Synchronization Control by Structural Modification of Nonlinear Oscillator Network

Tetsuro Funato, Daisuke Kurabayashi, and Masahito Nara

Tokyo Institute of Technology, Ookayama 2-12-1, Meguro-ku, Tokyo 152-8552, Japan, {funato,dkura,m-nara}@irs.ctrl.titech.ac.jp

Summary. The structural features of a system significantly affect the attributes and functions of the system. The effect of this phenomenon can be widely observed, from areas such as the WWW to the brains of animals. In the present paper, a method for controlling the behavior of a system by manipulating the structure is examined using a coupled nonlinear oscillator model. We first describe a property of the eigenfrequencies of coupled oscillators and show that convergent transition is possible by connecting oscillators with significantly different eigenfrequencies. Moreover, using the eigenvalues of a graph matrix, we reveal that a combination of distant oscillators can shift the converged state independent of the eigenfrequencies.

Key words: Network Structure, Nonlinear Oscillator, Synchronization

1 Introduction

Morphological approaches to biological motion represented by a passive dynamic walker[1] are attracting attention particularly with regard to the relationship between morphological properties and functional creation. Faculties derived from topology as such are not restricted only to motion. In the field of networks, The small world (SW) structure[2, 3] has a high transferring efficiency, and networks with an approximately power-law vertex degree distribution, such as the Internet, strongly resist the random removal of nodes[4]. These types of generation phenomena can also be seen in the brain. Synapses, the connections between neurons, are classified into electrical and chemical types according to their transmission mechanisms. Chemical transmission synapses are mediated by message-carrying chemicals, and the combination of chemicals can be changed by the amount of message-carrying chemicals. This transformation is used for memory and learning abilities. For instance, the tendency of crickets to change their behavior based on past experience is caused by the variation of message-carrying chemicals influenced by nitric oxide (NO)[5].

In order to understand the relationship between structure and function, an appropriate model is necessary. Nonlinear oscillators can serve this purpose because they can generate the forces for synchronization. Research regarding the nonlinear oscillator has been carried out for a long period. For example, Linkens[6, 7] analyzed a coupled van der Pol (VDP) oscillator system, particularly with respect to its convergence. Among recent studies, Kuramoto[8] proposed a model that has a phase as only one free parameter and indicated that synchronization is affected by the connection coefficient. Jadbabaie et al.[9] and Earl et al.[10] investigated the relationship between network structure and convergence by introducing geometric factors to the Kuramoto model.

Current oscillator models can be effectively used as a brain model; this is based on some biochemistry reports, such as one indicating that the synchronization of neurons is essential in the visual information processing of a cat[11]. In particular, oscillatory neural networks (ONNs) proposed by employing the oscillatory feature to a neural network (NN), the most popular information processing model of the brain, has attracted considerable attention. As an example, Hoppensteadt[12] realized associative memory with synchronization by introducing a Hebbian learning rule to the Kuramoto model.

Using ONNs, research on the structural features of the oscillator network may facilitate a discussion on the morphology of the brain. However, in order to consider the changes in synapses, the examination of a change in the synapse structure cannot be avoided. Nonetheless, there is little research on convergence in this regard, and current research mainly considers mainly the static properties. The purpose of this research is to elucidate the dynamic phenomenon of structural change and to control convergence by the manipulation of only the transition of structure.

We first understand a property of the phase gap of coupled oscillators using the Kuramoto model and propose a control method for the convergence state. Next, we confirm this method via simulation. Then, by using the eigenvalues of a graph matrix, we describe a geometric manipulation that enables a converged transition independent of the eigenfrequencies.

2 Models and Characteristics of Oscillators

2.1 Nonlinear oscillator

In order to consider the relationship between the system and structure, an appropriate model is required. In this study, we constructed a model by considering the characteristics of nonlinear oscillators as follows:

1. Connected nonlinear oscillators can cause the forces required for synchronization.
2. The converged states can be changed by these forces.

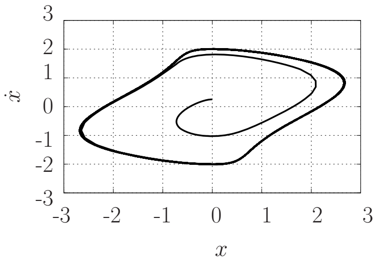

Fig. 1. Limit Cycle

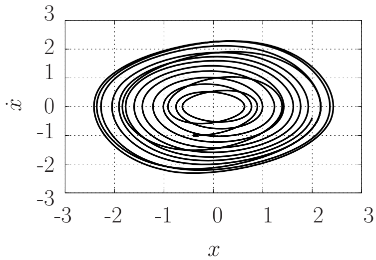

Fig. 2. Quasi-Periodic

The transition of converged states considered in this study is shown by the apparent change from the cyclic state (limit cycle) shown in Fig. 1 to the non-cyclic state (quasi-periodic oscillation) shown in Fig. 2.

By using nonlinear oscillators as models, a structural change produces forces between the oscillators in the system(property 1), and this effect can change the state of the oscillators, i.e., the state of the system (property 2).

2.2 Kuramoto model

We used the Kuramoto model[8] as nonlinear oscillators for this model. In this model, the phase of the ith oscillator θ_i is as follows:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\Gamma(\theta_j - \theta_i). \tag{1}$$

Here, K is the connection coefficient; ω_i, the eigenfrequency, Γ, the interaction function; and N, the total number of oscillators. We suppose that Γ is odd-symmetric ($\Gamma(\theta) = -\Gamma(-\theta)$) and $\Gamma(0) = 0$.

3 Convergent Transition via Properties of Eigenfrequency

3.1 Characteristics of coupled oscillators

We express a group of oscillators connected to the ith oscillator as O_i and that connected to the jth oscillator as O_j. When we consider further connections between the ith and jth oscillators, the transition can be represented by a change in γ (0→1); the equations of motion regarding these oscillators then become

$$\dot{\theta}_i = \omega_i + \frac{K}{N+\gamma}\left\{\sum_{s\in O_i}\Gamma(\theta_s - \theta_i) + \gamma\Gamma(\theta_j - \theta_i)\right\} \tag{2}$$

$$\dot{\theta}_j = \omega_j + \frac{K}{N+\gamma}\left\{\sum_{t\in O_j}\Gamma(\theta_t - \theta_j) + \gamma\Gamma(\theta_i - \theta_j)\right\}. \tag{3}$$

In this case, a phase gap between the ith and jth oscillators $\phi(= \theta_i - \theta_j)$ is expressed as

$$\dot{\phi} = \dot{\theta}_i - \dot{\theta}_j = \omega_i - \omega_j + \frac{K}{N+\gamma}\left\{\sum_{s\in O_i}\Gamma(\theta_s - \theta_i) - \sum_{t\in O_j}\Gamma(\theta_t - \theta_j)\right\} - \frac{2K\gamma}{N+\gamma}\Gamma(\phi) \tag{4}$$

$$\frac{\partial\phi(t,\gamma)}{\partial t} = \delta\omega + \frac{K}{N+\gamma}\left\{\sum_{s\in O_i}\Gamma(\theta_s - \theta_i) - \sum_{t\in O_j}\Gamma(\theta_t - \theta_j)\right\} - \frac{2K\gamma}{N+\gamma}\Gamma\left(\phi(t,\gamma)\right). \tag{5}$$

In the case widthout any connection,

$$\frac{\partial\phi(t,0)}{\partial t} = \delta\omega + \frac{K}{N}\left\{\sum_{s\in O_i}\Gamma(\theta_s - \theta_i) - \sum_{t\in O_j}\Gamma(\theta_t - \theta_j)\right\}. \tag{6}$$

When N is sufficiently large, by the approximation of $\frac{K}{N+\gamma} \simeq \frac{K}{N}\left(1 - \frac{\gamma}{N}\right) = \frac{K}{N} - \frac{\gamma}{N^2} \simeq \frac{K}{N}$,

$$\begin{aligned}\frac{\partial\phi(t,\gamma)}{\partial t} &\simeq \delta\omega + \frac{K}{N}\left\{\sum_{s\in O_i}\Gamma(\theta_s - \theta_i) - \sum_{t\in O_j}\Gamma(\theta_t - \theta_j)\right\} - \frac{2K\gamma}{N}\Gamma\left(\phi(t,\gamma)\right) \\ &= \frac{\partial\phi(t,0)}{\partial t} - \frac{2K\gamma}{N}\Gamma\left(\phi(t,\gamma)\right).\end{aligned} \tag{7}$$

This equation indicates that the addition of new combinations produces an effect of adding (or substituting) an integer multiple of the interaction function.

If the interaction function $\Gamma(\phi(t,\gamma)) = \sin\phi(t,\gamma)$,

$$\frac{\partial\phi(t,\gamma)}{\partial t} = -\frac{2K\gamma}{N}\sin\left(\phi(t,\gamma)\right) + \frac{\partial\phi(t,0)}{\partial t}, \tag{8}$$

which denotes a sine function with center $\frac{\partial\phi(t,0)}{\partial t}$ and amplitude $\frac{2K\gamma}{N}$.

From eq.8, following properties can be determined:

1. If $\left|\frac{\partial\phi(t,0)}{\partial t}\right| < \frac{2K\gamma}{N}$, there exists a certain ϕ that satisfies $\left|\frac{\partial\phi(t,0)}{\partial t}\right| = 0$.
2. If $\left|\frac{\partial\phi(t,0)}{\partial t}\right| > \frac{2K\gamma}{N}$, no ϕ satisfies $\left|\frac{\partial\phi(t,0)}{\partial t}\right| = 0$.

Similar to the concept of frequency locking[13], converged states become a limit cycle in case 1 because of the existence of an equivalent point and become a quasi-periodic oscillation in case 2. Figs. 3 and 4 display the graphs of eq.8 under the condition that $\frac{\partial\phi(t,0)}{\partial t}$ is constant.

As a result, we can control the converged states from the limit cycle to the quasi-periodic oscillation by coupling oscillators whose $\left|\frac{\partial\phi(t,0)}{\partial t}\right|$ is large, i.e., the gap in eigenfrequencies is large.

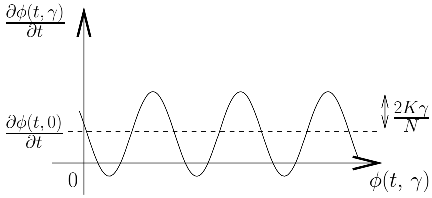

Fig. 3. $|\frac{\partial\phi(t,0)}{\partial t}| < \frac{2K\gamma}{N}$

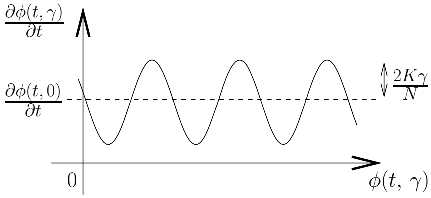

Fig. 4. $|\frac{\partial\phi(t,0)}{\partial t}| > \frac{2K\gamma}{N}$

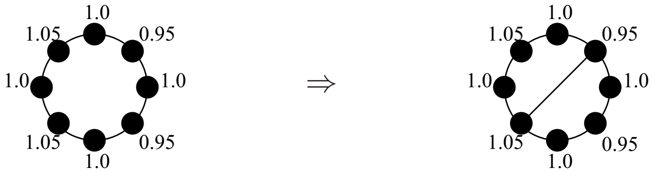

Fig. 5a. Structural transition

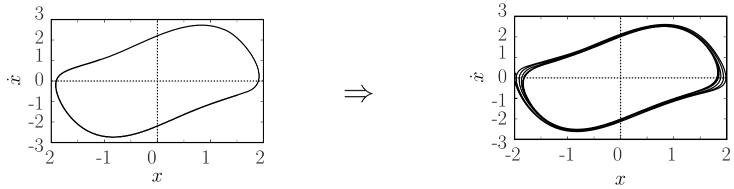

Fig. 5b. Convergent shift by structural transition

3.2 Simulation

We simulated the state of the oscillators by altering the connection structure based on the eigenfrequencies (Fig. 5a and 5b). We used VDP oscillators in this simulation. The equation of a VDP oscillator is $\ddot{x_i} - \epsilon_i(1 - x_i^2)\dot{x_i} + \omega_i^2 x_i = 0$, and the interaction forces for synchronization is expressed as $x_i(t+1) = x_i(t) + K\left\{\frac{1}{N_i(t)}\sum_{j=1}^{N_i(t)} x_j(t) - x_i(t)\right\}$. Here, $x_j(t)$ is the state of an oscillator that is connected to the ith oscillator and $N_i(t)$ is the total number of the connected oscillators.

In Fig. 5a, the vertices of the graph represent the oscillators, the edges express the relationship of the connections and the values of the vertices represent the eigenfrequencies.

We change the structure by creating a new connection between oscillators whose eigenfrequencies are significantly different. Through the simulation, converged states shift from the limit cycle to quasi-periodic oscillations; this is confirmed by Fig. 5b.

4 Convergent Transition via only Geometric Properties

4.1 Kuramoto model considering structural disposition

Jadbabaie et al.[9] proposed an oscillator model that builds on the Kuramoto model and includes connection relationship between oscillators.

$$\dot{\theta} = \omega - \frac{K}{N} B \sin\left(B^T \theta\right) \tag{9}$$

The $N \times e$ matrix B represents an oriented graph that has N vertices and e edges, and the following conditions hold:

- If edge j incoming to vertex i, $B_{ij} = 1$.
- If edge j outcoming from vertex i, $B_{ij} = -1$.
- If edge j and i are not connected, $B_{ij} = 0$.

In eq.9, θ and ω are N vectors expressing the phase and eigenfrequency, respectively.

4.2 Convergent condition via eigenfrequency

In addition, Jadbabaie showed that there is at least one convergent oscillator if the connection coefficient K satisfies

$$K > \left(\frac{\pi}{2}\right)^2 \frac{N\lambda_{\max}(L)}{\lambda_{\min}(L)^2} \|\omega\|_2. \tag{10}$$

Here, $L = BB^T$ is a matrix called the Laplacian and $\lambda(L)$ is the eigenvalue of L. $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues, respectively.

In eq.10, the geometric property is expressed only by

$$\frac{\lambda_{\max}(L)}{\lambda_{\min}(L)^2}, \tag{11}$$

therefore, convergent control is possible by considering the eigenvalues of the Laplacian.

Moreover, a similar conclusion can be gathered from the theses of Pecora et al.[14] and Barahona et al.[15]. They also modeled oscillator states using a geometric matrix and showed that oscillators can converge when the difference between the maximum eigenvalue and minimum eigenvalue goes below a certain constant value.

4.3 Eigenvalues of the Laplacian

In order to investigate the elemental property of structural transition, we consider connecting a pair of oscillators to a simple cycle-graph-shaped oscillator network.

$$B = \begin{bmatrix} -1 & 0 & & \cdots & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \vdots \\ & \text{\Large 0} & & 1 & -1 & 0 \\ & & & & 1 & -1 \end{bmatrix}, \quad L = BB^T = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & & & 0 \\ 0 & -1 & 2 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & & & & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{12}$$

We express the matrix of a graph that has an additional combination as $B' = [B|x]$ by using N vector x; $x = \left[0 \; \cdots \; 0 \; -1 \; \overbrace{0 \; \cdots \; 0}^{m} \; 1 \; \cdots \; 0\right]^T$. In this vector, m denotes the interval of the coupling oscillators. Because the current graph is ring-shaped, the variation in the starting point of the connection makes no difference. Therefore, x can be written as $x = \left[0 \; -1 \; \overbrace{0 \; \cdots \; 0}^{m} \; 1 \; \cdots \; 0\right]^T$ without a lack of generality. Then, the Laplacian of B' becomes

$$L' = BB^T + xx^T = \begin{bmatrix} 2 & -1 & 0 & \cdots & & 0 & 0 & \cdots & & 0 & -1 \\ -1 & 3 & -1 & & & \vdots & -1 & & & & 0 \\ 0 & -1 & 2 & \ddots & & & 0 & & & & \vdots \\ \vdots & & \ddots & \ddots & & 0 & \vdots & & & & 0 \\ 0 & \cdots & & 0 & -1 & \begin{matrix}-1\\2\end{matrix} & \begin{matrix}0\\-1\end{matrix} & 0 & \cdots & & 0 \\ 0 & -1 & 0 & \cdots & 0 & -1 & 3 & -1 & & & \vdots \\ \vdots & & & & & 0 & -1 & 2 & \ddots & & \\ & & & & & & & \ddots & \ddots & & 0 \\ 0 & & & & & \vdots & & & -1 & 2 & -1 \\ -1 & 0 & \cdots & & 0 & 0 & \cdots & & 0 & -1 & 2 \end{bmatrix}. \tag{13}$$

$$|L' - \lambda I| = \begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots & & \boldsymbol{m-1} & \boldsymbol{m} & \boldsymbol{m+1} & \cdots & \boldsymbol{N-1} & \boldsymbol{N} \\ \begin{matrix}-1\\2-\lambda\end{matrix} & \begin{matrix}0\\-1\end{matrix} & 0 & \cdots & & \begin{matrix}\cdots\\0\end{matrix} & 0 & \cdots & 0 & \begin{matrix}-1\\0\end{matrix} & \begin{matrix}2-\lambda\\-1\end{matrix} \\ -1 & 3-\lambda & -1 & & & \vdots & -1 & & & & 0 \\ 0 & -1 & 2-\lambda & \ddots & & & 0 & & & & \vdots \\ \vdots & & \ddots & \ddots & & 0 & \vdots & & & & 0 \\ 0 & \cdots & p & 0 & -1 & \begin{matrix}-1\\2-\lambda\end{matrix} & \begin{matrix}0\\-1\end{matrix} & 0 & \cdots & & 0 \\ 0 & -1 & 0 & \cdots & 0 & -1 & 3-\lambda & -1 & & & \vdots \\ \vdots & & & & & 0 & -1 & 2-\lambda & \ddots & & \\ & & & & & & & \ddots & \ddots & & 0 \\ 0 & & & \cdots & & & & 0 & -1 & 2-\lambda & -1 \end{vmatrix}. \tag{14}$$

We set $(m+1)\times(m+1)$ matrix S, $(m+1)\times(N-m-1)$ matrix T, $(N-m-1)\times(m+1)$ matrix U and $(N-m-1)\times(N-m-1)$ matrix V as follows:

$$
S = \begin{array}{c} \begin{array}{cccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots & & \boldsymbol{m-1} & \boldsymbol{m} & \boldsymbol{m+1} \end{array} \\ \left[\begin{array}{cccccccc}
-1 & 0 & & & & \cdots & 0 & 0 \\
2-\lambda & -1 & 0 & & & & 0 & \\
-1 & 3-\lambda & -1 & 0 & \cdots & 0 & -1 & \vdots \\
0 & -1 & 2-\lambda & \ddots & & & 0 & \\
\vdots & & \ddots & \ddots & & & \vdots & \\
 & & & & & -1 & 0 & \\
0 & \cdots & & 0 & -1 & 2-\lambda & -1 & 0 \\
0 & -1 & 0 & \cdots & 0 & -1 & 3-\lambda & -1
\end{array}\right] \end{array}, \quad
T = \left[\begin{array}{ccccc}
0 & \cdots & 0 & -1 & 2-\lambda \\
 & & & & -1 \\
 & & \ddots & & 0 \\
 & 0 & & & \vdots \\
 & & & & 0
\end{array}\right],
$$

$$
U = \left[\begin{array}{ccccc}
0 & \cdots & 0 & -1 & 2-\lambda \\
 & & & & -1 \\
 & & \ddots & & 0 \\
 & 0 & & & \vdots \\
 & & & & 0
\end{array}\right] \quad
V = \left[\begin{array}{cccccc}
-1 & & & & & \\
2-\lambda & -1 & & & 0 & \\
-1 & \ddots & \ddots & & & \\
0 & & & & & \\
\vdots & & & & & \\
0 & \cdots & 0 & -1 & 2-\lambda & -1
\end{array}\right].
$$

Based on these matrices, eq.14 becomes $|L'-\lambda I| = \left|\begin{smallmatrix} S & T \\ U & V \end{smallmatrix}\right| = |V|\,|S-TV^{-1}U|$. Because $|V|=1$, $|L'-\lambda I| = 0 \leftrightarrow |S-TV^{-1}U| = 0$.

If we set $V^{-1} = \{v_{ij}\}$ (i, j indicates row and column number respectively),

$$
\begin{cases}
[TV^{-1}U]_{1(N-1)} & = v_{(N-1)1} - (2-\lambda)v_{N1} \\
[TV^{-1}U]_{1N} & = -(2-\lambda)v_{(N-1)1} + (2-\lambda)^2 v_{N1} \\
 & \quad + v_{(N-1)2} - (2-\lambda)v_{N2} \\
[TV^{-1}U]_{2N-1} & = v_{N1} \\
[TV^{-1}U]_{2N} & = -(2-\lambda)v_{N1} + v_{N2} \\
[TV^{-1}U]_{other} & = 0
\end{cases} . \tag{15}
$$

Further,

$$
\begin{bmatrix} -1 & 0 \\ 2-\lambda & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ -(2-\lambda) & -1 \end{bmatrix}
$$

$$
\begin{bmatrix} -1 & 0 & 0 \\ 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 2-\lambda & -1 & 0 \\ -(2-\lambda)^2+1 & -(2-\lambda) & -1 \end{bmatrix}
$$

$$\vdots$$

indicates the following about V^{-1}:

$$
\begin{bmatrix} v_{1(N-1)} & v_{2(N-1)} & \cdots & v_{N(N-1)} \end{bmatrix}^T = \begin{bmatrix} 0 & \cdots & 0 & -1 & -(2-\lambda) \end{bmatrix}^T \tag{16}
$$

$$
\begin{bmatrix} v_{1N} & v_{2N} & \cdots & v_{NN} \end{bmatrix}^T = \begin{bmatrix} 0 & \cdots & 0 & -1 \end{bmatrix}^T, \tag{17}
$$

therefore,

$$TV^{-1}U = 0. \tag{18}$$

As a result, $|L' - \lambda I| = 0 \leftrightarrow |S| = 0$.

$$|S| = \begin{array}{|cccccccc|} \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots & & \boldsymbol{m-1} & \boldsymbol{m} & \boldsymbol{m+1} \\ -1 & 0 & & & & \cdots & 0 & 0 \\ 2-\lambda & -1 & 0 & & & & 0 & \\ -1 & 3-\lambda & -1 & 0 & \cdots & 0 & -1 & \vdots \\ 0 & -1 & 2-\lambda & \ddots & & & 0 & \\ \vdots & & \ddots & \ddots & & & \vdots & \\ & & & & & -1 & 0 & \\ 0 & \cdots & & 0 & -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 0 & \cdots & 0 & -1 & 3-\lambda & -1 \end{array} = 0 \tag{19}$$

$$\begin{array}{|cccc|} \mathbf{1} & \mathbf{2} & \cdots & \boldsymbol{m-3} \\ 2-\lambda & -1 & & 0 \\ -1 & 2-\lambda & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2-\lambda \end{array} = (-1)^m. \tag{20}$$

The size of this determinant is decided by the interval of the coupling oscillators. This result suggests that an even more distant connection induces a larger variation of the eigenvalues. Therefore, $\frac{\lambda_{\max}(L)}{\lambda_{\min}(L)^2}$ increases due to the remote oscillator connection, which breaks the condition of eq.10, and the oscillators become quasi-periodic.

Moreover, the right-hand side of eq.20 is -1 to the power of m. Therefore, the sign changes depending on whether the distance of the oscillators is an even or odd number; this causes a non-trivial difference in the convergent condition.

5 Conclusion

In this study, we examined the effect of structural transition on the behavior of systems and investigated a method whereby convergence can be controlled only by structural manipulation, using an oscillator network as the system model.

We first analyzed the behavior of a network from the viewpoint of phase gap and showed that it is possible to control the converged state by connecting oscillators with significantly different frequencies. In addition, we confirmed this phenomenon by simulation.

We then calculated the rate of maximum and minimum eigenvalues of a graph matrix to find a control method mainly via the structural properties based on the thesis by Jadbabie. Assuming that the oscillator network is constructed on a circular graph, we showed that the variation of eigenvalues depended on the distance of the additional coupled oscillators, i.e., a remote connection can induce convergent transition.

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References

1. T.McGeer. Passive dynamic walking. *The International Journal of Robotics Research*, Vol. 9, No. 2, pp. 62–82, 1990.
2. D.J.Watts and S.H.Strogatz. Collective dynamics of 'small-world' networks. *nature*, Vol. 393, pp. 440–442, 1998.
3. S.H.Strogatz. Exploring complex networks. *nature*, Vol. 410, pp. 268–276, 2001.
4. D.S.Callaway, M.E.J.Newman, S.H.Strogatz, and D.J.Watts. Network robustness and fragility: Percolation on random graphs. *Physical Review Letters*, Vol. 85, No. 25, pp. 5468–5471, 2000.
5. H.Aonuma, M.Iwasaki, and K.Niwa. Role of no signaling in switching mechanisms in the nervous system of insect. *SICE Proc. 2004*, pp. 2477–2482, 2004.
6. D.A.Linkens. Analytical solution of large numbers of mutually coupled nearly sinusoidal oscillators. *IEEE Trans. on Circuits and Systems*, Vol. CAS-21, No. 2, pp. 294–300, 1974.
7. D.A.Linkens. Stability of entrainment conditions for a particular form of mutually coupled van der pol oscillators. *IEEE Trans. on Circuits and Systems*, Vol. CAS-23, No. 2, pp. 113–121, 1976.
8. Y.Kuramoto. *Chemical Oscillations, waves, and Turbulence*. Springer, 1984.
9. A.Jadbabaie, N.Motee, and M.Barahona. On the stability of the kuramoto model of coupled nonlinear oscillators. In *the American Control Conference*, 2004.
10. M.G.Earl and S.H.Strogatz. Synchronization in oscillator networks with delayed coupling: A stability criterion. *Physical Review E*, Vol. 67, p. 036204, 2003.
11. C.M.Gray, P.Konig, A.K.Engel, and W.Singer. Oscillatory responses in cat visual cortex exhibit inter-columnar synchronization which reflects global stimulus properties. *nature*, Vol. 338, pp. 334–337, 1989.
12. F.C.Hoppensteadt and E.M.Izhikevich. Oscillatory neurocomputers with dynamic connectivity. *Physical Review Letters*, Vol. 82, No. 14, pp. 2383–2386, 1999.
13. H.Haken. *Advanced Synergetics*. Springer, 1983.
14. L.M.Pecora and T.L.Carroll. Master stability functions for synchronized coupled systems. *Physical Review Letters*, Vol. 80, No. 10, p. 2109, 1998.
15. M.Barahona and L.M.Pecora. Synchronization in small-world systems. *Physical Review Letters*, Vol. 89, No. 5, pp. 054101–1 – 054101–4, 2002.