sigma > 0$. We test the hypothesis that the two growth rate distributions have the same continuous distribution by using the two-sample Kolmogorov-Smirnov (K-S) test; the results shown in Table 1 mean that the test is not significant at the 5% level. These findings are

Table 1. Two-sample Kolmogorov-Smirnov test statistics and p-values for both output and equivalent income growth rate data for all the countries

Country	K-S test statistic	p-value
United States	0.0761	0.1133
United Kingdom	0.0646	0.6464
Germany	0.0865	0.2050

in quantitative agreement with results reported on the growth of firms and countries [19–26], leading us to the conclusion that the data are consistent with the assumption that a common empirical law might describe the growth dynamics of both countries and individuals.

Even if the functional form of the income distribution expressed as lognormal with power law tail seems stable, its parameters fluctuate within narrow bounds over the years for the same country. For example, the power law slope has a value $\alpha = [1.1, 3.34]$ for the US between 1980 and 2001, while the curvature of the lognormal fit, as measured by the Gibrat index $\beta = 1/\left(\sigma\sqrt{2}\right)$, ranges between approximately $\beta = 1$ and $\beta = 1.65$; for the UK between 1991 and 2001, $\alpha = [3.47, 5.76]$ and $\beta = [2.18, 2.73]$; for Germany between 1990 and 2002, $\alpha = [2.42, 3.96]$ and $\beta = [1.63, 2.14]$. The time pattern of these parameters is shown by the main panels of Fig. 5, which also reports in one of the insets the temporal change of inequality as measured by the Gini coefficient. As one can easily recognize, the information about inequality provided by the Gibrat index seems near enough to those provided by the Gini coefficient, which is a further confirmation of the fact that the lognormal law is a good model for the low-middle incomes of the distribution. The Pareto index is a rather strongly changing index. Among others, the definition of income we use in the context of our analysis contains asset flows. It is conceivable to assume that for the top 1% to 3% of the population returns on capital gains rather than labour earnings account for the majority share of the total income. This suggests that the stock market fluctuations might be an important factor behind the trend of income inequality among the richest, and that capital income plays an important role in determining the Pareto functional form of the observed empirical income distribution at the high income range [27]. The other insets of the pictures also show the time evolution of various parameters characterizing income distribution, such as the income separating the lognormal and Pareto regimes (selected as explained in Sect. 3.1), the fraction of population in the upper tail of the distribution, and the share of total income which this fraction accounts for.[10] One can observe that the fraction of population and the share of income in the Pareto tail move together in the opposite direction with respect to the cutoff value separating the body of the

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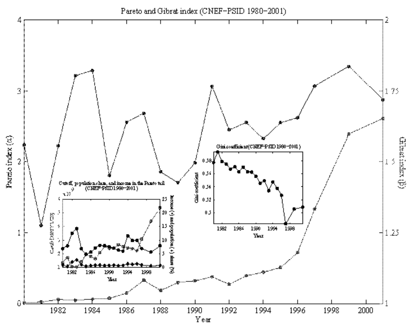
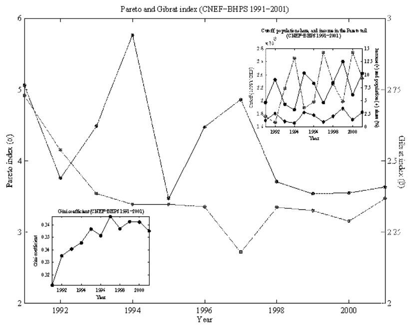
[10] The share of total income in the tail of the distribution is calculated as μ_α/μ, where μ_α is the average income of the population in the Pareto tail and μ is the average income of the whole population.



(a) US (1980–2001)



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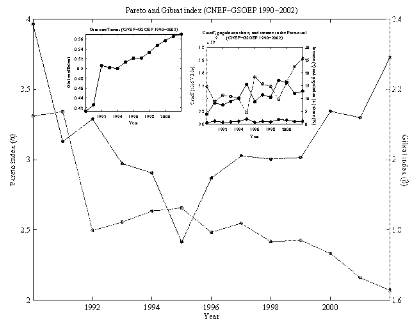

(c) Germany (1990–2002)

Fig. 5. Temporal evolution of various parameters characterizing the income distribution

distribution from its tail, and the latter seems to track the temporal evolution of the Pareto index. This fact means that a decrease (increase) of the power law slope and the accompanying decrease (increase) of the threshold value x_R imply a greater (smaller) fraction of the population in the tail and a greater (smaller) share of the total income which this population account for, as well as a greater (smaller) level of inequality among high income population.

4 Summary and Conclusions

Our analysis of the data for the US, the UK, and Germany shows that there are two regimes in the income distribution. For the low-middle class up to approximately 97%–99% of the total population the incomes are well described by a two-parameter lognormal distribution, while the incomes of the top 1%–3% are described by a power law (Pareto) distribution.

This structure has been observed in our analysis for different years. However, the distribution shows a rightward shift in time. Therefore, we analyze the output and individual income growth rate distribution from which we observe that, after scaling, the resulting empirical probability density functions appear similar for observations coming from different populations. This effect, which is statistically tested by means of a two-sample Kolmogorov-Smirnov test, raises the intriguing possibility that a common mechanism might characterize the growth dynamics of both output and individual income, pointing in this way to the existence of a correlation between these quantities. Furthermore, from the analysis of the temporal change of the parameters specifying the distribution, we find that these quantities do not necessarily correlate to each other. This means that different mechanisms are working in the distribution of the low-middle income range and that of the high income range. Since earnings from financial or other assets play an important role in the high income section of the distribution, one possible origin of this behaviour might be the change of the asset price, which mainly affects the level of inequality at the very top of the income distribution and is likely to be responsible for the power law nature of high incomes.

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Two-class Structure of Income Distribution in the USA: Exponential Bulk and Power-law Tail

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Summary. Personal income distribution in the USA has a well-defined two-class structure. The majority of population (97–99%) belongs to the lower class characterized by the exponential Boltzmann-Gibbs ("thermal") distribution, whereas the upper class (1–3% of population) has a Pareto power-law ("superthermal") distribution. By analyzing income data for 1983–2001, we show that the "thermal" part is stationary in time, save for a gradual increase of the effective temperature, whereas the "superthermal" tail swells and shrinks following the stock market. We discuss the concept of equilibrium inequality in a society, based on the principle of maximal entropy, and quantitatively show that it applies to the majority of population.

Attempts to apply the methods of exact sciences, such as physics, to describe a society have a long history [1]. At the end of the 19th century, Italian physicist, engineer, economist, and sociologist Vilfredo Pareto suggested that income distribution in a society is described by a power law [2]. Modern data indeed confirm that the upper tail of income distribution follows the Pareto law [3, 4, 5, 6, 7]. However, the majority of population does not belong there, so characterization and understanding of their income distribution remains an open problem. Drăgulescu and Yakovenko [8] proposed that the equilibrium distribution should follow an exponential law analogous to the Boltzmann-Gibbs distribution of energy in statistical physics. The first factual evidence for the exponential distribution of income was found in Ref. [9]. Coexistence of the exponential and power-law parts of the distribution was recognized in Ref. [10]. However, these papers, as well as Ref. [11], studied the data only for a particular year. Here we analyze temporal evolution of the personal income distribution in the USA during 1983–2001 [12]. We show that the US society has a well-defined two-class structure. The majority of population (97–99%) belongs to the lower class and has a very stable in time exponential ("thermal") distribution of income. The upper class (1–3% of population) has a power-law ("superthermal") distribution, whose parameters significantly change in time with the rise and fall of stock market. Using the principle of maximal entropy, we discuss the concept of equilibrium inequality in a society and quantitatively show that it applies to the bulk of population. Most of academic and government literature on income distribution and inequality [13, 14, 15, 16] does not attempt to fit the data by a simple formula. When fits are performed, usually the log-normal distribution [17] is used for the lower part of the distribution [5, 6, 7]. Only recently the exponential distribution started to be recognized in income studies [18, 19], and models showing formation of two classes started to appear [20, 21].

Let us introduce the probability density $P(r)$, which gives the probability $P(r)\,dr$ to have income in the interval $(r, r + dr)$. The cumulative probability $C(r) = \int_r^\infty dr' P(r')$ is the probability to have income above r, $C(0) = 1$. By analogy with the Boltzmann-Gibbs distribution in statistical physics [8, 9], we consider an exponential function $P(r) \propto \exp(-r/T)$, where T is a parameter analogous to temperature. It is equal to the average income $T = \langle r \rangle = \int_0^\infty dr' r' P(r')$, and we call it the "income temperature." When $P(r)$ is exponential, $C(r) \propto \exp(-r/T)$ is also exponential. Similarly, for the Pareto power law $P(r) \propto 1/r^{\alpha+1}$, $C(r) \propto 1/r^\alpha$ is also a power law.

We analyze the data [22] on personal income distribution compiled by the Internal Revenue Service (IRS) from the tax returns in the USA for the period 1983–2001 (presently the latest available year). The publicly available data are already preprocessed by the IRS into bins and effectively give the cumulative distribution function $C(r)$ for certain values of r. First we make the plots of $\log C(r)$ vs. r (the log-linear plots) for each year. We find that the plots are straight lines for the lower 97–98% of population, thus confirming the exponential law. From the slopes of these straight lines, we determine the income temperatures T for each year. In Fig. 1, we plot $C(r)$ and $P(r)$ vs. r/T (income normalized to temperature) in the log-linear scale. In these coordinates, the data sets for different years collapse onto a single straight line. (In Fig. 1, the data lines for 1980s and 1990s are shown separately and offset vertically.) The columns of numbers in Fig. 1 list the values of the annual income temperature T for the corresponding years, which changes from 19 k$ in 1983 to 40 k$ in 2001. The upper horizontal axis in Fig. 1 shows income r in k$ for 2001.

In Fig. 2, we show the same data in the log-log scale for a wider range of income r, up to about $300T$. Again we observe that the sets of points for different years collapse onto a single exponential curve for the lower part of the distribution, when plotted vs. r/T. However, above a certain income $r_* \approx 4T$, the distribution function changes to a power law, as illustrated by the straight lines in the log-log scale of Fig. 2. Thus we observe that income distribution in the USA has a well-defined two-class structure. The lower class (the great majority of population) is characterized by the exponential, Boltzmann-Gibbs distribution, whereas the upper class (the top few percent of population) has the power-law, Pareto distribution. The intersection point of the exponential and power-law curves determines the income r_* separating the two classes. The collapse of data points for different years in the lower, exponential part of the distribution in Figs. 1 and 2 shows that this part is very stable in

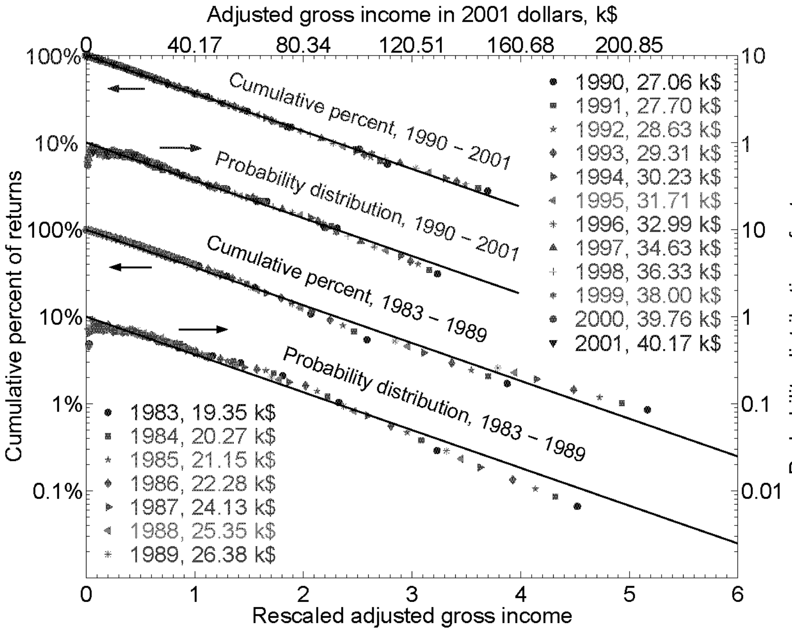

Fig. 1. Cumulative probability $C(r)$ and probability density $P(r)$ plotted in the log-linear scale vs. r/T, the annual personal income r normalized by the average income T in the exponential part of the distribution. The IRS data points are for 1983–2001, and the columns of numbers give the values of T for the corresponding years.

time and, essentially, does not change at all for the last 20 years, save for a gradual increase of temperature T in nominal dollars. We conclude that the majority of population is in statistical equilibrium, analogous to the thermal equilibrium in physics. On the other hand, the points in the upper, power-law part of the distribution in Fig. 2 do not collapse onto a single line. This part significantly changes from year to year, so it is out of statistical equilibrium. A similar two-part structure in the energy distribution is often observed in physics, where the lower part of the distribution is called "thermal" and the upper part "superthermal" [23].

Temporal evolution of the parameters T and r_* is shown in Fig. 3.A. We observe that the average income T (in nominal dollars) was increasing gradually, almost linearly in time, and doubled in the last twenty years. In Fig. 3.A, we also show the inflation coefficient (the consumer price index CPI from Ref. [24]) compounded on the average income of 1983. For the twenty years, the inflation factor is about 1.7, thus most, if not all, of the nominal increase in T is inflation. Also shown in Fig. 3.A is the nominal gross domestic product (GDP) per capita [24], which increases in time similarly to T and CPI. The ratio r_*/T varies between 4.8 and 3.2 in Fig. 3.A.

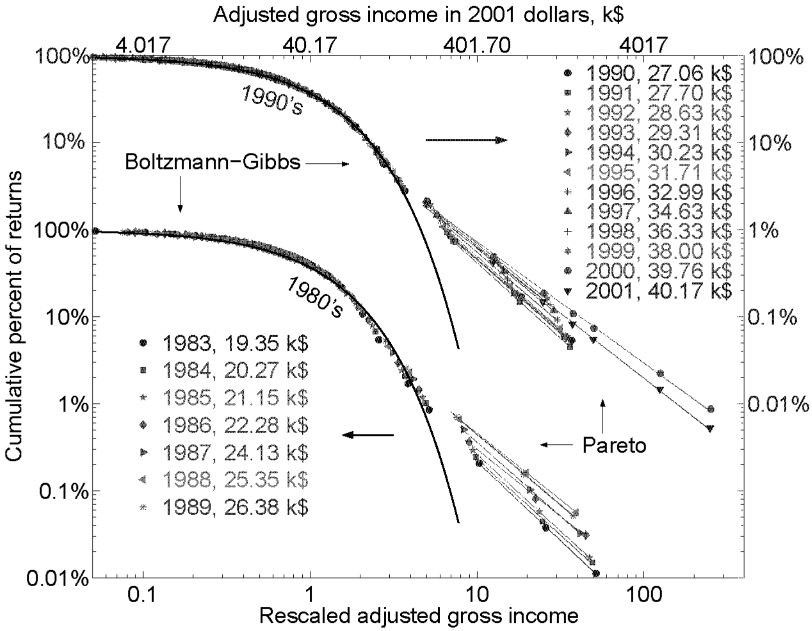

Fig. 2. Log-log plots of the cumulative probability $C(r)$ vs. r/T for a wider range of income r.

In Fig. 3.B, we show how the parameters of the Pareto tail $C(r) \propto 1/r^{\alpha}$ change in time. Curve (a) shows that the power-law index α varies between 1.8 and 1.4, so the power law is not universal. Because a power law decays with r more slowly than an exponential function, the upper tail contains more income than we would expect for a thermal distribution, hence we call the tail "superthermal" [23]. The total excessive income in the upper tail can be determined in two ways: as the integral $\int_{r_*}^{\infty} dr' r' P(r')$ of the power-law distribution, or as the difference between the total income in the system and the income in the exponential part. Curves (c) and (b) in Fig. 3.B show the excessive income in the upper tail, as a fraction f of the total income in the system, determined by these two methods, which agree with each other reasonably well. We observe that f increased by the factor of 5 between 1983 and 2000, from 4% to 20%, but decreased in 2001 after the crash of the US stock market. For comparison, curve (e) in Fig. 3.B shows the stock market index S&P 500 divided by inflation. It also increased by the factor of 5.5 between 1983 and 1999, and then dropped after the stock market crash. We conclude that the swelling and shrinking of the upper income tail is correlated with the rise and fall of the stock market. Similar results were found for the upper income tail in Japan in Ref. [4]. Curve (d) in Fig. 3.B shows the fraction of population in the upper tail. It increased from 1% in 1983 to 3% in 1999, but then decreased after the stock market crash. Notice, however, that the stock market dynamics had a much weaker effect on the average income T of the lower, "thermal" part of income distribution shown in Fig. 3.A.

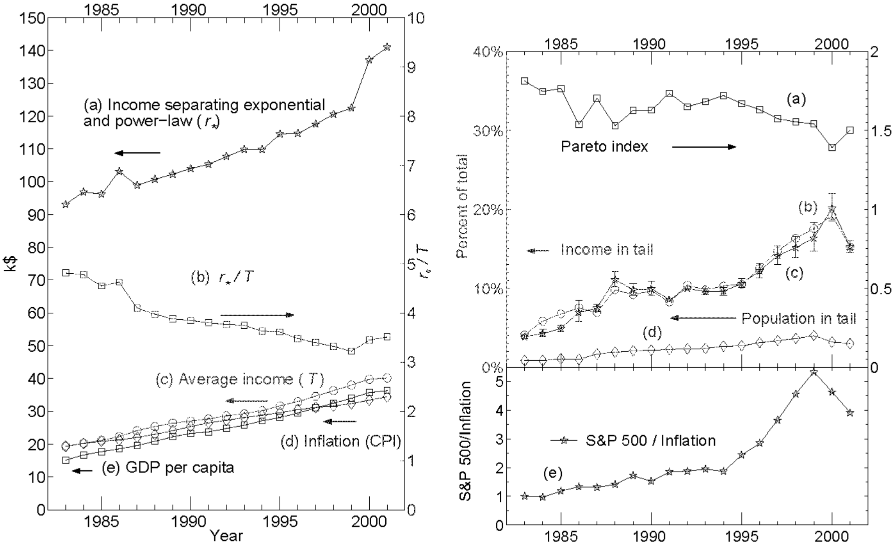

Fig. 3. Left panel A: Temporal evolution of various parameters characterizing income distribution. Right panel B: (a) The Pareto index α of the power-law tail $C(r) \propto 1/r^{\alpha}$. (b) The excessive income in the Pareto tail, as a fraction f of the total income in the system, obtained as the difference between the total income and the income in the exponential part of the distribution. (c) The tail income fraction f, obtained by integrating the Pareto power law of the tail. (d) The fraction of population belonging to the Pareto tail. (e) The stock-market index S&P 500 divided by the inflation coefficient and normalized to 1 in 1983.

For discussion of income inequality, the standard practice is to construct the so-called Lorenz curve [13]. It is defined parametrically in terms of the two coordinates $x(r)$ and $y(r)$ depending on the parameter r, which changes from 0 to ∞. The horizontal coordinate $x(r) = \int_0^r dr' P(r')$ is the fraction of population with income below r. The vertical coordinate $y(r) = \int_0^r dr' r' P(r') / \int_0^\infty dr' r' P(r')$ is the total income of this population, as a fraction of the total income in the system. Fig. 4 shows the data points for the Lorenz curves in 1983 and 2000, as computed by the IRS [16]. For a purely exponential distribution of income $P(r) \propto \exp(-r/T)$, the formula $y = x + (1-x)\ln(1-x)$ for the Lorenz curve was derived in Ref. [9]. This formula describes income distribution reasonably well in the first approximation [9], but visible deviations exist. These deviations can be corrected by taking into account that the total income in the system is higher than the income in the exponential part, because of the extra income in the Pareto tail. Correcting for this difference in the normalization of y, we find a modified expression

[11] for the Lorenz curve

$$y = (1 - f)[x + (1 - x)\ln(1 - x)] + f\Theta(x - 1), \tag{1}$$

where f is the fraction of the total income contained in the Pareto tail, and $\Theta(x-1)$ is the step function equal to 0 for $x < 1$ and 1 for $x \geq 1$. The Lorenz curve (1) experiences a vertical jump of the height f at $x = 1$, which reflects the fact that, although the fraction of population in the Pareto tail is very small, their fraction f of the total income is significant. It does not matter for Eq. (1) whether the extra income in the upper tail is described by a power law or another slowly decreasing function $P(r)$. The Lorenz curves, calculated using Eq. (1) with the values of f from Fig. 3.B, fit the IRS data points very well in Fig. 4.

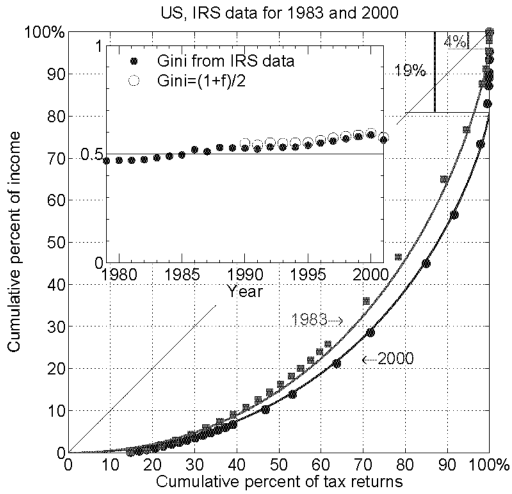

Fig. 4. Main panel: Lorenz plots for income distribution in 1983 and 2000. The data points are from the IRS [16], and the theoretical curves represent Eq. (1) with f from Fig. 3. Inset: The closed circles are the IRS data [16] for the Gini coefficient G, and the open circles show the theoretical formula $G = (1 + f)/2$.

The deviation of the Lorenz curve from the diagonal in Fig. 4 is a certain measure of income inequality. Indeed, if everybody had the same income, the Lorenz curve would be the diagonal, because the fraction of income would be proportional to the fraction of population. The standard measure of income inequality is the so-called Gini coefficient $0 \leq G \leq 1$, which is defined as the area between the Lorenz curve and the diagonal, divided by the area of the triangle beneath the diagonal [13]. It was calculated in Ref. [9] that $G = 1/2$ for a purely exponential distribution. Temporal evolution of the Gini coefficient, as determined by the IRS [16], is shown in the inset of Fig. 4. In the first approximation, G is quite close to the theoretically calculated value 1/2.

The agreement can be improved by taking into account the Pareto tail, which gives $G = (1 + f)/2$ for Eq. (1). The inset in Fig. 4 shows that this formula very well fits the IRS data for the 1990s with the values of f taken from Fig. 3.B. We observe that income inequality was increasing for the last 20 years, because of swelling of the Pareto tail, but started to decrease in 2001 after the stock market crash. The deviation of G below $1/2$ in the 1980s cannot be captured by our formula. The data points for the Lorenz curve in 1983 lie slightly above the theoretical curve in Fig. 4, which accounts for $G < 1/2$.

Thus far we discussed the distribution of individual income. An interesting related question is the distribution of family income $P_2(r)$. If both spouses are earners, and their incomes are distributed exponentially as $P_1(r) \propto \exp(-r/T)$, then

$$P_2(r) = \int_0^r dr' P_1(r') P_1(r - r') \propto r \exp(-r/T). \qquad (2)$$

Eq. (2) is in a good agreement with the family income distribution data from the US Census Bureau [9]. In Eq. (2), we assumed that incomes of spouses are uncorrelated. This assumption was verified by comparison with the data in Ref. [11]. The Gini coefficient for family income distribution (2) was found to be $G = 3/8 = 37.5\%$ [9], in agreement with the data. Moreover, the calculated value 37.5% is close to the average G for the developed capitalist countries of North America and Western Europe, as determined by the World Bank [11].

On the basis of the analysis presented above, we propose a concept of the *equilibrium inequality* in a society, characterized by $G = 1/2$ for individual income and $G = 3/8$ for family income. It is a consequence of the exponential Boltzmann-Gibbs distribution in thermal equilibrium, which maximizes the entropy $S = \int dr\, P(r) \ln P(r)$ of a distribution $P(r)$ under the constraint of the conservation law $\langle r \rangle = \int_0^\infty dr\, P(r)\, r = \text{const}$. Thus, any deviation of income distribution from the exponential one, to either less inequality or more inequality, reduces entropy and is not favorable by the second law of thermodynamics. Such deviations may be possible only due to non-equilibrium effects. The presented data show that the great majority of the US population is in thermal equilibrium.

Finally, we briefly discuss how the two-class structure of income distribution can be rationalized on the basis of a kinetic approach, which deals with temporal evolution of the probability distribution $P(r, t)$. Let us consider a diffusion model, where income r changes by Δr over a period of time Δt. Then, temporal evolution of $P(r, t)$ is described by the Fokker-Planck equation [25]

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial r}\left(AP + \frac{\partial}{\partial r}(BP)\right), \quad A = -\frac{\langle \Delta r \rangle}{\Delta t}, \quad B = \frac{\langle (\Delta r)^2 \rangle}{2\Delta t}. \qquad (3)$$

For the lower part of the distribution, it is reasonable to assume that Δr is independent of r. In this case, the coefficients A and B are constants. Then, the stationary solution $\partial_t P = 0$ of Eq. (3) gives the exponential distribution [8]

$P(r) \propto \exp(-r/T)$ with $T = B/A$. Notice that a meaningful solution requires that $A > 0$, i.e. $\langle \Delta r \rangle < 0$ in Eq. (3). On the other hand, for the upper tail of income distribution, it is reasonable to expect that $\Delta r \propto r$ (the Gibrat law [17]), so $A = ar$ and $B = br^2$. Then, the stationary solution $\partial_t P = 0$ of Eq. (3) gives the power-law distribution $P(r) \propto 1/r^{\alpha+1}$ with $\alpha = 1 + a/b$. The former process is additive diffusion, where income changes by certain amounts, whereas the latter process is multiplicative diffusion, where income changes by certain percentages. The lower class income comes from wages and salaries, so the additive process is appropriate, whereas the upper class income comes from investments, capital gains, etc., where the multiplicative process is applicable. Ref. [4] quantitatively studied income kinetics using tax data for the upper class in Japan and found that it is indeed governed by a multiplicative process. The data on income mobility in the USA are not readily available publicly, but are accessible to the Statistics of Income Research Division of the IRS. Such data would allow to verify the conjectures about income kinetics.

The exponential probability distribution $P(r) \propto \exp(-r/T)$ is a monotonous function of r with the most probable income $r = 0$. The probability densities shown in Fig. 1 agree reasonably well with this simple exponential law. However, a number of other studies found a nonmonotonous $P(r)$ with a maximum at $r \neq 0$ and $P(0) = 0$. These data were fitted by the log-normal [5, 6, 7] or the gamma distribution [19, 20, 26]. The origin of the discrepancy in the low-income data between our work and other papers is not completely clear at this moment. The following factors may possibly play a role. First, one should be careful to distinguish between personal income and group income, such as family and household income. As Eq. (2) shows, the later is given by the gamma distribution even when the personal income distribution is exponential. Very often statistical data are given for households and mix individual and group income distributions (see more discussion in Ref. [9]). Second, the data from tax agencies and census bureaus may differ. The former data are obtained from tax declarations of all taxable population, whereas the later data from questionnaire surveys of a limited sample of population. These two methodologies may produce different results, particularly for low incomes. Third, it is necessary to distinguish between distributions of money [8, 26, 27], wealth [20, 28], and income. They are, presumably, closely related, but may be different in some respects. Fourth, the low-income probability density may be different in the USA and in other countries because of different social security policies. All these questions require careful investigation in future work. We can only say that the data sets analyzed in this paper and our previous papers are well described by a simple exponential function for the whole lower class. This does not exclude a possibility that other functions can also fit the data [29]. However, the exponential law has only one fitting parameter T, whereas log-normal, gamma, and other distributions have two or more fitting parameters, so they are less parsimonious.

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Pareto-Zipf, Gibrat's Laws, Detailed-Balance and their Breakdown

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Summary. By employing exhaustive lists of personal income and firms, we show that the upper-tail of the distribution of income and firm size has power-law (Pareto-Zipf law), and that in this region their growth rate is independent of the initial value of income or size (Gibrat's law of proportionate effect). In addition, detailed balance holds in the power-law region; the empirical probability for an individual (a firm) to change its income (size) from a value to another is statistically the same as that for its reverse process in the ensemble. We prove that Pareto-Zipf law follows from Gibrat's law under the condition of detailed balance. We also show that the distribution of growth rate possesses a non-trivial relation between the positive and negative sides of the distribution, through the value of Pareto index, as is confirmed empirically. Furthermore, we also show that these properties break down in the non power-law region of distribution, and can possibly do so temporally according to drastic change in financial or real economy.

Key words. personal income, firm size, Pareto-Zipf distribution, Gibrat law, detailed-balance

1 Introduction

Flow and stock are fundamental concepts in economics. They refer to a certain economic quantity over a given period of time and its accumulation at a point in time respectively. Personal income and wealth can be regarded as the flow and stock of each *household* in a giant dynamical network of people, which is open to various economic activities. The same is true for *firms*.

High-income distribution follows a power-law: the probability $P_{>}(x)$ that a given individual has income equal to, or greater than x, obeys

$$P_{>}(x) \propto x^{-\mu}, \tag{1}$$

with a constant μ called the Pareto index. This phenomenon, now known as Pareto law, has been observed [5, 7, 2, 17, 8, 9] in different countries (see

Fig. 1 (a) for japanese data). On the other hand, low and middle-income distribution has been considered to obey log-normal distribution, Boltzman distribution or other functional form (see Clementi and Gallegati, Souma and Nirei, Willis, Yakovenko in this workshop).

A similar distribution has also been observed for firm size [4] (see Fig. 1 (b) for european data). μ is typically around 2 for personal income and around 1 for firm size distribution[1]. The latter is often refered to as Zipf law. We call the distribution (1) Pareto-Zipf law in this paper.

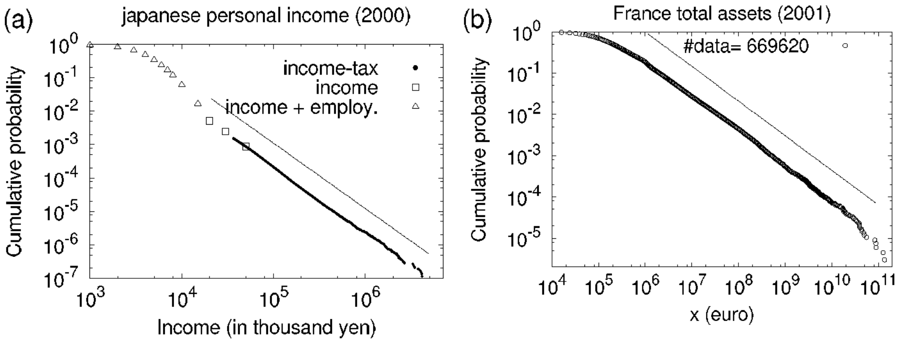

Fig. 1. (a) Cumulative probability distribution of japanese personal income in the year 2000. The line is simply a guide for eyes with $\mu = 1.96$ in (1). Note that the dots are income tax data of about 80,000 taxpayers. (See [17][9] for the details). (b) Cumulative probability distribution of firm size (total-assets) in France in the year 2001. Data consist of 669620 firms, which are exhaustive in the sense that firms exceeding a threshold are all listed. The line corresponds to $\mu = 0.84$ (See also [11]).

Understanding the origin of the law has importance in economics because of its linkage with consumption, business cycles, and other macro-economic activities. Also note that even if the range for which (1) is valid is a few percent in the upper tail of the distribution, it is often observed that such a small fraction of individuals (firms) occupies a large amount of total sum of income (size). Small idiosyncratic shock can make a considerable macro-economic impact.

Many researchers, recently including those in non-equilibrium statistical physics, have proposed models for power-law [7, 14, 15, 13, 6, 16]. Actually many kinds of proposed scenarios have predicted a power-law distribution as a static snapshot. However, in order to test models, it is highly desirable to have direct observation of the dynamical process of growth and fluctuations of personal income (or firm size). This is what we address in this paper.

[1] It would be interesting to note that debt distribution of *bankrupted* firms obeys Zipf law in accordance to firm size distribution [12].

2 Growth and Fluctuations

For the study of personal income, we employ japanese income-tax data, which is an exhaustive list of all taxpayers who paid 10 million yen (approx. 10 thousand euros/dollars) or more in a year. It is considered that the taxpayers cover most of the power-law region in Fig. 1 (a). Distribution of personal income has been studied by using the data [2, 17]. Furthermore, growth and fluctuations of *each* individual can be examined [9]. In fact, we examined a relatively stable period in japanese economy, namely 1997 and 1998.

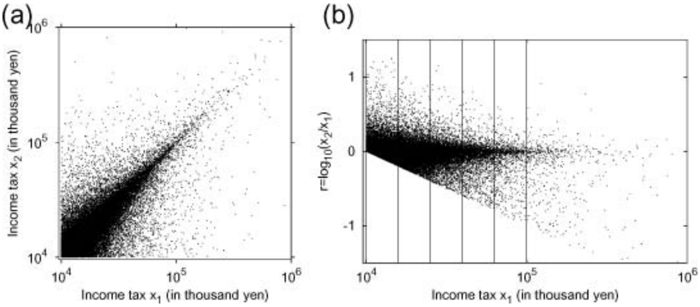

Fig. 2. (a) Scatter-plot of all individuals whose income tax exceeds 10 million yen in both 1997 and 1998. These points (52,902) were identified from the complete list of high-income taxpayers in 1997 and 1998, with income taxes x_1 and x_2 in each year. (b) The same as (a) with vertical axis for $r = \log_{10}(x_2/x_1)$. The segments are bins for $x_1 \in [10^{4+0.2(n-1)}, 10^{4+0.2n}]$ $(n = 1, \cdots, 5)$.

In the scatter plot in Fig. 2 (a) each point represents a person who is high-income taxpayers in both of the years, 1997 and 1998. The plot represents the joint distribution $P_{12}(x_1, x_2)$. The plot is consistent with *detailed-balance* in the sense that the joint distribution is invariant under the exchange of values x_1 and x_2, i.e. $P_{12}(x_1, x_2) = P_{12}(x_2, x_1)$[2]. Detailed-balance means that the empirical probability for an individual to change its income from a value to another is statistically the same as that for its reverse process in the ensemble.

Our concern is the annual change of individual income-tax. Growth rate is defined by $R = x_2/x_1$. It is customary to use its logarithm, $r \equiv \log_{10} R$. We examine the probability density for the growth rate $P(r|x_1)$ on the condition that the income x_1 in the initial year is fixed (see Fig. 2 (b)). The result shows that the distribution for growth rate r is *statistically independent* of the value of x_1, as shown in Fig. 3. This is known as *law of proportionate effect* or Gibrat's law (see [18]).

[2] Actually we can make a direct statistical test for the symmetry in the two arguments of $P_{12}(x_1, x_2)$. This can be done by two-dimensional Kolmogorov-Smirnov test, which is not widely known but has been developed by astrophysicists to test uniform distribution of galaxies appearing in the sky (see references in [11]) .

The phenomenological properties (A) detailed-balance, (B) Pareto-Zipf law, and (C) Gibrat's law are observed for firm size as well as for personal income. See [11] for such a study of large firms in European countries.

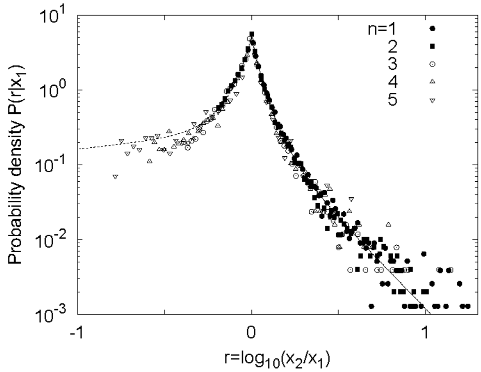

Fig. 3. Probability density $P(r|x_1)$ of growth rate $r \equiv \log_{10}(x_2/x_1)$ from 1997 to 1998. Note that due to the limit $x_1 > 10^4$ (in thousand yen), the data for large negative growth, $r < 4 - \log_{10} x_1$, are not available. Different bins of initial income-tax with equal size in logarithmic scale were taken as $x_1 \in [10^{4+0.2(n-1)}, 10^{4+0.2n}]$ $(n = 1, \cdots, 5)$ to plot probability densities separately for each such bin. The solid line in the portion of positive growth $(r > 0)$ is an analytic fit. The dashed line $(r < 0)$ on the other side is calculated by the relation in (7).

The probability distribution for the growth rate, such as the one observed in Fig. 3, contains information of dynamics. One can notice that it has a skewed and heavy-tailed shape with a peak at $R = 1$. How is such a functional form consistent with the detailed-balance shown in Fig. 2? And how these phenomenological facts are consistent with Pareto's law in Fig. 1? Answers to these questions are given in the next section.

3 Pareto-Zipf and Gibrat under detailed balance

Let x be a personal income or a firm size, and let its values at two successive points in time (i.e., two consecutive years) be denoted by x_1 and x_2. We denote the joint probability distribution for the variables x_1 and x_2 by $P_{12}(x_1, x_2)$. We define conditional probability, $P_{1R}(x_1, x_2/x_1) = x_1 P_{12}(x_1, x_2)$, where $P_1(x_1)$ is marginal, i.e., $P_1(x_1) = \int_0^\infty P_{1R}(x_1, R)dR = \int_0^\infty P_{12}(x_1, x_2)dx_2$.

The phenomenological properties can be summarized as follows.

(A) *Detailed Balance*:

$$P_{12}(x_1, x_2) = P_{12}(x_2, x_1). \qquad (2)$$

(B) *Pareto-Zipf's law*:

$$P_1(x) \propto x^{-\mu-1}, \tag{3}$$

for $x \to \infty$ with $\mu > 0$.

(C) *Gibrat's law*: The conditional probability $Q(R \,|\, x)$ is independent of x:

$$Q(R \,|\, x) = Q(R). \tag{4}$$

We note here that this holds only for x larger than a certain value. All the arguments below is restricted in this region.

Now we prove that the properties (A) and (C) lead to (B). Under the change of variables from (x_1, x_2) to (x_1, R), since $P_{12}(x_1, x_2) = (1/x_1)P_{1R}(x_1, R)$, one can easily see that $P_{1R}(x_1, R) = (1/R)P_{1R}(Rx_1, R^{-1})$. It immediately follows from the definition of $Q(R \,|\, x)$ that

$$\frac{Q(R^{-1} \,|\, x_2)}{Q(R \,|\, x_1)} = R\frac{P_1(x_1)}{P_1(x_2)}. \tag{5}$$

This equation is thus equivalent to detailed-balance condition.

If Gibrat's law holds, $Q(R \,|\, x) = Q(R)$, then

$$\frac{P_1(x_1)}{P_1(x_2)} = \frac{1}{R}\frac{Q(R^{-1})}{Q(R)}. \tag{6}$$

Note that while the left-hand side of (6) is a function of x_1 and $x_2 = Rx_1$, the right-hand side is a function of ratio R only. It can be easily shown that the equality is satisfied by and only by a power-law function (3)[3].

As a bonus, by inserting (3) into (6), we have a non-trivial relation:

$$Q(R) = R^{-\mu-2}Q(R^{-1}), \tag{7}$$

which relates the positive and negative growth rates, $R > 1$ and $R < 1$, through the Pareto index μ.

One can also show that $Q(R)$ has a cusp at $R = 1$; $Q'(R)$ is discontinuous at $R = 1$. Explicitly, $[Q^{+\prime}(1) + Q^{-\prime}(1)]/Q(1) = -\mu - 2$, where we denote the right and left-derivative of $Q(R)$ at $R = 1$ by the signs $+$ and $-$ in the superscript, respectively. This relation states that the shape of cusp in $Q(R)$ at $R = 1$ is determined by the Pareto index μ.

Summarizing this section, we have proved that under the condition of detailed balance (A), Gibrat's law (C) implies Pareto-Zipf law (B). The opposite (B) $\to$ (C) is not true. See [11] for several kinematic relations, and also [9, 3, 10] for the validity of our findings in personal income and firms data.

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[3] Expand (6) with respect to R around $R = 1$ and to equate the first-order term to zero, which gives an ordinary differential equation for $P_1(x)$.

4 Temporal breakdown of the laws

In economically unstable period such as "bubble" and "crash", income distribution deviates from power-law as shown in Fig. 4 for japanese case. The year 1991 coincides with the peak of speculative bubble of land price. One can observe that the 1991 data cannot be fitted by Pareto's law in the entire range of high income, while one year later the distribution went back to power-law.

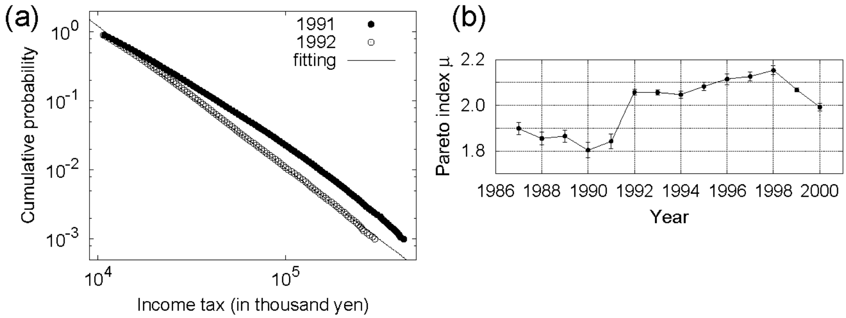

Fig. 4. (a) Cumulative probability distributions of income tax in 1991 and 1992. The fitted line is for $\mu = 2.057$. Note that the distribution does not obey power-law in 1991. (b) Annual change of Pareto index μ from the year 1987 to 2000. The abrupt change from 1991 to 1992 corresponds to abnormal rise and collapse of risky assets prices. (See [9] for estimation).

Sample survey on income earners provides information about income-sources. Picking those persons with total income exceeding 50 million yen (who are necessarily included in the exhaustive list described so far), it can be observed that in terms of the numbers chief income-sources are employment income, rental of real estate, and capital gains from lands and stock shares (Fig. 5 (a)), and that in terms of the amounts contribution comes largely from capital gains from lands and stocks (Fig. 5 (b)). It is expected that asymmetric behavior of price fluctuations in those risky assets and the accompanying increase in high-income persons cause the breakdown of detailed-balance and/or the statistical independence, which necessarily invalidates Pareto's law.

This can be verified in Fig. 6 showing the breakdown of Gibrat's law. By statistical test [11], the null hypothesis $P_{12}(x_1, x_2) = P_{12}(x_2, x_1)$ can be rejected for the pair of 1991 and 1992, but cannot be rejected for the pairs of 1992 and 1993, and of 1997 anf 1998 with significance level 95%.

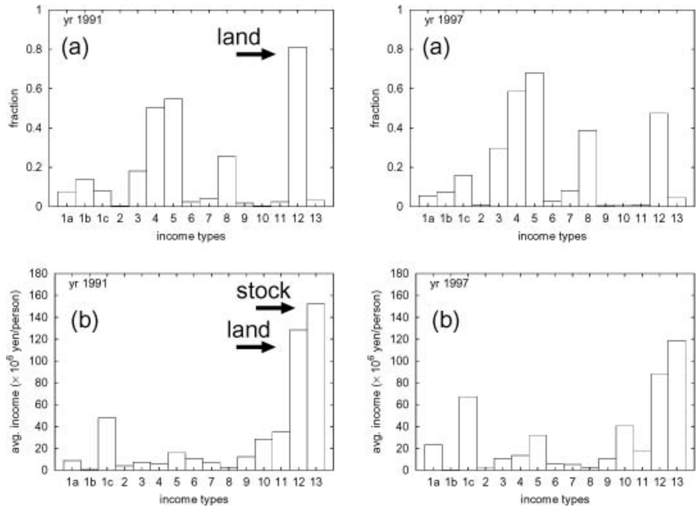

Fig. 5. See the list of income types below. *Left panels*: 1991. *Right panels*: 1992. (a) Fraction of the numbers for income earners with total income exceeding 50 million yen with a particular income type. A person can have more than a single income type. (b) Fraction of the amounts in average over all the earners. The numerals are 1a: business income, 1b: firm income (agricultural), 1c: other operating income (lawyers, doctor, entertainers, etc.), 2: interest income, 3: dividends, 4: rental income (mainly of real estate), 5: wages/salaries, 6: comprehensive capital gains, 7: sporadic income, 8: miscellaneous income (including public pension, etc.), 9: forestry income, 10: retirement income, 11: short-term separate capital gains (selling real estate possessed in 5 years), 12: long-term separate capital gains (selling real estate possessed over 5 years), 13: capital gains from stocks, etc.

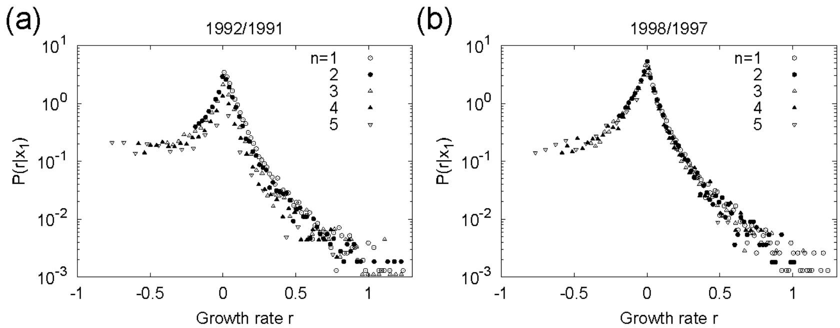

Fig. 6. (a) Probability density $P(r|x_1)$ of growth rate $r \equiv \log_{10}(x_2/x_1)$ from 1991 to 1992. It is obvious that $P(r|x_1)$ depends on x_1, thus breaking Gibrat's law. (b) The same plot for the successive years of 1997 and 1998, for which Gibrat's law holds.

5 Small and midsize firms

According to survey by statistics bureau, the number of japanese companies is approximately 1.6 million in the year 2001. Credit Risk Database (CRD) is a database of about one million japanese small-business firms. Small-business firms have qualitatively different characteristics of firm size growth from those for large firms [1]. The CRD covers the non-power-law regime and the transition region to Pareto-Zipf regime.

Fig. 7 (a)–(b) shows the breakdown of Gibrat's law by depicting the probability density function $P(r|x_1)$. The probability density has explicit dependence on x_1 showing the breakdown of Gibrat's law. In order to quantify the dependence, we examine how the standard deviation of r for each group of firms, whose size is $x_1 \sim dx_1$, scales as x_1 becomes larger. Let the standard deviation of r be denoted by σ. Fig. 7 (c)–(d) shows that σ scales as a function of x_1 ($\sigma \propto x_1^{-\beta}$), but asymptotically approaches non-scaling regime ($\sigma \sim$ const). The breakdown of Gibrat's law in the non-power-law regime and its validity in the power-law regime are consistent with what we showed in [11].

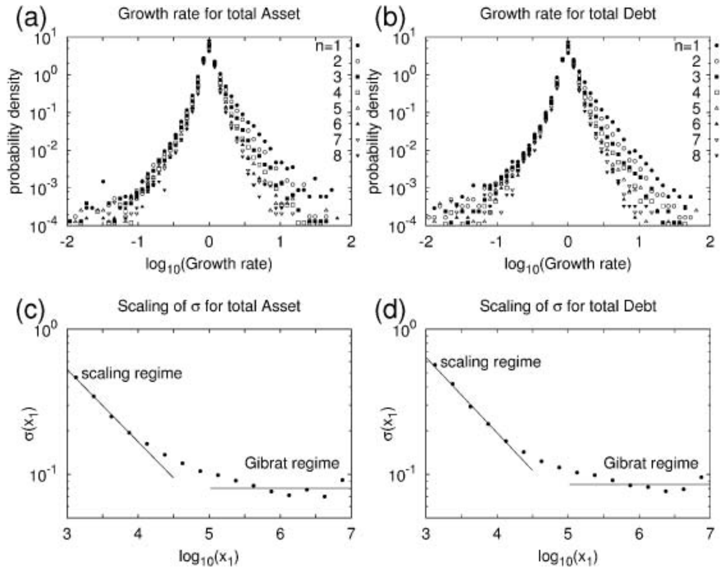

Fig. 7. *Upper panels:* Probability density function $P(r|x_1)$ for logarithmic growth-rate $r = \log_{10}(R)$. For conditioning x_1, we use different bins of initial firm size with equal interval in logarithmic scale as $x_1 \in [10^{4+0.25(n-1)}, 10^{4+0.25n}]$ ($n = 1, \cdots, 8$) for total-assets (a) and total-debts (b) (both in thousand yen). *Lower panels:* Standard deviation σ of r as a function of x_1 for total assets (c) and total debts (d).

6 Conclusion

We have shown the following stylized facts concerning distribution of personal income and firm size, their growth and fluctuations by studying exhaustive lists of high-income persons and firm sizes in Japan and in Europe.

- In power-law regime, detailed-balance and Gibrat's law hold.
- Under the condition of detailed-balance, Gibrat's law implies Pareto's law (but not *vice versa*).
- Growth-rate distribution has a non-trivial relation between its positive and negative growth sides through Pareto index. The distribution must have a cusp whose shape is related to the value of Pareto index.
- Power-law, detailed-balance and Gibrat's law break down according to abrupt change in risky asset market, such as japanese "bubble" collapse of real estate and stock.
- For firm size in non-power-law regime corresponding to small and mid-size firms, Gibrat's law does not hold. Instead, there is a scaling relation of variance in the growth-rates of those firms with respect to firm size, which asymptotically approaches to non-scaling region as firm size comes to power-law regime.

These stylized facts would serve to test models that explain personal income and firm size distributions.

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Empirical Study and Model of Personal Income

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Summary. Personal income distributions in Japan are analyzed empirically and a simple stochastic model of the income process is proposed. Based on empirical facts, we propose a minimal two-factor model. Our model of personal income consists of an asset accumulation process and a wage process. We show that these simple processes can successfully reproduce the empirical distribution of income. In particular, the model can reproduce the particular transition of the distribution shape from the middle part to the tail part. This model also allows us to derive the tail exponent of the distribution analytically.

Key words: Personal income, Power law, Stochastic model

1 Introduction

Many economists and physicists have studied wealth and income. About one hundred years ago, Pareto found a power law distribution of wealth and income [1]. However, afterwards, Gibrat clarified that the power law is applicable to only the high wealth and income range, and the remaining part follows a lognormal distribution [7]. This characteristic of wealth and income was later rediscovered [2][10][16][17]. Today, it is generally believed that high wealth and income follow a power law distribution. However, the remaining range of the distribution has not been settled. Recently an exponential distribution [5] and a Boltzmann distribution [20] has been proposed.

To explain these characteristics of wealth and income, some mathematical models have been proposed. One of them is based on a stochastic multiplicative process (SMP). For example, the SMP with lower bound [9], the SMP with additive noise [15][19], the SMP with wealth exchange [4], and the generalized Lotka-Voltera model [3][14].

This paper is organized as follows. In Sec. 2, we empirically study the personal income distribution in Japan. In Sec. 3, we propose a two-factor stochastic model to explain income distribution. The last section is devoted to a summary and discussion.

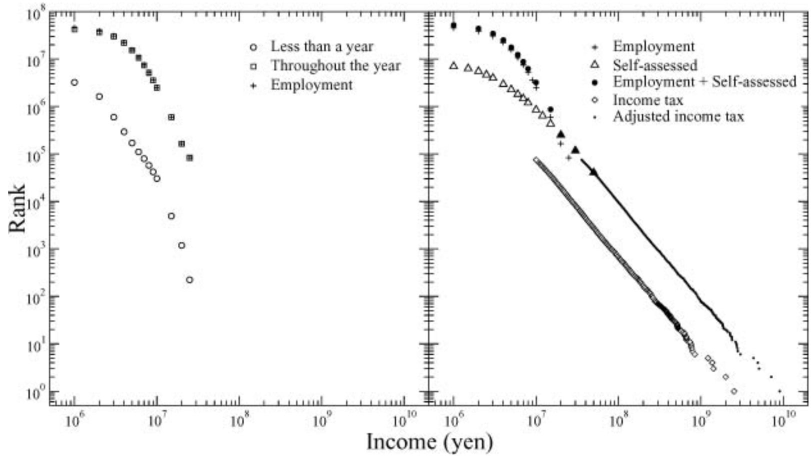

Fig. 1. A log-log plot of the distribution of employment income 1999 (left). A log-log plot of distributions in 1999 of self-assessed income, sum of employment income and self-assessed income, income tax data for top taxpayers, adjusted income tax data, and total income (right).

2 Empirical study of the personal income distribution

In this article we use three data sets. We call them employment income data, self-assessed income data, and income tax data for top taxpayers. The employment income data is coarsely tabulated data for the distribution of wages in the private sector. This is reported by the National Tax Agency of Japan (NTAJ) [11]. This is composed of two kinds of data. One is for employment income earners who worked for less than a year, and we can acquire the data since 1951. For example, a log-log plot of the rank-size distribution of the data in 1999 is shown by the open circles in the left panel of Fig. 1. The other is for employment income earners who worked throughout the year, and we can acquire the data since 1950. For example, the distribution in 1999 is shown by the open squares in the left panel of Fig. 1. In this figure the crosses are the sum of these two data, and are almost the same as the distribution of employment income earners who worked throughout the year.

The self-assessed income data is also reported by NTAJ. This is also coarsely tabulated data, and we can acquire this since 1887. The income tax law was changed many times, and so the characteristics of this data also changed many times. However, this data consistently contains high income earners. In Japan, in recent years, persons who have some income source, who earned more than 20 million yen, and who are not employees must declare their income. For example, the distribution in 1999 is shown by the open triangles in the right panel of Fig. 1. In this figure the filled circles are the sum of the employment income data and the self-assessed income data. However, we use only the self-assessed income data in the range greater than 20 million yen. This is because persons who earned more than 20 million yen must declare their income, even if they are employees and have only one income source. This figure shows that the distribution of middle and low income is almost the same as that of the employment income. This means that the main income source of middle and low income earners is wages.

In Japan, if the amount of one's income tax exceeds 10 million yen, the individual's name and the amount of income tax are made public by each tax office. Some data companies collect this and produce income tax data for top taxpayers. We obtained this data from 1987 to 2000. For example, the distribution in 1999 is shown by the open diamonds in the right panel of Fig. 1. To understand the whole image of distribution, we must convert income tax to income. We know from the self-assessed income data that the income of the 40,623th person is 50 million yen,. On the other hand we also know from the income tax data for top taxpayers that the income tax of the 40,623th person is 13.984 million yen Hence, if we assume a linear relation between income and income tax, we can convert income tax to income by multiplying 3.5755 by the income tax [1]. The dots in Fig. 1 represent the distribution of converted income tax. This clearly shows the power law distribution in the high income range, and the particular transition of the distribution shape from the middle part to the tail part.

2.1 Income sources

Understanding income sources is important for the modeling of the income process. As we saw above, the main income source of middle and low income earners is wages. We can also see the income sources of high income earners from the report of NTAJ. The top panel of Fig. 2 shows a number of high income earners who earned income greater than 50 million yen in each year from 2000 to 2003. In this figure income sources are divided into the 14 categories of business income, farm income, interest income, dividends, rental income, wages & salaries, comprehensive capital gains, sporadic income, miscellaneous income, forestry income, retirement income, short-term separate capital gains, long-term separate capital gains, and capital gains of stocks. The bottom panel of this figure shows the amount of income for each income source. These figures show that the main income sources of high income earners are wages and capital gains.

2.2 Change of distribution

The rank-size distribution of all acquired data is shown in the top panel of Fig. 3. The gap found in this figure reflects the change of the income tax law. We fit distributions in the high income range by the power law distribution, for which a probability density function is given by

$$p(x) = Ax^{-(\alpha-1)},$$

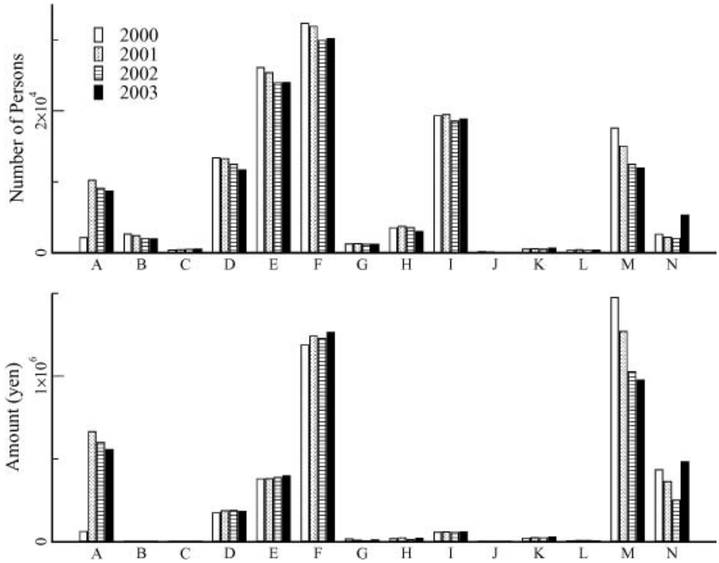

Fig. 2. Income sources of high income earners from 2000 to 2003. The top panel represents the number of high income earners , and the bottom panel represents the amount of income. In both panels, A: business income, B: farm income, C: interest income, D: dividends, E: rental income, F: wages & salaries, G: comprehensive capital gains, H: sporadic income, I: miscellaneous income, J: forestry income, K: retirement income, L: short-term separate capital gains, M: long-term separate capital gains, and N: capital gains of stocks.

where A is a normalization factor. Here α is called the Pareto index. The small α corresponds to the unequal distribution. The change of α is shown by the open circles in the bottom panel of Fig. 3. The mean value of the Pareto index is $\bar{\alpha} = 2$, and α fluctuates around it.

It is recognized that the period of modern economic growth in Japan is from the 1910s to the 1960s. It has been reported that the gross behavior of the Gini coefficient in this period looks like an inverted U-shape [18]. This behavior of the Gini coefficient is known as Kuznets's inverted U-shaped relation between income inequality and economic growth [8]. This postulates that in the early stages of modern economic growth both a country's economic growth and its income inequality rises, and the Gini coefficient becomes large. For developed countries income inequality shows a tendency to narrow, and the Gini coefficient becomes small. Figure. 3 shows that the gross behavior of the Pareto index from the 1910s to the 1960s is almost the inverse of that of the Gini coefficient, i.e., U-shaped. This means that our analysis of the Pareto index also supports the validity of Kuznets's inverted U-shaped relation.

We assume that the change of the Pareto index in the 1970s is responsible for the slowdown in the Japanese economic growth and the real estate boom. In Fig. 3 we can also see that α decreases toward the year 1990 and

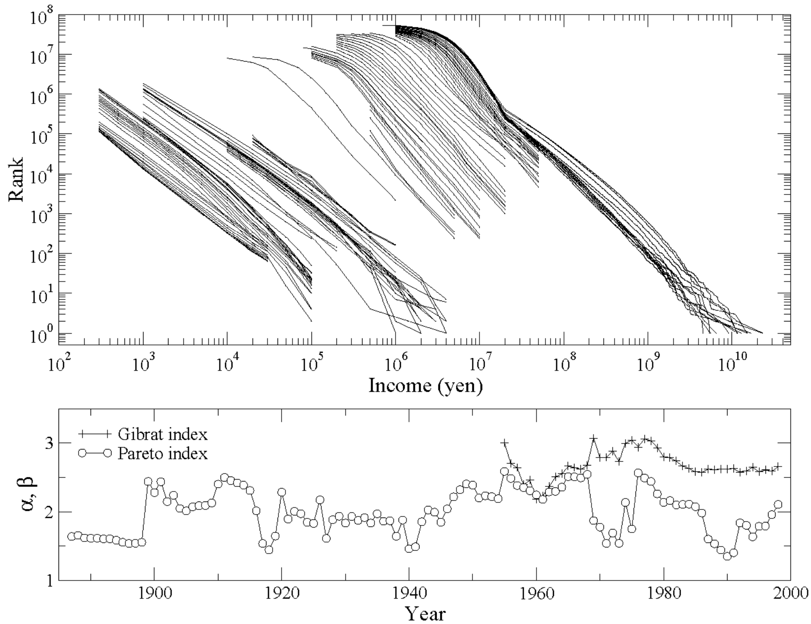

Fig. 3. A change of the personal income distribution (top) and that of the Pareto index and Gibrat index (bottom).

increases after 1990, i.e., V-shaped relation. In Japan, the year 1990 was the peak of the asset-inflation economic bubble. Hence the Pareto index decreases toward the peak of the bubble economy, and it increases after the burst of the economic bubble. The correlation between the Pareto index and risk assets is also clarified in Ref. [16].

We fit distributions in the low and middle income range by log-normal distribution, for which the probability density function is defined by

$$p(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\log^2(x/x_0)}{2\sigma^2}\right],$$

where x_0 is mean value and σ^2 is variance. Sometimes $\beta \equiv 1/\sqrt{2\sigma^2}$ is called the Gibrat index. Since the large variance means the global distribution of the income, the small β corresponds to unequal distribution. The change of β is shown by the crosses in the bottom panel of Fig. 3. This figure shows that α and β correlate with each other around the years 1960 and 1980. However, they have no correlation in the beginning of the 1970s and after 1985. Especially after 1985, β stays almost the same value. This means that the variance of the low and middle income distribution does not change. We assume that capital gains cause different behaviors of α and β, and α is more sensitive to capital gains than β.

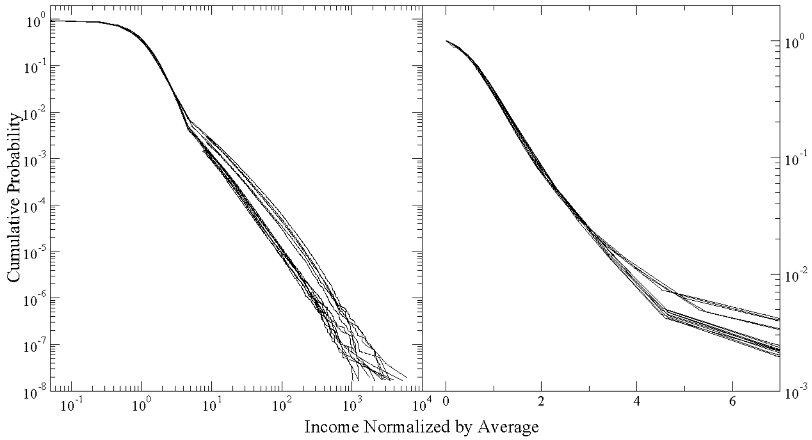

Fig. 4. A log-log plot of the cumulative distributions of normalized income from 1987 to 2000 (left) and a semi-log plot of them (right).

The top panel of Fig. 3 shows that the distribution moves to the right. This motivates us to normalize distributions by quantities that characterize the economic growth. Though many candidates exist, we simply normalize distributions by the average income. The left panel of Fig. 4 is a log-log plot of the cumulative distributions of normalized income from 1987 to 2000, and the right panel is a semi-log plot of them. These figures show that distributions almost become the same, except in the high income range. Though distributions in the high income range almost become the same, distributions of some years apparently deviate from the stational distribution. In addition the power law distribution is not applicable to such a case. This behavior happens in an asset-inflation economic bubble [6].

3 Modeling of personal income distribution

The empirical facts found in the previous section are as follows.

(i) The distribution of high income earners follows the power law distribution, and the exponent, Pareto index, fluctuates around $\alpha = 2$.
(ii) The main income sources of high income earners are wages and capital gains.
(iii) Excluding high income earners, the main income source is wages.
(iv) The distribution normalized by the average income is regarded as the stational distribution.

Hence, it is reasonable to regard income as the sum of wages and capital gains. However, to model capital gains, we must model the asset accumulation process. In the following we explain an outline of our model. Details of our model are found in Ref. [12].

3.1 Wage process

We denote the wages of the i-th person at time t as $w_i(t)$, where $i = 1 \sim N$. We assume that the wage process is given by

$$w_i(t+1) = \begin{cases} uw_i(t) + s\epsilon_i(t)\overline{w}(t) & \text{if } uw_i(t) + s\epsilon_i(t)\overline{w}(t) > \overline{w}(t), \\ \overline{w}(t) & \text{otherwise,} \end{cases} \quad (1)$$

where u is the trend growth of wage, and reflects an automatic growth in nominal wage. In this article we use $u = 1.0422$. This is an average inflation rate for the period from 1961 to 1999. In Eq. (1), $\epsilon_i(t)$ follows a normal distribution with mean 0 and variance 1, i.e., $N(0,1)$. In Eq. (1), s determines the level of income for the middle class. We choose $s = 0.32$ to fit the middle part of the empirical distribution. In Eq. (1), $\overline{w}(t)$ is the reflective lower bound, which is interpreted as a subsistence level of income. We assume that $\overline{w}(t)$ grows deterministically,

$$\overline{w}(t) = v^t \overline{w}(0).$$

Here we use $v = 1.0673$. This is a time average growth rate of the nominal income per capita.

3.2 Asset accumulation process

We denote the asset of the i-th person at time t as $a_i(t)$. We assume that the asset accumulation process is given by a multiplicative process,

$$a_i(t+1) = \gamma_i(t)a_i(t) + w_i(t) - c_i(t), \quad (2)$$

where the log return, $\log \gamma_i(t)$, follows a normal distribution with mean y and variance x^2, i.e., $N(y, x^2)$. We use $y = 0.0595$. This is a time-average growth rate of the Nikkei average index from 1961 to 1999. We use $x = 0.3122$. This is a variance calculated from the distribution of the income growth rate for high income earners. In Eq. (2), we assume that a consumption function, $c_i(t)$, is given by

$$c_i(t) = \overline{w}(t) + b\left\{a_i(t) + w_i(t) - \overline{w}(t)\right\}.$$

In this article we chose $b = 0.059$ from the empirical range estimated from Japanese micro data.

3.3 Income distribution derived from the model

We denote the income of the i-th person at time t as $I_i(t)$, and define it as

$$I_i(t) = w_i(t) + \mathrm{E}[\gamma_i(t) - 1]a_i(t).$$

The results of the simulation for $N = 10^6$ are shown in Fig. 5. The left panel of Fig. 5 is a log-log plot of the cumulative distribution for income normalized

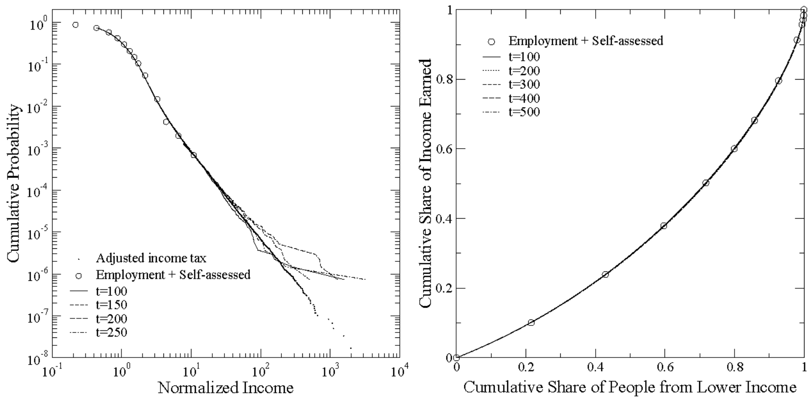

Fig. 5. A log-log plot of the cumulative distributions of normalized income in 1999 and simulation results (left), and the Lorenz curve in 1999 and simulation results (right).

by an average. The right panel of Fig. 5 is the simulation results for the Lorenz curve. These figures show that the accountability of our model is high.

In our model, the exponent in the power law part of the distribution is derived from the asset accumulation process. From Eq. (1), we can analytically derive

$$\alpha = 1 - \frac{2\log(1 - z/g)}{x^2} \approx 1 + \frac{2z}{gx^2}, \tag{3}$$

where z is a steady state value of $[w(t) - c(t)]/\langle a(t)\rangle$. Here $\langle a(t)\rangle$ is the average assets. In Eq. (3), g is a steady state value of the growth rate of $\langle a(t)\rangle$. Equation (3) shows that α fluctuates around $\alpha = 2$, if $2z \sim gx^2$.

4 Summary

In this article we empirically studied income distribution, and constructed a model based on empirical facts. The simulation results of our model can explain the real distribution. In addition, our model can explain the reason why the Pareto index fluctuate around $\alpha = 2$. However there are many unknown facts. For example, we have no theory that can explain the income distribution under the bubble economy, that can determine the functional form other than the high income range, and that can explain the shape of the income growth distribution, etc.

Acknowledgements

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Blockbusters, Bombs and Sleepers: The Income Distribution of Movies

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Summary. The distribution of gross earnings of movies released each year show a distribution having a power-law tail with Pareto exponent $\alpha \simeq 2$. While this offers interesting parallels with income distributions of individuals, it is also clear that it cannot be explained by simple asset exchange models, as movies do not interact with each other directly. In fact, movies (because of the large quantity of data available on their earnings) provide the best entry-point for studying the dynamics of how "a hit is born" and the resulting distribution of popularity (of products or ideas). In this paper, we show evidence of Pareto law for movie income, as well as, an analysis of the time-evolution of income.

1 Introduction

While the personal income distribution has been a subject of study for a long time [1], it is only recently that other kinds of income distribution, e.g., the income of companies [2], have come under close scrutiny. More than a century ago, Vilfredo Pareto had reported that the income distribution of individuals or households follows a power law with an universal exponent of $\alpha = 1.5$. While recent studies have shown this claim about universality to be untenable, it has indeed been extensively verified that the higher-end (i.e., the tail) of the income, as well as wealth, distribution follows a power law. Whether similar power laws occur for other types of income distribution is therefore of high topical interest.

The income (or gross) of movies released commercially in theaters every year provides an opportunity to study a very different kind of income distribution from those usually studied. Not only is movie income a very well-defined quantity, but high-quality data is publicly available from web-sites such as *The Numbers* [3] and *Movie Times* [4]. The income distribution, as well, as the time evolution of the income, can be empirically determined with high accuracy. Movie income distribution is also of theoretical interest because such a distribution clearly cannot be explained in terms of asset exchange models, one of the more popular class of models used for explaining the nature

of personal income distribution. As movies don't exchange anything between themselves, one needs a different theoretical framework to explain the observed distribution for movie income [5].
Even more significantly, movie income can be considered to be a measure of popularity [6]. Seen in this light, this distribution is a prominent member of the class of popularity distributions, that looks at how the success of various products (or ideas) in appealing to public taste is distributed. Examples of such distributions include the popularity of scientific papers as measured by the number of citations [7], books as measured by the sales figures from an online bookstore [8], etc. Of course, income is not the only measure of a movies' popularity; e.g., one possibility is to use the number of votes per film from registered users of IMDB [9]. However, such voting may not reflect the true popularity of movies as it costs nothing to give a vote. On the other hand, when one is voting with one's wallet, by going to see a movie in a theater, it is a far more reliable indicator of the film's popularity.

2 A Pareto Law for Movies

Previous studies of movie income distribution [10, 11, 12] had looked at limited data sets and found some evidence for a power-law fit. A more rigorous demonstration has been given in Ref. [6], where data for all movies released in theaters across USA during 1997-2003 were analysed. It was shown that the rank distribution of the opening gross as well as the total gross of the highest earning movies for all these years follow a power-law with an exponent close to $-1/2$. As the rank distribution exponent is simply the inverse of the cumulative gross distribution exponent [7], this gives a power-law tail for the income distribution with a Pareto exponent $\alpha \simeq 2$. It is very interesting that this value is identical to that of corresponding exponents for citations of scientific papers [7] and book sales [8], and is suggestive of an universal exponent for many different popularity distributions.
Fig. 1 (left) demonstrates the Pareto law of movie income for the movies released across theaters in USA in 2004. Both the opening gross, G_O, as well as the total gross, G_T, (scaled by their respective averages over all the movies released that year) show a power-law behavior with the same exponent. The similarity of these two curves can be partially explained from the inset figure, which shows that there is strong degree of correlation between the income of a movie at its opening, and its total income. Movies which open poorly but perform well later (*sleepers*) are relatively uncommon and are seen as the points deviating from the linear trend in the inset figure. Arguably, a better comparison with the Pareto distribution of personal income can be made by looking at the income distribution of movies running on a particular weekend [Fig. 1 (right)]. However, the smaller number of data points available for such a plot means that the scatter is larger. As a result, it is difficult to make a judgement on the nature of the weekend income distribution.

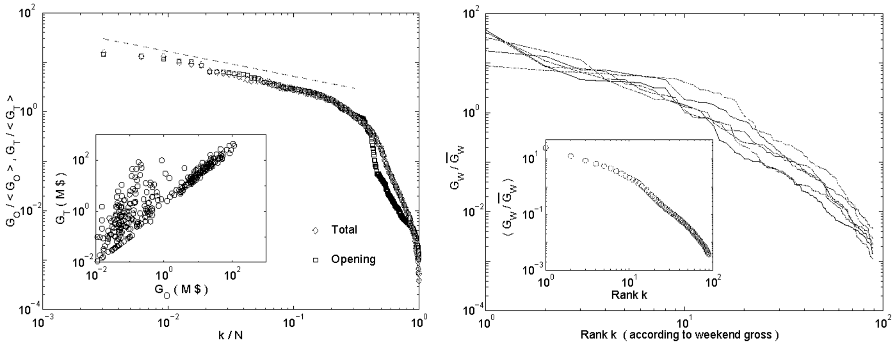

Fig. 1. Income distribution of movies released in theaters across USA for the year 2004: (Left) Scaled rank-ordered plot of movies according to opening gross (squares) and total gross (diamonds). The rank k has been scaled by the total number of movies released that year ($N = 326$) while the gross (G_O, G_T) has been scaled by its average. The broken line of slope -0.5 has been shown for visual reference. The inset shows the total gross earned by a movie, plotted against its opening gross (in millions of $). As indicated by the data, there is a high degree of correlation between the two. (Right) Scaled rank-ordered plot of movies according to weekend gross, G_W, for six arbitrarily chosen weekends. The top 89 movies in a weekend are shown, and the weekend gross of each movie has been scaled by the average weekend gross of all movies playing that weekend. The inset shows the average of the scaled rank-ordered plots for all the weekends in 2004.

3 Time-evolution of movie income

In this section, we focus on how the gross of a movie changes with time after its theatrical release, until it is withdrawn from circulation. Based on how they perform over this time, movies can be classified into ***blockbusters*** having both high opening and high total gross, ***bombs*** (or flops) having low opening as well as low total gross and ***sleepers*** that have low opening but high total gross. Not surprisingly, the corresponding theatrical lifespans also tend to be high to intermediate for blockbusters, low for bombs and high to very high for sleepers.

Consider a classic blockbuster movie, *Spiderman* (released in 2002). Fig. 2 (left) shows how the daily gross decays with time after release, with regularly spaced peaks corresponding to large audiences on weekends. To remove the intra-week fluctuations and observe the overall trend, we focus on the time series of weekend gross. This shows an exponential decay, a feature seen not only for almost all other blockbusters, but for bombs as well [Fig. 2 (right)]. The only difference between blockbusters and bombs is in their initial, or opening, gross. However, sleepers behave very differently, showing an increase in their weekend gross and reaching their peak performance (in terms of income) quite a few weeks after release, before undergoing an exponential decay.

To make a quantitative analysis of the relative performance of movies in a given year (say 2002), we define the persistence time τ of a movie as the time

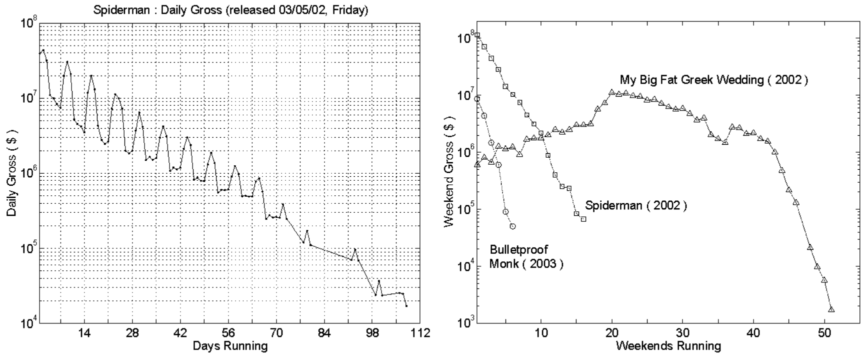

Fig. 2. Classifying movies according to time-evolution of the gross (income): (Left) Daily gross of a typical blockbuster movie (Spiderman) showing weekly periodic fluctuations (with gross peaking on weekends), while the overall trend is exponential decay. (Right) Comparing examples of blockbusters (Spiderman), bombs (Bulletproof Monk) and sleepers (My Big Fat Greek Wedding) in terms of the time-evolution of weekend gross. Time is measured in weekends to remove intra-week fluctuations.

(measured in number of weekends) upto which it is being shown at theaters. Fig. 3 (left) shows that most movies run for upto about 10 weekends, after which there is a steep drop in their survival probability. The tail is almost entirely composed of sleepers, the best performance being by *My Big Fat Greek Wedding* ($\tau = 51$ weekends). The inset shows the time-evolution of the average number of theaters showing a movie. It suggests an initial power-law decay followed by an exponential cut-off. We also look at the time-evolution of the gross per theater, g. This is a better measure of movie popularity, because a movie that is being shown in a large number of theaters has a bigger income simply on account of higher accessibility for the potential audience. Unlike the overall gross that decays exponentially with time, the gross per theater shows a power-law decay with exponent $\beta \simeq -1$ [Fig. 3 (right)].

4 Conclusions

To conclude, we have shown that movie income distribution has a power-law tail with Pareto exponent $\alpha \simeq 2$. This is suggestive of a possible universal exponent for many popularity distributions. The exponent is identical for the opening as as well as the total gross distribution. Since the Pareto tail appears at the opening week itself, it is unlikely that the mechanism for generating this behavior involves information exchange between moviegoers. Also, as mentioned before, conventional asset exchange models don't apply in this case. Therefore, explaining the Pareto tail of the income distribution, as well as the distribution of the time-evolution of movie income, is an interesting challenge to theories of distributions with power-law tails.

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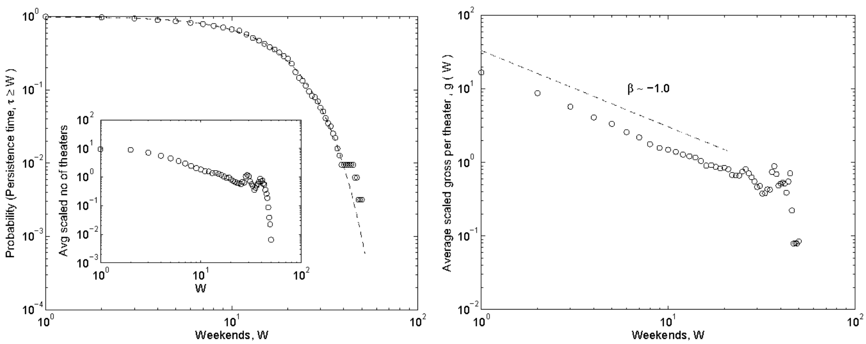

Fig. 3. Time evolution of movie income for all movies released across theaters in USA in the year 2002. (Left) Cumulative probability distribution of movie persistence time τ (in terms of weekends). The broken line shows fit with a stretched exponential distribution $P(x) = exp(-[x/x_0]^c)$, with $x_0 \simeq 16.5$ and $c \simeq 1.75$. The inset shows the number of theaters (scaled by the average number of theaters that a movie was shown in its theatrical lifespan) in which a movie runs after W weekends, averaged over the number of movies that ran for that long. (Right) Weekend gross per theater for a movie (scaled by the average weekend gross over its theatrical lifespan), $g(W)$, after it has run for W weekends, averaged over the number of movies that ran for that long. The initial decline follows a power-law with exponent $\beta \simeq -1$ (the fit is shown by the broken line).

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Part II

Models and theories

Emergent Statistical Wealth Distributions in Simple Monetary Exchange Models: A Critical Review

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Summary. This paper reviews recent attempts at modelling inequality of wealth as an emergent phenomenon of interacting-agent processes. We point out that recent models of wealth condensation which draw their inspiration from molecular dynamics have, in fact, reinvented a process introduced quite some time ago by Angle (1986) in the sociological literature. We emphasize some problematic aspects of simple wealth exchange models and contrast them with a monetary model based on economic principles of market mediated exchange. The paper also reports new results on the influence of market power on the wealth distribution in statistical equilibrium. As it turns out, inequality increases but market power alone is not sufficient for changing the exponential tails of simple exchange models into Pareto tails.

1 Introduction

Since the days of Vilfredo Pareto, the frequency distribution of wealth among the members of a society has been the subject of intense empirical research. Recent research confirms that power-law behaviour with an exponent between 1 and 2 indeed seems to characterize the right tail of the distribution (Levy and Solomon, 1997; Castaldi and Milakovic, 2005). However, when applied to the entire shape of the empirical distribution, the power law would produce a rather mediocre fit and would be outperformed by other candidate processes like the lognormal or Gamma distributions. As it seems to emerge from the literature, a transition occurs in the data from an exponential shape to power-law behavior somewhere above the 90 percent quantile again.

These and other findings should give rise to modelling efforts explaining the remarkably similar wealth distribution of many developed countries. Unfortunately, economic theory has been quite silent on this topic for a long time. Until recently, one had to go back the to literature of the fifties and sixties (e.g., Champernowne, 1953; Mandelbrot, 1961) to find stochastic models of wealth accumulation in modern societies. Recent advances in computer technology, however, open another avenue for analysis of the emergence of wealth distributions allowing this issue to be studied in a computational agent-based framework. Such a bottom-up approach could, in principle, be helpful in isolating the key mechanisms that apparently lead to a stratification of wealth in advanced economies. As it appears, this path has been pursued recently by physicists rather than economists (cf. Bouchaud and Mézard, 2000; Drăgulescu and Yakovenko, 2000; Chakraborti and Chakrabarti, 2000; Silver, Slad and Takamoto, 2002, among others). However, it has been entirely overlooked in the pertinent publications that these models have an important predecessor in the sociological literature. Investigating essentially the same structures already almost twenty years ago, Angle, 1986, might be considered as the first contribution to agent-based analysis of wealth formation. In the following, I will shortly review Angle's interesting work as the prototypical agent-based model of wealth dynamics, based on particle-like microscopic interactions of agents. I will point out aspects of this class of models (covering most of the econophysics contributions mentioned above) that would be considered to be problematic by economists (section 2). As an alternative framework, I will, then, review the contribution by Silver et. al. (2002) which much better fits into standard economic reasoning, but nevertheless provides a similarly simple formalization of an agent-based exchange model (section 3). Section 4 presents some additional results expanding on the seminal framework of Silver et. al. Conclusions are in section 5.

2 Angle's Surplus Theory of Social Stratification and the Inequality Process

In a long chain of papers covering more than 15 years, sociologist John Angle has elaborated on a class of stochastic processes which he first proposed in 1986 as a generating mechanism for the universal emergence of inequality in wealth distributions in human societies. His starting point is evidence he attributes to archeological excavations that inequality among the members of a community is typically first found with the introduction of agriculture and the ensuing prevalence of food abundance: While simpler hunter/gatherer societies appear to be rather egalitarian, production of a "surplus" beyond subsistence level immediately seems to lead to a "ranked society" or some kind of "chiefdom" (Angle, 1986, p. 298).

So as soon as there is some excess capacity of food, processes seem to be set into motion from which inequality emerges. Angle, surveying earlier narrative work in sociology, sees this as the result of redistribution by which some members of society succeed in grabbing some of the surplus wealth of others. The relevant empirical observations are summarized as follows:

"**Proposition 1:** *Where people are able to produce a surplus, some of the surplus would be fugitive and would leave the possession of the people who produce it.*

...

Proposition 2: *Wealth confers on those who possess it the ability to extract wealth from others. So netting out each person's ability to do this in a general competition for surplus wealth, the rich tend to take surplus away from the poor." (Angle, 1986, p. 298).*

According to Angle, the expropriation of the losers happens via (1) theft, (2) extortion, (3) taxation, (4) exchange coerced by unequal power between the participants, (5) genuinely voluntary exchange, or (6) gift (*ibid.*).

The process he designs as a formalisation of these ideas is a true interacting particle model: in a finite population, agents are randomly matched in pairs and try to catch part of the other's wealth. A random toss $D_t \in \{0, 1\}$ decides which of both agents is the winner of this conflict. Angle in various papers considers cases with equal winning probabilities 0.5 as well as others with probabilities being biased in favor of either the wealthier or poorer of both individuals. If the winner of this encounter is assumed to take away a fixed proportion of the other's wealth, ω, the simplest version of the "inequality process" leads to a stochastic evolution of wealth of individuals i and j who had bumped into each other according to:

$$\begin{aligned} w_{i,t} &= w_{i,t-1} + D_t \omega w_{j,t-1} - (1 - D_t)\omega w_{i,t-1}, \\ w_{j,t} &= w_{j,t-1} + (1 - D_t)\omega w_{i,t-1} - D_t \omega w_{j,t-1}. \end{aligned} \tag{1}$$

Time t is measured in encounters and one pair of agents from the whole population is chosen for this interaction in each period. Angle (1986) shows via simulations that this dynamics leads to a stationary distribution which can be reasonably well fitted by a Gamma distribution. Angle (1993) provides an argument for why the Gamma distribution approximates the equilibrium distribution of the process for empirically relevant values of its parameters. Later papers provide various extensions of the basic model. While the exponential decay of the Gamma distribution might not be in accordance with power law behavior at the high end of the richest individuals, Angle's model is the first agent-based approach matching several essential features of empirical wealth distributions which he carefully lists as desiderada (i.e. stylized facts) for a theory of inequality. Among other properties, he emphasizes the uni-modality with a mode above minimum income which could not be reproduced by a monotonic distribution function. Angle is also careful to point out that with binned data, realizations of his process would be hard to distinguish from realizations of Pareto random variables which he demonstrates via a few Monte Carlo runs.

Unfortunately, Angle's process might be hard to accept for economists as a theory of the emergence of inequality in market economies.

First, a glance at the list of the six mechanisms for appropriation of another agent's wealth might raise doubts about their relative importance in modern societies: for most countries of the world, "theft" should perhaps not be the most eminent mechanism for stratification of the wealth distribution. Note also that "genuinely voluntary exchange" is listed only at rank 5 and behind "exchange coerced by unequal power". However, voluntary exchange is at the heart of economic activity at all levels of development rather than being a minor facet.

However, despite being mentioned in the list of mechanisms of redistribution, voluntary exchange is not really considered in Angle's model in which an agent simply takes away part of the belongings of another. What is more, this kind of encounter would - in its literal sense - hardly be imaginable as both agents would rather prefer *not* to participate in this game of a burglar economy - at least if they possess a minimum degree of risk aversion. The model, thus, is not in harmony with the principle of voluntary participation of agents in the hypothesized process which economists would consider to be an important requirement for a valid theory of exchange activities. One should also note that another problem is the lack of consideration of the measurement of wealth (in terms of monetary units) and the influence of changes of the value of certain components of overall wealth.

Despite these problematic features from the viewpoint of economics, Angle's model deserves credit as the first contribution in which inequality results as an emergent property of an agent-based approach. A glance at the recent econophysics literature shows that the basic building blocks of practically all relevant contributions share the structure of the inequality process formalized by equation (1). The inequality process is, for example, practically identical to the process proposed by Bouchaud and Mézard (2000) and isomorphic to almost all other models mentioned above. This recent strand of research on wealth dynamics is, therefore, almost exemplary for the lack of coordination among research pursued on the same topic in different disciplines and for the unfortunate duplication of effort that comes along with it.

Interestingly, the above criticism concerning the structure of the exchange process had also been voiced in a review of monetary exchange models developed by physicists by Hayes (2002) who introduced the label of "theft and fraud" economies, but restricted it to variants in which the richer could lose more (in absolute value) than the poor. However, it is not clear why models which introduce a certain asymmetry to avoid this kind of exploitation should not also suffer from the lack of willingness of agents to participate in their exchange processes. It, therefore, appears that one might wish to reformulate the "burglar economies" in a way that brings elements of voluntary economic exchange processes into play. While the economics literature has not elaborated on wealth distributions emerging from exchange activities within a group of agents, a huge variety of approaches is available in economics that could be utilized for this purpose. An interesting start has been made in a recent paper by Silver, Slud and Takamoto (2002) which contains a two-good general equilibrium model of an economy with heterogenous agents. Somewhat ironically, the overall outcome of this model is the same as with the inequality process: the stationary wealth distribution turns out to be a Gamma distribution.

3 An Exchange Economy with Changing Preferences

Unlike the framework reviewed in the previous section, the setting of Silver et al. is an extremely familiar one for economists. Their economy consists of two goods, denoted x and y which necessitate the introduction of a relative price p being defined as the current value of a unit of good y in units of good x. Note that with this assumption, considerations of revaluation of wealth components come into play which are altogether neglected in the sociological/physical models. All agents of the economy have their preferences formalized by a so-called Cobb-Douglas utility function:

$$U_{i,t} = x_{i,t}^{f_{i,t}} \cdot y_{i,t}^{1-f_{i,t}}. \tag{2}$$

Here, i and t are indices for the individuals and time, respectively. $x_{i,t}$ and $y_{i,t}$ are, therefore, the possessions of good x and y by individual i at time t and $f_{i,t} \in [0,1]$ is a preference parameter which might differ among individuals and, for one and the same individual, might also change over time. $U_{i,t}$, then, is utility gained by individual i at time t. Individuals start with a given endowment in $t = 0$ and try to maximize their utility via transactions in a competitive market where one good is exchanged against the other. Given their possessions of both goods at some time $t-1$, it is a simple exercise to compute their demands for goods x and y at time t given the current preference parameter $f_{i,t}$:

$$\begin{aligned} x_{i,t} &= f_{i,t}(x_{i,t-1} + p_t y_{i,t-1}), \\ y_{i,t} &= (1 - f_{i,t}) \left(\frac{x_{i,t-1}}{p_t} + y_{i,t-1} \right). \end{aligned} \tag{3}$$

In (3), we have used the standard assumption that agents take the price as given in a competitive market. Note that this market, therefore, dispenses with any assumption of unequal exchange or even exploitation which is so central to the microscopic process of the previous chapter.

Summing up demand and supply by all our agents, we can easily calculate the equilibrium price which simultaneously clears both markets:

$$p_t = \frac{\sum_i (1 - f_{i,y})\, x_{i,t-1}}{\sum_i f_{i,t} y_{i,t-1}}. \tag{4}$$

After meeting in the market, each agent possesses a different bundle of goods and his wealth can be evaluated as:

$$w_{i,t} = x_{i,t} + p_t y_{i,t}. \quad (5)$$

The driving force of the dynamics of the model by Silver et al. is simply the assumption of stochastically changing preferences: all $f_{i,t}$ are drawn anew in each period independently for all individuals. In the baseline scenario, the $f_{i,t}$ are simply drawn from a uniform distribution over $[0, 1]$, but other distributions lead to essentially the same results. The dynamics is, thus, generated via the agents' needs to rebalance their possessions in order to satisfy their new preference ordering. With all agents attempting to change the composition of their "wealth", price changes are triggered because of fluctuations in the overall demand for x and y. This leads to a revaluation of agents previous possessions, $x_{i,t-1}$ and $y_{i,t-1}$, and works like a capital gain or loss.

To summarize, we have a model in which all agents are identical except for their random preference shocks and no market or whatsoever power is attributed to anyone. The resulting inequality (illustrated as the benchmark case $p_m = 0$ in Fig. 1) is, therefore, the mere consequence of the eventualities of the history of preference changes and ensuing exchanges of goods. We, therefore, do not have to impose any type of "power" in order to endogeneously generate a stratification of the wealth distribution that - like the model of section 2 - is able to capture all except the very end (the Pareto tail) of the empirical data.

4 Some Extensions of the Monetary Exchange Model

The model by Silver et al. demonstrates that stratification of wealth can result from an innocuous exchange dynamics without agents robbing or fleecing each other. It should, therefore, be a promising avenue to supplement the simpler dynamic models in the previous section. In some extensions, we, therefore, tried to explore the sensitivity of this approach to certain changes of its underlying assumptions. Among the many sensitivity tests we could imagine, we started with the following variations of the basic framework:

- replacement of market interaction by pairwise exchange,
- introduction of agents with higher bargaining power so that the outcome of pairwise matches could differ from a competitive framework,
- introduction of natural differences among agents of some kind: here we assumed that for part of the population, preference changes are less pronounced than for others,
- introduction of savings via a framework which allows for money as an additional component in the utility function.

Due to space limitations, we will not provide detailed results on all of these experiments, but will rather confine ourselves to one particularly interesting

variant: the introduction of market power.
Introducing market power of some sort is certainly interesting in light of the focus of the sociological and physics-inspired literature on issues of power of some individuals over others. Different avenues for implementing market power seem possible. Here, for the sake of a first exploration of this issue, we chose a very simple and extreme one. We assume that part of the population can act as *monopolists* in pairwise encounters: if they are matched with an agent from the complementary subset of non-monopolists, they can demand the monopoly price. If two non-monopolists are matched, we compute the competitive solution. We do the same when two monopolists meet each other assuming that their potential monopolistic power cancels out.

Although this is an almost trivial insight in economics, it should be noted that the monopolist is not entirely free in dictating any price/transaction combination, but has to observe the constraint that the other agent has to voluntarily participate in the transaction. Since the option to not agree on the transaction would leave the monopolist with a zero gain as well, even in this extreme market scenario "exploitation" is much more limited than in a world of "theft and fraud". Note also that although one could perhaps speak of exploitation (when comparing the monopoly setting with the competitive price), no *expropriation* is involved whatsoever since even the non-monopolist will still increase his utility by his transaction with the more "powerful" monopolist.

As it turns out, allowing for monopoly power indeed changes the resulting wealth distribution. Fig. 1 shows the pdf for (fixed) fractions of monopolists. Varying the proportion of monopolists from 0 (the former competitive scenario with pair-wise transactions) to 0.4 we see a slight change in the shape of the distribution. As it happens all distributions still show pronounced exponential decline and can be well fitted by Gamma distributions. However, the estimated parameters of the Gamma distribution show a systematic variation. In particular, the slope parameter decreases with the fraction of monopolists, p_m. A closer look at the simulation results also shows that the average wealth of monopolists exceeds that of other agents but the difference decreases with increasing p_m. Note that the Gini dispersion ratio (G) is a negative function of λ for the Gamma distribution: $G = \frac{\Gamma(\lambda+0.5)}{\pi^2 \Gamma(\lambda+1)}$, so that the increasing inequality would also be indicated by this popular statistics.

The result that monopoly power is not neutral with respect to the distribution of wealth is certainly reassuring. However, we may also note that its introduction in the present framework does not lead to a dramatic change of the shape of the distribution. In particular, it does not seem to lead to anything like a Pareto tail in place of the exponential tail of the more competitive society. Since we have already chosen the most extreme form of market power

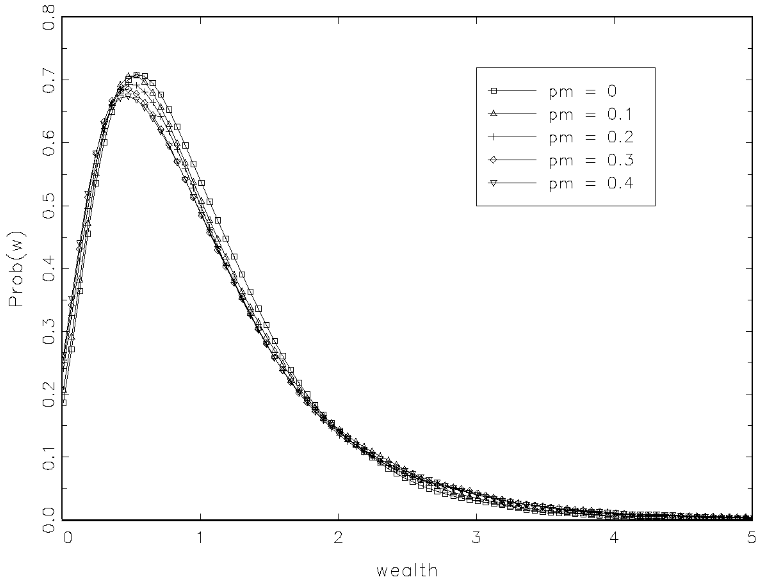

Fig. 1. Kernel estimates of statistical wealth distributions with different fractions of monopolistic agents p_m. Results are from simulations with 10,000 agents recorded after $5 * 10^5$ trading rounds.

in the above setting it seems also unlikely that one could obtain widely different results with milder forms of bargaining power.

5 Conclusions and Outlook to Future Research

What kind of conclusions can be drawn from this review of different approaches to agent-based models of wealth stratification? First, it is perhaps obvious that this author would like to advocate an approach in line with standard principles of economic modelling. If one is not willing to follow the emphasis of the sociological literature on all types of exertion of power, and if one tends to the view that wealth is influenced more by legal economic activity than by illegal theft and fraud, economic exchange should be explicitly incorporated in such models. This would also help to identify more clearly the sources of the changes of wealth. Note that despite the voluntary participation of agents in the exchange economy and the utility-improving nature of each trade, a change in the distribution of wealth comes with it. The difference to earlier models is that the changes in wealth are explained by deeper, underlying economic forces while they are simply introduced as such in the models reviewed in sec. 2. Market exchange models also allow to consider changes of monetary evaluation of goods and assets as a potentially important source of changes in an individual's nominal wealth.

Unfortunately, monetary exchange so far does not provide an explanation of the power-law characterizing the far end of the distribution. As we have shown above, even an extremely unequal distribution of market power within the population seems not sufficient to replicate this important empirical feature. Following recent proposals in the literature one could try additional positive feedback effects that give agents with an already high level of wealth an additional advantage (West, 2005; Sinha, 2005).

In the above model, one could argue that the more wealthy agents would also acquire more bargaining power together with their higher rank in the wealth hierarchy. Whether this would help to explain the outer region, remains to be analyzed. However, there are perhaps reasons to doubt that the Pareto feature might be the mere result of clever bargaining. A glance at the Forbes list of richest individuals (analyzed statistically by Levy and Solomon, 1997, and Castaldi and Milakovic, 2005) reveals that the upper end of the distribution is not populated by smart dealers who in a myriad of small deals succeeded to outwit their counterparts. Rather, it is the founders and heirs of industrial dynasties and successful companies operating in new branches of economic activity whom we find there[1]. The conjecture based on this anecdotal evidence would be that the upper end of the spectrum has its roots in risky innovative investments. Few of these succeed but the owners behind the succeeding ones receive an overwhelming reward. This would suggest that models without savings and investments should lack a mechanism for a power law tail. One would, therefore, have to go beyond such conservative models and combine their exchange mechanism (which works well for the greater part of the distribution) with an economically plausible process for the emergence of very big fortunes.

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[1] While the majority of entrants in the Forbes list might fall into that category, a few are, in fact, rather suggestive of "theft and fraud" avenues to big fortunes.

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Lagrange Principle of Wealth Distribution

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Summary. The Lagrange principle $L = f + \lambda g \rightarrow$ maximum! is used to maximize a function $f(x)$ under a constraint $g(x)$. Economists regard $f(x) = U$ as a rational production function, which has to be maximized under the constraint of prices $g(x)$. In physics $f(x) = lnP$ is regarded as entropy of a stochastic system, which has to be maximized under constraint of energy $g(x)$. In the discussion of wealth distribution it may be demonstrated that both aspects will apply. The stochastic aspect of physics leads to a Boltzmann distribution of wealth, which applies to the majority of the less affluent population. The rational approach of economics leads to a Pareto distribution, which applies to the minority of the super rich. The boundary corresponds to an economic phase transition similar to the liquid - gas transition in physical systems.

Key words: Lagrange principle, Cobb Douglas production function, entropy, Boltzmann distribution, Pareto distribution, econophysics.

1 Introduction

Since the work of Pareto [1] as long ago as 1897, it has been known that economic distributions strictly follow power law decays. These distributions have been observed across a wide variety of economic and financial data. More recently, Roegen [2], Foley [3], Weidlich,[4] Mimkes [5][6], Levy and Solomon [7][8], Solomon and Richmond [9], Mimkes and Willis [10], Yakovenko [11], Clementi and Gallegatti [12] and Nirei and Souma [13] have proposed statistical models for economic distributions. In this paper the Lagrange principle is applied to recent data of wealth in different countries.
The Lagrange equation

$$f - \lambda g \rightarrow maximum! \tag{1}$$

applies to all functions (f) that are to be maximized under constraints (g). The factor λ is called Lagrange parameter.

2 Calculation of the Boltzmann distribution

In stochastic systems the probability P is to be maximized under constraints of capital according to the Lagrange principle

$$lnP(x_j) - \lambda \Sigma_j w_j x_j \rightarrow maximum! \tag{2}$$

lnP is the logarithm of probability $P(x_j)$ or entropy that will be maximized under the constraints of the total capital in income $\Sigma w_j x_j$. The variable (x_j) is the relative number of people in the income class (w_j). The Lagrange factor $\lambda = 1/ < w >$ is equivalent to the mean income $< w >$ per person. Distributing N households to (w_j) property classes is a question of combinatorial statistics,

$$P = N!/\Pi(N_j!) \tag{3}$$

Using Sterlings formula $(lnN! = NlnN - N)$ and $x = N_j/N$ we may change to

$$-\Sigma_j x_j lnx_j - \lambda \Sigma_j w_j x_j \rightarrow maximum! \tag{4}$$

At equilibrium (maximum) the derivative of equation 4 with respect to x_j will be zero,

$$\partial lnP/\partial x_j = -(lnx_j + 1) = \lambda w_j \tag{5}$$

In this operation for x_j all other variables are kept constant and we may solve Eq.6 for $x = x_j$. The number $N(w)$ of people in income class (w) is given by

$$N(w) = A\, e^{-\frac{w}{<w>}} \tag{6}$$

In the physical model the relative number of people (x) in the income class (w) follows a Boltzmann distribution, Eq.6. The Lagrange parameter λ has been replaced by the mean income $< w >$. The constant A is determined by the total number of people N_1 with an income following a Boltzmann distribution and may be calculated by the integral from zero to infinity,

$$N_1 = \int N(w)dw = A < w > \tag{7}$$

The amount of capital K(w) in the property class (w) is

$$K(w) = A\, w\, e^{-\frac{w}{<w>}}. \tag{8}$$

The total amount K_1 is given by the integral from zero to infinity,

$$K_1 = A \int w\ x(w)\ dw = A < w >^2 \tag{9}$$

The ratio of total wealth, Eq.9 divided by the total number of households, Eq.8 leads to

$$K_1/N_1 =< w > \tag{10}$$

which is indeed the mean income $< w >$ per household. The Lorenz distribution $y = K_1(w)$ as a function of $x = N_1(w)$ in fig.1 may be calculated from the Boltzmann distribution, Eqs.8 and 6, and leads to

$$y = x + (1 - x)ln(1 - x) \tag{11}$$

This function will be applied to Lorenz distributions of wealth.

2.1 German wealth data 1993 and the Boltzmann distribution

Property data for Germany (1993) have been published [14] by the German Institute of Economics (DIW). The data show the number $N(w)$ of households and the amount of capital $K(w)$ in each property class (w). The data are generally presented by a Lorenz distribution. Fig.1 shows the Lorenz distribution of capital K vs. the number N of households in Germany 1993.

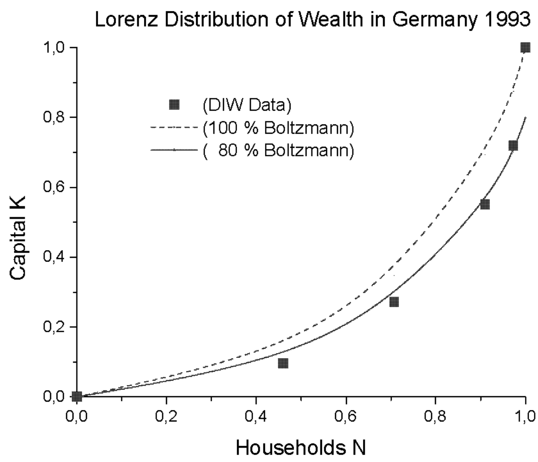

Fig. 1. Lorenz distribution, sum of capital K vs. sum of households N in Germany 1993, data from DIW [14], dotted line according to eq. 11. However, the Boltzmann approach only fits for 80 % of the total capital, solid line.

The data points are fitted by the dotted line for the Lorenz function of a Boltzmann distribution, eq. 11. The Boltzmann approach fits well only for 80% of the capital, solid line. A more detailed analysis of the super rich 20 % in Germany 1993 in fig.1 is not possible due to the few data points.

2.2 US wages data 1995 and the Boltzmann distribution

Wages like wealth may be expected to show a Boltzmann distribution according to eq.6. However, jobs below a wage minimum w_0 have the attrac-

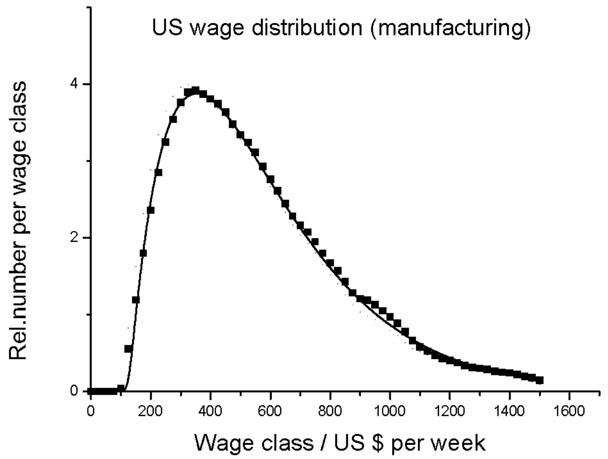

Fig. 2. Number of people in wage classes in manufacturing in the US [15]. The data have been fitted by the Boltzmann distribution, eq. 12 with $a(w, w_0,) = (w - w_0)$.

tiveness $a* = 0$. Accordingly, the distribution of income will be given by

$$N(w) = a(w, w_0)e^{-\frac{w}{<w>}} \tag{12}$$

The number of people earning a wage (w) will depend on the job attractiveness $a(w, w_0)$ and the Boltzmann function. Low wages will be very probable, high wages less probable. Fig. 2 shows the wage distribution for manufacturing in the US in 1995 [12]. Again the Boltzmann distribution seems to be a good fit. However, some authors prefer to fit similar data by a log normal distribution e.g. F. Clementi and M. Gallegatti [12]. Presently, both functions seem to apply equally well.

3 Calculation of the Pareto Distribution

Economic actions (f) are optimized under the constraints of capital, costs or prices (g). Economists are used to maximize the rational production function $U(x_j)$ under constraints of total income $\Sigma w_j dx_j$. The variable (x_j) is relative number of people in each income class (w_j). The Lagrange principle 27 is now given by

$$U(x_j) - \lambda \Sigma_j w_j x_j \rightarrow maximum! \tag{13}$$

At equilibrium (maximum) the derivative of equation 29 with respect to x_j will be zero,

$$\partial U/\partial x_j = \lambda w_j \tag{14}$$

In economic calculations a Cobb Douglas type Ansatz for the production function U is often applied,

$$U(x_j) = A\Pi x_j \alpha_j \tag{15}$$

A is a constant, the exponents α_j are the elasticity constants. Inserting Eq.15 into equation 14 we obtain

$$\partial U/\partial x_j = a_j A x_j^{\alpha_j - 1} = \lambda w_j \tag{16}$$

In this operation for x_j all other variables are kept constant and we may solve Eq.15 for $x = x_j$. The relative number $x(w)$ of people in income class (w) is now given by

$$N(w) = A(\lambda w/\alpha A)^{\frac{1}{(\alpha-1)}} = C(w_m/w)^{2+\delta} \tag{17}$$

According to this economic model the number of people $N(w)$ in the income class (w) follows a Pareto distribution! In eq. 17 the Lagrange parameter λ has been replaced by the minimum wealth class of the super rich, $w_m = 1/\lambda$, the constants have been combined to C. The Pareto exponent is given by $2 + \delta = 1/(1 - \alpha)$. The relative number of people $x(w)$ decreases with rising income (w). The total number of rich people N_2 with an income following a Pareto distribution is given by the integral from a minimum wealth w_m to infinity,

$$N_2 = \int N(w)dw = Cw_m/(1 + \delta) \tag{18}$$

The minimum wealth w_m is always larger than zero, $w_m > 0$. The amount of capital $K(w)$ in the property class (w) is

$$K(w) = N(w)w = Cw_m(w_m/w)^{1+\delta} \tag{19}$$

The total amount of capital K_2 of very rich people with an income following a Pareto distribution is given by the integral from a minimum wealth w m to infinity,

$$K_2 = \int N(w)wdw = Cw_m^2/\delta \tag{20}$$

For positive wealth the exponent δ needs to be positive, $\delta > 0$. For a high capital of the super rich the exponent δ is expected to be: $0 < \delta < 1$. The ratio of total wealth Eq.20 divided by the total number of super rich households Eq.19 leads to

$$w_m = (K_2/N_2)\delta/(1 + \delta) \tag{21}$$

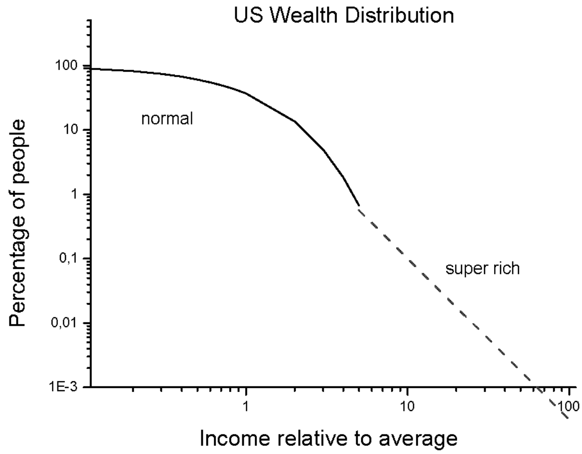

Fig. 3. shows the percentage of people in wage classes relative to average (USA 1983 – 2001) [Silva and Yakovenko] [11]: The distribution is clearly divided into two parts. The wealth of the majority of people (97%) follows a Boltzmann distribution, the wealth of the minority of super rich follows a Pareto law. The Pareto tail has a slope between 2 and 3.

3.1 Boltzmann and Pareto distribution in USA wealth data

Yakovenko [11] and others have presented data that follow a Boltzmann distribution for the majority of normal wages and a Pareto distribution for the income of the minority of very rich people, fig. 3.

A. Boltzmann distribution

A1. The wealth of the majority of the population in fig. 3 follows the Boltzmann distribution of Eq.6.

A2. The total number of normal rich people is $N_1 = 97\%$ of the total population,

$$N_1 = \int N(w)dw = A < w >= 0.97 \tag{22}$$

A3. The wealth of the normal population is given by

$$K_1 = \int N(w)wdw = N_1 < w >= 0.97 < w > \tag{23}$$

B. Pareto distribution

B1. The wealth of the super rich minority follows a Pareto law, Eq.17.

B2. The exponent of the Pareto tail in fig. 3 is between 2 and 3 or $0 < \delta < 1$, as required by Eq.20.

B3. The minimum wealth of the super rich according to fig. 3 is about eight

times the normal mean, $w_m = 8 < w >$.
B4. The total number of the super rich minority is $N_2 = 3\%$ of the total population,

$$N_2 = \int N(w)dw = Cw_m = 0,03 \tag{24}$$

B5. The total capital of the super rich minority (for $\delta = 0.5$) is

$$K_2 = N_2 w_m (1 + \delta)/\delta = 0.03 * 8 < w > *3 = 0.72 < w > \tag{25}$$

B6. The mean wealth of the super rich minority is

$$< w_2 >= (K_2/N_2) = w_m(1 + \delta)/\delta = 8 < w > *3 = 25 < w > \tag{26}$$

B7. In fig. 3 the super rich minority (3% of the population) owns $0.72 < w >=$ 40% of the national wealth and the normal majority (97% of the population) owns $0.97 < w >= 40\%$.

4 Boltzmann and Pareto phase transition

The normal rich majority and the super rich minority belong to two different states or phases. The majority is governed by the Boltzmann law, the minority by a Pareto law. This corresponds to two different phases, like liquid and gas in physical sciences, and may be calculated by the Lagrange principle,

$$L = f - \lambda g \to maximum! \tag{27}$$

$$L_1 = -xlnx - < w > \Sigma_j w_j x_j \to maximum! \tag{28}$$

$$L_2 = Ax^\alpha - < w > \Sigma_k w_k x_k \to maximum! \tag{29}$$

$$y = b - xm \tag{30}$$

Eq.27 is the general Lagrange equation, Eq.28 the Lagrange principle in stochastic systems and Eq.29 the Lagrange principle in rational systems. All may be considered linear equations of $< w >$. The $b-$ value in Eq.30 is given by the entropy or utility function, which is lower for the super rich population due to the small factor A, which is of the order of $A = 0.1$ in fig. 3. The slope "m" is given by the total wealth $\Sigma_j w_j x_j$, which is higher for the normal population (60%). The borderline between the normal and the super rich population is given by the intersection of the two lines at $< w >_c$ in 4. Below $< w >_c$ the solid line is higher (at maximum) and the normal phase dominates. Above $< w >_c$ the broken line is higher (at maximum) and the super rich phase dominates. The transition point $< w >_c$ is given by $L_1 = L_2$. However, the data are not yet sufficient to tell whether the transition "normal" - "super rich" really is of first order, as it is indicated by the sharp knee in fig.3 and the intersection in fig. 4. Other authors [13] find a smooth second

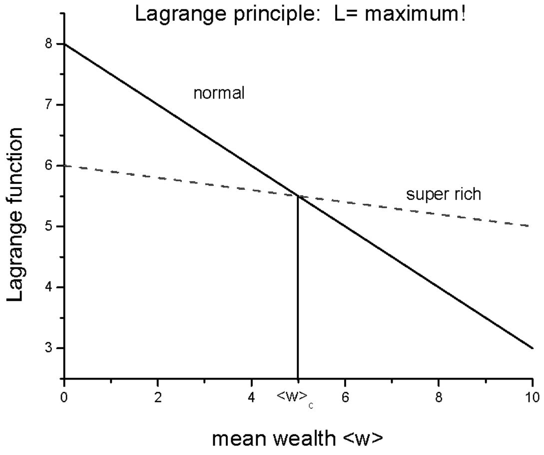

Fig. 4. For low mean income $< w >$ the Lagrange function L_1 (solid line) is at maximum, the normal rich phase dominates. For very large mean income $< w >$ the Lagrange function L_2 (broken line) is higher (at maximum), the super rich phase dominates.

order transition from the Boltzmann region of normal people to the Pareto region of the super rich. The point of transition is important for the full understanding of the system, but even more important is the mechanism that keeps normal and super rich people separated and drifting more and more apart. This topic will be discussed in a separate paper on the mechanism of economic growth.

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Carnot Process of Wealth Distribution

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Summary. Economic growth is a result of production and may be calculated by a differential form $\delta q(K, L)$, which depends on capital (K) and labor (L). Differential forms in two variables are generally not exact. Accordingly, a general production function q does not exist ex ante, the integral $\int \delta q$ may not be calculated, unless the path of integration is given. All production functions $q_x(K, L)$ depend on a given production process (x). This corresponds to the first law of thermodynamics. The not exact differential form δq of economic production will become exact by introducing an integrating factor (λ), $\delta q = \lambda df$. This corresponds to the second law of thermodynamics. The function f exists ex ante and is called entropy (S) in physics and production function (U) in economics. The factor λ will be a mean price level, the mean GDP per capita or standard of living. Production is a Carnot process, which always creates two levels (λ). In motors and refrigerators these levels (λ) are called hot and cold, in economic systems they are called rich and poor, which explains the two different functions of wealth for normal and for super rich people. The efficiency of the Carnot process grows with the difference in levels (λ). For this reason the gap between poor and rich tends to grow permanently.

Key words: Differential forms, economic growth, distribution of wealth, Boltzmann distribution, Pareto distribution, Carnot process, entropy, Cobb Douglas production function.

1 Introduction

In the last ten years new interdisciplinary approaches to economics have developed in natural science. First steps have been made by W. Weidlich 1972 [10], D. K. Foley 1994 [3], J. Mimkes 1995 [7], H. G. Stanley 1999 [9], Aruka 2000 [1] and others. In order to enhance the communication between different disciplines a number of international conferences have been carried out worldwide in the last five years, with topics on complexity in economics, econophysics and economic agents. In this paper differential forms are discussed as a basis for economic growth. In natural science the concept of not exact differential forms leads to the first and second laws of thermodynamics. Accordingly, the principles of calculus may be expected to lead to basic laws of macro economics and markets. In the first chapters we will discuss differential forms. One dimensional differential forms $df(x)$ are always exact, they may be calculated beforehand (ex ante), the integral from $x = A$ to $x = B$ will depend on the limits, only. (This is the calculus we learn in high school). A closed integral of an exact differential form will always be zero,

$$\oint df = \int_A^B df_1 + \int_B^A df_2 = \int_A^B df_1 - \int_A^B df_2 = 0 \tag{1}$$

However, in two dimensions differential forms $\delta g(x, y)$ are not necessarily exact and may not be calculated ex ante. A closed integral of not exact differential forms will generally not be zero, the value of the integral will depend on the limits A and B and on the path of integration, (see e.g. Kaplan [5])

$$\oint \delta g = \int_A^B \delta g_1 + \int_B^A \delta g_2 = \int_A^B \delta g_1 - \int_A^B \delta g_2 \neq 0 \tag{2}$$

The net output depends on the path of the integral and cannot generally be determined ex ante.

2 Differential Forms in Economics: the "First Law"

Economic growth (δq) is a result of periodic production (W). In farms, automobile plants or professional offices laborers may work for the same time, but the result will depend on the production process. Production may be modeled by the calculus of not exact differential forms,

$$-\oint \delta W = \oint \delta q \tag{3}$$

Periodic production depends on the path of integration, the production process. Eq.3 is equivalent to the first law of thermodynamics [4]. The negative sign indicates that work ($-W$) has to be invested in order to obtain a net output (Δq) of production.

2.1 Income (Y) and Consumption (C)

The cyclic process of economic production, Eq.3, may be split into two parts, the integral from A to B and back from B to A,

$$-\oint \delta W = \oint \delta q = \int_A^B \delta g_1 + \int_B^A \delta g_2 = \int_A^B \delta g_1 - \int_A^B \delta g_2 = Y - C = \Delta q \tag{4}$$

Sales returns (Y) and costs (C) of a company, income (Y) and consumption (C) of a private person are both part of the same production cycle and depend on the production and consumption process. Eq.4 states that the surplus or net output (Δq) may not be calculated beforehand (ex ante), unless the production process (x) or path of integration is known. Each production process leads to a specific production function q_X. A general production function (q) without a given production process does not exist, ex ante.

3 Differential Forms in Economics: the "Second Law"

The not exact differential form of economic net output δq may be turned into an exact differential form df - that may be calculated ex ante - by an integrating factor λ,

$$\delta q = \lambda df \tag{5}$$

This law corresponds to the second law of thermodynamics, $\delta Q = TdS$. In stochastic systems (thermodynamics) $\lambda = T$ is called temperature, $f = S$ is called entropy (see e.g. Fowler and Guggenheim [4]). In rational systems (economics) $f = U$ is called production function and λ is a mean price level or mean income level.

4 Production Function of Stochastic Systems

In stochastic systems the function $f = S$ is given by the entropy function. S measures the complexity of a stochastic (production) process and is defined by the logarithm of probability P,

$$S = lnP \tag{6}$$

Example 1. Production with heterogeneously skilled labour: Companies employ workers, engineers, secretaries, managers etc. If there are three types of labour - with N_1, N_2 and N_3 employees for each type, the entropy may be calculated by mathematics of combinations,

$$S = lnP = lnN!/N_1!N_2!N_3!. \tag{7}$$

Introducing the relative number of employees $x = N_1/N$, $y = N_2/N$, $z = N_3/N$ in each type of labour and the Stirling formula for $lnN! = NlnN - N$ we obtain the entropy of mixing,

$$S = lnP = -Nxlnx + ylny + zlnz \tag{8}$$

For constant λ Eq.5 may be integrated and we obtain the special production function q_λ (entropy of mixing in stochastic systems),

$$q_\lambda = \lambda \Delta S = -\lambda N ln(x^x, y^y, z^z) \quad (9)$$
$$x + y + z = 1$$
$$x, y, z > 0$$

This production function applies to a process with three production factors x, y, z at constant level λ. The production function 9 has been plotted for a new product, $q_T(x)$, in fig. 1. The first small numbers of a new product x obtain the highest marginal net productivity.

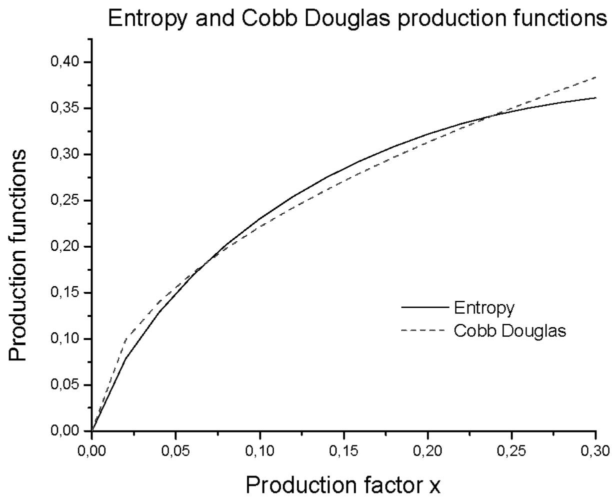

Fig. 1. The entropy function $q_\lambda(x)$ according to Eq.9 - solid line - differs only little from the Cobb Douglas production function $q_\lambda(x)$ according to Eq.10.

4.1 Production Function of Rational Systems

In rational systems (economics) the functions f in Eq.5 is called production function U,

$$f = U_{CD}(x, y, z) = A(x^\alpha y^\beta z^\gamma) \quad (10)$$

In the Cobb Douglas type function the values of power are variously specified according to the scale of production. At constant scale of production the condition for the elasticity coefficients is given by $\alpha + \beta + \gamma = 1$, $\alpha, \beta, \gamma > 0$. λ is again a mean price level or mean income level. In fig. 1 the Cobb Douglas type function $U = 0.7\sqrt{x}$ is given by the dashed line. Both functions, entropy of stochastic systems and the Cobb Douglas function of rational systems do not differ by much.

5 The Carnot Production Process

According to Carnot the first and second law may be combined and integrated for a specific closed path: along $\lambda = constant$ and $S = constant$ in the $\lambda - S$ diagram, as shown in 2,

$$\oint \delta q = \oint TdS = T_2 \Delta S - T_1 \Delta S = Y - C = \Delta q \tag{11}$$

$\lambda = constant$ is equivalent to production at constant price level or constant standard of living, $S = constant$ is equivalent to transport between two levels λ_1 and λ_2 without net output, $\delta q = 0$. The Carnot cycle is an idealized process. It is the basis of production, every economic unit like companies, banks, countries and single persons have to follow this process. The first and second law are universal and apply to economics, to biology, to natural science. In motors and refrigerators the Carnot process creates two temperature levels, the hot side and the cold side. In production, trade and finance the Carnot process creates two income levels, the rich side and the poor side. The efficiency of the Carnot process is proportional to the difference in price levels (λ).

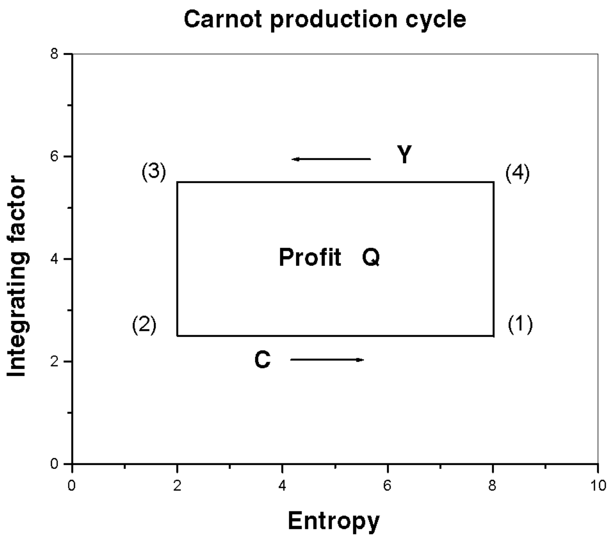

Fig. 2. In the Carnot cycle the closed integral of production Eq.11 is carried out at constant λ and constant S, starting and ending at point (1). The surplus of this production process $\Delta q = Y - C = \Delta\lambda * \Delta S$ corresponds to the enclosed area in 2.

In production we have earnings of owners (Y) and income of workers or labour costs (C), fig. 2. In trade we have import and export prices, in finance we find different interests for borrowing and lending. An example of the Carnot process is given for trade:

Example 2: Carnot cycle of trade (Dutch import of furniture from Indonesia):
(1) $\rightarrow$ (2): *A Dutch importer collects* ($\Delta S < 0$) *furniture from workers in Indonesia. At the low wage level* (λ_1) *in furniture producing in Indonesia the production costs are* $C = \lambda_1 * \Delta S$.
(2) $\rightarrow$ (3): *The furniture is transported to Holland without changing the distribution,* $S = constant$.
(3) $\rightarrow$ (4): *In Holland the furniture is sold and distributed* ($\Delta S > 0$) *to the customers. At the higher price level* (λ_2) *of furniture in Holland the earnings are* $Y = \lambda_2 * \Delta S$.
(4) $\rightarrow$ (1): *The cycle starts again. The surplus of the furniture production cycle is* $\Delta q = Y - C = N\Delta\lambda * \Delta S$ *and corresponds to the enclosed area in fig. 2.*
This example applies to the production processes of all commodities, to food markets and automobile companies, to financial markets and wealth distribution.

5.1 The two Levels in World Wealth Distribution

The 1995 world distribution of wealth, fig.3 is clearly divided into two parts. Fig.3 shows the GDP distribution of the world and the corresponding number of people. In the "third world" more than three billion people live below or close to a GDP of 2.000 US $ per capita. And in the "first world" about one billion people live between 12.000 and 16.000 US $ per capita. (The small dip at 14.000 US $ per capita is artificial and due to fluctuations of the US $ and EU currencies.) The separation into rich and poor countries may be explained by the Carnot cycle. In rich countries large companies, stocks and property are operating with high efficiency due to a well developed infra structure. The companies are big economic Carnot machines or capital pumps. The Carnot process is the basis of heat pumps, which may draw heat from a cold river to a warm house. And by the same mechanism capital pumps may draw capital from a poor countries and pump it to rich countries. An extreme example has been the 19th century colonialism, that has drawn immense wealth from the "third world" to the "first world". The efficiency (η) of a capital pump grows with the difference in income levels (λ),

$$\eta = (\lambda_2 - \lambda_1)/\lambda_1 \tag{12}$$

Example 3. Efficiency of trade: The GDP per capita in Holland and Indonesia are $\lambda_{Holland} = 12.000$ US $ per capita and $\lambda_{Indonesia} = 3.000$ US $ per capita. Importing local commodities like furniture from Indonesia to Holland leads to an (ideal) efficiency of $\eta = (12.000 - 3.000)/3.000 = 3$ or 300% revenues.

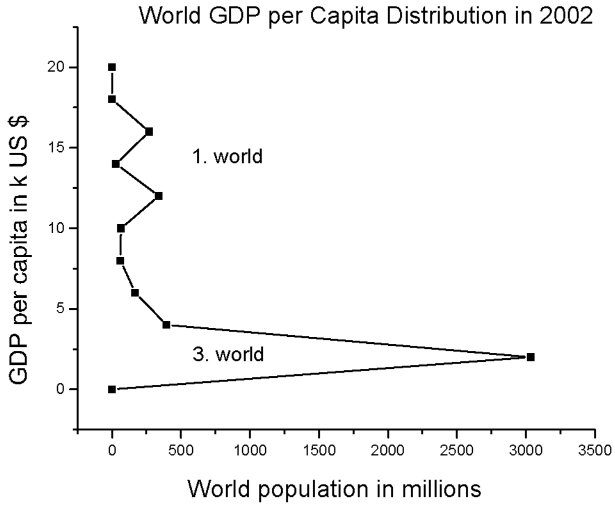

Fig. 3. Shows the distribution of wealth in the world [CIA World Factbook, USA, 2004][2]. The number of people in different income classes is given by the gross domestic product (GDP) per person. The distribution is clearly divided into two parts. The majority of people (about three billion) in the "third world" live below or close to 2.000 US $ per person. The minority of about one billion people in the "first world" have an income between 12.000 and 15.000 US $ per capita. (The gap at 14.000 US $ per capita is an artifact and due to US - EU currency fluctuations.) About two billion people, the "second world" live in between the two extremes.

The Carnot process stabilises the difference in levels λ in a motor, once it is running well. In the same way the Carnot process of trade may stabilise the difference in income levels between the first and third world.

5.2 The two Levels in National Wealth Distribution

The distribution of wealth within a nation like the USA 1993- 2001, fig.4, and other countries is again clearly divided into two parts. The wealth of less affluent majority follows a Boltzmann distribution, which may be derived from the entropy function 9. The wealth of the super rich minority follows a Pareto law, which may be derived from the Cobb Douglas function 10. The separation into less affluent and super rich people may be explained by the Carnot cycle. Super rich people run large companies, stocks and property. They are again owners of big economic Carnot machines or capital pumps. The efficiency of a capital pump grows with the difference in income levels. This is the reason for wage dumping and for first world companies to change to third world production. In a running motor the difference in temperature

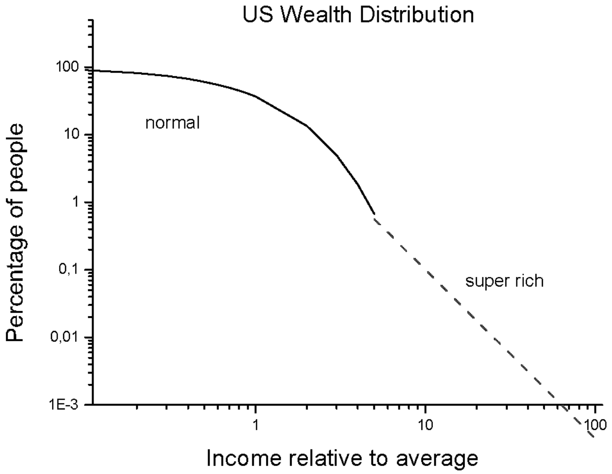

Fig. 4. shows the percentage of people in wage classes relative to average (USA 1983 – 2001) [Silva and Yakovenko]. The distribution is clearly divided into two parts. The wealth of the majority of people (97%) follows a Boltzmann distribution, the wealth of the minority of super rich follows a Pareto law.

levels inside and outside will grow. In the same way in economies the Carnot process will enhance the difference in income levels between the normal and the super rich.

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Ideal-Gas Like Markets: Effect of Savings

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Summary. We discuss the ideal gas like models of a trading market. The effect of savings on the distribution have been thoroughly reviewed. The market with fixed saving factors leads to a Gamma-like distribution. In a market with quenched random saving factors for its agents we show that the steady state income (m) distribution $P(m)$ in the model has a power law tail with Pareto index ν equal to unity. We also discuss the detailed numerical results on this model. We analyze the distribution of mutual money difference and also develop a master equation for the time development of $P(m)$. Precise solutions are then obtained in some special cases.

1 Introduction

The distribution of wealth among individuals in an economy has been an important area of research in economics, for more than a hundred years. Pareto [1] first quantified the high-end of the income distribution in a society and found it to follow a power-law

$$P(m) \sim m^{-(1+\nu)}, \tag{1}$$

where P gives the normalized number of people with income m, and the exponent ν is called the Pareto index.

Considerable investigations with real data during the last ten years revealed that the tail of the income distribution indeed follows the above mentioned behavior and the value of the Pareto index ν is generally seen to vary between 1 and 3 [2, 3, 4, 5]. It is also known that typically less than 10% of the population in any country possesses about 40% of the total wealth of that country and they follow the above law. The rest of the low income population, in fact the majority (90% or more), follow a different distribution which is debated to be either Gibbs [3, 6, 7] or log-normal [4].

Much work has been done recently on models of markets, where economic (trading) activity is analogous to some scattering process [6, 8, 9, 10, 11, 12, 13, 14, 15, 16] as in the kinetic theory [17] of gases or liquids.

We put our attention to models where introducing a saving propensity (or factor) [18] for the agents, a wealth distribution similar to that in the real economy can be obtained [8, 12]. Savings do play an important role in determining the nature of the wealth distribution in an economy and this has already been observed in some recent investigations [19]. Two variants of the model have been of recent interest; namely, where the agents have the same fixed saving factor [8], and where the agents have a quenched random distribution of saving factors [12]. While the former has been understood to a certain extent (see e.g, [20, 21]), and argued to resemble a gamma distribution [21], attempts to analyze the latter model are still incomplete (see however [22]). Further numerical studies [23] of time correlations in the model seem to indicate even more intriguing features of the model. In this paper, we intend to analyze the second market model with randomly distributed saving factor, using a master equation type approach similar to kinetic models of condensed matter.

We have studied here numerically a gas model of a trading market. We have considered the effect of saving propensity of the traders. The saving propensity is assumed to have a randomness. Our observations indicate that Gibbs and Pareto distributions fall in the same category and can appear naturally in the century-old and well-established kinetic theory of gas [17]: Gibbs distribution for no saving and Pareto distribution for agents with quenched random saving propensity. Our model study also indicates the appearance of self-organized criticality [24] in the simplest model so far, namely in the kinetic theory of gas models, when the stability effect of savings [18] is incorporated.

2 Ideal-gas like models

We consider an ideal-gas model of a closed economic system where total money M and total number of agents N is fixed. No production or migration occurs and the only economic activity is confined to trading. Each agent i, individual or corporate, possess money $m_i(t)$ at time t. In any trading, a pair of traders i and j randomly exchange their money [6, 7, 8], such that their total money is (locally) conserved and none end up with negative money ($m_i(t) \geq 0$, i.e, debt not allowed):

$$m_i(t) + m_j(t) = m_i(t+1) + m_j(t+1); \tag{2}$$

time (t) changes by one unit after each trading. The steady-state ($t \to \infty$) distribution of money is Gibbs one:

$$P(m) = (1/T)\exp(-m/T); T = M/N. \tag{3}$$

Hence, no matter how uniform or justified the initial distribution is, the eventual steady state corresponds to Gibbs distribution where most of the people have got very little money. This follows from the conservation of money and additivity of entropy:

$$P(m_1)P(m_2) = P(m_1 + m_2). \tag{4}$$

This steady state result is quite robust and realistic too! In fact, several variations of the trading, and of the 'lattice' (on which the agents can be put and each agent trade with its 'lattice neighbors' only), whether compact, fractal or small-world like [2], leaves the distribution unchanged. Some other variations like random sharing of an amount $2m_2$ only (not of $m_1 + m_2$) when $m_1 > m_2$ (trading at the level of lower economic class in the trade), lead to even drastic situation: all the money in the market drifts to one agent and the rest become truely pauper [9, 10].

2.1 Effect of fixed or uniform savings

In any trading, savings come naturally [18]. A saving propensity factor λ is therefore introduced in the same model [8] (see [7] for model without savings), where each trader at time t saves a fraction λ of its money $m_i(t)$ and trades randomly with the rest:

$$m_i(t+1) = m_i(t) + \Delta m; \quad m_j(t+1) = m_j(t) - \Delta m \tag{5}$$

where

$$\Delta m = (1-\lambda)[\epsilon\{m_i(t) + m_j(t)\} - m_i(t)], \tag{6}$$

ϵ being a random fraction, coming from the stochastic nature of the trading.

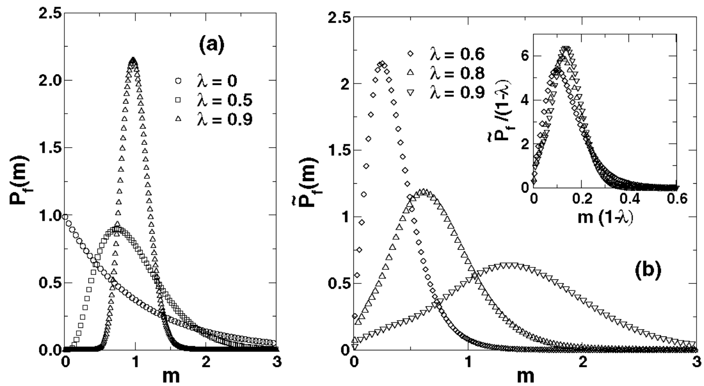

Fig. 1. Steady state money distribution (a) $P(m)$ for the fixed λ model, and (b) $\tilde{P}_f(m)$ for some specific values of λ in the distributed λ model. All data are for $N = 200$. Inset of (b) shows scaling behavior of $\tilde{P}_f(m)$.

Data and

The evolution of money in such a trading can be written as:

$$m_i(t+1) = \lambda_i m_i(t) + \epsilon_{ij} \left[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)\right], \qquad (7)$$

$$m_j(t+1) = \lambda_j m_j(t) + (1-\epsilon_{ij}) \left[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)\right] \qquad (8)$$

One again follows the same trading rules as before, except that

$$\Delta m = \epsilon_{ij}(1-\lambda_j)m_j(t) - (1-\lambda_i)(1-\epsilon_{ij})m_i(t) \qquad (9)$$

here; λ_i and λ_j being the saving propensities of agents i and j. The agents have fixed (over time) saving propensities, distributed independently, randomly and uniformly (white) within an interval 0 to 1 agent i saves a random fraction λ_i ($0 \le \lambda_i < 1$) and this λ_i value is quenched for each agent (λ_i are independent of trading or t). Starting with an arbitrary initial (uniform or random) distribution of money among the agents, the market evolves with the tradings. At each time, two agents are randomly selected and the money exchange among them occurs, following the above mentioned scheme. We check for the steady state, by looking at the stability of the money distribution in successive Monte Carlo steps t (we define one Monte Carlo time step as N pairwise interations). Eventually, after a typical relaxation time ($\sim 10^6$ for $N = 1000$ and uniformly distributed λ) dependent on N and the distribution of λ, the money distribution becomes stationary. After this, we average the money distribution over $\sim 10^3$ time steps. Finally we take configurational average over $\sim 10^5$ realizations of the λ distribution to get the money distribution $P(m)$. It is found to follow a strict power-law decay. This decay fits to Pareto law (1) with $\nu = 1.01 \pm 0.02$ (Fig. 2). Note, for finite size N of the market, the distribution has a narrow initial growth upto a most-probable value m_p after which it falls off with a power-law tail for several decades. This Pareto law (with $\nu \simeq 1$) covers the entire range in m of the distribution $P(m)$ in the limit $N \to \infty$. We checked that this power law is extremely robust: apart from the uniform λ distribution used in the simulations in Fig. 2, we also checked the results for a distribution

$$\rho(\lambda) \sim |\lambda_0 - \lambda|^{\alpha}, \quad \lambda_0 \ne 1, \quad 0 < \lambda < 1, \qquad (10)$$

of quenched λ values among the agents. The Pareto law with $\nu = 1$ is universal for all α. The data in Fig. 2 corresponds to $\lambda_0 = 0$, $\alpha = 0$. For negative α values, however, we get an initial (small m) Gibbs-like decay in $P(m)$ (see Fig. 3).

In case of uniform distribution of saving propensity λ ($0 \le \lambda < 1$), the individual money distribution $\tilde{P}_f(m)$ for agents with any particular λ value, although differs considerably, remains non-monotonic: similar to that for fixed λ market with $m_p(\lambda)$ shifting with λ (see Fig. 1). Few subtle points may be noted though: while for fixed λ the $m_p(\lambda)$ were all less than of the order of unity (Fig. 1(a)), for distributed λ case $m_p(\lambda)$ can be considerably larger and can approach to the order of N for large λ (see Fig. 1(b)). The other important

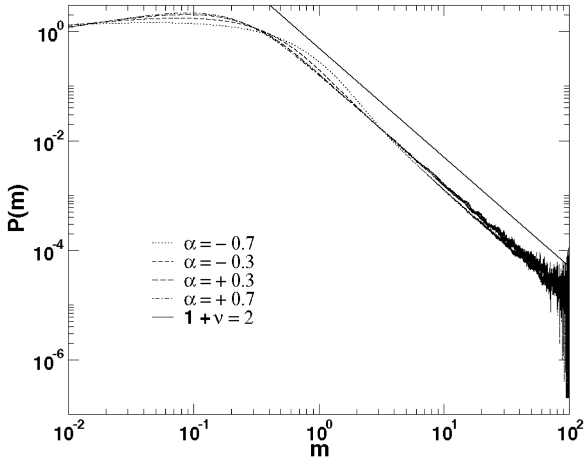

Fig. 3. Steady state money distribution $P(m)$ in the model with for a system of $N = 100$ agents with λ distributed as $\rho(\lambda) \sim \lambda^{\alpha}$, with different values of α. In all cases, agents play with average money per agent $M/N = 1$.

difference is in the scaling behavior of $\tilde{P}_f(m)$, as shown in the inset of Fig. 1(b). In the distributed λ ensemble, $\tilde{P}_f(m)$ appears to have a very simple scaling:

$$\tilde{P}_f(m) \sim (1-\lambda)\mathcal{F}(m(1-\lambda)), \tag{11}$$

for $\lambda \to 1$, where the scaling function $\mathcal{F}(x)$ has non-monotonic variation in x. The fixed (same for all agents) λ income distribution $P_f(m)$ do not have any such comparative scaling property. It may be noted that a small difference exists between the ensembles considered in Fig 1(a) and 1(b): while $\int mP_f(m)dm = M$ (independent of λ), $\int m\tilde{P}_f(m)dm$ is not a constant and infact approaches to order of M as $\lambda \to 1$. There is also a marked qualitative difference in fluctuations (see Fig. 4): while for fixed λ, the fluctuations in time (around the most-probable value) in the individuals' money $m_i(t)$ gradually decreases with increasing λ, for quenched distribution of λ, the trend gets reversed (see Fig. 4).

We now investigate on the range of distribution of the saving propensities in a certain interval $a < \lambda_i < b$, where, $0 < a < b < 1$. For uniform distribution within the range, we observe the appearance of the same power law in the distribution but for a narrower region. As may be seen from Fig. 5, as $a \to b$, the power-law behavior is seen for values a or b approaching more and more towards unity: For the same width of the interval $|b-a|$, one gets power-law (with same ν) when $b \to 1$. This indicates, for fixed λ, $\lambda = 0$ corresponds to Gibbs distribution, and one gets Pareto law when λ has got non-zero width of its distribution extending upto $\lambda = 1$. This of course indicates a crucial role of these high saving propensity agents: the power law behavior is truely valid upto the asymptotic limit if $\lambda = 1$ is included. Indeed, had we assumed $\lambda_0 = 1$

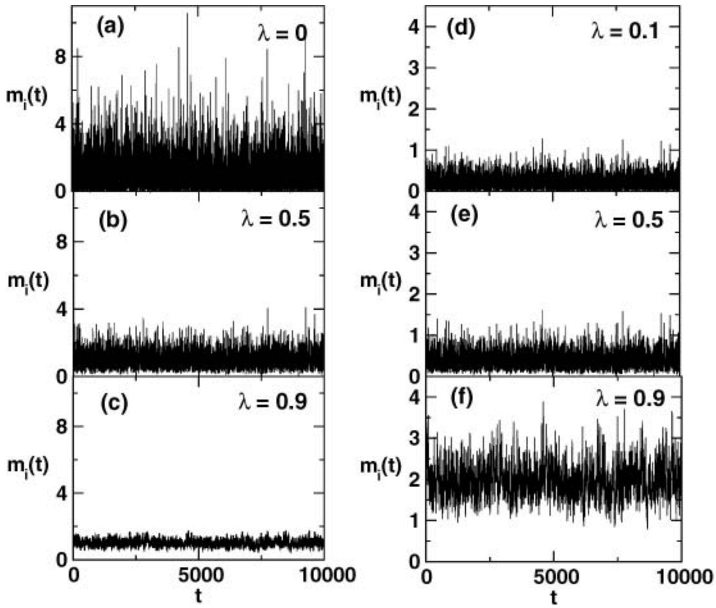

Fig. 4. Time variation of the money of the ith trader: For fixed λ market – (a), (b), (c); and for agents with specific values of λ in the distributed λ market – (d), (e), (f).

in (10), the Pareto exponent ν immediately switches over to $\nu = 1 + \alpha$. Of course, $\lambda_0 \neq 1$ in (10) leads to the universality of the Pareto distribution with $\nu = 1$ (irrespective of λ_0 and α). Indeed this can be easily rationalised from the scaling behavior (11): $P(m) \sim \int_0^1 \tilde{P}_f(m)\rho(\lambda)d\lambda \sim m^{-2}$ for $\rho(\lambda)$ given by (10) and $m^{-(2+\alpha)}$ if $\lambda_0 = 1$ in (10) (for large m values).

These model income distributions $P(m)$ compare very well with the wealth distributions of various countries: Data suggests Gibbs like distribution in the low-income range (more than 90% of the population) and Pareto-like in the high-income range [3] (less than 10% of the population) of various countries. In fact, we compared one model simulation of the market with saving propensity of the agents distributed following (10), with $\lambda_0 = 0$ and $\alpha = -0.7$ [12]. The qualitative resemblance of the model income distribution with the real data for Japan and USA in recent years is quite intriguing. In fact, for negative α values in (10), the density of traders with low saving propensity is higher and since $\lambda = 0$ ensemble yields Gibbs-like income distribution (3), we see an initial Gibbs-like distribution which crosses over to Pareto distribution (1) with $\nu = 1.0$ for large m values. The position of the crossover point depends on the value of α. It is important to note that any distribution of λ near $\lambda = 1$, of finite width, eventually gives Pareto law for large m limit. The same

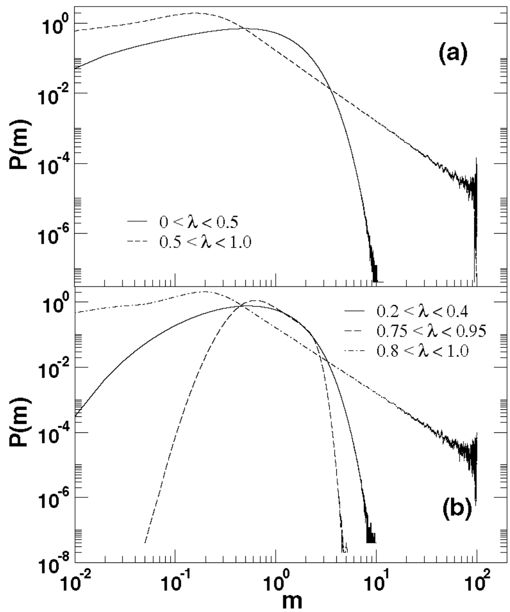

Fig. 5. Steady state money distribution in cases when the saving propensity λ is distributed uniformly within a range of values: (a) width of λ distribution is 0.5, money distribution shows power law for $0.5 < \lambda < 1.0$; (a) width of λ distribution is 0.2, money distribution shows power law for $0.7 < \lambda < 0.9$. The power law exponent is $\nu \simeq 1$ in all cases. All data shown here are for $N = 100$, $M/N = 1$.

kind of crossover behavior (from Gibbs to Pareto) can also be reproduced in a model market of mixed agents where $\lambda = 0$ for a finite fraction of population and λ is distributed uniformly over a finite range near $\lambda = 1$ for the rest of the population.

We even considered annealed randomness in the saving propensity λ: here λ_i for any agent i changes from one value to another within the range $0 \leq \lambda_i < 1$, after each trading. Numerical studies for this annealed model did not show any power law behavior for $P(m)$; rather it again becomes exponentially decaying on both sides of a most-probable value.

3 Dynamics of money exchange

We will now investigate the steady state distribution of money resulting from the above two equations representing the trading and money dynamics. We will now solve the dynamics of money distribution in two limits. In one case, we study the evolution of the mutual money difference among the agents and look for a self-consistent equation for its steady state distribution. In the other case, we develop a master equation for the money distribution function.

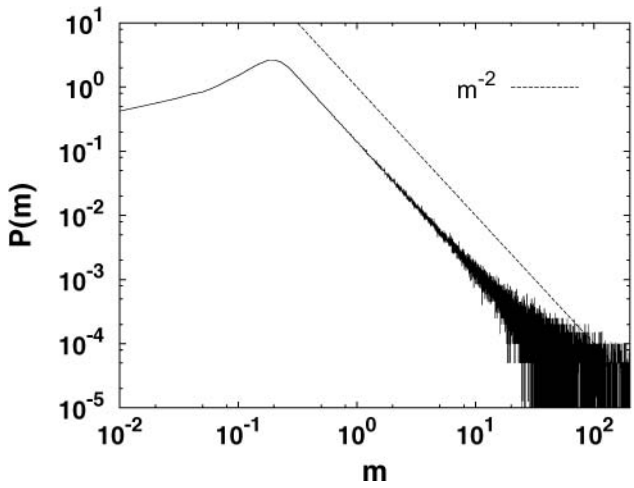

Fig. 6. Steady state money distribution $P(m)$ against m in a numerical simulation of a market with $N = 200$, following equations (7) and (8) with $\epsilon_{ij} = 1/2$. The dotted lines correspond to $m^{-(1+\nu)}$; $\nu = 1$.

3.1 Distribution of money difference

Clearly in the process as considered above, the total money $(m_i + m_j)$ of the pair of agents i and j remains constant, while the difference Δm_{ij} evolves as

$$\begin{aligned}(\Delta m_{ij})_{t+1} &\equiv (m_i - m_j)_{t+1} \\ &= \left(\frac{\lambda_i + \lambda_j}{2}\right)(\Delta m_{ij})_t + \left(\frac{\lambda_i - \lambda_j}{2}\right)(m_i + m_j)_t \\ &\quad +(2\epsilon_{ij} - 1)[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)]. \end{aligned} \tag{12}$$

Numerically, as shown in Fig. 2, we observe that the steady state money distribution in the market becomes a power law, following such tradings when the saving factor λ_i of the agents remain constant over time but varies from agent to agent widely. As shown in the numerical simulation results for $P(m)$ in Fig. 6, the law, as well as the exponent, remains unchanged even when $\epsilon_{ij} = 1/2$ for every trading. This can be justified by the earlier numerical observation [8, 12] for fixed λ market ($\lambda_i = \lambda$ for all i) that in the steady state, criticality occurs as $\lambda \to 1$ where of course the dynamics becomes extremely slow. In other words, after the steady state is realized, the third term in (12) becomes unimportant for the critical behavior. We therefore concentrate on this case, where the above evolution equation for Δm_{ij} can be written in a more simplified form as

$$(\Delta m_{ij})_{t+1} = \alpha_{ij}(\Delta m_{ij})_t + \beta_{ij}(m_i + m_j)_t, \tag{13}$$

where $\alpha_{ij} = \frac{1}{2}(\lambda_i + \lambda_j)$ and $\beta_{ij} = \frac{1}{2}(\lambda_i - \lambda_j)$. As such, $0 \leq \alpha < 1$ and $-\frac{1}{2} < \beta < \frac{1}{2}$.

The steady state probability distribution D for the modulus $\Delta = |\Delta m|$ of the mutual money difference between any two agents in the market can be obtained from (13) in the following way provided Δ is very much larger than the average money per agent $= M/N$. This is because, large Δ can appear from 'scattering' involving $m_i - m_j = \pm\Delta$ and when either m_i or m_j is small. When both m_i and m_j are large, maintaining a large Δ between them, their probability is much smaller and hence their contribution. Then if, say, m_i is large and m_j is not, the right hand side of (13) becomes $\sim (\alpha_{ij} + \beta_{ij})(\Delta_{ij})_t$ and so on. Consequently for large Δ the distribution D satisfies

$$\begin{aligned} D(\Delta) &= \int d\Delta' \; D(\Delta') \; \langle \delta(\Delta - (\alpha+\beta)\Delta') + \delta(\Delta - (\alpha-\beta)\Delta') \rangle \\ &= 2\langle \left(\frac{1}{\lambda}\right) \; D\left(\frac{\Delta}{\lambda}\right) \rangle, \end{aligned} \tag{14}$$

where we have used the symmetry of the β distribution and the relation $\alpha_{ij} + \beta_{ij} = \lambda_i$, and have suppressed labels i, j. Here $\langle \ldots \rangle$ denote average over λ distribution in the market. Taking now a uniform random distribution of the saving factor λ, $\rho(\lambda) = 1$ for $0 \leq \lambda < 1$, and assuming $D(\Delta) \sim \Delta^{-(1+\gamma)}$ for large Δ, we get

$$1 = 2\int d\lambda \; \lambda^{\gamma} = 2(1+\gamma)^{-1}, \tag{15}$$

giving $\gamma = 1$. No other value fits the above equation. This also indicates that the money distribution $P(m)$ in the market also follows a similar power law variation, $P(m) \sim m^{-(1+\nu)}$ and $\nu = \gamma$. We will now show in a more rigorous way that indeed the only stable solution corresponds to $\nu = 1$, as observed numerically [12, 13, 14].

3.2 Master equation and its analysis

We also develop a Boltzmann-like master equation for the time development of $P(m,t)$, the probability distribution of money in the market [25, 26]. We again consider the case $\epsilon_{ij} = \frac{1}{2}$ in (7) and (8) and rewrite them as

$$\begin{pmatrix} m_i \\ m_j \end{pmatrix}_{t+1} = \mathcal{A} \begin{pmatrix} m_i \\ m_j \end{pmatrix}_t \text{ where } \mathcal{A} = \begin{pmatrix} \mu_i^+ & \mu_j^- \\ \mu_i^- & \mu_j^+ \end{pmatrix}; \quad \mu^{\pm} = \frac{1}{2}(1 \pm \lambda). \tag{16}$$

Collecting the contributions from terms scattering in and subtracting those scattering out, we can write the master equation for $P(m,t)$ as

$$\frac{\partial P(m,t)}{\partial t} + P(m,t) = \langle \int dm_i \int dm_j \; P(m_i,t)P(m_j,t) \; \delta(\mu_i^+ m_i + \mu_j^- m_j - m) \rangle, \tag{17}$$

which in the steady state gives

$$P(m) = \langle \int dm_i \int dm_j \; P(m_i)P(m_j) \; \delta(\mu_i^+ m_i + \mu_j^- m_j - m) \rangle. \tag{18}$$

Assuming, $P(m) \sim m^{-(1+\nu)}$ for $m \to \infty$, we get [25, 26]

$$1 = \langle (\mu^+)^\nu + (\mu^-)^\nu \rangle \equiv \int \int d\mu^+ d\mu^- p(\mu^+) q(\mu^-) \left[(\mu^+)^\nu + (\mu^-)^\nu \right]. \tag{19}$$

Considering now the dominant terms ($\propto x^{-r}$ for $r > 0$, or $\propto \ln(1/x)$ for $r = 0$) in the $x \to 0$ limit of the integral $\int_0^\infty m^{(\nu+r)} P(m) \exp(-mx) dm$, we get from eqn. (19), after integrations, $1 = 2/(\nu + 1)$, giving finally $\nu = 1$ (details in Appendix).

4 Summary and Discussions

We have numerically simulated here ideal-gas like models of trading markets, where each agent is identified with a gas molecule and each trading as an elastic or money-conserving two-body collision. Unlike in the ideal gas, we introduce (quenched) saving propensity of the agents, distributed widely between the agents ($0 \leq \lambda < 1$). For quenched random variation of λ among the agents the system remarkably self-organizes to a critical Pareto distribution (1) of money with $\nu \simeq 1.0$ (Fig. 2). The exponent is quite robust: for savings distribution $\rho(\lambda) \sim |\lambda_0 - \lambda|^\alpha$, $\lambda_0 \neq 1$, one gets the same Pareto law with $\nu = 1$ (independent of λ_0 or α).

A master equation for $P(m,t)$, as in (17), for the original case (eqns. (7) and (8)) was first formulated for fixed λ (λ_i same for all i), in [20] and solved numerically. Later, a generalized master equation for the same, where λ is distributed, was formulated and solved in [22] and [25]. We show here that our analytic study clearly support the power-law for $P(m)$ with the exponent value $\nu = 1$ universally, as observed numerically earlier [12, 13, 14].

It may be noted that the trading market model we have talked about here has got some apparent limitations. The stochastic nature of trading assumed here in the trading market, through the random fraction ϵ in (6), is of course not very straightforward as agents apparently go for trading with some definite purpose (utility maximization of both money and commodity). We are however, looking only at the money transactions between the traders. In this sense, the income distribution we study here essentially corresponds to 'paper money', and not the 'real wealth'. However, even taking money and commodity together, one can argue (see [10]) for the same stochastic nature of the tradings, due to the absence of 'just pricing' and the effects of bargains in the market.

Apart from the observation that Gibbs (1901) and Pareto (1897) distributions fall in the same category and can appear naturally in the century-old and well-established kinetic theory of gas, that this model study indicates the appearance of self-organized criticality in the simplest (gas) model so far, when the stability effect of savings incorporated, is remarkable.

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A Alternative solution of the steady state master equation (18)

Let $S_r(x) = \int_0^\infty dm P(m) m^{\nu+r} \exp(-mx)$; $r \geq 0, x > 0$. If $P(m) = A/m^{1+\nu}$, then

$$\begin{aligned} S_r(x) &= A \int_0^\infty dm\; m^{r-1} \exp(-mx) \\ &\sim A\,\frac{x^{-r}}{r} \qquad \text{if } r > 0 \\ &\sim A \ln\left(\frac{1}{x}\right) \quad \text{if } r = 0. \end{aligned} \tag{20}$$

From eqn. (18), we can write

$$\begin{aligned} & S_r(x) = \\ & \langle \int_0^\infty dm_i \int_0^\infty dm_j\; P(m_i)P(m_j)(m_i\mu_i^+ + m_j\mu_j^-)^{\nu+r} \exp[-(m_i\mu_i^+ + m_j\mu_j^-)x] \rangle \\ \simeq & \int_0^\infty dm_i\; A m_i^{r-1} \langle \exp(-m_i\mu_i^+ x)\, (\mu_i^+)^{\nu+r} \rangle \left[\int_0^\infty dm_j\; P(m_j) \langle \exp(-m_j\mu_j^- x) \rangle \right] \\ + & \int_0^\infty dm_j\; A m_j^{r-1} \langle \exp(-m_j\mu_j^- x)\, (\mu_j^-)^{\nu+r} \rangle \left[\int_0^\infty dm_i\; P(m_i) \langle \exp(-m_i\mu_i^+ x) \rangle \right] \end{aligned} \tag{21}$$

or,

$$\begin{aligned} S_r(x) = & \int_{\frac{1}{2}}^1 d\mu_i^+\; p(\mu_i^+) \left(\int_0^\infty dm_i\; A m_i^{r-1} \exp(-m_i\mu_i^+ x) \right) (\mu_i^+)^{\nu+r} \\ & + \int_0^{\frac{1}{2}} d\mu_j^-\; q(\mu_j^-) \left(\int_0^\infty dm_j\; A m_j^{r-1} \exp(-m_j\mu_j^- x) \right) (\mu_j^-)^{\nu+r} \end{aligned} \tag{22}$$

since for small x, the terms in the square brackets in (21) approach unity. We can therefore rewrite (22) as

$$S_r(x) = 2\left[\int_{\frac{1}{2}}^{1} d\mu^+(\mu^+)^{\nu+r} S_r(x\mu^+) + \int_0^{\frac{1}{2}} d\mu^-(\mu^-)^{\nu+r} S_r(x\mu^-)\right]. \quad (23)$$

Using now the forms of $S_r(x)$ as in (20), and collecting terms of order x^{-r} (for $r > 0$) or of order $\ln(1/x)$ (for $r = 0$) from both sides of (23), we get (19).

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Kinetic Theory Models for the Distribution of Wealth: Power Law from Overlap of Exponentials

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Summary. Various multi-agent models of wealth distributions defined by microscopic laws regulating the trades, with or without a saving criterion, are reviewed. We discuss and clarify the equilibrium properties of the model with constant global saving propensity, resulting in Gamma distributions, and their equivalence to the Maxwell-Boltzmann kinetic energy distribution for a system of molecules in an effective number of dimensions D_λ, related to the saving propensity λ [M. Patriarca, A. Chakraborti, and K. Kaski, Phys. Rev. E 70 (2004) 016104]. We use these results to analyze the model in which the individual saving propensities of the agents are quenched random variables, and the tail of the equilibrium wealth distribution exhibits a Pareto law $f(x) \propto x^{-\alpha-1}$ with an exponent $\alpha = 1$ [A. Chatterjee, B. K. Chakrabarti, and S. S. Manna, Physica Scripta T106 (2003) 367]. Here, we show that the observed Pareto power law can be explained as arising from the overlap of the Maxwell-Boltzmann distributions associated to the various agents, which reach an equilibrium state characterized by their individual Gamma distributions. We also consider the influence of different types of saving propensity distributions on the equilibrium state.

1 Introduction

'A rich man is nothing but a poor man with money' — W. C. Fields.

If money makes the difference in this world, then it is perhaps wise to dwell on what money, wealth and income are, to study models for predicting the respective distributions, how they are divided among the population of a given country and among different countries. The most common definition of *money* suggests that money is the "Commodity accepted by general consent as medium of economics exchange" [1]. In fact money circulates from one economic agent (which can be an individual, firm, country, etc.) to another, thus facilitating trade. It is "something which all other goods or services are traded for" (for details see [2]) . Throughout history various commodities have been used as money, termed usually as "commodity money" which include rare seashells or beads, and cattle (such as cows in India). Since the 17th century the most common forms have been metal coins, paper notes, and book-keeping entries. However, this is not the only important point about money. It is worth recalling the four functions of money according to standard economic theory:

(i) to serve as a medium of exchange universally accepted in trade for goods and services
(ii) to act as a measure of value, making possible the determination of the prices and the calculation of costs, or profit and loss
(iii) to serve as a standard of deferred payments, i.e., a tool for the payment of debt or the unit in which loans are made and future transactions are fixed
(iv) to serve as a means of storing wealth not immediately required for use.

A main feature that emerges from these properties and that is relevant from the point of view of the present investigation is that money is the medium in which prices or values of all commodities as well as costs, profits, and transactions can be determined or expressed. As for the *wealth*, it usually refers to things that have economic utility (monetary value or value of exchange), or material goods or property. It also represents the abundance of objects of value (or riches) and the state of having accumulated these objects. For our purpose, it is important to bear in mind that wealth can be measured in terms of money. Finally, *income* is defined as "The amount of money or its equivalent received during a period of time in exchange for labor or services, from the sale of goods or property, or as profit from financial investments" [3]. Therefore, it is also a quantity which can be measured in terms of money (per unit time). Thus, money has a two-fold fundamental role, as (i) an exchange medium in economic transactions, and (ii) a unit of measure which allows one to quantify (movements of) any type of goods which would otherwise be ambiguous to estimate. The similarity with e.g., thermal energy (and thermal energy units) in physics is to be noticed. In fact, the description of the mutual transformations of apparently different forms of energy, such as heat, potential and kinetic energy, is made possible by the recognition of their equivalence and the corresponding use of a same unit. And it so happens that this same unit is also the traditional unit used for one of the forms of energy. For example, one could measure energy in all its forms, as actually done in some fields of physics, in degree Kelvin. Without the possibility of expressing different goods in terms of the same unit of measure, there simply would not be any quantitative approach to economy models, just as there would be no quantitative description of the transformation of the heat in motion and vice versa, without a common energy unit.

2 Multi-agent models for the distribution of wealth

In recent years several works have considered multi-agent models of a closed economy [4, 5, 6, 7, 8, 9, 10, 11, 14]. Despite their simplicity, these models predict a realistic shape of the wealth distribution, both in the low income part, usually described by a Boltzmann (exponential) distribution, as well in the tail, where a power law was observed a century ago by the Italian social economist Pareto [15]: the wealth of individuals in a stable economy follows the distribution, $F(x) \propto x^{-\alpha}$, where $F(x)$ is the upper cumulative distribution function, that is the number of people having wealth greater than or equal to x, and α is an exponent (known as the Pareto exponent) estimated to be between 1 and 2. In such models, N agents exchange a quantity x, that has sometimes been defined as wealth and other times as money. As explained in the introduction, here money must be interpreted all the goods that constitute the agents' wealth expressed in the same currency. To avoid confusion, in the following we will use only the term wealth. The states of agents are characterized by the wealths $\{x_n\}$, $n = 1, 2, \ldots, N$. The evolution of the system is then carried out according to a prescription, which defines a "trading rule" between agents. The evolution can be interpreted both as an actual time evolution or a Monte Carlo optimization procedure, aimed at finding the equilibrium distribution. At every time step two agents i and j are extracted at random and an amount of wealth Δx is exchanged between them,

$$\begin{aligned} x_i' &= x_i - \Delta x \,, \\ x_j' &= x_j + \Delta x \,. \end{aligned} \tag{1}$$

It can be noticed that in this way the quantity x is conserved during the single transactions, $x_i' + x_j' = x_i + x_j$. Here x_i' and x_j' are the agent wealths after the transaction has taken place. Several rules have been studied for the model.

2.1 Basic model without saving: Boltzmann distribution

In the first version of the model, so far unnoticed in later literature, the money difference Δx is assumed to have a constant value [4, 5, 6],

$$\Delta x = \Delta x_0 \;. \tag{2}$$

This rule, together with the constraint that transactions can take place only if $x_i' > 0$ and $x_j' > 0$, provides a Boltzmann distribution; see the curve for $\lambda = 0$

in Fig. 1. An equilibrium distribution with exponential tail is also obtained if Δx is a random fraction ϵ of the wealth of one of the two agents, $\Delta x = \epsilon x_i$ or $\Delta x = \epsilon x_j$. A trading rule based on the random redistribution of the sum of the wealths of the two agents has been introduced by Dragulescu and Yakovenko [7],

$$\begin{aligned} x_i' &= \epsilon(x_i + x_j)\,, \\ x_j' &= \bar{\epsilon}(x_i + x_j)\,, \end{aligned} \tag{3}$$

where ϵ is a random number uniformly distributed between 0 and 1 and $\bar{\epsilon}$ is the complementary fraction, i.e. $\epsilon + \bar{\epsilon} = 1$. Equations (3) are easily shown to correspond to the trading rule (1), with

$$\Delta x = \bar{\epsilon} x_i - \epsilon x_j \ . \tag{4}$$

In the following we will concentrate on the latter version of the model and its generalizations, though both the versions of the basic model defined by Eqs. (2) or (4) lead to the Boltzmann distribution,

$$f(x) = \frac{1}{\langle x \rangle} \exp\left(-\frac{x}{\langle x \rangle}\right) \,, \tag{5}$$

where the effective temperature of the system is just the average wealth $\langle x \rangle$ [4, 5, 6, 7]. The result (5) is found to be robust, in that it is largely independent of various factors. Namely, it is obtained for the various forms of Δx mentioned above, for pairwise as well as multi-agent interactions, for arbitrary initial conditions [8], and finally, for random or consecutive extraction of the interacting agents. The Boltzmann distribution thus obtained has been sometimes referred to as an "unfair distribution", in that it is characterized by a majority of poor agents and very few rich agents, as signaled in particular by a zero mode and by the exponential tail. The form of distribution (5) will be referred to as the Boltzmann distribution and is also known as Gibbs distribution.

2.2 Minimum investment model without saving

Despite the Boltzmann distribution is robust respect to the variation of several parameters, the way it depends on the details of the trading rule is subtle. For instance, in the model studied in [9], the equilibrium distribution can have a very different shape. In that model it is assumed that both economic agents i and j invest the same amount x_{min}, which is taken as the minimum wealth of the two agents, $x_{min} = \min\{x_i, x_j\}$. The wealths after the trade are $x_i' = x_i + \Delta x$ and $x_j' = x_j - \Delta x$, where $\Delta x = (2\epsilon - 1)x_{min}$.

We note that once an agent has lost all his wealth, he is unable to trade because x_{min} has become zero. Thus, a trader is effectively driven out of the market once he loses all his wealth. In this way, after a sufficient number of transactions only one trader survives in the market with the entire amount of wealth, whereas the rest of the traders have zero wealth.

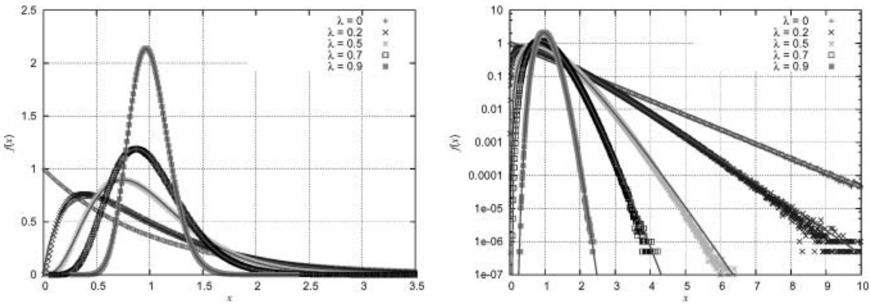

Fig. 1. Probability density for wealth x. The curve for $\lambda = 0$ is the Boltzmann function $f(x) = \langle x \rangle^{-1} \exp(-x/\langle x \rangle)$ for the basic model of Sec. 2.1. The other curves correspond to a global saving propensity $\lambda > 0$, see Sec. 2.3.

2.3 Model with constant global saving propensity: Gamma distribution

A step toward generalizing the basic model and making it more realistic is the introduction of a saving criterion regulating the trading dynamics. This can be achieved defining a saving propensity $0 < \lambda < 1$, that represents the fraction of wealth saved — and not reshuffled — during a transaction. The dynamics of the model is as follows [8, 9]:

$$\begin{aligned} x_i' &= \lambda x_i + \epsilon(1-\lambda)(x_i + x_j)\,, \\ x_j' &= \lambda x_j + \bar{\epsilon}(1-\lambda)(x_i + x_j)\,, \end{aligned} \tag{6}$$

with $\bar{\epsilon} = 1 - \epsilon$, corresponding to a Δx in Eq. (1) given by

$$\Delta x = (1-\lambda)[\bar{\epsilon} x_i - \epsilon x_j]\,. \tag{7}$$

This model leads to a qualitatively different equilibrium distribution. In particular, it has a mode $x_m > 0$ and a zero limit for small x, i.e. $f(x \to 0) \to 0$, see Fig. 1. The functional form of such a distribution has been conjectured to be a Gamma distribution on the base of an analogy with the kinetic theory of gases, which is consistent with the excellent fitting provided to numerical data [16, 17]. Its form can be conveniently written by defining the effective dimension D_λ as [17]

$$\frac{D_\lambda}{2} = 1 + \frac{3\lambda}{1-\lambda} = \frac{1+2\lambda}{1-\lambda}\,. \tag{8}$$

According to the equipartition theorem, one can introduce a corresponding temperature defined by the relation $\langle x \rangle = D_\lambda T_\lambda / 2$, i.e.

$$T_\lambda = \frac{2\,\langle x \rangle}{D_\lambda} = \frac{1-\lambda}{1+2\lambda}\,\langle x \rangle\ . \tag{9}$$

Then the distribution for the reduced variable $\xi = x/T_\lambda$ reads

$$f(\xi) = \frac{1}{\Gamma(D_\lambda/2)} \xi^{D_\lambda/2-1} \exp(-\xi) \equiv \gamma_{D_\lambda/2}(\xi) , \tag{10}$$

i.e. a Gamma distribution of order $D_\lambda/2$. For integer or half-integer values of $n = D_\lambda/2$, this function is identical to the equilibrium Maxwell-Boltzmann distribution of the kinetic energy for a system of molecules in thermal equilibrium at temperature T_λ in a D_λ-dimensional space (see Appendix A for a detailed derivation). For $D_\lambda = 2$, the Gamma distribution reduces to the Boltzmann distribution. This extension of the equivalence between kinetic theory and closed economy models to values $0 \leq \lambda < 1$ is summarized in Table 1. This equivalence between a multi-agent system with a saving propensity

Table 1. Analogy between kinetic and multi-agent model

	Kinetic model	Economic model
variable	K (kinetic energy)	x (wealth)
units	N particles	N agents
interaction	collisions	trades
dimension	integer D	real number D_λ [see Eq. (8)]
temperature	$k_B T = 2\langle K \rangle / D$	$T_\lambda = 2\langle x \rangle / D_\lambda$
reduced variable	$\xi = K/k_B T$	$\xi = x/T_\lambda$
equilibrium distribution	$f(\xi) = \gamma_{D/2}(\xi)$	$f(\xi) = \gamma_{D_\lambda/2}(\xi)$

$0 \leq \lambda < 1$ and an N-particle system in a space with effective dimension $D_\lambda \geq 2$ was originally suggested by simple considerations about the kinetics of a collision between two molecules. In fact, for kinematical reasons during such an event only a fraction of the total kinetic energy can be exchanged. Such a fraction is of the order of $1 - \lambda \approx 1/D$, to be compared with the expression $1 - \lambda = 3/(D/2 + 2)$ derived from Eq. (8) [17]. While λ varies between 0 and 1, the parameter D_λ monotonously increases from 2 to ∞, and the effective temperature T_λ correspondingly decreases from $\langle x \rangle$ to zero; see Fig. 2. It is to be noticed that according to the equipartition theorem only in $D_\lambda = 2$ effective dimensions ($\lambda = 0$) the temperature coincides with the average value $\langle x \rangle$, $T_\lambda = 2\langle x \rangle / 2 \equiv \langle x \rangle$, as originally found in the basic model [4, 5, 6, 7]. In its general meaning, temperature represents rather an estimate of the fluctuation of the quantity x around its average value. The equipartition theorem always gives a temperature smaller than the average value $\langle x \rangle$ for a number of dimensions larger than two. In the present case, Eqs. (8) or (9) show that this happens for any $\lambda > 0$.

The dependence of the fluctuations of the quantity x on the saving propensity λ was studied in [8]. In particular, the decrease in the amplitude of the fluctuations with increasing λ is shown in Fig. 3.

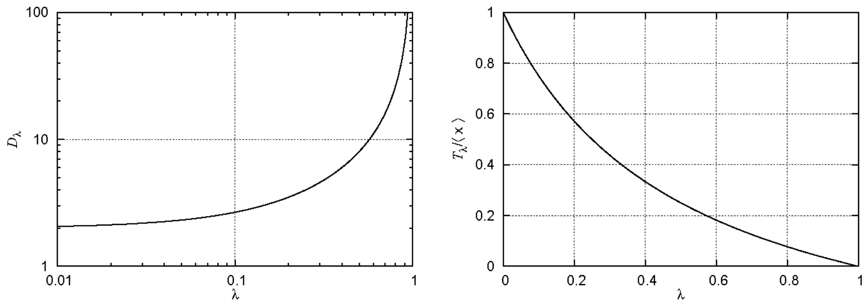

Fig. 2. Effective dimension D_λ, Eq. (8), and temperature, Eq. (9), as a function of saving propensity λ.

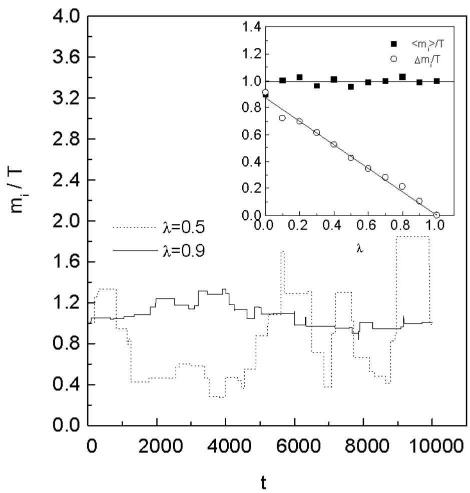

Fig. 3. Reproduced from [8] (only here $m \equiv x$ while $T \equiv 1$ is a constant). The continuous and the dotted curves are the wealths of two agents with $\lambda = 0.9$ and $\lambda = 0.5$ respectively: notice the larger fluctuations in correspondence of the smaller λ. The inset shows that $\Delta m \equiv \Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$ decreases with λ.

The fact that in general the market temperature T_λ decreases with λ means smaller fluctuations of x during trades, consistently with the saving criterion, i.e. with a $\lambda > 0$. One can notice that in fact $T_\lambda = (1 - \lambda) \langle x \rangle / (1 + 2\lambda) \approx (1 - \lambda) \langle x \rangle$ is of the order of the average amount of wealth exchanged during a single interaction between agents, see Eqs. (6).

2.4 Model with individual saving propensities: Pareto tail

In order to take into account the natural diversity between various agents, a model with individual propensities $\{\lambda_i\}$ as quenched random variables was studied in [10, 11]. The dynamics of this model is the following:

$$\begin{aligned} x'_i &= \lambda_i x_i + \epsilon[(1-\lambda_i)x_i + (1-\lambda_j)x_j]\,, \\ x'_j &= \lambda_j x_j + \bar{\epsilon}[(1-\lambda_i)x_i + (1-\lambda_j)x_j]\,, \end{aligned} \tag{11}$$

where, as above, $\bar{\epsilon} = 1 - \epsilon$. This corresponds to a Δx in Eq. (1) given by

$$\Delta x = \bar{\epsilon}(1-\lambda_i)x_i - \epsilon(1-\lambda_j)x_j\,. \tag{12}$$

Besides the use of this trading rule, a further prescription is given in the model, namely an average over the initial random assignment of the individual saving propensities: With a given configuration $\{\lambda_i\}$, the system is evolved until equilibrium is reached, then a new set of random saving propensities $\{\lambda'_i\}$ is extracted and reassigned to all agents, and the whole procedure is repeated many times. As a result of the average over the equilibrium distributions corresponding to the various $\{\lambda_i\}$ configurations, one obtains a distribution with a power law tail, $f(x) \propto x^{-\alpha-1}$, where the Pareto exponent has the value $\alpha = 1$. This value of the exponent has been predicted by various theoretical approaches to the modeling of multi-agent systems [12, 13, 14].

3 Further analysis of the model with individual saving propensities

On one hand, the model with individual saving propensities relaxes toward a power law distribution — with the prescription mentioned above to average the distribution over many equilibrium states corresponding to different configurations $\{\lambda_i\}$. On the other hand, the models with a global saving propensity $\lambda > 0$ and the basic model with $\lambda = 0$, despite being particular cases of the general model with individual saving propensities, relax toward very different distributions, namely a Gamma and a Boltzmann distribution, respectively. In this section we show that this difference can be reconciled by illustrating how the observed power law is due to the superposition of different distributions with exponential tails corresponding to subsystems of agents with the same value of λ.

3.1 The x-λ correlation

A key point which explains many of the features of the model and of the corresponding equilibrium state is a well-defined correlation between average wealth and saving propensity, which has been unnoticed so far in the literature

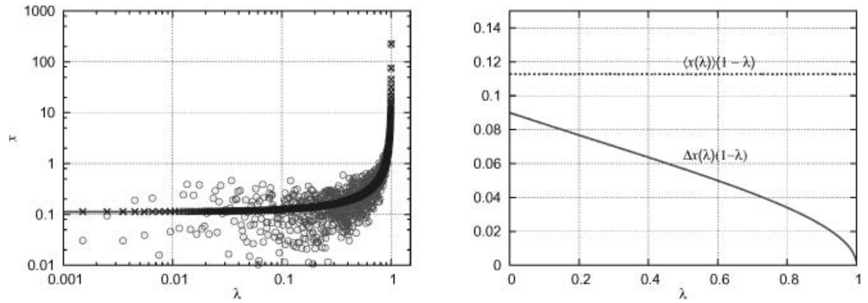

Fig. 4. Equilibrium state in the x-λ plane after $t = 10^9$ trades for a system of $N = 1000$ agents. Left: Circles ($\circ$) represent agents, crosses ($\times$) represent the average wealth $\langle x(\lambda)\rangle$, the continuous line is the function $\langle x(\lambda)\rangle = \kappa/(1-\lambda)$, with $\kappa = 0.1128$. Right: The product $\langle x(\lambda)\rangle\,(1-\lambda)$ (dotted line) is constant, in agreement with Eq. (14). The product $\Delta x(\lambda)(1-\lambda)$ (continuous line), where $\Delta x(\lambda)$ is the standard deviation, shows that $\Delta x(\lambda)$ grows slower than $\langle x(\lambda)\rangle$.

[18]. The existence of such a correlation can be related to the origin of the power law and its cut-off at high values of x. It also explains the paradox according to which a very rich agent may lose all his wealth when interacting with poor agents, as a consequence of the stochastic character of the trading rule defined by Eq. (11). Figure 4 shows the equilibrium state for a system with $N = 1000$ agents after $t = 10^9$ trades. Each agent is represented by a circle ($\circ$) in the wealth-saving propensity x_i-λ_i plane. The correlation between wealth x and saving propensity λ becomes very high at large values of x and λ. Namely, one observes that the average wealth $\langle x(\lambda)\rangle$ [crosses ($\times$) in Fig. 4] diverges for $\lambda \to 1$. The average $\langle x(\lambda)\rangle$ was obtained by computing the probability density $f(x, \lambda)$ in the x-λ plane (normalized so that $\int dx\, d\lambda\, f(x, \lambda) = 1$) and averaging for a fixed value of λ,

$$\langle x(\lambda)\rangle = \int dx\, x f(x, \lambda)\,. \tag{13}$$

The observed correlation naturally follows from the structure of the trade dynamics (11). We remind that initially every agent has the same wealth $x_0 = \langle x\rangle$. During the initial phase of the evolution, when all agents have approximately the same wealth $\langle x\rangle$, an agent i with a large saving propensity λ_i can save more — on average — and therefore accumulate more. Afterwards, the agent i will continue to enter trades by investing only a small fraction $(1-\lambda_i)x_i$ of his wealth x_i in the trade. Even when interacting with an agent j, with a smaller wealth $x_j < x_i$, agent i may still be successful in the trading, since agent j may have also a smaller saving propensity λ_j, so that the traded fraction of wealth $(1-\lambda_j)x_j$ is comparable with or even larger than $(1-\lambda_i)x_i$. Trading by agent i will very probably be successful on average with all agents j with a λ_j such that $(1-\lambda_j)x_j$ is smaller than $(1-\lambda_i)x_i$. These considerations

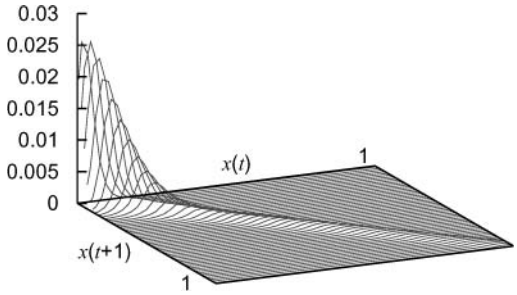

Fig. 5. Histogram of the wealths $x' = x(t+1)$ after a trade versus $x = x(t)$ before the trade for all agents and trades, in a system with $N = 1000$ agents and 10^9 trades. The distribution is narrower for large x (rich agents), implying that it is unlikely that a rich agent becomes poor within a single trade.

suggest that agent i will reach equilibrium (and his maximum possible wealth) when $(1-\lambda_i)x_i = \kappa \approx \langle(1-\lambda)x\rangle$. The ratio between the constant κ and the average $\langle(1-\lambda)x\rangle = \sum_j(1-\lambda_j)x_j/N$ is actually found to be of the order of magnitude of 10. The formula

$$\langle x(\lambda_i)\rangle = \frac{\kappa}{1-\lambda_i}, \tag{14}$$

however, shown as a continuous line in Fig. 4, provides an excellent interpolation of the average wealth $\langle x(\lambda)\rangle$ (also shown in the figure) computed numerically.

3.2 Variation of a single agent's wealth

The stability of the asymptotic state is also shown by the histogram in Fig. 5 of the wealths $x' \equiv x(t+1)$ after a trade versus $x \equiv x(t)$ before it defined in Eqs. (1). The distribution is narrower at larger values of x than at smaller ones, implying that the probability that an agent i will undergo a large relative variation of his wealth x_i within a single trade is much higher for poor agents. The situation at small x (corresponding to agents with smaller saving propensities) is instead more similar to that of the trading rule without saving ($\lambda = 0$), Eqs. (6): the distribution is broader, indicating a higher probability of a large wealth reshuffling during a trade.

3.3 Power laws at small x and t scales

A peculiarity of the model with individual saving propensities is noteworthy. On one hand, in the procedure used to obtain a power law in [10] all

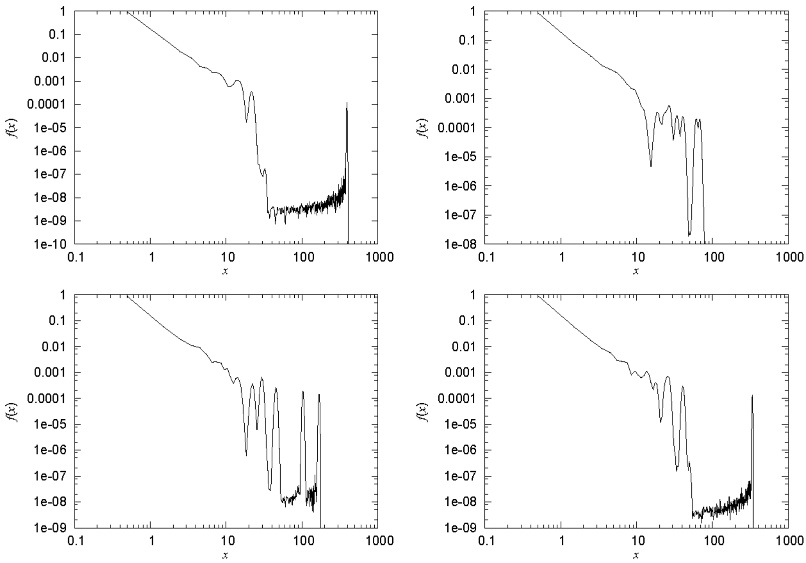

Fig. 6. The equilibrium configurations corresponding to four different random saving propensity sets $\{\lambda_i\}$, for a system with $N = 1000$ agents, differ especially at large x where the distribution deviates from a power law.

agents are equivalent to each other: they enter the dynamical evolution law on an equal footing, their saving propensities are reassigned randomly with the same uniform distribution between 0 and 1, and even their initial conditions can be set to be all equal to each other, $x_i = \langle x \rangle$, without loss of generality. Therefore the various equilibrium configurations, corresponding to different sets $\{\lambda_i\}$, are expected to be statistically equivalent to each other, in the sense that one should be able to obtain the power law distribution by a simple ensemble average for any fixed configuration of saving propensities $\{\lambda_i\}$, if the number of agents N is large enough. On the other hand, an averaging procedure over several $\{\lambda_i\}$ configurations is in practice necessary to obtain a power law distribution.

In order to understand this apparent paradox, we checked how the equilibrium distributions, corresponding to a given set of saving propensities $\{\lambda_i\}$, look like. One finds that every configuration $\{\lambda_i\}$ produces equilibrium distributions very different from each other; see Fig. 6 for some examples. The structures observed are very different from power laws, with well resolved peaks at large x. Only when an average over different $\{\lambda_i\}$ configurations is carried out, one obtains a smooth power law with Pareto exponent $\alpha = 1$. These same figures show, however, that for a given configuration $\{\lambda_i\}$ a power law is actually observed at small values of x. Another related interesting feature of simulations employing a single saving propensity configuration $\{\lambda_i\}$ is

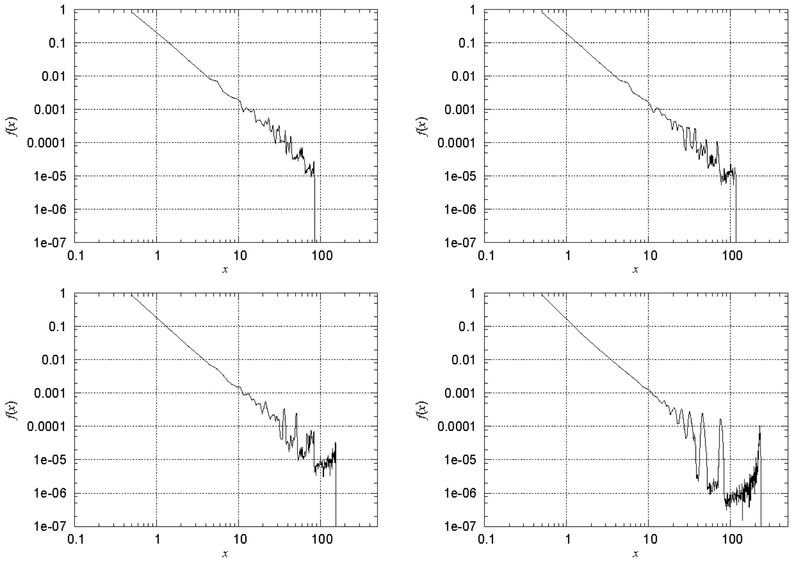

Fig. 7. Time evolution of the x distribution of a system with $N = 1000$ agents: $t = 2 \times 10^6$ (top left), 3×10^6 (top-right), 5×10^6 (bottom-left), and 2×10^7 trades (bottom-right). The distribution looks as a power law at small times, but develops into a structured distribution, maintaining a power law shape only at small x.

that a power law distribution is found only on a limited time scale, while it disappears partly or totally at equilibrium. Thus also in the time dimension one surprisingly finds a distribution much more similar to a power law at a smaller rather than larger scale. This is shown in the example in Fig. 7, where the distributions of a system of 1000 agents at four different times are compared to each other. These features suggest that the power law is intrinsically built into the dynamical laws of the model but that, for some reasons, it fades away at large x and t scales. The x-λ correlation discussed above in Sec. 3.1 can provide an explanation of these features, both for those in the x and in the time dimension, as discussed below.

3.4 Origin of the power law

The peculiar features illustrated above, the necessity of averaging over different configurations $\{\lambda_i\}$ as done in [10] to obtain a power law distribution, as well as the power law distribution itself, are here explained in terms of equilibrium states of suitably defined subsystems and the x-λ correlation illustrated above. This may seem odd since at first sight the averaging procedure of [10] defines a nonequilibrium process, the system being brought out of equilibrium from time to time by the reassignment of the saving propensities.

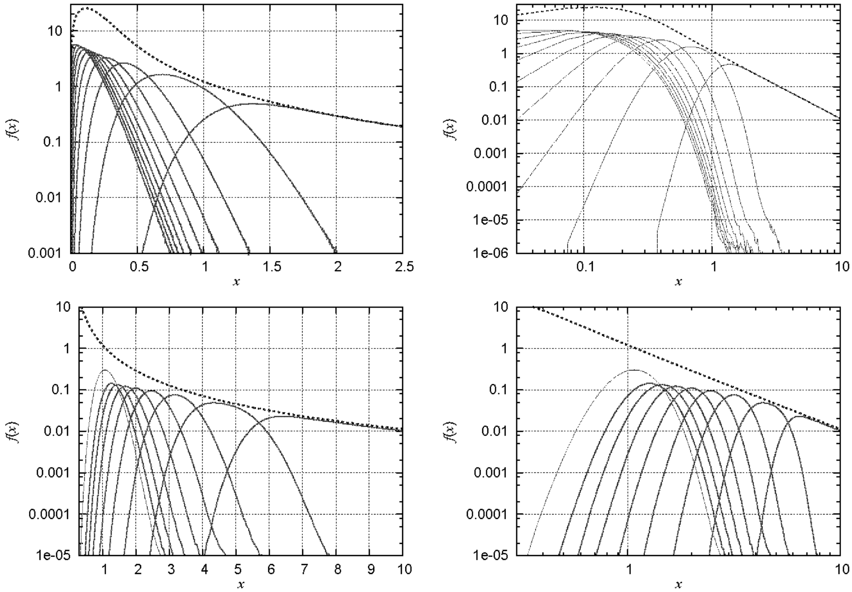

Fig. 8. Semi-log and log plots of partial distributions (continuous curves) and the resulting overlap (dotted line). Above: Partial distributions from the 10 intervals of width $\Delta\lambda = 0.1$ of the total λ range (0,1). Below: The last partial distribution (with a power law tail) above, corresponding to the interval $\lambda = (0.9, 1.0)$, has been further resolved into ten partial distributions for the sub-intervals of width $\Delta\lambda = 0.01$.

Correspondingly, one may attribute the power law to the underlying dynamical process, as it is often the case in nonequilibrium models (e.g. models of markets on networks [19]).

However, if one considers the partial distributions of agents with a certain value of λ, one finds an unexpected result. For numerical reasons we consider the subsets made up of those agents with saving propensity λ within a window $\Delta\lambda$ around a given value λ. Figure 8 (upper row) shows the partial distributions (continuous lines) of the ten subsystems obtained by dividing the λ range (0,1) into ten slices of width $\Delta\lambda = 0.1$ and average values 0.05,…,0.95 (curves from left to right respectively). Most of the partial distributions have an exponential tail, and only when summed up their overlap (dotted line) reproduces a power law. It can be noticed that the last partial distribution, corresponding to the interval $\lambda = (0.9, 1.0)$, is not of exponential form, but rather presents a power law tail, which overlaps with the total distribution at large x. However, its power law form is due only to the low resolution in λ employed. In fact even this partial distribution can in turn be shown to be given by the superposition of exponential tails. By increasing the resolution in λ, i.e. using a smaller interval $\Delta\lambda = 0.01$ to further resolve the interval

$\lambda = (0.9, 1.0)$ into subintervals with average values $\lambda = 0.905, \ldots, 0.995$, one obtains the partial distributions shown in the lower row of Fig. 8. It is to be noticed that also in this case the last partial distribution corresponding to the interval $\lambda = (0.99, 1.00)$ has a power law tail. The procedure can then in principle be reiterated to resolve also this partial distribution by increasing the resolution in λ.

These facts also explain the origin of the peaks visible at large x in the plots in Figs. 6 and 7. Due to the high wealth-saving propensity correlation at large values of x, see Fig. 4, these peaks are due to agents with high λ. The reason why these agents give rise to resolved peaks instead of contributing to extending the power law tail is that the partial distributions (i.e. the average values) of single agents get farther and farther from each other for $\lambda \to 1$, while the corresponding widths do not grow enough to ensure the overlap of the distributions of neighbor agents in λ-space. Eventually, each agent (or cluster of agents) at high values of x will be resolved as an isolated peak against the background of the total distribution. In greater detail, one finds that the average value $\langle x(\lambda) \rangle$ diverges for $\lambda \to 1$ as $1/(1-\lambda)$, as shown in Fig. 4. This implies that also the distance between two generic consecutive agents increases: if agents are labeled from $i = 1$ to $i = N$ in order of increasing λ ($\lambda_1 < \cdots < \lambda_N$) and the λ distribution is uniform, then $\Delta\lambda = \lambda_{i+1} - \lambda_i =$ constant. The distance between the average positions of the partial distributions of two consecutive agents is, from Eq. (14),

$$\delta\langle x(\lambda) \rangle = \langle x(\lambda + \Delta\lambda) \rangle - \langle x(\lambda) \rangle \approx \frac{\partial \langle x(\lambda) \rangle}{\partial \lambda} \Delta\lambda \approx \frac{\kappa \Delta\lambda}{(1-\lambda)^2}, \qquad (15)$$

where κ is a constant. Thus $\delta\langle x(\lambda) \rangle$ diverges even faster than $\langle x(\lambda) \rangle$. At the same time, the width of the partial distribution $\Delta x(\lambda)$, here estimated as $\Delta x(\lambda) = \sqrt{\langle x^2(\lambda) \rangle - \langle x(\lambda) \rangle^2}$, grows slower than $\langle x(\lambda) \rangle$, i.e. for $\lambda \to 1$ the ratio $\Delta x(\lambda) / \langle x(\lambda) \rangle \to 0$; see Fig. 4 (right). The breaking of the power law and the appearance of the isolated peaks takes place at a cutoff x_c where the distance $\delta\langle x(\lambda) \rangle$ between the peaks corresponding to consecutive agents i and $i+1$ becomes comparable with the peak width $\Delta x(\lambda)$.

Also the origin of the peculiarities in the time evolution of the distribution function, mentioned in Sec. 3.3, can now be explained easily. In order to reach the asymptotic equilibrium state, agents can rely on an income flux which is on average proportional to $x_i(1-\lambda_i)$. At the beginning, when all agents have the same wealth $x_i = x_0$, this quantity is smaller for agents with a larger λ_i; and with this smaller flux agents with large λ_i have to reach their higher asymptotic value $\propto 1/(1-\lambda_i)$. As a consequence, the relaxation time of an agent is larger for larger λ, a result already found in the numerical simulations of the multi-agent model with fixed global saving propensity (see Fig. 2 in [8]). Correspondingly, partial distributions of rich agents will reach their asymptotic form later (last frame in Fig. 7), while, at intermediate times, their

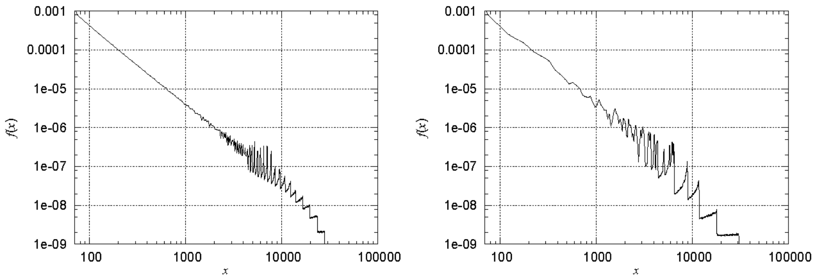

Fig. 9. Wealth distribution of a system of $N = 10^6$ agents after 10^{12} trades: the uniform λ distribution Eq. (16) produces a smoother distribution with a power law shape extending to higher x (left) than for a random λ distribution (right).

distributions will be spread at smaller values of x, contributing to smoothen the total distribution (first frame in Fig. 7).

It is also possible to explain why the averaging procedure of [10] is successful in producing a power law distribution. Averaging over different configurations $\{\lambda_i\}$ is equivalent to simulate a very dense distribution in λ — which has large relaxation time and number of agents — with an affordable number of agents and computer time. However, the procedure is not needed in principle, since the power law can be obtained also when a single configuration with a proper density in λ-space is used.

3.5 Checking different λ distributions

A practical way to avoid the appearance of the peaks at large x and obtain a distribution closer to a power law is to increase the density of agents, especially at values of λ close to 1. In a random extraction of $\{\lambda_i\}$, it is natural that consecutive values of λ_i will not be equally spaced. Even small differences will be amplified at large x and will result in the appearance of peaks. A deterministic assignment of the λ, e.g. a uniform distribution achieved through the following assignment,

$$\lambda_i = \frac{i}{N}, \qquad i = 0, N-1, \tag{16}$$

is a uniform distribution of λ in the interval [0,1) and will generate a smoother distribution of x. The comparison of the results for this distribution with those for a random distribution of λ is done in Fig. 9 (notice the high value of N). In the uniform case not only the power law extends to higher values of x but also that the distribution of peaks at large x is globally smoother, in the sense that on average the single peaks follow a power law better.

4 Conclusions

We have reviewed some multi-agent models for the distribution of wealth, in which wealth is exchanged at random in the presence of saving quantified by the saving propensity λ. We have shown how a distribution of λ generates a power law distribution of wealth through the superposition of Gamma distributions corresponding to particular subsets of agents. The physical picture for the model with individual saving propensities is thus more similar to that of the model with a constant global saving propensity than it may seem at first sight. In fact any subset of agents with the same value of the saving propensity λ equilibrates in a way similar to agents in the model with a global saving propensity, i.e. leading to a wealth distribution with an exponential tail. Correspondingly we have shown that both the noise in the power law tail and the cutoff in the power law depend on the coarseness of the λ distribution. This extends the analogy between economic and gas-like systems beyond the case of a global $\lambda \geq 0$, characterized by a Maxwell-Boltzmann distribution, to uniform continuous distributions in λ that span the whole interval $\lambda \in [0, 1)$.

Acknowledgment

This work was partially supported by the Academy of Finland, Research Centre for Computational Science and Engineering project nr. 44897 (Finnish Centre for Excellence Program 2000-2005). The work at Brookhaven National Laboratory was carried out under Contract nr. DE-AC02-98CH10886, Division of Material Science, U.S. Department of Energy.

A Maxwell-Boltzmann distribution in *D* dimensions

Here we show that for integer or half-integer values of the parameter n the Gamma distribution

$$\gamma_n(\xi) = \Gamma(n)^{-1} \, \xi^{n-1} \exp(-\xi) \,, \tag{17}$$

where $\Gamma(n)$ is the Gamma function, represents the distribution of the rescaled kinetic energy $\xi = K/T$ for a classical mechanical system in $D = 2n$ dimensions. In this section, T represents the absolute temperature of the system multiplied by the Boltzmann constant k_B.

We start from a system Hamiltonian of the form

$$H(\mathbf{P}, \mathbf{Q}) = \frac{1}{2} \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{m_i} + V(\mathbf{Q}) \,, \tag{18}$$

where $\mathbf{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_N\}$ and $\mathbf{Q} = \{\mathbf{q}_1, \dots, \mathbf{q}_N\}$ are the momentum and position vectors of the N particles, while $V(\mathbf{Q})$ is the potential energy contribution to the total energy. For systems of this type, in which the total energy factorizes as a sum of kinetic and potential contributions, the normalized probability distribution in momentum space is simply $f(\mathbf{P}) = \prod_i (2\pi m_i T)^{-D/2} \exp(-\mathbf{p}_i{}^2/2m_i T)$. Thus, since the kinetic energy distribution factorizes as a sum of single particle contributions, the probability density factorizes as a product of single particle densities, each one of the form

$$f(\mathbf{p}) = \frac{1}{(2\pi m T)^{D/2}} \exp\left(-\frac{\mathbf{p}^2}{2mT}\right) , \tag{19}$$

where $\mathbf{p} = (p_1, \dots, p_D)$ is the momentum of a generic particle. It is convenient to introduce the momentum modulus p of a particle in D dimensions,

$$p^2 \equiv \mathbf{p}^2 = \sum_{k=1}^{D} p_k^2 , \tag{20}$$

where the p_k's are the Cartesian components, since the distribution (19) depends only on $p \equiv \sqrt{\mathbf{p}^2}$. One can then integrate the distribution over the $D-1$ angular variables to obtain the momentum modulus distribution function, with the help of the formula for the surface of a hypersphere of radius p in D dimensions,

$$S_D(p) = \frac{2\pi^{D/2}}{\Gamma(D/2)} p^{D-1} . \tag{21}$$

One obtains

$$f(p) = S_D(p)\, f(\mathbf{p}) = \frac{2}{\Gamma(D/2)(2mT)^{D/2}} p^{D-1} \exp\left(-\frac{p^2}{2mT}\right) . \tag{22}$$

The corresponding distribution for the kinetic energy $K = p^2/2m$ is therefore

$$f(K) = \left[\frac{dp}{dK} f(p)\right]_{p=\sqrt{2mK}} = \frac{1}{\Gamma(D/2)T} \left(\frac{K}{T}\right)^{D/2-1} \exp\left(-\frac{K}{T}\right) . \tag{23}$$

Comparison with the Gamma distribution, Eq. (17), shows that the Maxwell-Boltzmann kinetic energy distribution in D dimensions can be expressed as

$$f(K) = T^{-1} \gamma_{D/2}(K/T) . \tag{24}$$

The distribution for the rescaled kinetic energy,

$$\xi = K/T , \tag{25}$$

is just the Gamma distribution of order $D/2$,

$$f(\xi) = \left[\frac{dK}{d\xi} f(K)\right]_{K=\xi T} = \frac{1}{\Gamma(D/2)} \xi^{D/2-1} \exp(-\xi) \equiv \gamma_{D/2}(\xi) . \tag{26}$$

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Detailed Simulation Results for Some Wealth Distribution Models in Econophysics

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Summary. In this paper we present detailed simulation results on the wealth distribution model with quenched saving propensities. Unlike other wealth distribution models where the saving propensities are either zero or constant, this model is not found to be ergodic and self-averaging. The wealth distribution statistics with a single realization of quenched disorder is observed to be significantly different in nature from that of the statistics averaged over a large number of independent quenched configurations. The peculiarities in the single realization statistics refuses to vanish irrespective of whatever large sample size is used. This implies that previously observed Pareto law is essentially a convolution of the single member distributions.

In a society different members possess different amounts of wealth. Individual members often make economic transactions with other members of the society. Therefore in general the wealth of a member fluctuates with time and this is true for all other members of the society as well. Over a reasonably lengthy time interval of observation, which is small compared to the inherent time scales of the economic society this situation may be looked upon as a stationary state which implies that statistical properties like the individual wealth distribution, mean wealth, its fluctuation etc. are independent of time.

More than a century before, Pareto observed that the individual wealth (m) distribution in a society is characterized by a power-law tail like: $P(m) \sim m^{-(1+\nu)}$ and predicted a value for the constant $\nu \approx 1$, known as the Pareto exponent [1]. Very recently, i.e., over the last few years, the wealth distribution in a society has attracted renewed interests in the context of the study of *Econophysics* and various models have been proposed and studied. A number of analyses have also been done on the real-world wealth distribution data in different countries [2, 3, 4]. All these recent data indeed show that Pareto like power-law tails do exist in the wealth distributions in the large wealth regime but with different values of the Pareto exponent ranging from $\nu = 1$ to 3. It has also been observed that only a small fraction of very rich members

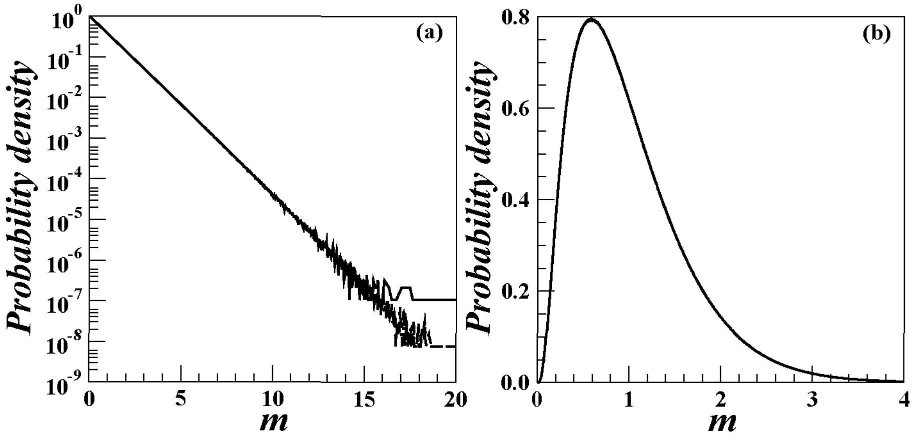

Fig. 1. The three probability densities of wealth distribution, namely $\mathrm{Prob}_1(m)$ (solid line), $\mathrm{Prob}_2(m)$ (dashed line) and $\mathrm{Prob}(m)$ (dot-dashed line) are plotted with wealth m for $N = 256$ in (a) for the DY model and in (b) for the CC model for $\lambda = 0.35$. The excellent overlapping of all three curves indicate that both the DY and CC models are ergodic as well as self averaging.

actually contribute to the Pareto behavior whereas the middle and the low wealth individuals follow either exponential or log-normal distributions.

In this paper we report our detailed simulation results on the three recent models of wealth distribution. The three models are: (i) the model of Drăgulescu and Yakovenko (DY) [5] which gives an exponential decay of the wealth distribution, (ii) the model of Chakraborti and Chakrabarti (CC) [6] with a fixed saving propensity giving a Gamma function for the wealth distribution and (iii) the model of Chatterjee, Chakrabarti and Manna (CCM) [7] with a distribution of quenched individual saving propensities giving a Pareto law for the wealth distribution.

All these three models have some common features. The society consists of a group of N individuals, each has a wealth $m_i(t), i = 1, N$. The wealth distribution $\{m_i(t)\}$ dynamically evolves with time following the pairwise conservative money shuffling method of economic transactions. Randomly selected pairs of individuals make economic transactions one after another in a time sequence and thus the wealth distribution changes with time. For example, let two randomly selected individuals i and j, $(i \neq j)$ have wealths m_i and m_j. They make transactions by a random bi-partitioning of their total wealth $m_i + m_j$ and then receiving one part each randomly:

$$\begin{aligned} m_i(t+1) &= \epsilon(t)(m_i(t) + m_j(t)) \\ m_j(t+1) &= (1 - \epsilon(t))(m_i(t) + m_j(t)). \end{aligned} \tag{1}$$

Here time t is simply the number of transactions and $\epsilon(t)$ is the t-th random fraction with uniform distribution drawn for the t-th transaction.

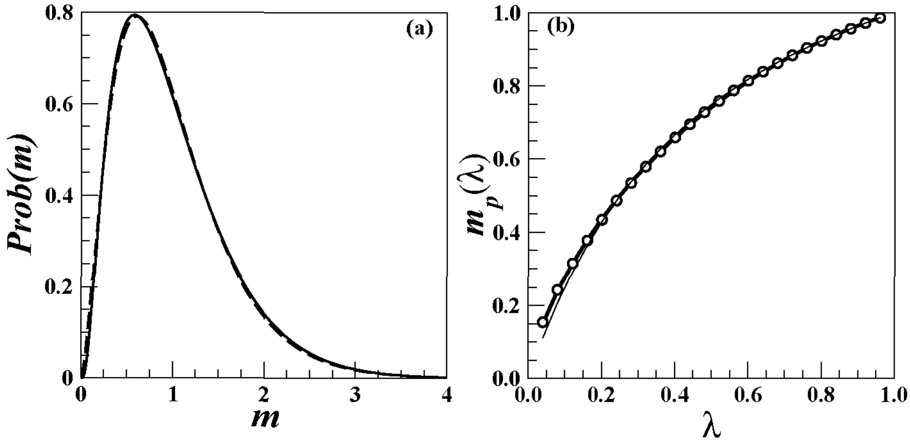

Fig. 2. For the CC model with $N = 256$ and $\lambda = 0.35$ these plots show the functional fits of the wealth distribution in (a) and the variation of the most probable wealth $m_p(\lambda)$ in (b). In (a) the simulation data of Prob(m) is shown by the solid black line where as the fitted Gamma function of Eqn. (5) is shown by the dashed line. In (b) the $m_p(\lambda)$ data for 24 different λ values denoted by circles is fitted to the Gamma function given in Eqn. (6) (solid line). The thin line is a comparison with the $m_p(\lambda)$ values obtained from the analytical expression of $a(\lambda)$ and $b(\lambda)$ in [10].

In all three models the system dynamically evolves to a stationary state which is characterized by a time independent probability distribution Prob(m) of wealths irrespective of the details of the initial distribution of wealths to start with. Typically in all our simulations a fixed amount of wealth is assigned to all members of the society, i.e. $\text{Prob}(m, t = 0) = \delta(m - \langle m \rangle)$. The model described so far is precisely the DY model in [5]. The stationary state wealth distribution for this model is [5, 8, 9]:

$$\text{Prob}(m) = \frac{1}{\langle m \rangle} \exp(-m/\langle m \rangle). \tag{2}$$

Typically $\langle m \rangle$ is chosen to be unity without any loss of generality.

A fixed saving propensity is introduced in the CC model [6]. During the economic transaction each member saves a constant λ fraction of his wealth. The total sum of the remaining wealths of both the traders is then randomly partitioned and obtained by the individual members randomly as follows:

$$\begin{aligned} m_i(t+1) &= \lambda m_i(t) + \epsilon(t)(1-\lambda)(m_i(t) + m_j(t)) \\ m_j(t+1) &= \lambda m_j(t) + (1-\epsilon(t))(1-\lambda)(m_i(t) + m_j(t)). \end{aligned} \tag{3}$$

The stationary state wealth distribution is an asymmetric distribution with a single peak. The distribution vanishes at $m = 0$ as well as for large m values. The most probable wealth $m_p(\lambda)$ increases monotonically with λ and the

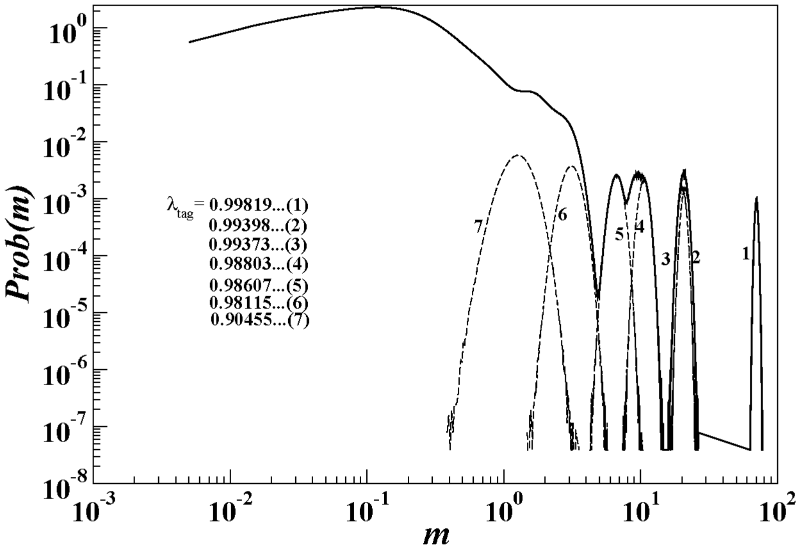

Fig. 3. The wealth distribution Prob(m) in the stationary state for the CCM model for a single initial configuration of saving propensities $\{\lambda_i\}$ with N=256 is shown by the solid line. Also the wealth distributions of the individual members with seven different tagged values of λ_{tag} are also plotted on the same curve with dashed lines. This shows that the averaged (over all members) distribution Prob(m) is the convolution of wealth distributions of all individual members.

distribution tends to the delta function again in the limit of $\lambda \to 1$ irrespective of the initial distribution of wealth.

In the third CCM model different members have their own fixed individual saving propensities and therefore the set of $\{\lambda_i, i = 1, N\}$ is a quenched variable. Economic transactions therefore take place following these equations:

$$m_i(t+1) = \lambda_i m_i(t) + \epsilon(t)[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)]$$
$$m_j(t+1) = \lambda_j m_j(t) + (1-\epsilon(t))[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)] \quad (4)$$

where λ_i and λ_j are the saving propensities of the members i and j. The stationary state wealth distribution shows a power law decay with a value of the Pareto exponent $\nu \approx 1$ [7].

In this paper we present the detailed numerical evidence to argue that while the first two models are ergodic and self-averaging, the third model is not. This makes the third model difficult to study numerically.

We simulated DY model with $N = 256, 512$ and 1024. Starting from an initial equal wealth distribution Prob(m) $= \delta(m-1)$ we skipped some transactions corresponding to a relaxation time $t_\times$ to reach the stationary state. Typically $t_\times \propto N$. In the stationary state we calculated the three different probability distributions, namely: (i) the wealth distribution $\text{Prob}_1(m)$

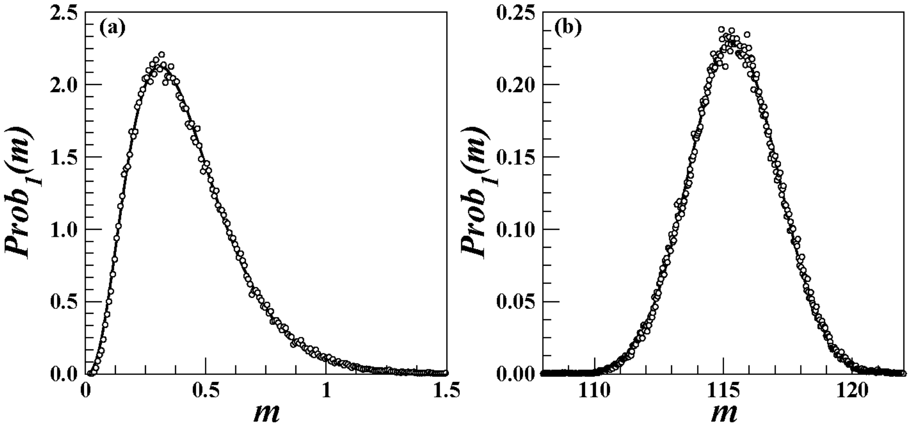

Fig. 4. The individual member's wealth distribution in the CCM model. A member is tagged with a fixed saving propensity λ_{tag}=0.05 in (a) and 0.999 in (b) for N=256. In the stationary state the distribution $\mathrm{Prob}_1(m)$ is asymmetric in (a) and is fitted to a Gamma function. However for very large λ the distribution in (b) is symmetric and fits very nicely to a Gaussian distribution.

of an arbitrarily selected tagged member (ii) the overall wealth distribution $\mathrm{Prob}_2(m)$ (averaged over all members of the society) on a long single run (single initial configuration, single sequence of random numbers) and (iii) the overall wealth distribution $\mathrm{Prob}(m)$ averaged over many initial configurations. In Fig. 1(a) we show all three plots for $N = 256$ and observe that these three plots overlap excellent, i.e., these distributions are same. This implies that the DY model is ergodic as well as self-averaging.

Similar calculations are done for the CC model as well (Fig. 1(b)). We see a similar collapse of the data for the same three probability distributions. This lead us to conclude again that the CC model is also ergodic and self-averaging. Further we fit in Fig. 2(a) the CC model distribution $\mathrm{Prob}(m)$ using a Gamma function as cited in [10] as:

$$\mathrm{Prob}(m) \sim m^{a(\lambda)} \exp(-b(\lambda)m) \tag{5}$$

which gives excellent non-linear fits by *xmgrace* to all values of λ in the range between say 0.1 to 0.9. Once fitting is done the most-probable wealth is estimated by the relation: $m_p(\lambda) = a(\lambda)/b(\lambda)$ using the values of fitted parameters $a(\lambda)$ and $b(\lambda)$. Functional dependences of a and b on λ are also predicted in [10]. We plot $m_p(\lambda)$ so obtained with λ for 24 different values of λ in Fig. 2(b). We observe that these data points fit very well to another Gamma function as:

$$m_p(\lambda) = A\lambda^{\alpha} \exp(-\beta\lambda). \tag{6}$$

The values of $A \approx 1.46$, $\alpha \approx 0.703$ and $\beta \approx 0.377$ are estimated for $N = 256$, 512 and 1024 and we observe a concurrence of these values up to three decimal

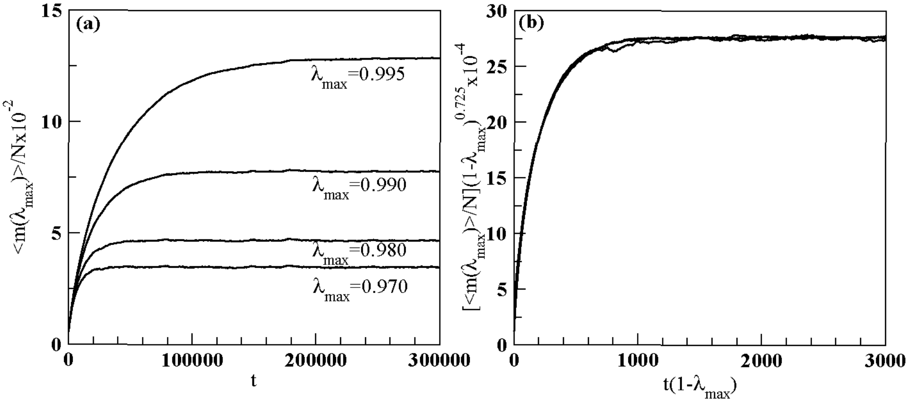

Fig. 5. (a) The mean wealth of a tagged member who has the maximal saving propensity is plotted as a function of time for four different values of λ_{max}. In (b) this data is scaled to obtain the data collapse.

places for the three different system sizes. While $m_p(0) = 0$ from Eqn. (6) is consistent, $m_p(1) = 1$ implies $A = \exp(\beta)$ is also consistent with estimated values of A and β. Following [10] we plotted $m_p(\lambda) = 3\lambda/(1+2\lambda)$ in Fig. 2(b) for the same values of λ and observe that these values deviate from our points for the small values of λ.

However, for the CCM model many inherent structures are observed. We argue that this model is neither ergodic nor self-averaging. For a society of $N = 256$ members a set of quenched individual saving propensities $\{0 \leq \lambda_i < 1, i = 1, N\}$ are assigned drawing these numbers from an independent and identical distribution of random numbers. The system then starts evolving with random pairwise conservative exchange rules cited in Eqn. (4). First we reproduced the Prob(m) vs. m curve given in [7] by averaging the wealth distribution over 500 uncorrelated initial configurations. The data looked very similar to that given in [7] and the Pareto exponent ν is found to be very close to 1.

Next we plot the same data for a single quenched configuration of saving propensities as shown in Fig. 3. It is observed that the wealth distribution plotted by the continuous solid line is far from being a nice power law as observed in [7] for the configuration averaged distribution. This curve in Fig. 3 has many humps, especially in the large wealth limit. To explain this we made further simulations by keeping track of the wealth distributions of the individual members. We see that the individual wealth distributions are significantly different from being power laws, they have single peaks as shown in Fig. 4. For small values of λ, the $\text{Prob}_1(m)$ distribution is asymmetric and has the form of a Gamma function similar to what is already observed for the CC model (Fig. 4(a)). On the other hand as $\lambda \to 1$ the variation becomes

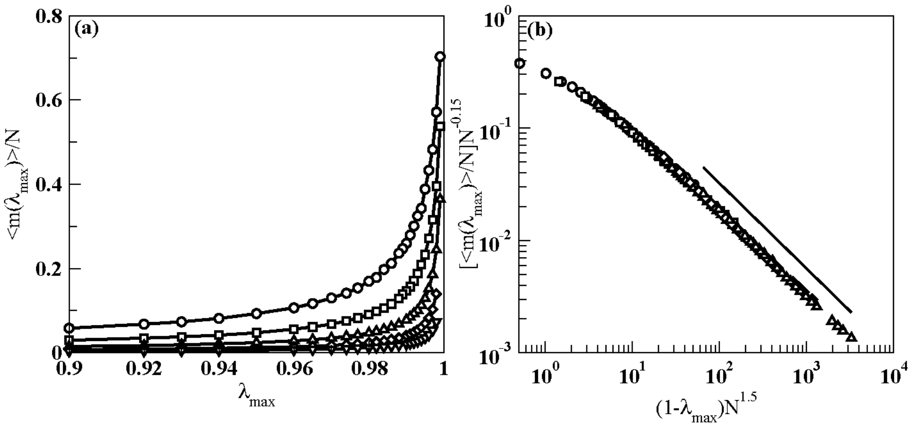

Fig. 6. In the stationary state the mean value of the wealth of the member with maximum saving propensity λ_{max} is plotted with λ_{max}. This value diverges as $\lambda_{max} \to 1$ for $N = 64$ (circle), 128 (square), 256 (triangle up), 512 (diamond) and 1024 (triangle down). (b) This data is scaled to obtain a data collapse of the three different sizes.

more and more symmetric which finally attains a simple Gaussian function (Fig. 4(b)). The reason is for small λ the individual wealth distribution does feel the presence of the infinite wall at $m = 0$ since no debt is allowed in this model, where as for $\lambda \to 1$ no such wall is present and consequently the distribution becomes symmetric. This implies that the wealth possessed by an individual varies within a limited region around an average value and certainly the corresponding phase trajectory does not explore the whole phase space. Therefore we conclude that the CCM model is not ergodic.

Seven individual wealth distributions have been plotted in Fig. 3. corresponding to six top most λ values and one with somewhat smaller value. We see that top parts of these $\text{Prob}_1(m)$ distributions almost overlap with the $\text{Prob}_2(m)$ distribution. This shows that $\text{Prob}_2(m)$ distribution is truly a superposition of N $\text{Prob}_1(m)$ distributions. In the limit of $\lambda \to 1$, large gaps are observed in the $\text{Prob}_2(m)$ distribution due to slight differences in the λ values of the corresponding individuals. These gaps remain there no matter whatever large sample size is used for the $\text{Prob}_2(m)$ distribution.

We further argue that even the configuration averaging may be difficult due to very slow relaxation modes present in the system. To demonstrate this point we consider the CCM model where the maximal saving propensity λ_{max} is continuously tuned. The N-th member is assigned λ_{max} and all other members are assigned values $\{0 \leq \lambda_i < \lambda_{max}, i = 1, N-1\}$. The average wealth $\langle m(\lambda_{max})\rangle/N$ of the N-th member is estimated at different times for $N = 256$ and they are plotted in Fig. 5(a) for four different values of λ_{max}. It is seen that as $\lambda_{max} \to 1$ it takes increasingly longer relaxation times to reach the stationary state and the saturation value of the mean wealth in the stationary state also increases very rapidly. In Fig. 5(b) we made a scaling of these plots like

$$[\langle m(\lambda_{max})\rangle/N](1-\lambda_{max})^{0.725} \sim \mathcal{G}[t(1-\lambda_{max})]. \tag{7}$$

This implies that the stationary state of the member with maximal saving propensity is reached after a relaxation time $t_\times$ given by

$$t_\times \propto (1-\lambda_{max})^{-1}. \tag{8}$$

Therefore we conclude that in CCM the maximal λ member takes the longest time to reach the stationary state where as rest of the members reach their individual stationary states earlier.

This observation poses a difficulty in the simulation of the CCM model. Since this is a problem of quenched disorder it is necessary that the observables should be averaged over many independent realizations of uncorrelated disorders. Starting from an arbitrary initial distribution of m_i values one generally skips the relaxation time $t_\times$ to reach the stationary state and then collect the data. In the CCM model the $0 \le \lambda_i < 1$ is used. Therefore if M different quenched disorders are used for averaging it means the maximal of all $M \times N$ λ values is around $1-1/(MN)$. From Eqn. (8) this implies that the slowest relaxation time grows proportional to MN. Therefore the main message is more accurate simulation one intends to do by increasing the number of quenched configurations, larger relaxation time $t_\times$ it has to skipp for each quenched configuration to ensure that it had really reached the stationary state.

Next, we calculate the variation of the mean wealth $\langle m(\lambda_{\rm max})\rangle/N$ of the maximally tagged member in the stationary state as a function of λ_{max} and for the different values of N. In Fig. 6(a) we plot this variation for $N = 64$, 128, 256, 512 and 1024 with different symbols. It is observed that larger the value of N the $\langle m(\lambda_{max})\rangle/N$ is closer to zero for all values of λ_{max} except for those which are very close to 1. For $\lambda_{max} \to 1$ the mean wealth increases very sharply to achieve the condensation limit of $\langle m(\lambda_{max} = 1)\rangle/N = 1$.

It is also observed that the divergence of the mean wealth near $\lambda_{max} = 1$ is associated with a critical exponent. In Fig. 6(b) we plot the same mean wealth with the deviation $(1-\lambda_{max})$ from 1 on a double logarithmic scale and observe power law variations. A scaling of these plots is done corresponding to a data collapse like:

$$[\langle m(\lambda_{max})\rangle/N]N^{-0.15} \sim \mathcal{F}[(1-\lambda_{max})N^{1.5}]. \tag{9}$$

Different symbols representing the data for the same five system sizes fall on the same curve which has a slope around 0.76. The scaling function $\mathcal{F}[x] \to x^{-\delta}$ as $x \to 0$ with $\delta \approx 0.76$. This means $\langle m(\lambda_{max})\rangle N^{-1.15} \sim (1-\lambda_{max})^{-0.76}N^{-1.14}$ or $\langle m(\lambda_{max})\rangle \sim (1-\lambda_{max})^{-0.76}N^{0.01}$. Since for a society of N traders $(1-\lambda_{max}) \sim 1/N$ this implies

$$\langle m(\lambda_{max})\rangle \sim N^{0.77}. \tag{10}$$

This result is therefore different from the claim that $\langle m(\lambda_{max})\rangle \sim N$ [7].

To summarize, we have revisited the three recent models of wealth distribution in Econophysics. Detailed numerical analysis yields that while the DY and CC models are ergodic and self-averaging, the CCM model with quenched saving propensities does not seem to be so. In CCM existence of slow modes proportional to the total sample size makes the numerical analysis difficult.

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Dynamics of Money and Income Distributions

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1 Introduction

We formulate a model of personal income distributions within society. Our aim is to understand the origin of the 'natural law' first proposed by Pareto who suggested that the high end part of the income distribution follows a power law where $f_1(w) \sim w^{-1-\alpha}$. We generalise a model proposed the so-called CC-CCM model introducing the possibility of exchanges of money at random times. Unlike the CC-CCM model this generalisation predicts tail exponents greater than unity in line with Pareto's prediction and empirical observation.

2 Kinetics of wealth distributions

We consider the Chakraborti, Chakrabarti - Chatterjee, Chakrabarti, Manna (CC-CCM) model first proposed by [2] and subsequently investigated numerically by [3, 4, 5] and analytically by [6] in different regimes of the parameters of the model. We modify the model appropriately in order to explain experimental findings concerning distributions of wealth in societies. In the model the time evolution of the distribution of wealth $w_i(t)$ is specified through dynamical rules that involve exchanges between two agents (two-agent exchanges) in two consecutive times t and $t - \delta t$. The rules (depicted in Fig. 1) read:

$$\begin{aligned} w_i\,(t) &= \lambda_i w_i(t-\delta t) + \epsilon\left[(1-\lambda_i)w_i(t-\delta t) + (1-\lambda_j)w_j(t-\delta t)\right] \\ w_j\,(t) &= \lambda_j w_j(t-\delta t) + \epsilon_1\left[(1-\lambda_i)w_i(t-\delta t) + (1-\lambda_j)w_j(t-\delta t)\right] \end{aligned} \tag{1}$$

where $\lambda_{i,j}$ describe the amount of money saved in the exchange process (saving propensities), $0 \leq \epsilon \leq 1$ is a uniformly distributed random number

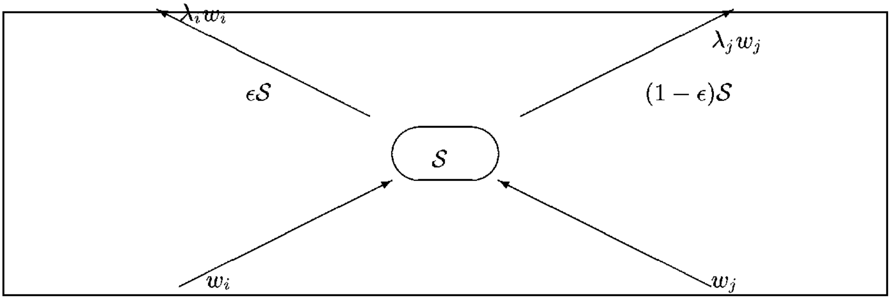

Fig. 1. Two agent exchange process described by the dynamical rules (1). Here λ_i and λ_j are saving propensities and $\mathcal{S} := [(1-\lambda_i)w_i(t) + (1-\lambda_j)w_j(t)]$ is the amount of money exchanged.

with $\langle\epsilon\rangle = 1/2$, $\epsilon_1 = 1-\epsilon$ and the time $\delta T = \delta t$ between two consecutive collision processes is assumed to be a random variable with a probability density function (pdf) $\rho_{\delta T}(\delta t)$ (waiting time pdf). The random waiting time is a novel element in CC-CCM model which we introduce to enforce a power law wealth distribution function with an arbitrary tail exponent in the high end.

We describe the model in terms of the two agent distribution function $f_2(v,w;t)$. This function specifies the probability density of an event in which two randomly chosen agents have wealth values $V_t = v$ and $W_t = w$ at time t. This means that $f_2(v,w;t) := P\left(V_t = v, W_t = w\right)$. The one-agent function $f_1(v;t)$ is obtained by integrating the two-agent function over values $w \geq 0$ at time t. The kinetic equation for the two-agent distribution function is obtained by conditioning on the occurrence of the collision at time $t-\xi$ for some $0 \leq \xi < t$. This means that

$$f_2(v,w;t) = \tag{2}$$

$$\int_0^t \rho_{\delta T}(\xi) d\xi \int_{\mathbb{R}_+^4} dv' dw' P\left(V_t = v, W_t = w \,|V_{t-\xi} = v', W_{t-\xi} = w'\right) f_2(v', w'; t-\xi) \tag{3}$$

where the integrations in (2) and (3) run from zero to infinity and the conditional probability in the integrand in (3) is expressed via a two-dimensional delta function (no dissipation of wealth assumption) as follows:

$$\delta\left(\begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} \lambda + \epsilon\lambda_1 & \epsilon\lambda_1 \\ \epsilon_1\lambda_1 & \lambda + \epsilon_1\lambda_1 \end{pmatrix} \begin{pmatrix} v' \\ w' \end{pmatrix}\right) \tag{4}$$

Here the saving propensities λ are assumed to be random variables distributed independently from the values of wealth. The terms in (3) correspond, from the left to the right, to following events, namely to the probability that the previous exchange took place at time $t-\xi$, the conditional probability that the agents have wealth values (v,w) given that they had wealth values (v',w') prior to the exchange, that occurred at time $t-\xi$, and to the unconditional probability that of the later event. Since the right hand side of (3) is a convolution both in the time and in the wealth arguments and since the time and wealth values are positive, we transform equation (3) into an algebraic equation by means of a Laplace-Laplace transform with respect to wealth and to time respectively. We take $\mathbf{x} := (x_1, x_2) \in \mathbb{R}_+^2$ and define:

$$\tilde{f}_2(\mathbf{x};s) := \mathcal{L}_{v,w}\mathcal{L}_t\left[f_2\right](\mathbf{x};s) := \int\limits_{\mathbb{R}_+^2} d\mathbf{x} \int\limits_{\mathbb{R}_+} dt f_2(v,w;t) e^{-(x_1 v + x_2 w)} e^{-st} \tag{5}$$

We multiply (3) by $e^{-(x_1 v + x_2 w)} e^{-st}$, integrate over $t \in \mathbb{R}_+$ and $(v,w) \in \mathbb{R}_+^2$ we obtain:

$$\tilde{f}_2(\mathbf{x};s) = \int\limits_{\mathbb{R}_+^2} d\mathbf{v} e^{-(\underline{\underline{A}}^T \mathbf{x})\mathbf{v}} \int\limits_{\mathbb{R}_+} dt \int\limits_0^t d\xi e^{-st} \rho_{\delta T}(\xi) f_2(v,w;t-\xi) = \tag{6}$$

$$\int\limits_{\mathbb{R}_+^2} d\mathbf{v} e^{-(\underline{\underline{A}}^T \mathbf{x})\mathbf{v}} \tilde{\rho}_{\delta T}(s) \tilde{f}_2(\mathbf{v},s) = \tilde{\rho}_{\delta T}(s) \tilde{f}_2((\underline{\underline{A}}^T \mathbf{x}), s) \tag{7}$$

where $\mathbf{x} = (x_1, x_2)$ the matrix $\underline{\underline{A}}$ is defined viz

$$\underline{\underline{A}} := \begin{pmatrix} \lambda + \epsilon\lambda_1 & \epsilon\lambda_1 \\ \epsilon_1\lambda_1 & \lambda + \epsilon_1\lambda_1 \end{pmatrix} \tag{8}$$

and $\tilde{\rho}_{\delta T}(s) := \int\limits_{\mathbb{R}_+} dt e^{-st} \rho_{\delta T}(t)$ is the Laplace transform of the waiting time pdf. Setting $x_2 = 0$ in (7) and using the fact that $\tilde{f}_2(x,0) = \tilde{f}_2(0,x) = \tilde{f}_1(x)$ we obtain the following equation for the one-point distribution function:

$$\tilde{f}_1(x;s) = \tilde{\rho}_{\delta T}(s) \tilde{f}_2((\lambda + \epsilon\lambda_1)x, \epsilon\lambda_1 x; s) \tag{9}$$

From (9) we see that the average wealth $\langle v \rangle := -\partial_x \tilde{f}_1(x;s)\Big|_{x=0}$ does not depend on time. Indeed, differentiating (9) by x at $x=0$ we get:

$$\langle v \rangle (s) = \tilde{\rho}_{\delta T}(s) \langle v \rangle (s) (\lambda + 2\epsilon\lambda_1) = \tilde{\rho}_{\delta T}(s) \langle v \rangle (s) \tag{10}$$

where the last equality in (10) follows from averaging over ϵ subject to the condition that $\langle \epsilon \rangle = 1/2$. Taking into account that $\tilde{\rho}_{\delta T}(0) = 1$, and inverting the Laplace transform in (10) yields:

$$\sum_{n=1}^{\infty} \frac{d^n \tilde{\rho}_{\delta T}(s)}{ds^n}\Bigg|_{s=0} \frac{(-1)^n}{n!} \frac{d^n}{dt^n} \left[\langle v \rangle (t)\right] = 0 \Leftrightarrow \langle v \rangle (t) = \mathrm{const}(t) \tag{11}$$

Since the average wealth is constant with time the one-point distribution function must have a stationary limit $f_1(w) = \lim_{t\to\infty} f_1(w,t)$ which we investigate in the following section.

2.1 The power law tail

Numerical simulations [4, 5] and analytical investigations [6] of the original CC-CCM model with not random waiting times suggest that the stationary one-point distribution conforms, in the high-end, to a power law $f_1(w) \sim 1/w^{1+\alpha}$ with the exponent (tail exponent) equal unity $\alpha = 1$. Measurements [8],[10] of wealth distributions yield, however, the tail exponent different from unity and dependent on the country or population group in question. Is the generalization of the CC-CCM model, that we develop in this paper and that involves random waiting times between exchange processes, consistent with experimental measurements at least as far the tail exponent is concerned? We show that this is indeed the case, the randomness of waiting times is a sufficient element to obtain the exponent α different from unity and dependent on the parameters of the model. Inverting the Laplace transform in (9) we obtain the stationary distribution of wealth in the Laplace domain. We have:

$$\tilde{f}_1(x) = \lim_{t\to\infty} \int_0^t d\xi \tilde{f}_2((\lambda + \epsilon\lambda_1)x, \epsilon\lambda_1 x, t-\xi)\rho_{\delta T}(\xi) \tag{12}$$

We expand the Laplace transform of the distribution function in a series in x as

$$\tilde{f}_1(x;t) = 1 - \langle v \rangle x + A(t)x^\alpha + O(x^{1+\alpha}) \tag{13}$$

where $A(t)$ is some function of time (tail amplitude), that depends on the initial conditions, and $\langle v \rangle$ is the average wealth, independent of time as shown in the previous section. We insert the expansion (13) into (12), we use the mean-field approximation, meaning that the two point function factorize-s into a product of one-point functions, and we obtain:

$$\begin{aligned} \tilde{f}_1(x) &= 1 - \langle v \rangle x \\ &+ \lim_{t\to\infty} \left(\int_0^t d\xi A(t-\xi)\rho_{\delta T}(\xi) \right) \left(\langle (\epsilon\lambda_1)^\alpha \rangle + \langle (\lambda + \epsilon\lambda_1)^\alpha \rangle \right) x^\alpha \\ &+ O(x^{\alpha+1}) \end{aligned} \tag{14}$$

Comparing (14) with (13) and averaging over the uniformly distributed ϵ subject to the condition $\langle \epsilon \rangle = 1/2$ we obtain a transcendental equation for the tail exponent α. We have:

$$\langle \lambda_1^\alpha \rangle + \left\langle \frac{1-\lambda^{\alpha+1}}{\lambda_1} \right\rangle = (1+\alpha)\lim_{t\to\infty} \frac{A(t)}{\int_0^t A(t-\xi)\rho_{\delta T}(\xi)d\xi} \tag{15}$$

$$= (1+\alpha)\lim_{t\to\infty} \frac{A(t)}{(A \otimes \rho_{\delta T})(t)} \tag{16}$$

where $(A \otimes \rho_{\delta T})(t)$ means a convolution of the tail amplitude $A(t)$ and the waiting time pdf $\rho_{\delta T})(t)$ at time t.

Note that the left hand side of (15) is equal to two at $\alpha = 0$ and $\alpha = 1$ respectively, it has a minimum and a maximum in the interval $\alpha \in [0, 2]$, is strictly increasing and decreasing for $\alpha > 2$ for $\langle\lambda\rangle > 1/2$ and for $\langle\lambda\rangle < 1/2$ respectively. The right hand side of (15) intersects the left hand side for some $\alpha \in [1, 2]$ and for some $\alpha \in [0, 1]$ in case if $\mathcal{N} := \frac{A(\infty)}{(A \otimes \rho_{\delta T})(\infty)}$ is smaller or is bigger than unity respectively. This depends on the form of the initial condition for the one-point wealth distribution and on the form of the waiting time pdf. For not random waiting times $\rho_{\delta T}(\xi) = \delta(\xi - dt)$, for some $dt > 0$, the large time limit of the convolution is equal to the large time limit of the tail amplitude $(A \otimes \rho_{\delta T})(\infty) = A(\infty)$ we have $\mathcal{N} = 1$, the only intersection of the left hand and right hand sides of (15) is $\alpha = 1$ and thus we retrieve the numerical and the analytical findings from [3, 4, 5] and from [6] respectively.

3 Conclusions

We have generalized the CC-CCM exchange of wealth model, by introducing random times between exchanges, in order to account for the fact that the tail exponent of the stationary one-agent wealth distribution is not unity. We have obtained variable tail exponents $\alpha \in [0, 2]$ dependent on the functional form of the tail amplitude, determined by the initial condition of the wealth distribution, and on the functional form of the waiting time pdf. We note that, in our generalization of the model, both the random waiting times, and the amounts of money exchanged are independent from one another and from the waiting times.

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Dynamic Process of Money Transfer Models

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Summary. We have studied numerically the statistical mechanics of the dynamic phenomena, including money circulation and economic mobility, in some transfer models. The models on which our investigations were performed are the basic model proposed by A. Drăgulescu and V. Yakovenko [1], the model with uniform saving rate developed by A. Chakraborti and B.K. Chakrabarti [2], and its extended model with diverse saving rate [3]. The velocity of circulation is found to be inversely related with the average holding time of money. In order to check the nature of money transferring process in these models, we demonstrated the probability distributions of holding time. In the model with uniform saving rate, the distribution obeys exponential law, which indicates money transfer here is a kind of Poisson process. But when the saving rate is set diversely, the holding time distribution follows a power law. The velocity can also be deduced from a typical individual's optimal choice. In this way, an approach for building the micro-foundation of velocity is provided. In order to expose the dynamic mechanism behind the distribution in microscope, we examined the mobility by collecting the time series of agents' rank and measured it by employing an index raised by economists. In the model with uniform saving rate, the higher saving rate, the slower agents moves in the economy. Meanwhile, all of the agents have the same chance to be the rich. However, it is not the case in the model with diverse saving rate, where the assumed economy falls into stratification. The volatility distribution of the agents' ranks are also demonstrated to distinguish the differences among these models.

Key words: Transfer model, Dynamic Process, Money Circulation, Mobility

1 Introduction

Recently, wealth or income distribution has attracted much attention in the field of econophysics [4, 5, 6]. More than 100 years ago, Italian economist Pareto first found that the income distribution follows an universal power law [7]. However, the economy has undergone dramatic transitions in last century, some researchers had doubted about if the law still holds in the modern stage and turned to reexamine the income distribution and its shift by employing income tax data [8, 9, 10, 11, 12]. The empirical analysis showed that in many countries the income distribution typically presents with a power-law tail, and majority of the income distribution can be described by an exponential law. This universal shape of distribution and its shift trigger an increasing interests in exploring the mechanism behind them. To solve this problem, several multi-agent models have been developed by applying principles of statistical mechanics [1, 2, 3, 13, 14, 15]. In these models, economic system is analogized to the ideal gas, where the agents can be regarded as particles, and money is just like energy. Therefore, the trading between agents can be viewed as collisions between particles in the gas. By using such analogy, the developed approach that applied to the ideal gas system now can be used to study this kind of economic system. Whatever the trading rule is set in these models, it is worthy noting that money is always transferred from one agent to another in the trading process. So this kind of models could be referred as money transfer models [16].

Leading the search into this issue was a paper by A. Drăgulescu and V. Yakovenko [1]. In their ideal-gas model, the economy is closed and the amount of money transferred in each round of trading is determined randomly. Their simulation analysis shows that the steady money distribution follows an exponential law. Several papers have extended the work by introducing different characteristics into the model and found that different trading rule may lead to different shapes of money distribution. A. Chakraborti and B.K. Chakrabarti examined the case where the agents do not take out all amount of money as they participate the exchange, but instead they save a part of their money [2]. This case is well grounded in reality, and the ratio they save is called saving rate hereafter. When the saving rate are the same for all agents, the money distribution obeys a Gamma law [17]. However, when the agents' saving rates are set randomly, the money distribution changes to a Power-law type [3]. A second extension looks at non-conservation. F. Slanina considered a case that the economy is not conserved but opened, and so he regarded it as inelastic granular gases [15]. Some further studies manage to seek for the exact mathematical solution by using a master equation [18, 19].

In fact, money transfer is a dynamic process. Besides the money distribution, it possess some other presentations. Thus, investigating the distribution only can not provide the whole picture of the relationship between the distribution and the trading rule. Some efforts have been put into the study on the dynamic mechanism behind the distribution, that opens more windows to observe how the economy works.

These works can be divided into two parts. One is about how the money moves in the assumed economy [20, 21, 22]. As we know, the money is not static even after the money distribution gets steady. They are always transferred among agents. Naturally, because of the randomness, whether in the simulations or in the reality, the time interval that money stays in one agent's pocket is a random variable which is named as holding time. The introduction of holding time opens a new path to understanding of the circulation velocity at micro level.

The other one is about how agents' positions shift in the economy [23]. Like the money, agents are not static in the transferring process. If the agents are sorted according to the amount of money they hold, it is found that the rank of any agent varies over time. This phenomenon is called mobility in economics. According to economists' argument, only analysis on the distribution is not sufficient especially when comparing the generating mechanism of income and the inequality[24, 25].

In addition, the study on the dynamic characters in the proposed models makes the evaluation criteria more complete. The aim of econophysicists to develop these models is to mimic the real economy by abstracting its essence. However, we cannot judge whether such abstraction is reasonable or not depending on the shape of distribution only. Thus, we must take the circulation and mobility into account when constructing a "good" multi-agent model.

In this paper, the dynamic processes of the transfer models are investigated by examining the holding time distribution and the degree of mobility. The models and simulations will be briefly presented in the next section. In the Sec. 3 and 4, we will show the nature of circulation and mobility in these models respectively. Finally, we will give our conclusion in Sec. 5.

2 Models and Simulations

We start with the transfer model proposed by A. Drăgulescu and V. Yakovenko, in which the economic system is closed, put it in another way, the total amount of money M and the number of economic agents N are fixed. Each of agents has a certain amount of money initially. In each round of trading process, two agents i, j are chosen to take part in the trade randomly. And it is also decided randomly which one is the payer or receiver. Suppose the amounts of money held by agent i and j are m_i and m_j, the amount of money to be exchanged Δm is decided by the following trading rule:

$$\Delta m = \frac{1}{2}\varepsilon(m_i + m_j), \tag{1}$$

where ε is a random number from zero to unit. If the payer cannot afford the money to be exchanged, the trade will be cancelled. This model is very simple and extensible which is named as the basic model in this paper.

When A. Chakraborti and B.K. Chakrabarti intended to extend the basic model, they argued that the agents always keep some of money in hand as saving when trading. The ratio of saving to all of the money held is denoted by s and called saving rate in this paper. For all the agents, the saving rates are set equally before the simulations. Like the trading pattern of the basic model, two agents i, j are chosen out to participate the trade in each round.

Suppose that at t-th round, agents i and j take part in trading, so at $t+1$-th round their money $m_i(t)$ and $m_j(t)$ change to

$$m_i(t+1) = m_i(t) + \Delta m; m_j(t+1) = m_j(t) - \Delta m, \tag{2}$$

where

$$\Delta m = (1-s)[(\varepsilon - 1)m_i(t) + \varepsilon m_j(t)], \tag{3}$$

and ε is a random fraction. It can be seen that Δm might be negative. That means agent i is probably the payer of the trade. This model degenerates into the basic model if s is set to be zero. In this model, all of agents are homogenous with the same saving rate. So we call it the model with uniform saving rate.

This model was further developed by B.K. Chakrabarti's research group by setting agents' saving rates randomly before the simulations and keeping them unchanged all through the simulations. Likewise, this is called the model with diverse saving rate. Correspondingly, the trading rule Equation (3) changes to

$$\Delta m = (1-s_i)(\varepsilon - 1)m_i(t) + (1-s_j)\varepsilon m_j(t), \tag{4}$$

where s_i, s_j are the saving rates of agent i and j respectively.

Our following investigations on the dynamic phenomena is based on these three models. The scale is the same for all the simulations: the number of agent N is $1,000$ and the total amount of money M is $100,000$.

3 Money Circulation

As the medium of exchange, money is held and transferred by people. In the process of money transferring, if an agent receives money from others at one moment, he will hold it for a period, and eventually pays it to another agent. The time interval between the receipt of the money and its disbursement is named as holding time. We introduce the probability distribution function of holding time $P_h(\tau)$, which is defined such that the amount of money whose holding time lies between τ and $\tau + d\tau$ is equal to $MP_h(\tau)d\tau$. In the stationary state, the fraction of money $MP_h(\tau)\, d\tau$ participates in the exchange after a period of τ. Then the average holding time can be expressed as

$$\bar{\tau} = \int_0^\infty P_h(\tau)\, \tau \, d\tau. \tag{5}$$

The velocity indicates the speed at which money circulates. Since money is always spent randomly in exchange, the transferring process can be deemed as a Poisson type, and the velocity of money can then be written as [20]

$$V = \frac{1}{\bar{\tau}}. \tag{6}$$

This is the statistical expression of the circulation velocity of money in terms of holding time.

Two caveats to this conclusion are in order. First, we need to observe the probability density function of holding time to check whether the transfer of money is a Poisson process. If the assumption is correct, the probability density function must take the following form

$$P(\tau) = \lambda e^{-\lambda\tau}, \tag{7}$$

where λ corresponds to the intensity of the Poisson process. We have carried out the measurement of holding time in our previous work [21]. In those simulations, the time interval between the moments when the money takes part in trade after t_0 for the first two times is recorded as holding time, supposing we start to record at round t_0. The data were collected after majority of money($> 99.9\%$) had been recorded and over 100 times with different random seeds.

The simulation results are shown in Fig.1. It can be seen the probability distributions of holding time decay exponentially in the model with uniform saving rate. This fact indicates that the process is a Poisson process. On the other case, when the saving rates are set diversely, the distribution changes to a power-law type.

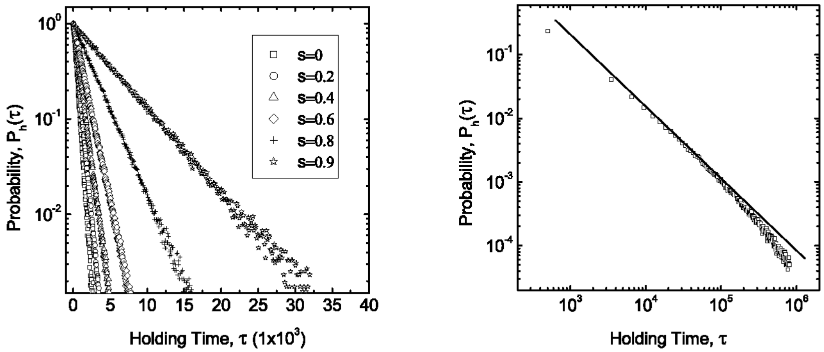

Fig. 1. The stationary distributions of holding time: (left panel) for the model with uniform saving rate in a semi-logarithmic scale, (right panel) for the model with diverse saving rate in a double-logarithmic scale, where the fitting exponent of the solid line is about -1.14. Note that in the figure the probabilities have been scaled by the maximum probability respectively.

In the model with uniform saving rate, the monetary circulation velocity corresponds to the intensity of Poisson process, which is negatively related to the saving rate. Form Fig. 1 we can see that the lower the saving rate is, the steeper the distribution curve. This result is also plotted in Fig. 2, which tells us the relation between the velocity and the saving rate is not linear.

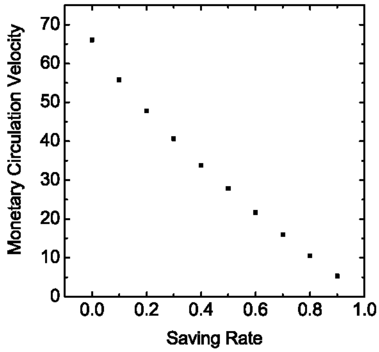

Fig. 2. The monetary circulation velocity versus the saving rate in the model with uniform saving rate.

Second, the relation between the velocity of money and the average holding time suggests that the velocity could be investigated by examining how economic agents make decisions on the holding time of money. There are many kinds of agents who may have different characters when they utilize money in an economic system, such as consumers, firms, and investors etc. We can choose one of them as a representative to examine how their spending decisions affect the velocity. The typical one is consumers whose behavior has always been depicted by the life-cycle model prevailed in economics. The model considers a representative individual who expects to live T years more. His object is to maximize the lifetime utility

$$U = \int_0^T u(C(t))\, dt, \tag{8}$$

subject to the budget constraint condition

$$\int_0^T C(t)\, dt \leq W_0 + \int_0^T Y(t)\, dt, \tag{9}$$

where $u(\cdot)$ is an instantaneous concave utility function, and $C(t)$ is his consumption in time t. The individual has initial wealth of W_0 and expects to earn labor income $Y(t)$ in the working period of his or her life. The main conclusion deduced from this optimal problem is that the individual wants to smooth his consumption even though his income may fluctuate in his life time. From this conclusion, we can also calculate the average holding time of money based on the time path of income and consumption as the following form

$$\bar{\tau} = \frac{\int_0^T [C(t) - Y(t)] t \, dt}{\int_0^T Y(t) \, dt}. \tag{10}$$

With a few manipulations in a simple version of the life-cycle model [22], we get

$$V = \frac{2}{T - T_0}. \tag{11}$$

This result tells us that the velocity of money depends on the difference between the expected length of life T and that of working periods T_0. It also implies that the velocity, as an aggregate variable, can be deduced from the individual's optimal choice. In this way, a solid micro foundation for velocity of money has been constructed.

4 Economic Mobility

It is the economists' consensus that static snapshots of income distribution alone is not sufficient for meaningful evaluation of wellbeing and the equality. This can be understood easily from a simple example. Suppose in an economy there are two individuals with money \$1 and \$2 initially. At the next moment, the amount of money held by them changes to \$2 and \$1. The distribution in this case is unchanged, but the ranks of both agents vary over time. Although the system seems unequal at either of the two moments in terms of the distribution, the fact is that the two individuals are quite equal combining these two moments. Besides, from this simple example, it can also been found that the structure of economy may vary heavily with an unchanged distribution. Thus the investigation on mobility is helpful not only to the measurement on equality but also to the exposure of the mechanism behind the distribution.

We investigated the mobility in the referred transfer models by placing emphasis on the "reranking" phenomenon. To show this kind of mobility, we sorted all of agents according to their money and recorded their ranks at the end of each round. All of data were collected after the money distributions get stationary and the sampling time interval was set to be 1000 rounds.

The time series of rank in these three models are shown in Fig.3. Then, we can compare the characters of rank fluctuation of these models. All of the agents in the basic model and the model with uniform saving rate can be the rich and be the poor. The rich have the probability to be poor and the poor also may be luck to get money to be the rich. The mobility in these two model are quite similar except the fluctuation frequency of the time series. The economy in the model with diverse saving rate is highly stratified (see Fig. 3c). The rich always keep their position, and the poor are doomed to be the poor. The agents in each level differ in their rank fluctuations. The higher the agent' rank, the smaller the variance of his rank.

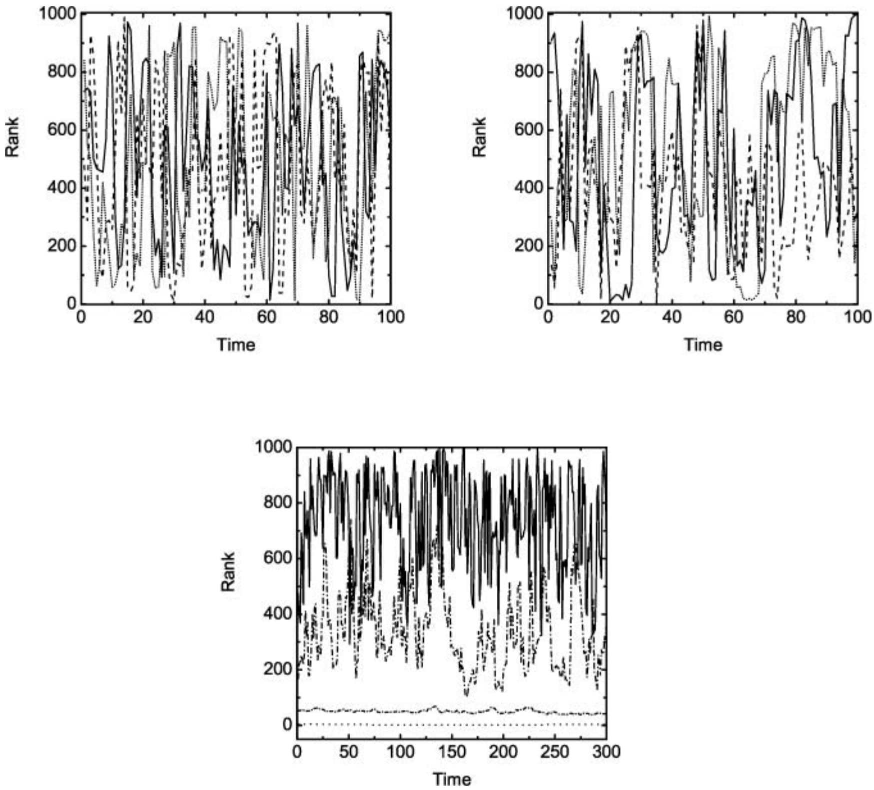

Fig. 3. The typical time series of rank (a) from basic model, (b) from the model with uniform saving rate $s = 0.5$ and (c) from the model with diverse saving rate where the saving rates of these typical agents are 0.99853, 0.9454, 0.71548 and 0.15798 (from bottom to top) respectively.

Table 1. Comparison of the Three Transfer Models in Mobility

	Mobility $l(t, t')$	Stratification
The Basic Model	0.72342	No
The Model with Uniform Saving Rate		No
$s = 0.1$	0.70269	
$s = 0.3$	0.65165	
$s = 0.5$	0.58129	
$s = 0.7$	0.4773	
$s = 0.9$	0.30212	
The Model with Diverse Saving Rate	0.19671	Yes

To compare the mobilities quantitatively, we applied the measurement index raised by G. S. Fields et al [26]. The mobility between the two sample recorded in different moments is defined as

$$l(t, t') = \frac{1}{N} \sum_{i=1}^{N} |\log(x_i(t)) - \log(x_i(t'))|, \tag{12}$$

where, $x_i(t)$ and $x_i(t')$ are the rank of agent i at t and t' respectively. It is obvious that the bigger the value of l, the greater the degree of mobility. To eliminate the effect of the randomness, we recorded more than 9000 samples continuously and calculated the value of mobility l between any two consecutive samples. The average value of ls in these models are shown in Table 1. It can be found that the degree of mobility decreases as the saving rate

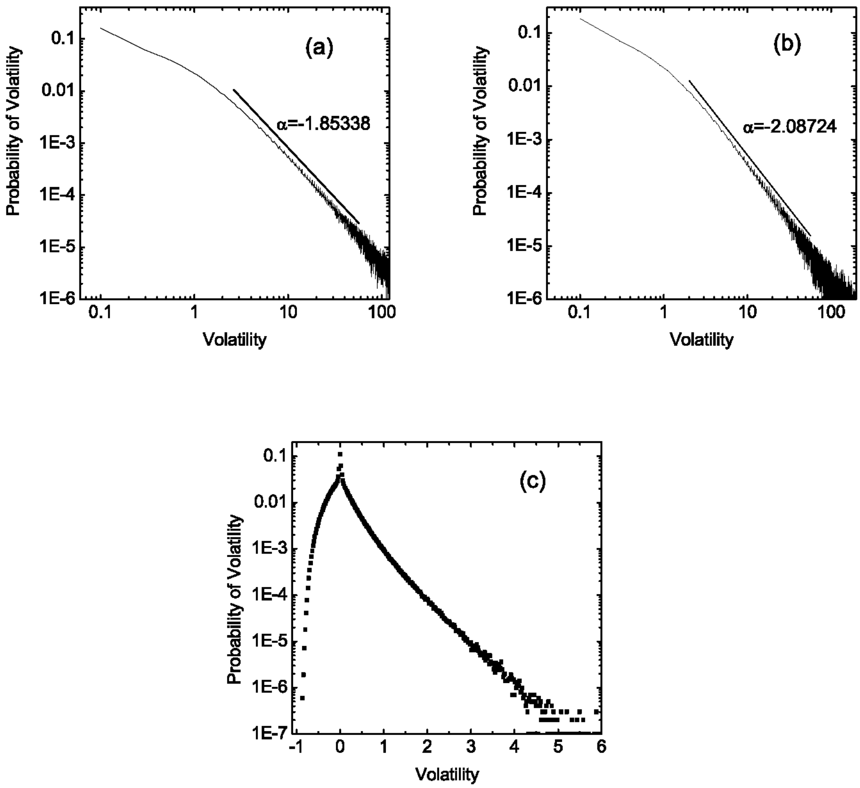

Fig. 4. The distribution of the volatility of agents' rank (a) for the basic model, (b) for the model with uniform saving rate $s = 0.5$ and (c) for the model with diverse saving rate.

increases in the model with uniform saving rate. The intuition for this result is straightforward. The larger the ratio agents put aside, the less money they take out to participate the trade. Then, the less money to lose or win. Thus, the higher saving rate, the less probability of change in rank or mobility. The very low degree of mobility in the model with diverse saving rate is due to its stratification.

To show more details of the mobility, we also obtain the distribution of the volatility ($\frac{x_i(t')-x_i(t)}{x_i(t)}$) which is shown in Fig.4. It is noted that the distributions of the rank variety ratio are quite similar and follow power laws in the basic model and the model with uniform saving rate. The exponent of the power-law distribution is found to decrease as the saving rate increases. This phenomenon is consistent with the alter trend of the index because the higher the saving rate, the little money is exchanged and the smaller the volatility of rank. Consequently, when the saving rate increases, the right side of volatility distribution will shift to the vertical axis, leading to a more steeper tail. From Fig.4c, we can see that the volatility distribution in the model with diverse saving rate ends with an exponential tail as the times of simulations increase.

5 Conclusion

The dynamic phenomena of three transfer models, including money circulation and economic mobility, are presented in this paper. The holding time distributions in these models are demonstrated, and the relation between the velocity of money and holding time of money is expressed. Studies on this dynamic process lead us to a good understanding the nature of money circulation process and provide a new approach to the micro-foundation of the velocity. The "reranking" mobilities in these models are compared graphically and quantitatively. This observation provide more information about the dynamic mechanism behind the distribution. Such investigations suggest that the characters of circulation and mobility should be considered when constructing a multi-agent model.

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A Stochastic Trading Model of Wealth Distribution

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Summary. We develop a stochastic model where the poorer end of the society engage in two-party trading while the richer end perform trade with gross entities. Using our model we are able to capture some of the essential features of wealth distribution in societies: the Boltzmann-Gibbs distribution at the lower end and the Pareto-like power law tails at the richer end. A reasonable scenario to connect the two ends of the wealth spectrum is presented. Also, we show analytically how different power law exponents can be obtained. Furthermore, a link with the models in macroeconomics is also attempted.

1 Introduction

In countries like United States, Japan, United Kingdom, Germany, Switzerland, New Zealand, etc., where data for wealth distribution is readily available, it is observed that the wealth is very unequally distributed and is highly concentrated. In a wealthy country like the United States various surveys over the past 30 years (in particular the Survey of Consumer Finances) show that a lions share of the total wealth is concentrated in the richest percentiles: the richest 1% owns one third of the wealth, and the top 5% holds more than half. At the other extreme, the bottom 10% own little or nothing at all.

Income is also unequally distributed and inequality in income leads to unequal wealth distribution. Income is defined as revenue from all sources before taxes but after transfers and thus includes labor earnings and income generated by wealth. However, income distribution is less skewed (and hence less unequal) than wealth distribution. The 1992 Survey of Consumer Finances revealed that the income of the income-rich top 1% was 18.5% of the total sample, and that of the top 5% was one third of the total income. In the same survey, the Gini index (whose value 0 corresponds to equal distribution and value 1 to wealth entirely in the hands of the richest) was shown to be 0.78 for the wealth distribution and only 0.57 for the income distribution.

There are many possible measures of wealth. In this paper we will concentrate on total net worth which includes all assets held by the households (financial wealth, real estate, vehicles) and net of all liabilities (mortgages and other debts). The degree of concentration of net worth held by various wealth percentile groups in various years is given in Table 1.

Table 1. Percent of net worth held by various percentile groups of the wealth distribution [1].

Percentile group	Year 1989	1992	1995	1998	2001
0-49.9	2.7	3.3	3.6	3.0	2.8
50-89.9	29.9	29.7	28.6	28.4	27.4
90-94.9	13.0	12.6	11.9	11.4	12.1
95-98.9	24.1	24.4	21.3	23.3	25.0
99-100	30.3	30.2	34.6	33.9	32.7

Inequality in the distribution of wealth in the population of a nation has provoked a lot of political debate. The observations that the top few percentage own a lions share of the wealth has been mathematically formulated as a power law by Pareto at the turn of the 19th century [2]. It is important for both economists, econophysicists, and policy makers to understand the root cause on this inequality: whether social injustice is the main culprit for such a lop-sided distribution. Over the past, economists have developed two models, namely, the dynastic model and the life-cycle model, to explain wealth distribution. In the dynastic model, where bequests are vehicles of transmission of wealth inequality, people save to improve the consumption of their descendants. On the other hand, in the life-cycle model, where wealth of an individual is a function of the age, people save to provide for their own consumption after retirement. Both these models and their hybrid versions have had only limited success quantitatively [3]. However, one of the ingredients that goes into these models, i.e., uninsurable shocks or stochasticity in income [4], has been exploited by econophysicists with remarkable success in reproducing power law tails qualitatively. It appears that randomness may very well be enough to explain the skewed wealth distribution and that a loaded dice may not be the root cause.

The wealth distribution of the poor (0-90 wealth percentile group) is exponential or Boltzmann-Gibb's like [5, 6], while that of the higher wealth group has a power law tail with exponent varying between 2 and 3. The Boltzmann-Gibb's law has been shown to be obtainable when trading between two people, in the absence of any savings, is totally random [7, 8, 9]. The constant finite savings case has been studied earlier numerically by Chakraborti and Chakrabarti [7] and later analytically by us [8]. As regards the fat tail in the wealth distribution, several researchers have obtained Pareto-like behavior using approaches such as random savings [10], inelastic scattering [11], generalized Lotka Volterra dynamics [12], asymmetric interactions between agents

[13], nonextensive Tsallis statistics [14], analogy with directed polymers in random media [15], and three parameter based trade-investment model [16]. As regards an egalitarian solution, there has also been an interesting model (conservative exchange market model) based on the Bak-Sneppen model that takes measures to improve the lot of the poorest [17]. Within this model the authors obtain a Gibbs type of wealth distribution with almost all agents above a finite poverty line.

In this article, we try to model the processes that produce the wealth distribution in societies. Our model involves two types of trading processes – tiny and gross [19]. The tiny process involves trading between two individuals while the gross one involves trading between an individual and the gross-system. The philosophy is that small wealth distribution is governed by two-party trading while the large wealth distribution involves big players interacting with the gross-system. The poor are mainly involved in trading with other poor individuals. Whereas the big players mainly interact with large entities/organizations such as government(s), markets of nations, etc. These large entities/organizations are treated as making up the gross-system in our model. The gross-system is thus a huge reservoir of wealth. Hence, our model invokes the tiny channel at small wealths while at large wealths the gross channel gets turned on. Our two types of trading model is motivated by the fact that a kink seems to be generic in the wealth/income distributions in real populations (as borne out by the empirical data in Fig. 9 of Ref. [5] and Fig. 1 of Ref. [18]) indicating that two different dynamics may be operative in the poor and the wealthy regimes.

2 Two Types of Trading Model

2.1 Model for Tiny-Trading

The model describes two-party trading between agents 1 and 2 whose respective wealths y_1 and y_2 are smaller than a cutoff wealth y_c. The two agents engage in trading where they put forth a fraction of their wealth $(1-\lambda_t)y_1$ and $(1-\lambda_t)y_2$ [with $0 \le \lambda_t < 1$]. Then the total money $(1-\lambda_t)(y_1+y_2)$ is randomly distributed between the two. The total money between the two is conserved in the two-party trading process. We assume that probability of trading by individuals having certain money is proportional to the number of individuals with that money.

We will now derive the equilibrium distribution function $f(y)dy$ which gives the probability of an agent having money between y and $y + dy$. We assume that, irrespective of the starting point, the system evolves to the equilibrium distribution after sufficient number of trading interactions. We will now consider interactions after the system has attained steady state. The joint probability that, before interaction, money of 1 lies between x and $x+dx$ and money of 2 lies between z and $z + dz$ is $f(x)dxf(z)dz$. Since the total money is conserved in the interaction, we let $L = x + z$ and analyze in terms of L. Then the joint probability becomes $f(x)dxf(L-x)dL$. We will now generate equilibrium distribution after interaction by noting that *at steady state the distribution is the same before and after interaction.* Probability that L is distributed to give money of 1 between y and $y + dy$ is

$$\frac{dy}{(1-\lambda_t)L} f(x)dxf(L-x)dL, \tag{1}$$

with $x\lambda_t \leq y \leq x\lambda_t + (1-\lambda_t)L$. Thus we see that $x \leq y/\lambda_t$ and $x \geq [y-(1-\lambda_t)L]/\lambda_t$. Actually x should also satisfy the constraint $0 \leq x \leq L$ because the agents cannot have negative money. Thus the upper limit on x is $\min\{L, y/\lambda_t\}$ (i.e., minimum of L and y/λ_t) and the lower limit is $_{\max}\{0, [y-(1-\lambda_t)L]/\lambda_t\}$. Now, we know that the total money L has to be greater than y so that the agents have non-negative money. Thus we get the following distribution function for the money of 1 to lie between y and $y + dy$

$$f(y) = \int_y^\infty dL \int_{a(y,L,\lambda_t)}^{b(y,L,\lambda_t)} dx\mathcal{F}(x, L, \lambda_t), \tag{2}$$

where

$$a(y, L, \lambda_t) \equiv_{\max} [0, \{y-(1-\lambda_t)L\}/\lambda_t],$$

$$b(y, L, \lambda_t) \equiv \min[L, y/\lambda_t],$$

and

$$\mathcal{F}(x, L, \lambda_t) \equiv \frac{f(x)f(L-x)}{(1-\lambda_t)L}.$$

The above result was obtained earlier by using Boltzmann transport theory [19].

On introducing an upper cutoff y_c for the two-party trading, the contribution to the distribution function $f(y)$ from the tiny channel becomes

$$\gamma \int_y^\infty dL \int_{a(y,L,\lambda_t)}^{b(y,L,\lambda_t)} dx\mathcal{F}(x, L, \lambda_t)\mathcal{H}(x, L, y_c). \tag{3}$$

In the above equation

$$\mathcal{H}(x, L, y_c) \equiv [1-\theta(x-y_c)][1-\theta(L-x-y_c)],$$

with $\theta(x)$ being the unit step function and $\gamma = 1/\int_0^{y_c} dxf(x)$ is a normalization constant introduced to account for the less than unity value of the probability of picking a person below y_c.

2.2 Model for Gross-Trading

Next, we will analyze the contribution to the distribution function $f(y)$ from gross-trading. An individual possessing wealth y_1 larger than a cutoff wealth (y_c) trades with a fraction $(1-\lambda_g)$ of his wealth y_1 with the gross-system. The latter puts forth an equal amount of money $(1-\lambda_g)y_1$. The trading involves the total sum $2(1-\lambda_g)y_1$ being randomly distributed between the individual and the reservoir. Thus on an average the gross-system's wealth is conserved. The probability that the individuals money after interaction lies between y and $y+dy$ is

$$\frac{dy}{2(1-\lambda_g)y_1} f(y_1)dy_1, \tag{4}$$

where $\lambda_g y_1 \leq y \leq (2-\lambda_g)y_1$. Then the distribution function $f(y)$ is given by

$$f(y) = \int_{y/(2-\lambda_g)}^{y/\lambda_g} \frac{dy_1 f(y_1)}{2y_1(1-\lambda_g)}. \tag{5}$$

Now it is interesting to note that the solution of the above equation is given by $f(y) = c/y^n$. Then, to obtain n one solves the equation

$$(2-\lambda_g)^n - \lambda_g^n = 2n(1-\lambda_g), \tag{6}$$

and obtains $n = 1, 2$. Only $n = 2$ is a realistic solution because it gives a finite cumulative probability. Surprisingly, the solution is *independent of* λ_g. Also, clearly the distribution function makes sense only for $y > 0$. On taking into account an upper cutoff y_c, the contribution to the distribution function $f(y)$ from the gross channel is

$$\int_{y/(2-\lambda_g)}^{y/\lambda_g} \frac{dy_1 f(y_1)}{2y_1(1-\lambda_g)} \theta(y_1 - y_c). \tag{7}$$

2.3 Hybrid Model

Here an individual possessing wealth larger than a cutoff wealth y_c does trading with the gross-system, while individuals possessing wealth smaller than y_c engage in two-party tiny-trading. Hence from Eqs. (3) and (7), the distribution function is obtained to be

$$\begin{aligned} f(y) = \; & \gamma \int_y^\infty dL \int_{a(y,L,\lambda_t)}^{b(y,L,\lambda_t)} dx \mathcal{F}(x,L,\lambda_t)\mathcal{H}(x,L,y_c) \\ & + \int_{y/(2-\lambda_g)}^{y/\lambda_g} \frac{dx f(x)}{2x(1-\lambda_g)} \theta(x - y_c). \end{aligned} \tag{8}$$

Now, it must be pointed out that when the savings $\lambda_t = 0$, $\lambda_g \neq 0$, and $y \to 0$, Eq. (8) yields (up to a proportionality constant) the following same result as the purely tiny-trading case without an upper cutoff [8]:

$$f'(y) \propto -f(y)f(0). \tag{9}$$

In obtaining the above equation we again assumed that the function $f(y)$ and its first and second derivatives are well behaved. Then the solution for small y is given by

$$f(y) \propto f(0)exp[-yf(0)]. \tag{10}$$

3 Results and Discussion

The distribution function $f(y)$ can be obtained by solving the nonlinear integral Eq. (8). To this end, we simplify Eq. (8) for computational purposes as follows:

$$\begin{aligned} f(y) = \ & \gamma\mathcal{G}(y,\lambda_t,y_c)\int_y^{2y_c} dL \int_{a(y,L,\lambda_t)}^{b(y,L,\lambda_t)} dx\mathcal{F}(x,L,\lambda_t)\mathcal{H}(x,L,y_c) \\ & +[1-\theta(y-y_{as})]\int_{y/(2-\lambda_g)}^{y/\lambda_g} \frac{dx f(x)}{2x(1-\lambda_g)}\theta(x-y_c) \\ & +\theta(y-y_{as})f(y_{as})\frac{y_{as}^2}{y^2}, \end{aligned} \tag{11}$$

where $\mathcal{G}(y,\lambda_t,y_c) \equiv 1-\theta[y-(2-\lambda_t)y_c]$ and $y > y_{as}$ gives the asymptotic behavior $f(y) \propto 1/y^2$. In our calculations, we have taken y_{as} to be at least $20y_c$ and obtained $f(y)$ for all y less than 2000 times the average wealth per person y_{av}. We solved Eq. (11) iteratively by choosing a trial function, substituting it on the RHS (right hand side) and obtaining a new trial function and successively substituting the new trial functions over and over again on the RHS until convergence is achieved. The criterion for convergence was that the difference between the new trial function f_n and the previous trial function f_p satisfies the accuracy test $\sum_i |f_n(y_i) - f_p(y_i)|/\sum_i f_p(y_i) \leq 0.002$ [20].

In Fig. 1, using a log-log plot we depict the distribution function $f(y)$ for the constant savings case $\lambda_t = \lambda_g = 0.5$ with the average money per person y_{av} being set to unity and with the values of the wealth cutoff $y_c = 3, 5, 10$. As expected, for larger values of y_c, the Pareto-like $1/y^2$ behavior sets in later. The transition to purely gross-trading occurs at $(2-\lambda_t)y_c$, while below $\lambda_g y_c$ it is purely two-party tiny-trading. Thus the transition from purely tiny-trading to purely gross-trading occurs in Fig. 1 over a region of width y_c. However, all the tails merge irrespective of the cutoff. At smaller values of y the behavior of $f(y)$, depicted in the inset, is similar to the purely two-party trading model studied earlier (see Ref. [8]). The curves in the inset appear to be close because here the trading is two-party and is governed by the same savings. Next, in Fig. 2 we plot $f(y)$ with the cutoff $y_c = 5$, $y_{av} = 1$, and for values of savings fraction $\lambda_t = \lambda_g = \lambda = 0.1, 0.5, 0.8$. Here the power-law behavior $(1/y^2)$ takes over for $y > (2-\lambda)y_c$ and hence at lower savings it sets in later. In the power-law region the curves merge together. As shown

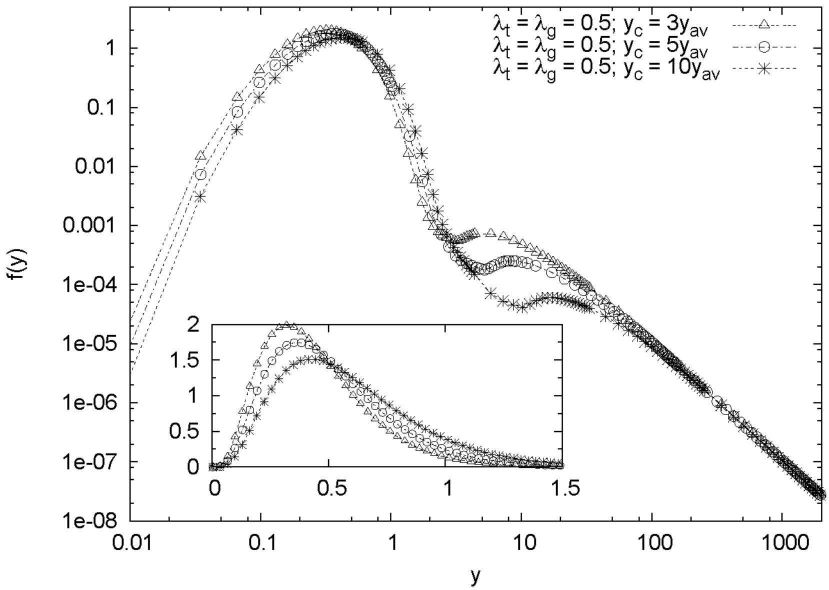

Fig. 1. Plot of the wealth distribution function for savings $\lambda_t = \lambda_g = 0.5$ and various wealth cutoff values $y_c = 3, 5, 10$. The average money per person y_{av} is set to unity. The dotted lines are guides to the eye.

in the inset of Fig. 2, at smaller values of y the $f(y)$s become zero with the higher peaked curves (corresponding to larger λs) approaching zero faster similar to the case of the purely two-party trading model in our earlier work [8]. Here the transition from purely tiny- to purely gross-trading at higher λ is sharper because the transition occurs over a region of width $2(1-\lambda)y_c$. Lastly, in Fig. 3, we show the distribution function $f(y)$ for the zero savings case in the tiny-channel ($\lambda_t = 0$) and for various savings $\lambda_g = 0.2, 0.5, 0.9$ in the gross-channel with $y_{av} = 1$ and $y_c = 5$. The distribution, as expected, decays exponentially (or Boltzmann-Gibbs-like) for small values of y and has power-law ($1/y^2$) behavior at large values. The curves merge in the Pareto-like region and, in fact, $f(y) \approx 0.1/y^2$ in all the three figures at large values of y. In Fig. 3 too, for reasons mentioned earlier, the transition is sharper at larger values of λ_g. Fig. 3 takes into account the fact that, in societies, the rich tend to have higher savings fraction (λ) compared to the poor. Actually, if the savings fraction were to increase gradually with wealth, one can expect a more gradual change in the transition region of the distribution rather than the sharp local maxima (around $y \approx 6.5$) shown by the $\lambda_g = 0.9$ curve.

In all the figures anomalous looking kinks/shoulders appear at the cross over between the Boltzmann-Gibbs-like and the Pareto-like regimes. This is due to the sharp cut-off at y_c that we introduced using a step function. However, as mentioned in the introduction, such kinks do occur in real in-

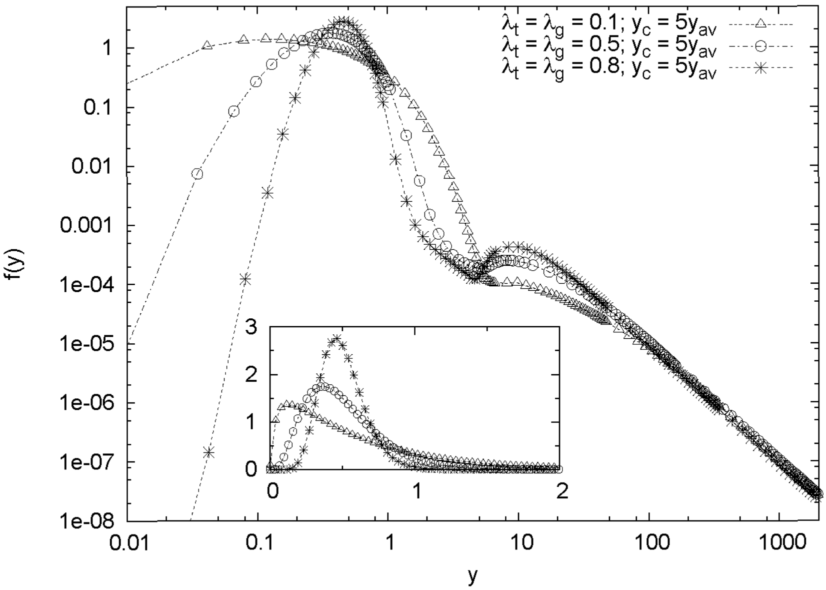

Fig. 2. Wealth distribution $f(y)$ at average wealth $y_{av} = 1$, wealth cutoff $y_c = 5$, and various values of savings $\lambda_t = \lambda_g = 0.1, 0.5, 0.8$.

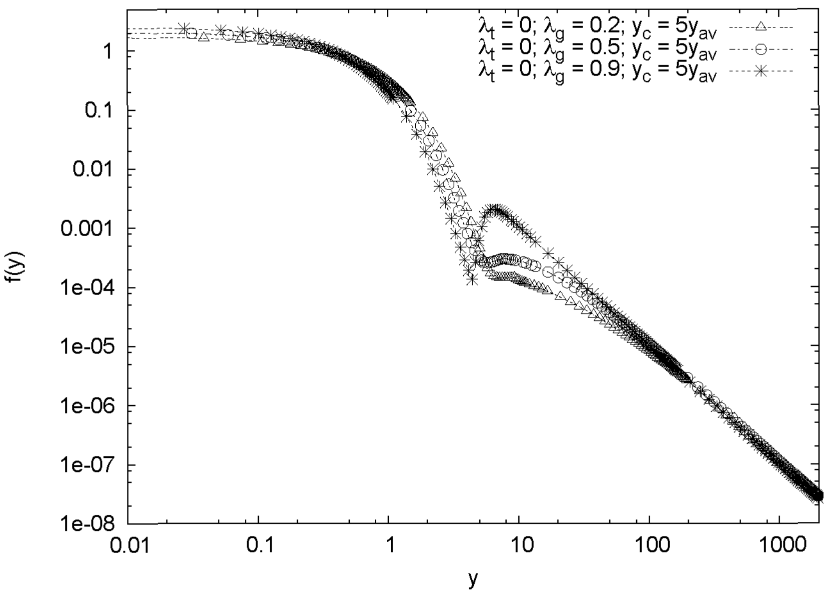

Fig. 3. Money distribution function at zero savings for tiny-trading and various savings values $\lambda_g = 0.2, 0.5, 0.9$ for the gross-trading. The average money $y_{av} = 1$ and the wealth cutoff $y_c = 5$.

come/wealth distributions [5, 18]. Different societies have the onset of Pareto-like behavior at different wealths which is indicative that the cut-off has to be obtained empirically based on various factors like the social structure, welfare policies, type of markets, form of government, etc. It is of interest to note that the analysis carried out on income classes in USA during 1983-2001 in Ref. [21] revealed that the Boltzmann-Gibbs part is quite stationary while the Pareto tail swells and shrinks (and thus changes with time). This is perhaps indicative that the poorer section corresponds to a system at equilibrium while the richer society represents a steady state system that is far from equilibrium. Thus perhaps some sort of a self-organized criticality is operative in the wealthier society where wealth generating ideas or new technology may be responsible for driving the system away from equilibrium.

In Japan the wealth/income distribution vanishes at zero wealth/income and then rises to a maximum (see Ref. [5]). In US the distribution seems to be a maximum at zero wealth/income (see Ref. [5]). Both these aspects can be covered in our model as the poor in general are known to save very little. If their savings are zero, one gets the Boltzmann-Gibbs behavior at the poor end. On the other hand, if the savings are small one gets a maximum close to zero and the distribution vanishes at zero wealth.

It would be interesting to deduce the savings pattern from the wealth distribution. While it has been observed that the rich tend to save more than the poor, how gradually the savings change as wealth increases can perhaps be inferred from the change in slope. However, as explained below, the middle region (involving the middle-class) has been modeled quite crudely by us and needs to be refined before a serious connection with savings pattern can be attempted.

We will now further discuss the motivation for using two different mechanisms to model the observed wealth distribution. The model is an approximation where the direct wealth exchange occurs between people who are in economic proximity. At the bottom of the spectrum, the poor, who have limited economic means and avenues, come in contact with a few poor and their economic activity is modeled in terms of two-party trading. At the other end of the wealth spectrum, the rich have access to various economic avenues (e.g., markets, know-how, work force, capital, credit facilities, contacts, wealthy society, etc.) due to which they can trade with huge organizations and are thus modeled to interact with a reservoir. As regards the middle-class that is between the rich and the poor, they trade amongst themselves as well as with the poor and the reservoir. As a first step towards realizing this scenario, we included in our earlier work [19] only the two extreme cases of interaction. What we had not taken into account is the interaction of the middle class with the reservoir. To rectify this, we have chosen the cutoff y_g for the interaction with the reservoir such that y_g lies below the two-party trading cutoff y_t. However, this did not seem to alter the calculated curves significantly [22]. Thus, we believe that our model is a reasonable one at the poor and rich ends and is a crude approximation for the middle class. In order to model the wealth distribution of the middle class better, one needs to produce a gradual transition from a two-party trading at the poorer end to the gross trading at the richer end.

Although it is true that the poor also come in market contact with wealthy organizations like a soft-drink company, nevertheless the contact is an indirect one mediated through intermediaries. For example, the poor person deals with a richer shop-keeper selling the drink who in turn deals with a richer local distributor who in turn deals with the big soft-drink company. Thus the middle-class act as intermediaries between the rich and the poor. Next, we will examine the validity for our type of two-party trading. We feel that in any trading there is a random fluctuation of the price around its true value. The total money put forth for trading corresponds to the amount of random fluctuation. However the poorer of the two puts forth less and makes the trading biased in his/her favor. This can be justified from the fact that the poor people are constantly looking for bargains to make ends meet.

Compared to other types of analysis involving two-party trading to explain Pareto law (see Ref. [10]), our gross-trading mechanism can make contact with the standard approach in macroeconomics as will be shown below. In macroeconomics, the objective is to maximize a cumulative utility function subject to a wealth constraint [23]. Mathematically this is formulated as

$$\max_{c_{t+i},y_{t+i}} E_t \sum_i \beta^i u(c_{t+i}), \tag{12}$$

subject to the constraint

$$y_{t+i} = (1+r)y_{t+i-1} + e_{t+i} - c_{t+i}, \tag{13}$$

where c_t, y_t, and e_t are consumption, wealth, and labor earnings respectively at time t, r is the interest rate on wealth y, $0 < \beta < 1$ is the time-discount factor, $u(c_t)$ is the concave utility function, E_t is the expectation value based on the available information at time t. Using the method of Lagrange multipliers, the conditions of optimality yield

$$E_t[u'(c_t) - (1+r)\beta u'(c_{t+1})] = 0, \tag{14}$$

where $u'(c_t)$ is the derivative of $u(c_t)$ with respect to c_t. From the above equation we see that consumption at different times are related. In our work [see Eq. (4)], we introduced the stochasticity

$$y_{t+1} - y_t = \epsilon(1-\lambda_g)y_t, \tag{15}$$

where ϵ is a random number such that $-1 \leq \epsilon \leq 1$, which implies that

$$ry_t + e_{t+1} - c_{t+1} = \epsilon(1-\lambda_g)y_t. \tag{16}$$

The above equation can be made consistent with the optimal consumption relation given by Eq. (14). In fact if it is assumed that $(1+r)\beta = 1$, which is anyway approximately true, then consumption smoothing of the form $c_{t+1} = c_t$ [which is consistent with Eq. (14)] implies that all the stochasticity given by the RHS of Eq. (16) lies in the income only. Thus our model (for the power-law tail) is consistent with the standard approaches in macroeconomics using uninsurable shocks in income.

It is of interest to note that if we modify the stochasticity as

$$y_{t+1} - y_t = \epsilon(1 - \lambda_g)y_t^{1-\delta}, \tag{17}$$

with $0 \leq \delta \leq 1$, then the asymptotic behavior of the distribution function has two power-law solutions with exponents $2 - 2\delta$ and $1 - 2\delta$ [24]. Such solutions are obtained by solving the integral equation

$$f(y) = \int_{x_-}^{x_+} \frac{dx f(x)}{2(1 - \lambda_g)x^{1-\delta}}, \tag{18}$$

where the limits of integration $x_\pm$ are obtained iteratively in terms of y, from the equation

$$x_\pm = y \pm (1 - \lambda_g)x_\pm^{1-\delta}, \tag{19}$$

as a power series with a typical term in the series being $y^{1-n\delta}$ with $n = 0, 1, 2, ...$ Thus one can obtain different exponents for the power-law tail.

In conclusion, we introduced interaction of the rich with huge entities (a model that is consistent with main models in macroeconomics) and obtained a Pareto-like power-law. On the other hand, the Boltzmann-Gibbs-like wealth distribution, corresponding to the bulk of the society, is understood through a two-party trading mechanism. All in all, we show that stochasticity can explain the observed skewness in the wealth distribution in societies.

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Wealth Distribution in a Network with Correlations Between Links and Success

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1 Introduction

A study of the distribution of the income of workers, companies and countries was presented, a little more than a century ago, by Italian economist Vilfredo Pareto. He investigated data of personal income for different European countries and found a power law distribution that seems not to be dependent on the different economic conditions of the countries. In his book *Cours d'Economie Politique* [1] he asserted that in all countries and times the distribution of income and wealth follows a power law behaviour where the cumulative probability $P(w)$ of people whose income is at least w is given by $P(w) \propto w^{-\alpha}$, where the exponent α is named today Pareto index, while the power law is known as Pareto law. The exponent α for several countries was $1.2 \leq \alpha \leq 1.9$. However, recent data indicates that, even though Pareto's distribution provides a good fit to the distribution of high range of income, it does not agree with observed data over the middle and low range of income. For instance, data from Japan [2, 3], Italy [4], India [5], Brazil [6], the United States of America and the United Kingdom [7, 8, 9] are fitted by a lognormal or Gibbs distribution with a maximum in middle range plus a power law for high income one.

The existence of these two regimes may be justified in a qualitative way by stating that in the low and middle income class the process of accumulation of wealth is additive, causing a Gaussian-like distribution, while in the high income class the wealth grows in a multiplicative way, generating the power law tail [3]. However, it is not clear if the difference between this two-regime behaviour and the original Pareto law is the results of an historical change of the income profile during the last century, or a characteristic of the data analysed.

Different models of capital exchange among economic agents have been proposed trying to obtain the power law distribution for the wealthiest strata. Most of these models consider an ensemble of interacting economic agents that exchange a fixed or random amount of a quantity called "wealth". This wealth parameter represents the welfare of the agents. The exact choice of this parameter is not straightforward. For instance, when thinking of countries in the world economy, the GNP (Gross National Product) or some function of macroeconomic indicators could be a reasonable choice. In the case of companies, equity, share price or some suitable combination of them with outstanding debt are reasonable candidates. In the model of Dragulescu and Yakovenko [7] this parameter is associated with the amount of money a person has available to exchange. Within this model the amount of money corresponds to a kind of economic "energy" that may be exchanged by the agents in a random way and the resulting wealth distribution is a Gibbs exponential distribution, as it would be expected. An exponential distribution as a function of the square of the wealth is also obtained in a model where some action is taken, at each time step, on the poorest agent, trying to improve its economic state [10]. In the case of this last model a poverty line with finite wealth is also obtained, describing a way to diminish inequalities in a wealth distribution [11].

In order to try to obtain the power law tail several methods have been proposed. Keeping the constraint of wealth conservation a detailed studied proposition is that each agent saves a fraction - constant or random - of their resources [13, 14, 15, 16, 19, 20, 12, 17, 18], fraction that introduces a multiplicative factor in the exchanges. One possible result of that model is condensation, i.e. the concentration of all the available wealth in just one or a few agents. To overcome this situation different rules of interaction have been applied, for example increasing the probability of favouring the poorer agent in a transaction [19, 18], or introducing a cut-off that separates interactions between agents below and above this cut-off [21]. Most of these models are able to obtain a power law regime for the high-income class, but for some values of the parameters, while for the low income, the regime can be approximately fit by an exponential or lognormal function. Finally it is worth quoting that Slanina [22] proposed a non-conservative version of the "gas" model[7], where the agents win or lose some extra wealth in the interaction, and he is able to obtain a power law regime for the high income class. One interesting point of this model is the non-conservation of wealth (or money) that makes it more realistic; on the other hand, the model is deterministic, not stochastic.

Here we would like to address the point that in all those models possible correlations between wealth and probability of interaction are not considered. That means that there are no correlations between the wealth of the agents and the probability of interaction between them. This seems to be at odds with the idea that people tend to strongly interact mainly with others of their own social and economic class[21] and also the fact that success in business is awarded with more business. One example are the internet based e-shops that are beginning to substitute traditional shops. When presented with different choices, people prefer to buy in e-shops that have better "references", i.e. in the shops that have made transactions with more customers. In that way successful traders are rewarded with more links. Another example: Inaoka et al [23] analyse the exchanges in Japanese banks, concluding that the bigger ones have more interactions between them and with the others than the small banks. The resulting network of interactions is very different for big banks (almost fully connected) than for small ones (a kind of star-like network).

Recently we have presented a model including correlations between wealth and the possibility of having an exchange[24]. In this model agents can trade just if they belong to the same economic class (i.e. their wealth difference is within a given range u) and the result is an extreme class polarization with the decline of the middle class. Here we present a different approach, correlating the success of an agent in their economics exchanges with its degree of connectivity. A model is considered where each agent possess a given amount of wealth, randomly chosen between the arbitrary values of 0 and w_{max}. A different level of a randomly distributed risk aversion parameter is also attributed to each agent, as in previously discussed models, being this individual risk aversion level constant during the simulation. The agents are initially placed on a random lattice, with a given average connectivity $\overline{\nu}$. When the exchange of wealth between agents take place, every time an agent increase its wealth, it also increases its connectivity, that is, the number of neighbours that are linked to it. In the next section we describe the details of the model and the simulations, and in the last section we present the results and conclusions.

2 Dynamic network model

We consider a set of economic agents characterized by two parameters: a wealth w_i and a risk aversion factor $\beta(i)$, with $0 \leq \beta(i) \leq 1$. The last parameter remains fixed during the whole process, and allows us to define the quantity $[1-\beta(i)]$ as the percentage of wealth that agent i is disposed to risk. Agents are the nodes of a random network (i.e. a network having a Poisson distribution of connectivities) with average connectivity $\overline{\nu}$ and interactions are only allowed between connected agents.

The dynamics of the system consist first in choosing at random two agents connected by a link, which will exchange resources. Then, we put them to interact with the following rules: we establish that no agent can win more than the amount he puts at risk. This means that the amount that will be exchanged is the minimum value of the available resources of both agents, $dw = \min[(1-\beta_1)w_1; (1-\beta_2)w_2]$.Finally, we introduce a probability $p \geq 0.5$ of favoring the poorer of the two partners, because increasing the probability of favoring the poorer agent is a way to simulate the action of the state or of some type of regulatory policy that tries to redistribute the resources [17, 11]. Also, several authors[13, 14, 12, 18] have shown that without this prescription the system condensates, i.e. just one or a few agents concentrate the total wealth of the system. Here we determine this probability using a formula proposed by Scafetta et. al. [19, 20, 18],

$$p = \frac{1}{2} + f \times \frac{|w_1 - w_2|}{w_1 + w_2} \tag{1}$$

being w_1 and w_2 the respective wealths of the two partners in the exchange; f is a factor going from 0 (equal probability for all agents) to 1/2. Thus, in each exchange the poorer agent has probability p of receiving the quantity dw whereas the richer one has probability $1 - p$.

Moreover, at the same time that the winner in the exchange increases his wealth by dw, he is also rewarded with a given number of links, proportional to the amount dw. These additional links could come from the loser agent (version A of the model), or could be taken at random from any point of the lattice (version B).

We performed numerical simulation with these rules and found that, after a transient, the system arrives to a stationary state where the wealth has been distributed but also the network has changed from a random one to a web where the richer agents concentrate most of the links. This represents a society where the more successful agents obtain also better trade conditions, thus improving the opportunities of making more money. On the other side the situation will be not so unfair as expected for the poorer strata of the population. The smaller connectivity creates a kind of “protective screening” for the less favored agents, preventing them from losing more money.

3 Results and Conclusions

We consider a number of agents N ranging from 5000 to 10000 and a number of average exchanges big enough to guarantee a stationary state (10^3 to 10^4 exchanges per agent). The initial wealth for each agent is chosen at random from an uniform distribution where between 0 and w_{max}, being here $w_{max} = 500$. We investigate several values for the average number of links per agent, going from 5 to 80 links per agent in the case $N = 5 \times 10^5$ agents. The initial distribution of links is a Poisson distribution.

In order to update the lattice at each exchange, we divide the total wealth of the system by the total number of links, attributing a “monetary” value to each link. The winner in a transaction also wins the equivalent number of links, rounded by elimination of any fractionary number. Finally the value of f used to determine the probability p of favoring the poorer agent has been set equal to 0., 0.1, 0.3 and 0.5.

We show here three different cases: the static lattice, in order to have a reference for comparison, the case A – where after the exchange the winner takes links from the loser up to a limit of leaving the loser connected by at least one link –, and the case B – where the winner takes a link at random from any agent –. Notice that in all three cases the total number of links remains constant throughout the evolution.

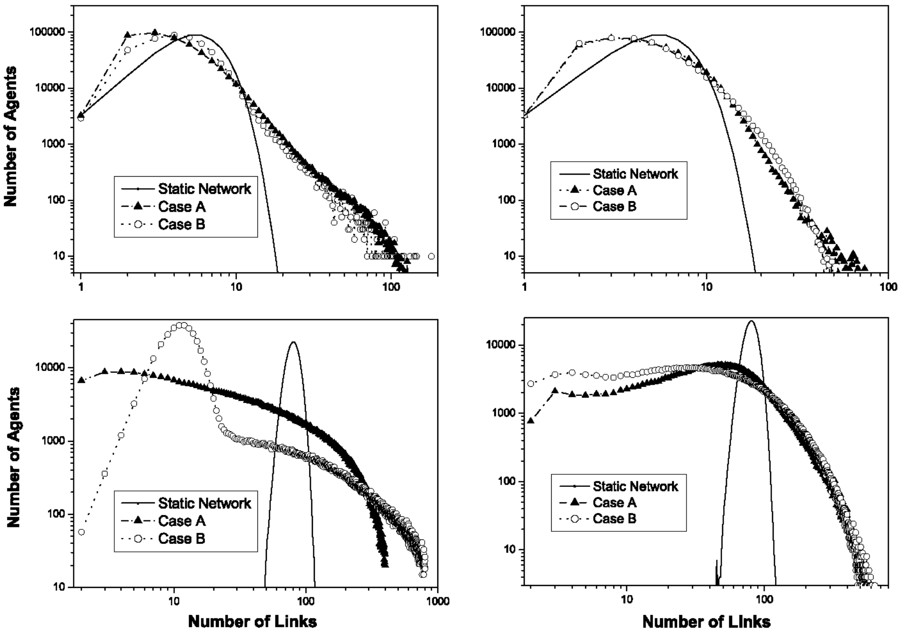

Fig. 1. Cumulative histogram of the asymptotic link distribution for $f = 0.1$ (left column) and $f = 0.5$ (right column), $\overline{\nu} = 5$ (first row) and $\overline{\nu} = 80$ (second row). The results are taken from 100 runs. The lines joining the symbols are only guides to the eye.

In Fig. 1 we show the asymptotic distributions of links for $f = 0.1$ and $f = 0.5$ (the poorer agent is maximally favored at each transaction) and for $\overline{\nu} = 5$ and $\overline{\nu} = 80$. In all cases the full curve corresponds to the initial distribution, that is also the static one, as the latter is not modified by the dynamics. It can be seen that in all cases the resulting distribution deviates significantly from the initial one: a few agents end up having a number of links much higher than the average, whereas most of the population has very few links. The maximum is always shifted to the left. This effect is most dramatic in the case $f = 0.1$, $\overline{\nu} = 80$: for case B the maximum is shifted from 80 links to only 10, whereas some agents are connected to up to ~ 800 other agents. In the case A, for these same parameters, the resulting distribution is rather different: the maximum is much less pronounced but has been shifted to very low values $\nu \approx 1$, while the maximal number of links is also much smaller. Finally, in the case $f = 0.5$, the effect of favoring the poorer agents seems to smooth out almost completely the differences between the dynamics A and B.

The most interesting results concern the asymptotic wealth distribution. In Fig. 2 we present results for $f = 0.1$ and $f = 0.5$, $\overline{\nu} = 5$ and $\overline{\nu} = 80$. For $f = 0.1$, (but also for small values of f), there appears a very high peak for low values of income: about 60 per cent of the agents own about one tenth of the average wealth. On the other hand, most of the wealth is owned by a few very rich agents. The personal wealth of these agents is about ten times greater than the average wealth. The differences between the different cases concern mainly the number of people in the middle class, loosely defined as the wealth interval between $w_{max}/10$ and w_{max}, and the number of people in the upper class ($w > w_{max}$). One striking feature observed for $f = 0.1$ is that in the high class the asymptotic distribution for case A follows a power law, whose exponent is ~ -2 (corresponding to a Pareto exponent -1). Also, it is rather surprising that the distribution for case A and the static one are almost identical for $f = 0.1$, $\overline{\nu} = 80$, even though the underlying lattices are very different (see Fig. 1).

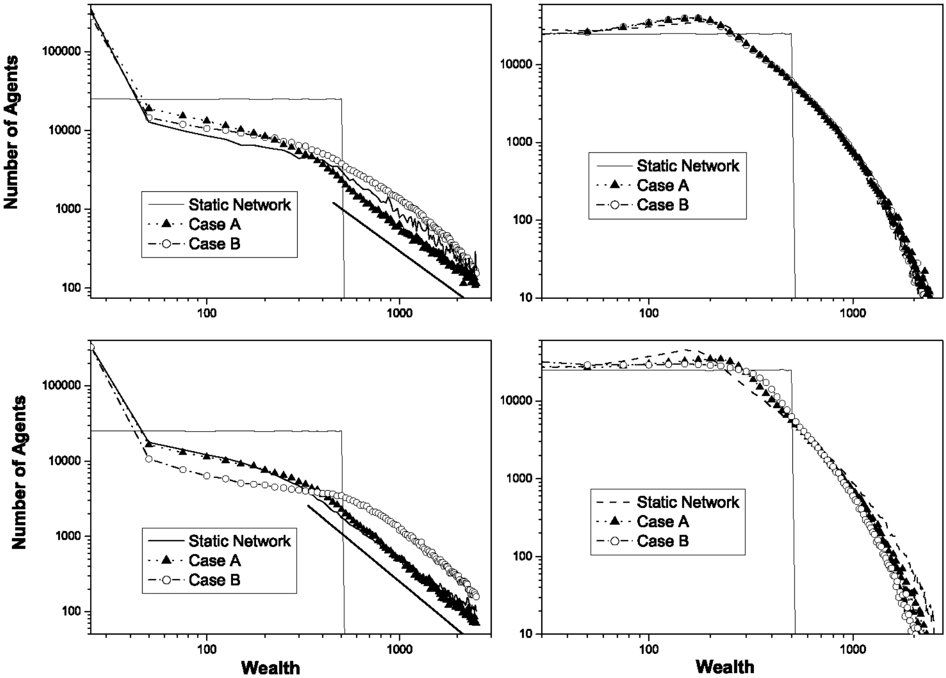

Fig. 2. Cumulative histogram of the asymptotic wealth distribution for $f = 0.1$ (left column) and $f = 0.5$ (right column), $\overline{\nu} = 5$ (first row) and $\overline{\nu} = 80$ (second row). The results are taken from 100 runs. For f=0.1 the straight lines correspond to fits with a power law, whose exponents are -1.8. The lines joining the symbols are only guides to the eye.

As it has been observed for the links distribution, the differences between the three cases are smoothed out for high values of f. But the wealth distribution depends on the dynamics of the network for low values of f. It seems that the effect of the dynamics of links is important when there are no regulations in the exchanges $f \approx 0.$, and the result is that the number of agents in the middle class decreases while the number of agents in the very low or in the high income class increases, but the effect is not as pronounced as in ref. [24]. In the case of $f = 0.5$ the wealth distribution looks similar to that of developed countries like Japan[2] or England [9]: A maximum in the distribution is observed for a "middle class" and for high income a power law may be drawn, but on a relatively narrow strip of wealth. The income of that "middle class" is almost the same average initial value of the wealth, while the number of very rich people is smaller by a factor of 10 compared to what happens for $f = 0.1$.

In order to compare the different distributions between them and with empirical data, it is useful to determine the values of the Gini coefficients. As it can be observed in Table 1 differences among the different cases are only significant for low values of f, but in these cases the Gini coefficients are very far from being realistic. It is only for high values of f that we obtain Gini indexes that are close to those observed in real societies. For $f = 0.$ and $f = 0.1$ unfairness clearly increases with connectivity in case A and for the same parameters it also increases when switching from the static case to both dynamic lattices. On the other hand, for higher values of f ($f = 0.3$ and $f = 0.5$), it seems that the reconnection of the lattice induces a kind of protective screening of the lower classes, being the Gini exponents slightly lower for both dynamic networks than for the static one (with the exception of case A, $\overline{\nu} = 5$). Moreover, the Gini indices are even lower for case B, when the links are cut at random, than for case A, when they are taken from the loser agent. However, in all cases the changes are small, meaning that the reconnection of the lattice has little effect on inequalities.

	Static			Case A			Case B		
	5	20	80	5	20	80	5	20	80
0	0.816	0.9213	0.955	0.964	0.981	0.983	0.980	0.987	0.985
0.1	0.793	0.878	0.91	0.884	0.897	0.915	0.89	0.868	0.873
0.3	0.609	0.651	0.666	0.62	0.622	0.623	0.603	0.59	0.593
0.5	0.443	0.466	0.473	0.441	0.432	0.428	0.433	0.422	0.424

Table 1. Gini coefficients for the three dynamics treated in the article. The columns in each case correspond to the different values of $\overline{\nu}$ whereas the rows correspond to the different values of f.

We have also analyzed the correlation between the number of links of each agent and the wealth he has accumulated. For the static case we find that there is no correlation between connectivity and wealth, as it is expected i.e. we find that for all wealth classes the average connectivity coincides with its population average. For case B we always find that there is a linear relationship: wealthier agents tend to be the more connected. On the other hand, in the case A we find a clear linear relation only for high values of f. For small values of f and not too large connectivity we find that the average connectivity is almost constant but, unlike the static case, this constant is *smaller* than that of the population average. There are, of course, agents with more links than the average and these are, surprisingly, very low income agents. Probably this is one of the reasons that for low values of f the low income class has practically zero wealth: they have success in their exchanges but, as they can only get the same amount they risk, the average gain is negligible.

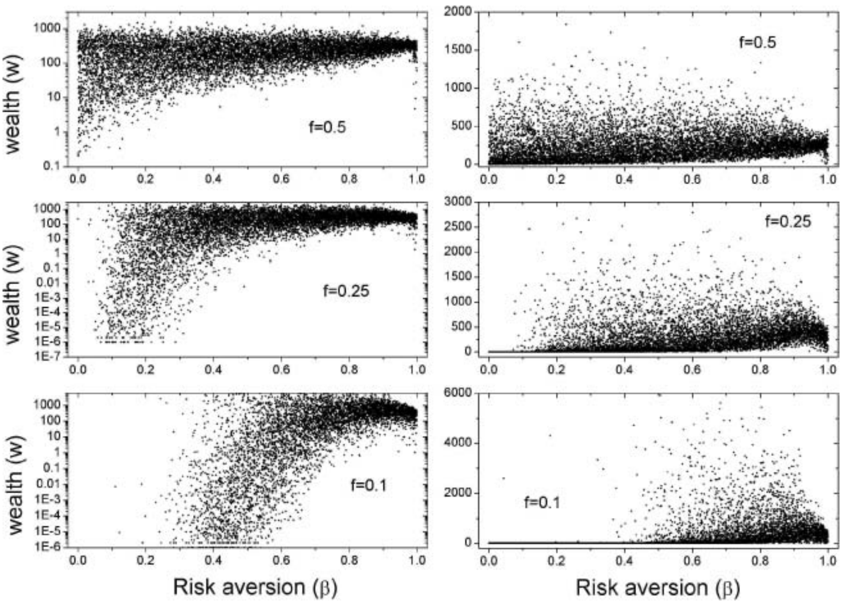

Fig. 3. Wealth vs. Risk aversion. Each point represents an individual, and the whole population has 10000 agents. The two columns only differ in the way the graphs are presented: a logarithmic ordinate scale in the left and normal ordinate in the right.

Finally, the correlation between wealth and risk aversion also presents some interesting features. In Fig. 3 we show some results for case B (no noticeable differences are observed for different cases and connectivities)(.) For low values of f, only the most risk averting individuals have a significant chance to get rich. But, to get rich, β should be of the order of 0.6 to 0.8, as bigger values of β imply very low sums put at stake. For high values of f the situation changes to become more uniform; even very risk-prone individuals are able to get very wealthy. However, for high values of f the wealthier individuals are significantly poorer than their counterparts for low values of f: the richer agents have a wealth of the order of $4 \times w_{max}$ for $f = 0.5$ and $12 \times w_{max}$ for $f = 0.1$. Concerning the lower classes, it is possible to see on the logarithmic representation that a risk aversion of the order of 0.5 or higher guarantees a finite wealth of the order of $1/10 \times w_{max}$ for $f = 0.25$.

We conclude that the introduction of a correlation between connectivity and success in "commercial" exchanges produces a wealth distribution and Gini coefficients different than that of an static social lattice, but the effects are not as evident as expected. The more important parameter is still f, the probability of favoring the poorer agent in each exchange, and, in a minor degree, the average connectivity of the lattice. For low values of f the wealth distribution is very unfair, still worse than in real societies. A finite fraction of the order of 2/3 of the population has almost zero wealth while for the richer classes one obtains a rather robust power law. For high values of f the distribution presents a maximum for finite values of the wealth, of the order of the average wealth, but there is still a finite fraction of agents with almost zero wealth. The effect of the average connectivity is also more evident for low values of f but in all cases it seems that increasing the average connectivity is the best way to obtain power laws in the different situations studied. The effect of a dynamical network can be boosted either considering a correlation between the wealth of the agents, as in ref. [24], or modifying the rule that no agent can win more than the amount he puts at stake, because this strong constraint limits social and economical mobility.

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The Monomodal, Polymodal, Equilibrium and Nonequilibrium Distribution of Money

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Summary. The distribution of money for several countries is analyzed according to the Bolztmann-Gibbs distribution with explicit consideration of the degeneracy of states. At high values of money the experimental data are systematically larger than the values corresponding to the BG statistics. The use of Tsallis non extensive statistics results in a good fit in the whole range of income values, converging to Paretos law in the high money limit and indicating the fractal nature of the distribution. In some cases, the distribution has two or more components, which, according to model calculations, arise from the different degeneracy of each ensemble. Criteria to determine whether this situation corresponds to equilibrium are analysed.

1 Introduction

The shape and origin of the income distribution is of utmost importance in order to develop models to explain it and to analyse the causes of inequality [1].

The higher end of the distribution of money seems to follow a power law of universal character, as shown by Pareto more than a century ago [2]. Several attempts were made in order to explain this intriguing behaviour [3-7] as well as the low and medium region income [7-11]. Recently, a Boltzmann-Gibbs (BG) distribution has been proposed to account for the income distribution for several countries, that, notwithstanding, does not follow Paretos law in the high income limit [12-14]. Therefore, the behaviour of the distribution in the whole range of money requires the use of two functions, one for the high and one for the low and medium income region. As shown in the present communication, this discrepancy can be settled using the non extensive statistics proposed by Tsallis [15,16], thus indicating the fractal nature of the distribution. Tsallis statistics has also been used by other authors in connection with the distribution of money [17,18].

In addition to this difficulty, the income distribution of several countries shows the presence of more than one component in the intermediate region.

While this polymodal character of the distribution can be easily accounted for by a superposition of functions it should also be considered whether this behaviour corresponds to a real equilibrium or to an intermediate state in the temporal evolution of the system. The purpose of this work is to show that the Tsallis function can account for the distribution of money in the whole range of money values, to find criteria to establish if the distribution corresponds to equilibrium and to analyze bimodal cases to obtain information on the equilibrium and its social consequences.

2 The distribution of money

In a previous report the income distributions of the UK, Japan and New Zealand were shown to follow quite closely the BG function, when the degeneracy of states is considered [14]. Assuming that the degeneracy is proportional to money, m, the BG equation becomes the Gamma function, i.e.

$$P_i(m) = Nm^{(\alpha-1)} \exp(-m/\beta) \tag{1}$$

The data for several countries, exemplified by those of Japan, New Zealand and the UK in selected years, could be well reproduced by this function.

However, more detailed consideration, as evidenced in a log plot, shows that this distribution strongly deviates from the experimental data in the high income limit (Fig 1).

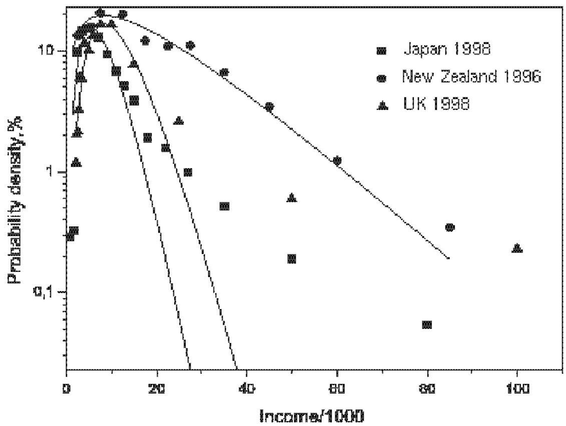

Fig. 1. Probability density vs. money for the income data of Japan and UK, 1998 and New Zealand 1996. The solid lines correspond to the Gamma function. The income axis has been scaled according to the following factors: New Zealand: 1; UK: 2; Japan: 500.

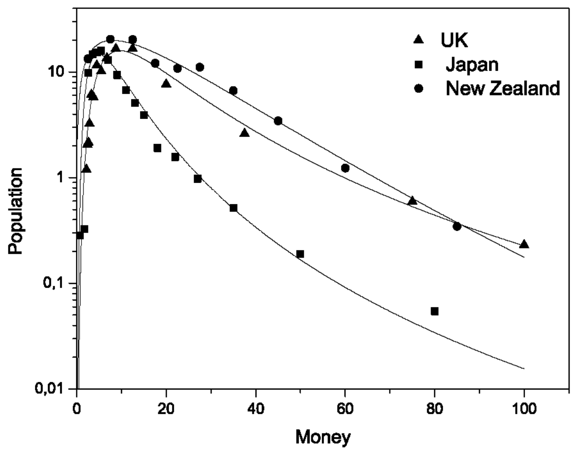

Fig. 2. Probability density vs. money for the income data of Japan and UK, 1998, and New Zealand 1996. The solid lines correspond to the Tsallis function.

However, the data could be fit to a Tsallis function in the whole money range, with a value of q close to 1.1 (Fig. 2). The lines shown in that figure corresponds to $q = 1.13$, 1.13 and 1.10, for Japan, UK and New Zealand, respectively.

Tsallis equation for the probability density is:

$$P_i = Ng_i[1 - (1-q)Bx]^{1/1-q} \tag{2}$$

and reduces to the usual BG equation

$$P_i = Ng_i \exp(-x/\beta) \tag{3}$$

for $q = 1$. In Eq.2, g_i is the degeneracy of states, B is a constant and q is a parameter associated with the dimension of the system. This equation has been successfully applied to a variety of problems. For $x = m$, at high values of m Tsallis function becomes $P_i = Nm^n$, where $n = \alpha - 1 + 1/1 - q$, which, for $q > 1$ and $(\alpha - 1) < |1/(q-1)|$ becomes Paretos law.

Therefore, non extensive statistics not only accounts for the distribution of money in the whole income range, when the degeneracy of states is properly considered but also shows its fractal nature. However, taking into account that Eq.2 is more difficult to use than Eq. 3 and that the BG statistics produces results in satisfactory agreement with the Tsallis function if the small fraction associated with the tail of the distribution is neglected, we will assume, for the present purposes, than Eq. 3 applies.

Another point to be considered is that the income distribution in many cases shows bimodal (or, in general, polymodal) behaviour. One example is obtained from the data of Japan for fiscal year 1998. These data are presented in Fig. 3, together with the fit to a double gamma function and the individual components of the distribution.

Since β is related to temperature [12, 13, 19], polymodal distributions for systems in equilibrium should be characterised by a unique value of β. This seems to be the case for the income distribution of Japan 1998, shown in Fig.3, where the value β for both components are the same, within the experimental error ($\beta_1 = 0.8 \pm .3$ and $\beta_2 = 1.1 \pm 2.1$).

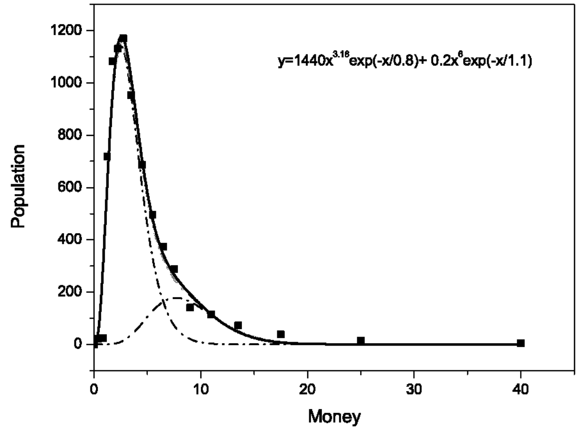

Fig. 3. Income distribution for Japan 1998, showing the two components of the distribution

A notable example of polymodal distributions is provided by the data from Argentina, during the economic crisis at the beginning of 2002 (Fig. 4) [20]. In this case, the components cannot be satisfactorily fit to a combination of BG functions, but they are well reproduced by the addition of Gaussian functions.

3 The evolution of the distribution

In an attempt to gain a further insight on the nature of multiple components in the income distribution, we made model calculations based on the

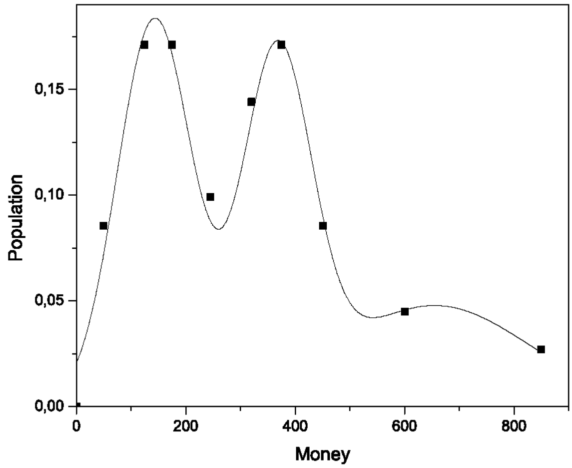

Fig. 4. Income distribution for Argentina, May 2002, showing the three components of the distribution

rate equations for the transference on money between pairs of agents in a society, following the same method than in our previous work [14]. However, we have now considered the case that an initial BG distribution, characterized by single values of α and β could change to two different ensembles, A and B, with the same value of β but different values of α, $alpha_A$ and α_B at constant total energy.

In these calculations we used $\beta = 40$ a.u. and an initial monomodal gamma distribution with a value of $\alpha = 3$, which yields an average money $< M >= \alpha\beta = 120$ a.u. The final statecharacterized by $\alpha_A = 2$ and $\alpha_B = 5$, that is, $< M_A >= 80$ a.u. and $< M_B >= 200$ a.u., that, in order to keep the total money constant requires a fractional final population for A, $P_A = 2/3$ and for B, $P_B = 1/3$.

In the absence of any flow of money in and out of the ensemble, the populations of A and B in money level i change in time according to the following master equations:

$$\frac{dn_i^A}{dt} = \omega_{AA} \sum_j P_{ij}^{AA} n_j^A + \omega_{AB} \sum_j P_{ij}^{AB} n_j^A - \omega_{BA} \sum_j P_{ij}^{BA} n_j^A - \omega_{AA} \sum_j P_{ij}^{AA} n_j^A - k_{AB} n_i^A + k_{BA} n_I^B \quad (4)$$

$$\frac{dn_i^B}{dt} = \omega_{BB} \sum_j P_{ij}^{BB} n_j^B + \omega_{AB} \sum_j P_{ij}^{AB} n_j^B - \omega_{BA} \sum_j P_{ij}^{BA} n_j^B - \omega_{BB} \sum_j P_{ij}^{BB} n_j^B - k_{BA} n_i^B + k_{AB} n_i^A \quad (5)$$

where P_{ij}^{XZ} is, in general, the probability of money transference from level j to i of X by interaction with Z, with interaction frequency ω_{XZ} and population n_i^X. The coefficients k_{zx} stand to account the rate for the rate of change between both ensembles, X and Z.

Integration of the set of equations 3 and 4 requires values for P_{ij}^{XZ}. The values of these elements for the gain of money are related to those for money loss by detailed balance, i.e.

$$\frac{P_{ji}}{P_{ij}} = \left(\frac{n_j}{n_i}\right)_{eq} = \left(\frac{g_j}{g_m}\right) \exp(-(M_j - M_i)/\beta) \quad (6)$$

This restriction, together with the condition of detailed balance for the back and forward rate of conversion of A into B, assures that the composition of equilibrium will be that of the BG distribution. A similar equation could also be imposed on the Tsallis distribution, if it were used instead of the BGs.

Therefore, the final state to be reached is determined by detailed balance, while the instantaneous value of the population distribution will depend on the values of the transition probabilities and the rate constants.

Several different calculations were made. In all of them, the elements P_{ij}^{XZ} were calculated using a normalized exponential model

$$P_{ij}^{XZ} = N \exp[-(M_i - M_j)/ < \Delta M >] \quad (7)$$

so that the transference of small amounts of money prevails.

In most of the calculations the rate coefficients were taken as constant, independent of the level of money and with the condition $k_{AB} = k_{BA}/2$. In a few cases, not presented here, these coefficients were assumed to increase with i. The average amount of money transferred per interaction, $< M >$, was set equal to 10.

A representative calculation is shown in Fig.5. The initial BG distribution separates into two different sets to finally reach the corresponding BG equilibrium composition. A similar calculation but starting from two well separated initial distributions should merge into the same final state, if the same parameters were used.

It should be noted that the curves presented in Fig. 5 for the intermediate states in the evolution to equilibrium can be very well reproduced by BG functions although with different values of β. Thus, the curves whose maxima are shifted to lower money values show a decrease of β from the initial

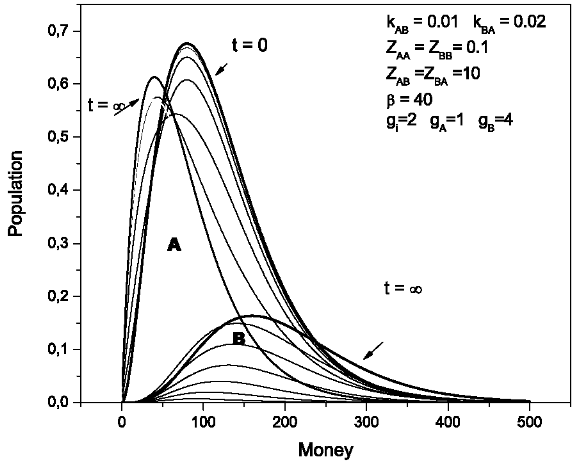

Fig. 5. Dissociation of a single initial distribution into two different ensembles, A and B.

value of 40 a.u. to around 20 a.u., followed by a relatively slower increase to the equilibrium value. In correlation with this, the value of β for the other ensemble initially increases and then decreases as money is equilibrated. This behaviour arises from the fast rate of change of A into B, as compared with money transference, so that an initial disequilibrium appears.

These results show that an equilibrium society could dissociate in two different groups, while still maintaining equilibrium, if a change if the properties of its components takes place. This change is evidenced by the value of α, that is, the degeneracy of states. A larger value of α increases the ability of the agents in that group to accommodate the money they have and the distribution moves to larger money values, while a decrease of α produces the opposite effect. The segregation results in a broader total distribution, given by the addition of the distributions of A and B, with more differences between the rich and the poor. Note however that both ensembles have poor and rich components, even thought in different proportions.

The same argument applies if the two groups were two different countries. The conclusion is that the key to a richer and egalitarian society (world) depends on the ability to increase $< m >$ by increasing the value of α. On the contrary, an increase in richness as a result of a larger value of β causes a broader distribution, with more differences between the poor and the rich.

An additional difficulty, not easy to overcome, is the lack of experimental data of the nonequilibrum distribution of income a various times. One recent example could be obtained from the evolution of the economy in Argentina around the end of the 20th century and the beginning of the present. The crisis attained its maximum intensity between the end of 2001 and the first quarter of 2002, which is clearly evidenced in the income distribution. According to the data available, the money distribution in Argentina was variable and showed certain bimodality. A few representative examples are shown in Figs.

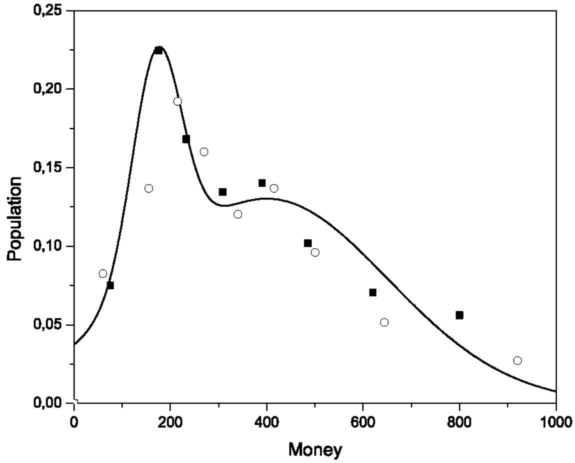

Fig. 6. Income distribution for Argentina, (open circles) October 2001, (filled squares) May 2004. The solid curve represents the fit to the data of October.

The data corresponding to May 2002 are fitted by three Gaussian functions, in agreement with the previous study that a system far from equilibrium evolves to the BG distribution through Gaussian distributions.

The income distribution in previous years could be fit to a single Gamma function, although bimodality was always present. However, as the crisis developed, the low and medium region of the data could only be fit to Gaussian functions. The distortion reached its maximum in May 2003 and seemed to tend to return to a more normal shape in 2004.

The appearance of a Gaussian shape in the distribution is expected according to model calculations presented before, for the evolution of a system far from equilibrium.

4 Conclusions

The main findings reported in this work are:

1. Monomodal distributions can be reproduced by the sole use of Tsallis non extensive statistics, with scaling factors close to 1.1. While others studies required the addition of two functions which separately fit the medium and

high income regions, where Paretos law is obeyed, Tsallis function fits both regions simultaneously.

2. Bimodal distributions *per se* do not indicate a deviation from equilibrium. Equilibrium is characterised by a single value of β for all the ensembles of the system. A rate equation analysis of the evolution of the populations indicate that a society with a monomodal BG distribution could dissociate in separate ensembles, to attain a new equilibrium.

3. The income distribution of Argentina during the economy crisis in the period 2001-2004 shows polymodal components, with Gaussian shapes, which is one of the characteristics of a system out of equilibrium. The other criterium is observing BG functions but with different values of β.

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Relieving Poverty by Modifying Income and Wealth Distributions

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Summary. The paper starts with the assumption that income and wealth distributions are composed of Boltzmann distributions with power low tails at the high end. Examples of alternative energy distributions found in physical systems are discussed, and how they could be used to construct economic models that might allow alternative overall distributions of wealth and income in society. These ideas are further expanded to show alternative ways in which poverty could be tackled, both within individual countries, and globally.

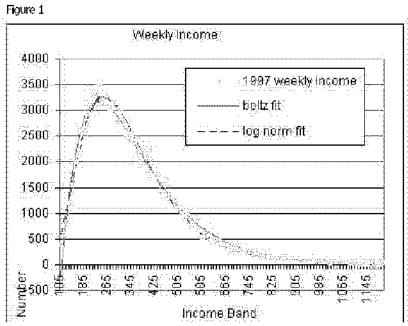
1 Background: Income Distribution and Statistical Physics

Since the work of Pareto as long ago as 1897, it has been known that distributions of wealth or income have appeared to be log normal distributions that follow power law decays at the high end. These distributions have been observed across a wide variety of different economies over long periods of time. Traditionally this has been an economic puzzle; as intuitively different income distributions would be expected in differently structured economies.

In recent years, the study of income distributions has gone through a small renaissance with new interest in the field shown by physicists with an interest in economics. The work of many in this field has demonstrated that income and wealth distributions are Maxwell-Boltzmann / Gamma distributions at low and medium level, with power law tails at the high end. Support for these theories come from raw data [1, 2, 3, 4], theoretical [5, 6, 7, 8, 9, 10, 11] and modeling approaches [12, 13, 14, 15, 16]. This is also discussed at length in many of the other papers in this volume.

As an example; Figure 1 demonstrates the close correlation that can be seen between actual economic data and Boltzmann distributions in the UK.

Although this has not been formally accepted in the wider economics community; this paper takes as a starting point the assumption that income and wealth are distributed on an econodynamic basis, and that the overwhelming majority of income is distributed as a Maxwell-Boltzmann function as a result of maximum entropy considerations.



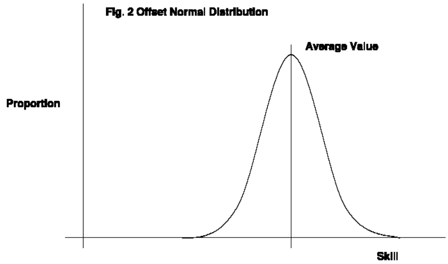


Fig. 1. Boltzmann Distribution Fitted to Income Data.
Fig. 2. Offset Normal Distribution.

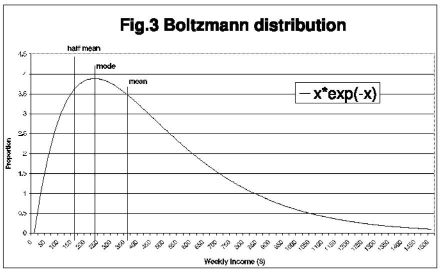
For physicists and others that are familiar with a maximum entropy approach it is not then a surprise that the same distributions of income are seen in widely different economic systems. From a statistical mechanical point of view; as the number of participants in a system increases, the underlying mechanisms of exchange (whether this is of energy or wealth) become irrelevant; and the resulting distribution is simply the one that is statistically most likely. This has very important consequences, both for the effects that such distributions have on human life, and the ways that human beings can affect these distributions.

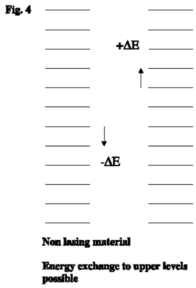
Firstly it is worth considering the appropriateness of the Boltzmann distribution as a method for sharing wealth amongst humanity. Most human abilities are found to be distributed on the basis of a normal distribution as shown in Fig. 2. The tails of this distribution decline to zero rapidly, and the mode usually has a large offset from zero when describing human qualities. In such a distribution the mean, median and mode averages coincide very closely.

The result is that for most human skills the variation in ability between the top decile and the bottom decile is only of the order of a factor of two or three or so, very rarely by factors of ten or more.

The above is not true of course for learnt skills, it is however generally true of the ability to learn these skills. Given these ranges of human abilities it is possible to construct an argument that a "fair" economic distribution would be one similar to an offset normal distribution.

The Boltzmann distribution (see Fig. 3.) however is markedly different in two important respects. Firstly it is skewed; the mode average is considerably below the mean average, with the median somewhere between these two. Secondly it has a long tail with significant numbers of extreme events populated at levels considerably above either the median or mean averages. In a Boltzmann distribution the bottom decile lies close to the zero axis and has wealth significantly less than the average wealth. The top decile has wealth considerably in excess of the average wealth. Two other things can be noted with regard to the distribution. Firstly, that the displacement between the mean and the median results in a significant majority of individuals having less than the mean value of wealth. Secondly there is automatically a portion, roughly 15% of all individuals, who are permanently below half the mean average income; that is, the normally defined poverty level. (It should be noted that adding a power law tail to this distribution further skews the distribution, so increasing the disparities between rich and poor).





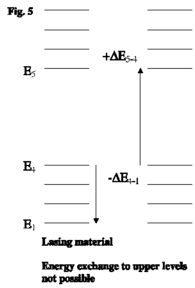


Fig. 3. Boltzmann Distribution.
Fig. 4 and 5. Energy Levels in Non-lasing & Lasing Materials.

It is possible from the above to construct an argument that the Boltzmann distribution is not a "fair" way for wealth or income to be distributed in a society. Certainly social democrats, socialists and communists have constructed such arguments and have offered differing solutions to solving this perceived problem.

In communist states strict, and active, microeconomic control was the normal way of attempting to prevent large discrepancies in wealth. In democratic countries this has generally been avoided, because of the stunting effects on economic growth. Instead these countries have instituted large scale systems of taxation and welfare in an attempt to transfer wealth from the rich to the poor. Meanwhile trade unions and professional societies also attempt to modify wealth distributions for there own members.

From an econophysics point of view, the above methods of attempting to influence wealth distribution are deeply flawed. In a system of a large number of freely interacting particles the Boltzmann distribution is inevitable and methods of exchange, even ones such as tax and welfare, are largely irrelevant.

From an econodynamic perspective, an approach that does make some sense is that of the trade unionists and professional societies. By tying together the interests of thousands, or even millions, of individuals their members are no longer "freely interacting" and are able to release themselves from the power of entropy to a limited extent. (Monopolistic companies attempt to subvert entropy by similar means).

Traditional methods of taxation and welfare seem to have much less justification. It is common experience that such transfers give little long-term benefit to the poor. Transfers need to be massive and continuous to be effective, and there is a wealth of data to suggest that many welfare programmes result in the giving of benefit to those of medium income, rather than to the poor. This is of course exactly what an econodynamic analysis would predict.

Given the power of entropy to force the overall distribution regardless of different sorts of microeconomic interactions, it would initially seem that attempting to modify income distribution will be futile. This is not necessarily the case.

A possible approach is to look at analogies from other physical systems, which could be used as alternative economic models. I intend to follow this approach in the next two sections of this paper.

2 Alternative Distributions

The distinctive skewed shape of the Boltzmann distribution is a result of the particular boundary conditions found in most energetic systems. In an ideal gas the positive energy of any individual molecule is effectively unlimited; the molecule can go as fast as it wants. There is however a very clear boundary condition in the other direction, it is impossible for any molecule to have negative energy, once it is stopped it can not go any slower. It is this boundary condition that forces the skewness in the Boltzmann distribution. For a few of the molecules to have a lot of energy it is not possible for a few to have a lot of negative energy. Instead a summation over all possible assemblies dictates that a lot of molecules must have a little energy to compensate for the few with a lot of energy.

(In economic systems this zero boundary condition is due to the difficulty any human being has in maintaining significant long term values of negative wealth; a growing problem with the increasing levels of communication between credit agencies.)

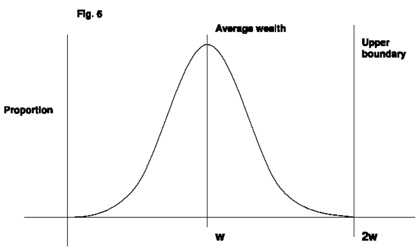
There are a small number of systems that do not show the typical Boltzmann distribution of energy, the most obvious of which is the laser.

A non-lasing material has closely spaced energy gaps. The exchange of photons between molecules can result in free interchange of energy states, with one molecule increasing in energy and the other decreasing; as shown in Fig. 4.

A lasing material typically has a closely spaced band of lower energy levels with a large gap between this band the next one above. If a molecule is already at the top of this lower band (at E_4) it is unable to go to the next energy level up because there is no other molecule available in bands $E_1 - E_4$ that can make an equal large jump down. In Fig. 5. ΔE_{4-5} is greater than ΔE_{1-4}, so no molecule can jump from E_4 to E_5, because their is no matching drop available to keep total energies balanced. If the material is kept isolated, this lower band can be given a very high occupation of energy levels "pumped" by an external source. Again, if kept isolated; this "inverted" distribution can be maintained for a significant time. Such an inverted distribution is not inherently thermodynamically stable; which is part of the reason that the release of energy is so intense when a laser is allowed to interact with its external environment.

The reason for the inversion of the distribution is the existence of an effective upper limit on the distribution. It is possible that such an approach could be used in an economic system.

Given a hypothetical isolated economic population N with total Wealth W and average wealth $w = W/N$, let us assume that a law is introduced that dictates that any individual that has more than double the average wealth is committing a criminal offence and is jailed. So the range of assemblies over which the total possible distributions is to be calculated is now limited at $2w$ instead of infinity. Any distribution that has a person with wealth greater than $2w$ must be discarded from the total of assemblies.



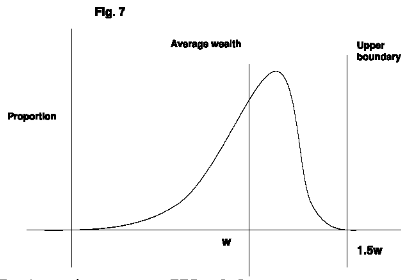


Fig. 6. Wealth Limit at Twice Average Wealth.
Fig. 7. Wealth Limit at 1.5 Times Average Wealth.

By symmetry this would result in bell shape running from zero at zero, to zero at $2w$, and having a maximum at w; Something like Fig. 6, and similar to a "wide" normal distribution offset from zero to w.

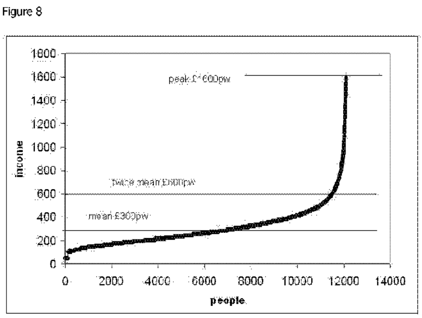
Such a distribution would move a significant group of people out of the lower wealth levels and could be perceived as being more fair in its overall sharing of wealth.

It is possible to go further; if the maximum wealth were set at 1.5 times the average wealth say, then a laser like inverted distribution of the form shown in Fig. 7. would result. Statistically the occupation of the lower levels would be very small indeed.

While these ideas are theoretically sound they have very fundamental flaws as practical ways of running modern economies.

The first obvious problem is that of isolation. If a maximum wealth (or maximum income) law was introduced in a typical western economy then the individuals affected by the law would simply move themselves or their excess wealth to another economy without such a law. To be effective such a maximum wealth law must either apply to a nation that is isolated with strict controls on both emigration and financial transfers, or the law would have to be applied simultaneously to all interacting economies without the exemption of a single offshore tax haven.

The second obvious problem is the process of actually moving from a free Boltzmann distribution to a "capped" distribution. This would involve a substantial reduction in wealth for a significant (and influential) proportion of the population. Perhaps more importantly it might create a perceived loss of opportunity for a much larger portion of the population, and would almost certainly be seen as an infringement of liberty in most societies.



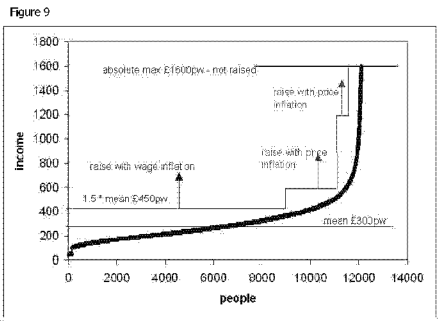


Fig. 8. Ranked Income in the UK. **Fig. 9.** Salary Caps on Income.

Figure 8 shows the income distribution of people in the UK from the NES earnings survey for 2002. Each point on the graph represents ten people in the survey, which in turn is one percent of all taxpayers (12 million people). The graph can be thought of as a very finely divided bar chart ranked from the lowest to the highest wage earners. It should be noted that the cut off of £1600 per week is a consequence of the survey, and that a small percentage of high earners have not been caught in this data. Notwithstanding this, the graph is representative of the typical shape of earnings in a free market society.

In this example the average wage is £297pw. If a maximum wage was set at twice this average at £600, then fully 5% of the population would be above this cut off. Reducing these peoples salaries by half to two-thirds would be very politically difficult.

A possible solution to this would be to introduce 'stepped-caps'. Figure 9 shows a simplistic example. Here the top cap would be set at the existing maximum, and would not be changed, or even increased with inflation. The bottom cap is set at one and a half times the average wage, and would be increased with wage inflation. The intermediate caps would increase with price inflation, which is generally less than wage inflation. After very many years the caps would eventually catch up with each other to achieve a single cap. This would slowly pressurise the system and produce a more equitable distribution.

Again these proposals would be slow acting, and are likely to be seen as politically unacceptable in a free society.

3 Creating Local Isolated Systems - Laser Welfare

In this section a welfare system is proposed that would operate within an economy but be isolated from the economy, apart from the subsidies needed to keep the isolated system functioning.

The system to be created will consist of one thousand unemployed people. These people are assumed to be "priced out of the market" their existing skills are not productive enough for them to be attractive to an employer in the open market. It is intended to provide subsidies to these people to help them into jobs. These subsidies will be equivalent to the "pumping" that takes place in a laser.

To build the system a number of items are needed. The first is a separate system to represent financial wealth. In this system these will be referred to as "coupons". Each coupon (C1) will be redeemable from the government for $1 cash.

The second item needed is a way of limiting the number of coupons an individual is allowed to earn. This is achieved by issuing each person in the scheme an allowance book. Each week the beneficiary will be allowed to claim cash from the government against any coupons they have gained that week up to a maximum of each week's allowance. Allowances are strictly non-transferable, whilst coupons are freely tradable.

The government uses subsidies and the free market to keep the system circulating. It is assumed that the beneficiaries are only 50% of the efficiency of a typical person employed on low wages. It is also assumed that the government wants each beneficiary to earn around $100/week; this is deemed to be sufficient to meet their basic needs. Suppose there are 1000 beneficiaries in the scheme, then each beneficiary is given an allowance book that allows the beneficiary to cash in up to, but no more than, 120 coupons ($C120$) each week. Therefore, each week the total of available allowances will be $C120,000$.

However each week the government will only release $C100,000$. These coupons will be released by auctioning them to a number of registered employers taking part in the subsidised labour scheme. The employers will purchase the coupons from the government each week for cash. The employers will then exchange them with the beneficiaries in exchange for their (inefficient) labour.

This may seem a very complex way of getting money into the hands of unemployed people, but it does have some positive effects. A micro economy has been created in which different employers compete to buy coupons from the government, and different beneficiaries compete with each other. However the competition for the beneficiaries is not so fierce. A closed system has been created in which average occupancy is 83% (100/120), an inverted distribution will therefore result, and the number of beneficiaries earning less than say $80 will be very small.

Initially the price paid by the employers to the government would probably be very low, $0.1/Coupon or something of this order, but competition should drive this price up to around $0.5/Coupon, as we assumed our beneficiaries were 50% efficient (this ignores bureaucratic costs). This is very useful for the government as they are now only paying out a net $0.5 for each $1 that reaches the pocket of the beneficiary. However the process should not stop there. With competition in place it is to the advantage of both the employer and the beneficiary to improve the efficiency of the beneficiary. The employer that is able to make the best use of the employee is the one that will make the most profit at a certain auction price for coupons. The beneficiary that can increase the value of his skills to his employer is the one that is likely to earn closer to 120 coupons rather than only end up with 80 coupons. Healthy competition has advantages for both parties.

A more detailed account of the workings of such a scheme are given in a previous paper [17].

Clearly the above is very simplistic and will need substantial research and development before it could be worked into a real life welfare program. Like any welfare or taxation program there will be opportunities for fraud; ways in which human beings can breakdown the barrier between the two "isolated" systems. Also, as in any scheme that subsidises labour, there are likely to be significant problems with displacement of jobs.

However the main point that is being made is that a knowledge of the principles of econophysics / econodynamics may be potentially used to create alternative and effective financial systems.

4 Conclusions

Econophysics is a relatively new science; while dramatic intellectual insights have been made, progress to date has largely been observational, with explanations being given for existing phenomena.

In this paper an engineering approach has been taken that uses assumed knowledge of the underlying mechanisms of wealth / income distributions to propose possible effective ways of changing economic systems.

It is highly likely that these initial ideas are far too simplistic to be practical in the forms described above. It is also the case that the decisions to make such changes would be essentially political. It is hoped however that the ideas above show a possible line of enquiry that could prove more practical in redistributing income in the long term than current policies of transfers of taxation to welfare.

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The Rich *Are* Different ! Pareto Law from Asymmetric Interactions in Asset Exchange Models

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Fitzgerald: The rich are different from you and me
Hemingway: Yes, they have more money

Summary. It is known that asset exchange models with symmetric interaction between agents show either a Gibbs/log-normal distribution of assets among the agents or condensation of the entire wealth in the hands of a single agent, depending upon the rules of exchange. Here we explore the effects of introducing asymmetry in the interaction between agents with different amounts of wealth (i.e., the rich behave differently from the poor). This can be implemented in several ways: e.g., (1) in the net amount of wealth that is transferred from one agent to another during an exchange interaction, or (2) the probability of gaining vs. losing a net amount of wealth from an exchange interaction. We propose that, in general, the introduction of asymmetry leads to Pareto-like power law distribution of wealth.

1 Introduction

"The history of all hitherto existing society is a history of social hierarchy" – *Joseph Persky* [1]

As is evident from the above quotation, the inequality of wealth (and income) distribution in society has long been common knowledge. However, it was not until the 1890s that the nature of this inequality was sought to be quantitatively established. Vilfredo Pareto collected data about the distribution of income across several European countries, and stated that, for the high-income range, the probability that a given individual has income greater than or equal to x is $P_>(x) \sim x^{-\alpha}$, α being known as the Pareto exponent [2]. Pareto had observed α to vary around 1.5 for the data available to him and believed $\alpha \simeq 1.5$ to be universal (i.e., valid across different societies). However, it is now known that α can vary over a very wide range [3]; furthermore,

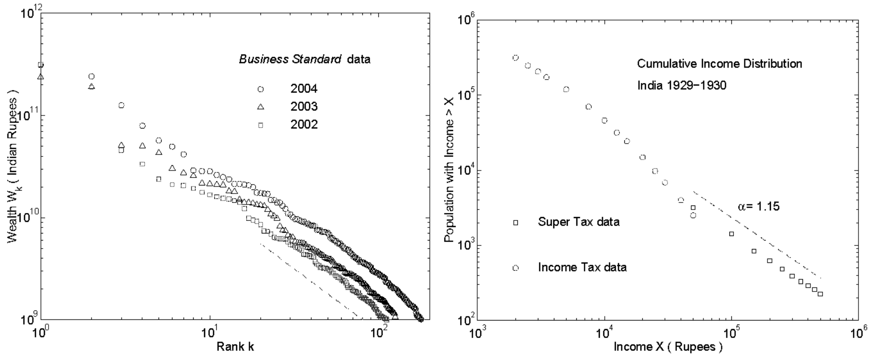

Fig. 1. Wealth and income distribution in India: (Left) Rank ordered wealth distribution during the period 2002-2004 plotted on a double-logarithmic scale, showing the wealth of the k-th ranked richest person (or household) in India against the rank k (with rank 1 corresponding to the wealthiest person) as per surveys conducted by *Business Standard* [7] in Dec 31, 2002 (squares), Aug 31, 2003 (triangles) and Aug 31, 2004 (circles). The broken line having a slope of -1.23 is shown for visual reference. (Right) Cumulative income distribution during the period 1929-30 as per information obtained from Income Tax and Super Tax data given in Ref. [8]. The plot has Gibbs/log-normal form at the lower income range, and a power law tail with Pareto exponent $\alpha \simeq 1.15$ for the highest income range.

for the low-income end, the distribution follows either a log-normal [4] or exponential distribution [5]. Similar power law tails have been observed for the wealth distribution in different societies. While wealth and income are obviously not independent of each other, the exact relation between the two is not very clear. While wealth is analogous to *voltage*, being the net value of assets owned at a given point of time, income is analogous to *current*, as it is the net flow of wages, dividends, interest payments, etc. over a period of time. In general, it has been observed that wealth is more unequally distributed than income. Therefore, the Pareto exponent for wealth distribution is smaller than that for income distribution.

Most of the empirical studies on income and wealth distribution have been done for advanced capitalist economies, such as, Japan and USA. It is interesting to note that similar distributions can be observed even for India [6], which until recently had followed a planned economy. As income tax and other records about individual holdings are not publicly available in India, we had to resort to indirect methods. As explained in detail in Ref. [6], the Pareto exponent for the power-law tail of the wealth distribution was determined from the rank-ordered plot of wealth of the richest Indians [Fig. 1 (left)]. This procedure yielded an average Pareto exponent of $\simeq 1/1.23 = 0.82$. A similar exercise carried out for the income distribution in the highest income range produced a Pareto exponent $\alpha \simeq 1.51$. Surprisingly, this is identical to what Pareto had thought to be the universal value of α. Comparing this with historical data of income distribution in India [8], we again observe the power-law tail although with a different exponent [Fig. 1 (right)]. In addition, we note that the low-income range has a log-normal or Gibbs form very similar to what has been observed for advanced capitalist economies [4]. In the subsequent sections, we will try to reproduce these observed features of wealth & income distributions through models belonging to the general class of asset exchange models.

2 Asset exchange models

Asset exchange models belong to a class of simple models of a closed economic system, where the total wealth available for exchange, W, and the total number of agents, N, trading among each other, are fixed [9, 10, 11, 12, 13]. Each agent i has some wealth $W_i(t)$ associated with it at time step t. Starting from an arbitrary initial distribution of wealth $(W_i(0), i = 1, 2, 3, \ldots.)$, during each time step two randomly chosen agents i and j exchange wealth, subject to the constraint that the combined wealth of the two agents is conserved by the trade, and that neither of the two has negative wealth after the trade (i.e., debt is not allowed). In general, one of the players will gain and the other player will lose as a result of the trade. If we consider an arbitrarily chosen pair of agents (i, j) who trade at a time step t, resulting in a net gain of wealth by agent i, then the change in their wealth as a result of trading is:

$$W_i(t+1) = W_i(t) + \Delta W; W_j(t+1) = W_j(t) - \Delta W,$$

where, ΔW is the net wealth exchanged between the two agents.
Different exchange models are defined based on how ΔW is related to $[W_i(t), W_j(t)]$. For the *random exchange* model, the wealth exchanged is a random fraction of the combined wealth $[W_i(t) + W_j(t)]$, while for the *minimum exchange* model, it is a random fraction of the wealth of the poorer agent, i.e., $min[W_i(t), W_j(t)]]$. The asymptotic distribution for the former is exponential, while the latter shows a condensation of the entire wealth W into the hands of a single agent [Fig. 2 (left)]. Neither of these reflect the empirically observed distributions of wealth in society, discussed in the previous section.
Introducing savings propensity in the exchange mechanism, whereby agents don't put at stake (and are therefore liable to lose) their entire wealth, but put in reserve a fraction of their current holdings, does not significantly change the steady state distribution [10]. By increasing the savings fraction (i.e., the fraction of wealth of an agent that is not being put at stake during a trade), one observes that the steady-state distribution becomes non-monotonic, although the tail still decays exponentially. However, randomly assigning different savings fractions (between [0,1]) to agents lead to a power-law tail in the asymptotic distribution [13].
This result raises the question of whether it is the differential ability of agents to save that gives rise to the Pareto distribution. Or, turning the question

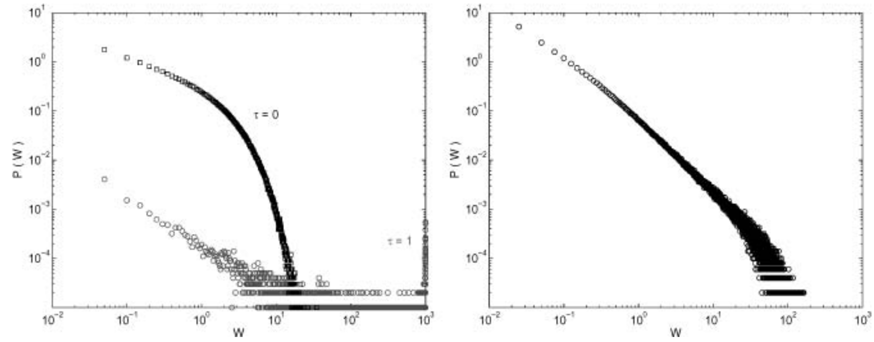

Fig. 2. (Left) Asymptotic wealth distribution for the random exchange model ($\tau = 0$: exponential distribution) and the minimum exchange model ($\tau = 1$: condensation). (Right) Power law wealth distribution with exponent $\simeq -1.5$ for the asymmetric exchange model with $\tau = 0.99$. All figures shown for $N = 1000$, $t = 1 \times 10^7$ iterations, averaged over 2000 realizations.

around, we may ask whether the rich save more. This question has been the subject of much controversy, but recent work seems to have answered this in the affirmative [14]. As mentioned in a leading economics textbook, savings is the greatest luxury of all [15] and the amount of savings in a household rises with income. In terms of the asset exchange models, one can say that an agent with more wealth is more likely to save (or saves a higher fraction of its wealth). Implementing this principle appropriately in the exchange rules, one arrives at the *asymmetric exchange* model.

3 Asymmetric exchange model

The model is defined by the following exchange rules specifying the change in wealth, $W_A(t+1) - W_A(t)$, of agent A who wins a net amount of wealth after trading with agent B $[W_B(t+1) - W_B(t) = W_A(t) - W_A(t+1)]$:

$$W_A(t+1) = W_A(t) + \epsilon(1 - \tau[1 - \frac{W_A(t)}{W_B(t)}])W_B(t), \text{ if } W_A(t) \leq W_B(t),$$

$$= W_A(t) + \epsilon W_B(t), \text{otherwise},$$

where ϵ is a random number between 0 and 1, specifying the fraction of wealth that has been exchanged. For $\tau = 0$, this is the random exchange model, while for $\tau = 1$, it is identical to the minimum exchange model [Fig. 2 (left)]. In the general case, $0 < \tau < 1$, the relation between the agents trading with each other is asymmetric, the richer agent having more power to dictate terms of trade than the poorer agent. The parameter τ (*thrift*) measures the degree to which the richer agent is able to use this power.

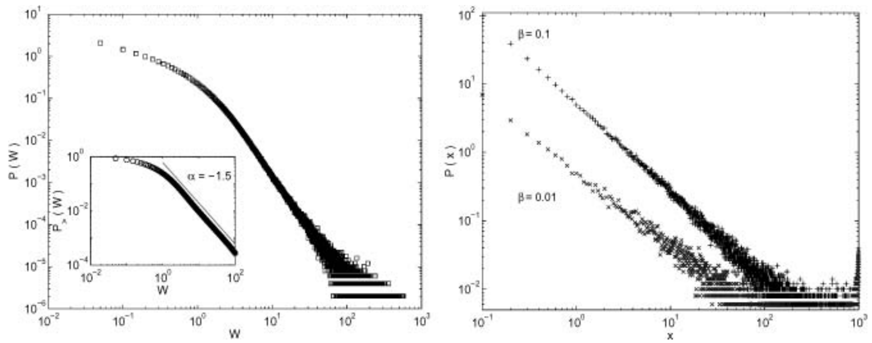

Fig. 3. (Left) Asymptotic wealth distribution (inset shows the cumulative distribution) with a power-law tail having Pareto exponent $\alpha \simeq 1.5$, for the asymmetric exchange model with τ distributed uniformly over the unit interval $[0, 1]$ among N agents ($N = 1000$, $t = 1 \times 10^7$ iterations, averaged over 10^4 realizations). (Right) Asymptotic wealth distribution for model having asymmetric winning probability with $\beta = 0.1$ [pluses] (slope of the power-law curve is 1.30 ± 0.05) and $\beta = 0.01$ [crosses] (slope of the power-law curve is 1.27 ± 0.05). ($N = 1000$, $t = 1.5 \times 10^7$ iterations, averaged over 5000 realizations).

As τ is increased from 0 to 1, the asymptotic distribution of wealth is observed to change from exponential to a condensate (all wealth belonging to a single agent). However, at the transition between these two very different types of distribution ($\tau \to 1$) one observes a power-law distribution ! As seen in Fig. 2 (right), the power-law extends for almost the entire range of wealth and has a Pareto exponent $\simeq 0.5$. This is possibly the simplest asset exchange model that can give rise to a power-law distribution. Note that, unlike other models [13], here one does not need to assume the distribution of a parameter among agents.
However, the Pareto exponent for this model is quite small compared to those empirically observed in real economies. This situation is remedied if instead of considering a fixed value of τ for all agents, we consider the heterogeneous case where τ is distributed randomly among agents according to a quenched distribution. For an uniform distribution of τ, the steady-state distribution of wealth has a power-law tail with $\alpha = 1.5$ [Fig. 3 (left)], which is the value predicted by Pareto, while at the region corresponding to low wealth, the distribution is exponential. By changing the nature of the random distribution, one observes power-law tails with different exponents. For example, for $P(\tau) \sim \tau$, the resulting distribution has a Pareto exponent $\alpha \sim 1.3$, while for $P(\tau) \sim \tau^{-2/3}$, one obtains $\alpha \sim 2.1$. A non-monotonic U-shaped distribution of τ yields $\alpha \sim 0.73$. However, the fact that even with these extremely different distributions of τ one always obtains a power-law tail for the wealth distribution, underlines the robustness of our result.

4 Asymmetric Winning Probability Model

Asymmetry in the interaction between agents (as a function of their wealth) can also be introduced through the probability that an agent will gain net wealth from an exchange. Consider a variant of the minimum exchange model where the probability that agent A (wealth W_A) will win a net amount in an exchange with B (wealth W_B) is

$$p(A|A,B) = \frac{1}{1 + exp(\beta[\frac{W_A(t)}{W_B(t)} - 1])},$$

where $\frac{1}{\beta}$ is the indifference to relative wealth (for details see Ref. [12]). For $\beta = 0$, i.e., $p(A|A,B) = \frac{1}{2}$, the minimum exchange model is retrieved, where, in the steady state, the entire wealth belongs to a single agent (condensation). However, for a finite value of β, the poorer agent has a higher probability of winning. For large β, the asymptotic distribution is exponential, similar to the random exchange model. At the transition between these two very different types of distributions (condensate and exponential) we observe a power-law distribution of wealth [Fig. 3 (right)].

5 Discussion

The two models discussed here for generating Pareto-like distribution of wealth are both instances of the "Rich Are Different" principle, implemented in the formalism of asset exchange models. It is interesting to note that other recently proposed models for generating Pareto law also use this principle, whether this is in terms of kinetic theory as in the present paper [16, 17] or in a network context [18, 19]. This leads us to conclude that asymmetry in agent-agent interactions is a crucial feature of models for generating distributions having power-law tails.

To conclude, we have presented two models illustrating the general principle of how Pareto-like distribution of wealth (as observed in empirical observations in society) can be reproduced by implementing asymmetric interactions between agents in asset exchange models. In the models presented here the asymmetry is based on wealth of agents, with the rich agents behaving differently from the poor, either in terms of net wealth changing hands, or the probability of gaining net wealth out of a trade. One of the models is possibly the simplest asset exchange model that gives a power-law distribution. The results are also very robust, the power law being observed for a wide variety of parameter distributions. The different values of α obtained for different parameter distributions is a possible explanation of why different Pareto exponents have been measured in different societies, as well as in the same society at different times.

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Do We All Face the Same Constraints?

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1 A Brief Taxonomy of Wealth

There seems to be widespread agreement on the functional form of wealth and income distributions observing a power law tail [3, 13, 9, 14], while the left part of wealth and income distributions is somewhat more controversial and typically found to follow an exponential or Gamma-like distribution [1, 3, 13, 11]. The presentations by Clementi, Galegatti, Fujiwara, Souma, Sinha, and Yakovenko at this conference certainly point in the same direction but the evidence on income distributions clearly outweighs that on wealth—which is hardly surprising because it is much easier to observe income flows than the stock of wealth. None the less, we would like to focus our attention on the distribution of wealth and argue that the observed distributional regularities are statistical equilibrium outcomes of two distinct economic processes.

So what exactly is wealth? The *economic sources* of wealth are income, inheritance, and the revaluation of assets or liabilities. Savings are a theoretical accounting tool, essentially describing the mediation from income flows to the stock of wealth. The *economic uses* of wealth are expressed in the *composition of wealth portfolios* and are reflected in the accounting definitions of wealth: Marketable wealth, or net worth, is composed of (1) the gross value of owner-occupied housing; (2) other real estate owned by the household; (3) cash and demand deposits; (4) time and savings deposits, certificates of deposit (CDs), and money market accounts; (5) bonds (government, corporate, foreign) and other financial securities; (6) the cash surrender value of life insurance plans; (7) the cash surrender value of pension plans; (8) corporate stock, including mutual fund holdings; (9) net equity in unincorporated businesses; and (10) equity in trust funds. Subtracting the current value of mortgage debt, consumer debt, and other debt yields a household's marketable wealth. When items (6) and (7) are included, the measure is sometimes also referred to as *augmented* wealth, while the definition of *financial* wealth subtracts the net equity position in owner-occupied housing, i.e. the difference between the property value and outstanding mortgage debt.

Unfortunately, personal wealth data cannot be measured with high accuracy [2]. Nevertheless, a clear qualitative picture emerges from household survey data regarding the composition of wealth. First, the overall composition of wealth has remained fairly stable over the last two decades [15, 16]. Second, the composition of wealth by wealth class has also remained very stable over time. The rich hold most of their wealth in financial assets, investment real estate, and stakes in unincorporated businesses, while the vast majority of the population holds their wealth primarily in the form of owner-occupied housing, deposits, and pension and life insurance plans [2].

A few figures illustrate the stark contrast in portfolio compositions. In 1989, the top one percent of US wealth holders had 52 percent of their wealth invested in investment real estate and unincorporated businesses, 29 percent in traditional financial securities, 11 percent in liquid assets and only 8 percent in owner-occupied housing. In contrast, the bottom 80 percent of households held 63 percent of their wealth in the form of owner-occupied housing, 21 percent in the form of liquid assets, 10 percent in real estate and business equity, and only 6 percent in traditional financial assets. During the same year the richest one percent of the US population held 45 percent of all nonresidential real estate, 62 percent of all business assets, 49 percent of all publicly held stock, and 78 percent of all bonds. The richest 10 percent of families held 80 percent of all nonresidential real estate, 91 percent of all business assets, 85 percent of all stocks, and 94 percent of all bonds, while the bottom 90 percent of wealth holders accounted for 64 percent of all principal residences, 55 percent of the value of life insurances, 40 percent of deposits, and 38 percent of the value of pension accounts [4, 15]. In the UK, evidence from estate data confirms the qualitative picture observed in the US [12].

The starting point for our model will be the pronounced difference in portfolio compositions between the very wealthy and the rest. Different households are subject to different economic processes that govern their possibilities of accumulating personal wealth. We want to argue that the vast majority of households engages in a life-cycle type of saving in order to provide themselves with housing and financial claims that will ensure their economic viability beyond working age. Hence their wealth will be roughly proportional to earned income, describing an *additively* driven process designed to realize a return in the distant future. In contrast, the very wealthy accumulate their fortunes by re-investing returns into assets that typically yield a return in each period. Essentially, then, large fortunes are accumulated for their own sake and the underlying accumulation process has a *multiplicative* character.

2 Maximum Entropy Principle

Market economies consist of a large number of heterogeneous agents whose interactions produce—possibly unintended and regularly unforseen—aggregate consequences that feed back into agents' behavior and the environment they interact in. The vast amount of information in such a complex system makes it impractical, if not impossible, to model the distribution of wealth by tracing the microscopic fate of all agents. The economic concept of statistical equilibrium [5, 10] acknowledges this difficulty from the start and consequently curbs its methodological ambition to more modest levels, being content with describing the statistical properties of aggregate outcomes as a probability distribution of economic agents over possible outcomes.

The mathematical formalism underlying statistical equilibrium analysis is known as the *maximum entropy principle.* Building on entropy concepts from statistical mechanics and information theory, Jaynes [6] generalized the principle of entropy maximization into a theory of probabilistic inference that has found numerous applications across the natural and social sciences [8]. Based on the premise of incorporating solely knowledge that has been given to us and scrupulously avoiding probabilistic statements that would imply more information than we actually have, the maximum entropy principle derives probability distributions from known moment constraints. Virtually all known distributions can be derived from the maximum entropy principle [7]. Let us denote the number of theoretically admissible values of our variable of interest x by $i = 1, \dots, n$; then the maximum entropy principle prescribes to maximize (informational) entropy $H \equiv -\sum_i p_i \log p_i$ subject to the natural constraint $\sum_i p_i = 1$ and $m < n$ observed moment constraints $\sum_i p_i g_k(x_i) = \overline{g_k}$ for all $k = 1, \dots, m$. Applying Lagrange's multiplier technique yields probability distributions of the generic form $p_i = Z^{-1}(\lambda_1, \dots, \lambda_m) \exp(-\lambda_1 g_1(x_i) - \dots - \lambda_m g_m(x_i))$, where $Z(\lambda_1, \dots, \lambda_m) \equiv \sum_i \exp(-\lambda_1 g_1(x_i) - \dots - \lambda_m g_m(x_i))$ is the *partition function* that normalizes the distribution and $\lambda_1, \dots, \lambda_m$ are the Lagrange multipliers chosen so as to satisfy the moment constraints, which is the case when $\overline{g_k} = -\partial \log Z / \partial \lambda_k$ for all $k = 1, \dots, m$. Concavity of the objective function and linear (or a convex set of) constraints ensure that the resulting probability distribution is unique and attained at a global entropy maximum, while the exponential form of the generic distribution only admits positive probabilities. The results readily carry over to a continuous state space and we need not bother here with some minor subtleties that arise from the continuum concept and are discussed in [6, 7]. The advantage of the continuous maximum entropy program lies in its ability to derive closed-form solutions for the parameters in many cases of interest.

Moreover, according to Jaynes' *concentration theorem* [6], the distribution of maximum entropy is not only 'most likely' in the combinatorial sense of being achievable in the largest number of ways—but the overwhelming majority of possible distributions compatible with the constraints will have entropy very close to the maximum. Thus inference from observed constraints to resulting frequency distribution becomes exceptionally robust *and* vice versa: suppose our variable of interest is distributed with a specific functional form; then the concentration theorem assures us of the extreme improbability that constraints other than those implied by the maximum entropy principle are responsible for the observed outcome, and probability distributions and aggregate constraints become two sides of the same coin. Hence, as far as the distribution of wealth is concerned, we should ask which economic constraints produce the observed distributional regularities.

3 Wealth Distribution in Statistical Equilibrium

As we will see in the following subsections, it is more convenient to model the power law tail of the wealth distribution starting from the economic uses of wealth. When we turn to the exponential distribution of fortunes, it will be easier to incorporate economic intuitions by considering the sources of wealth. Proper accounting ensures that the sources and the uses of wealth are equal and therefore the outcome of the maximum entropy models does not depend on whether one route or the other is chosen—what is important is the nature of the constraints in the maximum entropy program.

3.1 The Multiplicative Case: Mixing of Returns

A power law distribution results from the decentralized investment activity of wealthy agents who combine assets with uncertain returns in their portfolios while being constrained by the aggregate growth rate of wealth. The statistical equilibrium wealth distribution then defines a probability field over returns from available combinations of investment opportunities under the most decentralized (or entropy maximizing) investment activity of wealthy agents. We will summarize the power law tail model very briefly here since an extended version of model can be found in [10].

Conceptualize the economy as a set $\mathbf{K} = \{1, \ldots, K\} \subseteq \mathbf{N}$ of *investment opportunities*. For all $k \in \mathbf{K}$, let $V^k(t)$ denote the time t value of economic activity k, and for all $h \in \{1, \ldots, n\}$, $n < \infty$, let $a_h^k(t)$ denote the *position* of household h in activity k, with the interpretation that $a_h^k(t) > 0$ indicates a long position at time t (k is an *asset*) and $a_h^k(t) < 0$ a short position (k is a *liability*) while $a_h^k(t) = 0$ allows for the absence of activity k in the portfolio of household h. Then the time t value of the wealth portfolio of household h, denoted $w_h(t)$, follows from the household's combination of the K different investment opportunities $w_h(t) \equiv \sum_{k \in \mathbf{K}} a_h^k(t) V^k(t)$ for all $h \in \{1, \ldots, n\}$. Changes in the value of a portfolio are either the result of a revaluation of economic activities, or of changes in the behavior of the household—expressed as changes in the household's positions. Instead of putting forward a specific theory of portfolio choice and asset pricing, our statistical equilibrium model assumes that we observe a well-defined macroscopic average, namely the growth rate of (or average return to) wealth. Assuming that returns are compounded continuously and that there is a fictional initial period t_0 where all portfolios start out with the same wealth level w_0, wealth levels and returns r will be proportional such that $w_h(T) = w_0 \exp(T r_h)$, where $r_h = (\log w_h(T) - \log w_0)/T$ is the average return of portfolio h over the time interval $[t_0, T]$. Since we want to allow for negative rates of return, we also define $r_m > -\infty$ as the minimum return that is observed among the portfolios over the given period. Switching to a continuous state space, we maximize the entropy measure $-\int_Z \mathrm{d}r\, f(r) \log f(r)$ subject to the natural constraint $\int_Z \mathrm{d}r f(r) = 1$ and the arithmetic mean constraint on returns, $\int_Z \mathrm{d}r f(r)\, r = \bar{r}$, where returns now take values on the support $Z = [r_m, \infty)$.

The maximum entropy distribution of hypothetical returns is an exponential distribution, $f(r) = \lambda \exp(-\lambda(r - r_m))$ and the continuous formulation allows to explicitly determine that $\lambda = 1/(\bar{r} - r_m)$. Recalling the definition of returns as $r = (\log w - \log w_0)/T$, we obtain the corresponding distribution of wealth levels from the theorem of densities of a function of a random variable, yielding

$$f(w) = \phi {w_m}^{\phi} w^{-(\phi+1)} \tag{1}$$

where w_m is the wealth level corresponding to the minimum return r_m and $\phi \equiv \lambda/T$. Notice that it would have been possible to obtain a power law directly from the maximum entropy program by postulating a logarithmic mean constraint on wealth levels instead of an arithmetic mean constraint on returns but the interpretation becomes more cumbersome because the logarithmic mean carries no time dimension. Focusing explicitly on returns illustrates that the essentially random experience of the mixing of portfolios over the returns distribution is sufficient by itself to explain the power law distribution of wealth.

3.2 The Additive Case: Life-Cycle Saving

Our analysis starts again from the stylized fact concerning the composition of wealth portfolios that account for the vast majority of agents, where the net position in owner-occupied housing, deposits, and life insurance and pension plans are the main assets being held by those agents. We assume, in contrast to our previous analysis where we only cared about the uses of wealth and not its sources, that these assets are financed from earned income that will mostly flow from wages and salaries. Other possible sources are government transfer payments, rents, and profits arising from unincorporated businesses and financial assets. Regardless of the source of income, and this is the crucial point, we postulate that additions to wealth will *not* be re-invested as in the case of very wealthy agents. Instead, the existing level of wealth will be augmented by additions out of current income such that wealth remains roughly proportional to income.

Suppose that for all individuals $j \in \{1, \ldots, n\}$, $n < \infty$, who accumulate wealth in such fashion $y_j^e(t)$ designates the disposable income from source $e \in \mathbf{E} = \{1, \ldots, E\}$ at time period t. Moreover, if $w_j(t)$ denotes the wealth of agent j in period t then the change in wealth between periods will depend on how much of the agent's income has been 'saved' from the different sources, $\Delta w_j(t) = \sum_{e \in \mathbf{E}} s_j^e(t) y_j^e(t)$ for all j, where $s_j^e(t)$ represents the proportion of income from source e that agent j utilizes to augment wealth at time t. The stock of wealth $w_j(\tau) = \sum_t \Delta w_j(t)$ that agent j has accumulated up to period τ, when we observe the personal wealth distribution, will depend on the agent's accumulation behavior s_j^e, the agent's fortunes in the (labor) market y_j^e, and on the number of periods $T_j \equiv T_j^e + T_j^r$ in which the agent has an income either earned during T_j^e periods of working life or flowing during the T_j^r periods after retirement, with $t \in \{1, \ldots, T_j\}$.

Thus $s_j^e(t)$ should not be understood in the classical sense of a 'saving propensity' since we want to allow for a negative $s_j^e(t)$, for example to represent the decrease of wealth that occurs when an agent retires and spends previously accumulated pension income for consumption.[1] The macroscopic constraint on the average wealth accumulated by the population at time τ is given by

$$\overline{w_\tau} = n^{-1} \sum_{j=1}^{n} \sum_{t=1}^{T_j} \sum_{e \in \mathbf{E}} s_j^e(t) y_j^e(t) \,. \tag{2}$$

We denote the set of theoretically possible wealth levels by $W = [0, w_m)$, where w_m is the wealth level that separates the empirically observed exponential and power law regimes.[2] Let $i \in \{1, \ldots, z\}$ run over the set of discrete wealth levels $w_i \in W$ and let n_i be the fraction of agents with wealth w_i. Then it must also be true that $\overline{w_\tau} = \sum_i w_i n_i / n \equiv \sum_i w_i p_i$ and, to ensure that all agents are assigned to a wealth level, the natural constraint $\sum_i p_i = 1$ must hold as well. The wealth distribution that allows for the largest number of individual destinies and behaviors consistent with the observed average stock of wealth in W will be given by the solution to the maximum entropy program under the arithmetic mean constraint $\overline{w_\tau} = \sum_i w_i p_i$ and the natural constraint. This yields again a (discrete version of the) exponential distribution,

$$p_i = \frac{\mathrm{e}^{-\mu w_i}}{Z(\mu)}, \tag{3}$$

with the partition function $Z(\mu) \equiv \sum_i \exp(-\mu w_i)$. It is straightforward to show that on the continuous support $W = [0, w_m)$ the relationship between μ and $\overline{w_\tau}$ is given by $\overline{w_\tau} = 1/\mu - w_m/(\exp(\mu w_m) - 1)$, which for a reasonably large w_m approximates the familiar result $\mu = 1/\overline{w_\tau}$.

In summary, the distribution of smaller fortunes in market economies can be linked to earned income that is accumulated for precautionary reasons and retirement purposes. The close link to earned income also suggests that the

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[1] Consumption should be understood in the broad sense of an economic use not included in the categories that count as personal wealth.

[2] The exponential law cannot account for individuals with negative wealth. Thus, for the sake of completeness, we should also have a constraint that prevents negative wealth. Similar to the conventional life-cycle model, this boils down to the postulate that life-time earnings should be greater than or equal to life-time 'consumption.'

distribution of wealth should be quite similar to the distribution of earned income itself and the empirical evidence presented at this conference certainly points in this direction as well (see also [3, 13]). A final remark concerns the possibility that the majority of households are described by a Gamma instead of an exponential distribution. Pension and life insurance schemes and sources of income other than wages and salaries, like rents or profits, also provide returns. If we believe that the magnitude of these returns is significant in the accumulation process, we would need to incorporate a multiplicative constraint in addition to the additive constraint on the average stock of wealth. So far we have not been able to model the constraint in a convincing manner but if it were possible to formulate it as a logarithmic mean, the maximum entropy program would yield a Gamma distribution in the simultaneous presence of arithmetic and logarithmic means [7].

4 Conclusion and Outlook

The composition of wealth portfolios depends on the level of wealth and expresses two different economic behaviors in the process of wealth accumulation. At the same time, the functional form of wealth distributions shows two distinct regimes, the right tail following a power law and the left part generally belonging to the family of Gamma distributions. Statistical equilibrium views these phenomena as reflecting two distinct processes in the accumulation of wealth. We have argued that the composition of wealth portfolios by wealth class can explain the different distributional regularities effectively by itself. While smaller fortunes, accounting for the vast majority of agents, are regulated by the adjustment of the stock of wealth to the flow of income, larger fortunes are regulated by the adjustment of the flow of income to the stock of wealth. If large fortunes were subject to a constraint on the stock of wealth, corresponding to an arithmetic mean constraint in the maximum entropy program, the combinatorially most likely or informationally least biased distribution of wealth would not be a power law. In other words, the power law distribution implies the complete absence of any aggregate constraints other than a logarithmic mean. Contrary to the jargon and intuition of thermodynamics, we detect the absence of a 'conservation principle' in the personal wealth of the mighty rich.

While the intuition behind our results should be quite clear, our argument in favor of the Gamma law has obviously not been developed in a formally satisfactory way so far. Once we have a better statistical equilibrium formulation to account for the Gamma distribution, the remaining task will be to elaborate the relationship between the parameters of the Gamma distribution and the observed arithmetic and logarithmic means of wealth, with the aim of calibrating the model from empirical data.

The basic puzzle of the distribution of wealth is how the pronounced difference in the composition of wealth portfolios between the majority of the population and the mighty rich is upheld. After all, it would seem that reasonably wealthy agents who do not fall into the power law tail region could afford to allocate substantial parts of their wealth in asset classes that yield a return in each period. Moreover the demarcation wealth level w_m cannot be determined from our models, which consider it as an exogenous parameter, but it obviously plays a central role in the processes of wealth accumulation.

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Part III

Related studies

A Stochastic Model of Wealth Distribution

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1 Introduction

A major research focus in economics and econophysics is on the distribution of wealth in societies at different stages of development. Wealth includes money, material goods and assets of different kinds. Knowledge of the monetary equivalent of the latter two components is required in order to quantify wealth. A related and easier to measure distribution is that of income. The major motivation of theoretical models is to provide insight on the microscopic origins of income/wealth distributions. Such distributions are expected to provide good fits to the empirical data. In the context of incomes, Champernowne [2] has commented "The forces determining the distribution of incomes in any community are so varied and complex and interact and fluctuate so continuously, that any theoretical model must either be unrealistically simplified or hopelessly complicated." The statement highlights the desirability of finding a middle ground between the unrealistically simple and the hopelessly complicated.

A number of distribution functions has been proposed so far to describe income and wealth distributions. Theoretical models based on stochastic processes, have been formulated to explain the origins of some of the distributions [1, 2, 3, 4, 5, 6, 7]. One proposed distribution, mention of which is found in economic literature, is the beta distribution [8, 9]. In this paper, we describe a simple stochastic model of wealth distribution and show that the beta distribution is obtained in the non-equilibrium steady state.

2 Stochastic model

In the model, each economic agent (can be an individual, a family or a company) may be in two states: inactive (E) and active (E^*). We determine the probability distribution of the wealth of an agent randomly selected from a population of agents. Let the agent possess wealth M at time t. Increase in the wealth of the agent can occur in two ways: at a steady rate and at random time intervals. In state E, the agent's wealth increases at rate b_m and in state E*, the rate is given by $b_m + j_m$. In both E and E*, the agent's wealth decreases at rate $k_m M$. The decay rate is proportional to the current wealth with k_m being the decay rate constant. Transitions between the states E and E*occur at random time intervals. The rate of change of wealth is governed by the equation

$$\frac{dM}{dt} = j_m z + b_m - k_m M = f(M, z) \tag{1}$$

where $z = 1\,(0)$ when the agent is in the state E*(E). Let $p_j(M,t)$ $(j = 0, 1)$ be the probability density function for wealth distribution when $z = j$. The rate of change of the probability density is given by

$$\frac{\partial p_j(M,t)}{\partial t} = -\frac{\partial}{\partial M}[f(M,z)p_j(M,t)] + \sum_{k \neq j}[W_{kj}p_k(M,t) - W_{jk}p_j(M,t)] \tag{2}$$

where W_{kj} is the transition rate from state k to state j. The first term in Eq.(2) is the "transport" term representing the net flow of probability density and the second term represents the gain/loss in the probability density due to random transitions between the state j and the other accessible state. One can define the activation and deactivation rates, k_a and k_d respectively, to be $k_a = W_{01}$, and $k_d = W_{10}$. From Eq. (2),

$$\frac{\partial p_0}{\partial t} = -\frac{\partial}{\partial M}\{(b_m - k_m M)p_0\} + k_d p_1 - k_a p_0 \tag{3}$$

$$\frac{\partial p_1}{\partial t} = -\frac{\partial}{\partial M}\{(j_m + b_m - k_m M)p_1\} + k_a p_0 - k_d p_1 \tag{4}$$

with $p = p_0 + p_1$. The minimum and the maximum amounts of wealth possessed by the agent are given by $M_{min} = b_m/k_m$ and $M_{max} = (b_m + j_m)/k_m$. Define $m = M/M_{max}$, $m_{min} = M_{min}/M_{max}$, $r_1 = k_a/k_m$ and $r_2 = k_d/k_m$. In the steady state, $\partial p_0/\partial t = 0$ and $\partial p_1/\partial t = 0$. The steady state solution turns out to be the beta distribution

$$p(m, r_1, r_2) = \frac{(m - m_{min})^{r_1 - 1}(1 - m)^{r_2 - 1}}{B(r_1, r_2)(1 - m_{min})^{r_1 + r_2 - 1}} \tag{5}$$

The normalization constant $B(r_1, r_2)$ is

$$B(r_1, r_2) = \int_{m_{min}}^{1} \frac{(m - m_{min})^{r_1 - 1}(1 - m)^{r_2 - 1}}{(1 - m_{min})^{r_1 + r_2 - 1}}\, dm \tag{6}$$

In Eqs. (5) and (6), $r_1 > 0$, $r_2 > 0$ and $m_{min} < m < 1$. Let $m_{min} = 0$ and $r_1 > 1$ and $r_2 > 1$. In this case,

$$B(r_1, r_2) = \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1 + r_2)} \tag{7}$$

is the well-known beta function. The mean wealth m_{av} and the variance m_{var} are given by

$$m_{av} = \frac{r_1}{r_1 + r_2}, \; m_{var} = \frac{r_1 r_2}{(r_1 + r_2)^2} \frac{1}{r_1 + r_2 + 1} \tag{8}$$

The quantities depend on the ratios r_1 and r_2 rather than on the individual values of k_a, k_d and k_m.

3 Results and discussion

Societies are traditionally divided into three classes: poor, middle and rich. Figs. 1(a), (b), and (c) show the $p(m)$ versus m distributions in the three cases. One can obtain similar curves when $m_{min} \neq 0$. The Gini coefficient G, a measure of wealth inequality, is expected to be small for each separate class. For example, $G = 0.2$ in the case of Fig. 1(a) describing wealth distribution for the poor class. The two-parameter beta distribution is flexible and can take a variety of shapes. The precision in fitting data is, however, limited in this case.

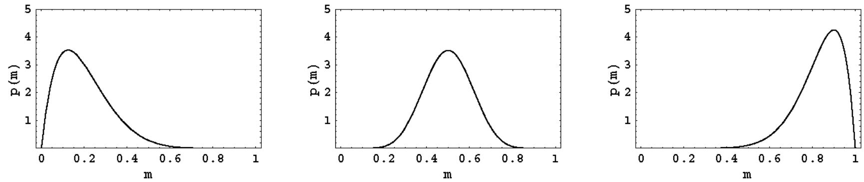


Fig. 1. Probability density function $p(m)$ as a function of m for (a) $r_1 = 2,\ r_2 = 8$, (b) $r_1 = 10,\ r_2 = 10$, (c) $r_1 = 10,\ r_2 = 2$

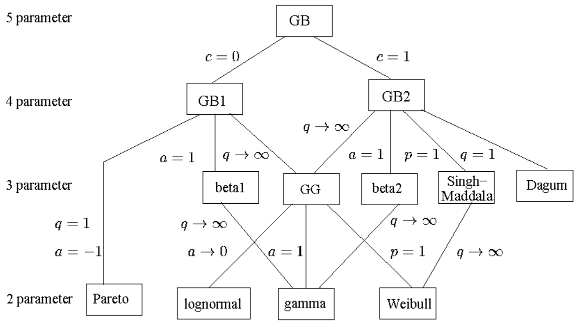
McDonald and Xu [10] have proposed a five-parameter generalised beta distribution

$$GB(y;\ a, b, c, p, q) = \frac{|a| y^{ap-1} \{1 - (1 - c)(\frac{y}{b})^a\}^{q-1}}{b^{ap} B(p, q)(1 + c(\frac{y}{b})^a)^{p+q}} \tag{9}$$

where $0 < y^a < b^a$ and is zero otherwise. Also, $0 \leq c \leq 1$ and $b,\ p,\ q > 0$. $B(p, q)$represents the normalisation constant. The beta distribution (Eq. (5)) is a special case of GB(y; a,b,c,p,q) with $m_{min} = 0$, $c = 0$, $a = 1$, and $b = 1$.

Many well-known distribution functions are limiting cases of the generalised beta distribution GB. Some examples are shown in Table 1. The beta1 distribution reduces to the beta distribution ($m_{min} = 0$) with $b = 1$. GG refers to the generalised gamma distribution. The special cases of $GB(y)$, $GB1$ and $GB2$ have been shown to outperform other distributions in providing good

Table 1. Generalised beta distribution (GB) and its special cases



quantitative fits to the income data from various countries and segments of society. The beta distribution, considered in the paper, is a special case of $GB1$.

In this paper, we have provided a stochastic model of wealth distribution leading to the beta distribution in the non-equilibrium steady state. It will be of interest to formulate stochastic models of generalised beta distributions GB, $GB1$ and $GB2$. An understanding of the microscopic origins of income/wealth distributions may provide insight on the policies required to ensure that the benefits of economic growth reach all sections of society.

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1

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How the Rich Get Richer

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Summary. In our model, n traders interact with each other and with a central bank; they are taxed on the money they make, some of which is dissipated away by corruption. A generic feature of our model is that the richest trader always wins by 'consuming' all the others: another is the existence of a *threshold* wealth, below which all traders go bankrupt. The two-trader case is examined in detail, in the *socialist* and *capitalist* limits, which generalise easily to $n > 2$. In its mean-field incarnation, our model exhibits a two-time-scale *glassy dynamics*, as well as an astonishing *universality*. When preference is given to local interactions in finite neighbourhoods, a novel feature emerges: instead of at most one overall winner in the system, finite numbers of winners emerge, each one the *overlord* of a particular region. The patterns formed by such winners (*metastable states*) are very much a consequence of initial conditions, so that the fate of the marketplace is ruled by its past history; *hysteresis* is thus also manifested.

1 Introduction

The tools of statistical mechanics [1] are increasingly being used to analyse problems of economic relevance [2]. Our model below, although originally formulated to model the evolution of primordial black holes [3, 4], is an interesting illustration of the *rich-get-richer* principle in economics. It is inherently disequilibrating; individual traders interact in such a way that the richest trader *always* wins.

2 The model

In this model, n traders are linked to each other, as well as to a federal reserve bank; an individual's money accrues interest at the rate of $\alpha > 1/2$ [3] but is also taxed such that it is depleted at the rate of $1/t$, where t is the time. The interaction strength g_{ij} between traders i and j is a measure of how much of their wealth is invested in trading; income from trading is also taxed at the rate of $t^{1/2}$. There is a *threshold* term such that the less a trader has, the more he loses; additionally the model is *non-conservative* such that some of the wealth disappears forever from the local economy. These last terms can have different interpretations in a macro- or a micro-economic context. In the former case (where the traders could all be citizens of a country linked by a federal bank), the threshold term could represent the plight of the (vanishing) middle classes, while the non-conservative nature of the model could represent the contribution of *corruption* to the economy - some of the taxed money disappears forever from the region, to go either to the black economy or to foreign shores. In a more micro-economic context (where traders linked by a bank are a subset of the major economy), the interpretation is the reverse: the non-conservative nature of the model would imply money lost irretrievably by taxation (to go to social benefits from which the traders do not themselves benefit), while the threshold term could represent the effect of corruption (poorer traders lose more by graft than richer ones). Including all these features, we postulate that the wealth $m_i(t)$ for $i = 1, \ldots, n$ of each trader evolves as follows [4]:

$$\frac{\mathrm{d}m_i}{\mathrm{d}t} = \left(\frac{\alpha}{t} - \frac{1}{t^{1/2}} \sum_j g_{ij} \frac{\mathrm{d}m_j}{\mathrm{d}t} \right) m_i - \frac{1}{m_i}. \tag{1}$$

In the following,we use units of reduced time $s = \ln \frac{t}{t_0}$ (to renormalise away the effect of initial time t_0), reduced wealth $x_i = \frac{m_i}{t^{1/2}}$ and reduced square wealth $y_i = x_i^2 = \frac{m_i^2}{t}$. In these units, we recall the result for an *isolated* trader [3]. A trader whose initial wealth y_0 is greater than $y_\star$, (with $y_\star(t_0) = \left(\frac{2t_0}{2\alpha-1}\right)$), is a *survivor* who keeps getting richer forever: a trader with below this threshold wealth goes bankrupt and disappears from the marketplace in a finite time. The influence of this initial threshold $y_\star$ will be seen to persist throughout this model: in every case we examine, surviving winners will all be wealthier than this.

3 A tale of two traders: socialist vs capitalist?

We examine the two-trader case in the *socialist* and *capitalist* limits. In the socialist limit, the initial equality of wealth is maintained forever by symmetry: their common wealth $x(s)$ obeys:

$$x' = \frac{(2\alpha - 1)x^2 - 2 - gx^3}{2x(1 + gx)}. \tag{2}$$

This equation is analytically tractable: it has fixed points given by $(2\alpha-1)x^2 - 2 - gx^3 = 0$. A critical value of the interaction strength g, $g_c = \left(\frac{2(2\alpha-1)^3}{27}\right)^{1/2}$, separates two qualitatively different behaviours. For $g > g_c$, there is no fixed point; overly heavy trading (insufficient saving) causes both traders to go quickly bankrupt, independent of their initial capital. In the opposite case of sensible trading, $g < g_c$, there are two positive fixed points, $y_\star^{1/2} < x_{(1)}$ (unstable) $< (3y_\star)^{1/2} < x_{(2)}$ (stable). If both traders are initial equally poor with wealth $x_0 < x_{(1)}$, this is dynamically attracted by $x = 0$ – the traders go rapidly bankrupt! For initially rich traders with $x_0 > x_{(1)}$, their wealth is dynamically attracted by $x_{(2)}$ – they grow richer forever as $m(t) \approx x_{(2)} t^{1/2}$, a growth rate which is *less* than that for an isolated trader! This case, where equality and overall prosperity prevail even though there are no individual winners, could correspond to a (modern) Marxist vision.

In the *capitalist* case, with traders who are initially unequally wealthy, any small differences always diverge exponentially early on: the details of this transient behaviour can be found in [5]. However, the asymptotic behaviour is such that richer trader wins, while the poorer one goes bankrupt: *the survival of the richest is the single generic scenario for two unequally wealthy traders.* At this point, we are back to the case of an isolated trader referred to in Section 3: he may, depending on whether his wealth at this point is less or greater than $y_\star$, also go bankrupt or continue to get richer forever.

All of the above generalises easily to any finite number $n \geq 2$ of interacting traders.

4 Infinitely many traders in a soup - the mean field limit

We now examine the limit $n \to \infty$: we first explore the *mean field behaviour* where every trader is connected to every other by the same dilute interaction $g = \frac{\overline{g}}{n}$. For fixed $\overline{g}$, the limit $n \to \infty$ leads to the *mean field equations* [5]:

$$y'(s) = \gamma(s)y(s) - 2 \tag{3}$$

When additionally, $\overline{g}$ is small (weak trading), a *glassy* dynamics [1] with two-step relaxation is observed. In Stage I, individual traders behave as if they were isolated, so that the survivors are richer than threshold ($y_\star$), exactly as in the one-trader case of Section 2. In Stage II, all traders interact *collectively*, and *slowly* [5]. All but the richest trader eventually go bankrupt during this stage.

The model also manifests a striking *universality*. For example, with an exponential distribution of initial wealth, the survival probability decays asymptotically as $S(t) \approx \frac{2\alpha-1}{\overline{g}} \left(C \ln \frac{t}{t_0}\right)^{-1/2}$; additionally, the mean wealth of the surviving traders grows as $\langle\langle m \rangle\rangle_t \approx \left(C t \ln \frac{t}{t_0}\right)^{1/2}$. In both cases, $C = \pi$ *irrespective* of α, $\overline{g}$ and the parameters of the exponential distribution. The universality we observe goes further than this, in fact: it can be shown [5] that C only depends on whether the initial distribution of wealth is bounded or not and on (the shape of) the tail of the wealth distribution.

5 Infinitely many traders with *local interactions* - the emergence of overlords

Still in the $n \to \infty$ limit, we now introduce local interactions: traders interact preferentially with their $z = 2D$ nearest neighbours on a D-dimensional lattice: once again we look at the limit of weak trading ($g \ll 1$). The dynamics once again consists of two successive well-separated stages with fast individual Stage I dynamics, whose survivors are richer than threshold, exactly as before (Section 4). The effects of going beyond mean field are only palpable in Stage II: the effect of local interactions lead to a slow dynamics which is now very different from the mean-field scenario above. The survival probability $S(s)$ in fact decays from its plateau value $S_{(1)}$ (number of Stage I survivors) to a non-trivial limiting value $S_{(\infty)}$; unlike the mean field result, a *finite* fraction of traders now survive forever!

Figure 1 illustrates this two-step relaxation in the decay of the survival probability $S(s)$. While the (non-interacting) decay to the plateau at $S_{(1)} = 0.8$ is (rightly) independent of g, the Stage II relaxation shows *ageing*; the weaker the interaction, the longer the system takes to reach the (non-trivial) limit survival probability $S_{(\infty)} \approx 0.4134$.

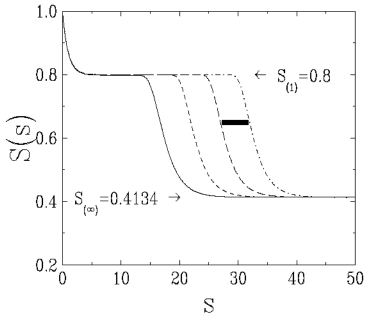

Fig. 1. Plot of the survival probability $S(s)$ on the chain with $S_{(1)} = 0.8$ (after reference [5]. Left to right: Full line: $g = 10^{-3}$. Dashed line: $g = 10^{-4}$. Long-dashed line: $g = 10^{-5}$. Dash-dotted line: $g = 10^{-6}$.

At the end of Stage II, the system is left in a non-trivial *attractor*, which consists of a pattern where each surviving trader is *isolated*, an *overlord* who keeps getting richer forever. We call these attractors *metastable states*, since they form valleys in the existing random energy landscape; the particular metastable state chosen by the system (corresponding to a particular choice of pattern) is the one which can most easily be reached in this landscape[1]. The number $\mathcal{N}$ of these states generically grows exponentially with the system size (number of sites) N as $\mathcal{N} \sim \exp(N\Sigma)$ with Σ the configurational entropy or *complexity*. The limit survival probability $S_{(\infty)}$ (Figure 1) is just the density of a typical attractor, i.e., the fraction of the initial clusters which survive forever.

We now examine in some more detail the fate of a set of $k \geq 1$ surviving traders: this depends on k as follows.

- ⋆ $k = 1$: If there is only one trader, he survives forever, trading with the reserve and getting richer.
- ⋆ $k = 2$: If a pair of neighbouring traders (represented as ••) survive Stage I, the poorer dies out, while the richer is an overlord, leading to •∘ or ∘•.
- ⋆ $k \geq 3$: If three or more traders survive Stage I, they may have more than one fate. Consider for instance (•••): if the middle trader goes bankrupt first (•∘•), the two end ones are isolated, and both will become overlords. If on the other hand the trader at the 'end' first goes bankrupt (e.g. ••∘), only the richer among them will become an overlord (e.g. •∘∘). The pattern of these immortal overlords, and even their number, therefore *cannot* be predicted a priori.

Finally, we present some of the observed patterns. If $S_{(\infty)} = 1/2$ on, say, a square lattice, (i.e. the highest density of surviving traders is reached), there are only two possible 'ground-state' configurations of the system; the two possible patterns of immortal overlords are each perfect checkerboards of one of two possible parities. This allows for an interesting possibility: we can define a checkerboard index for each site, which classifies it according to its *parity* [5].

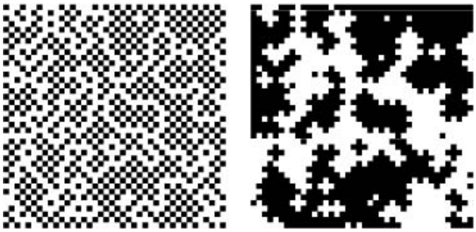

Fig. 2. Two complementary representations of a typical pattern of surviving clusters on a 40^2 sample of the square lattice, with $S_{(1)} = 0.9$, so that $S_{(\infty)} \approx 0.371$ (after reference [5]. **Left**: Map of the survival index. Black squares represent overlords for which $\sigma_{\mathbf{n}} = 1$, while white squares represent bankrupt sites for which $\sigma_{\mathbf{n}} = 0$. **Right**: Map of the checkerboard index. Black squares represent positive, while white squares represent negative, parity

Figure 2 shows a map of the survival index and of the checkerboard index for the same attractor for a particular sample of the square lattice. The *local* checkerboard structure, with random frozen-in defects between patterns of different parities is of course entirely inherited from the initial conditions. The overlords in the left-hand part of the figure are surrounded by rivulets of poverty ; in the right-hand figure, the deviation from a perfect checkerboard structure (all black or all white) is made clearer. Neighbouring sites are fully anticorrelated, because each overlord is surrounded by paupers: however, at least close to the limit $S_{(\infty)} = 1/2$, overlords are very likely to have next-nearest neighbours who are likewise overlords. The detailed examination of survival and mass correlation functions made in a longer paper [5] confirms these expectations.

To conclude, we have presented a model where traders interact through a reserve; we are able to model the effects of corruption and taxation via the non-conservative, threshold nature of our model. These could have different implications for micro- and macroeconomic situations. Our main results are that, in the presence of global interactions, typically only the wealthiest trader survives (provided he was born sufficiently rich); however, if traders interact locally, finite numbers of local overlords emerge by creating zones of poverty around them.

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Power-Law Distribution in an Emerging Capital Market

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Summary. In an emerging (Indian) capital market, power-law distribution emerges at high market capitalization level and the values of the exponent of the power-law are found to be consistent with wealth distribution of the individuals in the economy. The high growth of the firms does not change the wealth distribution and likewise the value of the exponent. However, negative growth of the firms affects the wealth distribution of the firms and is accompanied by the reduction in the value of the exponent. Since, mean difference in the ranks for the negative growth is much higher and statistically significant than the high growth firms, we conclude that it may have led to larger change in the exponents value.

1 Introduction

It was empirically proposed by Pareto in 1897 that wealth of individuals in an economy follows a power-law distribution at high wealth levels. The characteristic form of the distribution is:

$$y \sim x^{-\gamma}, \tag{1}$$

where y is the number of people having wealth of x more and γ is an exponent, which varies between 1 and 2 and is generally applicable across countries regardless of their social, political and fiscal conditions, with some differences in the value of exponent. The Pareto wealth distribution is generated through a stochastic process. It is well known that in the capital market the wealth (market capitalization of the firms) is generated through a stochastic process. If the underlying concept is true, higher wealth level of the firms, would lead to power law distribution in capital market, with exponents value converging to the value as observed for wealth distribution of the individuals at high wealth level. In addition, the firms wealth generation, i.e. growth or decline of market capitalization will have direct linkage with the emergence of power law. Stated differently, high growth firms will likely to have greater fits in power law distribution over time in a given wealth level, while firms with declining growth will show lesser accuracy in power law distribution belonging to the same wealth level. Accordingly purpose of this paper is to examine:

– Whether a power-law distribution emerges at high market capitalization of the firms similar to individual wealth distributions.

– How exponent of power-law distribution behaves across different level of firms performance in the capital market over time.

Pareto distribution can be expressed in various forms and one of the forms is:

$$x_n = C.n^{-1/\gamma} \qquad (2)$$

where x_n is he wealth of the n-th ranked individual. Since, we are measuring firms wealth (market capitalization); x_n is the n-th ranked firm in terms of market capitalization. Taking logarithm of (1) yields the following equation:

$$\ln x_n = \ln C - (1/\gamma) \ln n \qquad (3)$$

The equation (2) is a linear equation of the form:

$$\ln x_n = a + b \ln n \qquad (4)$$

where, x_n is the market capitalization of the firm with rank n, and can be used as a regression equation to estimate the value of the exponent γ by finding out the value of regression coefficient b.

We use the database containing the highest market capitalization of the top 500 companies, listed in the stock market, National Stock Exchange in India. The market capitalization is reckoned as the average market capitalization between the period March 16 and March 31 for 2004, and between the period July 16 and August 31, for 2003(1), while for 2003(2) the market capitalization is the average calculated over the period between March 16 and March 31, 2003.

In table 4 and in table 6, we present the regression results as well as the values of the exponent at different points of time. We also present the values of exponent for top 2% through 9% of the ranked firms in table 5. The coefficient of determination R2 and adjusted R2 are observed to be high, indicating excellent fit of the regression equation. Figure below shows regression fit in respect of 2003(1).

In addition, the t-values are shown in the parentheses and they are significant at 1% level. It may be noted that 10% of the firms comprise of 74% of the total market capitalization, while bottom 10% of the firm constitute 1.3% of the total market capitalization. With the growth of market capitalization over time, there are changes in the ranking of the firms. In order to find out whether the growth has any linkage or association in influencing the power law distribution, we choose the firms which have shown high increase ($>$ 200%) of their market capitalization over time, i.e. in 2004 over 2003(1). We also carry out similar test for the firms with declining wealth levels. The results are shown in table 7. Since the stock prices exhibit non-stationarity, we apply

Augmented Dicky Fuller (ADF) statistic to test the hypothesis of existence of unit root. The reported values in the table are found to be significant for rejection of unit root.

The power-law distribution emerges at higher market capitalization belonging to the upper 10%. The values of the exponent under different percentile of the firms occupying the ranks within upper 10% of market capitalization depict a decreasing trend. The power law becomes less conspicuous as the market capitalization of the firms decreases. The reduction in value of the exponent is consistent with other research findings. For example, Steindl (1987) and Persky (1992) found that power law fits the upper tail of the wealth distribution but becomes less accurate in the lower tail. The higher growth of the market capitalization does not seem to influence the exponent as may be observed from the Table 4. We observe that faster growth rate in capital market does not change distribution of the wealth of the firms and correspondingly, we do not observe any noticeable change in the value of the exponent. We however notice that the value of exponent reduces for 0.805 in 2003(1) to 0.702 in 2004, in those cases where the firms undergo negative growth. A possible reason for unchanged wealth distribution for high growth ($> 200\%$) firms could be that the changes in the ranks of these firms are not significant between beginning and end of the period of growth. To test this hypothesis that there is no significant difference in the ranks of the firms between before and after growth period, we apply paired-sample t-test, which yields the following results. The table shows that 2-tailed significance of the

Table 1. Paired Difference

Mean difference	Std. Deviation	t-value	df	2-tail significance
-119.3077	80.64	-5.335	12	.000

test is .000. As a result we reject the hypothesis at 5% significance level and conclude that there is significant difference in the ranks. The above results contradict the argument that faster growth of wealth portfolio on average would result in more unequal distribution. A similar phenomenon is also observed for the top 10% firms, where the average growth rate is 59.73% in 2004 over 2003(1), but the exponent exhibits a very small change in value. A plausible explanation for this phenomenon may be related to the strict-sense stationarity of a stochastic process and in such conditions, the pdf is invariant under a time shift. In the case of negatively growing firms (table 7), the value of the exponent is observed to decrease from 0.805 to 0.702. We conduct the paired-sample t-test on a null hypothesis that there is no significant difference in the ranks and the results are as follows: The table shows that the 2-tailed significance of the test is .000 and as a result we reject the null hypothesis at 5% significance level and conclude that there is significant difference in the ranking. The reason for observed change in the wealth distribution of the

Table 2. Paired Difference

Mean difference	Std. Deviation	t-value	df	2-tail significance
63.1837	24.65	17.941	48	.000

declining firms over time could be related to the higher difference in ranks as against growing firms. In order to test the hypothesis, whether difference in ranks is significantly different between growing market capitalization and declining market capitalization of the firms, we apply independent sample t-test and the results are given below: Levenes tests for equality of variances

Table 3. t-test for Equality of Means

.	t-value	2-tail significance
Assumed Equal Variance	-3.796	.000
Assumed Unequal Variance	-2.520	.027

yields $F = 13.269$, $p = .000$ Since, the 2-tail significance value is less than .05, we conclude that rank differences between growing and declining firms are different. This difference in ranks leads to changes in wealth distribution and in the value of the exponent of the declining firms.

2 Conclusion

The purpose of this paper is to examine whether power-law distribution emerges in an emerging capital market like, Indian capital market, similar to the wealth distribution of individuals in an economy and how the exponent of power-law behaves at various high wealth levels as well as how the relative growth and decline of firms over time affects the distribution. The wealth of the firm is equivalent to the market capitalization of the firm. Using a database of 500 companies having highest market capitalization, we observe that the power-law distribution emerges at the top 10% wealth level of the firms. As we go down on the ranks, the power-law becomes less conspicuous, which is consistent with other research findings. The value of the exponent is observed to compare well with respect to wealth distribution of the individuals in the economy. The behaviour of exponent in respect of high growth and declining firms is also investigated . We observe that the value of exponent does not change but the value reduces in the cases of firms showing negative growth. The t-test shows that as the rank difference is high and statistically significant for the declining firms, it may have caused the exponent to change its value.

Table 4. Regression Results (Firms Market Capitalization belonging to Top 10%)

.	2004	2003(1)	2003(2)
a	11.853 (199.85) *	11.397 (224.591) *	5.00 (138.84) *
b	-0.843 (-43.958) *	-0.847 (-51.371) *	-0.973 (-36.47) *
R^2	.976	.983	.966
Adj R^2	.975	.982	.965
γ	1.186	1.181	1.028
ADF Statistic	-3.252 *	-2.859 *	-4.153 *

Table 5. Value of Powerlaw Exponent at High Wealth Level

Market capitalization of the top (.)% firms of γ Exponent

Year %	2	3	4	5	6	7	8	9
2004	1.333	1.287	1.328	1.346	1.325	1.304	1.269	1.227
2003(1)	1.408	1.221	1.211	1.239	1.241	1.233	1.230	1.205
2003(2)	1.404	1.199	1.164	1.155	1.155	1.151	1.107	1.104

Table 6. Value of Powerlaw Exponent at Lower Wealth Level

	2004			**2003(1)**		
Firms Capitalization between	b	Adj R^2	γ	b	Adj R^2	γ
51 and 150	-1.517 (-94.986)*	.989	0.659	-1.326 (-123.923)*	.994	0.754
151 and 250	-1.585 (-125.346)*	.994	0.631	-1.454 (-153.256)*	.996	0.688
251 and 350	-1.870 (-105.955)*	.991	0.535	-1.523 (-175.440)*	.997	0.657
351 and 450	-1.979 (-150.563)*	.996	0.505	-1.699 (-189.416)*	.997	0.589
451 and 500	-2.136 (80.924)*	.992	0.468	-1.979 (60.523)*	.987	0.505

Table 7. Regression Results: Growth of Market Capitalization of firms in 2004 over 2003(1)

	High ($\geq$ 200%) Growth		**Negative Growth**	
.	2004	2003(1)	2004	2003(1)
b	-1.254 (-23.591)*	-1.256 (-8.970)*	-1.424 (-41.338)*	-1.241 (-47.404)*
Adj R^2	.979	.869	.973	.979
γ	0.797	0.796	0.702	0.805

* t-value in parenthesis and significant at 1% level. (in all the tables given above).

Statistical Analysis on Bombay Stock Market

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Summary. We have made extensive studies on the Daily Close Price Index of Bombay Stock Exchange (BSE) for the period of 1997 to 2004. Our analysis revealed the fact that the returns of daily close price index of BSE follow Levy stable disrtibution and exhibit randomness.

1 Introduction

The complexity of Financial market has attracted attention of many physicists in recent years [1] . Consequently, the newly developed methods of analysis in the field of statistical physics and Nonlinear dynamics have been successfully applied in the field of Economics. India being the country with the second largest population in the world the study on the Indian stock market is always very important and significant from the economic point of view.

2 Data

We have analyzed the daily close price index of the Bomay Stock Exchange (BSE) for the period 1997 to 2004.The variation of the BSE price indeces and the returns are shown in Fig. 1. (a) and (b) respectively. The return of the price index time series X(t) is calculated by the formula:

$$R(t) = log((X(t+1)/X(t))$$

Fig. 2 shows the probabilty distribution of the return. The statistical parameters of the distribution are listed in Table 1. The negative value of the skewness indicates the assymetric property of the return distribution. The large value of the kurtosis compared to the Gaussian kurtosis (k=0), shows that the tail of the return distribution are very fatter than the Gaussian ones. We have fit the return distribution with the Levy stable distribution, which is defined by its characteristic function as :

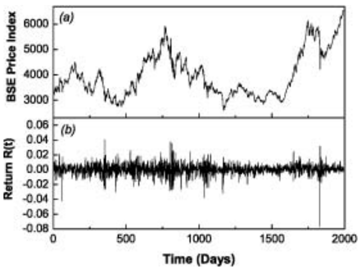

Fig. 1. Variation of (a) the daily close price index and (b) the reutrns of Bombay stock market for the period 1997 to 2004.

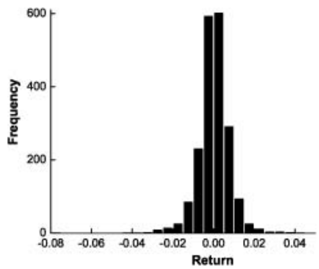

Fig. 2. Probability distribution of the returns.

Table 1. Mean, standard deviation, skewness and kurtosis of the BSE price index return.

Mean	Std. Dev.	Skewness	Kurtosis
0.0002	0.0076	-0.6103	7.9582

$$ln(\varphi(q) = i\mu q - \gamma|q|^{\alpha}\left[1 + i\beta\frac{q}{|q|}tan(\frac{\pi}{2}\alpha)\right] for[\alpha \neq 1]$$

From the fit we have obtained the value of α to be 1.69. Which means that the daily close price index of the return follow Levy stable distribution.

3 Methods of Analysis

We have used two newly developed methods of scaling analysis, namely (i) Finite Variance Scaling Method (FVSM) and (ii) Diffusion Entropy Analysis (DEA) [2] to detect the exact scaling behavior of the daily close price index of the BSE. These methods are based on the prescription that numbers in a time series R(t) are the fluctuations of a diffusion trajectory see Refs[1] for details. Therefore we shift our attention from the time series R(t) to probability density function (PDF) *p(x,t)* of the corresponding diffusion process. Here x denotes the variable collecting the fluctuation and is referred to as the diffusion variable. The scaling property of *p(x,t)* takes the form:

$$p(x,t) = \frac{1}{t^\delta} F(\frac{x}{t^\delta}) \tag{1}$$

3.1 Finite Variance Scaling Method

In the FVSM one examines the scaling properties of the second moment of the diffusion process generated by a time series. One version of FVSM is the standard deviation analysis (SDA), which is based on the eavaluation of the standard deviation *D(t)* of the variable x, and yeilds

$$D(t) = \sqrt{\langle x^2; t\rangle - \langle x; t\rangle^2} \propto t^\gamma \tag{2}$$

The exponent γ is interpreted as the scaling exponent.

3.2 Diffusion Entropy Analysis

DEA introduced recently by Scafetta *et. al.* [2] focuses on the scaling exponent δ evaluated through the Shannon entropy *S(t)* of the diffusion generated by the fluctuations R(t) of the time series using the scaling PDF. Here, the PDF of the diffusion process *p(x,t)* is evaluated by means of the subtrajectories $x_n(t) = \sum R_{i+n}(t)$ with n=0,1,.... Using the scaling PDF (1) we arrive at the expression for $S(t)$ as

$$S(t) = -A + \delta ln(t) \tag{3}$$

where A is a constant. Eq. (3) indiactes that in the case of a diffusion process with a scaling PDF, its entropy *S(t)* increases linearly with ln(t). Finally we compare γ and δ. For fractional Brownian motion the scaling exponent δ coincided with the γ. For random noise with finite variance, the PDF p(x,t) will converge to a Gaussian distribution with $\gamma = \delta = 0.5$. If $\gamma \neq \delta$ the scaling represents anomalous behavior.

4 Results

The plots of SDA and DEA for returns of daily price index of BSE are shown in Fig. 3. and Fig. 4. respectively. The values of the scaling exponents are also depicted in the figures. It is seen that the values of both the scaling

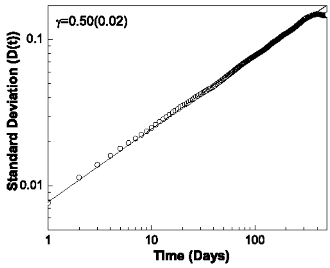

Fig. 3. SDA of the returns of the BSE daily price index.

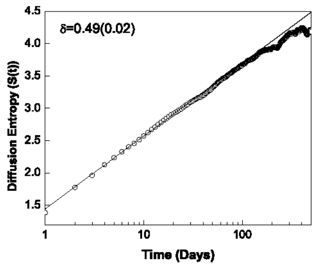

Fig. 4. DEA of the returns of the BSE daily price index.

exponents γ and δ are nearly equal to 0.5, sgnifying the absence of scaling i.e. randomness in the returns of daily close price index of BSE.

Thus our analyses revealed the fact that BSE daily close proce index returns follow Levy stable distribution with index 1.69 and exhibit randomness or scale independence.

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Funds Management by Banks in India: Solution to a Persisting Optimization Problem

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1 Banking in India: Cash Credit system

In recent times commercial banks in India have started functioning as universal banks. As of now, banks - whether public sector, private sector or foreign, can offer comprehensive financial services under one roof. Their functional diversity encompasses project appraisal, project financing, lease financing, extending working capital loans and quasi-credit facilities like bank guarantees and letters of credit, offering derivative products such as forwards, swaps and options, as well as syndication and consultancy services.

While the role of commercial banks has undergone a substantial change in the post-liberalization era, working capital loan still continues to be a major functional area for the commercial banks. Conventionally, working capital finance has been extended by commercial banks in India in the form of cash credit facility. Under this system, the lending bank sanctions a loan limit up to which the customer may be allowed to draw subject to availability of adequate security pledged or hypothecated to the lending bank. The amount of loan outstanding can vary freely and at times the balance in the cash credit account can even be in credit (i.e. the bank is indebted to the customer). Interest is payable based on the actual level of loan enjoyed on a daily product basis. While the borrower has the option to draw up to the limit without any prior notice, he has no corresponding obligation either to compensate the banker for this option or to ensure an optimum utilization of the facility. In such a situation, funds management and financial planning become relatively low priority issues for the borrowers, who can pass on the consequences of inadequate planning and inefficient management on their part to the banking system, where the problem manifests itself as a serious strain on cash management. This is a major drawback of the cash credit system. In India, credit is considered a scarce commodity and need-based financing continues to be a main plank underlying the central bank's credit policy even in the liberalized regime. An arbitrary break-up of working capital facility into fixed and variable components is thus not in line with the central bank's approach.

2 Tandon Committee - Style of Credit

Tandon Committee, which did the most important work on working capital finance in India, aimed at enforcing an effective financial planning by the borrower through a system of reward and penalty. It suggested a bifurcation of the working capital facility into two components: a fixed or demand loan component at a certain interest rate throughout the year, and a variable component bearing a somewhat higher rate of interest. The variable component indicates the excess of borrowing over the demand loan component.

If a borrower tries to project the demand loan component at a "higher than necessary" level, it would end up paying interest on amounts not actually required. On the other hand, if it projects the demand loan at an inappropriately low level, much of its withdrawals will attract a higher rate of interest and the overall interest cost over the year would not be minimized. The borrowing company should, therefore, ensure an efficient financial planning and make correct projections for its requirements, with which the lending bank has to select an optimum bifurcation in order to minimize the borrower's annual interest burden.

3 Formulation of the problem

As we have observed, implementation of Tandon Committee's recommendation requires us to work out the optimum level of demand loan that will minimize the annual interest burden for a borrower company. While there can be no two opinions about the usefulness of the style of credit recommended by the Tandon Committee, the question arises how a practical banker is to go about the exercise once a customer submits the pattern of the working capital requirement over the next one year. The task is one of bifurcation of the working capital requirement into two components - a fixed component and a variable component, which will minimize the annual interest burden for the borrowing company.

If $W(t)$ = working capital finance required by a borrowing company, expressed as a function of time, over the next one year,
x= level of demand loan or fixed component,
$W(t) - x$ [where $W(t) > x$] = level of variable component,
I = Interest burden for the company for the next one year
a = Rate of interest for the fixed component
$(a + b)$ = Rate of interest for the variable component

$$I = \int a \, .x \, . \, dt + \int [W(t) - x] \, . \, \theta[W(t) - x] \, .(a + b)dt, \qquad (1)$$

where $\theta[W(t) - x]$ is the well-known step-function. Or,

$$I = \sum_{i=1}^{12} \frac{a}{12} \, .x + \sum_{i=1}^{12} \frac{a + b}{12} \, . \, [W_i - x] \, . \, \theta[W_i - x] \qquad (2)$$

where W_i = level of working capital finance for the ith month. Conventionally the customer will submit his monthwise requirement to the lending bank.

4 The solution

If a company works out its requirement of working capital finance for the next twelve months i.e W_i for $i = 1, 2, \cdots, 12$, then the total interest burden for the company during the next twelve months works out to:

$$\begin{aligned} I(x) &= \sum_{i=1}^{12} \frac{a}{12} .x + \sum_{i=1}^{12} [W_i - x] \, . \, \theta[W_i - x] \, . \frac{a+b}{12} \\ &= a \, . \, x + \frac{a+b}{12} \sum_{i=1}^{12} (W_i - x) \, . \, \theta(W_i - x). \end{aligned} \quad (3)$$

Then,

$$\begin{aligned} \frac{dI}{dx} &= a - \frac{a+b}{12} \sum_{i=1}^{12} [\theta(W_i - x) + (W_i - x)\delta(W_i - x)] \\ &= a - \frac{a+b}{12} n - \frac{a+b}{12} \sum_{i=1}^{12} (W_i - x)\delta(W_i - x), \end{aligned} \quad (4)$$

where $\delta(W_i - x)$ is the Dirac Delta function and n = number of months for which $W_i > x$.

Or,

$$\frac{n}{12} = \frac{a}{a+b} \quad for \quad \frac{dI}{dx} = 0. \quad (5)$$

In other words, for I to be minimum, X is to be so chosen that for n months $W_i > x$, where $\frac{n}{12} = \frac{a}{a+b}$. Let such solution(s) be designated by x_0.

5 The implementation

A. Let the levels of borrowing projected by a company for the next twelve months be 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 and 52 units (the levels of borrowing need not occur in this chronological order). Also, let $a = 10\%$ p.a. and $b = 2\%$ p.a.

In such a situation, $n/12 = 10/(10+2)$, or $n = 10$. In other words, the demand loan component should be set at such a level that the level of borrowing would exceed the demand loan for 10 months. Thus, $x_0 = 42$ units or more but less than 43 units. The actual solution may be chosen (though its is not essential) to be the lowest value of x_0, i.e. $x_0 = 42$ units.

Part I

Data and analysis

Income Distribution in the Boltzmann-Pareto Framework

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Summary. The physical concept of entropy is used to develop a model of entropy distribution of income. The Pareto model is compared with the Boltzmann model and their implications are analysed in income distribution. Expenditure data of India and Sri Lanka are used to verify the workability of the Pareto Law.

Key words: Entropy, Pareto Law, Boltzmann Law, Expenditure Distribution, Truncation of Distribution

1 Introduction

In every economic system a set of agents participate in the process of an operation just as a set of particles (e.g., a collection of gas molecules forming a box of gas) join together to form a physical system. Both the systems are described by a set of parameters. A gaseous system may be described by the size of the box in which the gas moves. The production system of an economy may be described by the types of activities (services, industry, agriculture etc.) in which economic agents take part. A person may have access to some or all the activities and acquire the capabilities (energy) to enter into the market. The total income-capability of the economic system is distributed among the persons depending upon their actual participation in the diverse processes of production. But how is the distribution process in the economy? Does it follow some regularity? Can it be approximated by some functional process like the Pareto or the Boltzmann process? Has the physical concept of entropy any use to measure the degree of concentration in the distribution? The present paper attempts to examine all these issues in the econophysics framework, using expenditure data of India and Sri Lanka.

2 Concept of entropy in economics

Consider an economy (system) with n economic agents. The agents may be producers or consumers. Let the total income (energy) generated in the economy be $\sum Y_i$ units. And the agent i shares Y_i units out of total $\sum Y_i$ units. The incomes may be generated from different sources like agriculture, industry, services etc. Define $y_i = Y_i / \sum Y_i$. Clearly, $0 \leq y_i \leq 1$ and $\sum y_i = 1$. Let us assume that the macro-state of the economy is denoted by $(Y_1, \ldots, Y_n)$ in some income space. Agent i commands Y_i units of income (energy) in this space and it spends to have utility (welfare). Following Varian's (1992) money-metric measure of satisfaction we write economy's total welfare:

$$S = F(Y_1, \ldots, Y_n) \tag{1}$$

which is assumed to be onedegree homogeneous so that $s = S/Y = f(Y_1/\sum Y_i, \ldots, Y_n/\sum Y_i) = f(y_1, \ldots, y_n)$, $f_i' > 0$ and $f_i'' < 0$. Assume f to be additive. Then

$$s = \sum w_i f_i(y_i) \tag{2}$$

where w_i's are weights for aggregation. Assume the utility function to be logarithmic: Therefore

$$s = \sum w_i \ln y_i. \tag{3}$$

Weights are normalized so that $0 \leq w_i \leq 1$ and $w_i = 1$. Equation (3) may be viewed an equation for decomposition of total satisfaction among the agents. The ith component, $w_i \ln y_i$ measures the contribution of agent i to the overall satisfaction. Here the issue is: distribute total income among the agents so that the distribution becomes optimal in the sense of generating maximum satisfaction subject to the given constraints. To put it mathematically,

$$Max \quad s = \sum w_i \ln y_i \quad \text{subject to} \quad w_i = 1 \quad \text{and} \quad y_i = 1. \tag{4}$$

For solution, form the Lagrange function: $L = w_i \ln y_i - \lambda_1(\sum w_i - 1) - \lambda_2(\sum y_i - 1)$; λ_1, λ_2: Lagrange multipliers. Set the partial derivatives equal to zero:

$$\delta L/\delta w_i = \ln y_i - \lambda_1 = 0 \tag{5}$$

$$\delta L/\delta y_i = w_i/y_i - \lambda_2 = 0 \tag{6}$$

$$\delta L/\delta \lambda_1 = -(\sum w_i - 1) = 0 \tag{7}$$

$$\delta L/\delta \lambda_2 = -(\sum y_i - 1) = 0 \tag{8}$$

for all $i = 1, 2, \ldots, n$. Solving eqns. (5), (6), (7), (8) we get

$$y_i = w_i \text{and} y_i = 1/n, \text{for all} i. \tag{9}$$

When $y_i = w_i$, equation (3) becomes

$$s = \sum y_i \ln y_i \tag{10}$$

It is exactly Boltzmann's H-function used in his statistical analysis of Thermodynamics (Georgescu-Roegen 1971). For $y_i = 1/n$, for all i, $s = -\ln n$. Also $s = 0$, when $y_i = 1$, $y_j = 0$, $j \neq i$. Thus $-\ln n < s < 0$. Alternatively, define $s^* = -s$ so that $0 < s^* < \ln n$. Or $0 < s^* < \ln n < 1$.

Observations:

(i) The identification of s defined above with Boltzmann's H- function shows that it can be treated as entropy conventionally defined as in equation (10). The condition for maximum entropy indeed comes out to be exactly similar to that found from the normal definition in the physical system.

(ii) The entropy s of the system is uniquely determined, given the microscopic constitution (description) of the system.

(iii) Absolute entropy of the distribution rises (falls) more and more, as total energy(income) is being distributed among more and more (less and less) persons (agents) equally. The economic system becomes more and more egalitarian, as more and more persons share income equally: the market (capitalistic) economy turns more and more to one of socialistic economy. This is expected from the analogy of the economic system with a physical system. The system attains maximum entropy when all y_i's are equal, meaning that each agent in the system earns the mean level of income. The mean incomes may be thought to be the temperatures of the sub-systems in the larger system. Thus using our definition of entropy we arrive at the natural conclusion that for the system to attain equilibrium (ideal economic state), the temperature throughout the system must be equal.

(iv) The similarity in the characteristics of "economic entropy" defined above with "physical entropy" however presents some difficulty: physical systems attain equilibrium by maximizing the entropy; economic systems are never found in the state of maximum entropy (as defined here), for no economic system is absolutely egalitarian. This means the characteristic distributions of income (discussed in the next section) observed for economic systems are not, in fact, equilibrium distributions in the physical sense.

The concept of entropy has been extensively used in economics in analyzing the distribution structures. Theil (1967) has used it as a measure of inequality. Subsequently researchers (Pal and Pal, 1981; Pal 1987) have applied it as a measure of diversification.

3 Theoretical Models: Pareto and Boltzmann

We now examine whether the actual distributions may be parameterized by some theoretical model distributions. Based on empirical results Pareto (1896) suggested a functional form of income distribution which exhibits some sort of statistical regularity. The form is:

$$N_Y \propto (Y-\theta)^{-\alpha}, \alpha > 0 \tag{11}$$

where θ and α are parameters. N_Y is the number of persons with income $\geq Y$. Take $\theta = 0$. The corresponding distribution function in terms of proportion: $F(Y) = 1 - {}_N Y/N \propto e^{-\beta Y}$; the probability density function:

$$f(Y) \propto \alpha Y^{-(1+\alpha)} \tag{12}$$

In contrast, the Boltzmann function is

$$N_Y \propto e^{-\beta Y}, \beta > 0 \tag{13}$$

which yields

$$F(Y) = 1 - N_Y/N \propto e^{-\beta Y} \quad \text{and} f(Y) \propto e^{-\beta Y} Y \tag{14}$$

$Y^{-\alpha}$ and $e^{-\beta Y}$ are respectively the Pareto and the Boltzmann factors. Both are declining with the rise in income but the former declines less rapidly than the latter.

In Pareto $(dN_Y/N_Y)/dY/Y = -\alpha$ which entails that the relative fall in the number of persons as income rises becomes smaller and smaller and declines in proportion to the income. In other words, the percentage fall in the number of persons is proportional to the percentage rise income. This is the famous Pareto Law. But in Boltzmann $(dN_Y/N_Y)/dY = -\beta$ which entails that the percentage fall in the number of persons is proportional to the amount of rise in income. This may be called the Boltzmann Law.

Actual income distributions are not at all entirely represented either by the Boltzmann or by the Pareto distribution. Studies (Silva and Yakovenko, 2005) reveal that the lower part of the distribution is Boltzmann while the upper part is Pareto. In fact, there exits some income, say Y^*, which truncates the distribution into two parts. We can use R^2 to identify the value of Y^*. In the first stage Least-squares (LS) fitting is performed for all (n) values and R^2 is noted. In the second stage LS fitting is performed to the upper k values $(k < n)$ and R_1^2 is noted. $R_1^2 > R_2^2$. In the 3rd stage the upper $(k+1)$ values are taken in estimation and R_2^2 is noted. If $R_2^2 < R_1^2$, the k-th value (Y^*) is identified to truncate the upper part of the distribution consisting of upper k values. If $R_2^2 > R_1^2$, the process goes on until $R_j^2 < R_{j-1}^2$.

Pareto observed α to lie between 1.2 and 1.9, the average being 1.5. He claimed his law to operate in all conditions in spite of the social systems being changed. $\alpha = 1.5$ is thought to be a state for social equilibrium. If α deviates from 1.5, social tensions occur. People cry for change and ultimately α restores to 1.5.

Pareto did not use the parameter α to measure the degree of inequality in the distribution. Entropy of Pareto distribution may be computed and used to examine the nature of (in) equality prevailing in the system. Entropy is defined in the first section in terms of discrete individuals. In Pareto distribution, individuals are specified for an interval of income. So the following adjustment is made for the expression of entropy for the Pareto distribution: Let μ be the mean income of N persons. Then $y_i \ln y_i = (Y_i/N\mu)\ln(Y_i/N\mu)Nf(Y)$. Therefore, $s = \int_0^\infty (Y/\mu)\ln(Y/\mu)f(Y)dY$. For the Pareto distribution $\mu = \alpha/(\alpha-1)$ and $s_p = \int_0^\infty (Y/\mu)\ln(Y/\mu)\alpha Y^{-(1+\alpha)}dY = 1/(\alpha-1) - \ln(N\alpha/\alpha-1)$ (obtained after successive substitutions of $v = \ln Y$ and $u = (\alpha-1)v$) and $s_p^* = -s_p$.

Estimates (Table 1) of Pareto exponent (α) based on Monthly Per Capita Consumption Expenditure (MPCE) in India and Sri Lanka reveal that (i) the upper part of the distribution in both the countries follows Pareto, (ii) India's rural society is relatively more egalitarian and (iii) compared to India, Sri Lanka is relatively less egalitarian in MPCE.

Table 1. Pareto Exponent (α) based on Monthly Per Capita Consumption Expenditure (MPCE).

Country and Year	Region	No. of MPCE Class	α	R^2	$s^*/\ln N$
India: 1994-95	Rural	12	1.28	0.76	0.70
		Top 8	2.34	0.98	0.97
	Urban	12	1.08	0.84	-
		Top 9	1.60	0.98	0.90
Sri Lanka: 1986-87	Overall	12	0.60	0.63	-
		Top 7	1.46	0.95	0.70

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Part IV

Comments and Discussions

Econophys-Kolkata: A Short Story

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Following the early studies of the Traveling Salesman and other multivariate optimization problems, employing classical statistical [1] and quantum mechanical [2] tricks, during 1985-1990, the Kolkata group made some of the earliest modelling investigations regarding the nature of wealth and income distribution in societies and its comparison with the energy distribution in some (quantum) gases. In the 1994 Kolkata Conference, many Indian economists (mainly from Indian Statistical Institute campuses) and physicists discussed about the possible formulations of some of the economic problems and their solutions using tricks from physics [3]. In fact, in one of these papers in the proceedings, possibly the first published joint paper involving both physicist and economist (Sugata Marjit) Indian co-authors [4], the possibility of ideal-gas like model of trading market was discussed. Among other things, it tried to identify, from the known effects of various fiscal policies, the equivalence of the kinetic energy of the gas molecules with the money of the agents in the market and of temperature with the average money in the market. Such a 'finite temperature' gas model of the market was first noted by Dietrich Stauffer (Cologne) [5]. With the possibility of putting more than one agent in the same microstate, identified by the price or money income of the agent in the market, the likely distribution was concluded there [4] to be Bose-Einstein like, rather than Gibbs like. This study of course had the limitation of absence of any comparison with real income distributions in any market or country. In 1995, in the second 'Statphys-Kolkata' series of Conferences (being held in Kolkata for the last one and a half decade now [6]), Gene Stanley (Boston) first introduced the term 'Econophysics' to describe such researches [7]. Since then, Kolkata (erstwhile Calcutta) is considered to be the formal birthplace of this new term: "*The term econophysics was ... first used in 1995 at an international conference ... in Calcutta*", as mentioned in the successive Symposium homepages of the Nikkei Econophysics Symposia [8], and also elsewhere.

The general features of the observed income/wealth distributions in any society, namely the initial rise of the distribution and then exponential decay (or a log-normal/Gamma function decay) for the majority middle income region (apart from the final Pareto tail for the rich), was taken as an indication that a simple Markov scattering, as in kinetic theory of gas, is insufficient to capture the full trading picture. It was immediately clear that a saving propensity (fraction) for each agent would give the desired feature of a dip at the low income: an agent with some initial money cannot now become pauper in one scattering or trading as a finite fraction will be saved and can become so only if he/she loses in every successive trading. This study was first done with Anirban Chakraborti [9]. Actually, a little before its publication, Victor Yakovenko and his collaborators (Maryland) [10] had put their seminal paper on the ideal (classical) gas model of income distribution in the cond-mat (electronic) archive and later published (also giving the US data to support their ideal gas model). This observation stimulated the Kolkata group very much and noting the advantage of the saving factor in explaining the initial dip in the distribution, over the Gibbs distribution in the ideal gas model of market, several extensions were made: Srutarshi Pradhan and coworkers analyzed the self-organizing property of such models [11], Sitabhra Sinha (Chennai) made a detailed investigation [12] on the stochastic map equivalents of such models, and Anirban Chakraborti, together with Marco Patriarca (Helsinki) and Kimmo Kaski (Helsinki), made an extensive numerical study of the ideal ideal gas model with fixed savings and proposed [13] the Gamma distribution for the steady state income distribution in the model. However, a simple, yet profound, observation by Arnab Chatterjee in late 2002, introducing randomly distributed saving propensity in the same ideal gas model, proved very successful in capturing all the important features of the observed income/wealth distributions: dip for low income group, exponential (Gibbs) decay for the middle income group and power-law (Pareto) tail for the rich people! This was first reported in the 2nd Nikkei Econophysics Symposium in Tokyo in November 2002 [14]. This model immediately attracted a lot of attention from physicists (and also from economists; see, e.g., the next, rather critical, comment by Paul Anglin of Univ. Windsor).

In the meantime, there were several regular and 'popular science' articles which tried to explain and also justify the use of stochasticity in such gas models of markets: for example, Brian Hayes (American Scientist) argued how a little mismatch over the 'just price' of any commodity, as induced by common bargain capacity of the agents in the market, eventually leads to a stochastic gas model he "had accidentally created", which he discovered "to be the same as the" Kolkata model [15]. This kind of spontaneous rediscovery of the gas model for the market independently by several groups indicate perhaps the inevitability of the model.

The fixed saving propensity gas model was later analyzed and improved by several groups (a few of them reporting in this workshop). Arnab Das and Sudhakar Yarlagadda here wrote a Boltzmann-like equation for the income probability density, which they solved numerically for the steady state [16]. With Subhrangshu Manna, extensive numerical studies were made on the distributed savings model [17] and the Pareto behavior of the large income tail was established. In fact, together with Debashish Chowdhury (Kanpur), Kimmo Kaski and Janos Kertész (Budapest), a Conference on "Unconventional Applications of Statistical Physics" was held in Kolkata in early 2003 [18], where several groups (a few of them reporting in this workshop as well) made further numerical and analytical studies on the Kolkata models, and established several robust features. The data for the Indian income distribution has also been analyzed recently by Sitabhra Sinha [19]. Robin Stinchcombe (Oxford) joined recently (in his latest June-July 2004 visit to Kolkata) in solving analytically the master equation for the random saving gas model of the market [20].

Jenny Hogan (New Scientist) in her very recent report [21] on these developments described briefly the Kolkata models and mentioned that this "more sophisticated model" (with saving factor) has some added desirable features over the ideal gas model of markets. She additionally reported some interesting (and a few inspiring) opinions of several distinguished economists and physicists on these developments. She also described this Kolkata workshop as "the first ever conference on Econophysics of Wealth Distributions" where "economists will join physicists to discuss these issues" (her report was published just before the workshop). We indeed believe that some of these wishful successes and develpoments have already started taking place!

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Econophysics of Wealth Distribution: A Comment

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Summary. Some recent papers have proposed models of trading which produce realistic-looking distributions of wealth. This Comment argues that, while the models are correct arithmetically and the papers claim that the empirical support is persuasive, they involve undeclared simplifications that limit their application and that point to empirical propositions that are easily refuted. Since many of the simplifications involve dismissing the economists' favorite price mechanism, it is important to realize why the models do not propose a coherent alternative. This Comment suggests several directions for future research on this important topic. [1], [2]

1 An Interesting Question

Econophysicists have proposed explanations for the distribution of wealth using a class of models that the authors describe as "simple", "rich", "intriguing" and "generic". The authors and certain commentators (Ball [1], Hayes [2], Buchanan [3]) find it striking that such models can reproduce a statistical regularity which economists appear to overlook. They also use the models to comment on government policies that might improve the welfare of society. In my opinion, while the arithmetic is correct and the conclusions are seductive, the models are not sufficiently reliable to extrapolate their conclusions to the real world confidently.

These models are interesting because they use a previously-unconsidered micro-process to explain a macro-phenomenon: that the distribution of wealth

[1] The author could not come to the workshop for unavoidable reasons, but sent this comment for discussion there and for inclusion in the Proc. Vol. – Editors.

[2] Though these people do not necessarily agree with what is written, this paper has benefited from comments by Richard Arnott, Bill Baylis, Stefan Bornholdt, J.-P. Bouchaud, Bikas Chakrabarti, Eric Nodwell, Nicola Scafetta and Y. Sudhakar. Research assistance by Xuzhen Zhang is appreciated. An extended version of this Comment with more examples, more discussion and a longer bibliography can be found at http://www.uwindsor.ca/PaulAnglin.

displays a power law. The associated equations are sufficiently simple, and computers have become sufficiently fast, that it is easy to simulate these processes. Many papers verify the resulting regularity (e.g. Chakrabarti and Chatterjee [4], Reed [5], Souma [6]) based sometimes on the entire distribution and sometimes on a part of the distribution. This Comment does not offer new mathematical insights or data because referees and editors have already determined the validity of the proofs presented in published papers.

I focus on a different aspect of the puzzle which may also be useful: the validity and relevance of the assumptions being used. As isolated models in a broader research program, their ultimate contribution is not yet known. The facts that the authors compare the outcome of these models to the distribution of wealth in specific countries and that they consider the implications of policy variables, such as taxes, suggest that the authors think that the models are almost realistic. Given that existing research has demonstrated that many things affect the distribution of wealth (e.g. Champernowne [7], Sutton [8] or Sattinger [9]) and that wealth affects many other things, this Comment questions the sense in which these models appear realistic.

I focus on three lines of reasoning. First, some authors are confused or careless when using some key words. Second, the models overlook five specific principles, each of which has been refined by at least a century of economic debates. For this reason and to introduce readers to debates amongst economists that appear to have been overlooked, the bibliography is relatively long.

The importance of the third reason depends on a difference between the methods of economists and physicists. Stanley et al [10] noted that physicists are "fundamentally empirical", in contrast to economists. I argue that the data analysis used in this literature uses a standard that ignores differences between physical objects and economic objects and that is especially relevant to the proposition advanced by this literature. A simpler explanation, which must be at least part of the answer, has been overlooked and can generate a not-unrealistic distribution.

2 Five Basic Principles of Economics

The models focus on the evolution of a single variable, called "wealth" or "money",[3] based on a dynamic process described as scale-free. When trying to

[3] Some papers perpetuate misunderstandings about the nature of money. For example, the first sentence of Bornholdt and Wagner [11] claimed that Debreu's book [12] studied the role of money in an economy. Debreu (p. 28 and endnote 3 to Ch. 2) claimed the opposite. Claims of equivalence between money, income and/or wealth can be found in Chatterjee, Chakrabarti and Manna ([13], p. 161-162) and Patriarca, Chakraborti, and Kaski ([14], first paragraph). Dragulescu and Yakovenko [15] is more careful in its introduction but not consistently. Anybody who uses a credit card quickly learns that there are differences between the concepts of "money" "income" and "wealth".

relate the model to the real world, the importance of the measure is unclear. It is certainly true that many economists attempt to summarize personal or national well-being in terms of wealth or gross domestic product (GDP) but even introductory textbooks note problems with the attempt.

I think that one source of confusion is that the authors view economic issues from a Mercantilist perspective. This perspective opposes what Adam Smith and David Ricardo identified as the true source of wealth in a nation: trading of goods for more-preferred goods, not the holding of money gold or wealth, enables a trader to become better off.[4] Many authors criticize this conclusion by noting that few markets display the conditions needed for the famous Invisible Hand result, or its precise formulation as the "First Welfare Theorem." Such authors seem to forget that trading can be mutually beneficial even when markets are not perfectly efficient.

I think that the basic problem is that the models obscure ideas which would show whether their conclusions are robust. To answer my challenge, I think that the research program needs to be explicit about five Basic Principles: Opportunity Cost, Gains from Trade, Margins, Equilibrium, Comparative Statics. The following discussion illustrates how these models fail each of these principles in ways that are empirically relevant.

2.1 Opportunity Cost and Gains from Trade

The principle of Opportunity Cost is so fundamental to economic analysis that its importance cannot be overstated. When differences between individual are summarized by a single good, called money or wealth, the models do not permit differences in taste or different uses of a given type of good. Such differences are important since economists often advocate a decentralized price mechanism because it enables traders with different tastes to consume different bundles of goods. If differences in taste are empirically relevant and if the models do not allow different traders to have different tastes, then the relevance of the models' predictions seems limited. To put this idea another way, without a sensible reason to trade, it becomes easy to "conclude" that any trading mechanism produces a bad outcome.

2.2 Margins

Merely recognizing the existence of more than one good, and including a measure of tastes to account for each trader's trade off between them, points to an empirical inconsistency in the models: when the "price" of the consumable increases, a trader can reduce consumption without consumption falling to zero. In many cases, quantity demanded of a good is so sensitive to its price

[4] Paul Krugman has written on why this important idea is so difficult to understand and to explain to non-economists: http://www.pkarchive.org then click on "International Trade," then on "Ricardo's Difficult Idea 3-96".

that expenditure on a good is *negatively* correlated with the price. (An increase in the price may cause a trader to want to sell more and I will comment on the selling dimension next.)

While these models reject the economists' traditional ideas of a market process, it is unfortunate that many models also reject the notion and the implications of a price. Including price explicitly would make it easier to evaluate the empirical relevance of a model which assumes that "the amount of money earned or spent by each economic agent is proportional to its wealth" ([16], p. 537).[5] Similarly, it is difficult to evaluate the assertion that the value of an item is constant when there are many familiar counter-examples: e.g. a cup of coffee (first in the morning versus a second or third cup). For all these reasons, noting the possibility of a variable consumption margin implies that the models should be interpreted as focusing on a special case.

2.3 Equilibrium

Compared to other critics of economics, these models are careful about the concept of an equilibrium, usually meaning a stable distribution. But the models rarely represent an equilibrium in the sense of quantity supplied equalling quantity demanded.[6] Since the models propose that trading occurs in pairs, the models must assume that any trader can satisfy the demands of any other trader at any time. This aspect of a model creates at least two problems. First, if a trader can always produce enough stuff for any other trader then they should be able to produce enough for themselves. Clarifying the model in this way would have the unintended implication that poverty is not a barrier to consumption. Second, if a rich trader meets a poor trader,

[5] The reasoning used to justify scale-free-ness fails to distinguish "real" and "nominal" wealth. Economists recognize that neutral inflation has no real effect but any reasoning concerning inflation reveals little about interaction if some of us become rich in the sense that Bill Gates is rich. The essential problem is that many processes are scale-free, as demonstrated in the extended version of this Comment, and this ambiguity shows one of the dangers of arguing by "analogy" (quoted in Dragulescu and Yakovenko [15], p. 723 and in Pianegonda et al [17], p. 668).

[6] Many authors assert that markets rarely attain a perfectly competitive equilibrium and use this assertion to justify an alternative model. Some models apply this assertion inconsistently (e.g. [18], p. 449 vs p. 446). In other models, an economist would predict either that *any* price could be an equilibrium price or that there is no perfectly competitive equilibrium because quantity demanded and supplied do not vary with price.

Economists continue to research this topic: e.g. Barro [19], Benassy [20] and Fisher [21] offer three very different approaches. Krueger's [22] presidential address to the American Economic Association gives some ideas of how thinking amongst economists has evolved. It may also be interesting to note that, in part, Vernon Smith is a Nobel Prize Laureate for using controlled experiments to demonstrate that markets are more competitive than they should be in theory.

then the poor trader would realize that the rich trader wants to buy large quantity and the poor trader could improve his or her bargaining position by saying "Sorry, I haven't got any".

Different economists have proposed many different pricing mechanisms, and the debate amongst economists has not ended, but all economists agree that the process which determines prices cannot be separated from the process which determines quantities. A popular alternative is to suppose that any imbalance is rationed in a way that is consistent with a formal bargaining process (e.g. [23] or [24]).

2.4 Comparative Statics

Comparative static analysis emphasizes the idea that careful study of the effect of a change in a parameter ("exogenous variable") requires comparing the equilibrium solution ("endogenous variables") before the change with the equilibrium solution after the change. Comparative statics analysis helps economists to answer the kinds of questions that people ask: Is "globalization" or the information revolution responsible for the *changes* in the distribution of income/wealth? Do free markets, central planning or some third way *increase* total wealth?

Some authors, e.g. [16], claim that their mechanism permits many interpretations. Scafetta, Picozzi and West [25] claim that their model provides more support for a classical model rather than a neoclassical model. Before accepting these claims, I wonder if this class of models permits the economist's textbook model as a special case. Comparative statics analysis could be used to study the effects of different trading mechanisms. Being able to isolate these effects from other aspects of the model would show which parts of the simplified models are important.

The textbook model would be useful as a reference point, if only because it is commonly discussed. Consider a model which includes a parameter measuring the "degree of market imperfection." A costly comparison shopping process, where each buyer (seller) meets many sellers (buyers) but selects only low price sellers (high price buyers) as trading partners, is an empirically relevant alternative model of an imperfect trading mechanism that seems to have been overlooked. This margin of adjustment creates a kind of market power that is easy to exercise. Identifying the effects of a change in the cost of comparison would be a truer test of how imperfections affect the distribution of wealth.[7]

[7] This kind of process has been used to study labour markets [23], financial markets [28] and real estate markets [29]. Diamond [30] showed that the market equilibrium is not necessarily a continuous function of the imperfection.

3 Reinterpreting the Data: Wealth and Age

The models noted above are simple and simple models have the advantage of summarizing key information with a few parameters. Unfortunately, the key parameter in many of these models is not observed directly and the range of acceptable parameter values may be wide, if the model is the only explanation permitted. The power of estimating a parameter indirectly depends on the realism of the model. Other models link an observable parameter (e.g. Chakraborti and Chakrabarti [26] and Chatterjee, Chakrabarti and Manna [13] focus on the savings rate) to the skewness of wealth. Unfortunately, a test involving differences in the savings rate would require comparing the distributions of wealth in different countries and countries differ in many ways, other than the savings rate.

People save for many reasons but one of the most important reasons is to create wealth for retirement. Empirically, knowing how wealth is distributed across different ages is at least as important as knowing the overall distribution of wealth: Kennickell ([27] esp. Figure 3) showed that the percentage difference between the wealth of the median 30- year-old and the wealth of the median 50-year-old American is about the same as the difference between the 10th percentile of 50-year-olds and the 90th percentile of 50-year-olds.

More formally, suppose that each person starts working at age 20 and lives for a maximum of 100 years. Income, y, is the same for all workers. While working, the person saves S per year. At date T, they retire and plan to spend their wealth at a constant rate R. Two ideas familiar to economists show how R, S and T are linked. First, since the saving are intended to pay for consumption after retirement, the present value of post-retirement consumption should equal wealth at the date of retirement. Second, the trade off between current savings and current consumption can be resolved by invoking the idea of consumption smoothing. To apply these conditions, suppose that an investment grows exponentially and without risk at a rate of r: saving S at age t becomes S exp(r(T- t)) at age T and a constant flow of saving means that wealth at an age of a is

$$W(a) = S(\exp(r(a-20)) - 1)/r. \tag{1}$$

After age T, wealth is used to pay for consumption until age 100; the present value of such consumption at age T is

$$R(1 - \exp(-r(100 - T)))/r. \tag{2}$$

Thus, lifetime consumption is self-financing if and only if

$$S(\exp(rT) - 1)/r = R(1 - \exp(-rT))/r = W(T). \tag{3}$$

Smoothing consumption over time implies that

$$y - S = R. \tag{4}$$

If r= 0.065 and T= 65 then computation shows that a savings rate of 0.048 during a working life enables a constant rate of consumption over a lifetime. A change in r or in the savings rate would produce different solutions for R, S and T but these numbers are broadly consistent with observed data.

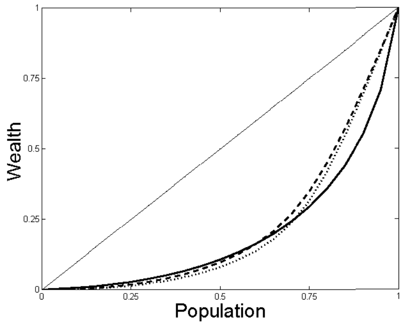

Fig. 1. Comparing Distributions

The dashed line in Figure 1 shows the Lorenz Curve for the distribution of wealth under a Cumulative Savings plan using information on the current age distribution of the US ([31]). As expected, it is skewed with the wealthiest people being near age 65. Many of the models in this literature seek to compare the observed distribution of wealth to a Pareto distribution. The solid line shows the Lorenz Curve for that distribution using the parameter 1.5 which, some people claim, represents a reasonable value for the US. It is possible to compare these two distributions more precisely but, using an eyeball metric, the Cumulative Saving distribution looks more skewed for lower levels of wealth and less skewed for the higher levels of wealth.

A reader would be justified in believing that differences between the distributions might be explained by one or more aspects omitted from both models.[8] The principle of comparative statics shows how to investigate the significance of an omission. To take a simple example, if the rate of return on investment had been 7.5 percent instead of 6.5 percent then those who are

[8] Sometimes, simplified models omit some relevant ideas. For example, the assumption that income does not vary with age begs to be improved but most changes to this assumption would raise a question of whether it is proper to distinguish between wealth disguised as financial assets and wealth disguised as human capital. Or, people may prefer to use pensions in place of personal wealth to finance consumption during their retirement. Wealthier people may invest in riskier investments which have a higher average rate of return or wealthier people may have higher annual income and save more. Rather than offering a more complex model, and attempt to resolve the implied measurement issues, my intention is to offer a model with more realistic behavioural foundations that also produces a realistic-looking distribution.

already wealthy benefit disproportionately. But, since the reason to generate wealth is to fund consumption during retirement and since r is higher for a lifetime, there is also an indirect effect: the savings rate falls to 0.032. The resulting Lorenz Curve is shown with the dotted line.

Using the eyeball metric again and to quote some authors out of context, these data "compare very well" or may "encourage" further work in this direction. And, while it is true that an eyeball metric is not very precise, it is also true that the power and size of the statistical tests used by others are rarely reported. More generally, this example help to explain why the distribution of wealth would be stable regardless of why it is skewed and why an economic process which produces a skewed distribution is not necessarily unfair.

4 Concluding Thoughts

Tools employed by physicists have created attention-grabbing profits for Wall Street firms, as discussed in Matenga and Stanley [32], but the recent research on the distribution of wealth strays into areas where previously successful tools become unreliable. The ideas that economists have over-simplified their models of buying and selling and that a more realistic trading rule produces a more realistic distribution of wealth are interesting. The arithmetic of the models is simple, is correct and the results can be described with surprising ease.

Any one model is part of a larger research program where simple models are replaced by better models. As Einstein is supposed to have said: Things should be made as simple as possible but no simpler. The success of the models encourages further research but this Comment has tried to argue that, to study the behaviour of real people who trade when they are willing and able, these models unintentionally propose a theory where traders are forced to trade and always can. I noted several instances where this class of models ignores other explanations as well as historically significant and empirically relevant basic principles. The history of economic science offers examples of empirical regularities that misled because of a weak behavioural foundation: e.g. The failure of large scale Keynesian models of the 1960s and 1970s, at about the time that they were used widely, led to the insight now known as the Lucas Critique [33]. For these reasons and others, it is difficult for this economist to determine the power of the empirical tests used to investigate the hypothesis offered by this new literature.

Given the number of times that the name of Adam Smith is invoked, I am surprised that the models seem to reject his intellectual contribution. Only sprinkling words like "money" and "wealth" in the text make these models sound analogous to a real economy. I offer a process that is consistent with many of the ideas that neoclassical economists like, that helps to explain an equally important phenomenon and that has been overlooked as an explanation: the relationship between wealth and age. This process also creates a realistic-looking distribution. Moreover, analysing it illustrates why questions concerning the distribution of wealth should not be isolated from other questions. Is the Cumulative Savings process too simple? Of course. Both this model and the other models omit behavioral and technological features of an economy whose significance has been repeatedly confirmed.

A final conclusion of this Comment should be that nobody needs to confuse the techniques of analysis with the principles that those techniques embody. Many academics and ordinary people claim that modern economics is merely mathematized ideology (e.g. McCauley [34]) or that it is not a real science. Researchers investigating economic questions are less able to control the initial conditions of their experiments. Economic analysis also suffers from the fact that the actors being studied exercise freedom of choice. These facts make human behaviour less predictable. They also imply that a human being can participate in a system *selectively* in a way that an atom cannot. Hence, certain methods of analysis become inappropriate and results which are arithmetically correct, but not robust to perturbations from previously identified sources of ignorance, can be unimportant. The five basic principles discussed above are easy to identify in the models which economists have found useful in the past [35]. They can also guide future research.

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Econophysics of Wealth Distributions: Workshop Summaries

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1 Summary by Peter Richmond and Jürgen Mimkes

Econophysics has emerged in the past 10 years as an important and active area of research for physicists and economistsl. Within Europe, the USA and Australasia, there have been a number of important meetings dedicated to the area. We note for example, the European Physical Society's series of APFA meetings (Application of Physics to Financial Analysis). held in Dublin (1999), Liege (2000) London (2001) and Warsaw (2003). A fifth meeting is being planned.

The 2005 workshop in Kolkata brought together physicists and economists from Europe, North and South America, China, Japan and India to focus specifically on the important topic of wealth distributions - a topic of great importance to policy makers, government regulators and taxation authorities. A number of important issues emerged from the many interesting presentation and subsequent discussion:
(1) In line with Pareto's 100 year old conjecture, wealth distribution does appear to be similar in form across all nations and times studied so far. Exception do arise but apparently during periods of economic stress and turmoil. Data compiled during the recent crisis in Argentine and presented at the meeting illustrated this clearly.
(2) The is apparently stationary distribution seems to follow from a few basic principles. However more detailed data for both stationary and dynamical

behaviour is needed to enable extensive verification. Such data should span both more countries and times. Data such as that presented for Japan that identifies over time very wealthy citizens offers scope for development of micro models and detailed testing of predictions of the model dynamics and time series as well as stationary distributions..

The dialogue between economists and physicists is clearly vital for this research effort to achive its full potential. In this sense the Kolkata workshop fulfilled a valuable purpose and one can hope that other workshops can build on these first steps. Within Europe, a number of EC sponsored networks are being developed. One such, established in late 2003 is the COST network, physics of risk. This spans over 20 EC states together with Australia. It is possible that India could become a member for the remainder of the period to end 2006.

One issue discussed at the workshop was that of the standing of young students who persue a PhD in this area. Would they be considered by their peers to be professional physicists? Would they be employable?

It is clear that the nature of the problems in econophysics can be formly rooted within the broad area of statistical physics. The systems are complex and exhibit all the richness of traditional many body systems. A physicist needs all the tools traditionally taught within statistical physics and field theory - and argualbly more to deal with socio-economic systems. Employment statistics may merit further study. However we note that a recent UK survey by the Insititute of Physics stated that each year upwards of 20% of new physics graduates now enter the finance industry. Furthermore as a result of this demand, EPSRC, the UK research funding agency has recently invited bids from consortia of universities who wish to establish new centres of both research and training of physicist in areas relevant to the finance industries. The UK Insititute of Physics has initiated a new subject group 'Physics of Finance'; German Physical Society has initiated a section on "physics of socio-economic systems". Both aim to promote scientific and professional activity of physicists in this area. Across Europe it is likely that more activity will emerge that supports econo and sociophysics under the broad heading of complex systems research.

The situation in India may merit further study, howver, the participants at this meeting would encourage further development of these transdisciplinary research areas that links physics not only to economics but also offers scope for other links to finance, business, psychology and sociology. A nation that aspires to be a part of the knowledge community cannot stand idly by and let others monopolize these developments.

2 Summary by Mauro Gallegati

The day of the opening of the "Econophys symposium", *The Times of India* under the headline "Nation jumps to 3rd place in Asia", publishes 2 articles on wealth distribution. They emphasize, with a moderate tone of self-accomplishment, the extraordinary progress made by India in recent years: "8th in the world in terms of the number of billionaires and 9th in terms of the total wealth of super rich." Nevertheless, they point out the extreme difference in wealth distribution *among* Indian citizens: an average Indian billionaires wealth is equivalent to almost 9 million times the countrys per capita GDP, while the same figures are in the order of ten-thousand *times* for the most developed countries, and over a million for the third world countries.

One of the main results of the conference was the consensus reached by economists and physicists on the shape of the distribution curve: log-normal or exponential for the first 97-98th percentiles of it, and Pareto for the very upper tail. Moreover, the functional form of the wealth distribution is stable over many years and countries, although the parameters fluctuate within narrow bounds. These fluctuations explain the wealth differences *among* citizens of countries at different levels of development and concern the very nature of the economic phenomena, which cannot be reduced to physics by simply substituting some concepts of the two fields, such as money instead of energy. In a nutshell, while most of the physicists in this conference assume the existence of time reversal symmetry, the economists in Kolkata seem to believe that *the* economy is characterized by "self organized criticality".

On the steps of classical mechanics, mainstream economists try to explain the behavior of the aggregate through the analysis of its single components. The methodological "reductionism" in fact assumes that, since the aggregate is nothing but the sum of its components, aggregate dynamics is determined by individual dynamics. This is true if and only if the action of an agent *is* not affected by the actions of the others, i.e. if there is *no* interaction, or *if* information is complete and agents process it rationally. In presence of interaction in fact, the aggregate is different from the sum of its components. The only interaction of agents possible in the mainstream economic model is an indirect one, through the price system only.

Quantum revolution dismayed the reductionism hypothesis: the characteristics of the single particles are not intrinsic properties but they can be understood only by analyzing the aggregate. Analogously to quantum physics, economic agents do not exist if not connected *to* each other. While in classical mechanics the property and behavior of the parts determine those of the aggregate, in *quantum* mechanics the opposite is true: the whole determines the behavior of the parts.

Our hope is that econophysics will help *mainstream* economics to leave the *"straightjacket of reductionism"*.

3 Summary by Thomas Lux

The distribution of wealth is an outcome of all the complex interactions of modern economies. Unlike with certain stock market phenomena one can, therefore, not restrict the analysis to a subsystem (may be at time disconnected from the rest of the economy). Rather, in some sense, empirical distributions of wealth and income reflect the entirety of production, distributions (via factor inputs to production and their compensations), and redistribution (by the government). It is the more remarkable that through the empirical work of recent years one has obtained a very strong characterization of the main stylized facts, of the distribution of wealth.

As the main contributions of physics-oriented approaches represented in the workshop I would, therefore, like to point out the following:
(1) a clear description of the universal features of wealth distributions for almost all countries for which data are available, namely, a Gibbs-Boltzmann (or Gamma) distribution for the bulk of the data and a distinct Pareto tail for the extremal part. This establishes the main empirical features to be explained. This behavior of the data also proves that explaining wealth distributions is far from trivial,
(2) attempts at explaining the distribution via agent interaction. The findings of Pareto have for a long time be simply taken as empirical facts for which no satisfactory explanation could be provided. It is important to point out that such distributional features can be explained via appropriate models of large ensembles of interacting units (which is what an economy consists of),
(3) as expressed in various ways, the bulk of the distribution with its exponential or Gamma distribution can be explained by chance events or maximum entropy considerations. It, therefore, seems genuinely less interesting than the Pareto tail. Expressed differently, one needs a distinctly different mechanism to cover the tail data in a theoretical model.

Now come the more critical remarks: one disturbing observation is the almost complete negligence of available knowledge which has been accumulated in economics over one or two centuries. This leads to attempts of modeling economic interactions in a 'naive' way according to simple ideas of how one could conceive economic activity. It is very *likely* that these first ideas should already have been proposed in the history of economic thought and have been overcome by more elaborate theories and models. The received body of knowledge is not only available in mongraphs and articles, but had already been summarized in handbooks[7] and is documented in specialized journals[8], which shows the state of development of this area of research. The total absence of any reference to this whole body of knowledge would almost certainly make econophysics contributions conspicuous for economists (imagine a new paper on quantum physics without any reference to received literature). However, in all the voluminous literature, a perspective of explaining the shape of the distribution from interactions of agents is almost entirely absent. Adding such

[7] Atkinson A, Bourguignon F (2000) Eds., Handbook of Income Distribution, Amsterdam, Elsevier.
[8] Journal of Income Distribution, Review of Income and Wealth.

a perspective would, therefore, be the most important contribution of research inspired by statistical physics.

By the way, modern economic theory is thought to commence with Adam Smith s The Wealth of Nations (1776). His main point was that bilateral exchange (trade) is mutually advantageous. In this, he turned against earlier theories according to which one party looses and the other gains in a bilateral exchange. Needless to say, Smith s position is universally accepted today.

To quote Dragulescu and Yakovenko, one should, therefore, try to simulate and analyse big ensembles of economic agents following realistic deterministic strategies (Eur. Phy. J. B 17, 2000, 729). Models should be based on economically plausible assumptions and mechanisms, should incorporate markets, voluntary exchange and prices. Wealth is not a primary economic concept, but needs to be computed from more elementary variables (quantities of goods, assets and their prices).

As a more long-term goal one should envisage the identification of further stylized facts (outof- equilibrium dynamics, e.g. changes of the shape of the distribution of wealth during industrialization) and try to construct appropriate models for their explanation.

Any successful attempt at deciphering the underlying forces behind the universal laws of wealth distributions would certainly have to built upon available knowledge in economics and combine it with the methodology of statistical physics developed for multi-agent problems.

4 Summary by Victor M Yakovenko

Here I try to summarize major achievements of this conference and highlight unresolved issues and directions for future studies. As with experiment and theory in physics, establishing close connection between empirical data and theoretical modeling is vitally important for economic science.

4.1 Analysis of Empirical Data

This conference has firmly established that income distribution has a two-class structure — it was even shown on the conference poster, to which many speakers referred. The upper tail follows the Pareto power law, whereas the distribution for the lower-class majority is similar to the Boltzmann-Gibbs distribution of energy in physics. This understanding needs to be propagated to a broader community and recognized as an important "stylized fact". Many researchers still have misconceptions about the structure of the actual distribution.

The lower-class distribution was fitted to the exponential, gamma, and log-normal distributions by different researchers. We need to find out whether these differences are real and reflect social structure of different countries, or they are spurious effects. For example, it is important to distinguish between distributions of individual income vs. family or household income. The exponential distribution of individual income results in the gamma distribution of family income, and both fit the corresponding sets of data for USA. Different ways of collecting income data (e.g. taxes vs. surveys) may have different sampling rate of the low income population.

It would be very interesting to expand the empirical study of income distribution to more countries and different time periods. Having established what the equilibrium distribution is, we can study deviations from it under various circumstances. For example, very interesting non-equilibrium data for Argentina were presented by Juan Ferrero. Are income distributions in developing and post-socialist countries evolving toward the maximal-entropy equilibrium shape? Have the governments of some developed countries succeeded in changing the shape of the distribution by social engineering?

Unlike for income, only limited data is collected on wealth distribution, and virtually no data on money distribution. Assuming that people keep their money in bank accounts, the distribution of bank deposits would give some information about distribution of money. A big bank could, in principle, collect such statistics, and it would be very interesting to compare it with the theoretical predictions for money distribution. The distribution of money would give information about purchasing power — money that people already have and can, in principle, spend without going into debt.

4.2 Theoretical Modeling

Statistical distributions in various models with a conserved quantity (analogous to energy in physics) were presented at the conference in great detail. Now we have a very good understanding of mathematical behavior in these models. However, much less attention was paid to the economic interpretation of the conserved quantity. I believe that the conserved quantity should be identified with money, because the ordinary people can only receive and give, but not manufacture money. Of course, central banks and governments can inject money in the economy, but these can be treated as external sources without violation of the local conservation principle. Going into debt may "create money", but it also creates debt obligations, so conservation laws can be formulated in this case too. Models with debt were not presented at this conference, but there are interesting and elegant econophysics papers that naturally incorporate debt. Surprisingly, we heard from some economists that money is "illusory" — a statement in sharp contrast with the everyday experience for most of us, lower-class people.

However, most of the conservative models called the conserved quantity not "money", but "wealth". One can define wealth as money + property, the latter being the number of assets multiplied by their prices. When an agent pays money for intangible service, like in the barbershop model of Peter Richmond, his wealth changes. However, when an agent receives tangible assets in exchange for money, his wealth does not change. On the other hand, his wealth may change because of material production, which changes the number of objects, or because the price of objects changes ("paper wealth" generation). These processes are not captured in many conservative models, so applying them to wealth distribution raises questions. However, it depends to which class the models are applied. The lower-class may have little assets besides money, so their wealth distribution would be the same as money distribution, whereas the upper class wealth would be determined primarily by assets dynamics. Physicists should also pay more attention to economic justification of the model transaction rules. Some of the proportional rules may be unrealistic, because they imply that people are charged different prices depending how much money they have.

While most of theoretical models deal with money or wealth, most of empirical data is available on distribution of income, which is the flux of money. Physicists often use income distribution as a proxy for money or wealth distribution. This is not because they are so ignorant that they do not understand the difference, but because the data on money or wealth distribution is typically unavailable. The idea is that, because all three quantities are closely inter-related, income distribution should generally reflect the major qualitative features of money and wealth distribution. However, it is time to develop models specifically dedicated to income. These models can be based on diffusion in the income space — the so-called income mobility. The models should include the age of agents, because young people enter economy with low income, and old people leave it with higher income, so that the statistical distribution results from a constant demographic flux.

The real challenge for modeling is to show how the two-class structure develops in a society that initially consists of equal agents. The problem goes beyond obtaining the power-law and exponential distributions, but concerns with identifying the functions of the two classes in the "social ecology", e.g. as employers and employees, perhaps by analogy with the predator and prey models.

Differences in money and income temperature between different countries have important consequences for the world economy. These were described in the talk by Jurgen Mimkes on the basis of thermodynamic analogy. More attention of the econophysics community should be attracted to the international aspects of money and wealth distribution. Here we deal with strong space and time gradients of control parameters, which require non-equilibrium theories. As we know from physics, developing good understanding of systems in statistical equilibrium is only the first step and a reference point on the way to understanding non-equilibrium behavior of global and local economy.

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