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Charles Sturm and the Development of Sturm-Liouville Theory in the Years 1900 to 1950

W. Norrie Everitt

This paper is dedicated to the achievements and memory
of
Charles François Sturm
1803 to 1855

Abstract. The first joint publication by Sturm and Liouville in 1837 introduced the general theory of Sturm-Liouville differential equations.

This present paper is concerned with the remarkable development in the theory of Sturm-Liouville boundary value problems, which took place during the years from 1900 to 1950.

Whilst many mathematicians contributed to Sturm-Liouville theory in this period, this manuscript is concerned with the early work of Sturm and Liouville (1837) and then the contributions of Hermann Weyl (1910), A.C. Dixon (1912), M.H. Stone (1932) and E.C. Titchmarsh (1940 to 1950).

The results of Weyl and Titchmarsh are essentially derived within classical, real and complex mathematical analysis. The results of Stone apply to examples of self-adjoint operators in the abstract theory of Hilbert spaces and in the theory of ordinary linear differential equations.

In addition to giving some details of these varied contributions an attempt is made to show the interaction between these two different methods of studying Sturm-Liouville theory.

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Contents

1. References

The early papers of Sturm on ordinary linear differential equations, and their initial value and boundary value problems, date from 1829 to 1836; see [21], [22] and [23].

The first joint paper by Sturm and Liouville on boundary problems is given in [24]; the results from this remarkable paper are discussed in Section 2 below.

The place and significance of these Sturm and Liouville results in the history of mathematics in the 19^{th} century are considered in detail in the paper of Lützen [16].

In this present paper there are detailed discussions of contributions to Sturm-Liouville theory from: Weyl [32], [33] and [34]; Dixon [5]; Stone [20]; Titchmarsh [26], [27], [28], [29] and [30].

There are later accounts of the Titchmarsh-Weyl theories in the works of Coddington and Levinson [4]; Everitt [6]; Hellwig [9]; Hilb [10]; Hille [11]; Jörgens [12]; Kodaira [14]; Titchmarsh [30]; Yoshida [36].

The theory of Sturm-Liouville differential operators in Hilbert function spaces is developed in the works of Akhiezer and Glazman [1]; Hellwig [9]; Jörgens and Rellich [13]; Naimark [17]; von Neumann [18]; Stone [20].

In view of the significance for Sturm-Liouville theory of the 1910 paper [33] by Weyl special mention is made of the M.Sc. thesis of Race, see [19]. This thesis contains a translation from the German into English of the major part of the Weyl paper; in particular, there is a complete translation of Chapters I and II, together with the translation of the more significant results and remarks from the remaining Chapters, III and IV.

The numerical treatment of Sturm-Liouville boundary value problems has been extensively developed; a summary of results, references together with information on the SLEIGN2 computer code, is given in [2].

Finally there is an epilogue from Weyl in his paper of 1950 [35].

This work is not to be counted as a history of Sturm-Liouville theory for the period 1900 to 1950; such a history should contain reference to many more individual contributions from mathematicians, other than those named at the end of this paper. This paper is an attempt to view the development of Sturm-Liouville theory in the light of advancing techniques in mathematical analysis over this period: the theories of real and complex functions, measure and integration, and linear operators in function spaces.

2. Sturm and Liouville and the paper of 1837

As mentioned in the previous section the history of Sturm-Liouville theory is presented in detail in the scholarly paper of Lützen [16]. The main Sturm and Liouville contributions listed in the references are:

(i) The Sturm papers [21], [22] and [23].
(ii) The Sturm and Liouville paper [24].

For a discussion of the results in the three papers listed in (i) see [16].

The Sturm and Liouville paper [24] in (ii) is totally remarkable; it is four pages long but, in almost modern notation, presents the essentials of a Sturm-Liouville boundary value problem on a compact interval, with separated boundary conditions.

The boundary value problem studied by Sturm and Liouville in [24], see also [16, Introduction], is, in their notation,

$$-\frac{d}{dx}\left(k\frac{dV}{dx}\right)+lV=rgV \text{ on the interval } [\mathbf{x},\mathbf{X}] \tag{2.1}$$

with the imposed separated boundary conditions

$$\frac{dV}{dx}-hV=0 \text{ for } x=\mathbf{x} \tag{2.2}$$

$$\frac{dV}{dx}+HV=0 \text{ for } x=\mathbf{X}. \tag{2.3}$$

Here the coefficients k, l, g are positive on the interval $[\mathbf{x}, \mathbf{X}]$, h and H are given positive numbers and r is a real-valued parameter.

It is shown that the initial value problem at the endpoint $\mathbf{x}$, determined by the differential equation (2.1) and the initial boundary condition (2.2), has a non-null solution V for all values of the parameter r.

This boundary value problem (2.1), (2.2) and (2.3) only allows for non-null solutions (now called eigenfunctions) for certain values (now called eigenvalues) of the parameter r in (2.1); these values are determined in [24, Page 221] as roots of a transcendental equation, involving the solutions of the equation,

$$\omega(r) = 0; \tag{2.4}$$

namely the equation obtained by inserting the general solution V of (2.1) and (2.2) into the remaining boundary condition (2.3). In the earlier Sturm papers [21], [22] and [23] it is shown that the transcendental equation (2.4) has an infinity of real simple roots which are positive and denoted in [24, Page 221] by $r_1, r_2, \ldots, r_n, \ldots$ arranged in increasing order of magnitude. These are the eigenvalues of the boundary value problem; likewise the associated solution functions (eigenfunctions) are denoted by $V_1, V_2, \ldots, V_n, \ldots$

It is remarked that the transcendental function ω in (2.4) has the property that $\omega'(r_n) \neq 0$ for all $n \in \mathbb{N}$; in fact it is shown that

$$\int_{\mathbf{x}}^{\mathbf{X}} g(x) V_n^{\,2}(x)\ dx = -k(\mathbf{X}) V_n(\mathbf{X}) \omega'(r_n) \tag{2.5}$$

where the numbers $k(\mathbf{X})$ and $V_n(\mathbf{X})$ are both non-zero; it is this result that yields the proof that the zeros of the transcendental function ω are all simple. (Note the use of the prime $'$ notation in (2.5) for the derivative of the function ω; this is the Lagrange notation for the derivative, see [16, Introduction, Page 310].)

The formulae given also show, in effect, that the solution functions

$$V_1, V_2, \ldots, V_n, \ldots$$

have the orthogonality properties

$$\int_{\mathbf{x}}^{\mathbf{X}} g(x) V_m(x) V_n(x)\ dx = 0 \tag{2.6}$$

for all $m, n \in \mathbb{N}$ with $m \neq n$.

Given a function f defined on the interval $[\mathbf{x}, \mathbf{X}]$ the following formulae are obtained (recall that V is the solution of the initial value problem (2.1) and (2.2))

$$\int_{\mathbf{x}}^{\mathbf{X}} g(x) V(x) f(x)\ dx = \sum_{n=1}^{\infty} \left\{ \frac{\int_{\mathbf{x}}^{\mathbf{X}} g(x) V(x) V_n(x)\ dx \cdot \int_{\mathbf{x}}^{\mathbf{X}} g(x) V_n(x) f(x)\ dx}{\int_{\mathbf{x}}^{\mathbf{X}} g(x) V_n^{\,2}(x)\ dx} \right\}, \tag{2.7}$$

and if F on $[\mathbf{x}, \mathbf{X}]$ is given by

$$F(x) = \sum_{n=1}^{\infty} \left\{ \frac{V_n(x) \int_{\mathbf{x}}^{\mathbf{X}} g(y) V_n(y) f(y)\ dy}{\int_{\mathbf{x}}^{\mathbf{X}} g(y) V_n^{\,2}(y)\ dy} \right\} \tag{2.8}$$

then

$$\int_{\mathbf{x}}^{\mathbf{X}} g(x)V(x)F(x)\ dx = \sum_{n=1}^{\infty} \left\{ \frac{\int_{\mathbf{x}}^{\mathbf{X}} g(x)V(x)V_n(x)\ dx \cdot \int_{\mathbf{x}}^{\mathbf{X}} g(x)V_n(x)f(x)\ dx}{\int_{\mathbf{x}}^{\mathbf{X}} g(x)V_n^2(x)\ dx} \right\}.$$

From these results it follows that

$$\int_{\mathbf{x}}^{\mathbf{X}} g(x)V(x)[F(x) - f(x)]\ dx = 0 \tag{2.9}$$

and leads to the conclusion that

$$F(x) - f(x) = 0 \text{ for all } x \in [\mathbf{x}, \mathbf{X}]. \tag{2.10}$$

Finally then the series expansion is obtained

$$f(x) = \sum_{n=1}^{\infty} \left\{ \frac{V_n(x) \int_{\mathbf{x}}^{\mathbf{X}} g(y)V_n(y)f(y)\ dy}{\int_{\mathbf{x}}^{\mathbf{X}} g(y)V_n^2(y)\ dy} \right\} \text{ for all } x \in [\mathbf{x}, \mathbf{X}]. \tag{2.11}$$

Remark 2.1. We enter three remarks:

(a) We have followed the outline details of the proof of the critical result (2.10) from the paper [24]; however there seems to be a difficulty in deducing (2.10) from (2.9), since the function V may not be of one sign on the interval $[\mathbf{x}, \mathbf{X}]$; for clarification on this point see the remarks by Lützen [16, Section 49, Page 348].

(b) At the end of the Sturm and Liouville paper [24, Page 223] there is a footnote written by Liouville indicating that complete details of the analysis of the results announced are to be published in a following mémoire; however, Lützen remarks in his paper [16, Section 49, Page 349, Line 5] that this work has been lost.

(c) In modern terminology this last result (2.11) is then the eigenfunction expansion of a continuous function f in terms of these eigenfunctions, within the weighted Hilbert function space $L^2([\mathbf{x}, \mathbf{X}]; g)$.

Sturm, in his first large paper wrote, see [22, Page 106] and [16, Section II, Page 315],

"La résolution de la plupart des problèmes relatifs à la distribution de la chaleur dans des corps de formes diverses et aux petits mouvements oscillatoires des corps solides élastiques, des corps flexibles, des liquides et des fluides élastiques, conduit à des équations différentielles linéaires du second ordre…".

As an example Sturm discussed heat conduction in an inhomogeneous thin bar; in this case the temperature is governed by the linear partial differential equation

$$g\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) - lu. \tag{2.12}$$

Applying the method of solution by separation of variables leads to ordinary boundary value problems of the form (2.1), (2.2) and (2.3) with the coefficients g, k, l. The expansion in terms of the solution functions of these boundary value problems, as given above, then led to formal solutions of boundary and initial value problems associated with partial differential equations of the form (2.12).

Remark 2.2. With the advantage of hindsight the following remarks can be made:

(i) As to be expected, given the period when these early results were obtained there are no continuity or differentiability conditions on the three coefficient functions k, l, g.
(ii) Essentially, in the earlier results of Sturm, see [21], [22] and [23], there are underlying assumptions of continuity needed to obtain the existence of the real, simple roots $\{r_n : n \in \mathbb{N}\}$ of the transcendental equation (2.4).
(iii) The sign convention in the differential equation (2.1), in particular the negative sign in the derivative term, comes from consideration of the separation of variables technique applied to such partial differential equations as (2.12).
(iv) The positivity of the coefficients k, l and g in the differential equation (2.1), and the positive values of the boundary numbers h and H, are responsible for the parameter values of r, obtained from the transcendental equation (2.4), being all non-negative and so bounded below on $\mathbb{R}$.

3. Notations

The symbols $\mathbb{N}$ and $\mathbb{N}_0$ represent the positive and the non-negative integers, respectively.

The real and complex number fields are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. Lebesgue integration is denoted by L; the Hilbert function space $L^2((a,b);w)$, given the interval (a,b) and the weight w, is the collection (of equivalence classes) of complex-valued, Lebesgue measurable functions f defined on (a,b) such that

$$\int_a^b w(x)\,|f(x)|^2\,dx < +\infty. \tag{3.1}$$

The class of Cauchy entire (integral) complex-valued functions defined on $\mathbb{C}$ is denoted by $\mathbf{H}$.

The Sturm and Liouville differential equation in [24] may be rewritten in the form, in modern notation,

$$-\frac{d}{dx}\left(k(x)\frac{dV(x)}{dx}\right) + l(x)V(x) = rg(x)V(x) \text{ for all } x \in [a,b] \tag{3.2}$$

where:

(a) the coefficients $k, l, g : [a,b] \to \mathbb{R}$, with $l, g \in C[a,b]$ and $k \in C^{(1)}[a,b]$
(b) $k, l, g > 0$ on $[a,b]$
(c) the parameter $r \in \mathbb{R}$
(d) the dependent variable $V : [a,b] \to \mathbb{R}$.

To compare with another widely used modern notation for the Sturm-Liouville differential equation, now involving the concept of quasi-derivative (see [17] and [6]), we have (here the prime $'$ denotes classical differentiation on $\mathbb{R}$)

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (a,b), \tag{3.3}$$

where a wider class of coefficients is admitted as follows:

(α) for the interval (a,b) the endpoints, in general, satisfy $-\infty \le a < b \le +\infty$

(β) the coefficients $p, q, w : (a,b) \to \mathbb{R}$ and $p^{-1}, q, w \in L^1_{\text{loc}}(a,b)$, where $L^1_{\text{loc}}(a,b)$ is the local Lebesgue integration space

(γ) the weight $w(x) > 0$ for almost all $x \in (a,b)$; there are no sign restrictions on the coefficients p, q

(δ) the spectral parameter $\lambda \in \mathbb{C}$.

In all the cases which follow, Sturm-Liouville differential equations are given in the form (3.3), with restrictions on the coefficients p, q, w and the interval (a,b).

4. Mathematical analysis

The four main areas in mathematical analysis that influenced the development of Sturm-Liouville theory, and were in part influenced by this theory in its own right, are:

(i) The Lebesgue integral
(ii) Integrable-square Hilbert function spaces
(iii) Complex function theory on the plane $\mathbb{C}$
(iv) Spectral theory of unbounded operators in Hilbert spaces.

This influence from within and without Sturm-Liouville theory is to be seen in the sections which follow in this paper.

5. Hermann Weyl and the 1910 paper

This paper [33] has now long been regarded as one of the most significant contributions to mathematical analysis in the 20^{th} century; whilst not the first paper to consider the singular case of the Sturm-Liouville differential equation it is the first structured consideration of the analytical properties of the equation. The range of new definitions and results is remarkable and set the stage for the full development of Sturm-Liouville theory in the 20^{th} century, as to be seen in the later theory of differential operators in the work of von Neumann [18] and Stone [20], and in the application of complex variable techniques by Titchmarsh [30].

The paper considers the equation (3.3) with the restrictions:

(i) The interval is $[0,\infty)$ and $p, q : [0,\infty) \to \mathbb{R}$; $w(x) = 1$ for all $x \in [0,\infty)$
(ii) The coefficients satisfy: $p, q \in C[0,\infty)$; $p > 0$ on $[0,\infty)$
(iii) The spectral parameter $\lambda \in \mathbb{C}$.

The differential equation is then

$$-(p(x)y'(x))' + q(x)y(x) = \lambda y(x) \text{ for all } x \in [0,\infty). \tag{5.1}$$

Before listing the main results from this paper there are two comments to be made:

1. At the start of the paper, see [33, Chapter I, Page 221, Footnote †)], Weyl points out that no assumption is made concerning the differentiability of the leading coefficient p; it is sufficient to require only the continuity of p on $[0,\infty)$; this assumption has the consequence that any term of the form py' has to be considered as a single symbol; in particular the derivative y' may not exist separately at any point of the interval $[0,\infty)$. The initial conditions at the regular endpoint 0 for the existence theorem in [33, Chapter I, Section 1] are of the form, for numbers $\alpha, \beta \in \mathbb{C}$,

$$y(0) = \alpha \qquad (py')(0) = \beta \tag{5.2}$$

which fits in with the existence result that both y and (py') are continuous on $[0,\infty)$. In this respect Weyl is working with the quasi-derivative (py') in place of the classical derivative y' many years before the general introduction of quasi-derivatives, see [1, Appendix 2, Section 123] and [17, Chapter V, Section 15].

2. Throughout the paper, but not always stated in theorems and other results, Weyl assumes a boundary condition, at the regular endpoint 0, on the solutions of the equation (5.1), of the form

$$\cos(h)y(0) + \sin(h)(py')(0) = 0 \tag{5.3}$$

where h is a given real number; see [33, Chapter I, Section 2, (10)].

The main results from this paper are:

(a) Chapter I, Theorem 1: the introduction of the circle method for the differential equation (5.1), and the definition of the limit-circle and limit-point classification of the equation for any point $\lambda \in \mathbb{C}$.
(b) Chapter I, Theorem 2: for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the differential equation (5.1) has at least one non-null solution in the Hilbert function space $L^2(0,\infty)$.
(c) Chapter II, Theorem 5: the limit-circle/limit-point classification of the equation (5.1) is independent of the spectral parameter λ, and depends only on the choice of the two coefficients p and q.
(d) Chapter II, Theorem 5: in the limit-circle case all solutions of the differential equation (5.1) are in the Hilbert function space $L^2(0,\infty)$, for all $\lambda \in \mathbb{C}$.
(e) Chapter II, Theorem 5: in the limit-point case there is at most one non-null solution of the equation in the Hilbert function space $L^2(0,\infty)$, for all $\lambda \in \mathbb{C}$.
(f) Chapter II, Corollary to Theorem 5: if the coefficient q is bounded below on $[0,\infty)$ then for all coefficients p the differential equation is in the limit-point case.

(g) Chapter III: at the start of this chapter there are the Weyl definitions of the point spectrum (Punktspektrum), and the continuous spectrum (Streckenspektrum) involving eigendifferentials; the latter definition agrees with the earlier definition of continuous spectrum given by Hellinger in the paper [8].

(h) Chapter III: the Hellinger/Weyl definition of continuous spectrum introduces essentially the concept of eigenpackets as later studied by Rellich [13, Chapter II, Sections 1 and 2] and Hellwig [9, Chapter 10, Section 10.4].

(i) Chapter II, Theorem 4: this theorem gives the eigenfunction expansion in the limit-circle case for the boundary value problem consisting of the Sturm-Liouville differential equation (5.1), a boundary condition (5.3) at the regular endpoint 0, and a boundary condition [33, Chapter II, (41)] at the endpoint $+\infty$; the spectrum of this problem consists only of the point spectrum of real, simple eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ with an accumulation point at $+\infty$ on the real axis $\mathbb{R}$ of the complex spectral plane $\mathbb{C}$; the corresponding eigenfunctions $\{\varphi_n : n \in \mathbb{N}\}$ form a complete, orthogonal set in the Hilbert function space $L^2(0,\infty)$; there is a pointwise expansion of a function $f \in L^2(0,\infty)$, subject to additional smoothness and integrability conditions on the function f, in terms of the eigenfunctions where the infinite series is absolutely convergent and locally uniformly convergent.

(j) Chapter III, Theorem 7: this theorem gives the eigenfunction expansion in the limit-point case for the boundary value problem consisting of the Sturm-Liouville differential equation (5.1), and a boundary condition (5.3) at the regular endpoint 0; in this case no boundary condition is required at the endpoint $+\infty$; the spectrum of this problem consists of a point spectrum, which may be empty, and a continuous spectrum, which may be empty; the point spectrum gives rise to eigenfunctions and a series expansion; the continuous spectrum gives rise to eigendifferentials and an integral expansion; there is a pointwise expansion of a function $f \in L^2(0,\infty)$, subject to additional smoothness and integrability conditions on f, in terms of the eigenfunctions and eigendifferentials, where the series and integrals are, respectively, absolutely convergent and locally uniformly convergent.

(k) Chapter IV, Theorem 8: in the limit-point case the continuous spectrum is independent of the choice of the boundary condition at the regular endpoint 0; the point spectrum is different for each particular boundary condition at the regular endpoint 0.

(l) Chapter IV, Theorem 9: if the coefficient q satisfies the condition

$$\lim_{x\to\infty} q(x) = +\infty \tag{5.4}$$

then, for all coefficients p, the limit-point case holds and the spectrum for any boundary value problem consists only of the point spectrum of real, simple eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ with accumulation only at $+\infty$ on the real axis $\mathbb{R}$ of the complex spectral plane $\mathbb{C}$; the corresponding eigenfunctions $\{\varphi_n : n \in \mathbb{N}\}$ form a complete, orthogonal set in the Hilbert function space $L^2(0,\infty)$; for all $n \in \mathbb{N}$ the eigenfunction φ_n has exactly n zeros in the open interval $(0,\infty)$.

(m) Chapter IV, Theorem 11: under the following conditions on the coefficients p,q

$$p(x) = 1 \text{ for all } x \in [0,\infty) \qquad \int_0^\infty x\,|q(x)|\ dx < +\infty \tag{5.5}$$

the limit-point case holds and the spectrum for any boundary value problem consists of a finite number of strictly negative eigenvalues for the point spectrum, and the half line $[0,\infty)$ for the continuous spectrum.

(n) Chapter IV, Section 22: here there is a remarkable example which illustrates the effectiveness of the Weyl definition of the continuous spectrum of Sturm-Liouville differential equations. Let the coefficients p,q be given by

$$p(x) = 1 \text{ and } q(x) = -x \text{ for all } x \in [0,\infty). \tag{5.6}$$

The resulting Sturm-Liouville differential equation is

$$-y''(x) - xy(x) = \lambda y(x) \text{ for all } x \in [0,\infty) \tag{5.7}$$

which has solutions that can be expressed in terms of the classical Bessel functions. Weyl gives a proof that this equation is in the limit-point case at the singular endpoint $+\infty$; also that, in terms of his definition of the continuous spectrum, any boundary value problem, determined by a boundary condition at the regular endpoint 0, has no eigenvalues and the whole real line $\mathbb{R}$ as continuous spectrum.

(o) Closing remark: at the end of the paper Weyl remarks that all the main results and theorems can be extended to the case when a weight function w is included in the Sturm-Liouville differential equation; that is the results extend to the general equation (3.3)

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in [0,\infty). \tag{5.8}$$

Here the weight function w is positive-valued and continuous on the half line $[0,\infty)$; in these circumstances the Hilbert function space is

$$L^2((0,\infty);w),$$

see Section 3 above.

6. A.C. Dixon and the paper of 1912

This paper is significant in the development of the Sturm-Liouville differential equation for one reason; it seems to be the first paper in which the continuity conditions on the coefficients p,q,w are replaced by the Lebesgue integrability conditions; these latter conditions are the minimal conditions to be satisfied by p,q,w within the environment given by the Lebesgue integral, see Section 3 above.

The paper uses the same notation [5, Section 1, (1)] of the Sturm-Liouville differential equation as given in the original paper of Sturm and Liouville, *i.e.*,

$$-\frac{d}{dx}\left(k\frac{dV}{dx}\right) + lV = rgV \text{ on the interval } [a,b]; \tag{6.1}$$

however there is no direct reference to the paper [24].

In the notation of Section 3 this Dixon paper considers the equation (3.3) with the assumptions:

(i) The interval $[a,b]$ is compact and $p,q,w:[a,b]\to\mathbb{R}$.
(ii) The coefficients p,q,w satisfy the Lebesgue minimal conditions $p^{-1},q,w\in L^1[a,b]$, and both $p,w>0$ almost everywhere on $[a,b]$.

The paper discusses the existence of solutions of this Sturm-Liouville differential equation

$$-(p(x)y'(x))'+q(x)y(x)=\lambda w(x)y(x) \text{ for all } x\in[a,b] \tag{6.2}$$

under these coefficient conditions; the existence proof is based on replacing, formally, the differential equation (6.1) with the two integral equations, see [5, Section 1, (2)],

$$U=\int V(l-gr)\ dx \qquad V=\int \frac{1}{k}U\ dx. \tag{6.3}$$

However, boundary conditions at the endpoints a and b, which will determine the associated Sturm-Liouville boundary value problem, are difficult to locate.

Certain expansion theorems are given, see for example [5, Section 19]; however again it is difficult to relate such results to the original series type of expansions associated with regular Sturm-Liouville boundary value problems.

This paper by Dixon raises a number of interesting remarks:

1. In effect, the Dixon existence theorem, see [5, Section 3], is a special case of the existence theorem for linear differential systems, with locally integrable coefficients, see [17, Chapter V, Section 16.1]. Note that there seems to be no reference in the paper [5] to the fact that the quasi-derivative kV' exists for solutions of 6.1 (to see this point differentiate the second term in (6.3)) but that the classical derivative V' may not exist.
2. The paper [5] was published two years after the Weyl paper [33] but no reference is given to these earlier results on Sturm-Liouville differential equations. Nevertheless, the Dixon conditions on the coefficients make for a remarkable advance in the study of such differential equations.
3. The years from 1910 onwards saw the introduction of the Lebesgue integral into mathematics; it is interesting to compare the use of integration in the Weyl paper [33], seemingly generalized Riemann integration, with that of the Dixon paper, Lebesgue integration.

7. M.H. Stone and the book of 1932

The general theory of unbounded linear operators in Hilbert spaces was developed by John von Neumann from 1927 onwards, and independently by M.H. Stone from 1929 onwards.

The book [20] appeared in the year 1932; it is a remarkable compendium of results and properties of Hilbert spaces. With regard to Sturm-Liouville theory there is in [20, Chapter X, Section 3] a detailed study of Sturm-Liouville differential operators; this study seems to be the first extended account of the properties of Sturm-Liouville differential operators in Hilbert function spaces, under the Lebesgue minimal conditions on the coefficients of the differential equation.

In respect of the standard form of the Sturm-Liouville differential equation given in Section 3 above, see (3.3), the conditions adopted in [20, Chapter X, Section 3] are:

(i) The open interval $(a, b) \subseteq \mathbb{R}$ is arbitrary, so that $-\infty \leq a < b \leq +\infty$.
(ii) The coefficient w is restricted to $w(x) = 1$ for all $x \in (a, b)$.
(iii) The coefficients $p, q : (a, b) \to \mathbb{R}$ and satisfy $p^{-1}, q \in L^1_{\mathrm{loc}}(a, b)$.
(iv) The Sturm-Liouville differential equation and operators are studied in the Hilbert function space $L^2(a, b)$.

Thus the differential equation studied in [20, Chapter X, Section 3] is

$$-(p(x)y'(x))' + q(x)y(x) = \lambda y(x) \text{ for all } x \in (a, b). \tag{7.1}$$

Remark 7.1. Four remarks are important:

1. The general weight coefficient w, under the conditions of Section 3, can be included in the differential equation to yield all the results in [20, Chapter X, Section 3], with only some additional technical details to the proofs of the stated lemmas and theorems; thus the Stone theory of Sturm-Liouville differential operators applies to the general differential equation

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (a, b), \tag{7.2}$$

working now in the weighted Hilbert function space $L^2((a, b); w)$.
2. Stone in the 1932 book [20] makes only marginal reference to the results of Weyl given in the 1910 paper [33]; in the following discussion of the Stone results these two contributions to Sturm-Liouville theory are brought closer together.
3. In his paper [33] of 1910 Weyl introduced his classification of the singular endpoint as limit-point or limit-circle; this classification is replaced in the Stone book [20] by the concept of the deficiency index of the minimal closed symmetric operator in the Hilbert function space; reference is made to this connection in some of the statements made below.
4. The spectral theory of self-adjoint operators is considered in detail in [20, Chapter V, Section 5], and is based on the properties of the resolution of the identity of the operator; although not discussed in the book [20] it can be shown that the definition of the continuous spectrum (Streckenspektrum) in the Weyl paper [33] is consistent with the definition of continuous spectrum in the book, see [20, Chapter V, Section 5, Theorem 5.11].

The main results from the Stone book, see [20, Chapter X, Section 3], are:

(a) Lemma 10.1: the existence theorem for solutions of the differential equation (7.2) determined by initial conditions at any point $c \in (a,b)$; this existence result involves the requirement to use the quasi-derivative py' in stating the initial conditions for any solution.

(b) Theorem 10.11: this theorem defines and gives the essential properties of the minimal and maximal differential operators in the Hilbert function space $L^2(a,b)$, generated by the differential expression $-(pf')' + qf$; these definitions involve the use of the bilinear form, from the Green's formula for the differential expression, to determine the domain of the minimal operator; the minimal operator is closed and symmetric in $L^2(a,b)$ and its adjoint operator is the maximal operator; the entries in the deficiency index (m,m) of the minimal operator are equal, since this operator is real in $L^2(a,b)$, and take the values $m = 0, 1$ or 2.

(c) Theorem 10.15: this theorem considers the special case of Theorem 10.11 when one endpoint of the interval, say a, satisfies $a \in \mathbb{R}$ and the coefficients then satisfy $p^{-1}, q \in L^1_{\text{loc}}[a,b)$, *i.e.*, this is the case when one endpoint is regular; the maximal operator is then defined as before; the domain of the minimal operator consists of all elements f of the maximal domain satisfying the boundary conditions

$$f(a) = 0 \text{ and } (pf')(a) = 0, \tag{7.3}$$

in addition to a boundary condition at the endpoint b; again the minimal operator is closed and symmetric in $L^2(a,b)$ and its adjoint operator is the maximal operator; the entries in the deficiency index (m,m), in $L^2(a,b)$, of the minimal operator are equal since this operator is real in $L^2(a,b)$, and take the values $m = 1$ or 2. This case is equivalent to the singular problem considered in the 1910 paper [33] of Weyl; the connection here is that the deficiency index $(1,1)$ is equivalent to the Weyl limit-point case, and the deficiency index $(2,2)$ is equivalent to the Weyl limit-circle case.

(d) Theorem 10.16: this theorem considers the properties of the Sturm-Liouville differential operators in the special case of Theorem 10.15, when it is assumed that the deficiency index is $(1,1)$; this is the limit-point case as defined and considered in the Weyl paper [33].

(e) Theorem 10.17: this theorem considers the properties of the Sturm-Liouville differential operators in the special case of Theorem 10.15, when it is assumed that the deficiency index is $(2,2)$; this is the limit-circle case as defined and considered in the Weyl paper [33].

(f) Theorem 10.18: this theorem considers properties of self-adjoint Sturm-Liouville differential operators when the interval (a,b) is bounded and the coefficients then satisfy $p^{-1}, q \in L^1_{\text{loc}}[a,b]$, *i.e.*, the so-called regular case of Sturm-Liouville theory; here the deficiency index of the minimal closed symmetric operator is $(2,2)$; the results of this theorem show how to construct the domains of all self-adjoint extensions of this minimal operator by imposing symmetric, separated or coupled boundary conditions, at the regular endpoints a and b, on elements of the domain of the maximal operator.

(g) Theorem 10.19: this theorem returns to the case of the general theorem given above as Theorem 10.11, but when the deficiency index is $(2, 2)$; it is remarked that this assumption on the index is equivalent to assuming that for some $\lambda \in \mathbb{C}$ the Sturm-Liouville differential equation (7.1) has all solutions in the space $L^2(a, b)$ (this solution property then holds for all $\lambda \in \mathbb{C}$); this condition is equivalent to assuming that the differential equation is in the limit-circle case at both endpoints a and b; the results show how to construct the resolvent operator of any self-adjoint extension of the minimal operator, which resolvent is shown to be a Hilbert-Schmidt integral operator in the space $L^2(a, b)$; the point spectrum of this self-adjoint extension is a denumerable infinite point set, with no finite accumulation point; these points are the eigenvalues of the operator, none of which has multiplicity greater than 2; the continuous spectrum of this self-adjoint operator is empty.

(h) Theorem 10.20: this theorem returns to the case of the general theorem given above as Theorem 10.11, but when the deficiency index is $(1, 1)$; if the point $c \in (a, b)$, then this index situation can arise when one only of the following two cases holds:

1. the deficiency index in the space $L^2(a, c)$ for the interval $(a, c]$ is $(2, 2)$, and the deficiency index in the space $L^2(c, b)$ for the interval $[c, b)$ is $(1, 1)$
2. the deficiency index in the space $L^2(a, c)$ for the interval $(a, c]$ is $(1, 1)$, and the deficiency index in the space $L^2(c, b)$ for the interval $[c, b)$ is $(2, 2)$.

Self-adjoint extensions of the minimal operator, in these circumstances, may have eigenvalues but only of multiplicity 1; the continuous spectrum of such a self-adjoint operator need not be empty.

(i) Theorem 10.21: this theorem returns to the case of the general theorem given above as Theorem 10.11, but when the deficiency index is $(0, 0)$; if the point $c \in (a, b)$, then this index situation can arise only when the deficiency index in the space $L^2(a, c)$ for the interval $(a, c]$ is $(1, 1)$, and the deficiency index in the space $L^2(c, b)$ for the interval $[c, b)$ is $(1, 1)$, *i.e.*, when the differential equation is in the limit-point case at both endpoints a and b. Self-adjoint extensions of the minimal operator, in these circumstances, may have eigenvalues but only of multiplicity 1, and the continuous spectrum may not be empty.

8. E.C. Titchmarsh and the papers from 1939

The Titchmarsh contributions to Sturm-Liouville theory began about 1938 and concerned the analytic properties of the differential equation

$$-y''(x) + q(x)y(x) = \lambda y(x) \text{ for all } x \in [0, \infty) \tag{8.1}$$

under the coefficient conditions

(i) $q : [0, \infty) \to \mathbb{R}$
(ii) q is continuous on $[0, \infty)$
(iii) the spectral parameter $\lambda \in \mathbb{C}$.

This is a special case of the general Sturm-Liouville differential equation (3.3); however, as in the work of both Weyl and Stone some, but not all, of the Titchmarsh analysis extends to this general form of the equation, and to the case when the coefficients p, q, w satisfy the local integrability conditions given in Section 3.

Both the regular and singular cases of Sturm-Liouville boundary value problems are considered in the Titchmarsh literature; for the requirements of this present paper the three 1941 contributions [26], [27] and [28] are significant; for the consolidated results from Titchmarsh in Sturm-Liouville theory see the second edition of the volume *Eigenfunction expansions* I, [30], and the relevant chapter in the volume *Eigenfunction expansions* II, [31, Chapter XX].

The main thrust of the Titchmarsh method is to apply the extensive theory of functions of a single complex variable to the study of Sturm-Liouville boundary value problems; in the singular case this method involves the existence proof of the complex analytic form of the Weyl integrable-square solution of the differential equation (8.1). This proof of the basic Titchmarsh result dates from 1941, introduces the m-coefficient as a Nevanlinna (Herglotz, Pick, Riesz) analytic function which plays such a significant part in the eigenfunction expansion theory of singular Sturm-Liouville boundary value problems. This structure of the Weyl integrable-square solution enables the definition of the Titchmarsh resolvent function Φ and this step leads to the classical proof of the eigenfunction expansion theorem by contour integration in the complex λ-plane.

8.1. The regular case

The regular Sturm-Liouville case concerns the differential equation (8.1) when considered on a compact interval $[a, b]$

$$-y''(x) + q(x)y(x) = \lambda y(x) \text{ for all } x \in [a, b] \tag{8.2}$$

see [30, Chapter I, Section 1.5]. The starting point is the existence of a solution $\varphi : [a, b] \times \mathbb{C} \to \mathbb{C}$ determined by the initial conditions, for some $\alpha \in [0, \pi)$,

$$\varphi(a, \lambda) = \sin(\alpha) \qquad \varphi'(a, \lambda) = -\cos(\alpha) \qquad \text{for all } \lambda \in \mathbb{C}; \tag{8.3}$$

it follows that $\varphi(x, \cdot) \in \mathbf{H}$ for all $x \in [a, b]$. Similarly for the solution χ determined, for some $\beta \in [0, \pi)$, by

$$\chi(b, \lambda) = \sin(\beta) \qquad \chi'(b, \lambda) = -\cos(\beta) \qquad \text{for all } \lambda \in \mathbb{C}. \tag{8.4}$$

Then the boundary value problem determined by the equation (8.2) and the separated boundary conditions, see [30, Chapter V, Section 5.3],

$$y(a)\cos(\alpha) + y'(a)\sin(\alpha) = 0 \tag{8.5}$$

$$y(b)\cos(\beta) + y'(b)\sin(\beta) = 0 \tag{8.6}$$

has a discrete, simple, real spectrum with eigenvalues $\{\lambda_n : n \in \mathbb{N}_0\}$ determined by the zeros of the entire function $\omega \in \mathbf{H}$, where

$$\omega(\lambda) := W(\chi, \varphi)(\lambda). \tag{8.7}$$

Here $W(\chi, \varphi)$ is the Wronskian of the solutions φ and χ which is independent of the variable $x \in [a, b]$.

For any zero λ of ω the solutions $\varphi(\cdot, \lambda)$ and $\chi(\cdot, \lambda)$ are linearly dependent and so there exists a real number $k \neq 0$ such that

$$\chi(x, \lambda) = k\varphi(x, \lambda) \text{ for all } x \in [a, b]. \tag{8.8}$$

The zeros of ω are all real, see [30, Chapter I, Section 1.8].

It then follows that

$$0 \neq k \int_a^b \varphi(x, \lambda)^2 \, dx = \omega'(\lambda) \tag{8.9}$$

so that all the zeros of ω are not only real but also simple. (At this stage it is interesting to return to the original paper [24] of Sturm and Liouville; this last result echoes the earlier result given above in Section 2, see (2.5).)

The asymptotic properties of the solutions φ and χ, for fixed x and large values of $|\lambda|$, show that ω is an entire function on $\mathbb{C}$ which is of order $1/2$, see [30, Chapter I, Section 1.7]; this implies that ω has a denumerable number of zeros, see [25, Chapter VIII, Section 8.6]; let these zeros (eigenvalues) be denoted by $\{\lambda_n : n \in \mathbb{N}_0\}$.

The resolvent function $\Phi : [a, b] \times \mathbb{C} \times L^2(a, b) \to \mathbb{C}$ is defined by

$$\Phi(x, \lambda; f) := \frac{\chi(x, \lambda)}{\omega(\lambda)} \int_a^x \varphi(t, \lambda) f(t) \, dt + \frac{\varphi(x, \lambda)}{\omega(\lambda)} \int_x^b \chi(t, \lambda) f(t) \, dt. \tag{8.10}$$

From this definition it follows that, for almost all $x \in [a, b]$,

$$-\Phi''(x, \lambda; f) + q(x)\Phi(x, \lambda; f) = \lambda\Phi(x, \lambda; f) + f(x) \tag{8.11}$$

and that $\Phi(\cdot, \lambda; f)$ satisfies the boundary conditions (8.5) and (8.6) at the endpoints a and b.

For $x \in [a, b]$ and $f \in L^2(a, b)$, the resolvent $\Phi(x, \cdot; f)$ is a Cauchy analytic function, regular on $\mathbb{C} \backslash \{\lambda_n : n \in \mathbb{N}_0\}$, with simple poles at the eigenvalues $\{\lambda_n : n \in \mathbb{N}_0\}$. With the corresponding values of k in (8.8) given by $\{k_n : n \in \mathbb{N}_0\}$, the residues are

$$\frac{k_n}{\omega'(\lambda_n)} \varphi(x, \lambda_n) \int_a^b \varphi(t, \lambda_n) f(t) \, dt. \tag{8.12}$$

If now the function $\Phi(x, \cdot; f)$ is integrated around a closed contour Γ_N in the complex plane which avoids any of the zeros of ω but contains the finite number of eigenvalues $\{\lambda_n : n = 0, 1, 2, \ldots, N\}$, then the Cauchy calculus of residues gives

$$\frac{1}{2\pi i} \int_{\Gamma_N} \Phi(x, \lambda; f) \, d\lambda = \sum_{n=0}^{N} \frac{k_n}{\omega'(\lambda_n)} \varphi(x, \lambda_n) \int_a^b \varphi(t, \lambda_n) f(t) \, dt. \tag{8.13}$$

The sequence of contours $\{\Gamma_N : N \in \mathbb{N}_0\}$ is then chosen so to extend to infinity over the complex plane; an argument based on the asymptotic properties of the solutions φ and χ then shows that, for suitable conditions on the function $f \in L^2(a,b)$ and for certain values of the variable $x \in [a,b]$,

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\Gamma_N} \Phi(x,\lambda;f)\ d\lambda = f(x). \tag{8.14}$$

Formally then this argument gives the classical eigenfunction expansion for regular Sturm-Liouville boundary value problems, see [30, Chapter I, Section 1.6, (1.6.5)],

$$f(x) = \sum_{n=0}^{\infty} \frac{k_n}{\omega'(\lambda_n)} \varphi(x,\lambda_n) \int_a^b \varphi(t,\lambda_n) f(t)\ dt, \tag{8.15}$$

or

$$f(x) = \sum_{n=0}^{\infty} \psi_n(x) \int_a^b \psi_n(t) f(t)\ dt \tag{8.16}$$

where, for each $n \in \mathbb{N}$, ψ_n is the real-valued normalized eigenfunction

$$[k_n/\omega'(\lambda_n)]^{1/2}\, \varphi(\cdot,\lambda_n),$$

using (8.9).

In [30, Chapter I, Theorem 1.9] Titchmarsh shows how these formal results can be made rigorous to prove that:

1. The infinite series of eigenfunctions (8.16) converges in the topology of $\mathbb{C}$ to $f(x)$, under Fourier type convergence conditions on the function f.
2. The normal orthogonal set of eigenfunctions $\{\psi_n : n \in \mathbb{N}_0\}$ is complete in the Hilbert function space $L^2(a,b)$.
3. If given the element $f \in L^2(a,b)$ the generalized Fourier coefficients $\{c_n : n \in \mathbb{N}_0)$ are defined by

$$c_n := \int_a^b \psi_n(x) f(x)\ dx \text{ for all } n \in \mathbb{N}_0 \tag{8.17}$$

then the Parseval identity holds

$$\int_a^b |f(x)|^2\, dx = \sum_{n=0}^{\infty} |c_n|^2\,. \tag{8.18}$$

Remark 8.1. Two remarks are important:

1. The use of complex variable techniques in [30, Chapter I] illustrates the use of classical analysis to study this regular Sturm-Liouville boundary value problem, without resource to operator theoretic methods; note that there is no mention of the underlying self-adjoint operator in the Hilbert function space $L^2(a,b)$ although the completeness of the eigenfunctions in $L^2(a,b)$ is established. Moreover the methods used enable a proof of the pointwise eigenfunction expansion on the interval (a,b) of a function $f \in L^2(a,b)$,

subject to f satisfying the same conditions that give direct convergence, *i.e.*, convergence in $\mathbb{C}$, of the classical Fourier series.

2. However, it has to be accepted that these complex variable methods do not extend to the analysis of the general regular Sturm-Liouville differential equation

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in [a,b] \tag{8.19}$$

when the minimal coefficient conditions $p^{-1}, q, w \in L^1(a,b)$ only are satisfied. In general, with these conditions, it is impossible to apply the transformation, known as the Liouville transformation, to reduce the equation (8.19) to the Titchmarsh form (8.2); the coefficient p may change sign essentially on the compact interval $[a,b]$, and all three coefficients may be unbounded at endpoints and interior points of this interval.

Moreover, whilst the corresponding solutions φ and χ and their Wronskian ω have the same holomorphic properties on $\mathbb{C}$, it is impossible, in general, to obtain similar asymptotic properties of these functions for large values of $|\lambda|$; however, the reality of the eigenvalues and the orthogonality of the eigenfunctions can be established. The resolvent function Φ is defined in the same manner, and formal results such as (8.15) and (8.16) follow as above.

8.2. The singular case

For the singular case we return to the differential equation (8.1), to be studied in the Hilbert function space $L^2(0,\infty)$,

$$-y''(x) + q(x)y(x) = \lambda y(x) \text{ for all } x \in [0,\infty) \tag{8.20}$$

with the given conditions on the coefficient q. Note that if y is a solution of this equation then y, y', y'' are all continuous on the interval $[0,\infty)$.

The main problem of extending the Titchmarsh analysis of the regular case to the singular case is to find the equivalent of the boundary function χ; in general there is no method to use a boundary condition at the singular endpoint $+\infty$ in order to determine such a solution as χ; however such an extension is essential to defining a resolvent function Φ for the singular case.

This problem was resolved by Titchmarsh by using:

1. The existence of the Weyl integrable-square solution of the equation (8.20) for complex values of the parameter λ, see [33, Chapter I, Theorem 2] and item (*b*) of Section 5 above.
2. The definition of the m-coefficient to give a complex analytic structure to this Weyl solution.

The relevant Titchmarsh work for this programme is to be found in the three 1941 papers [26], [27] and [28]. In the original notation the l-coefficient is introduced in [26, Section 2]; the analytic properties of this coefficient are given in [26, Section 5], noting the use that is made of the Vitali convergence theorem. However this original l-coefficient notation was later altered to the present m-coefficient notation in the first edition of *Eigenfunction Expansions* I, see [29, Chapters II and III], and the second edition [30, Chapters II and III].

Let the solutions $\theta, \varphi : [0, \infty) \times \mathbb{C} \to \mathbb{C}$ of (8.20) be defined by the initial conditions, for some $\alpha \in [0, \pi)$,

$$\begin{cases} \theta(0,\lambda) = \cos(\alpha) & \theta'(0,\lambda) = \sin(\alpha) \\ \varphi(0,\lambda) = -\sin(\alpha) & \varphi'(0,\lambda) = \cos(\alpha) \end{cases} \tag{8.21}$$

and for all $\lambda \in \mathbb{C}$. Then the pair θ, φ forms a basis for solutions of (8.20), for all $\lambda \in \mathbb{C}$, and $\theta(x,\cdot), \theta'(x,\cdot), \varphi(x,\cdot), \varphi'(x,\cdot)$ are all entire (integral) functions on $\mathbb{C}$, for all $x \in [0, \infty)$. Note that this definition of the initial values of the pair θ, φ at 0 yields the Wronskian condition, for all $x \in [0, \infty)$ and $\lambda \in \mathbb{C}$,

$$W(\theta, \varphi)(x, \lambda) \equiv \theta(x,\lambda)\varphi'(x,\lambda) - \theta'(x,\lambda)\varphi(x,\lambda) = 1; \tag{8.22}$$

this sign convention differs from the Titchmarsh convention of the Wronskian given in [30, Chapter II, Section 2.1, (2.1.4)]; this change is to adopt the now standard sign convention for the m-coefficient as a Nevanlinna analytic function, see item (iii) below.

Weyl, see [33], proved that either

(i) in the limit-point case

$$\theta(\cdot,\lambda) \notin L^2(0,\infty) \text{ and } \varphi(\cdot,\lambda) \notin L^2(0,\infty) \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{8.23}$$

or

(ii) in the limit-circle case

$$\theta(\cdot,\lambda) \in L^2(0,\infty) \text{ and } \varphi(\cdot,\lambda) \in L^2(0,\infty) \text{ for all } \lambda \in \mathbb{C}. \tag{8.24}$$

In both cases Titchmarsh showed, see [26] and later in [30, Chapter II, Sections 2.1 and 2.2], that there exists at least one analytic function (the m-coefficient) with the properties (note the sign change that has been effected from the formulae in [30, Chapter II, Sections 2.1 and 2.2], see (8.22) above):

(i) m is regular on $\mathbb{C} \setminus \mathbb{R}$
(ii) $\overline{m}(\lambda) = m(\overline{\lambda})$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$
(iii) $\mathrm{Im}(m(\lambda)) > 0$ for all λ with $\mathrm{Im}(\lambda) > 0$;
$\mathrm{Im}(m(\lambda)) < 0$ for all λ with $\mathrm{Im}(\lambda) < 0$
(iv) the analytic function $m(\cdot)$ considered in the upper half plane

$$\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) > 0\}$$

of $\mathbb{C}$ may or may not have a continuation into the lower half plane

$$\mathbb{C}_- := \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) < 0\}$$

of $\mathbb{C}$; if it does so continue the continuation may or may not be the analytic function $m(\cdot)$ in the lower half plane $\mathbb{C}_-$

(v) the solution $\psi(\cdot,\lambda)$ of the equation (8.20) defined by

$$\psi(x,\lambda) := \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda) \text{ for all } x \in [0,\infty) \text{ and all } \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{8.25}$$

satisfies

$$\int_0^\infty |\psi(x,\lambda)|^2 \, dx = \frac{\mathrm{Im}(m(\lambda))}{\mathrm{Im}(\lambda)} < +\infty \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{8.26}$$

The existence of this solution ψ, and the analytic m-coefficient are fundamental to the Titchmarsh eigenfunction analysis as developed in the text [30]. The existence proofs so concerned involve the introduction of the Weyl circle method, see [33, Chapter I, Theorem 1], but with the additional use of complex function theory, see [30, Chapter II, Sections 2.1 and 2.2].

For the m-coefficient the following cases occur, see [30, Chapter II, Section 2.1],

1. If the differential equation (8.20) is in the limit-point case at the singular endpoint $+\infty$ then for each choice of the boundary condition parameter $\alpha \in [0,\pi)$ there is a unique m-coefficient, which depends upon α, with the above properties; for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the unique value $m(\lambda)$ is the limit-point of the circles for that value of λ.
2. If the differential equation (8.20) is in the limit-circle case at the singular endpoint $+\infty$ then for each choice of the boundary condition parameter $\alpha \in [0,\pi)$ there is a continuum of m-coefficients, each continuum depending upon α; the determination of any particular m-coefficient depends upon the limit-circle process, but see the application of the Vitali convergence theorem in [30, Chapter II, Section 2.2].

Although not part of the Titchmarsh theory it is well to remark that the properties (i), (ii) and (iii) above imply that the analytic coefficient $m(\cdot)$ is a Nevanlinna (Herglotz, Pick, Riesz) function and so has a representation of the form, see [1, Chapter 6, Section 69, Theorem 2], where $\gamma, \delta \in \mathbb{R}$ with $\delta \geq 0$,

$$m(\lambda) = \gamma + \delta\lambda + \int_{-\infty}^{+\infty} \left\{ \frac{1}{t-\lambda} - \frac{t}{t^2+1} \right\} d\rho(t) \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{8.27}$$

Here the function $\rho : \mathbb{R} \to \mathbb{R}$ is monotonic non-decreasing on $\mathbb{R}$ and satisfies the growth restriction

$$\int_{-\infty}^{+\infty} \frac{1}{1+t^2} \, d\rho(t) < +\infty; \tag{8.28}$$

this function ρ is the spectral function for the m-coefficient. The integrals in (8.27) and (8.28) are best interpreted as Lebesgue-Stieltjes integrals with the symbol ρ representing a Borel measure.

The resolvent function $\Phi : [0,\infty) \times \mathbb{C} \setminus \mathbb{R} \times L^2(0,\infty)$ is now defined by

$$\Phi(x,\lambda;f) := \psi(x,\lambda) \int_0^x \varphi(t,\lambda) f(t) \, dt + \varphi(x,\lambda) \int_x^\infty \psi(t,\lambda) f(t) \, dt. \tag{8.29}$$

The Sturm-Liouville boundary value problem considered by Titchmarsh in the singular case is best formulated by requiring that any solution y of the differential equation (8.20) is to satisfy the following conditions, see [30, Chapter II, Section 2.7, Theorem 2.7 (i)],

$$\begin{cases} \text{(i)} & y \in L^2(0,\infty) \\ \text{(ii)} & W(y,\varphi)(0) \equiv y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0 \\ \text{(iii)} & \lim_{x\to\infty} W(y,\psi(\cdot,\lambda))(x) = 0 \text{ for all } \lambda \in \mathbb{C}\setminus\mathbb{R}. \end{cases} \tag{8.30}$$

The condition (iii) is the required boundary condition at the singular endpoint $+\infty$; it was introduced by Weyl in 1910, see [33, Chapter II, Section 8, (41)], and later by Titchmarsh in 1941, [26, Section 6, (6.2)]; this form of boundary condition heralded the introduction of structured boundary conditions for classical and quasi-differential operators, see [1, Appendix 2, Section 127, Theorem 2] and [17, Chapter V, Section 18.1, Theorem 4].

As in the regular case, see (8.10), the resolvent function Φ of (8.29) satisfies the boundary conditions (8.30); see [30, Chapter II, Sections 2.8 and 2.9].

The Titchmarsh eigenfunction expansion for the singular Sturm-Liouville boundary value problem (8.20) and (8.30) is considered in two separate cases; the series case when it is assumed that the m-coefficient is meromorphic on $\mathbb{C}$, see [30, Chapter II], and the general case in [30, Chapter III].

These two cases both concern the situation when the interval for the differential equation (8.20) is the closed half-line $[0,\infty)$; for both the series and general case Titchmarsh also considers expansion theorems when the interval is the whole real line $(-\infty,\infty)$, see [30, Chapter II, Section 2.18] and [30, Chapter III, Section 3.8].

8.2.1. The singular case: series expansion. In this case there is a significant additional assumption in that, given $\alpha \in [0,\infty)$, the m-coefficient is assumed to be a meromorphic analytic function on the complex λ-plane $\mathbb{C}$; this property for m can arise in the limit-point case (for an example see [30, Chapter IV, Section 4.12]); it is always satisfied in the limit-circle case, see [30, Chapter V, Section 5.12].

Suppose that m has a denumerable set of poles at the points $\{\lambda_n : n \in \mathbb{N}_0\}$; then $\lambda_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$; it is shown in [30, Chapter II, Section 2.2] that all these poles are simple; let the residue of m at λ_n be r_n for all $n \in \mathbb{N}_0$. The analysis in [30, Chapter II, Section 2.5] shows that if the sequence of functions $\{\psi_n : n \in \mathbb{N}_0\}$ is defined by

$$\psi_n(x) := |r_n|^{1/2}\,\varphi(x,\lambda_n) \text{ for all } x \in [0,\infty) \text{ and } n \in \mathbb{N}_0, \tag{8.31}$$

then $\{\psi_n : n \in \mathbb{N}_0\}$ is a normal orthogonal set in the space $L^2(0,\infty)$. From this result it follows that, see [30, Chapter II, Section 2.6], the resolvent function $\Phi(x,\cdot;f)$ is meromorphic on the complex plane $\mathbb{C}$, with simple poles at the points $\{\lambda_n : n \in \mathbb{N}_0\}$; the residue at the pole λ_n is

$$r_n\varphi(x,\lambda_n)\int_0^\infty \varphi(t,\lambda_n)f(t)\ dt = \psi_n(x)\int_0^\infty \psi_n(t)f(t)\ dt = c_n\psi_n(x), \tag{8.32}$$

where, given $f \in L^2(0,\infty)$, the generalized Fourier coefficients $\{c_n : n \in \mathbb{N}_0\}$ are defined by

$$c_n := \int_0^\infty \psi_n(t) f(t)\ dt \text{ for all } n \in \mathbb{N}_0.$$

It is now possible to prove, following the analysis in [30, Chapter II, Section 2.6], that the solution $\varphi(\cdot, \lambda_n)$ of the differential equation (8.20), with $\lambda = \lambda_n$, satisfies the boundary conditions (8.30); this λ_n is an eigenvalue of the singular Sturm-Liouville boundary value problem (8.20) and (8.30), and $\varphi(\cdot, \lambda_n)$ is the associated eigenfunction.

The Titchmarsh analysis, see [30, Chapter II, Section 2.7], now continues to prove that if $f : [0,\infty) \to \mathbb{C}$ satisfies the conditions

$$\begin{cases} \text{(i)} & f, f' \in AC_{\text{loc}}[0,\infty) \\ \text{(ii)} & f, f'' - qf \in L^2(0,\infty) \\ \text{(iii)} & W(f,\varphi)(0) \equiv f(0)\cos(\alpha) + f'(0)\sin(\alpha) = 0 \\ \text{(iv)} & \lim_{x\to\infty} W(f, \psi(\cdot,\lambda))(x) = 0 \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R} \end{cases} \tag{8.33}$$

then

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) \text{ for all } x \in [0,\infty), \tag{8.34}$$

where the infinite series converges absolutely for all $x \in [0;\infty)$ and is locally uniformly convergent on $[0,\infty)$.

Further analysis then shows that for any element $f \in L^2(0,\infty)$ we have the Parseval identity

$$\int_0^\infty |f(x)|^2\ dx = \sum_{n=0}^{\infty} |c_n|^2 . \tag{8.35}$$

These last results represent the classical solution to the singular Sturm-Liouville boundary value problem determined by the differential equation (8.20) and the boundary conditions (8.30).

Remark 8.2. The Parseval identity (8.35) shows that the normal orthogonal set $\{\psi_n : n \in \mathbb{N}_0\}$ is complete in the Hilbert function space $L^2(0,\infty)$; this result implies that the meromorphic m does have a denumerable number of poles on the real line $\mathbb{R}$; this property was assumed to hold at the beginning of Section 8.2.1.

8.2.2. The singular case: the general expansion. Let all the previous definitions concerning the solutions θ and φ of the equation (8.20) and initial conditions (8.21) hold; let an m-coefficient be chosen, which implies that the properties (8.25) and (8.26) are satisfied.

To consider the general singular case, *i.e.*, when no additional assumptions are made on the m-coefficient, Titchmarsh introduced the k function; originally this function was defined in the 1941 paper [27, Section 4] but here quoted from [30, Chapter III, Section 3.3, Lemma 3.3].

Let $k : \mathbb{R} \to \mathbb{R}$ be defined by (again there is a sign change from the original definition)

$$k(t) := \lim_{\delta \to 0^+} \int_0^t \operatorname{Im}(m(u + i\delta))\ du \text{ for all } t \in \mathbb{R}. \tag{8.36}$$

The analysis in [30, Chapter III, Section 3.3] shows that this limit exists for all $t \in \mathbb{R}$ and that k is a non-decreasing function on $\mathbb{R}$ which satisfies

$$k(t) = \tfrac{1}{2}\{k(t+0) + k(t-0)\} \text{ for all } t \in \mathbb{R}. \tag{8.37}$$

The function k defines a non-negative Borel measure on the real line $\mathbb{R}$ to give the Lebesgue-Stieltjes integrable-square space $L^2(\mathbb{R}; k(\cdot))$ with elements

$$F : (-\infty, +\infty) \to \mathbb{C}$$

satisfying

$$\int_{(-\infty,+\infty)} |F(t)|^2\, dk(t) < +\infty.$$

To obtain the eigenfunction expansion of any element $f \in L^2(0, \infty)$ Titchmarsh gives the following definitions and properties, see [30, Chapter III, Sections 3.4 to 3.6],

1. Let $\chi : [0, \infty) \times (-\infty, +\infty) \to \mathbb{R}$ be defined by the Lebesgue-Stieltjes integral

$$\chi(x, t) := \int_{[0,t]} \varphi(x, s)\ dk(s) \text{ for all } x \in [0, \infty) \text{ and } t \in (-\infty, +\infty); \tag{8.38}$$

   then

$$\chi(\cdot, t) \in L^2(0, \infty) \text{ for all } t \in (-\infty, +\infty). \tag{8.39}$$

2. Given $f \in L^2(0, \infty)$ let $\mathcal{F} : (-\infty, +\infty) \to \mathbb{R}$ be defined by

$$\mathcal{F}(t) := \int_0^\infty \chi(x, t) f(x)\ dx \text{ for all } t \in (-\infty, +\infty); \tag{8.40}$$

   then it can be shown that $\mathcal{F} \in BV_{\mathrm{loc}}(-\infty, +\infty)$, the space of complex-valued functions, defined on $\mathbb{R}$, which are of bounded variation on all compact intervals of $\mathbb{R}$.
3. Now let f additionally satisfy the boundary conditions (8.33); then

$$f(x) = \frac{1}{\pi} \int_{(-\infty,+\infty)} \varphi(x, t)\ d\mathcal{F}(t) \text{ for all } x \in [0, \infty) \tag{8.41}$$

   where the integral is taken in the sense of Lebesgue-Stieltjes and so is absolutely convergent for all $x \in [0, \infty)$.

The result (8.41) is then the general singular Sturm-Liouville eigenfunction expansion for the Titchmarsh differential equation (8.20) when the function f satisfies the boundary conditions (8.33).

In [30, Chapter III, Section 3.7] Titchmarsh gives the Parseval identity for this eigenfunction expansion:

1. Let $f \in L^2(0, \infty)$; then the sequence of functions $\{F_n : n \in \mathbb{N}_0\}$, where
$$F_n : (-\infty, +\infty) \to \mathbb{C},$$
is defined by
$$F_n(t) := \int_0^n \varphi(x, t) f(x)\ dx \text{ for all } t \in (-\infty, +\infty). \tag{8.42}$$
2. Then it may be shown that $F_n \in L^2(\mathbb{R}; k(\cdot))$ for all $n \in \mathbb{N}_0$, that the sequence $\{F_n : n \in \mathbb{N}_0\}$ converges in mean to, say, $F \in L^2(\mathbb{R}; k(\cdot))$ in this space, and
$$\int_0^\infty |f(x)|^2\ dx = \int_{(-\infty,+\infty)} |F(t)|^2\ dk(t). \tag{8.43}$$

9. The Titchmarsh-Weyl contributions

In this section we review some aspects of the Weyl and Titchmarsh contributions to the development of Sturm-Liouville boundary value problems in the years 1900 to 1950.

9.1. The regular case

From the viewpoint of classical analysis the Titchmarsh theory of the regular case, see [30, Chapter I] is still a significant contribution to Sturm-Liouville theory. The spectrum of the boundary value problem is proved to be discrete with a denumerable number of eigenvalues, the eigenfunctions are complete in the Hilbert space $L^2(a, b)$, and the Parseval identity is established. Of course, all these results also follow from the properties of the associated self-adjoint operators in $L^2(a, b)$, see again [20, Chapter X, Section 3, Theorem 10.18].

However, the additional contribution in [30, Chapter I, Section 1.9, Theorem 1.9] is that of the pointwise convergence result of the eigenfunction expansion under Fourier type conditions on the function $f \in L^2(a, b)$, as in the classical theory of Fourier series. Such results are not in general possible using the operator methods of Sturm-Liouville theory.

In the Titchmarsh theory with the differential equation
$$-y''(x) + q(x)y(x) = \lambda y(x) \text{ for all } x \in [a, b],$$
the coefficient q is required to be continuous; however, this condition can be relaxed to $q \in L^1(a, b)$ to achieve the same results.

9.2. The singular case: general remarks

We make the following general remarks on the singular Sturm-Liouville case as to be seen in the work of Weyl and Titchmarsh up to the year 1950.

9.2.1. Pointwise convergence theorems. The eigenfunction expansions as envisaged originally by Sturm and Liouville [24], and later developed by Weyl [33], Dixon [5] and Titchmarsh [30] are all modelled on the classical theory of Fourier series.

The pointwise convergence, in $\mathbb{C}$, for a function $f : [0, 2\pi) \to \mathbb{C}$ in Fourier series is given in detail in [25, Chapter XIII, Section 8.2]; in addition to starting with $f \in L^1(0, 2\pi)$ some form of smoothness on f is required. However, if $f \in L^2(0, 2\pi)$ then convergence in this space requires no additional restrictions on f; the main tool here is the Bessel inequality; the expansion result is seen in the form of the Parseval identity, see [25, Chapter XIII, Section 13.6].

The original problem of Sturm and Liouville [24] in 1837 was to consider pointwise convergence, as viewed at that time, of the series of solution functions. The Weyl paper of 1910 [33] considers both pointwise and L^2 convergence; the Dixon paper of 1912 [5] considers only some form of pointwise convergence; of course, in both these theories the concept of convergence has been made rigorous.

The Titchmarsh theory, as now gathered together in the text [30], is influenced throughout by classical Fourier theory. In both the regular and singular cases we have:

(i) direct or pointwise convergence of the eigenfunction expansion requiring some form of second derivative integrability on the function f, in addition to the initial requirement that $f \in L^2(a, b)$ or $L^2(0, \infty)$

(ii) integrable-square convergence involving only that $f \in L^2(a, b)$ or $L^2(0, \infty)$ where the main methods are in obtaining the Bessel inequality and, in particular, the Parseval identity.

There are additional pointwise convergence results under Fourier conditions in [30, Chapter IX]; these results relax the conditions on the function f but require additional constraints on the coefficient q.

For the Stone book [20] the only convergence considered is that involved with abstract Hilbert space theory; there are no pointwise convergence results.

9.2.2. Operator theory. The theory of Sturm-Liouville differential operators is fully developed in the Stone treatise [20, Chapter X, Section 3]; see Section 7 above.

There are interesting connections between this operator theory and some of the classical convergence results in the work of Weyl [33] and Titchmarsh [30].

In [33] the two main pointwise expansion theorems are [33, Chapter II, Theorem II] and [33, Chapter III, Theorem 7]. In both theorems the domains of functions in the space $L^2(0, \infty)$ for which the expansion results are valid are virtually the domains of the corresponding self-adjoint operators in the Stone theory, see respectively the statements of the theorems [20, Chapter X, Section 3, Theorem 10.17] and [20, Chapter X, Section 3, Theorem 10.16].

These remarks also apply, respectively, to the Titchmarsh expansion results given above in Section 8.2.1 and 8.2.2; the set of functions (8.33) for which these expansions are valid are, in effect, the domains of the corresponding Stone self-adjoint differential operators.

In general, well-posed Sturm-Liouville boundary value problems generate self-adjoint differential operators in $L^2(a,b)$ for which the generalized Parseval identity holds. However, if a pointwise expansion theorem is required for the same boundary value problem, then the function in $L^2(a,b)$, to be expanded, has to satisfy additional smoothness conditions equivalent to the function belonging to the domain of the corresponding self-adjoint differential operator.

In one respect Titchmarsh came much closer to the operator theory than did Weyl. The Titchmarsh k function, see the definition in (8.36), introduces the Lebesgue-Stieltjes Hilbert function space $L^2(\mathbb{R};k(\cdot))$. Now the canonical form of the self-adjoint Stone differential operator in $L^2(a,b)$ is simply the self-adjoint multiplication operator in $L^2(\mathbb{R};k(\cdot))$; these two self-adjoint operators are unitarily equivalent and so the spectrum of the Sturm-Liouville operator can be read off from the jump and continuity properties of the monotonic non-decreasing function k. Although, seemingly, Titchmarsh was not aware of this operator theoretic connection, he successfully defines the spectrum of his singular Sturm-Liouville boundary value problem in terms of the k function, see [30, Chapter III, Section 3.9] and Section 9.2.3 below.

9.2.3. The spectrum. The definition of the spectrum of singular Sturm-Liouville boundary value problems is best seen from the operator theoretic viewpoint; for self-adjoint operators this definition concerns the resolution of the identity of the operator, see [20, Chapter V, Section 5, Definition 5.2 and Theorem 5.11].

From the classical viewpoint, such as is involved with the results and work of Weyl and Titchmarsh, the definitions are equivalent to the operator theoretic definitions but this statement has to be justified analytically. It should be remembered that Weyl, see [33, Chapter III] and items (g) and (h) of Section 5 above, formulated his definitions some twenty years before the Hilbert space definitions were in place. In the case of Titchmarsh, seemingly, he framed his definition of the spectrum of his Sturm-Liouville boundary value problems, see [30, Chapter III, Section 3.9], solely in analytical terms of his k function, as derived from the m-coefficient.

In the case of the Weyl definition of the spectrum the connection with the operator theoretic definition is given by Hellwig, see [9, Chapter 10, Sections 10.4 and 10.5]

As mentioned above, the Titchmarsh definition of the spectrum is made in terms of the monotonic non-decreasing function k; see [30, Chapter III, Section 3.9].

Given a singular Sturm-Liouville boundary value problem as discussed in Section 8.2.2 above, let the function k be defined as in (8.36). If k is constant over any open interval of $\mathbb{R}$ then it follows from the formulae (8.38) and (8.40) that this open interval makes no contribution to the expansion formula (8.41). The spectrum of the boundary value problem is then defined as the complement in the real line $\mathbb{R}$ of the set of all such open intervals of constancy of the function k. Thus the spectrum of the boundary value problem is a closed subset of the real line $\mathbb{R}$.

Points of $\mathbb{R}$ being being points of discontinuity of k belong to the spectrum and represent the eigenvalues of the boundary value problem; points of continuity but where k is increasing are in the continuous spectrum; also the limit points of these two sets are in the spectrum.

The connection between the Titchmarsh definition and the operator theory definition of the spectrum is best considered in terms of the self-adjoint multiplication operator in the Hilbert space $L^2(\mathbb{R}; k(\cdot))$; see the remarks in the last paragraph of Section 9.2.2 above.

The Titchmarsh definition of the spectrum can also be made in terms of the properties of the m-coefficient on the real line $\mathbb{R}$ of the complex plane $\mathbb{C}$. For the definitions concerned and the connection with the spectrum defined by the k function see the paper by Chaudhuri and Everitt [3].

Finally, the Titchmarsh spectral properties can be determined, or defined, from the spectral function ρ of the m-coefficient, see (8.27). There is a connection between the Titchmarsh k function and the Nevanlinna ρ function

$$k(t) = \pi\rho(t) \text{ for all } t \in \mathbb{R};$$

see [30, Chapter VI, Section 6.7, (6.7.5)], so that spectral properties may be deduced equally well from k as from ρ.

10. Aftermath

From 1950 onwards all these properties and results of the Sturm-Liouville differential equations and boundary value problems formed the basis of the spectral theory of ordinary and quasi-differential equations, and the associated differential operators, of arbitrary integer order with real and complex coefficients.

For some details of the progress made in the study of Sturm-Liouville differential equations and boundary value problems, following soon after the years 1900 to 1950, see the texts of Akhiezer and Glazman [1, Appendix 2], Naimark [17], Glazman [7] and Coddington and Levinson [4], and the papers by Kodaira [14] and [15]. For a new proof of the expansion theorem for Sturm-Liouville equations, see the article by Bennewitz and Everitt in this volume.

For a final word from Hermann Weyl see his epilogue [35] written in 1950.

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12. Salute

When Charles Sturm died in 1855 Liouville said, at the side of the grave,

"Adieu, Sturm, Adieu".

At my lecture to the Sturm Bicentennial meeting at the University of Geneva, in September 2003, I finished with the words

"Merci, Sturm-Liouville, Merci Bien".

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W. Norrie Everitt
School of Mathematics and Statistics
University of Birmingham
Edgbaston
Birmingham B15 2TT
England, UK
e-mail: `w.n.everitt@bham.ac.uk`