Operator Theory:
Advances and Applications, Vol. 162, 19–45

Reproducing Kernel Spaces of Series of Fueter Polynomials

Daniel Alpay, Michael Shapiro and Dan Volok

Abstract. We study reproducing kernel spaces of power series of Fueter polynomials and their multipliers. In particular we prove a counterpart of Beurling–Lax theorem in the quaternionic Arveson space and we define and characterize counterparts of the Schur–Agler classes. We also address the notion of rationality in the hyperholomorphic setting.

Introduction

Reproducing kernel Hilbert spaces of analytic functions in one complex variable play an important role in operator theory, in particular in operator models and in interpolation theory, to name two instances. An important case is that of reproducing kernels of the form $c(z\overline{w})$ where c is a function analytic in a neighborhood of the origin with power series expansion $c(t) = \sum_{n=0}^{\infty} c_n t^n$ such that $c_n \geq 0$ for all $n \in \mathbb{N}$. The function $K(z,w) = c(z\overline{w})$ is positive and the associated reproducing kernel Hilbert space is the set of functions

$$f(z) = \sum_{\substack{n=0 \\ c_n \neq 0}}^{\infty} f_n z^n,$$

with norm

$$\|f\|^2 = \sum_{\substack{n=0 \\ c_n \neq 0}}^{\infty} \frac{|f_n|^2}{c_n} < \infty.$$

The research of Michael Shapiro was partially supported by CONACYT projects as well as by Instituto Politécnico Nacional in the framework of COFAA and CGPI programs. The stay of Michael Shapiro at Ben–Gurion University and the postdoctoral fellowship of Dan Volok were supported by the Center for Advanced Studies in Mathematics of Ben Gurion University of the Negev.

Typical examples include the Hardy space and the Bergman space, corresponding respectively to the functions

$$c(t) = \frac{1}{1-t} \quad \text{and} \quad c(t) = \frac{1}{(1-t)^2}.$$

These spaces of power series also have counterparts in the setting of several complex variables: $K(z,w) = c(\langle z, w\rangle)$ with $z, w \in \mathbb{C}^N$ and

$$t = \langle z, w\rangle = \sum_{i=1}^{N} z_i \overline{w_i}.$$

Examples include the Hardy space of the unit ball $\mathbb{B}_N$, the Bergman space and the Arveson space and the corresponding functions c are respectively

$$c(t) = \frac{1}{(1-t)^N}, \qquad c(t) = \frac{1}{(1-t)^{N+1}}, \qquad \text{and} \quad c(t) = \frac{1}{(1-t)}.$$

Other spaces of interest correspond to kernels of the form

$$K(z,w) = \sum_{\substack{n \in \mathbb{N}^N \\ c_n \neq 0}} c_n z^n \overline{w}^n,$$

where we have used the multi-index notation and where the $c_n \geq 0$. The corresponding reproducing kernel Hilbert spaces are sometimes called *weighted power series spaces*. The spaces under consideration will now include the Hardy space of the polydisk $\mathbb{D}^N$, corresponding to $c_n \equiv 1$.

We study in the present paper the counterparts of these weighted power series spaces in the quaternionic setting; power series are now replaced by series of Fueter polynomials. See the discussion at the beginning of Section 3.

This paper is written with two audiences in mind and is at the intersection of two different fields; on the one hand, people familiar with the theory (or one should say, theories) of reproducing kernel Hilbert spaces of power series in one and several complex variables and on the other hand, people familiar with hypercomplex analysis.

The paper intends to be of a review nature and also to contain new results: among the new results presented we mention:

1. Another approach to quaternionic rational functions; see Theorem 2.8.
2. The fact that the quaternionic Cauchy kernel is rational. See Corollary 2.10.
3. A characterisation of the Leibenson shift operators. See Theorem 3.5.
4. A Beurling type theorem in the quaternionic Arveson space.
5. A definition and study of Schur–Agler type classes in the quaternionic setting. See Section 4.2.

The setting which we present contains both a non-commutative and an analytic aspects. It is different from the non-commutative theory (but some formulas are quite similar; see, e.g., formula (3.12) for the realization of a Schur multiplier) and it is also quite different from the analytic setting. The Fueter polynomials (see

Definition 2.1) play now the role of the usual monomials $z_1^{n_1}\cdots z_N^{n_N}$ and, although similar in notation, have quite different properties.

Before considering the hyperholomorphic case we discuss briefly in the next section the case of several complex variables.

1. The case of several complex variables

1.1. Rational functions

Quaternionic hyperholomorphic rational functions play an important role in this paper and we begin by reviewing some facts for the corresponding objects in the setting of one and several complex variables. A *rational function* of one complex variable is just a quotient of polynomials with complex coefficients. A matrix-valued function is rational if its entries are rational. Equivalently, it is rational if it is the quotient of a matrix polynomial (a matrix function with polynomial entries) with a scalar polynomial. Originating with the theory of linear systems, another representation of rational function proved to be very useful:

Proposition 1.1. *A matrix-valued function $r(z)$ analytic in a neighborhood of the origin is rational if and only if it can be written as*

$$r(z) = D + C(I_n - zA)^{-1}zB \tag{1.1}$$

where I_n denotes the identity matrix of order n and where A, B, C and D denote matrices of appropriate sizes.

An expression of the form (1.1) is called a *realization.* See, e.g., [9], where functions analytic at infinity rather than at the origin are considered.

In several complex variables, rational functions are also defined as quotient of polynomials, and the realization result extend: a matrix-valued function of several complex variables analytic in a neighborhood of the origin is rational if and only if it can be written as

$$r(z) = D + C(I_n - \sum_{i=1}^{N} z_iA_i)^{-1}\left(\sum_{i=1}^{N} z_iB_i\right) \tag{1.2}$$

where $A_1, \ldots, A_N, B_1, \ldots, B_N, C$ and D are matrices of appropriate sizes. We can rewrite (1.2) as (1.1) by setting

$$z = \begin{pmatrix} z_1I_n & \cdots, z_NI_n \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix} \quad B = \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}.$$

For a recent proof of this well-known realization result, see [3]. We refer to the papers [18] and [29] for connections with the theory of linear systems (note that the paper [29] considers a different kind of realization).

1.2. Some Hilbert spaces of power series

In this section we review some results from the theory of reproducing kernel Hilbert spaces of power series in several complex variables.

Definition 1.2. *Let* $\mathbf{c} = \{c_\nu\}$ *be a sequence of positive numbers indexed by* $\mathbb{N}^N$ *and let its support* supp $(\mathbf{c})$ *be defined by*

$$\text{supp }(\mathbf{c}) = \left\{\nu \in \mathbb{N}^N \mid c_\nu \neq 0\right\}. \tag{1.3}$$

We denote by $\mathcal{H}(\mathbf{c})$ *the space of power series of the form*

$$f(x) = \sum_{\nu\in\text{supp }(\mathbf{c})} z^\nu f_\nu$$

where the $f_\nu \in \mathbb{C}$ *are such that*

$$\|f\|_{\mathbf{c}}^2 := \sum_{\nu\in\text{supp }(\mathbf{c})} \frac{|f_\nu|^2}{c_\nu} < \infty. \tag{1.4}$$

In the sequel we use the notion of *lower inclusive sets*. These sets were introduced in the work [8] of Ball, Li, Timotin and Trent (the term itself was coined in Woerderman's paper [37]) and used in [37] to solve the Carathéodory–Féjer interpolation problem in the polydisk.

Define a partial order $\leq_p$ on $\mathbb{N}^N$ as follows: For $k = (k_1, k_2, \ldots, k_N) \in \mathbb{N}^N$ and $\ell = (\ell_1, \ell_2, \ldots, \ell_N) \in \mathbb{N}^N$, we say that $k \leq_p \ell$ if and only if $k_i \leq \ell_i \quad i = 1, 2, \ldots, N$.

Definition 1.3. *A set* $\mathcal{K} \subseteq \mathbb{N}^N$ *is said to be lower inclusive if the following condition holds:*
if $k \in \mathcal{K}$ *and* $\ell \leq_p k$*, then* $\ell \in \mathcal{K}$*.*

1.3. Gleason's problem and the Leibenson's shift operators

What is now called Gleason's problem was considered by Hefer; the paper [25] was published in 1950. In a footnote, Behnke and Stein state that the author died in 1941 and that the paper is part of his 1940 Munster dissertation. For a related result, see also [11].

Problem 1.4. *Let* $\mathcal{M}$ *be a set of functions analytic in a subset* $\Omega \subset \mathbb{C}^N$ *and let* $a \in \Omega$*. Given* $f \in \mathcal{M}$*; to find functions* $g_1(z,a), \ldots, g_N(z,a) \in \mathcal{M}$ *such that*

$$f(z) - f(a) = \sum_{j=1}^N (z_j - a_j) g_j(z,a).$$

A more restrictive requirement is to ask that there are bounded operators $T_{j,a}$ such that $g_j(z,a) = (T_{j,a}f)(z)$. We then say that the $T_{j,a}$ solve Gleason's problem. When $N = 1$ and $a = 0$, we get back to the well-known notion of backward-shift invariance: is $\mathcal{M}$ invariant under the backward-shift operator $\frac{f(z)-f(0)}{z}$?

Let us take $a = 0$ and $N \geq 1$. Differentiating the function $t \mapsto f(tz)$ (with $t \in [0,1]$) and integrating back one obtains

$$f(z) - f(0) = \sum_{\ell=1}^{N} z_j(\mathcal{R}_j f)(z)$$

where

$$\mathcal{R}_j f(z) = \int_0^1 \frac{\partial f}{\partial z_j}(tz)dt.$$

One has

$$\mathcal{R}_j z^\alpha = \begin{cases} \frac{\alpha_j}{|\alpha|} z^{\alpha - e_j} & \text{if } \alpha_j > 0, \\ 0 & \text{if } \alpha_j = 0, \end{cases}$$

where e_j denotes the row vector with all components equal to 0, besides the jth one equal to 1, and so:

Lemma 1.5. *A necessary condition for a space $\mathcal{H}(\mathbf{c})$ to be $\mathcal{R}_j$-invariant is that the set* supp $(\mathbf{c})$ *is lower inclusive.*

The $\mathcal{R}_j$ are the generalized backward-shift operators introduced by Leibenson and one version of Gleason's problem is to ask whether the space $\mathcal{M}$ is invariant under the $\mathcal{R}_j$. Of course a negative answer does not mean that Gleason's problem is not solvable in $\mathcal{M}$.

The functions $g_j(z, a)$ are not uniquely defined in general; on the other hand, when $a = 0$, the choice $g_j(z, 0) = R_j f(z)$ is unique under appropriate hypothesis. A first set of such hypothesis was given in [15], where E. Doubtsov proved that the Leibenson solution is a minimal solution (in an appropriate sense). In [2] it is shown that if the space is $\mathcal{R}_j$-invariant the $\mathcal{R}_j$ are the only commutative solution to Gleason's problem.

Theorem 1.6. *Let $\mathcal{P}$ be a space of $\mathbb{C}^p$-valued functions analytic on a domain $\Omega \subset \mathbb{C}^N$ containing the origin, and which is invariant under the multiplication operators M_{z_j} for $j = 1, \ldots, N$. The set of commuting, bounded operators solving Gleason's problem in $\mathcal{P}$, if it exists, is unique, and is given by*

$$T_j := T_{j,0} \colon f(z) \mapsto g_j(z)$$

where $g_j(z)$ is the uniquely determined element of $\mathcal{P}$ having Taylor expansions with center point at the origin given by

$$g_j(z) = \sum_{\alpha \in \mathbb{N}^N \colon \alpha_j \geq 1} z^{\alpha - e_j} \frac{\alpha_j}{|\alpha|} f_\alpha$$

if $f(z)$ has Taylor expansion at the origin given by

$$f(z) = \sum_{\alpha \in \mathbb{N}^N} z^\alpha f_\alpha.$$

In Section 3.2 we prove a similar result in the setting of hyperholomorphic functions.

1.4. Schur multipliers

Definition 1.7. *Let $K(z,w)$ be a function positive on the set Ω. The function $s : \Omega \longrightarrow \mathbb{C}$ is called a Schur multiplier if the operator M_s of multiplication by s is a contraction from $\mathcal{H}(K)$ (the reproducing kernel Hilbert space with reproducing kernel K) into itself.*

It is well known (but, as we will see, the hyperholomophic counterparts of these formulas are more involved; see, e.g., formula (3.2)) that

$$M_s^* K(\cdot, w) = K(\cdot, w)s(w)^* \tag{1.5}$$

and that s is a Schur multiplier if and only if the function

$$(1 - s(z)s(w)^*)K(z,w)$$

is positive in Ω.

1.5. The Hardy space of the ball

The Hardy space of the ball $\mathbf{H}_2(\mathbb{B}_N)$ is the reproducing kernel Hilbert space with reproducing kernel $\frac{1}{(1-\langle z,w\rangle)^N}$. Since

$$\frac{1}{(1-\langle z,w\rangle)^N} = \sum_{\alpha\in\mathbb{N}^N} \frac{(N+|\alpha|-1)!}{\alpha!(N-1)!} z^\alpha \overline{w^\alpha}$$

the space $\mathbf{H}_2(\mathbb{B}_N)$ is a weighted power series space and its elements can be characterized via (1.4). A function s analytic in the ball is a contractive multiplier for the Hardy space if and only if the kernel

$$\frac{1 - s(z)s(w)^*}{(1-\langle z,w\rangle)^N} \tag{1.6}$$

is positive in $\mathbb{B}_N$.

The norm (1.4) has also a geometric interpretation as

$$\|f\|^2 = \sup_{0<r<1} \int_{|z|=1} |f(rz)|^2 d\lambda(z).$$

Thanks to this interpretation, Schur multipliers of the Hardy space are readily seen (as in the case $N = 1$) to be exactly the set of functions analytic and contractive in $\mathbb{B}_N$. For $N = 1$ interpolation theory and realization theory of these functions is a very well developed topic, known as Schur analysis; see, e.g., [19], [21] for some references. For $N > 1$ these same questions (interpolation theory and realization theory) seem beyond the scope of current methods of several complex variables and operator theory.

1.6. The Arveson space and de Branges–Rovnyak spaces

The kernel $\frac{1}{1-\langle z,w\rangle}$ is positive in the open unit ball. When $N>1$ the associated reproducing kernel Hilbert space is strictly and contractively included in the Hardy space of the ball. This space was introduced by Drury in [16] and studied further by Arveson [7]. Following other authors we will call it the Arveson space.

A function s analytic in the ball is a contractive multiplier for the Arveson space if and only if the kernel

$$\frac{1-s(z)s(w)^*}{1-\langle z,w\rangle}$$

is positive in $\mathbb{B}_N$. Note the difference with (1.6). We note that there are functions analytic and contractive in the ball and which are not Schur multipliers of the Arveson space. In the statement (and in the sequel of the paper), a co-isometric operator is an operator whose adjoint is isometric.

Theorem 1.8. *A function s analytic in the ball is a Schur multiplier of the Arveson space if and only if there exists a Hilbert space $\mathcal{H}$ and a co-isometric operator*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_N & B_N \\ C & D \end{pmatrix} : \mathcal{H}^N \oplus \mathbb{C} \Longrightarrow \mathcal{H} \oplus \mathbb{C}$$

such that

$$s(z) = D + C(I_{\mathcal{H}} - zA)^{-1} zB \tag{1.7}$$

where

$$zA = z_1A_1 + \cdots + z_NA_N, \quad zB = z_1B_1 + \cdots + z_NB_N.$$

Remark 1.9. *It follows from formula* (1.7) *that we have the power series expansion*

$$s(z) = D + \sum_{k=1}^{N} \sum_{\nu\in\mathbb{N}^N} \frac{|\nu|!}{\nu!} z_k z^\nu CA^\nu B_k,$$

where

$$A^\nu = A_1^{\times\nu_1} \times \cdots \times T_N^{\times\nu_N}$$

and

$$A_1 \times A_2 \times \cdots \times A_n = \frac{1}{n!} \sum_{\sigma\in S_n} A_{\sigma(1)}A_{\sigma(2)} \cdots A_{\sigma(n)}.$$

Remark 1.10. *The knowledgeable reader will have noticed that in the above statements it is not necessary to assume that s is* analytic *in the ball. It is enough to assume that s is defined on a uniqueness set in the ball. This is one instance of a general principle where positivity forces analyticity.*

We presented the definition and characterization of Schur multipliers in the scalar case, but these also make sense in the case of operator-valued functions.

1.7. The polydisk and the Schur–Agler classes

The Hardy space of the polydisk is the reproducing kernel Hilbert space with reproducing kernel

$$k(z,w)=\frac{1}{\prod_1^N(1-z_j\overline{w_j})}.$$

It would seem natural to define in the setting of the polydisk the class of Schur multipliers, that is functions analytic in the polydisk and such that the kernel

$$\frac{1-s(z)\overline{s(w)}}{\prod_1^N(1-z_j\overline{w_j})} \tag{1.8}$$

is positive in $\mathbb{D}^N$. Unfortunately, as soon as $N>2$, these are not classes for which there is a nice characterization in terms of realization. J. Agler introduced (see [1]) the class of functions s such that

$$1-s(z)\overline{s(w)}=\sum_{j=1}^N(1-z_j\overline{w_j})K_j(z,w)$$

for some (in general not uniquely defined) functions $K_1,\ldots,K_N$ positive in $\mathbb{D}^N$.

Dividing both sides of the above equality by $\prod_1^N(1-z_j\overline{w_j})$ we see that the kernel (1.8) is positive. In particular the function s is a contractive multiplier of the Hardy space of the polydisk and is thus automatically analytic there. For $N>2$ the Schur–Agler class is strictly smaller than the class of contractive multipliers of the Hardy space of the polydisk. As in Section 1.6 we focus on the scalar case.

Theorem 1.11. *A function s analytic in the polydisk is in the Schur–Agler class if and only if it can be written as*

$$s(z)=D+C(I-d(z)A)^{-1}d(z)B,$$

where in the expression $d(z)=\operatorname{diag}\,(z_jI_{\mathcal{H}_j})$ for some Hilbert spaces $\mathcal{H}_j$ and

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}:\quad\begin{pmatrix}\mathcal{H}_1\\ \vdots\\ \mathcal{H}_N\\ \mathbb{C}\end{pmatrix}\longrightarrow\begin{pmatrix}\mathcal{H}_1\\ \vdots\\ \mathcal{H}_N\\ \mathbb{C}\end{pmatrix}$$

is a co-isometric operator.

One of the main results of this paper is the definition and characterization of the Schur–Agler classes in the quaternionic setting; see Section 4.2.

2. Hyperholomorphic functions

2.1. Quaternions and quaternionic hyperholomorphic functions

The building of the skew-field of quaternions has a fascinating history; see for instance [14]. For our present purposes it is enough to define directly the quaternions as

$$\mathbb{H} = \left\{ q = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}, \quad z, w \in \mathbb{C} \right\}.$$

This is readily seen to be a skew-field. Writing $z = x_0 + ix_1$ and $w = x_2 + ix_3$ we have that

$$x = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \tag{2.1}$$

where

$$\mathbf{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We will denote $\mathbf{e}_0 = 1$ and note that the $\mathbf{e}_j$ satisfy the Cayley multiplication table

	$\mathbf{e_0}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_0}$	$\mathbf{e_0}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_1}$	$\mathbf{e_1}$	$-\mathbf{e_0}$	$\mathbf{e_3}$	$-\mathbf{e_2}$
$\mathbf{e_2}$	$\mathbf{e_2}$	$-\mathbf{e_3}$	$-\mathbf{e_0}$	$\mathbf{e_1}$
$\mathbf{e_3}$	$\mathbf{e_3}$	$\mathbf{e_2}$	$-\mathbf{e_1}$	$-\mathbf{e_0}$

. (2.2)

In the sequel we identify the space $\mathbb{R}^3$ with the set of purely vectorial quaternions, that is quaternions x such that $x_0 = 0$.

The function $f : \Omega \subset \mathbb{R}^4 \to \mathbb{H}$ is called *left-hyperholomorphic* if

$$D\ f := \frac{\partial}{\partial x_0} f + \mathbf{e_1}\frac{\partial}{\partial x_1} f + \mathbf{e_2}\frac{\partial}{\partial x_2} f + \mathbf{e_3}\frac{\partial}{\partial x_3} f = 0. \tag{2.3}$$

Write $f = f_0 + \mathbf{e}_1 f_1 \mathbf{e}_2 + f_2 + \mathbf{e}_3 f_3$. The components f_j of f satisfy the system

$$\begin{aligned}
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} &= 0, \\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} &= 0, \\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} &= 0, \\
\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} &= 0.
\end{aligned} \tag{2.4}$$

See, e.g., [20, equations (2a) p. 76]. The system of equations (2.4) when there is no dependence on x_0 appears in [27, (5) p. 985]; nowadays it bears the name of the Moisil–Theodoresco system and there is a long list of works about its properties. A curious reader can find it useful to look into the books [17] and [23] as well as into the papers [22], [31], [33] and [36].

The case where $f_0 \equiv 0$ and where f_1, f_2 and f_3 do not depend on x_0 is of special interest; see [20, p. 78]. The system (2.4) can be re-written now as

$$\begin{aligned}
\operatorname{div}\ \vec{f} &= 0, \\
\operatorname{rot}\ \vec{f} &= 0,
\end{aligned} \tag{2.5}$$

where $\vec{f} = (f_1, f_2, f_3)$; hence (2.5) being a particular case of (2.4) has both purely mathematical and physical developments. Again, a long list of references could be composed from which we indicate a few instances: [10, pp. 81–96], [38], [24], [30].

A solution of (2.5), and more generally of its generalization to any dimension, is called a system of conjugate harmonic functions; see [28, p. 18]. The paper [32] can be useful for a first acquaintance and to understand the main ideas.

We now introduce a family of hyperholomorphic polynomials, and describe the counterpart of the Taylor series at the origin. Let f be left-hyperholomorphic. The chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}\,t} f(tx) = \sum_{\ell=0}^{3} x_\ell \frac{\partial f}{\partial x_\ell}(tx). \tag{2.6}$$

Since the function is left-hyperholomorphic we have

$$\frac{\partial f}{\partial x_0} = -\mathbf{e_1}\frac{\partial}{\partial x_1} f - \mathbf{e_2}\frac{\partial}{\partial x_2} f - \mathbf{e_3}\frac{\partial}{\partial x_3} f.$$

Replacing $\frac{\partial f}{\partial x_0}$ by this expression in (2.6) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t} f(tx) &= x_0\left(-\mathbf{e_1}\frac{\partial}{\partial x_1} f(tx) - \mathbf{e_2}\frac{\partial}{\partial x_2} f(tx) - \mathbf{e_3}\frac{\partial}{\partial x_3} f(tx)\right) + \\ &\quad + \sum_{\ell=1}^{3} x_\ell \frac{\partial f}{\partial x_\ell}(tx) \\ &= \sum_{\ell=1}^{3}(x_\ell - x_0\mathbf{e}_\ell)\frac{\partial f}{\partial x_\ell}(tx).\end{aligned}$$

Integrating with respect to t we obtain

$$f(x) - f(0) = \sum_{\ell=1}^{3}(x_\ell - x_0\mathbf{e}_\ell)\int_0^1 \frac{\partial f}{\partial x_\ell}(tx)dt. \tag{2.7}$$

It remains to show that the functions $g_\ell(x) = \int_0^1 \frac{\partial f}{\partial x_\ell}(tx)dt$ are left-hyperholomorphic. This follows from the fact that, for a given t, Df evaluated at the point tx is equal to 0 and that we can interchange integration and derivation when computing Dg_ℓ.

We note that the functions $\zeta_\ell(x) = x_\ell - x_0\mathbf{e}_\ell$ are hyperholomorphic. Iterating formula (2.7) we get

$$f(x) - f(0) = \sum_{\nu\in\mathbb{N}^3} \zeta^\nu f_\nu$$

where $f_\nu \in \mathbb{H}$ and ζ^ν are non-commutative homogeneous hyperholomorphic polynomials in the ζ_j given by the formula

$$\zeta^\nu(x) = \zeta_1(x)^{\times\nu_1} \times \zeta_2(x)^{\times\nu_2} \times \zeta_3(x)^{\times\nu_3}, \tag{2.8}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ and where the symmetrized product of $a_1, \dots, a_n \in \mathbb{H}$ is defined by

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}, \tag{2.9}$$

where S_n is the set of all permutations of the set $\{1, \dots, n\}$.

Definition 2.1. *The polynomials $\zeta^\nu(x)$ defined by* (2.8) *are called the Fueter polynomials.*

2.2. The Cauchy–Kovalevskaya extension and product

The pointwise product of two hyperholomorphic functions is not in general hyperholomorphic. The Cauchy–Kovalevskaya product allows to remedy this situation. Let $\varphi(x_1, x_2, x_3)$ be a real analytic function from some open domain of $\mathbb{R}^3$ into $\mathbb{H}$, that is φ is given by four coordinate real analytic real-valued functions

$$\varphi(x_1, x_2, x_3) = \varphi_0(x_1, x_2, x_3) + \sum_1^3 \mathbf{e}_i \varphi_i(x_1, x_2, x_3).$$

The Cauchy–Kovalevskaya theorem (in fact, in its simplest form; see [26, Section 1.7]; see also [26, Section 1.10] and [13, Section I.7] for the general version) implies that the system of equations (2.4) with initial conditions

$$f_i(0, x_1, x_2, x_3) = \varphi_i(x_1, x_2, x_3)$$

admits a unique real analytic solution in a neighborhood of the origin in $\mathbb{R}^4$. This solution

$$f(x_0, x_1, x_2, x_3) = f_0(x_0, x_1, x_2, x_3) + \sum_1^3 \mathbf{e}_i f_i(x_0, x_1, x_2, x_3)$$

is hyperholomorphic by definition and is called the Cauchy–Kovalevskaya extension of the function φ.

Example 2.2. *The Cauchy–Kovalevskaya extension of the polynomial x^α is the Fueter polynomial ζ^α.*

More generally, the Cauchy–Kovalevskaya extension of the $\mathbb{H}$-valued real-analytic function $\sum_{k=0}^\infty \sum_{|\nu|=k} x^\alpha f_\nu$ (where $f_\nu \in \mathbb{H}$) is the function

$$f(x) = \sum_{k=0}^\infty \sum_{|\nu|=k} \zeta^\nu f_\nu.$$

Consider now two hyperholomorphic functions f and g and let φ and ψ be their restrictions to $\mathbb{R}^3$ (that is, when setting $x_0 = 0$). The functions φ and ψ are real analytic and so is their (pointwise) product. The Cauchy–Kovalevskaya extension of $\varphi\psi$ is called the *Cauchy–Kovalevskaya product* of f and g. It was first introduced by F. Sommen in [35].

Consider now a function f hyperholomorphic in a neighborhood of the origin. The function $f(0,x_1,x_2,x_3)$ is real analytic in a neighborhood of the origin of $\mathbb{R}^3$ and thus we can write

$$f(0,x_1,x_2,x_3)-f(0,0,0,0)=x_1h_1(x_1,x_2,x_3)+x_2h_2(x_1,x_2,x_3)+x_3h_3(x_1,x_2,x_3),$$

where the h_j are $\mathbb{H}$-valued and real analytic in a neighborhood of the origin of $\mathbb{R}^3$. Taking the Cauchy–Kovaleskaya extension of this expression we get to

$$f(x)-f(0)=\zeta_1(x)\odot g_1(x)+\zeta_2(x)\odot g_2(x)+\zeta_3(x)\odot g_3(x), \tag{2.10}$$

where the g_j are hyperholomorphic in a neighborhood of the origin.

The Cauchy–Kovalevskaya product can be defined also directly in terms of the power series expansions at the origin of the two functions. More precisely we have:

Theorem 2.3. *Let f and g be two functions hyperholomorphic in a neighborhood of the origin, with power series expansions*

$$f(x)=\sum_{k=0}^{\infty}\sum_{|\nu|=k}\zeta^{\nu}f_{\nu} \quad \textit{and} \quad f(x)=\sum_{k=0}^{\infty}\sum_{|\nu|=k}\zeta^{\nu}g_{\nu}.$$

Then,

$$(f\odot g)(x)=\sum_{n=0}^{\infty}\sum_{|\eta|=n}\zeta^{\eta}\sum_{0\leq\nu\leq\eta}f_{\nu}g_{\eta-\nu}.$$

The proof of Theorem 2.3 can be found in [12]. In view of Example 2.2, it follows from the right $\mathbb{H}$-linearity of the equation (2.3).

2.3. The quaternionic Cauchy kernel

Neither the quaternionic variable (2.1) nor its powers x^n (with $n=\pm1,\pm2,\ldots$) are hyperholomorphic. As noted by Fueter [20, p. 77] the functions $\Delta_{\mathbb{R}^4}x^n$ are both left- and right-hyperholomorphic.

The quaternionic Cauchy kernel is defined by the formula: for $x\neq 0$,

$$K(x):=-\frac{1}{2\,\mathrm{vol}\,\mathbb{S}^3}\overline{D}\frac{1}{|x|^2}=\frac{1}{\mathrm{vol}\,\mathbb{S}^3}\frac{\overline{x}}{|x|^4}=-\frac{1}{4\,\mathrm{vol}\,\mathbb{S}^3}\Delta_{\mathbb{R}^4}x^{-1}.$$

For a function $f(x)$ left hyperholomorphic in a neighborhood of the ball $B(0,R)$ the following Cauchy formula holds:

$$f(x)=\int_{|y|=R}K(y-x)d\sigma f(y).$$

Finally we note the expansion

$$\Delta\ (y-x)^{-1}=y^{-1}\sum_{n=0}^{\infty}\Delta\ (xy^{-1})^{n+2}=\sum_{n\in\mathbb{N}^3}\alpha_n(y)\beta_n(x)$$

valid for $|x|<|y|$. See [20, p. 81]. The β_n are hyperholomorphic polynomials, and are in fact the Fueter polynomials defined above.

2.4. Rational functions

In [4] we defined matrix-valued rational hyperholomorphic functions as functions obtained from Fueter polynomials after a finite number of operations of the following type: addition, Cauchy–Kovalevskaya multiplication, and if the the function is invertible at the origin, say $R(x) = R(0)(I_p - T(x))$, with $T(0) = 0$ and $R(0)$ invertible, then the Cauchy–Kovalevskaya inversion is defined by

$$R(x)^{-1} = \{I_p + T(x) + T(x) \odot T(x) + T(x) \odot T(x) \odot T(x) + \cdots\} R(0)^{-1}.$$

We proved that:

Proposition 2.4. *A matrix-valued function hyperholomorphic in a neighborhood of the origin is rational if and only if it can be written as*

$$\begin{aligned} R(x) = D + C \odot (I - (\zeta_1(x)A_1 + \zeta_2(x)A_2 + \zeta_3(x)A_3))^{-\odot} \odot \\ \odot (\zeta_1(x)B_1 + \zeta_2(x)B_2 + \zeta_3(x)B_3) \end{aligned} \tag{2.11}$$

where $A_1, A_2, A_3, B_1, B_2, B_3, C$ and D are matrices of appropriate sizes.

Compare (2.11) with (1.2).

We now give another characterization of hyperholomorphic rational functions. As a corollary we will obtain that the quaternionic Cauchy kernel is rational see also [5].

We first define rational functions of three real variables and whose values are matrices with quaternionic entries. Since the variables and the coefficients commute the notion of polynomials makes no difficulty. We call matrix-polynomial any finite sum

$$p(x) = \sum x^{\alpha_1} x^{\alpha_2} x^{\alpha_3} p_{(\alpha_1,\alpha_2,\alpha_3)} \tag{2.12}$$

where the $p_\alpha \in \mathbb{H}^{p\times q}$.

Definition 2.5. *A rational function of three real variables and with quaternionic coefficients is any function obtained from polynomials of the form* (2.12) *after a finite number of the following operations: addition, pointwise multiplication and inversion.*

In the sequel we focus on the case of functions which are real analytic in a neighborhood of the origin.

Proposition 2.6. *A function of three real variables, real analytic in a neighborhood of the origin and with quaternionic coefficients is rational if and only if it can be represented as*

$$r(x_1, x_2, x_3) = D + C(I - (x_1A_1 + x_2A_2 + x_3A_3))^{-1}(x_1B_1 + x_2B_2 + x_3B_3) \tag{2.13}$$

where $A_1, A_2, A_3, B_1, B_2, B_3, C$ and D are matrices of appropriate sizes.

Proof. The proof follows a classical argument and proceeds in a number of steps (we omit the proofs):

Step 1. Constant matrices and monomials of the form $x_i M$ have realizations of the form (2.13).

Step 2. If r and s admit realizations of the form (2.13) and if the product rs (resp. the sum $r+s$) makes sense, then the product (resp. the sum) admits also a realization of the form (2.13).

We note that the result on the sum follows from the result on the product since (with identities of appropriate sizes)

$$r + s = \begin{pmatrix} r & I \end{pmatrix} \begin{pmatrix} I \\ s \end{pmatrix}.$$

Step 3. If r admits a realization and $r(0)$ is invertible, then r^{-1} also admits a realization.

Proposition 2.7. *A function of three real variables, real analytic in a neighborhood of the origin and with quaternionic coefficients is rational if and only if it can be represented as*

$$r(x_1, x_2, x_3) = \frac{q(x_1, x_2, x_3)}{p(x_1, x_2, x_3)},$$

where q is a polynomial with quaternionic coefficients and p is a polynomial with real coefficients, such that $p(0) \neq 0$.

Proof. This is an immediate consequence of the inversion formula

$$q^{-1} = \frac{\overline{q}}{|q|^2} \quad \forall q \in \mathbb{H} \setminus \{0\}.$$

□

We now turn to the main result of this section:

Theorem 2.8. *A function defined in an open set Ω of $\mathbb{R}^4$ containing the origin is hyperholomorphic rational if and only if its restriction to $\Omega \cap \mathbb{R}^3$ is a rational $\mathbb{H}$-valued function of the three real variables x_1, x_2, x_3.*

Proof. Assume that R is hyperholomorphic and rational, that is, admits a realization of the form (2.11). Then, setting $x_0 = 0$ in (2.11), the Cauchy–Kovalevskaya products become usual products and we obtain

$$R(0, x_1, x_2, x_3) = D + C(I - (x_1 A_1 + x_2 A_2 + x_3 A_3))^{-1}(x_1 B_1 + x_2 B_2 + x_3 B_3),$$

and so the restriction $R(0, x_1, x_2, x_3)$ of R to $\Omega \cap \mathbb{R}^3$ is rational.

Conversely, (2.13) defines a function which is real analytic in a neighborhood of the origin. Taking the Cauchy–Kovalevskaya extension of both sides of (2.13) we obtain by definition on the left a hyperholomorphic function. On the right, by definition of the Cauchy–Kovalevskya product we obtain an expression of the form (2.11). □

Remark 2.9. *There is a fundamental difference between the expressions* (2.11) *and* (2.13). *The former is local* (*that is valid only in a neighborhood of the origin*), *while the latter is global. It makes sense for every* $(x_1, x_2, x_3) \in \mathbb{R}^3$ *where the matrix* $I - (x_1A_1 + x_2A_2 + x_3A_3)$ *is invertible.*

Corollary 2.10. *Let* $y \neq 0 \in \mathbb{H}$. *The quaternionic Cauchy kernel* $x \mapsto \frac{1}{2\pi^2}\frac{\overline{x}-\overline{y}}{|x-y|^4}$ *is rational.*

Proof. It suffices to note that the restriction of the function $x \mapsto \frac{\overline{x}-\overline{y}}{|x-y|^4}$ to $\mathbb{R}^3$ is rational in x_1, x_2, x_3 and has no singularities in a neighborhood of 0 when $y \neq 0$. □

In a similar way the quaternionic Bergman kernel for the unit ball is rational; indeed, this kernel is shown in [34, p. 10] to be equal to

$$K(x,y) = \frac{2}{\pi^2}\frac{(1-2\langle y,x\rangle + |y|^2|x|^2)(1-2\overline{x}y) + (\overline{y}-\overline{x}|y|^2)(x-y|x|^2)}{(1-2\langle y,x\rangle + |y|^2|x|^2)^3}.$$

3. Reproducing kernel spaces of power series of Fueter polynomials

We now define the counterparts of the spaces $\mathcal{H}(\mathbf{c})$ in the setting of hyperholomorphic functions. Our motivation for studying such spaces came from the quaternionic Arveson space, which we defined and studied in [4], [6].

3.1. Generalities

Theorem 3.1. *Let* $\mathbf{c} = \{c_\nu\}$ *be a sequence of positive numbers indexed by* $\mathbb{N}^3$ *with support* supp $(\mathbf{c})$ (*defined by* (1.3)). *Let*

$$k_{\mathbf{c}}(x,y) = \sum_{k=0}^{\infty}\sum_{\substack{\nu\in\text{supp }(\mathbf{c})\\|\nu|=k}} c_\nu\zeta^\nu(x)\overline{\zeta^\nu(y)}.$$

and

$$\Omega_{\mathbf{c}} = \left\{x\in\mathbb{R}^4 \mid \sum_{k=0}^{\infty}\sum_{\substack{\nu\in\text{supp }(\mathbf{c})\\|\nu|=k}} c_\nu|\zeta^\nu(x)|^2 < \infty\right\}.$$

Then $k_{\mathbf{c}}$ *is positive for* $x, y \in \Omega_{\mathbf{c}}$ *and the associated reproducing kernel Hilbert space of left hyperholomorphic functions is the set of functions*

$$f(x) = \sum_{\nu\in\text{supp }(\mathbf{c})} \zeta^\nu(x)f_\nu$$

where the $f_\nu \in \mathbb{H}$ *are such that*

$$\|f\|_{\mathbf{c}}^2 := \sum_{\nu\in\text{supp }(\mathbf{c})}\frac{|f_\nu|^2}{c_\nu} < \infty. \tag{3.1}$$

We will denote by $\mathcal{H}(\mathbf{c})$ the reproducing kernel Hilbert space of left hyperholomorphic functions with reproducing kernel $k_{\mathbf{c}}$. Its norm is given by (3.1). We note that $\mathcal{H}(\mathbf{c})$ contains the span of the ζ^ν where $\nu \in \text{supp}\ (\mathbf{c})$.

We will say that the function s hyperholomorphic in $\Omega_{\mathbf{c}}$ is a multiplier (resp. a Schur multiplier) if the operator of Cauchy–Kovalevskaya multiplication by s on the left is bounded (resp. is a contraction) from $\mathcal{H}(\mathbf{c})$ into itself. We now present the counterpart of formula (1.5).

Proposition 3.2. *Let s be a multiplier of $\mathcal{H}(\mathbf{c})$. Then it holds that:*

$$\left(M_s^*\left(k_y a\right)\right)(x) = \sum_{k=0}^{\infty} \sum_{|\nu|=k} c_\nu \zeta^\nu(x) \overline{(s \odot \zeta^\nu(y))} a. \tag{3.2}$$

Proof. Let $a, b \in \mathbb{H}$ and $x, y \in \Omega_{\mathbf{c}}$. We have:

$$\begin{aligned}
\langle M_S^* k_y a, k_x b\rangle_{\mathcal{H}(\mathbf{c})} &= \langle k_y a, s \odot (k_x b)\rangle_{\mathcal{H}(\mathbf{c})} \\
&= \langle k_y a, \sum_{k=0}^{\infty} \sum_{|\nu|=k} c_\nu (s \odot \zeta^\nu) \overline{\zeta^\nu(x)} b\rangle_{\mathcal{H}(\mathbf{c})} \\
&= \sum_{k=0}^{\infty} \sum_{|\nu|=k} c_\nu \overline{\left(\overline{a}\left(s \odot \zeta^\nu\right)(y) \overline{\zeta^\nu(x)} b\right)} \\
&= \sum_{k=0}^{\infty} \sum_{|\nu|=k} c_\nu \overline{b} \zeta^\nu(x) \overline{(s \odot \zeta^\nu(y))} a
\end{aligned}$$

and hence we obtain the formula (3.2). □

As a corollary we obtain that the function s is a Schur multiplier if and only if the kernel

$$\sum_{k=0}^{\infty} \sum_{|\nu|=k} c_\nu \left(\zeta^\nu(x) \overline{\zeta^\nu(y)} - (s \odot \zeta^\nu)(x) \overline{(s \odot \zeta^\nu)(y)}\right) \tag{3.3}$$

is positive in $\Omega_{\mathbf{c}}$.

3.2. Gleason's problem and Leibenson's shift operators

Proposition 3.3. *The operator*

$$\mathcal{R}_j f(z) = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \zeta^{\nu - e_j} \frac{\nu_j}{|\nu|} f_\nu \tag{3.4}$$

is bounded in $\mathcal{H}(\mathbf{c})$ if and only if the following two conditions hold: the set supp $(\mathbf{c})$ *is lower inclusive and it holds that*

$$\sup_j \left(\frac{\nu_j}{|\nu|}\right)^2 \frac{c_\nu}{c_{\nu_j}} < \infty. \tag{3.5}$$

When it is bounded it is equal to the Leibenson backward-shift operator

$$\mathcal{R}_j f(x) = \int_0^1 \frac{\partial f}{\partial x_j}(tx)dt. \tag{3.6}$$

Proof. The operator $\mathcal{R}_j$ is bounded if and only if there exists a constant $K > 0$ such that

$$\sum_{k=0}^{\infty} \sum_{|\nu|=k} \left(\frac{\nu_i}{|\nu|}\right)^2 \frac{|f_\nu|^2}{c_{\nu-e_i}} \leq K \left(\sum_{k=0}^{\infty} \sum_{|\nu|=k} \frac{|f_\nu|^2}{c_\nu}\right).$$

The result follows easily. □

The Leibenson backward-shift operator operators (3.6) have previously appeared in (2.7) which can be viewed as Gleason problem (see Problem 1.4) with respect to hyperholomorphic variables. However, we are mainly interested in the following version of this problem, formulated in terms of the Cauchy–Kovalevskaya product:

Problem 3.4. *Let $\mathcal{M}$ be a set of functions left hyperholomorphic in a neighborhood Ω of the origin. Given $f \in \mathcal{M}$; to find functions $p_1, \ldots, p_3 \in \mathcal{M}$ such that*

$$f(x) - f(0) = \sum_{j=1}^{3} (\zeta_j \odot p_j)(x).$$

Theorem 3.5. *Under hypothesis* (3.5) *Problem* 3.4 *is solvable in the spaces $\mathcal{H}(\mathbf{c})$ and the Leibenson type operators* (3.6) *are the only commutative solution of the problem.*

Proof. The proof parallels the proof of the similar fact in $\mathbb{C}^N$ presented in [2]. Here we consider the special case of power series expansions at the origin.

First of all, we note that the operators $\mathcal{R}_j$ given by (3.4) commute:

$$\mathcal{R}_j \mathcal{R}_\ell f = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \zeta^\nu \frac{(\nu_j+1)(\nu_\ell+1)}{(|\nu|+1)(|\nu|+2)} f_{\nu+e_j+e_\ell} = \mathcal{R}_\ell \mathcal{R}_j f,$$

and solve the Gleason problem:

$$\sum_{j=1}^{3} \zeta_j \odot (\mathcal{R}_j f) = \sum_{j=1}^{3} \zeta_j \odot \sum_{k=0}^{\infty} \sum_{|\nu|=k} \zeta^\nu \frac{\nu_j+1}{|\nu|+1} f_{\nu+e_j} = \sum_{k=1}^{\infty} \sum_{|\nu|=k} \sum_{j=1}^{3} \frac{\nu_j}{|\nu|} \zeta^\nu f_\nu = f - f(0).$$

Furthermore, let us assume that T_1, T_2, T_3 are some commuting bounded operators on $\mathcal{H}(\mathbf{c})$ which solve the Gleason problem, as well. Then we have for $f \in \mathcal{H}(\mathbf{c})$ and x in a neighborhood of the origin

$$\begin{aligned} f(x) &= f(0) + \sum_{j=1}^{3} (\zeta_j \odot (T_j f))(x) \\ &= f(0) + \sum_{j=1}^{3} \zeta_j(x) T_j f(0) + \sum_{j,\ell=1}^{3} (\zeta_j \odot \zeta_\ell \odot (T_\ell T_j f))(x). \end{aligned}$$

Continuing to iterate this formula and taking into account that T_j commute, we obtain the Taylor series for $f(x)$ in the form

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \frac{|\nu|!}{\nu!} \zeta^\nu(x)(T^\nu f)(0),$$

where

$$T^\nu = T_1^{\nu_1} T_2^{\nu_2} T_3^{\nu_3}.$$

In particular,

$$(T^\nu f)(0) = \frac{|\nu|!}{\nu!} f_\nu.$$

Now we write the Taylor expansion for $T_j f$:

$$(T_j f)(x) = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \frac{|\nu|!}{\nu!} \zeta^\nu(x)(T^{\nu+e_j} f)(0) = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \frac{|\nu|!}{\nu!} \zeta^\nu(x) f_{nu+e_j} = (\mathcal{R}_j f)(x).$$

□

3.3. The quaternionic Arveson space

The quaternionic Arveson space $\mathcal{A}$ corresponds to the choice $c_\nu = \frac{|\nu|!}{\nu!}$.

Proposition 3.6. *Assume that $c_\nu = \frac{|\nu|!}{\nu!}$. Then $\Omega_{\mathbf{c}}$ is the ellipsoid*

$$\Omega_{\mathbf{c}} = \left\{ x \in \mathbb{H} \mid 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1 \right\}$$

and

$$K_{\mathcal{A}}(x,y) := K_{\mathbf{c}}(x,y) = (1 - \zeta_1(x)\overline{\zeta_1(y)} - \zeta_2(x)\overline{\zeta_2(y)} - \zeta_3(x)\overline{\zeta_3(y)})^{-\odot}.$$

A proof can be found in [6].

The choices $s(x) = \zeta_j(x)$ for $j = 1, 2, 3$ in (3.3) leads to:

Theorem 3.7. *Let C be the operator of evaluation at the origin. It holds that*

$$I - \sum_{j=1}^{3} M_{\zeta_j} M_{\zeta_j}^* = C^* C \tag{3.7}$$

if and only if $c_\nu = \frac{|\nu|!}{\nu!}$, that is, if and only if we are in the setting of the quaternionic Arveson space $\mathcal{A}$.

Proof. Applying on both sides of the operator identity (3.7) to the kernel $k_{\mathbf{c}}$ we obtain

$$c_{\nu-e_1} + c_{\nu-e_2} + c_{\nu-e_3} = c_\nu.$$

The only solution of this equation with $c_0 = 1$ is $c_\nu = \frac{|\nu|!}{\nu!}$. □

Theorem 3.8. *The operators M_j defined by $f \mapsto f \odot \zeta_j$ are continuous in the quaternionic Arveson space $\mathcal{A}$ and their adjoints are given by $M_j^* = \mathcal{R}_j$. The Arveson space is the only space of hyperholomorhpic functions with these two properties.*

Proof. The result follows from (3.7) and from Theorem 3.5. □

3.4. $\mathcal{H}^*$-valued hyperholomorphic functions

Let $\mathcal{H}$ be a right linear quaternionic Hilbert space and let $\mathcal{H}^*$ denote the (left) dual space of bounded $\mathbb{H}$-linear functionals on $\mathcal{H}$. Let Ω be a domain in $\mathbb{R}^4$ containing the origin and let $f : \Omega \mapsto \mathcal{H}^*$ be a mapping such that $\forall h \in \mathcal{H}$, $f(\cdot)h$ is a left-hyperholomorphic function in Ω. Such a mapping f is said to be an $\mathcal{H}^*$-valued left-hyperholomorphic function in Ω.

Theorem 3.9. *Let $f(x)$ be an $\mathcal{H}^*$-valued left-hyperholomorphic function in a ball $B(0,R)$. Then f can be represented as the series*

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \zeta^{\nu}(x) f_{\nu}, \quad f_{\nu} \in \mathcal{H}^*,$$

which converges normally in $B(0,R)$ with respect to the operator norm.

Proof. First we note that for every $R' \in (0,R)$ the family of functionals $\{f(x) : |x| \leq R'\}$ is uniformly bounded: $\sup_{|x|\leq R'} \|f(x)\| < \infty$. Let $h \in \mathcal{H}$ and let

$$f(x)h = \sum_{k=0}^{\infty} P_k(x,h)$$

be the expansion of $f(x)h$ into the series of homogeneous polynomials of x. Then it follows from the Cauchy formula for the hyperholomorphic functions that for $x < R'$

$$|P_k(x,h)| \leq C_1(k+2)(k^2+1)\left(\frac{|x|}{R'}\right)^k \|h\| \sup_{|x|\leq R'} \|f(x)\|,$$

which can be proved like in [12, p.82], and where C_1 is a positive real constant independent of k. By uniqueness of the Taylor expansion, $P_k(x,h)$ is linear with respect to h, hence we can write $P_k(x,h) = P_k(x)h$ where $P_k(x) \in \mathcal{H}^*$ and the series $\sum_{k=0}^{\infty} P_k(x)$ converges to $f(x)$ normally in $B(0,R)$ with respect to the operator norm. Furthermore, since the polynomial $P_k(x)h$ is hyperholomorphic, we have

$$P_k(x)h = \sum_{|\nu|=k} \zeta^{\nu}(x) f_{\nu}(h),$$

where $f_{\nu}(h)$ is linear with respect to h and satisfies for $x < R'$ the estimate

$$|f_{\nu}(h)| \leq C_2 \frac{1}{(R')^k} \frac{(k+2)!}{\nu!} \|h\| \sup_{|x|\leq R'} \|P_k(x)\| \leq C(\nu, R')\|h\| \sup_{|x|\leq R'} \|f(x)\|.$$

Hence $f_{\nu} \in \mathcal{H}^*$. □

Corollary 3.10. *A positive kernel $k(x,y)$ can be represented as*

$$k_{\mathbf{c}}(x,y) = g(x)g(y)^*,$$

where $g(x)$ is an $\mathcal{H}(k)^*$-valued hyperholomorphic function.

3.5. de Branges–Rovnyak spaces

We shall say that an $\mathcal{H}^*$-valued hyperholomorphic function $s(x)$ is a Schur multiplier if the kernel

$$K_s(x,y) = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \frac{|\nu|!}{\nu!} \left(\zeta^\nu(x)\overline{\zeta^\nu(y)} - (\zeta^\nu \odot s)(x)(\zeta^\nu \odot s)(y)^* \right)$$

is positive (compare with (3.3)). Note that, in view of Corollary 3.10, this condition is in force if and only if there exist a quaternionic Hilbert space $\mathcal{G}$ and a $\mathcal{G}^*$-valued hyperholomorphic function $g(x)$ such that

$$1 - s(x)s(y)^* = g(x)g(y)^* - \sum_{\ell=1}^{3} (\zeta_\ell \odot g)(x)(\zeta_\ell \odot g)(y)^*.$$

Our terminology can be explained as follows. Let us denote by $\ell^2(\mathcal{H})$ the quaternionic Hilbert space of sequences $(f_\nu : \nu \in \mathbb{N}^3, f_\nu \in \mathcal{H})$ such that $\sum \frac{\nu!}{|\nu|!}|f_\nu|^2 < \infty$. Then $s = \sum \zeta^\nu s_\nu$ is a Schur multiplier if and only if the operator M_s defined by

$$M_s(f_\nu) = \sum_\nu \zeta^\nu \left(\sum_{\mu \leq \nu} s_\mu f_{\nu-\mu} \right)$$

is a contraction from $\ell^2(\mathcal{H})$ into the quaternionic Arveson space $\mathcal{A}$. In this case $K_s(\cdot, y) = (I - M_s M_s^*)K_{\mathcal{A}}$ and we denote by

$$\mathcal{H}(s) := (I - M_s M_s^*)^{\frac{1}{2}} \mathcal{A}$$

the quaternionic Hilbert space with the reproducing kernel K_s. This is the de Branges–Rovnyak space in the present setting.

Theorem 3.11. *Let s be an $\mathcal{H}^*$-valued hyperholomorphic Schur multiplier. Then there exists a co-isometry*

$$V = \begin{pmatrix} T_1 & F_1 \\ T_2 & F_2 \\ T_3 & F_3 \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{H}(s) \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H}(s)^3 \\ \mathbb{H} \end{pmatrix}$$

such that

$$\left(\sum_{k=1}^{3} \zeta_k \odot (T_k f) \right)(x) = f(x) - f(0); \tag{3.8}$$

$$\left(\sum_{k=1}^{3} \zeta_k \odot (F_k h) \right)(x) = (s(x) - s(0))h; \tag{3.9}$$

$$Gf = f(0); \tag{3.10}$$

$$Hh = s(0)h. \tag{3.11}$$

Furthermore, $s(x)$ admits the representation

$$s(x)h = Hh + \sum_{k=1}^{3} \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} (\zeta_k \odot \zeta^\nu)(x) G T^\nu F_k h, \quad x \in \Omega,\ h \in \mathcal{H}, \tag{3.12}$$

where we use the notation

$$T^\nu = T_1^{\times \nu_1} \times T_2^{\times \nu_2} \times T_3^{\times \nu_3}.$$

Proof. Let us denote by $\mathcal{H}(s)_3$ the closure in $\mathcal{H}(s)^3$ of the linear span of the elements of the form

$$w_y = \begin{pmatrix} (I - M_s M_s^*)\mathcal{R}_1 K_{\mathcal{A}}(\cdot, y) \\ (I - M_s M_s^*)\mathcal{R}_2 K_{\mathcal{A}}(\cdot, y) \\ (I - M_s M_s^*)\mathcal{R}_3 K_{\mathcal{A}}(\cdot, y) \end{pmatrix}, \quad y \in \Omega.$$

Define

$$\left(\hat{T} w_y q\right)(x) = \left(K_s(x, y) - K_s(x, 0)\right) q, \quad \hat{F} w_y q = \left(s(y)^* - s(0)^*\right) q, \tag{3.13}$$

$$\left(\hat{G} q\right)(x) = K_s(x, 0) q, \quad \hat{H} q = s(0)^* q, \tag{3.14}$$

then it follows from Theorems 3.7, 3.8 that

$$\left\langle \begin{pmatrix} \hat{T} w_{y_1} q_1 + \hat{G} p_1 \\ \hat{F} w_{y_1} q_1 + \hat{H} p_1 \end{pmatrix}, \begin{pmatrix} \hat{T} w_{y_2} q_2 + \hat{G} p_2 \\ \hat{F} w_{y_2} q_2 + \hat{H} p_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} w_{y_1} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} w_{y_2} q_2 \\ p_2 \end{pmatrix} \right\rangle$$

for any $y_1, y_2 \in \Omega$ and $p_1, p_2, q_1, q_2 \in \mathbb{H}$. Hence the operator matrix $\hat{V} = \left(\begin{smallmatrix} \hat{T} & \hat{G} \\ \hat{F} & \hat{H} \end{smallmatrix}\right)$ can be extended as an isometry from $\left(\begin{smallmatrix} \mathcal{H}(s)_3 \\ \mathbb{H} \end{smallmatrix}\right)$ into $\left(\begin{smallmatrix} \mathcal{H}(s) \\ \mathcal{H} \end{smallmatrix}\right)$. Let us set $V = \left(\begin{smallmatrix} T & F \\ G & H \end{smallmatrix}\right) = \hat{V}^*$. Then the relations (3.13), (3.14) imply (3.8)–(3.11). Now, iterating (3.8) as in the proof of Theorem 3.5 we obtain (3.12). $\square$

Theorem 3.12. *Let $\mathcal{G}, \mathcal{H}$ be right quaternionic Hilbert spaces and let*

$$V = \begin{pmatrix} T_1 & F_1 \\ T_2 & F_2 \\ T_3 & F_3 \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{G} \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{G}^3 \\ \mathbb{H} \end{pmatrix}$$

be a co-isometry. Then

$$s_V(x) = H + \sum_{k=1}^{3} \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} (\zeta_k \odot \zeta^\nu)(x) G T^\nu F_k$$

is an $\mathcal{H}^*$-valued Schur multiplier.

Proof. Define

$$A_\mu(x) = \sum_{\nu\in\mathbb{N}^3} \frac{|\nu|!}{\nu!}\times$$
$$\times \left((\zeta^{\mu+\nu}\odot\zeta_1)(x)GT^\nu \quad (\zeta^{\mu+\nu}\odot\zeta_2)(x)GT^\nu \quad (\zeta^{\mu+\nu}\odot\zeta_3)(x)GT^\nu\right),$$
$$B_\mu(x) = \sum_{\nu\in\mathbb{N}^3} \frac{|\nu|!}{\nu!}\zeta^{\mu+\nu}(x)GT^\nu, \quad C(x) = \sum_{\nu\in\mathbb{N}^3} \frac{|\nu|!}{\nu!}\zeta^{\nu}(x)GT^\nu.$$

Then

$$\begin{aligned} A_\mu(x)A_\mu(y)^* + \zeta^\mu(x)\overline{\zeta^\mu(y)} &= \left(A_\mu(x) \quad \zeta^\mu(x)\right)VV^*\left(A_\mu(y) \quad \zeta^\mu(y)\right)^* \\ &= \left(B_\mu(x) \quad (\zeta^\mu\odot s_V)(x)\right)\left(B_\mu(y) \quad (\zeta^\mu\odot s_V)(y)\right)^* \\ &= B_\mu(x)B_\mu(y)^* + (\zeta^\mu\odot s_V)(x)(\zeta^\mu\odot s_V)(y)^*. \end{aligned}$$

Hence

$$\begin{aligned} K_{s_V}(x,y) &= \sum_{\mu\in\mathbb{N}^3}\frac{|\mu|!}{\mu!}\left(\zeta^\mu(x)\overline{\zeta^\mu(y)} - (\zeta^\mu\odot s_V)(x)(\zeta^\mu\odot s_V)(y)^*\right) \\ &= \sum_{\mu\in\mathbb{N}^3}\frac{|\mu|!}{\mu!}\left(B_\mu(x)B_\mu(y)^* - A_\mu(x)A_\mu(y)^*\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\sum_{\mu\in\mathbb{N}^3}\frac{|\mu|!}{\mu!}\sum_{n=1}^3(\zeta^{\mu+\nu}\odot\zeta_n)(x)GT^\nu(T^\eta)^*G^*\overline{(\zeta^{\mu+\eta}\odot\zeta_n)(y)} \\ &\qquad = \sum_{n=1}^3\sum_{\mu:\mu_n>0}\frac{\mu_n}{|\mu|}\frac{|\mu|!}{\mu!}\zeta^{\mu+\nu}(x)GT^\nu(T^\eta)^*G^*\overline{\zeta^{\mu+\eta}(y)} \\ &\qquad = \sum_{|\mu|>0}\zeta^{\mu+\nu}(x)GT^\nu(T^\eta)^*G^*\overline{\zeta^{\mu+\eta}(y)}, \end{aligned}$$

and thus $K_{s_V}(x,y) = C(x)C(y)^*$ is positive. □

Theorem 3.13. *Let $\mathcal{H}$ be a right linear quaternionic reproducing kernel Hilbert space of functions, left-hyperholomorphic in a neighborhood of the origin. Assume that there exist bounded operators T_1, T_2 T_3 from $\mathcal{H}$ into itself such that*

$$\left(\sum_{k=1}^3\zeta_k\odot(T_kf)\right)(x) = f(x) - f(0)$$

and

$$\sum_{k=1}^3\|T_kf\|^2 \le \|f\|^2 - |f(0)|^2.$$

Then there exist a quaternionic Hilbert space $\mathcal{G}$ and a $\mathcal{G}^*$-valued Schur multiplier s such that $\mathcal{H} = \mathcal{H}(s)$.

Proof. Since $T^*T + G^*G \leq I$, there exist F, H such that $V = \begin{pmatrix} T & F \\ G & H \end{pmatrix}$ is a co-isometry. Hence s_V is a Schur multiplier, and in particular

$$K_{s_V}(x,y) = C(x)C(y)^*$$

where

$$C(x) = \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu(x) G T^\nu.$$

But then for $f \in \mathcal{H}$ we have $C(x)f = f(x)$, hence $\mathcal{H} = \mathcal{H}(s_V)$. □

4. The analogue of the Hardy space of the polydisk

4.1. The Hardy type space

The counterpart of Hardy space here corresponds to the case $c_\nu \equiv 1$. Now

$$\Omega_{\mathbf{c}} = \left\{ x \in \mathbb{H} \mid \sup_{i=1,2,3} (|x_0|^2 + |x_i|^2) < 1 \right\}.$$

4.2. Quaternionic Schur–Agler spaces and realization theory

Definition 4.1. *Let s be hyperholomorphic in a neighborhood of the origin. Then s is said to belong to the Schur–Agler class, if there exist hyperholomorphic operator-valued $g_1(x), g_2(x), g_3(x)$ such that*

$$1 - s(x)\overline{s(y)} = \sum_{\ell=1}^{3} \left(g_\ell(x) g_\ell(y)^* - \zeta_\ell \odot g_\ell(x) (\zeta_\ell \odot g_\ell(y))^* \right).$$

Theorem 4.2. *Let s be in the Schur–Agler class and for $\ell = 1,2,3$ let $\mathcal{H}_\ell$ be the Hilbert space with the reproducing kernel $g_\ell(x)g_\ell(y)^*$. Then there exists a co-isometry*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \bigoplus_{\ell=1}^3 \mathcal{H}_\ell \\ \mathbb{H} \end{pmatrix} \mapsto \begin{pmatrix} \bigoplus_{\ell=1}^3 \mathcal{H}_\ell \\ \mathbb{H} \end{pmatrix},$$

such that for arbitrary $q \in \mathbb{H}$ and $h_\ell \in \mathcal{H}_\ell$, $\ell = 1,2,3$

$$\zeta \odot Ah = \sum_{\ell=1}^{3} h_\ell - h_\ell(0), \quad \zeta \odot Bq = (s - s(0))q,$$

$$Ch = \sum_{\ell=1}^{3} h_\ell(0), \quad Dq = s(0)q.$$

In terms of the operators A, B, C, D the function s admits the realization

$$s = D + \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu \odot \sum_{\ell=1}^{3} \zeta_\ell C A^{[\nu]} \pi_\ell B, \tag{4.1}$$

where π_ℓ is the orthogonal projection onto $\mathcal{H}_\ell$ in $\bigoplus_{\ell=1}^3 \mathcal{H}_\ell$ and

$$A^{[\nu]} = (\pi_1 A)^{\times \nu_1} \times (\pi_2 A)^{\times \nu_2} \times (\pi_3 A)^{\times \nu_3}.$$

Proof. The proof is analogous to that of Theorem 3.11. We consider

$$\mathcal{H} = \overline{\text{span}} \left\{ \begin{pmatrix} (\zeta_1 \odot g_1)(y)^* a \\ (\zeta_2 \odot g_2)(y)^* a \\ (\zeta_3 \odot g_3)(y)^* a \\ b \end{pmatrix} \right\}.$$

and define

$$\hat{A}\left(\zeta_\ell \odot g_\ell(y)^* a\right) = \left(g_\ell(y)^* a - g_\ell(0)^* a\right), \quad \hat{C} b = g_\ell(0)^* b,$$
$$\hat{B}\left(\zeta_\ell \odot g_\ell(y)^* a\right) = \overline{s(y)} a - \overline{s(0)} a, \quad \hat{D} b = \overline{s(0)} b.$$

Then $\begin{pmatrix} \hat{A} & \hat{C} \\ \hat{B} & \hat{D} \end{pmatrix}$ can be extended as an isometry from $\begin{pmatrix} \mathcal{H} \\ \mathbb{H} \end{pmatrix}$ into $\begin{pmatrix} \oplus_{\ell=1}^3 \mathcal{H}_\ell \\ \mathbb{H} \end{pmatrix}$ and the adjoint operator matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{C} \\ \hat{B} & \hat{D} \end{pmatrix}^*$ possesses the desire properties. □

Theorem 4.3. *Let s be of the form (4.1), where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a co-isometry. Then s belongs to the Schur–Agler class.*

Proof. Note that

$$\begin{pmatrix} \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu \odot \sum_{\ell=1}^3 \zeta_\ell C A^{[\nu]} \pi_\ell & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu C A^{[\nu]} & s \end{pmatrix}.$$

Hence

$$\sum_{\ell=1}^3 \left(\zeta_\ell \odot \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu C A^{[\nu]} \pi_\ell \right)(x) \left(\zeta_\ell \odot \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu C A^{[\nu]} \pi_\ell \right)(y)^* + 1 =$$
$$+ \left(\sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu C A^{[\nu]} \pi_\ell \right)(x) \left(\sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu C A^{[\nu]} \pi_\ell \right)(y)^* + s(x)\overline{s(y)}.$$

□

References

[1] J. Agler. *On the representation of certain holomorphic functions defined on a polydisk*, volume 48 of *Operator Theory: Advances and Applications*, pages 47–66. Birkhäuser Verlag, Basel, 1990.

[2] D. Alpay and C. Dubi. On commuting operators solving Gleason's problem. *PAMS*, 133:3285–3293, 2005.

[3] D. Alpay and C. Dubi. A realization theorem for rational functions of several complex variables. *System & Control Letters*, 49:225–229, 2003.

[4] D. Alpay, B. Schneider, M. Shapiro, and D. Volok. Fonctions rationnelles et théorie de la réalisation: le cas hyper-analytique. *Comptes Rendus Mathématiques*, 336:975–980, 2003.

[5] D. Alpay, M. Shapiro, and D Volok. Rational hyperholomorphic functions in $\mathbb{R}^4$. *Journal of Functional Analysis*, 221:122–149, 2005.
Available at http://arxiv.org/pdf/math.FA/0312504.

[6] D. Alpay, M. Shapiro, and D. Volok. Espaces de de Branges Rovnyak: le cas hyper-analytique. *Comptes Rendus Mathématiques*, 338:437–442, 2004.

[7] W. Arveson. The curvature of a Hilbert module over $\mathbf{c}[z_1, \ldots, z_d]$. *Proc. Natl. Acad. Sci. USA*, 96(20):11096–11099 (electronic), 1999.

[8] J.A. Ball, W.S. Li, D. Timotin, and T.T. Trent. A commutant lifting theorem on the polydisc. *Indiana Univ. Math. J.*, 48(2):653–675, 1999.

[9] H. Bart, I. Gohberg, and M. Kaashoek. *Minimal factorization of matrix and operator functions*, volume 1 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1979.

[10] S. Bergman. *Integral operators in the theory of linear partial differential equations.* Second revised printing. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 23. Springer-Verlag New York Inc., New York, 1969.

[11] M.F. Bessmertnyĭ. On realizations of rational matrix functions of several complex variables. In *Interpolation theory, systems theory and related topics (Tel Aviv/Rehovot, 1999)*, volume 134 of *Oper. Theory Adv. Appl.*, pages 157–185. Birkhäuser, Basel, 2002. Translated from the Russian by D. Alpay and V. Katsnelson.

[12] F. Brackx, R. Delanghe, and F. Sommen. *Clifford analysis*, volume 76. Pitman research notes, 1982.

[13] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. II: Partial differential equations.* (Vol. II by R. Courant.). Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962.

[14] M.J. Crowe. *A history of vector analysis.* Dover Books on Advanced Mathematics. Dover Publications Inc., New York, 1985. The evolution of the idea of a vectorial system, Reprint of the 1967 edition.

[15] E. Doubtsov. Minimal solutions of the Gleason problem. *Complex Variables Theory Appl.*, 36(1):27–35, 1998.

[16] S.W. Drury. A generalization of von Neumann's inequality to the complex ball. *Proc. Amer. Math. Soc.*, 68(3):300–304, 1978.

[17] A. Dzhuraev. *Methods of singular integral equations*, volume 60 of *Pitman Monographs and Surveys in Pure and Applied Mathematics.* Longman Scientific & Technical, Harlow, 1992. Translated from the Russian; revised by the author.

[18] E. Fornasini and G. Marchesini. State-space realization theory of two-dimensional filters. *IEEE Trans. Automatic Control*, AC–21(4):484–492, 1976.

[19] B. Fritzsche and B. Kirstein, editors. *Ausgewählte Arbeiten zu den Ursprüngen der Schur-Analysis*, volume 16 of *Teubner-Archiv zur Mathematik.* B.G. Teubner Verlagsgesellschaft, Stuttgart–Leipzig, 1991.

[20] R. Fueter. Die Theorie der regulären Funktionen einer quaternionen Variablen. In *Comptes rendus du congrès international des mathématiciens, Oslo 1936, Tome I*, pages 75–91, 1937.

[21] I. Gohberg, editor. *I. Schur methods in operator theory and signal processing*, volume 18 of *Operator theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1986.

[22] Yu.M. Grigor′ev and V.V. Naumov. Approximation theorems for a Moisil-Theodorescu system. *Sibirsk. Mat. Zh.*, 25(5):9–19, 1984.

[23] K. Gürlebeck and W. Sprössig. *Quaternionic analysis and elliptic boundary value problems*, volume 56 of *Mathematical Research*. Akademie-Verlag, Berlin, 1989.

[24] K. Gürlebeck and W. Sprössig. *Quaternionic and Clifford calculus for physicists and engineers*, volume 1 of *Mathematical methods in practice*. John Wiley and Sons, 1997.

[25] H. Hefer. Zur Funktionentheorie mehrerer Veränderlichen. Über eine Zerlegung analytischer Funktionen und die Weilsche Integraldarstellung. *Math. Ann.*, 122:276–278, 1950.

[26] S.G. Krantz and H.P. Parks. *A primer of real analytic functions*, volume 4 of *Basler Lehrbücher [Basel Textbooks]*. Birkhäuser Verlag, Basel, 1992.

[27] G. Moisil. Sur l'équation $\Delta u = 0$. *C. R. Acad. Sci. Paris Sér. I Math.*, 191:984–986, 1930.

[28] B. Muckenhoupt and E.M. Stein. Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.*, 118:17–92, 1965.

[29] R. Roesser. A discrete state-space model for linear image processing. *IEEE Trans. Automatic Control*, AC–20:1–10, 1975.

[30] M.V. Shapiro. Structure of the quaternionic modules and some properties of the involutive operators. *J. Natur. Geom.*, 1(1):9–37, 1992.

[31] M.V. Shapiro. Some remarks on generalizations of the one-dimensional complex analysis: hypercomplex approach. In *Functional analytic methods in complex analysis and applications to partial differential equations (Trieste, 1993)*, pages 379–401. World Sci. Publishing, River Edge, NJ, 1995.

[32] M.V. Shapiro. On the conjugate harmonic functions of M. Riesz–E. Stein–G. Weiss. In S. Dimiev and K. Sevigawa, editor, *Topics in complex analysis, differential geometry and mathematical physics (St. Konstantin, 1996)*, pages 8–32. World Sci. Publishing, River Edge, NJ, 1997.

[33] M.V. Shapiro and N.L. Vasilevski. Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I. ψ-hyperholomorphic function theory. *Complex Variables Theory Appl.*, 27(1):17–46, 1995.

[34] M.V. Shapiro and N.L. Vasilevski. On the Bergman kernel function in hyperholomorphic analysis. *Acta Appl. Math.*, 46(1):1–27, 1997.

[35] F. Sommen. A product and an exponential function in hypercomplex function theory. *Applicable Anal.*, 12(1):13–26, 1981.

[36] N.N. Tarkhanov. A remark on the Moisil-Teodorescu system. *Sibirsk. Mat. Zh.*, 28(3):208–213, 225, 1987.

[37] H. Woerdeman. Positive Carathéodory interpolation on the polydisc. *Integral Equations Operator Theory*, 42(2):229–242, 2002.

[38] M. Zhdanov. *Integral transforms in geophysics*. Springer-Verlag, Berlin, 1988. Translated from the Russian by Tamara M. Pyankova.

Daniel Alpay
Department of Mathematics
Ben–Gurion University of the Negev
Beer-Sheva 84105
Israel
e-mail: dany@math.bgu.ac.il

Michael Shapiro
Departamento de Matemáticas
E.S.F.M del I.P.N.
07300 México, D.F.
México
e-mail: shapiro@esfm.ipn.mx

Dan Volok
Department of Mathematics
Ben–Gurion University of the Negev
Beer-Sheva 84105
Israel
e-mail: volok@math.bgu.ac.il