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Sturm's Theorems on Zero Sets in Nonlinear Parabolic Equations

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Abstract. We present a survey on applications of Sturm's theorems on zero sets for linear parabolic equations, established in 1836, to various problems including reaction-diffusion theory, curve shortening and mean curvature flows, symplectic geometry, etc. The first Sturm theorem, on nonincrease in time of the number of zeros of solutions to one-dimensional heat equations, is shown to play a crucial part in a variety of existence, uniqueness and asymptotic problems for a wide class of quasilinear and fully nonlinear equations of parabolic type. The survey covers a number of the results obtained in the last twenty-five years and establishes links with earlier ones and those in the ODE area.

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1. Introduction: Sturm's theorems for parabolic equations

In 1836 C. Sturm published two celebrated papers in the first volume of J. Liouville's Journal de Mathématique Pures et Appliquées. The first paper [125] on zeros of solutions $u(x)$ of second-order ordinary differential equations such as

$$u'' + q(x)u = 0, \quad x \in \mathbb{R}, \tag{1.1}$$

very quickly exerted a great influence on the general theory of ODEs. Then and nowadays Sturm's oscillation, comparison and separation theorems can be found in most textbooks on ODEs with various generalizations to other equations and systems of equations. In general, such theorems classify and compare zeros and zero sets $\{x \in \mathbb{R} : u(x) = 0\}$ of different solutions $u_1(x)$ and $u_2(x)$ of (1.1) or solutions of equations with different continuous ordered potentials $q_1(x) \geq q_2(x)$. We refer to other papers of the present volume containing a detailed survey of this classical theory.

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The second paper [126] was devoted to the *evolution* analysis of zeros and zero sets $\{x : u(x,t) = 0\}$ for solutions $u(x,t)$ of partial differential equations of parabolic type, for instance,

$$u_t = u_{xx} + q(x)u, \quad x \in [0, 2\pi], \; t > 0, \tag{1.2}$$

with the same ordinary differential operator as in (1.1) and the Dirichlet boundary condition $u = 0$ at $x = 0$ and $x = 2\pi$ and given smooth initial data at $t = 0$. Two of Sturm's results on PDEs like (1.2) can be stated as follows:

First Sturm Theorem: nonincrease with time of the number of zeros (or sign changes) of solutions;

Second Sturm Theorem: a classification of blow-up self-focusing formations and collapses of multiple zeros.

We will refer to both of Sturm's Theorems together as the *Sturmian argument* on zero set analysis. Most of Sturm's PDE paper [126] was devoted to the second Theorem on striking evolution "dissipativity" properties of zeros of solutions of linear parabolic equations, where a detailed backward-forward continuation analysis of the collapse of multiple zeros of solutions was performed. The first Theorem was formulated as a consequence of the second one (it is a form of the strong Maximum Principle (MP) for parabolic equations). As a by-product of the first Theorem, Sturm presented an evolution proof of bounds on the number on zeros of eigenfunction expansions. For finite Fourier series

$$f(x) = \textstyle\sum_{L \le k \le M} (a_k \cos kx + b_k \sin kx), \quad x \in [0, 2\pi], \tag{1.3}$$

by using the PDE (1.2), $q \equiv 0$ (with periodic boundary conditions), it was proved that $f(x)$ has at least $2L$ and at most $2M$ zeros.[1] Sometimes the lower bound on zeros is referred to as the Hurwitz Theorem, which was better known than the first Sturm PDE Theorem. This Sturm-Hurwitz Theorem is the origin of many striking results, ideas and conjectures in topology of curves and symplectic geometry.

Unlike the classical Sturm theorems on zeros of solutions of second-order ODEs, Sturm's evolution zero set analysis for parabolic PDEs did not attract much attention in the nineteenth century and, in fact, was forgotten for almost a century. It seems that G. Pólya (1933) [112] was the first person in the twentieth century to revive interest in the first Sturm Theorem for the heat equation. (The earlier extension by A. Hurwitz (1903) [71] of Sturm's result on zeros of (1.3) to infinite Fourier series with $M = \infty$ did not use PDEs.) Since the 1930s the Sturmian argument has been rediscovered in part several times. For instance, a key idea of the Lyapunov monotonicity analysis in the famous KPP-problem, by A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov (1937) [82] on the stability of travelling waves (TWs) in reaction-diffusion equations, was based on the first Sturm Theorem in a simple geometric configuration with a single intersection between solutions. This was separately proved there by the Maximum Principle.

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[1] Sturm also presented an ODE proof.

From the 1980s the Sturmian argument for PDEs began to penetrate more and more into the theory of linear and nonlinear parabolic equations and was found to have several fundamental applications. These include asymptotic stability theory for various nonlinear parabolic equations, orbital connections and transversality of stable-unstable manifolds for semilinear parabolic equations such as Morse-Smale systems, unique continuation theory, Floquet bundles and a Poincaré-Bendixson theorem for parabolic equations and problems of symplectic geometry and curve shortening flows. A survey on Sturm's ideas in PDEs will be continued in Section 2, where we present the statements of both of Sturm's Theorems, and in Section 3, where we describe further related results and generalizations achieved in the twentieth century.

2. Sturm's theorems for linear parabolic equations

2.1. First Sturm Theorem: nonincrease of the number of sign changes

Let D and J be open bounded intervals in $\mathbb{R}$. Consider in $S = D \times J$ the linear parabolic equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u. \tag{2.1}$$

Given a constant $\tau \in J$, we denote the parabolic boundary of the domain $S_\tau = S \cap \{t < \tau\}$, i.e., the lateral sides and the bottom of the boundary of S_τ, by ∂S_τ. Given a solution u defined on S_τ, the positive and negative sets of u are defined as follows:

$$U^+ = \{(x,t) \in S_\tau : \ u(x,t) > 0\}, \quad U^- = \{(x,t) \in S_\tau : \ u(x,t) < 0\}. \tag{2.2}$$

A *component* of U^+ (or U^-) is a maximal open connected subset of U^+ (or U^-).

Given a $t \in \overline{J}$, the number (finite or infinite) of components of $\{x \in D : u(x,t) \neq 0\}$ minus one is called the *number of sign changes* of $u(x,t)$ and is denoted by $Z(t,u)$. Alternatively, let K be the supremum over all natural numbers k such that there exist k points from D, $x_1 < x_2 < \cdots < x_k$, satisfying

$$u(x_j,t) \cdot u(x_{j+1},t) < 0 \quad \text{for all } \ j = 1,2,\ldots,k-1,$$

then $Z(t,u) = K - 1$.

Theorem 2.1 (First Sturm Theorem: sign changes). *Let a, b, c be continuous, bounded and $a \geq \mu > 0$ in S for some constant μ. Let $u(x,t)$ be a solution of* (2.1) *in S that is continuous on $\overline{S}$.*

(i) *Suppose that on ∂S_τ there are precisely n (respectively m) disjoint intervals where u is positive (respectively negative). Then U^+ (resp. U^-) has at most n (resp. m) components in S_τ and the closure of each component must intersect ∂S_τ in at least one interval.*

(ii) *The number of sign changes $Z(\tau,u)$ of $u(x,\tau)$ on D is not greater than the number of sign changes of u on ∂S_τ.*

The first Sturm Theorem is formulated on p. 431 in [126]. The present proof of Theorem 2.1 is taken from [118] (similar to that in [105]).

Proof. The proof is based on the strong Maximum Principle.
(i) Let $I \subset \partial S_\tau$ be a maximal interval where $u > 0$. Suppose that two open connected subsets F_1, $F_2 \subset U^+$ intersect ∂S_τ in disjoint open intervals I_1, $I_2 \subset I$. Since u is continuous in $\overline{S}_\tau$, there exists an open set $G \subset U^+$ whose closure in $\overline{S}_\tau$ contains I. Then G must contain points of both F_1 and F_2 so that these must belong to the same open component of U^+. Thus, at most one component of U^+ intersects each of the n open intervals on ∂S_τ where $u > 0$. The same result holds for the components of U^-. Therefore, it suffices to show that every component of U^+ (or U^-) intersects ∂S_τ in one or more intervals.

We can assume that $c \le 0$ in S_τ. Otherwise, we set $u = e^{\lambda t} v$ ($U^\pm$ stay the same for v), where v then solves equation (2.1) with the last coefficient c on the right-hand side replaced by $c - \lambda$ and we can choose the constant $\lambda \ge \sup c$.

Let $F \subset U^+$ be a component in S_τ. Since u is continuous, it must attain a positive maximum on $\overline{F}$. Then $c \le 0$ implies $u_t \le a u_{xx} + b u_x$ in F, and, by continuity, $u = 0$ at any boundary point of F which is interior to S_τ. By the MP, u cannot attain its maximum at an interior point of F or on the line $\{t = \tau\}$. Hence, F must have a boundary point $Q \in \partial S_\tau$ such that $u(Q) > 0$ and by continuity u is positive in an interval of ∂S_τ about Q.
(ii) is a straightforward consequence of (i). □

The first Sturm Theorem is true for wider classes of linear parabolic equations that are sufficiently regular (so the strong MP can be applied). An important example is the radial parabolic equation in $\mathbb{R}^N$ with continuous coefficients and $a \ge \mu > 0$,

$$u_t = a(r,t)\Delta u + b(r,t)u_r + c(r,t)u, \tag{2.3}$$

where $r = |x| \ge 0$ denotes the radial variable and $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}$ is the radial Laplace operator. Bearing in mind that we consider smooth bounded solutions satisfying the symmetry condition at the origin, $u_r(0,t) = 0$ for $t \in J$, the MP applies to equation (2.3) in $S = D \times J$, where $D = \{r < R\}$ is a ball in $\mathbb{R}^N$, and the first Sturm Theorem holds.

2.2. Second Sturm Theorem: formation and collapse of multiple zeros

Results in the class of analytic functions. We consider parabolic equations with analytic coefficients admitting analytic solutions. Then any zero of $u(x,t)$ has finite multiplicity. Under this assumption, the following result is true:

Theorem 2.2 (Second Sturm Theorem: multiple zeros). *Let* $O = (0,0) \in S$ *and* $u \in C^\infty(S) \cap C(\overline{S})$ *be a solution of equation* (2.1) *with* C^∞*-coefficients* a, b, c*, where* $a \ge \mu > 0$ *in* S*. Assume that* $u(x,t)$ *does not change sign on the lateral boundary of* S*, and* $u(x,0)$ *has a zero of order* $m \ge 2$ *at the origin* $x = 0$*, i.e.,*

$$D_x^k u(0,0) = 0 \quad \text{for } k = 0,1,\dots,m-1 \;\text{ and }\; D_x^m u(0,0) = m!A \ne 0. \tag{2.4}$$

Then $Z(t,u)$ decreases at $t=0$, and for any $t_1<0<t_2$ near $t=0$, there holds

$$Z(t_1,u)-Z(t_2,u)\geq \{m,\ \textit{if } m \textit{ is even};\quad m-1,\ \textit{if } m \textit{ is odd}\}. \tag{2.5}$$

In the proof of Theorem 2.2 we will follow Sturm's original computations and analysis in [126], pp. 417–427, which was done for the following semilinear parabolic equation on a bounded interval

$$gu_t=(ku_x)_x-lu,\quad x\in(\mathrm{x},\mathrm{X}),\ t>0, \tag{2.6}$$

with smooth functions g,k and l depending on x and t. The main calculations were performed for g,k,l depending on x only. A comment on p. 431 extends the results to allow dependence on t. Third type (Robin) boundary conditions were incorporated:

$$ku_x-hu=0\quad \text{at } x=\mathrm{x},\qquad ku_x+Hu=0\quad \text{at } x=\mathrm{X}, \tag{2.7}$$

where h,H are constants but also can depend on t, see p. 431. (Zero Dirichlet boundary conditions are also mentioned there.) Sturm's analysis on pp. 428–430 includes the case of multiple zeros occurring at boundary points x or X.

Proof. By Taylor's formula near the origin we have

$$u(x,0)=Ax^m+O(x^{m+1}). \tag{2.8}$$

Using a Taylor expansion in t, we have

$$u(x,t)=u(x,0)+u_t(x,0)t+\tfrac{1}{2!}\,u_{tt}(x,0)t^2+\cdots+\tfrac{1}{n!}\,D_t^n u(x,0)t^n+O(t^{n+1}), \tag{2.9}$$

where $n=m/2$ if m is even and $n=(m-1)/2$ if m is odd. Let us estimate the coefficients. Let $d_j=m!/(m-2j)!$ for $j=0,1,\dots,n$. It follows from the parabolic equation (2.1) and (2.8) that $u_t(x,0)=a(x,0)u_{xx}(x,0)+b(x,0)u_x(x,0)+c(x,0)u(x,0)=a_0Ad_1x^{m-2}+O(x^{m-1})$, where $a_0=a(0,0)$ and $a(x,0)=a_0+O(x)$. Differentiating the equation and using expansion (2.8) again, we obtain, keeping the leading terms only,

$$u_{tt}(x,0)=au_{txx}+\cdots=a_0^2Ad_2x^{m-4}+O(x^{m-3}),$$

and finally $D_t^n u(x,0)=D_t^{n-1}a_0u_{xx}(0,0)+\cdots=a_0^nAd_nx^{m-2n}+O(x^{m-2n+1})$. The Taylor expansion in both independent variables, x and t, takes the form

$$u(x,t)=A(x^m+a_0d_1x^{m-2}t+\tfrac{1}{2!}\,a_0^2d_2x^{m-4}t^2+\cdots+\tfrac{1}{n!}\,a_0^nd_nx^{m-2n}t^n)+O(\cdot) \tag{2.10}$$

with the remainder $O(\cdot)=O(|x|^{m+1}+|x|^{m-1}|t|+\cdots+|x|^{m-2n+1}|t|^n+|t|^{n+1})$.

(i) Backward continuation. Consider the behavior for $t\approx 0^-$. The dimensional structure of the right-hand side of (2.10) suggests rewriting this expansion in terms of the rescaled *Sturm backward continuation variable*

$$z=x/\sqrt{a_0(-t)}\quad \text{for } t<0. \tag{2.11}$$

Substituting $x=z\sqrt{a_0(-t)}$, we obtain that

$$A^{-1}a_0^{-m/2}(-t)^{-m/2}u(x,t)=P_m(z)+O((-t)^{1/2}(1+|z|^{m+1})), \tag{2.12}$$

where $P_m(z) = \sum_{j=0}^{n}(-1)^j \frac{d_j}{j!} z^{m-2j}$. The mth order polynomial $P_m(z)$ is the Hermite polynomial $H_m(z)$ (up to a constant multiplier which we omit in what follows). Each orthogonal polynomial $H_m(z)$ has exactly m simple zeros $\{z_i,\ i = 1,\dots,m\}$ with $H_m'(z_i) \neq 0$. Sturm proved this separately on p. 426. This is the classical theory of orthogonal polynomials, see G. Szegö's book [128], Chapter 6.

A similar expansion for the derivative $u_x(x,t)$ shows that (2.12) can be differentiated in x giving the derivative $P_m'(z)$ in the right-hand side. It follows from the expansions of $u(x,t)$ and $u_x(x,t)$ near the multiple zero that for any $t \approx 0^-$, the solution $u(x,t)$ has m simple zeros $\{x_i(t),\ i = 1,\dots,m\}$, $u_x(x_i(t),t) \neq 0$, with the following asymptotic behavior: $x_i(t) = z_i(-t)^{1/2} + O(-t) \to 0$ as $t \to 0^-$, so that exactly m smooth zero curves intersect each other at the origin $(0,0)$.

(ii) Forward continuation. Following Sturm's analysis, we consider the behavior of the solution $u(x,t)$ as $t \to 0^+$. Introducing the heat kernel rescaled variable of the *forward continuation*

$$z = x/\sqrt{a_0 t} \quad \text{for } t > 0, \tag{2.13}$$

instead of (2.12) we obtain another polynomial on the right-hand side

$$A^{-1} a_0^{-m/2} t^{-m/2} u(x,t) = Q_m(z) + O(t^{1/2}(1 + |z|^{m+1})), \tag{2.14}$$

where $Q_m(z) = \sum_{j=0}^{n} \frac{d_j}{j!} z^{m-2j}$. The mth order polynomial $Q_m(z)$ has positive coefficients. If m is odd, then it is strictly increasing with $Q_m(0) = 0$. If m is even, then it has a single positive minimum at $z = 0$. Therefore, (2.14) implies that for small $t > 0$ on compact subsets $\{|x| \leq ct^{1/2}\}$ with any $c > 0$, the solution $u(x,t)$ has a unique simple zero $\tilde{x}_1(t) = O(t)$ if m is odd, and no zeros if m is even. This is Sturm's analysis on p. 423.

In order to complete the proof, it suffices to observe that if m is even and, say, $A > 0$, by continuity and the strong MP, there exists a small interval $(-\varepsilon, \varepsilon)$ such that $u(x,t)$ becomes strictly positive on $(-\varepsilon, \varepsilon)$ for all small $t > 0$. This means that at least m zero curves disappear at $(0,0)$. If m is odd and $A > 0$, then applying Theorem 2.1 to the domain $S = (-\varepsilon, \varepsilon) \times (0, \varepsilon)$ we have that on $(-\varepsilon, \varepsilon)$ for $t > 0$ there exists a unique continuous curve of simple zeros $\tilde{x}_1(t)$ starting from $(0,0)$. In this case at least $m - 1$ zero curves disappear at the origin as $t \to 0^-$. □

Such a complete analysis of the evolution of multiple zeros in 1D applies to more general parabolic equations. In particular, in N-dimensional geometry similar results are true for radial solutions $u = u(r,t)$ of parabolic equations (2.3) with analytic coefficients; see the next section.

Sturm's proof, consisting of two parts (i) and (ii), exhibits typical features of the asymptotic evolution analysis for general linear uniformly parabolic equations:

(i) A finite-time formation of a multiple zero as $t \to 0^-$ as a *singularity formation* (single point blow-up self-focusing of zero curves);
(ii) Disappearance of multiple zeros at $t = 0^+$, i.e., instantaneous *collapse of a singularity* and a unique continuation of the solution beyond the singularity.

Regarding this part of Sturm's analysis, we present the result separately as follows.

Corollary 2.1. *Under the assumptions of Theorem 2.2, the following results hold:*

(i) *As $t \to 0^-$, the rescaled solution converges uniformly on any compact subset $\{|z| \le$ const.$\}$ to the mth order Hermite polynomial with finite oscillations:*

$$A^{-1} a_0^{-m/2} (-t)^{-m/2} u(x,t) \to H_m(z). \tag{2.15}$$

(ii) *As $t \to 0^+$, the rescaled solution converges uniformly on compact subsets to the non-oscillating mth order polynomial:*

$$A^{-1} a_0^{-m/2} t^{-m/2} u(x,t) \to Q_m(z). \tag{2.16}$$

Phenomena of singularity blow-up formation, collapse and proper solution extensions beyond singularities are important subjects of general PDE theory. In applications to semilinear and quasilinear parabolic equations of reaction-diffusion type, the perturbation techniques for infinite-dimensional dynamical systems plays a key role; see various examples in [58]. We briefly comment on Sturm's analysis using the perturbation theory of linear operators.

(i) Formation of multiple zeros: backward continuation. Using Sturm's backward rescaled variable (2.11), we introduce the rescaled solution

$$u(x,t) = \theta(z,\tau), \quad z = x/\sqrt{a_0(-t)}, \tag{2.17}$$

where $\tau = -\ln(-t) \to +\infty$ as $t \to 0^-$ is the new time variable. Substituting (2.17) into equation (2.1) yields the rescaled equation

$$\theta_\tau = \mathbf{B}\,\theta + \mathbf{C}(\tau)\theta, \tag{2.18}$$

where $\mathbf{B}$ is the linear operator

$$\mathbf{B} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{1}{2} z \frac{\mathrm{d}}{\mathrm{d}z} \equiv \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}z}\left(\rho \frac{\mathrm{d}}{\mathrm{d}z}\right), \quad \text{where } \rho(z) = e^{-z^2/4}, \tag{2.19}$$

which is symmetric in $L^2_\rho(\mathbb{R}^N)$ (see below). The non-autonomous perturbation in (2.18) has the form

$$\mathbf{C}(\tau)\theta = \left(\frac{a - a_0}{a_0}\right) \theta_{zz} + e^{-\tau/2} \frac{b}{\sqrt{a_0}}\, \theta_z + e^{-\tau} c\, \theta,$$

where for the regular coefficient a, $(a(x,t) - a_0)/a_0 \equiv (a(z[a_0(-t)]^{1/2}, t) - a_0)/a_0 = O(e^{-\tau/2})$. This means that for smooth solutions, the perturbation

$$\mathbf{C}(\tau)\theta = e^{-\tau/2}[\theta_{zz}\, O(1) + b a_0^{-1/2} \theta_z + e^{-\tau/2} c\, \theta]$$

is exponentially small as $\tau \to \infty$. Equation (2.18) is an exponentially small perturbation of the autonomous equation

$$\theta_\tau = \mathbf{B}\,\theta. \tag{2.20}$$

The operator $\mathbf{B}$ is known to be self-adjoint in the weighted space $L^2_\rho(\mathbb{R})$ with the inner product $(v,w)_\rho = \int_{-\infty}^{\infty} \rho(z) v(z) w(z) dz$. Its domain $\mathcal{D}(\mathbf{B}) = H^2_\rho(\mathbb{R})$ is a Hilbert space of functions v satisfying $v, v', v'' \in L^2_{\mathrm{loc}}(\mathbb{R})$ with the inner product

$\langle v, w\rangle_\rho = (v, w)_\rho + (v', w')_\rho + (v'', w'')_\rho$ and the induced norm $\|v\|_\rho^2 = \langle v, v\rangle_\rho$. Moreover, $\mathbf{B}$ has compact resolvent and its spectrum only consists of eigenvalues:

$$\sigma(\mathbf{B}) = \{\lambda_k = -\tfrac{k}{2},\ \ k = 0, 1, \dots\}.$$

The eigenfunctions are orthonormal Hermite polynomials $\tilde{H}_k(z) = c_k H_k(z)$, c_k being normalization constants. These are classical results of the theory of linear self-adjoint operators in Hilbert spaces. We refer to the first chapters of the book [22] (see p. 48 on Hermite polynomials in $\mathbb{R}^N$). Using eigenfunction expansions and semigroup estimates (see Section 3) yields that the exponentially perturbed dynamical system (2.18) on $L^2_\rho(\mathbb{R}^N)$ admits a discrete subset of asymptotic patterns. These coincide with those for the unperturbed equation (2.20) exhibiting the asymptotic behavior on tangent stable ($\lambda_m < 0$) eigenspaces of $\mathbf{B}$. Hence (2.15) holds. As $\tau \to \infty$, uniformly on compact subsets we have

$$\theta(z, \tau) = Ce^{\lambda_m \tau} H_m(z) + O(e^{\lambda_{m+1}\tau}) \quad \text{with a constant } C \neq 0. \tag{2.21}$$

(ii) Collapse of multiple zero on the spatial structure of adjoint polynomials: forward continuation. For $t > 0$, we use the forward rescaled variable (2.13). Similarly, we deduce that the rescaled function $u(x, t) = g(z, s)$, where the time variable is $s = \ln t \to -\infty$ as $t \to 0^+$, solves the exponentially perturbed equation as $s \to -\infty$

$$g_s = (\mathbf{B}^* - \tfrac{1}{2}I)g + \mathbf{C}(s)g, \tag{2.22}$$

where I denotes identity and $\mathbf{B}^*$ is the adjoint differential operator

$$\mathbf{B}^* = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{1}{2}z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}I \equiv \frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}z}\left(\nu\frac{\mathrm{d}}{\mathrm{d}z}\right) + \frac{1}{2}I \quad \text{with weight } \nu(z) = e^{z^2/4}.$$

As in the backward analysis, the perturbation term $\mathbf{C}(s)g = O(e^{s/2}) \to 0$ as $s \to -\infty$ and is exponentially small for smooth solutions on compact subsets. $\mathbf{B}^*$ is self-adjoint in $L^2_\nu(\mathbb{R})$, $\mathcal{D}(\mathbf{B}^*) = H^2_\nu(\mathbb{R})$, with the point spectrum $\sigma(\mathbf{B}^*) = \sigma(\mathbf{B})$ and a complete set of orthonormal eigenfunctions.

Unlike the phenomenon of the *evolution* blow-up formation of multiple zeros, in the asymptotic analysis as $s \to -\infty$ spectral properties and eigenfunctions of $\mathbf{B}^*$ play no role. The limit $t \to 0^+$ corresponds to the collapse of the *initial singularity* created by the preceding singularity formation as $t \to 0^-$. The behavior of $u(x, t)$ as $t \to 0^+$ is uniquely determined by the initial data $u(x, 0)$. Consider (2.21) for $|z| \gg 1$. Since $P_m(z) \equiv H_m(z) = z^m + \cdots$ as $z \to \infty$, it can be shown (a compactness argument is necessary at this step to extend the behavior from compact subsets $\{|z| \leq c\}$ to $\{0 < |x| \ll 1\}$) that passing to the limit $t \to 0^-$ gives $u(x, 0)$ as follows:

$$u(x, t) = C(-t)^{-\lambda_m} x^m a_0^{-m/2} (-t)^{-m/2} + \cdots \to C a_0^{-m/2} x^m + \cdots. \tag{2.23}$$

The solution $g(z, s)$ of the rescaled equation (2.22) with initial data calculated in (2.23) has the expansion

$$g(z, s) = \tilde{C}e^{-\lambda_m s} Q_m(z) + \cdots, \quad \tilde{C} \neq 0, \tag{2.24}$$

where Q_m is the polynomial solution of the linear equation $(\mathbf{B}^* - \frac{1}{2}I)Q_m = \frac{m}{2}Q_m$. We thus arrive at the linear problem for the "adjoint" polynomials $\{Q_m\}$. Notice that these have nothing to do with the orthogonal subset of eigenfunctions $\{\exp(-z^2/4)\,H_m(z)\}$ of the adjoint operator $\mathbf{B}^*$. Moreover $Q_m \notin L^2_\nu(\mathbb{R})$. In order to match (2.24) and the initial condition (2.23), by a similar local extension to $\{0 < |x| \ll 1\}$ we have that

$$g(z,s) = \tilde{C}t^{-\lambda_m}x^m a_0^{-m/2}t^{-m/2} + \cdots \to \tilde{C}a_0^{-m/2}x^m + \cdots \quad \text{as } t \to 0^+.$$

By matching with (2.23), this uniquely determines the constant $\tilde{C} = C$ in (2.24) and completes the asymptotic analysis of both the backward and forward evolution of multiple zeros.

Results in classes of finite regularity. Fix finite $T > 0$ and let $J = (0,T)$. If $u(x,t) \not\equiv 0$ is a solution, analytic in x, of the linear parabolic equation (2.1) with analytic coefficients a, b, c, then for any $t \in (0,T)$, all the zeros of $u(x,t)$ are isolated and hence the number of sign changes $Z(t,u)$ is finite even if $Z(0,u) = \infty$. A similar result holds in classes of solutions and equations of finite regularity. We present without proofs two results by S. Angenent [7]; more references are given in Section 3. We begin with initial-boundary value problems.

Theorem 2.3. *Let u be a bounded solution of (2.1) in $S = D \times (0,T)$ which does not change sign on the lateral boundary of S. Assume that the coefficients a, b and c of the equation are such that*

$$a, a^{-1}, a_x, a_{xx}, b, b_t, b_x, c \in L^\infty(S).$$

Then the number of sign changes of $u(\cdot,t)$ satisfies:

(i) *$Z(t,u)$ is finite and nonincreasing on $(0,T)$;*

(ii) *If $x = x_0 \in D$ is a multiple zero of $u(x,t_0)$ for some $t_0 \in (0,T)$, then for all $0 < t_1 < t_0 < t_2 < T$ the strict inequality $Z(t_1,u) > Z(t_2,u)$ holds, so that $Z(t,u)$ is strictly decreasing at $t = t_0$.*

As a consequence, any global solution $u(x,t)$ defined in $S = D \times \mathbb{R}_+$ has only simple zeros for all $t \gg 1$. A similar result is valid for parabolic equations in unbounded domains if we restrict the analysis to classes of functions with a fixed growth at infinity, similar to Tikhonov's classes of uniqueness. Let $D = \mathbb{R}$, and consider the following linear parabolic equation:

$$u_t = u_{xx} + q(x,t)u \quad \text{in } S = \mathbb{R} \times (0,T). \tag{2.25}$$

Theorem 2.4. *Let $q \in L^\infty(S)$, and let $u(x,t)$ be a solution of (2.25) in the class $\{|u(x,t)| \le Ae^{Bx^2}$ in $S\}$ for some positive constants A and B. Then for each $t \in (0,T)$, the zero set of the solution $\{x \in \mathbb{R} : u(x,t) = 0\}$ is a discrete subset of $\mathbb{R}$.*

As a direct consequence of this we have that if $x = \pm\infty$ are not accumulation points of zeros of $u(x,0)$, then statements (i) and (ii) of Theorem 2.3 hold. Theorem 2.4 is true for more general equations like (2.1) in unbounded domains in suitable classes of uniqueness. Equation (2.1) can be reduced to (2.25) by the Liouville transformation. Using the new spatial coordinate $y = \int_0^x (a(s,t))^{-1/2}\, ds$, we have that $u = u(y,t)$ satisfies the equation

$$u_t = u_{yy} + \tilde{b}(y,t)u_y + \tilde{c}(y,t)u.$$

Substituting $v(y,t) = \exp\{\frac{1}{2}\int_0^y \tilde{b}(s,t)\, ds\}u(y,t)$, yields equation (2.25) for $v(y,t)$ with a potential $\tilde{q}(y,t)$. Checking necessary properties of $\tilde{q}(y,t)$ one deduces that Sturm's results are valid in the corresponding uniqueness classes.

3. Survey on Sturm's theorems and ideas in parabolic PDEs

We begin our survey with those ODE results that fall into the scope of the PDE theory or can admit a PDE treatment or proof. The rest is devoted to applications of Sturm's Theorems in areas where parabolic PDEs occur.

3.1. On some ODE results

Classical Sturm results on zeros for a single second-order ODE like

$$y'' + q(t)y = 0, \quad t \in (0, 2\pi), \tag{3.1}$$

can be stated in a topological form describing rotations in the phase space of equations (this form is convenient for extensions to higher-order equations). Let

$$Y(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \quad \text{satisfying } Y(0) = E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be a matrix solution of (3.1), where $y_1(t)$ and $y_2(t)$ are linearly independent solutions. Then the vector $z(t) = y_1(t) + iy_2(t)$ moves counterclockwise in the complex plane. Indeed, since by construction the Wronskian $W(y_1, y_2)(t) = \det Y(t) \equiv 1$, we have that $\arg z(t) = \tan^{-1}(y_2(t)/y_1(t))$ satisfies $\frac{d}{dt}\arg z = W(y_1, y_2)/(y_1^2 + y_2^2) = 1/(y_1^2 + y_2^2) > 0$. Sturm's theorems follow from this monotonicity property.

The first generalizations of Sturm's theorems to the case of vector-valued operators and to systems (3.1) with symmetric matrices $q(t)$ are due to M. Morse (1930) [101], [102], where variational methods are applied. Oscillatory theorems for general canonical systems of $2k$th order were first established by V.B. Lidskii (1955) [88] for the equation

$$y' = IH(t)y, \quad I = \begin{pmatrix} 0 & E_k \\ -E_k & 0 \end{pmatrix},$$

where E_k is the $k \times k$ identity matrix and $H(t)$ is a $2k \times 2k$ real continuous symmetric matrix (the Hamiltonian). We present brief comments on these results. Let $Y(t)$ with $Y(0) = E_{2k}$ be a matrix solution. Then $Y(t)$ is symplectic: $Y^*IY \equiv I$. Denote

$$H(t) = \begin{pmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{pmatrix} \quad \text{and} \quad Y(t) = \begin{pmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{pmatrix},$$

where $h_{ij}(t)$ and $y_{ij}(t)$ are $k \times k$ blocks. Consider the non-singular matrix $z(t) = y_{11}(t) + iy_{12}(t)$ (cf. the case $k = 1$ above), and set $u(t) = (\overline{z}(t))^{-1}z(t)$. Then $u(t)$

is unitary and symplectic. The alternation theorem of Lidskii is as follows. Let $h_{22}(t) > 0$ (for (3.1) with $k = 1$, $h_{22} \equiv 1$). Then the eigenvalues $\rho_1(t), \ldots, \rho_k(t)$ of $u(t)$ move counterclockwise around the unit circle: $\frac{d}{dt}\arg\rho_s(t) > 0$ for $s = 1, \ldots, k$. For $\rho_s(t) = -1$ (resp., $\rho_s(t) = +1$) the matrix $u(t)$ has the same zero subspace as $y_{11}(t)$ (resp., $y_{12}(t)$), i.e., the "zeros" of the matrices $y_{11}(t)$ and $y_{12}(t)$ alternate. Lidskii also proved an analogue of the Sturm comparison theorem. Consider two canonical systems

$$Y_1' = IH_1(t)Y_1 \quad \text{and} \quad Y_2' = IH_2(t)Y_2, \quad \text{where } H_1(t) > H_2(t).$$

Then specially enumerated eigenvalues $\rho_s^{(1)}(t)$ and $\rho_s^{(2)}(t)$ of the unitary matrices $u^{(1)}(t)$ and $u^{(2)}(t)$ satisfy $\arg\rho_s^{(1)}(t) > \arg\rho_s^{(2)}(t)$, $s = 1, \ldots, k$, i.e., $\rho_s^{(1)}(t)$ moves "ahead" of $\rho_s^{(2)}(t)$.

Variational approaches to Sturm's theorems for self-adjoint linear $2k$th order systems were also developed by R. Bott (1959) [24] and by H.H. Edwards (1964) [39]. (See the books [115] and [20] for a detailed presentation.) These results were related to the Maslov index [95]. In 1985 V.I. Arnold [14] characterized this as follows: "... numerous authors writing on the Maslov index, symplectic geometry, geometric quantization, Lagrangian analysis, etc., starting with [13], have not noticed the earlier works by Lidskii [88], as well as the earlier works of Bott [24] and Edwards [39], in which a Hermitian version of the theory of the Maslov index and Sturm intersections were constructed."

A survey of earlier results concerning distribution and alternation of zeros for nth order linear ODEs can also be found in [87], where, as well as in the books mentioned above, various links to other related subjects are described in detail. These include S.A. Chaplygin's comparison theorem (1932) [30] closely connected with the theory of positive operators, W.A. Markov's theorem (1916) [94] on the conservation of the alternation of zeros of polynomials under differentiation, C. de la Vallée-Poussin's theorem (1929) [38] and G. Pólya's (1924) [111] criterion on non-oscillation (the first non-oscillation test of best-possible character is due to N.E. Zhukovskii (1892) [136]), F.R. Gantmakher (1936) [59] and M.G. Krein's (1939) [83] theory of oscillating kernels [60] (a direction originated with O.D. Kellogg's work (1922) [78] on symmetric kernels), S.N. Bernstein results (1938) [21] on connections between Chebyshev and Cartesian systems, etc. See also Hinton's survey [69].

Sturmian methods for ODEs can be applied to investigations in the complex plane, see [68], Chapter 8. The classical Sturm comparison theorem for ODEs admits special extensions to linear and quasilinear elliptic and parabolic PDEs, see first results in [108], the book [127] and [2], as well as to ODEs in Hilbert spaces [75]. More recent extensions of Sturm's comparison theorems to quasilinear elliptic equations can be found in [3], [4], where extra references are available.

Sturm's Theorem on the number of distinct real roots of polynomials by computing the number of sign changes in Sturm sequences (1835) [124] is well known in algebra, see, e.g., [86] and [23]. In constructing Sturm sequences the first step is differentiation, establishing a link to ODEs (Sturm's comparison or oscillation theorems).

As with respect to ODEs, Sturm's ideas have applications in the classical problem on zeros of complete Abelian integrals defined by means of a planar Hamiltonian flow, which is closely related to Hilbert's 16th problem (the so-called weakened, infinitesimal or tangential Hilbert problem). Abelian integrals were known to satisfy a system of Picard-Fuchs ODEs [61], see also [72] for further references. This is a part of a general problem on zeros of Pfaffian functions and the fewnomials theory, [79], [80], where the eventual reduction to polynomial structures is used. In particular, algorithmic consistency problems for systems of Pfaffian equations and inequalities occur (with applications to computer sciences); see [50] and references therein.

Let us return to the Sturm-Hurwitz theorem establishing that the finite Fourier series (1.3) has at least $2L$ and at most $2M$ zeros. On pp. 436–444 of the PDE paper [126], Sturm presented an ODE proof of the result. Sturm's ODE proof, as well as Liouville's one in [89] published in the same volume, exhibit certain features of a discrete evolution analysis (to be compared with Sturm's PDE proof via parabolic evolution equation with continuous time variable). A. Hurwitz (1903) [71] extended this result to Fourier series with $M = \infty$.

Further extension is due to O.D. Kellog (1916) [77] who proved oscillation theorems for linear combinations of real continuous functions $\phi_0(x), \phi_1(x), \dots, \phi_n(x)$ that are orthonormal in $L^2((0,1))$. These are not eigenfunctions of a Sturm-Liouville problem. The main assumption is as follows (we keep the original notation). For any $n \geq 1$, let the determinants

$$D(x_0, x_1, \dots, x_n) = \begin{vmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_n(x_1) \\ \dots & \dots & \dots & \dots \\ \phi_0(x_n) & \phi_1(x_n) & \dots & \phi_n(x_n) \end{vmatrix}$$

be positive for any $0 < x_0 < x_1 < \dots < x_n < 1$ ($D_0(x_0)$ being understood as $\phi_0(x_0)$). Let

$$\Phi_{m,n}(x) = c_m \phi_m(x) + \dots + c_n \phi_n(x).$$

Then, among other results, it is established that:

(i) $\Phi_{0,n}(x)$ cannot vanish at $n+1$ distinct points in $(0,1)$ without vanishing identically;
(ii) $\phi_n(x)$ vanishes exactly n times and changes sign at each zero;
(iii) every continuous function $\psi(x)$ orthogonal to $\phi_0(x), \dots, \phi_n(x)$ changes sign at least $n+1$ times;
(iv) $\Phi_{m,n}(x)$ changes sign at least m times and at most n times.

The infinitesimal version of the discriminants with $x_{k+1} - x_k \to 0$, $k = 0, 1, \dots, n-1$, defines the Wronskians of the given functions. Hence some of the assumptions are valid for eigenfunctions of regular Sturm-Liouville problems. On the other hand, Kellogg's results do not cover those of Sturm, see p. 5 in [77].

The Sturm-Hurwitz Theorem plays a fundamental role in topological problems in wave propagation theory (topology of caustics and wave fronts), the geometry of plane and spherical curves and in general symplectic geometry and topology, see [14], [16], [17], [19] and references therein. Alternating, oscillating and non-oscillating Sturm theorems have multi-dimensional symplectic analogues and describe rotation of a Lagrangian subspace of the phase space [14]. For instance, the Sturm-Hurwitz theorem proves a generalization [129] of the classical four vertex theorem by S. Mukhopadyaya [103] and A. Kneser [81] asserting that a plane closed non-self-intersecting curve has at least four vertices (critical points of the curvature). It is pointed out in [17] that the same minimal number occurs in:

(i) theorems on four cusps of general caustics on every convex surface of positive curvature (the related conjecture goes back to C.G.J. Jacobi (1884) [74]),
(ii) four cusps of the envelope of the family of perturbed Larmor orbits of given energy,
(iii) the tennis-ball theorem (a closed curve on the sphere without self-intersections, a smooth embedding $S^1 \to S^2$, dividing the sphere into two parts of equal area, has at least four points of spherical inflection with zero curvature),
(iv) the four equilibrium points theorem,
(v) the four flattening points theorem for perturbed convex curves of positive curvature on a plane lying in three-dimensional space, etc.

Infinitesimal versions of such topological theorems (for infinitely small perturbations of curves) follow from the Sturm-Hurwitz theorem. For finite perturbations, some of these results can be proved by means of evolution Sturm theorems on zeros for parabolic PDEs to be discussed later on.

Half of Arnold's third lecture in the Fields Institute (1997) [18] was devoted to Sturm's theory on Fourier series, which "provides one of the manifestations of the general principle of economy in algebraic geometry" (related to Arnold's conjecture (1965) and the symplectification of topology). In particular, the Morse inequality (in the simplest version it says that the number of critical points of functions on the circle is at least 2) is the Sturm-Hurwitz theorem with $L = 1$.

The Sturm-Hurwitz theorem was first proved by the PDE method [126], pp. 431–436, in the general form including any (finite) series composed from eigenfunctions of a Sturm-Liouville problem. These extensions of Sturm's ideas have many other applications to be discussed below.

Extensions of Sturm's results on zeros (nodal sets) of linear combinations of eigenfunctions to standard self-adjoint elliptic operators (e.g., the Laplacian Δ) in bounded smooth domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$, are unknown; see [17] and [18]. In particular, the so-called Herrmann theorem announced in [37], p. 454: a linear combination of the first n eigenfunctions divides the domain, by means of its nodes (piecewise smooth nodal surfaces), into not more than n subdomains, fails to hold for the spherical Laplacian [18]. Courant's Theorem on p. 452 asserts that the nodes of the nth eigenfunction divide the domain into no more than n subdomains.

In dimensions $N \geq 2$, given a linear combination $f(x)$ of eigenfunctions of Δ, the structure of the nodal set itself $\mathcal{N}(f) = \{x \in \Omega : f(x) = 0\}$ is not sufficient to define a kind of a Sturmian "index" of the surface $z = f(x)$, similar to the number of zeros in 1D, which can inherit a certain numerical property (say, a lower bound) from the lowest harmonic of the series. Such an index should depend on global properties of $f(x)$ at all points $x \in \Omega$ including those far away from $\mathcal{N}(f)$. It seems reasonable that for a proper definition of a Sturmian index, it is necessary to control the intersections of the graph of the function f with the graphs of the functions in the finite-dimensional set $B = \{V_\nu(x)\}$ containing functions associated with the operator Δ. Roughly speaking, this would mean that such a "local" characteristic as the number of zeros of $f(x)$ on an interval from $\mathbb{R}$ cannot work in $\mathbb{R}^N$, where any possible nonincreasing property of, say, the number of maximal connected subdomains of the positivity subset $\{f(x) > 0\}$ should include some global properties of the function formulated in an unknown way. In any case, a proper definition of Sturmian index of surfaces governed by parabolic equations in $\mathbb{R}^N$ is not expected to admit a simple formulation or such easy and effective applications as it has in the 1D case.

3.2. Parabolic PDEs and Sturm's theorems

The Sturmian argument for 1D parabolic equations turns out to be an extremely effective technique in the study of different aspects of the theory of nonlinear parabolic equations. In the twentieth century the argument was partially and independently rediscovered several times. We will mention some of the papers published at least twenty years ago, but of course there are many other interesting and important papers published more recently, which are not referred to here.

G. Pólya (1933) [112] paid special attention to Sturm's zero set properties of periodic solutions to the heat equation. He studied the number of "Nullstellen" of $u(x,t)$, i.e., the number of $x \in [0, 2\pi]$ such that $u(x,t) = 0$, on the basis of Sturm's approach with a reference to [126]. Radial and cylindrical solutions were considered and zero properties of convolution integrals were also studied.

The celebrated KPP-paper (1937) [82] was devoted to the stability analysis of the minimal travelling wave (TW) for a semilinear heat equation

$$u_t = u_{xx} + f(u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

with the typical nonlinearity $f(u) = u(1-u)$. There the construction of a geometric Lyapunov function in Theorem 11 was based on the following intersection comparison argument: the initial 1-step function $u_0(x) = 1$ for $x > 0$ and 0 for $x \leq 0$ intersects any smooth travelling wave profile exactly at a single point and there exists a unique intersection curve for $t > 0$. In our notation this means that the number of intersections $\mathrm{Int}(t, V) \equiv 1$ for any TW $V(x,t) = g(x - \lambda_0 t + a)$ and any $t > 0$, where $\lambda_0 > 0$ is the minimal speed. In general, the number of intersections can be treated as a discrete nonincreasing Lyapunov function. On the other hand, it gives a standard monotone Lyapunov function: on any fixed level $\{u(x,t) = c \in (0,1)\}$ the derivative $u_x(x,t) < 0$ is monotone increasing in

t and bounded above. Then passage to the limit $t \to \infty$ establishes the convergence to the minimal TW profile in the hodograph plane $\{u, u_x\}$ or in the moving coordinate system in the $\{x, u\}$-plane.

K. Nickel's paper (1962) [105] (see also [106]) established nonincrease of the number of sign changes of solutions of parabolic equations (more precisely, of the number of relative maxima of a solution profile, i.e., the number of zeros of the derivative $u_x(x, t)$). Nickel's results are explained in detail relative to general fully nonlinear parabolic equations in W. Walter's books [134] and [133], Section 27. R.M. Redheffer and W. Walter (1974) [114] extended such results to more general classes of equations. For particular linear parabolic equations in $\mathbb{R}$, these results were proved by S. Karlin (1964) [76], whose analysis was based on ideas of total positivity of Green's functions and applied to Brownian motion processes. Related questions and techniques were discussed by I.K. Ivanov (1965) [73] (the number of changes of sign was considered), by E.K. Godunova and V.I. Levin (1966) [62] (a proof of existence of a single maximum was based on the theory of probabilistic distributions; eventual single maximum distribution and eventual concavity of solutions were also established) and by E.M. Landis (1966) [85] (properties of evolution of level sets for (2.1) were investigated). D.H. Sattinger's results (1969) [118] on sign changes for linear parabolic equations are similar to those obtained by Nickel and Walter. Observe that in the proof of Theorem 7 on exponential decay of total variation, Sattinger uses a reflection technique and studies zeros of the differences $u(x, t)$ and the reflected solution $u(2l - x, t)$, see p. 88 in [118]. Such a combination of Sturm's theorems and A.D. Aleksandrov's Reflection Principle and ideas (1960) [1] later became a powerful tool in the asymptotic theory for nonlinear singular parabolic equations. Papers by A.N. Stokes (1977) [122] and [123] used the nonincrease of zero number with application to stability analysis of travelling waves. Here the basic idea of proving a Lyapunov monotonicity property in the hodograph plane is essentially the same as in the KPP-analysis [82]. A general stability analysis of TWs in analytic semilinear parabolic equations via zero set properties was performed in [12].

H. Matano (1978) [96] proved the first Sturm Theorem and applied it to establishing that the ω-limit set of any bounded solution to a semilinear parabolic equation $u_t = (a(x)u_x)_x + f(x, u)$ on $(0, L) \times \mathbb{R}_+$, $a \geq a_0 > 0$, with smooth coefficients and Robin boundary conditions contains at most one stationary point. At that time such a result was already known [135] for smooth uniformly parabolic equations $u_t = a(x, u, u_x)u_{xx} + b(x, u, u_x)$ with general nonlinear boundary conditions. It was proved by constructing a standard (integral) Lyapunov functional by the method of characteristics, a fruitful idea which applies to 1D quasilinear parabolic equations. The geometric proof by Matano is more general and can be applied to fully nonlinear parabolic equations

$$u_t = F(x, u, u_x, u_{xx}). \tag{3.2}$$

More detailed results related to the first Sturm Theorem were published in [97]. A finite difference approach to some of these Sturmian properties was developed earlier by M. Tabata (1980) [130]. An application of intersection comparison to blow-up solutions of quasilinear parabolic equations $u_t = (k(u)u_x)_x + Q(u)$ was given in [52].

Computations similar to those of Sturm in the proof of Theorem 2.2 in Section 2 can be found in [12], Section 5. For radial equations (2.3) with $N > 1$ such computations for $t < 0$ lead to Laguerre polynomials $L_m^\gamma(z)$ of order $\gamma = N/2$, see Section 3 in [8]. Perturbation techniques for the operator (2.19) were developed in [65], [7], [32]. Sturm's backward parabolic rescaling with $z = x/(-t)^{1/2}$ plays an important role in continuation theorems and topology of nodal sets for linear parabolic equations in $\mathbb{R}^N$ [32]. A weak form of the continuation analysis [113] based on a monotonicity formula and weighted inequalities (this idea goes back to T. Carleman (1939) [29] with applications to elliptic equations), which are convolutions with the backward heat kernel, uses the same Sturm backward variable.

The evolution proof of the Sturm-Hurwitz Theorem on zeros of (finite) linear combinations of eigenfunctions $\{V_k(x),\ k = 1, 2, \ldots\}$, where each V_k has exactly $k - 1$ simple transversal zeros, of a Sturm-Liouville operator given by (2.6), (2.7),

$$Y(x) = C_i V_i(x) + C_{i+1} V_{i+1}(x) + \cdots + C_p V_p(x)$$

is given on pp. 431-444 in [126] and is as follows (we keep the original notation). Consider the solution

$$u(x,t) = C_i V_i(x) e^{-\rho_i t} + C_{i+1} V_{i+1(x)} e^{-\rho_{i+1} t} + \cdots + C_p V_p(x) e^{-\rho_p t} \tag{3.3}$$

of the parabolic equation (2.6) with $u(x, 0) \equiv Y(x)$, where the sequence of eigenvalues $\{-\rho_k\}$ is strictly decreasing. Then for $t \gg 1$, the first harmonic is dominant and hence $u(x,t)$ has exactly $i - 1$ zeros. Since the number of zeros of $u(x,t)$ does not increase, $u(x,t)$ has at least $i-1$ zeros for all $t \in \mathbb{R}$, and hence at $t = 0$. On the other hand, for $t \ll -1$ the last harmonic in (3.3) is dominant, $u(x,t)$ has exactly $p-1$ zeros, so that by Sturm's Theorem, $u(x,t)$ has at most $p-1$ zeros for all $t \in \mathbb{R}$.

On p. 436 Sturm compares his proof with that by J. Liouville [89] "... without using consideration of the auxiliary variable t ..." (by means of an ODE argument). In Section XXVI Sturm presents his own ODE proof. Corollary 2.1 is a paraphrase of Sturm's calculations. The proof of Theorems 2.3 and 2.4 are given in [7]. Finiteness of $Z(t,u)$ on $(0,1)$ for $t > 0$ was also established in [84] for coefficients $a \in H^1$, $b \in W^{1,\infty}$ and $c \in L^\infty$ depending on x only. The second Sturm Theorem on formation of multiple zeros remains valid for $W^{2,1}_{p,\mathrm{loc}}$ solutions ($p > 1$) from Tikhonov's uniqueness class for linear uniformly parabolic equations in $\mathbb{R}^N$ with bounded coefficients [32] (the proof uses Sturmian backward rescaling). The analytic case was treated in [12]. Eventual simplicity of zeros was first observed in [26].

An evolution approach to connections of equilibria for semilinear parabolic equations was introduced by D. Henry [65], where such a time-dependent Sturm-Liouville theory was rigorously established (including completeness of asymptotic limits in Theorem 4 proved by Agmon's estimates). This theory was used in completing the proof that, under some hypotheses, a general semilinear parabolic equation $u_t = u_{xx} + f(x, u, u_x)$ in $(0, 1) \times \mathbb{R}_+$, with Dirichlet or nonlinear boundary conditions, represents a Morse-Smale system. It is established that given a heteroclinic connection $\bar{u}(x, t)$ of two hyperbolic (linearly nondegenerate) equilibria $\phi_\pm$, $u(x, -\infty) = \phi_-(x)$ and $u(x, +\infty) = \phi_+(x)$, the stable manifold $W^s(\phi_+)$ and the unstable one $W^u(\phi_-)$ meet transversally at $\bar{u}(\cdot, t)$ for each t. See also [6] for the case $f = f(x, u) \in C^2$. This transversality result was used in [65] to describe all connecting orbits between equilibria for the Chafee-Infante problem with $f = f(u)$, $f(0) = 0$. For earlier results on connections for parabolic equations see [64] and [25]. For more general $f \in C^2$ such connections were established in [27]. See also the survey [44].

A spectrum of Hermite polynomials occurred in the zero set analysis by D. Henry [65] and S.B. Angenent [7]. Zero set results played a role in the analyticity study of solutions of the porous medium equation (PME) [8]. A few years after papers [65], [6] and [7] on parabolic Morse-Smale systems, the same linearized operators, with eigenfunctions composed from Hermite polynomials, were obtained in the center and stable manifold behavior in the study of blow-up solutions of the semilinear parabolic equations from combustion theory $u_t = \Delta u + u^p$, $p > 1$ and $u_t = \Delta u + e^u$ (the nonstationary Frank-Kamenetskii equation), see [132], [47], [66], [131] and [99].

Sturm's Theorems play a key role in the analysis of other aspects of behaviour in infinite-dimensional dynamical systems associated with nonlinear parabolic equations. These are convergence to periodic solutions and related questions for periodic equations [33], [28] (results apply to general 1D fully nonlinear equations), [43], [31] (transversality properties), [109], [67] and [34] (applied to N-dimensional semilinear parabolic equations by means of symmetrization and moving plane techniques), [120] (almost periodicity). Zero set analysis is a leading ingredient of a Poincaré-Bendixson theorem for semilinear heat equations, [12], [98], [45], and in the construction of G. Floquet bundles (see [48] and results by A.M. Lyapunov [91]) for linear parabolic equations in periodic and nonperiodic cases (solutions $u_n(x, t)$ having exactly n zeros for all $t \in \mathbb{R}$) [35], [36] (a generalization of Sturm-Liouville theory to the time-dependent case, results include exponential dichotomies and other estimates). Such Floquet-type solutions $\{u_n(x, t), t > 0\}$ exist for the semilinear heat equation $u_t = u_{xx} - |u|^{p-1}u$ in $\mathbb{R} \times \mathbb{R}_+$ with exponential decay as $t \to \infty$ depending on n [100]. The nonincreasing number of zeros plays a key role in the problems of Morse decomposition [92] and connections of Morse sets [46] for the monotone feedback differential delay equation

$$\dot{u}(t) = f(u(t), u(t-1)), \quad u \in \mathbb{R}.$$

Nonincrease of the number of zeros per unit interval for such linear equations was first established by A.D. Myschkis (1955), see Theorem 32 in [104]. It is also true for monotone cyclic feedback systems [93] $\dot{u}_i = f_i(u_i, u_{i-1})$, $u_i \in \mathbb{R}$, i mod n.

Sturm's intersection ideas play a fundamental role in *curve shortening* or *flows by mean curvature* problems for curves on surfaces. For curves on a surface M with a Riemannian metric g, such a motion is described by the curve shortening equation

$$v^{\perp} = V(t, k), \tag{3.4}$$

where $v^{\perp}$ is the normal velocity of the curve, k is the curvature and V is a $C^{1,1}$ function satisfying $\partial V / \partial k > 0$. The reason that Sturm's results apply to such evolution problems (though some of the properties are intuitively obvious for intersections of curves) is that (3.4) reduces to a nonlinear parabolic equation for the curvature k or, after a suitable parametrization, for a function $u(x, t)$ satisfying a fully nonlinear parabolic equation (3.2), where F depends on V. [See the first results in [70], [119] and [51] (a parabolic PDE for curvature $k_\tau = k^2(k_{\theta\theta} + k)$ was derived for the flow $v^{\perp} = k$), and [41], [63].] A general approach to curve shortening flows via 1D parabolic equations was developed in [9], [10] (where Sturm's intersection theory is described), see also [117]. The mean curvature flows can generate different types of singularities.

Parabolic properties of a curve shortening evolution can be used in a number of well-known problems concerning plane curves. As a first example, a Birkhoff curve shortening evolution was a basic idea in proving the theorem of the three geodesics (any Riemannian 2-sphere has at least three simple closed geodesics) by L.A. Lusternik and L.G. Schnirelman (1929) [90]. A smooth evolution via curvature was used in [63] based on Uhlenbeck's suggestion of using the curvature flow.

Sturm's evolution PDEs approach on zero sets can give a new insight to a number of topological problems of plane and spherical curves, caustics, and related topics of symplectic geometry briefly outlined above. For instance, three of Arnold's theorems [15] on the number of inflection points (at least four for any embedded curve in S^2, the "tennis ball theorem"; and at least three for any noncontractible embedded curve in $\mathbb{RP}^2$) and extatic points (at least six for any plane convex curve) can be proved by using a suitable parabolic mean curvature evolution (the affine one for extatic points), see [11] and comments in [18]. Namely, the asymptotic expansion of the solution $u(x, t)$ as $t \to \infty$ describing the convergence to limiting geodesics via a 1D parabolic equation determines a minimally possible number of critical points. Then the result follows from Sturm's result on the nonincrease with time of the number of such points (e.g., inflections which are zeros of the curvature). While the Sturm-Hurwitz theorem can deal with infinitesimal perturbations of curves (see above), Sturm's evolution analysis extends the results to any finite perturbation. It follows that the statements from [17], p. 14, " The tennis ball theorem asserts that the result remains true for finite perturbations, even very large ones," and "... the tennis-ball theorem may be considered as a generalization of Hurwitz' theorem to the case of multi-valued functions" are covered by the first Sturm Theorem on zeros of single-valued functions (solutions of the PDE) since a suitable parabolic 1D evolution is available. The case of finite perturbations reduces via parabolic evolution to the infinitesimal one, and then Sturm's Theorem establishes that the number of critical points (zeros, inflections, extatic points, etc.) cannot be less than the eventual, infinitesimal one for arbitrarily small perturbations where a standard linearization applies. If a suitable parabolic evolution exists, the Sturm-Hurwitz theorem guarantees that the "infinitesimal geometric characteristic" of convergence (the number of critical points) is the optimal lower bound for any finite, arbitrarily large perturbation.

After a suitable surface parametrization, the quasilinear parabolic equation

$$u_t = u_{xx}/[1 + (u_x)^2] - (N - 2)/u$$

describes the evolution of cylindrically symmetric hypersurfaces moving by mean curvature in $\mathbb{R}^N$, $N \geq 3$, [42], [121], [5]. A similar singular lower-order term occurs in the Prandtl boundary layer equations, which by von Mises non-local transformation reduce to the PME with an extra term $u_t = (uu_x)_x + g(t)/u$ where g depends on the velocity of the potential flow (though in the original setting no singularities occur); see Section 30 in [134].

It is known that the first Sturm Theorem cannot be generalized to parabolic equations in $\mathbb{R}^N$ in the sense that such a general "order structure" does not exist; see [49] and the detailed survey [110].

The Sturmian classification of multiple zeros holds for a system of parabolic inequalities. Rescaling by Sturm's backward variable shows that Sturm's Theorems are true for $W^{2,1}_{p,\mathrm{loc}}$ solutions (from Tikhonov's class) of a system of parabolic inequalities

$$|u_t - u_{xx}| \leq M_1|u_x| + M_0|u|, \quad x \in \mathbb{R}, \; t \in J.$$

See [32], where such rescaling detailed analyses of nodal sets were carried out for equations in $\mathbb{R}^N$, namely the heat equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times (-\infty, 0).$$

In terms of Sturm's backward variable $z = x/(-t)^{1/2}$ this reduces to the rescaled equation

$$u_\tau = \mathbf{B}u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } \tau = -\ln(-t) \to \infty \;\text{ as }\; t \to 0^-, \tag{3.5}$$

with the symmetric second-order operator

$$\mathbf{B}u = \Delta u - \tfrac{1}{2}\, z \cdot \nabla u \equiv \tfrac{1}{\rho}\, \nabla \cdot (\rho \nabla u), \;\; \rho(z) = e^{-|z|^2/4}. \tag{3.6}$$

It is self-adjoint in $L^2_\rho(\mathbb{R}^N)$ with the domain $H^2_\rho(\mathbb{R}^N)$ and a point spectrum $\sigma(\mathbf{B}) = \{\lambda_\beta = -|\beta|/2, \, |\beta| = 0, 1, \ldots\}$ ($\beta = (\beta_1, \ldots, \beta_N)$ is a multiindex, $|\beta| = \beta_1 + \cdots + \beta_N$) and the eigenfunctions $\Phi = \{H_\beta(z) = \rho^{-1}(z) D^\beta \rho(z)\}$ are Hermite polynomials in $\mathbb{R}^N$; see [22], p. 48. The asymptotic structures $Ce^{\lambda_\beta \tau} H_\beta(z)$ with any eigenvalue $\lambda_\beta < 0$ describe for $\tau \to \infty$ all possible types of multiple zeros of the heat equation in $\mathbb{R}^N$. This makes it possible to study general properties (e.g., Hausdorff dimension) of nodal sets of general solutions [32].

The main principles of Sturm's evolution analysis of multiple zeros also remain valid for $2m$th order linear parabolic equations. Since the analysis is essentially local in a shrinking neighborhood of zero (according to Sturm's variable $z = x/(-t)^{1/2}$), without loss of generality, we consider the canonical $2m$th order parabolic equation with constant coefficients

$$u_t = -(-\Delta)^m u \quad \text{in } \mathbb{R}^N \times (-\infty, 0).$$

Sturm's backward variable takes the form

$$z = x/(-t)^{1/2m}$$

and we arrive at the equation (cf. (3.5))

$$u_\tau = \mathbf{B}u, \quad \text{where } \mathbf{B} = -(-\Delta)^m - \tfrac{1}{2m}\, z \cdot \nabla, \quad \tau = -\ln(-t). \tag{3.7}$$

For any $m > 1$, this operator is not self-adjoint in any weighted space $L^2_\rho(\mathbb{R}^N)$ unlike the second-order case $m = 1$. We introduce the space $L^2_\rho(\mathbb{R}^N)$ with the exponential weight $\rho(z) = e^{-a|z|^\alpha} > 0$ in $\mathbb{R}^N$, where $\alpha = 2m/(2m-1) \in (1, 2)$ and $a = a(m, N) > 0$ is a sufficiently small constant. For $m = 1$ we have $\alpha = 2$, $a = 1/4$ and $\rho(z) = e^{-|z|^2/4}$ is the rescaled Gaussian kernel as in (3.6). In $L^2_\rho(\mathbb{R}^N)$ the operator $\mathbf{B}$, with domain $H^{2m}_\rho(\mathbb{R}^N)$ being a weighted Sobolev space, admits the point spectrum $\sigma(\mathbf{B}) = \{\lambda_\beta = -|\beta|/2m \le 0,\ |\beta| = 0, 1, \dots\}$. The subset of eigenfunctions $\{\psi_\beta(z)\}$ (Kummer's polynomials in $\mathbb{R}^N$ of order $|\beta|$) is complete in $L^2_\rho(\mathbb{R}^N)$ [40], [54]. For $m = 1$, these are the Hermite polynomials. In view of completeness of polynomials, in the existence class $\{|u(x,t)| \le Ae^{a|x|^\alpha}\}$, $a, A > 0$, any solution of (3.5), (3.7) has the eigenfunction expansion $u(z,\tau) = \sum C_\beta e^{\lambda_\beta \tau} \psi_\beta(z)$. As a consequence, the complete subset of polynomials $\{\psi_\beta(z)\}$ describes in the rescaled form possible types of formation of multiple zeros occurring for this higher-order parabolic equation and describing local properties of nodal sets, [55]. Of course, the first Sturm Theorem in 1D (nonincrease of the number of zeros) is no longer available for $2m$th order equations, where new zeros can occur with evolution.

Finally, we notice that Sturm's zero-set ideas often play a crucial role in the asymptotic analysis of nonlinear parabolic PDEs admitting finite-time singularities or free boundaries of different types. A large amount of mathematical literature was devoted to these subjects during the last twenty years. An extensive list of references on geometric Sturmian approaches to nonlinear parabolic equations with applications to singularity formation phenomena (like blow-up, extinction or focusing) and regularity analysis of free-boundary problems are available in books [58] and [56] and in the survey papers [57] and [53].

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