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Can Damage Mechanics Explain Temporal Scaling Laws in Brittle Fracture and Seismicity?

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Abstract—Time delays associated with processes leading to a failure or stress relaxation in materials and earthquakes are studied in terms of continuum damage mechanics. Damage mechanics is a quasi-empirical approach that describes inelastic irreversible phenomena in the deformation of solids. When a rock sample is loaded, there is generally a time delay before the rock fails. This period is characterized by the occurrence and coalescence of microcracks which radiate acoustic signals of broad amplitudes. These acoustic emission events have been shown to exhibit power-law scaling as they increase in intensity prior to a rupture. In case of seismogenic processes in the Earth's brittle crust, all earthquakes are followed by an aftershock sequence. A universal feature of aftershocks is that their rate decays in time according to the modified Omori's law, a power-law decay. In this paper a model of continuum damage mechanics in which damage (microcracking) starts to develop when the applied stress exceeds a prescribed yield stress (a material parameter) is introduced to explain both laboratory experiments and systematic temporal variations in seismicity.

Key words: Fracture, seismicity, damage mechanics, aftershocks, power-law scaling.

1. Introduction

Time delays are generally observed in processes of rock fracture and earthquake occurrences. The time delay associated with the initiation and propagation of a single fracture can be attributed to stress corrosion and a critical stress intensity factor (DAS and SCHOLZ 1981; FREUND 1990). Usually, however, the fracture of a brittle material, such as rock, results from the coalescence and growth of microcracks (MOGI, 1962; HIRATA, 1987; HIRATA *et al.*, 1987; LOCKNER *et al.*, 1992; LOCKNER, 1993). In these processes it is possible to distinguish two types of time delays. One type is associated with processes leading to the failure of a material and is realized as a process of accumulation and coalescence of microcracks and microdefects or in the case of earthquakes the occurrence of foreshocks before a main shock. The second type of

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time delays is associated with the relaxation processes in materials or in the Earth's crust. A specimen subjected to a sufficient external load yields and, as a result, its mechanical properties degrade over time. Time delays are also associated with the occurrence of aftershocks that is a relaxation process.

In the case of fracture phenomena, time delays were observed prior to the failure of a material (SORNETTE and ANDERSEN, 1998; JOHANSEN and SORNETTE, 2000; GLUZMAN and SORNETTE, 2001). We will consider in some detail experiments on the fracture of fiberboard panels (GUARINO *et al.*, 1998, 1999, 2002). When the panels were subjected to rapid loading, microcracks developed randomly and then coalesced until a throughgoing rupture developed. Experiments with very rapid loading, showed a systematic power-law decrease in the delay time to failure as a function of the increasing difference between the applied stress and a yield stress. These experiments also gave a power-law increase in the energy associated with acoustic emissions as a function of time prior to rupture. Also, the frequency-strength statistics of the acoustic emissions satisfied the power-law Gutenberg-Richter scaling applicable to earthquakes.

Another example of systematic delays in rock failure is the occurrence of earthquake aftershocks. Aftershocks are attributed to the increase in stress in some regions near an earthquake rupture. This increase is very rapid, on the scale of the earthquake rupture which is typically a few minutes. They also could result from the changes in pore fluid pressure associated with the migration of water in the damaged zone after a main shock. However, aftershocks occur days, months, and years later. This temporal decay of the rate of occurrence of aftershocks is extremely systematic and satisfies the modified Omori's law to a good approximation (UTSU *et al.*, 1995; SCHOLZ, 2002).

The principal purpose of this paper is to demonstrate that the temporal scaling laws in material fracture and earthquakes described above can be explained by continuum damage mechanics. We will first provide a brief description of a damage-mechanics model that captures essential aspects of processes leading to the failure of a specimen and also processes of relaxation.

2. A Model of Continuum Damage Mechanics

Damage refers to the irreversible deformation of solids. Some examples include plasticity, brittle microcracking, and thermally activated creep [KRAJCINOVIC, 1996]. In order to quantify the deformation of solids associated with microcracking, several empirical continuum damage mechanics models were introduced and are widely used in civil and mechanical engineering [KACHANOV 1986; KRAJCINOVIC, 1989, 1996; HILD, 2002; KATTAN and VOYIADJIS 2002; SHCHERBAKOV and TURCOTTE, 2003). Damage mechanics has also been applied to the brittle deformation of the Earth's crust by a number of authors (LYAKHOVSKY *et al.*, 1997, 2001; BEN-ZION and LYAKHOVSKY, 2002; TURCOTTE *et al.*, 2003).

The application of continuum damage mechanics can be illustrated by considering a rod in a state of uniaxial stress $\sigma_{xx} \neq 0$, $\sigma_{yy} = \sigma_{zz} = 0$. For an elastic material Hooke's law is applicable and is written in the form

$$\sigma = E_0 \epsilon, \tag{2.1}$$

where ϵ is a strain and E_0 is the Young's modulus of the undamaged material.

In this paper we will consider a model of continuum damage mechanics as introduced by (SHCHERBAKOV*et al.*, 2005). If the stress is less than the yield stress $\sigma \leq \sigma_y$, (2.1) is assumed to be valid. If the stress is greater than the yield stress, $\sigma > \sigma_y$, a damage variable α is introduced according to

$$\sigma - \sigma_y = E_0(1 - \alpha)(\epsilon - \epsilon_y), \tag{2.2}$$

where $\sigma_y = E_0 \epsilon_y$. When $\alpha = 0$, (2.2) reduces to (2.1) and linear elasticity is applicable; as $\alpha \to 1 (\epsilon \to \infty)$ failure occurs. Increasing values of α in the range $0 \leq \alpha < 1$ quantify the weakening (decreasing E) associated with the increase in the number and size of microcracks in the material. Several authors (KRAJCINOVIC, 1996; TURCOTTE *et al.*, 2003) have shown a direct correspondence between the damage variable α in a continuum material and the number of surviving fibers N in a fiber bundle that originally had N_0 fibers, $\alpha = 1 - N/N_0$.

To complete the formulation of the damage problem it is necessary to specify the kinetic equation for the damage variable. In analogy to Lyakhovsky *et al.* (1997) we take

$$\frac{d\alpha(t)}{dt} = 0, \quad \text{if} \quad 0 \leq \sigma \leq \sigma_y \tag{2.3}$$

$$\frac{d\alpha(t)}{dt} = \frac{1}{t_d}\left[\frac{\sigma(t)}{\sigma_y} - 1\right]^{\rho}\left[\frac{\epsilon(t)}{\epsilon_y} - 1\right]^2, \quad \text{if} \quad \sigma > \sigma_y, \tag{2.4}$$

where t_d is a characteristic time scale for damage and ρ is a constant to be determined from experiments. The power-law dependence of $d\alpha/dt$ on stress (and strain) given above must be considered empirical in nature. However, a similar power dependence of dN/dt on stress is widely used in the analysis of fiber failures in fiber bundles (NEWMAN and PHOENIX, 2001). It is important to note the introduction of a yield limit in (2.3) and (2.4). If the stress is less than the yield stress, $\sigma < \sigma_y$, no damage occurs. The monotonic increase in the damage variable α given by (2.4) represents the weakening of the brittle solid by the nucleation and coalescence of microcracks. Microcracks are initiated only when $\sigma > \sigma_y$.

As a first example we considered a rod to which a constant uniaxial tensional stress $\sigma_0 > \sigma_y$ is applied instantaneously at $t = 0$ and held constant until the sample fails. The applicable kinetic equation for the rate of increase of damage with time is obtained from (2.2)–(2.4) with the result

$$\frac{d\alpha}{dt} = \frac{1}{t_d}\left(\frac{\sigma_0}{\sigma_y} - 1\right)^{\rho+2} \frac{1}{[1-\alpha(t)]^2}. \tag{2.5}$$

Integrating with the initial condition $\alpha(0) = 0$ gives

$$\alpha(t) = 1 - \left[1 - 3\frac{t}{t_d}\left(\frac{\sigma_0}{\sigma_y} - 1\right)^{\rho+2}\right]^{1/3}. \tag{2.6}$$

Substituting (2.6) into (2.2) gives the strain in the sample as a function of time t

$$\frac{\epsilon(t)}{\epsilon_y} = 1 + \frac{\left(\sigma_0/\sigma_y - 1\right)}{\left[1 - 3\frac{t}{t_d}\left(\sigma_0/\sigma_y - 1\right)^{\rho+2}\right]^{1/3}}. \tag{2.7}$$

This behavior is illustrated in Fig. 1 for the case $\sigma_0/\sigma_y = 2.0$. Initially, at $t = 0$, we have $\epsilon(0)/\epsilon_y = 2.0$ at point A. The state of the rod moves along the constant stress path AB until it fails. Failure occurs at the time t_f when $\alpha \to 1(\epsilon \to \infty)$. From (2.7) this failure time is given by

$$t_f = \frac{t_d}{3}\left(\frac{\sigma_0}{\sigma_y} - 1\right)^{-(\rho+2)}. \tag{2.8}$$

The time to failure tends to infinity as a power-law as the applied stress approaches the yield stress $\sigma_0 \to \sigma_y$. Substitution of (2.8) into (2.6) gives the time evolution of damage as

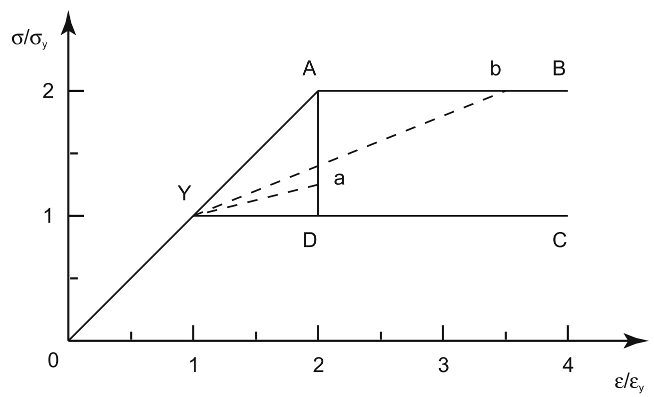

Figure 1
Dependence of the nondimensional stress σ/σ_y on nondimensional strain ϵ/ϵ_y (σ_y and ϵ_y are the yield stress and strain). Path OA is linear elasticity. If a stress $\sigma_0/\sigma_y = 2.0$ is applied instantaneously failure and damage occur along the path AB. If the stress is removed instantaneously the sample follows the path bYO. At point b the damage α is the slope of the straight line connecting point b with the yield point Y as illustrated. If a strain $\epsilon_0/\epsilon_y = 2.0$ is applied instantaneously stress relaxation and damage occur along the path AD. If the stress on the sample is removed instantaneously at the point a the sample follows the path aYO.

$$\alpha(t) = 1 - \left(1 - \frac{t}{t_f}\right)^{1/3} \tag{2.9}$$

and the corresponding time dependence of strain from (2.7)

$$\frac{\epsilon(t)}{\epsilon_y} = 1 + \frac{(\sigma_0/\sigma_y - 1)}{(1 - t/t_f)^{1/3}}. \tag{2.10}$$

The approach to failure is in the form of a power-law.

A particularly interesting set of experiments on brittle failure were carried out by GUARINO *et al.* (1999). These authors studied the failure of circular panels (222 mm diameter, 3–5 mm thickness) of chipboard panels. A differential pressure was applied rapidly across a panel and it was held constant until failure occurred. For these relatively thin panels, bending stresses were negligible and the panels failed under tension (a mode I fracture). Initially the microcracks appeared to be randomly distributed across the panel, as the number of microcracks increased they tended to localize and coalesce in the region where the final rupture occurred. The times to failure t_f of these chipboard panels were found to depend systematically on the applied differential pressure P. Taking a yield pressure (stress) $P_y = 0.038$ MPa their results are reinterpreted in Fig. 2 assuming $\sigma/\sigma_y = P/P_y$. Their results correlate very well with our failure condition (2.8) taking $\rho = 0.25$ and $t_d = 168$ s.

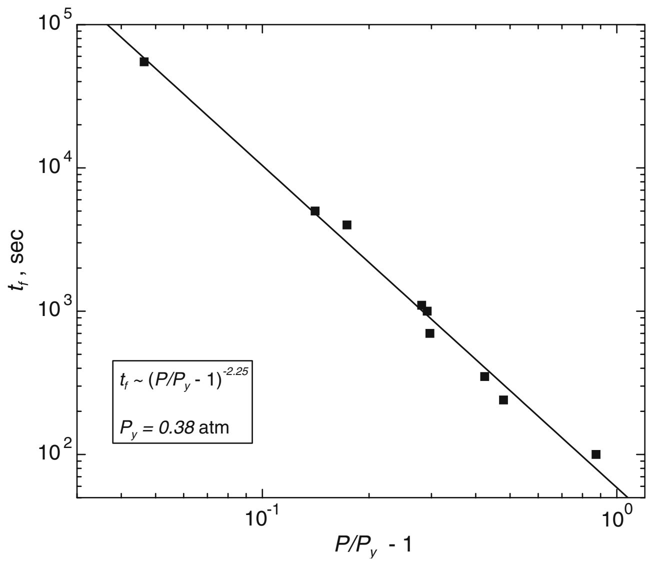

Figure 2

Times to failure of chipboard panels subjected to differential pressure (GUARINO *et al.*, 1999). Failure times t_f are given as a function of the ratio of the excess stress over the yield stress $\sigma/\sigma_y - 1$ (assumed to be equal to $P/P_y - 1$ with $P_y = 0.038$ MPa. The straight line correlation is with (2.8) taking $\rho = 0.25$ and $t_d = 168$ s.

GUARINO *et al.* (1999) also determined the cumulative energy in the acoustic emission (AE) events e_{AE} as a function of the time t. The total AE energy at the time of rupture is e_{tot}. They studied the dependence of e_{AE}/e_{tot} on $(1 - t/t_f)$. After an initial transient period good power-law scaling was observed that can be correlated with $e_{AE} \propto (1 - t/t_f)^{-0.27}$. This equivalent to having $de_{AE}/dt \propto (1 - t/t_f)^{-1.27}$.

We now relate the rate of increase of the damage variable $d\alpha/dt$ to the rate of AE events. The rod extends under the constant stress $\sigma_0 = 2\sigma_y$ from point A to point b as shown in Figure 1. The slope of the straight line connecting the point b and the yield point Y is $1 - \alpha$. We assume that when the stress at point b is removed instantaneously the sample will follow the path *bYO*. Thus the energy lost in acoustic emissions e_{AE} is equal to the work done on path *Ab* so that we have

$$e_{AE} = \frac{1}{2}(\sigma_0 - \sigma_y)(\epsilon_b - \epsilon_A) = \frac{\sigma_y \epsilon_y}{2}\left(\frac{\sigma_0}{\sigma_y} - 1\right)^2 \left[\frac{1}{(1 - t/t_f)^{1/3}} - 1\right], \tag{2.11}$$

where we used (2.9). Taking the derivative of (2.11) with respect to t we obtain the rate of AE

$$\frac{de_{AE}}{dt} = \frac{\sigma_y \epsilon_y}{6t_f}\left(\frac{\sigma_0}{\sigma_y} - 1\right)^2 \left(1 - \frac{t}{t_f}\right)^{-4/3}. \tag{2.12}$$

This result compares with the observed time dependence of the rate of the acoustic energy release in experiments on chipboard panels described above.

It should be noted that the relaxation path from b to Y to O in Figure 1 is not unique. An alternative path would be parallel to the initial path O to A. The path we have chosen corresponds to the closure of type I extensional microcracks along the path b to Y and the subsequent elastic behavior from Y to O. The alternative parallel path would be appropriate for shear displacements that do not relax. The entire relaxation process is elastic. It should be noted however, that the time dependence given in (2.12) can be obtained taking either path.

As a second example we consider a rod under a tensional stress that increases linearly in time, that is

$$\sigma(t) = \sigma_y \frac{t}{t_y}. \tag{2.13}$$

Damage begins to occur when $\sigma = \sigma_y$ at $t = t_y$. The applicable equation for the subsequent increase in damage with time is obtained from (2.2)–(2.4) and (2.13) with the result

$$\frac{d\alpha}{dt} = \frac{1}{t_d}\left(\frac{t}{t_y} - 1\right)^{\rho+2} \frac{1}{[1 - \alpha(t)]^2}. \tag{2.14}$$

Integrating with the initial condition $\alpha(t_y) = 0$ gives

$$\alpha(t) = 1 - \left[1 - \frac{3t_y}{(\rho+3)t_d}\left(\frac{t}{t_y} - 1\right)^{\rho+3}\right]^{1/3}. \tag{2.15}$$

Substituting (2.13) and (2.15) into (2.2) gives the time dependence of the strain

$$\frac{\epsilon(t)}{\epsilon_y} = 1 + \frac{(t/t_y - 1)}{\left[1 - \frac{3t_y}{(\rho+3)t_d}\left(t/t_y - 1\right)^{\rho+3}\right]^{1/3}}. \tag{2.16}$$

Failure occurs at the time t_f when $\alpha = 1(\epsilon \to \infty)$. From (2.15) this failure time is given by

$$\frac{t_f}{t_y} = 1 + \left[\frac{(\rho+3)t_d}{3t_y}\right]^{\frac{1}{\rho+3}} \tag{2.17}$$

and the corresponding time dependent strain from (2.16) is given by

$$\frac{\epsilon(t)}{\epsilon_y} = 1 + \frac{(t/t_y - 1)}{\left[1 - \left(\frac{t-t_y}{t_f-t_y}\right)^{\rho+3}\right]^{1/3}}. \tag{2.18}$$

Again the approach to failure takes the form of a power law.

We will now apply the results derived above to the problem of seismic activation prior to earthquakes. Systematic increases in intermediate levels of seismicity prior to large earthquakes have been documented by several authors (SYKES and JAUMÉ, 1990; KNOPOFF et al., 1996; JAUMÉ and SYKES, 1999). It has also been observed that an increase in the seismic activity prior to large earthquakes takes the form of a power law. This was first proposed by BUFE and VARNES (1993). They considered the cumulative Benioff strain in a region defined as

$$\epsilon_{\mathrm{B}}(t) = \sum_{i=1}^{N(t)} \sqrt{e_i}, \tag{2.19}$$

where e_i is the seismic energy release in the ith precursory earthquake and $N(t)$ is the number of precursory earthquakes considered up to time t.

The precursory increase in seismicity is referred to as accelerated moment release (AMR). In terms of the cumulative Benioff strain it is quantified as

$$\epsilon_{\mathrm{B}}(t) = \epsilon_{\mathrm{B}}(t_f) - B\left(1 - \frac{t}{t_f}\right)^{s}, \tag{2.20}$$

where $\epsilon_{\mathrm{B}}(t_f)$ is the final cumulative Benioff strain when the large earthquake occurs, t_f is time measured forward from the beginning of AMR, B is a constant, and s is the exponent. Examples of AMR have been given by BOWMAN *et al.* (1998), BUFE *et al.* (1994), VARNES and BUFE (1996), BREHM and BRAILE (1998, 1999a,b), ROBINSON (2000), ZÖLER *et al.* (2001), MAIN (1999), BOWMAN and KING (2001), YANG *et al.* (2001), KING and BOWMAN (2003), BOWMAN and SAMMIS (2004), and SAMMIS *et al.* (2004). RUNDLE *et al.* (2000) found that the distribution of values of the power-law exponent s for 12 earthquakes was $s = 0.26 \pm 0.15$.

We assume that the stress in a seismological zone increases linearly with time as given by (2.13). This increase in the tectonic stress is very slow compared with the development of damage so that it is appropriate to assume $t_y/t_d \gg 1$. We associate the AMR energy e_{AMR} with the energy added to the rod by the time t. Using (2.13) and (2.18) we can obtain the rate of energy release

$$\frac{de_{\mathrm{AMR}}}{dt} = \frac{1}{2}\frac{d}{dt}[\sigma(t)\epsilon(t)] = \frac{\epsilon_y\sigma_y}{2t_y}\left\{1 + \frac{6t - 3t_y + [3t_y + (\rho-3)t]\left(\frac{t-t_y}{t_f-t_y}\right)^{\rho+3}}{3t_y\left[1 - \left(\frac{t-t_y}{t_f-t_y}\right)^{\rho+3}\right]^{4/3}}\right\}. \tag{2.21}$$

It is also possible to analyze the behavior of the rate near the failure time t_f. The expansion of (2.21) around $(t - t_f)$ gives

$$\begin{aligned}\frac{de_{\mathrm{AMR}}}{dt} = \frac{\epsilon_y\sigma_y}{2t_y}\Bigg\{1 &+ \frac{t_f}{3t_y(\rho+3)^{1/3}}\frac{1}{\left(\frac{t-t_f}{t_y-t_f}\right)^{4/3}} \\ &+ \frac{t_f(10-\rho) - 6t_y}{9t_y(\rho+3)^{1/3}}\frac{1}{\left(\frac{t-t_f}{t_y-t_f}\right)^{1/3}} + O(t-t_f)^{2/3}\Bigg\}.\end{aligned} \tag{2.22}$$

The result $de_{\mathrm{AMR}}/dt \propto (1 - t/t_f)^{-4/3}$ was previously derived from damage mechanics by BEN-ZION and LYAKHOVSKY (2002) who used the assumption of the constant applied stress.

To make the association with Benioff strain we use a derived expansion for the energy release rate (2.22)

$$\frac{d\epsilon_{\mathrm{B}}}{dt} = \sqrt{\frac{de_{\mathrm{AMR}}}{dt}}. \tag{2.23}$$

The cumulative Benioff strain can be obtained by integrating (2.23) with the result

$$\epsilon_{\mathrm{B}}(t) = \epsilon_{\mathrm{B}}(t_f) - \int_t^{t_f} \frac{d\epsilon_{\mathrm{B}}}{dt} dt \tag{2.24}$$

$$= \epsilon_{\mathrm{B}}(t_f) - \sqrt{\frac{3t_f(t_y - t_f)^{4/3}\sigma_y\epsilon_y}{2t_y^2(\rho+3)^{1/3}}}(t_f - t)^{1/3} - O(t_f - t)^{4/3}, \tag{2.25}$$

where we used (2.22). The exponent of $s = 1/3$ is in reasonably good agreement with the range of values associated with AMR $s = 0.26 \pm 0.15$ as given above. This agreement was also noted by BEN-ZION and LYAKHOVSKY (2002).

As our final example we will consider a rod to which a constant uniaxial tensional strain $\epsilon_0 > \epsilon_y$ has been applied instantaneously at $t = 0$ and is held constant. The applicable equation for the rate of increase of damage with time is obtained from (2.2)–(2.4) with the result

$$\frac{d\alpha}{dt} = \frac{1}{t_d}\left(\frac{\epsilon_0}{\epsilon_y} - 1\right)^{\rho+2}[1 - \alpha(t)]^{\rho}. \tag{2.26}$$

Integrating with the initial condition $\alpha(0) = 0$, we find

$$\alpha(t) = 1 - \left[1 + (\rho - 1)\frac{t}{t_d}\left(\frac{\epsilon_0}{\epsilon_y} - 1\right)^{\rho+2}\right]^{-\frac{1}{\rho-1}}. \tag{2.27}$$

This result is valid as long as $\rho > 1$. The damage increases monotonically with time and as $t \to \infty$ the maximum damage is $\alpha(\infty) = 1$. Using (2.27) with (2.2) one derives the stress relaxation in the material as a function of time t

$$\frac{\sigma(t)}{\sigma_y} = 1 + \left(\frac{\epsilon_0}{\epsilon_y} - 1\right)\left[1 + (\rho - 1)\frac{t}{t_d}\left(\frac{\epsilon_0}{\epsilon_y} - 1\right)^{\rho+2}\right]^{-\frac{1}{\rho-1}}. \tag{2.28}$$

At $t = 0$ we have linear elasticity corresponding to $\alpha = 0$. In the limit $t \to \infty$ the stress relaxes to the yield stress $\sigma(\infty) = \sigma_y$ below which no further damage can occur. This behavior is illustrated in Figure 1 for the case $\epsilon_0/\epsilon_y = 2.0$. Initially, at $t = 0$, we have $\sigma/\sigma_y = 2.0$ at point A. The sample then moves along the constant strain path AD until the stress has relaxed to the yield stress σ_y.

This stress relaxation process has been applied to the understanding of the aftershock sequence that follows an earthquake. During an earthquake some regions in the vicinity of the fault rupture experience a rapid increase of strain (stress). This is in direct analogy to the instantaneous application of strain considered above. Just as the microcracks associated with damage relax stress in our model, aftershocks relax stresses applied during the main shock. We recognize that all aftershocks have secondary aftershocks and so forth but all aftershocks contribute to stress relaxation similarly.

A universal scaling law is applicable to the temporal decay of aftershock activity following an earthquake. This is known as the modified Omori's law and can be written in the form (UTSU *et al.*, 1995, SCHOLZ, 2002)

$$\frac{dN}{dt} = \frac{1}{\tau[1 + t/c]^p}, \tag{2.29}$$

where dN/dt is the rate of occurrence of aftershocks with magnitudes greater than m, t is the time that has elapsed since the main shock, τ and c are characteristic times, and the exponent p has a value near unity. The total number of aftershocks with magnitudes greater than m, $N_{\mathrm{tot}}(\geq m)$, was obtained by integrating (2.29) with the result (SHCHERBAKOV *et al.*, 2004)

$$N_{\mathrm{tot}}(\geq m) = \int_0^\infty \frac{dN}{dt} dt = \int_0^\infty \frac{dt}{\tau(1 + t/c)^p} = \frac{c}{\tau(p-1)}. \tag{2.30}$$

If the modified Omori's law is assumed to be valid for large times (no cutoff) then the total number of aftershocks is finite only for $p > 1$. Combining (2.29) and (2.30) gives

$$\frac{1}{N_{\mathrm{tot}}} \frac{dN}{dt} = \frac{p-1}{c} \frac{1}{(1 + t/c)^p}, \tag{2.31}$$

We will next show that this result can be derived using continuum damage mechanics.

In order to relate our continuum damage mechanics model to aftershocks we determine the rate of energy release in the relaxation process considered above. In order to do this we use the approach applied to the AE experiments considered in the previous section. Since the strain is constant during the stress relaxation, no work is done on the sample. We hypothesize that if the applied strain (stress) is instantaneously removed during the relaxation process then the sample will return to a state of zero stress and strain following a linear stress-strain path (aY) with slope $E_0(1-\alpha)$ to stress σ_y and following the path (YO) with slope E_0 to zero stress (Fig. 1). We assume that the difference between the energy added e_{YA} and the energy recovered e_{aY} is lost in aftershocks and find that this energy e_{as} is given by

$$e_{\mathrm{as}} = e_{YA} - e_{aY} = \frac{1}{2} E_0 \left(\epsilon_0 - \epsilon_y\right)^2 \alpha(t). \tag{2.32}$$

The rate of energy release is obtained by substituting (2.27) into (2.32) and taking the time derivative with the result

$$\frac{de_{\mathrm{as}}}{dt} = \frac{E_0 \epsilon_y^2}{2t_d}\left(\frac{\epsilon_0}{\epsilon_y} - 1\right)^{\rho+4}\left[1 + (\rho - 1)\left(\frac{\epsilon_0}{\epsilon_y} - 1\right)^{\rho+2}\left(\frac{t}{t_d}\right)\right]^{-\frac{\rho}{\rho-1}}. \tag{2.33}$$

The total energy of aftershocks e_{ast} is obtained by substituting $\alpha(\infty) = 1$ into (2.32) with the result

$$e_{\mathrm{ast}} = \frac{1}{2}E_0\left(\epsilon_0 - \epsilon_y\right)^2. \tag{2.34}$$

Combining (2.33) and (2.34) gives

$$\frac{1}{e_{\mathrm{ast}}}\frac{de_{\mathrm{as}}}{dt} = \frac{\frac{1}{t_d}\left(\epsilon_0/\epsilon_y - 1\right)^{\rho+2}}{\left[1 + (\rho - 1)\left(\epsilon_0/\epsilon_y - 1\right)^{\rho+2}\left(\frac{t}{t_d}\right)\right]^{\frac{\rho}{\rho-1}}}. \tag{2.35}$$

This result is clearly similar to the aftershock relation given in (2.31). To demonstrate this further let us make the substitutions

$$\rho = \frac{p}{p-1}, \tag{2.36}$$

$$c = \frac{t_d}{(\rho - 1)\left(\epsilon_0/\epsilon_y - 1\right)^{\rho+2}}. \tag{2.37}$$

Substitution of (2.36) and (2.37) into (2.35) gives

$$\frac{1}{e_{\mathrm{ast}}}\frac{de_{\mathrm{as}}}{dt} = \frac{p-1}{c}\frac{1}{(1 + t/c)^p}. \tag{2.38}$$

This damage mechanics result is identical in form to the modified Omori's law for aftershocks given in (2.31).

We next consider a specific example. In Fig. 3 the rate of occurrence of aftershocks with $m \geq 2.5$ following the $m_{\mathrm{ms}} = 7.3$ Landers (California) earthquake, June 28, 1992, is given as a function of the time after the earthquake. Also shown is the correlation with (2.31) taking $\tau = 2.22 \times 10^{-3}$ days, $c = 2.072$ days, and $p = 1.22$. From (2.36) we find that the corresponding power law exponent is $\rho = 5.55$. Further assuming that $\epsilon_0/\epsilon_y = 1.2$ we find from (2.37) that the damage time is $t_d = 4.3$ s.

3. *Discussion*

In this paper we have presented a damage mechanics model with a yield stress. We have also applied the solutions of the model to several problems in the fracture of materials and earthquakes. This model is derived from several damage mechanics models that have been applied to problems in mechanical and civil engineering (ANIFRANI *et al.*, 1995; KRAJCINOVIC, 1996). The approach based on specifying the

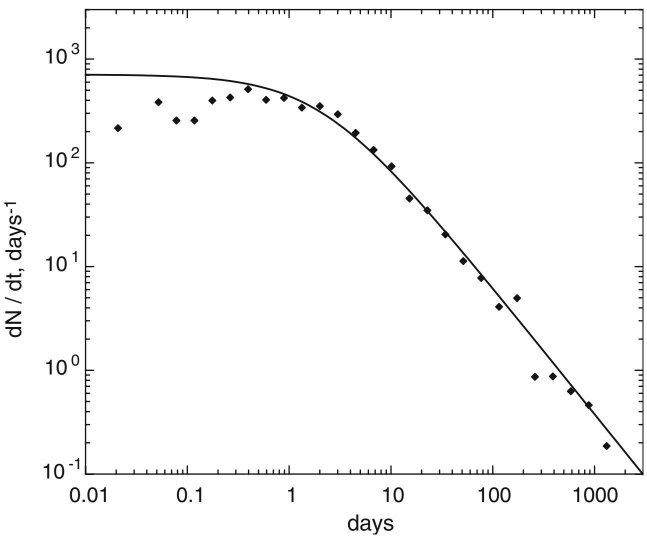

Figure 3
The rate of occurrence of aftershocks as a function of time following the June 28, 1992 Landers earthquake ($m_{ms} = 7.3$). The magnitude cutoff for aftershocks is $m = 2.5$. Also shown is the correlation with (29) taking $\tau = 2.22 \times 10^{-3}$ days, $c = 2.072$ days, and $p = 1.22$.

kinetic equation for damage evolution is empirical, however, it is justified by experimental observations. Although the physical mechanisms of time delays leading to fracture of materials remain unclear.

It should be pointed out that a similar time-to-failure model is routinely applied to composite materials. In the time-dependent fiber-bundle model the failure statistics of the individual fibers that make up the fiber bundle are specified (COLEMAN, 1957, 1958; SMITH and PHOENIX, 1981; NEWMAN and PHOENIX, 2001). It has been shown by KRAJCINOVIC (1996) and TURCOTTE *et al.* (2003) that the damage variable α can be introduced into the studies of fiber-bundle models where it defines the fraction of broken fibers. In the dynamic fiber-bundle models the time delays of failures of individual fibers are specified through an empirical distribution function.

The description of the damage evolution based on the empirical equation is valid for a gradual deterioration of ductile materials under creep (KACHANOV, 1986). This should be distinguished from the case of brittle-elastic behavior where damage can be defined as a density of microcracks and cavities and can be calculated exactly for certain geometries of these defects. It was noted by KACHANOV (1994) that the use of a kinetic equation for the scalar damage variable in the case of brittle-elastic solids can lead to inconsistencies. Our proposed kinetic equation is based on the studies of the fiber-bundle model where a similar power-law dependence on stress is observed (TURCOTTE *et al.*, 2003) and can be considered as an approximation. This model is also one-dimensional and extrapolations to higher dimensions should be done with care.

We have compared the predictions of our damage model with laboratory studies of the time delays associated with the rupture of chipboard panels and also with the time delays associated with the occurrence of aftershocks. In both cases the predicted power-law dependence of damage on time is consistent with the observations. The damage mechanics model basically has two parameters: the power-law exponent ρ and the characteristic time t_d. For the failure of the chipboard we find $\rho = 0.25$ and $t_d = 168$ s. For the aftershock sequence we have $\rho = 5.55$ and $t_d = 4.3$ s. Although both behave brittley, the chipboard is a much ''softer'' material than the rock in which aftershocks occur. We attribute the differences in the parameter values to this difference in material properties.

Some forms of damage are clearly thermally activated. The irreversible deformation of a solid by diffusion or dislocation creep is an example. The ability of vacancies and dislocations to move through a crystal is governed by an exponential dependence on absolute temperature with a well-defined activation energy. Time delays associated with fracture have been attributed to stress corrosion which is also a thermally activated process (DAS and SCHOLZ, 1981). However, GUARINO *et al.* (1998) varied the temperature in their experiments on the fracture of chipboard and found no effect. An alternative explanation for the time delay associated with microcracking has been given by SCORRETTI *et al.* (2001) and CILIBERTO *et al.* (2001). An effective ''temperature'' can be attributed to the spatial disorder (heterogeneity) of the solid. The spatial variability of stress in the solid is caused by the microcracking itself, not by thermal fluctuations.

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