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Sturm-Liouville Theory: Past and Present, 1–27

Sturm's 1836 Oscillation Results Evolution of the Theory

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This paper is dedicated to the memory of Charles François Sturm

Abstract. We examine how Sturm's oscillation theorems on comparison, separation, and indexing the number of zeros of eigenfunctions have evolved. It was Bôcher who first put the proofs on a rigorous basis, and major tools of analysis where introduced by Picone, Prüfer, Morse, Reid, and others. Some basic oscillation and disconjugacy results are given for the second-order case. We show how the definitions of oscillation and disconjugacy have more than one interpretation for higher-order equations and systems, but it is the definitions from the calculus of variations that provide the most fruitful concepts; they also have application to the spectral theory of differential equations. The comparison and separation theorems are given for systems, and it is shown how they apply to scalar equations to give a natural extension of Sturm's second-order case. Finally we return to the second-order case to show how the indexing of zeros of eigenfunctions changes when there is a parameter in the boundary condition or if the weight function changes sign.

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1. Introduction

In a series of papers in the 1830's, Charles Sturm and Joseph Liouville studied the qualitative properties of the differential equation

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right) + GV = 0, \quad \text{for} \quad x \geq \alpha \tag{1.1}$$

where K, G, and V are real functions of the two variables x, r. Their work began research into the qualitative theory of differential equations, i.e., the deduction of properties of solutions of the differential equation directly from the equation and without benefit of knowing the solutions. However, it was half a century before significant interest in the qualitative theory took hold. In (1.1) and elsewhere, we consider only real solutions unless otherwise indicated.

In more modern notation (for spectral theory it is convenient to have the leading coefficient negative; for the oscillation results of Sections 2 and 3, we return to the convention of positive leading coefficient), (1.1) would be written as

$$-(py')' + qy = 0, \quad x \in I, \tag{1.2}$$

or as (when eigenvalue problems are studied)

$$-(py')' + qy = \lambda wy, \quad x \in I, \tag{1.3}$$

where the real functions p, q, w satisfy

$$p(x),\ w(x) > 0 \text{ on } I\ , 1/p, q, w \in L_{\mathrm{loc}}(I), \tag{1.4}$$

where $L_{\mathrm{loc}}(I)$ denotes the locally Lebesgue integrable functions on I. These are the minimal conditions the coefficients must satisfy for the initial value problem,

$$-(py')' + qy = 0, \quad x \in I, \quad y(a) = y_0, \quad y'(a) = y_1,$$

to have a unique solution. Sturm imposed no conditions on his coefficients, but was perhaps thinking of continuous coefficients. It is fair to say that thousands of papers have been written concerning the properties of solutions of (1.2), and hundreds more are published each year. Tony Zettl has called (1.2) the world's most popular differential equation. A recent check in math reviews shows 8178 entries for the word "oscillatory", 3284 entries for "disconjugacy", 1412 entries for "non-oscillatory", and even 62 for "Picone identity". The applications of (1.2) and (1.3) are ubiquitous. Their appearance in problems of heat flow and vibrations were well known since the work of Fourier. They play an important role in quantum mechanics where the problems are singular in the sense that I is an interval of infinite extent or where at a finite endpoint a coefficient fails to satisfy certain integrability conditions. Today we can find numerically with computers the solutions of (1.2) or the eigenvalues and eigenfunctions associated with (1.3). However, even with current technology, there are still problems which give computational difficulty such as computing two eigenvalues which are close together. Codes such as SLEIGN2 [9] (developed by Bailey, Everitt, and Zettl) or the NAG routines give quickly and accurately the eigenvalues and eigenfunctions of large classes of Sturm-Liouville problems. The recent text by Pryce [85] is devoted to the numerical solution of Sturm-Liouville problems.

For (1.1), Sturm imposed a condition ($h(r)$ is a given function),

$$\frac{K(\alpha, r)}{V(\alpha, r)} \frac{\partial V(\alpha, r)}{\partial x} = h(r), \tag{1.5}$$

and obtained the following central result [94] (after noting that when the values of $V(\alpha, r)$, $\partial V(\alpha, r)/\partial x$ are given, the solution $V(x, r)$ is uniquely determined). We have also used Lützen's translation [74].

Theorem A. *If V is a nontrivial solution of* (1.1) *and* (1.5), *and if for all $x \in [\alpha, \beta]$,*

1. $K > 0$ *for all r and K is a decreasing function of r,*
2. G *is an increasing function of r,*
3. $h(r)$ *is a decreasing function of r,*

then $\left(\frac{K}{V}\frac{\partial V}{\partial x}\right)$ is a decreasing function of r for all $x \in [\alpha, \beta]$.

Here decreasing or increasing means strictly. If $V(\alpha, r) = 0$, then $h(r)$ decreasing means $\partial V/\partial x \cdot \partial V/\partial r < 0$ at $x = \alpha$. Sturm's method of proof of Theorem A was to differentiate (1.1) with respect to r, multiply this by V, and then subtract this from $\partial V/\partial r$ times (1.1). After an integration by parts over $[\alpha, x]$, the resulting equation obtained is

$$\left(-V^2 \frac{\partial}{\partial r}\left(\frac{K}{V}\frac{\partial V}{\partial x}\right)\right)(x) = \left(-V^2(\alpha, r)\frac{dh}{dr}\right) + \int_\alpha^x \left[\frac{\partial G}{\partial r}V^2 - \frac{\partial K}{\partial r}\left(\frac{\partial V}{\partial r}\right)^2\right], \quad (1.6)$$

where we have used

$$-V^2 \frac{\partial}{\partial r}\left(\frac{K}{V}\frac{\partial V}{\partial x}\right) = K\frac{\partial V}{\partial x}\frac{\partial V}{\partial r} - V\frac{\partial}{\partial r}\left(K\frac{\partial V}{\partial x}\right). \quad (1.7)$$

If we solve this equation for the term $\frac{\partial}{\partial r}\left(\frac{K}{V}\frac{\partial V}{\partial x}\right)(x)$, then we get

$$\frac{\partial}{\partial r}\left(\frac{K}{V}\frac{\partial V}{\partial x}\right)(x, r) < 0, \quad (1.8)$$

which completes the proof.

An examination of the above proof shows that the same conclusion can be reached with less restrictive hypotheses. With $K > 0$, an examination of the right-hand side of (1.6) shows that it is positive, and hence (1.8) holds under any one of the following three conditions.

$$\frac{\partial G}{\partial r} > 0, \ \frac{\partial K}{\partial r} \leq 0, \frac{dh}{dr} \leq 0, \quad (1.9)$$

$$\frac{\partial G}{\partial r} \geq 0, \ \frac{\partial K}{\partial r} \leq 0, \frac{dh}{dr} < 0, \quad (1.10)$$

$$\frac{\partial G}{\partial r} \geq 0, \ \frac{\partial K}{\partial r} < 0, \frac{dh}{dr} \leq 0, \ V \text{ is not constant.} \quad (1.11)$$

Theorem A has immediate consequences. The first is that if $x(r)$ denotes a solution of $V(x, r) = 0$, then by implicit differentiation, we get from (1.7) and (1.8) that

$$\frac{dr}{dx} = -\frac{\partial V}{\partial x} / \frac{\partial V}{\partial r} < 0. \quad (1.12)$$

Note that this implies under the conditions of Theorem A, that the roots $x(r)$ of $V(x, r)$ are decreasing with respect to r. With $K > 0$ the same conclusion may be reached by replacing the hypothesis of Theorem A with (1.9), (1.10), or (1.11).

By considering two equations, $(K_i V_i')' + G_i V_i = 0$, $i = 1, 2$, with $G_2(x) \geq G_1(x)$, $K_2(x) \leq K_1(x)$ and embedding the functions h_1, h_2, G_1, G_2 and K_1, K_2 into a continuous family, e.g., one can define

$$\hat{G}(r, x) = rG_2(x) + (1 - r)G_1(x),\ 0 \leq r \leq 1,$$

and similarly for K, Sturm was able to prove comparison theorems. In particular he proved

Theorem B (Sturm's Comparison Theorem). *For $i = 1, 2$ let V_i be a nontrivial solution of $(K_i V_i')' + G_i V_i = 0$. Suppose further that with $h_i = (K_i V_i'/V_i)(\alpha)$,*

$$h_2 < h_1, \quad G_2(x) \geq G_1(x), \quad K_2(x) \leq K_1(x), \quad x \in [\alpha, \beta].$$

Then if α, β are two consecutive zeros of V_1, the open interval (α, β) will contain at least one zero of V_2.

In case $V_i(\alpha) = 0$, the proper interpretation of infinity must be made.

This version of comparison corresponds to using the hypothesis (1.10). Other versions may be proved by using either (1.9) or (1.11). Perhaps the most widely stated version of Sturm's comparison theorem (not the version he proved) may be stated as follows.

Theorem B*. *For $i = 1, 2$ let V_i be a nontrivial solution of $(K_i V_i')' + G_i V_i = 0$ on $\alpha \leq x \leq \beta$. Suppose further that the coefficients are continuous and for $x \in [\alpha, \beta]$,*

$$G_2(x) \geq G_1(x), \text{ with } G_2(x_0) > G_1(x_0) \text{ for some } x_0, \quad K_2(x) \leq K_1(x).$$

Then if α, β are two consecutive zeros of V_1, the open interval (α, β) will contain at least one zero of V_2.

Sturm's methods also yielded (in modern terminology):

Theorem C (Sturm's Separation Theorem). *If V_1, V_2 are two linearly independent solutions of $(KV')' + GV = 0$ and a, b are two consecutive zeros of V_1, then V_2 has a zero on the open interval (a, b).*

The final result of Sturm that we wish to quote concerns the zeros of eigenfunctions and is proved in his second memoir [95]. Here he considered the eigenvalue problem,

$$(k(x)V'(x))' + [\lambda g(x) - l(x)]V(x) = 0, \quad \alpha \leq x \leq \beta, \tag{1.13}$$

with separated boundary conditions,

$$k(\alpha)V'(\alpha) - hV(\alpha) = 0, \quad k(\beta)V'(\beta) + HV(\beta) = 0. \tag{1.14}$$

Further the functions k, g, and l are assumed positive. Some properties he established are:

Theorem D. *There are infinitely many real simple eigenvalues $\lambda_1, \lambda_2, \ldots$ of (1.13) and (1.14), and if $V_1, V_2, \ldots$ are the corresponding eigenfunctions, then for $n = 1, 2, \ldots,$*

1. *V_n has exactly $n - 1$ zeros in the open interval (α, β),*
2. *between two consecutive zeros of V_{n+1} there is exactly one zero of V_n.*

Theorem D relates to the spectral theory of the operator associated with (1.13) and (1.14). For (1.2) considered on an infinite interval $I = [a, \infty)$, an eigenvalue problem, in order to define a self-adjoint operator, may only require one boundary condition at a (limit point case at infinity), or it may require two boundary conditions involving both a and infinity (limit circle case at infinity). This dichotomy was discovered by Weyl. In the limit point case with $w \equiv 1$, a self-adjoint operator is defined in the Hilbert space $L^2(a, \infty)$ of Lebesgue square integrable functions by

$$L_\alpha[y] = -(py')' + qy, \quad y \in \mathcal{D},$$

where

$$\mathcal{D} = \{y \in L^2(a, \infty) : y,\, py' \in AC_{loc}, L_\alpha[y] \in L^2(a, \infty), \\ y(a) \sin\alpha - (py')(a) \cos\alpha = 0\}, \quad (1.15)$$

and AC_{loc} denotes the locally absolutely continuous functions.

Unlike the case (1.13) and (1.14) for the compact interval, the spectrum for the infinite interval may contain essential spectrum, i.e., numbers λ such that $L_\alpha - \lambda I$ has a range that is not closed, and Theorem D does not apply. However in the case of a purely discrete spectrum bounded below, a version of Theorem D carries over to the operator L_α above in the relation of the index of the eigenvalue to the number of zeros of the eigenfunction in (a, ∞) [22]. In general, one can say that the number of points in the spectrum of L_α below a real number λ_0 is infinite if and only if the equation $-(py')' + qy = \lambda_0 y$ is oscillatory, i.e., the solutions have infinitely many zeros on $[a, \infty)$. This same result carries over to self-adjoint equations of arbitrary order if the definition of oscillation in Section 4 is used [80, 99]. This basic connection has been used extensively in spectral theory. Note that if $-(py')' + qy = \lambda_0 y$ is non-oscillatory for every λ_0, then the spectrum of L_α consists only of a sequence of eigenvalues tending to infinity. Theorem D and its generalizations have also important numerical consequences. When an eigenvalue is computed, it allows one to be sure which eigenvalue it is, i.e., just count the zeros of the eigenfunction. It also allows the calculation of an eigenvalue without first calculating the eigenvalues that precede it. This feature is built into some eigenvalue codes.

A number of monographs deal almost exclusively with the oscillation theory of linear differential equations and systems. The books of Coppel [24] and Reid [88] emphasize linear Hamiltonian systems, but also contain substantial material on the second-order case. Coppel contains perhaps the most concise treatment of Hamiltonian systems; Reid is the most comprehensive development of Sturm theory. The book of Elias [29] is based on the oscillation and boundary value problem theory for two term ordinary differential equations, while Greguš [38] deals entirely with third-order equations. The text by Kreith [62] includes abstract oscillation theory as well as oscillation theory for partial differential equations. Finally the classic book by Swanson [96] has special chapters on second, third, fourth-order ordinary differential equations as well as results for partial differential equations. The reader is also referred to the survey papers of Barrett [10] and Willett [100]. The books by Atkinson [8], Glazman [37], Hartman [44], Ince [53], Kratz [61], Müller-Pfeiffer [80], and Reid [86] contain many results on oscillation theory.

As noted, the literature on the Sturm theory is voluminous. There are extensive results on difference equations, delay and functional differential equations, and partial differential equations. The Sturm theory for difference equations is similar to that of ordinary differential equations, but contains many new twists. The book by Ahlbrandt and Peterson [6] details this theory (see also the text by B. Simon in the present volume). Oscillation results for delay and functional equations as well as further work on difference equations can be found in the books by Agarwal, Grace, and O'Regan [1, 2], I. Gyori and G. Ladas [39], and L. Erbe, Q. Kong, and B. Zhang [31]. We confine ourselves to the case of ordinary differential equations and at that we are only able to pursue a few themes.

The comparison and oscillation theorems of Sturm have remained a topic of considerable interest. While the extensions and generalizations have much intrinsic interest, we believe their continued relevance is due in no small part to their intimate connection with problems of physical origin. Particularly the connections with the minimization problems of the calculus of variations and optimal control as well as the spectral theory of differential operators are important. We will discuss some of these connections below. We will trace some of the developments that have occurred with respect to the comparison and separation theorems as well as other developments related to Theorem D. The tools introduced by Picone, Prüfer, and the variational methods will be discussed and their applications to second-order equations as well as to higher-order equations and systems. Sample results will be stated and a few short and elegant proofs will be given. The problem of extending Sturm's results to systems was only considered about one hundred years after Sturm; the work of Morse was fundamental in this development. It is interesting that it was variational theory which gave the most natural and fruitful generalization of the definitions of oscillation. In a very loose way, we show that the theme of largeness of the coefficient q in $(py')' + qy = 0$ leads to oscillation in not only the second-order, but also higher-order equations, while $q \leq 0$, or $|q|$ small leads to disconjugacy.

2. Extensions and more rigor

Sturm's proofs of course do not meet the standards of modern rigor. They meet the standards of his time, and are in fact correct in method and can without too much trouble be made rigorous. The first efforts to do this are due to Bôcher in a series of papers in the Bulletin of the AMS [17] and are also contained in his book [18]. Bôcher [17] remarks that "the work of Sturm may, however, be made perfectly rigorous without serious trouble and with no real modification of method". The conditions placed on the coefficients were to make them piecewise continuous. Bôcher used Riccati equation techniques in some of his proofs; we note that Sturm mentions the Riccati equation, but does not employ it in his proofs. Riccati equation techniques in variational theory go back at least to Legendre who in 1786 gave a flawed proof of his necessary condition for a minimizer of an integral functional. A correct proof of Legendre's condition using Riccati equations can be found in Bolza's 1904 lecture notes [19]. Bolza attributes this proof to Weierstrass.

Bôcher was also motivated by the oscillation theorem of Klein [58] which is a multiparameter version of Sturm's existence proof for eigenvalues. Bôcher [17] noted that Klein "had given rough geometrical proofs which however made no pretence at rigor". The general form of Klein's problem may be stated as follows, see Ince [53, p. 248]. Suppose in (1.2), q is of the form

$$q(x) = -l(x) + [\lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n]g(x),$$

where p, l, g are continuous with $p(x), g(x) > 0$. Further let there be $n+1$ intervals $[a_0, b_0], \ldots, [a_n, b_n]$ with $a_0 < b_0 < a_1 < \cdots < a_n < b_n$. Suppose m_s, $s = 0, \ldots, n$ are given nonnegative integers and on each interval $[a_s, b_s]$, separated boundary conditions of the form (1.14) are given. Then there exist a set of simultaneous characteristic numbers $\lambda_0, \ldots, \lambda_n$ and corresponding functions $y_0, \ldots, y_n$ such that on each $[a_s, b_s]$, y_s has m_s zeros in (a_s, b_s) and satisfies the boundary conditions for $[a_s, b_s]$. Klein was interested in the two parameter Lamé equation

$$y'' + \frac{1}{2}\left[\frac{1}{x-e_1} + \frac{1}{x-e_2} + \frac{1}{x-e_3}\right]y' - \frac{Ax+B}{4(x-e_1)(x-e_2)(x-e_3)}y = 0$$

because of its application to physics. The text by Halvorsen and Mingarelli [40] deals with the oscillation theory of the two parameter case.

The proofs of Sturm's theorems depend on existence-uniqueness results for (1.2), and Norrie Everitt has brought to our attention that it was Dixon [25] who first proved that these are valid under only the assumption that the coefficients $1/p$, q are Lebesgue integrable functions. The details of Dixon's work may be found in N. Everitt's text in the present volume. Later Carathéodory generalized the concept of a solution of a system of differential equations to only require the equation hold almost everywhere. When (1.2) is written in system form, the Dixon and Carathéodory conditions are the same. Richardson [89, 90] extended the results of counting zeros of eigenfunctions further by allowing the weight $g(x)$ in (1.13) to not be of constant sign and called this the non-definite case. We will return to his case in Section 5. Part (1) of Theorem D, which is for the separated boundary conditions (1.14), was extended by Birkhoff [16] to the case of arbitrary self-adjoint boundary conditions.

To simplify our discussion, we will henceforth assume that all coefficients and matrix components are real and piecewise continuous unless otherwise stated.

Thinking of examples like $y'' + ky = 0, k > 0$, whose solutions are sines and cosines or the Euler equation $y'' + kx^{-2}y = 0$ which has oscillatory solutions if and only if $k > 1/4$, it is natural to pose the problem:

$$\text{When are all solutions of } (py')' + qy = 0 \text{ oscillatory on } I? \tag{2.1}$$

We use the term *oscillatory* (*non-oscillatory*) here in the sense of infinitely (finitely) many zeros for all nontrivial solutions. Because of the Sturm separation theorem, if one nontrivial solution has infinitely many zeros, then all do, but this property fails for nonlinear equations. A second problem, not quite so obvious, but which arose naturally from the calculus of variations, is

$$\text{When is the equation } (py')' + qy = 0 \text{ disconjugate on } I? \tag{2.2}$$

The term *disconjugate* is used here to mean that no nontrivial solution has more than one zero on I. If a nontrivial solution of $(py')' + qy = 0$ has a zero at a, then the first zero of y to the right of a is called the first right *conjugate point* of a; if there are no zeros to the right of a, then we say the equation is right disconjugate. Successive zeros are isolated and hence yield a counting of conjugate points. If y satisfies $y'(a) = 0$, then the first zero of y to the right of a is called the first right *focal point* of a. If y has no zeros to the right of a, then $(py')' + qy = 0$ is called right *disfocal*. Similar definitions are made to the left. The simplest criterion for both right disconjugate and disfocal is for $q(x) \le 0$, for then an easy argument shows y is monotone if $y(a) \ge 0$, $y'(a) \ge 0$. On a compact or open interval I disconjugacy is equivalent to there being a solution of $(py')' + qy = 0$ with no zeros on I [24, p.5]. For a half-open interval $(py')' + qy = 0$ can be disconjugate without there being a solution with no zeros as is shown by the equation $y'' + y = 0$ on $[0, \pi)$ which is disconjugate, but every solution has a zero in $[0, \pi)$.

A major advance was made by Picone [83] in his 1909 thesis. He discovered the identity

$$\left[\frac{u}{v}\left(vpu' - uPv'\right)\right]' = u(pu')' - \frac{u^2}{v}(Pv')' + (p - P)\,u'^2 + P\left(u' - \frac{u}{v}v'\right)^2 \tag{2.3}$$

which holds when u, v, pu', and Pv' are differentiable and $v(x) \neq 0$. In case u, v are solutions of the differential equations

$$(pu')' + qu = 0, \qquad (Pv')' + Qv = 0,$$

(2.3) reduces to

$$\left[\frac{u}{v}\left(vpu' - uPv'\right)\right]' = (Q - q)\,u^2 + (p - P)\,u'^2 + P\left(u' - \frac{u}{v}v'\right)^2. \tag{2.4}$$

With this identity one can give an elementary proof of Sturm's comparison Theorem B* which we now give. Suppose $p(x) \ge P(x)$, $Q(x) \ge q(x)$ with $Q(x_0) > q(x_0)$ at some x_0, α, β are consecutive zeros of a nontrivial solution u of $(pu')' + qu = 0$, and that v is a solution of $(Pv')' + Qv = 0$ with no zeros in the open interval (α, β). Note the quotient $u(x)/v(x)$ has a limit at the endpoints. For example the limit at α is zero if $v(\alpha) \neq 0$, and the limit is $u'(\alpha)/v'(\alpha)$ if $v(\alpha) = 0$. Integration of (2.4) over $[\alpha, \beta]$ yields that the left-hand side integrates to zero while the right-hand side integrates to a positive number. This contradiction proves the theorem.

Another major advance was made by Prüfer [84] with the use of trigonometric substitution. In the equation $(pu')' + (q + \lambda w)u = 0$, he made the substitution

$$u = \rho \sin\theta, \qquad pu' = \rho\cos\theta,$$

and then proved that ρ, θ satisfy the differential equations

$$\theta' = \frac{1}{p}\cos^2\theta + (q + \lambda w)\sin^2\theta, \quad \rho' = (\frac{1}{p} - q - \lambda w)(\sin\theta\cos\theta)\rho.$$

The zeros of the solution u are given by the values of x such that $\theta(x) = n\pi$ for some integer n. The equation for θ is independent of ρ, and by using a first-order comparison theorem for nonlinear equations, it is possible to establish Sturm's comparison theorem. Prüfer used the equation for θ to establish the link stated in Theorem D between the number of zeros of an eigenfunction and the corresponding eigenvalue. These equations can also be used to prove the existence of infinitely many eigenvalues. This is the method used in most textbooks today for the proof of Theorem D.

Note that with Prüfer's transformation, the equation $(py')' + qy = 0$, $a \leq x < \infty$, is oscillatory if and only if $\theta(x) \to \infty$ as $x \to \infty$. It also follows easily from this transformation that

$$\int_a^\infty \left[\frac{1}{p} + |q|\right] dx < \infty \Rightarrow \text{ non-oscillation,}$$

$$\int_a^\infty \left[\frac{1}{p} + |q|\right] dx < \pi \Rightarrow \text{ disconjugacy.}$$

Kamke [56] used the trigonometric substitution technique to prove a Sturm type comparison theorem for a system of first-order equations

$$y' = Py + Qz, \quad z' = Ry + Sz$$

where the coefficients are continuous functions.

Klaus and Shaw [57] used the Prüfer transformation to study the eigenvalues of a Zakharov-Shabat system. One of their results shows that the first-order system

$$v_1' = sv_1 + q(t)v_2, \quad v_2' = -sv_2 - q(t)v_1,$$

is (in our terminology) right disfocal on $-d \leq t \leq d$ if $\int_{-d}^{d} |q(t)|dt \leq \pi/2$; moreover the constant $\pi/2$ is sharp. Extension is then made to the interval $(-\infty, \infty)$ and for complex-valued q. Application is made to the nonexistence of eigenvalues (s is the eigenparameter) of the Zakharov-Shabat system, and hence to the nonexistence of soliton solutions of an associated nonlinear Schrödinger equation.

Sturm's comparison Theorem B* has been generalized to include integral comparisons of the coefficients. Consider the two equations, for $a \leq x < \infty$,

$$y'' + q_1(x)y = 0, \tag{2.5}$$

$$y'' + q_2(x)y = 0. \tag{2.6}$$

Then we may phrase Sturm's comparison theorem by:

If $q_1(x) \leq q_2(x)$, $a \leq x < \infty$, then (2.6) disconjugate $\Rightarrow$ (2.5) disconjugate.

This result was extended by Hille [50] (as generalized by Hartman [44, p. 369]) to read:

$$\text{If } \int_t^\infty q_1(x)dx \le \int_t^\infty q_2(x)dx,\ a \le t < \infty,$$
$$\text{then (2.6) disconjugate} \Rightarrow \text{ (2.5) disconjugate.}$$

Further results of this nature were given by Levin [67] and Stafford and Heidel [92].

3. Some basic oscillation results

The first major attack on problem (2.1) seems to have been made in 1883 by Kneser [59] who studied the higher-order equation $y^{(n)} + qy = 0$, and proved that all solutions oscillate an infinite number of times provided that $x^m q(x) > k > 0$ for all sufficiently large values of x, where $n \ge 2m > 0$ and n is even. Of course for $n = 2$, this follows immediately from the Sturm comparison theorem applied to the oscillatory Euler equation $y'' + kx^{-2}y = 0$, $k > 1/4$, since $k/x^2 \le k/x$ for $x \ge 1$. Hubert Kalf has noted that Weber [98] refined Kneser's result to decide on oscillation or non-oscillation in the case where $x^2 q(x)$ tends to a limit as x tends to infinity. The Kneser criterion has recently been extended by Gesztesy and Ünal [36].

A result which subsequently received a lot of attention was proved by Fite [33] in studying the equation $y^{(n)} + py^{(n-1)} + qy = 0$ on a ray $x \ge x_1$. Fite's result was if $q \ge 0$, $\int_{x_1}^\infty q dx = \infty$ and y is a solution of $y^{(n)} + qy = 0$, then y must change sign an infinite number of times in case n is even, and in case n is odd such a solution must either change sign an infinite number of times or not vanish at all for $x \ge x_1$. For $n = 2$ we then have a sufficient condition for (2.1), i.e.,

$$q(x) \ge 0, \quad \int_{x_1}^\infty q(x)dx = \infty \Rightarrow y'' + qy = 0 \text{ is oscillatory.}$$

This theme of $q(x)$ being sufficiently large has reoccurred in oscillation theory in many situations. The first improvement of the Fite result was due to Wintner [101] who removed the sign restriction on $q(x)$ and proved the stronger result

$$t^{-1}\int^t q(x)(t-x)dx \to \infty \text{ as } t \to \infty \Rightarrow y'' + qy = 0 \text{ is oscillatory.}$$

Independently Leighton [64] proved, for $(py')' + qy = 0$, that

$$\int^\infty \frac{dx}{p(x)} = \infty, \quad \int^\infty q(x)dx = \infty \Rightarrow (py')' + qy = 0 \text{ is oscillatory.}$$

Again there is no sign restriction on $q(x)$.

An elegant proof of this Fite-Wintner-Leighton result has been given by Coles [23]. We give this proof since it a good illustration of Riccati equation techniques.

Suppose that $\int^\infty p^{-1}\,dx = \infty$, $\int^\infty q\,dx = \infty$, and that u is a non-oscillatory solution of $(pu')' + qu = 0$, say $u(x) > 0$ on $[b, \infty)$. Define $r = pu'/u$. Then a calculation shows that $r' = -q - r^2/p$, and hence for large x, say $x \geq c$,

$$r(x) + \int_b^x \frac{r^2}{p}\,dt = r(b) - \int_b^x q\,dx < 0.$$

This implies that $r(x) < -\int_b^x p^{-1}r^2\,dt$. Thus defining $R(x) = \int_b^x p^{-1}r^2\,dt$, one has that for $x \geq c$, $R' = r^2/p \geq R^2/p$. Integration of this inequality gives

$$\int_c^x \frac{1}{p}\,dt \leq \int_c^x \frac{R'}{R^2}\,dt = \frac{1}{R(c)} - \frac{1}{R(x)} \leq \frac{1}{R(c)}$$

which is contrary to $\int^\infty p^{-1}\,dx = \infty$.

Related to the above result of Wintner is that of Kamenev [55] who showed that if for some positive integer $m > 2$,

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_a^t (t-s)^{m-1} q(s)\,ds = \infty,$$

then the equation $y'' + qy = 0$ is oscillatory on $[a, \infty)$. The Kamenev type results have been extended to operators with matrix coefficients and Hamiltonian systems by Erbe, Kong, and Ruan [30], Meng and Mingarelli [75], and others.

The mid-twentieth century saw a large number of papers written on problems (2.1) and (2.2). We mention a small sampling of these results.

Theorem 3.1 (Hille, 1948). *If $q(x) \geq 0$ is a continuous function on $I = [a, \infty)$, such that $\int_a^\infty q < \infty$, and*

$$g_* := \liminf_{x\to\infty} x \int_x^\infty q(t)dt, \quad g^* := \limsup_{x\to\infty} x \int_x^\infty q(t)dt,$$

then $g^* > 1$ or $g_* > 1/4$ implies $y'' + qy = 0$ is oscillatory, and $g^* < 1/4$ implies $y'' + qy = 0$ is non-oscillatory.

Hille's results have been extended to equations with matrix coefficients and linear Hamiltonian systems by Sternberg [93] and Ahlbrandt [3].

Theorem 3.2 (Hartman, 1948). *If $y'' + qy = 0$ is non-oscillatory on $[a, \infty)$, then there are solutions u, v of $y'' + qy = 0$ such that*

$$\int^\infty u^{-2}(t)\,dt < \infty \quad \text{and} \quad \int^\infty v^{-2}(t)\,dt = \infty.$$

Theorem 3.3 (Wintner, 1951). *The equation $y'' + qy$ is non-oscillatory on $[a, \infty)$ if $\int_x^\infty q(t)\,dt$ converges and either $-3/4 \leq x\int_x^\infty q(t)\,dt \leq 1/4$ or $[\int_x^\infty q(t)\,dt]^2 \leq q(x)/4$.*

Theorem 3.4 (Nehari, 1954). *If $I = [a, \infty)$ and $\lambda_0(b)$ is the smallest eigenvalue of*

$$-y'' = \lambda c(x) y, \quad y(a) = y'(b) = 0,$$

where $c(x) > 0$ is continuous on I, then $y'' + c(x)y = 0$ is non-oscillatory on I iff $\lambda_0(b) > 1$ for all $b > a$.

Theorem 3.5 (Hartman-Wintner, 1954). *The equation* $y''+qy=0$ *is non-oscillatory on* $[a,\infty)$ *if* $f(x)=\int_x^\infty q(t)dt$ *converges and the differential equation* $v''+4f^2(x)v=0$ *is non-oscillatory.*

Theorem 3.6 (Hawking-Penrose, 1970). *If* $I=(-\infty,\infty)$ *and* $q(x)\geq 0$ *is a continuous function on* I *such that* $q(x_0)>0$ *for some* x_0*, then* $y''+q(x)y=0$ *is not disconjugate on* I*.*

A particularly simple proof of this result has been given by Tipler [97] which we now present. Suppose y is the unique solution of $y''+q(x)y=0$ with the initial conditions $y(x_0)=1$, $y'(x_0)=0$. Then $y''(x_0)=-q(x_0)y(x_0)<0$, and further $y''(x)\leq 0$ as long as $y(x)\geq 0$. Since $y'(x_0)=0$, this concavity of y implies that y eventually has a zero both to the right and to the left of x_0.

Many results on oscillation can be expanded by making a change of independent and dependent variables of the form $y(x)=\mu(x)z(t)$, $t=f(x)$, where $\mu(x)$ and $f'(x)$ are nonzero on the interval I. In the case of $(py')'+qy$, this leads to

$$(py')'+qy=(\gamma/\mu)[\dot{w}+Qz], \quad w=P\dot{z}, \quad \gamma(x)=f'(x),$$

where $\dot{z}=dz/dt$ and

$$P(t)=p(x)\mu^2(x)\gamma(x), \quad Q(t)=\frac{\mu(x)}{\gamma(x)}\left[(p\mu')'+q\mu\right].$$

Applications of these ideas can be found in Ahlbrandt, Hinton, and Lewis [5].

To return to the concept of disconjugacy and the link to the calculus of variations, it was in 1837 that Jacobi [54] gave his sufficient condition for the existence for a (weak) minimum of the functional

$$J[y]=\int_a^b f(x,y,y')dx \tag{3.1}$$

over the class of admissible functions y defined as those sufficiently smooth y satisfying the endpoint conditions $y(a)=A$, $y(b)=B$. A necessary condition for an extremal is the vanishing of the first variation, $dJ(y+\epsilon\eta)/d\epsilon\rfloor_{\epsilon=0}$, for sufficiently smooth variations η satisfying $\eta(a)=\eta(b)=0$. This leads to the Euler-Lagrange equation $f_y-d(f_{y'})/dx=0$ for y. A sufficient condition for a weak minimum is that the second variation

$$\delta^2 J(\eta)=\int_a^b\left[p\eta'^2+q\eta^2\right]dx \tag{3.2}$$

be positive for all nontrivial admissible η where $p=f_{y'y}$ and $q=f_{yy}-d(f_{y'y})/dx$. Jacobi discovered that the positivity of (3.2) was related to the oscillation properties of $-(py')'+qy=0$. In particular he discovered (3.2) is positive if $-(py')'+qy=0$ has a solution y which is positive on $[a,b]$. The condition of (3.2) being positive is equivalent to $-(py')'+qy=0$ being disconjugate on $[a,b]$. This is the principal connection of oscillation theory to the calculus of variations. This connection may be proved with Picone's identity as we now demonstrate.

First suppose (1.2) is disconjugate on $[a, b]$; hence there is a solution v of (1.2) which is positive on [a,b]. Then (2.3) with $p = P$ yields for the variation η,

$$\left[\eta(p\eta') - \frac{\eta^2}{v}(Pv')\right]' = \eta(p\eta')' - \frac{\eta^2}{v}(Pv')' + p\left(\eta' - \frac{\eta}{v}v'\right)^2.$$

Simplifying this expression yields that

$$p\eta'^2 - \left[\frac{\eta^2}{v}(pv')\right]' = -q\eta^2 + p\left(\eta' - \frac{\eta}{v}v'\right)^2,$$

which one can verify directly using only one derivative for η. An integration and applying $\eta(a) = \eta(b) = 0$ gives that

$$0 \leq \int_a^b p\left(\eta' - \frac{\eta}{v}v'\right)^2 dx = \int_a^b \left(p\eta'^2 + q\eta^2\right) dx$$

with equality if and only if $\eta' = \eta v'/v$. But $\eta' = \eta v'/v$ implies $(\eta/v)' = 0$ or η/v is constant. This is contrary to $\eta(a) = 0$, $v(a) \neq 0$. Hence $\delta^2 J(\eta)$ is positive. On the other hand if (1.2) is not disconjugate, there is a nontrivial solution u with $u(c) = u(d) = 0$, $a \leq c < d \leq b$. By defining $\eta(x) = u(x)$, $c \leq x \leq d$, and $\eta(x) = 0$ otherwise, it follows that $\delta^2 J(\eta) = 0$ so that $\delta^2 J$ fails to be positive for all nontrivial admissible functions.

Leighton was able to exploit this equivalence to obtain comparison theorems, e.g., as in [65]. One of his results is that if there is a nontrivial solution u in $[a, b]$ of $(pu')' + qu = 0$ such that $u(a) = u(b) = 0$, and

$$\int_a^b \left[(p - P)u'^2 + (Q - q)u^2\right] dx > 0,$$

then every solution of $(Pv')' + Qv = 0$ has at least one zero in (a,b). This has as a corollary Sturm's Comparison Theorem B*. Angelo Mingarelli has pointed out that the monotonicity condition on the G coefficient in Sturm's comparison theorem has been replaced by a convexity condition by Hartman [45].

When the equivalence of disconjugacy of (1.2) to positivity of (3.2) is used to show oscillation, it is frequently done by a construction. That is, if (1.2) is considered on $I = [a, \infty)$, and it can be shown that for each $b > a$ there is a function η_b with compact support in $[b, \infty)$ such that $\delta^2 J(\eta_b) \leq 0$, then (1.2) is oscillatory. When the equivalence is used to show disconjugacy, it is usually done by the use of inequalities which bound the integral $\int_a^b q\eta^2\, dx$ in terms of the integral $\int_a^b p\eta'^2\, dx$. For example, the Hardy inequality $\int_a^b x^{-2}u^2\, dx \leq (1/4)\int_a^b u'^2\, dx$ for functions u satisfying $u(a) = u(b) = 0$, $a > 0$, can be used to show that $u'' + qu = 0$ is disconjugate on $[a, \infty)$, $a > 0$, if $|x\int_x^\infty q(t)dt| \leq 1/4$.

Oscillation theory in the complex domain, i.e., for an equation of the form

$$w''(z) + G(z)w(z) = 0, \quad z \in \mathcal{D}, \tag{3.3}$$

where $w(z)$ is a function analytic in the domain $\mathcal{D}$, did not begin until the end of the nineteenth century. The earliest work dealt with special functions which are themselves solutions of second-order linear differential equations. Hurwitz [52] in 1889 investigated the zeros of Bessel functions in the complex plane. Work soon followed on other special functions. The definitions of disconjugate and nonoscillatory are the same as in the real case although now there is no simple ordering of the zeros. The location of complex zeros has found recent application in the quantum mechanical problem of locating resonances and anti-bound states as in Brown and Eastham [20], Eastham [28], or Simon [91]. A fairly extensive analytic oscillation theory has been developed by Hille [49], Beesack [11], London [73], Nehari [81], and others. We state two such results.

Theorem 3.7 (Nehari, 1954). *If $G(z)$ is analytic in $|z| < 1$, then* (3.3) *is disconjugate in $|z| < 1$ if $|G(z)| \leq (1 - |z|^2)^{-2}$ in $|z| < 1$.*

Theorem 3.8 (London, 1962). *If $G(z)$ is analytic in $|z| < 1$, then* (3.3) *is disconjugate in $|z| < 1$ if*

$$\iint_{|z|<1} |G(z)| dxdy \leq \pi.$$

It is surprising that the oscillation theory on the real axis, especially the comparison theory, plays an important role in the analytic oscillation theory, cf., Beesack [11]. Analytic oscillation theory is also connected with the theory of univalent functions. If $f(z)$ is analytic in $\mathcal{D}$ and $G(z) = \{f(z), z\}/2$ where $\{f(z), z\}$ is the Schwarzian derivative of f, then the univalence of f in $\mathcal{D}$ is equivalent to the disconjugacy of (3.3) in $\mathcal{D}$ [11]. A summary of the analytic oscillation theory can be found in the books by Hille [51] and Swanson [96].

A notable result on disconjugacy was given by Lyapunov in 1893 [71].

Theorem 3.9 (Lyapunov). *The equation $y'' + q(x)y = 0$ is disconjugate on $[a, b]$ if $(b - a) \int_a^b |q(x)| dx \leq 4$.*

Extensions of Lyapunov's theorem to systems in the Stieltjes integral setting have been made by Brown, Clark, and Hinton [21]; further the $L[a, b]$ norm on q has been replaced by an $L_p[a, b]$ norm for $1 \leq p \leq 2$.

Disconjugacy theorems play an important role in the stability of differential equations with periodic coefficients. For $-y'' + qy = \mu y$ on $[0, \infty)$ with $q(t)$ periodic of period T, the equation is called stable if all solutions are bounded. This occurs if $\lambda_0 < \mu < \lambda_0^*$, where λ_0 is the first eigenvalue of $-y'' + qy = \lambda y$ with periodic boundary conditions, and where λ_0^* is the first eigenvalue of $-y'' + qy = \lambda y$ with semi-periodic boundary conditions. The criterion of Krein/Borg [103, II, p. 729] (see also Eastham [27, p. 49]) states that $-y'' + qy = 0$ is stable if $\int_0^T q \leq 0$, $q \neq 0$, and $T \int_0^T q_- \leq 4$, where $q_-(t) = \max\{-q(t), 0\}$. The proof of this uses the fact that $T \int_0^T q_- \leq 4$ and q periodic implies the spacing of zeros of solutions of $-y'' + qy = 0$ is greater than T. Much of the work on stability of solutions of periodic equations and systems can be found in the Russian literature; in particular, see Yakubovich and Starzhinskii [103].

Thus we see from these theorems that q sufficiently large in $(py')' + qy = 0$ will give oscillation, and that $q \le 0$ or $|q|$ sufficiently small will give disconjugacy.

4. Higher-order equations and systems

For higher-order differential equations, what is the "correct" extension of the definition of oscillatory? of disconjugate? Consider for example the two-term fourth-order equation $y^{(iv)}+q(x)y = 0$. For the distribution of four zeros of a nontrivial solution, there are seven possibilities, 3-1 (meaning $y(a) = y'(a) = y''(a) = y(b) = 0$ for some $a < b$), and with similar meanings the distributions 2-2, 1-3, 2-1-1, 1-2-1, 1-1-2, 1-1-1-1. Hence one could define seven different kinds of disconjugacy.

A widely studied point of view is that an nth-order linear ordinary differential equation is disconjugate if no nontrivial solution has n zeros containing multiplicities. This was the definition used by Levin [68, 69] and others. For the differential expression

$$l[y] = y^{(n)} + \sum_{i=1}^{n} a_i(x)y^{(n-i)} = 0, \quad \alpha \le x \le \beta, \tag{4.1}$$

one defines the first conjugate point $\delta(\alpha)$ as the supremum of all γ such that no nontrivial solution of (4.1) has more than $n-1$ zeros, counting multiplicities, on $[\alpha, \gamma]$. One result of Levin is that if $\delta(\alpha) < \infty$, then there is a nontrivial solution of (4.1) which is positive on $(\alpha, \delta(\alpha))$, and for some k, $1 \le k \le n-1$, it has a zero of order not less than k at α and a zero of order not less than $n-k$ at $\delta(\alpha)$. Green's functions are useful in establishing disconjugacy criteria in this sense. One such result by Levin is that $y^{(4)} + q(x)y = 0$ is disconjugate on $[\alpha, \beta]$ if $q(x) \ge 0$, and $\int_\alpha^\beta q(x)\,dx \le 384(\beta - \alpha)^{-3}$.

For oscillation one could again say that the equation is oscillatory if all nontrivial solutions have infinitely many zeros. However for the equation $y^{(iv)} - y = 0$ some solutions have infinitely many zeros and others have none, so some modification of the definition is required. There has been much research on the structure of somewhat special equations. In a classic paper on fourth-order equations, Leighton and Nehari [66] studied the oscillatory structure of the equations

$$(ry'')'' + qy = 0, \tag{4.2}$$

$$(ry'')'' - qy = 0, \tag{4.3}$$

where r, q are positive continuous functions on an interval $I = [a, \infty)$. Typical of their results are:

(1) If u and v are linearly independent solutions of (4.3) on $[a, \infty)$ such that $u(a) = u'(a) = v(a) = v'(a) = 0$, then the zeros of u and v separate each other in (a, ∞).

(2) If u and v are nontrivial solutions of (4.2), the number of zeros of u on any closed interval $[\alpha, \beta]$ cannot differ by more than 4 from the number of zeros of v on $[\alpha, \beta]$. In particular the nontrivial solutions of (4.2) all have infinitely many zeros on $[a, \infty)$ or none do.

(3) Suppose that $r(x) \geq R(x)$ and $q(x) \leq Q(x)$ in (4.3) and in $(Ry'')'' - Qy = 0$. Let u and v be nontrivial solutions of (4.3) and $(Ry'')'' - Qy = 0$, respectively, such that $u(\alpha) = v(\alpha) = u(\beta) = v(\beta) = 0$. If n, m denote the number of zeros of u, v respectively on $[\alpha, \beta]$ $(n \geq 4)$, then $m \geq n - 1$.

This type of separation, where the zeros of one solution of a scalar equation have interlacing properties with another solution, has been developed by Hanan [41] for third-order equations.

However, we will concentrate here on the definition of oscillation and disconjugacy that comes from the calculus of variations and has other applications such as in optimal control and spectral theory of differential equations. If in (3.1) the functions f and y are n-vector-valued, the Euler-Lagrange equation is a coupled system of n second-order differential equations. The quadratic form of the second variation is (where * indicates transpose)

$$J[\eta, \xi] = \int_a^b (\xi^*[R(x)\xi + Q(x)\eta] + \eta^*[Q^*(x)\xi + P(x)\eta])dx$$

which arises from the vector equation

$$(Ru' + Qu)' - (Q^*u' + Pu) = 0 \tag{4.4}$$

with $R(x)$, $P(x)$ hermitian and $R(x)$ nonsingular. The functions $P(x)$, $R(x)$, $Q(x)$ are expressed in terms of the partial derivatives of the components of f. Equation (4.4) can be written in the linear Hamiltonian system form

$$u' = Au + Bv, \quad v' = Cu - A^*v \tag{4.5}$$

with $A = -R^{-1}Q$, $B = R^{-1}$, and $C = P - Q^*R^{-1}Q$.

Symmetric scalar differential equations can also be put in the form (4.5). For example, the equation

$$(ry'')'' + (py')' + q(x)y = 0, \tag{4.6}$$

has the system form (4.5) with

$$u = \begin{bmatrix} y \\ y' \end{bmatrix}, \; v = \begin{bmatrix} -(ry'')' - py' \\ ry'' \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 0 & 1/r \end{bmatrix}, \; C = \begin{bmatrix} q & 0 \\ 0 & -p \end{bmatrix}.$$

In analogy to the scalar case, the vector minimization problem with fixed endpoints leads to admissible perturbations with $\eta(a) = \eta(b) = 0$. Thus we say that a solution u, v of (4.5) has a zero at a provided $u(a) = 0$, and we say $b > a$ is *conjugate* to a if there is a nontrivial solution u, v of (4.5) such that $u(a) = u(b) = 0$. Note that as applied to the scalar equation (4.6), $u(a) = 0$ is equivalent to $y(a) = y'(a) = 0$. We will say the system (4.5) is *disconjugate* on $[a, b]$ provided that there do not exist $c < d$ in $[a, b]$ such that d is conjugate to c. Otherwise we say (4.5) is *oscillatory* on [a,b]. The definition of oscillatory on a ray $I = [a, \infty)$ that turns out to be useful for spectral theory is that (4.5) is *oscillatory* on $[a, \infty)$ if for every $b > a$ there exist $b \leq c < d$ such that d is conjugate to c. Analogous to problems (2.1) and (2.2) are the questions of when (4.5) is oscillatory or disconjugate on an interval. The definitions of disfocal are similar to those in the second-order case. The system (4.5) is called *identically normal* on an interval I if $u \equiv 0$ on a subinterval of I implies also $v \equiv 0$ on the subinterval. This is a controllability condition, cf. [24].

The theme of a sufficiently large coefficient that is in the Fite-Leighton-Wintner Theorem of Section 3 has continued in the case of scalar equations of order greater than 2 and for Hamiltonian systems. Some of these results are described below.

Theorem 4.1 (Byers, Harris, Kwong, 1986). *If $Q(x)$ is a continuous symmetric $n \times n$ matrix function on $I = [a, \infty)$, and*

$$\max \text{ eigenvalue } \int_a^x Q(t)dt \longrightarrow \infty \text{ as } x \longrightarrow \infty,$$

then the equation $y'' + Q(x)y = 0$ is oscillatory on $[a, \infty)$.

Note that the scalar condition $\int_a^\infty q(x)dx = \infty$ has been replaced by the maximum eigenvalue condition.

Glazman [37] proved that the scalar equation $(-1)^{n+1}y^{(2n)} + q(x)y = 0$ is oscillatory on $[a, \infty)$ if $\int_a^\infty q(x)dx = \infty$. Various extensions of this have been made. In particular we quote the result:

Theorem 4.2 (Müller-Pfeiffer, 1982). *The equation $(-1)^{n+1}(p(x)y^{(n)})^{(n)} + q(x)y = 0$ is oscillatory on $[a, \infty)$ if*

1. *$p(x) > 0$ and for some m, $0 \le m \le n-1$, $\int_a^\infty x^{2m}[p(x)]^{-1}\,dx = \infty$,*
2. *$\int_a^\infty q(x)Q^2(x)dx = \infty$ for some polynomial Q of degree $\le n-m-1$.*

For two-term equations, the theory of reciprocal equations has been fruitful. Using the results of Ahlbrandt [4], it follows that the equation $(-1)^n(r^{-1}y^{(n)})^{(n)} - py = 0$ is non-oscillatory on $[a, \infty)$ if and only if $(-1)^n(p^{-1}y^{(n)})^{(n)} - ry = 0$ is non-oscillatory on $[a, \infty)$. Using these ideas, Lewis [70] was able to answer affirmatively an open question posed by Glazman that the condition $\lim_{x\to\infty} x^{2n-1}\int_x^\infty 1/r = 0$ was a necessary condition for the equation $(-1)^n(ry^{(n)})^{(n)} = \lambda y$ to be non-oscillatory on $[a, \infty)$ for all λ. The condition was known to be sufficient.

As noted in the theorems for second-order equations, the equation is disconjugate if the coefficient of y is sufficiently small. A theorem of this type for scalar equations is

Theorem 4.3 (Ashbaugh, Brown, Hinton, 1992). *The scalar equation $(x^\delta y^{(n)})^{(n)} + q(x)y = 0$, δ not in $\{-1, 1, \ldots, 2n-1\}$, is non-oscillatory on $I = [a, \infty)$, $a > 0$, if there is an s, $1 \le s < \infty$, such that $\int_a^\infty x^{2n-\delta-1/s}|q(x)|^s dx < \infty$.*

Associated with the system (4.5) is the matrix system

$$U' = AU + BV, \quad V' = CU - A^*V \tag{4.7}$$

where U, V are $n \times n$ matrix functions. When U is nonsingular, the function $W = VU^{-1}$ satisfies the Riccati equation

$$W' = C - WA - A^*W - WBW. \tag{4.8}$$

A solution of (4.7) is called *conjoined or isotropic* if $U^*V = V^*U$. When U is nonsingular, it is easy to show $W = W^*$ if and only if the solution U, V is conjoined. All of these concepts can be brought together in what Calvin Ahlbrandt calls the Reid Roundabout Theorem [88, p. 285].

Theorem 4.4. *Suppose on $I = [a, b]$ the coefficients A, B, C are Lebesgue integrable with C, B hermitian and B positive semi-definite and the system* (4.5) *is identically normal on I. Define $\mathcal{D}_0[a, b]$ to be the set of all n-dimensional vector functions η on $[a, b]$ which are absolutely continuous, satisfy $\eta(a) = \eta(b) = 0$, and for which there is an essentially bounded function ξ such that $\eta'(x) = A(x)\eta(x) + B(x)\xi(x)$ a.e. on $[a, b]$. For $\eta \in \mathcal{D}_0[a, b]$ define*

$$J(\eta, a, b) = \int_a^b [\xi^*(x)B(x)\xi(x) + \eta^*(x)C(x)\eta(x)]\, dx. \tag{4.9}$$

Then the following statements are equivalent.

1. *There is a conjoined solution U, V of* (4.7) *such that U is nonsingular on $[a, b]$.*
2. *If $\eta \in \mathcal{D}_0[a, b]$ and η is not the zero function, then $J(\eta, a, b) > 0$.*
3. *The system* (4.5) *is disconjugate on $[a, b]$.*
4. *The equation* (4.8) *has a hermitian solution on $[a, b]$.*

The proof of Theorem 4.4 is greatly facilitated by the Legendre or Clebsch transformation of the functional (4.9) which we now state. Suppose U, V are $n \times n$ matrix solutions of (4.7) on an interval [a,b] and U is nonsingular on $[a, b]$. If $\eta \in \mathcal{D}_0[a, b]$ with corresponding function ξ, and $W = VU^{-1}$, then

$$[\eta^* W\eta]' + [\xi - W\eta]^* B [\xi - W\eta] = \eta^* C\eta + \xi^* B\xi.$$

This follows by differentiation and substitution from (4.8).

A general Picone identity for the system (4.7) may be stated. Suppose for i=1,2 we have on an interval matrix solutions U_i, V_i of

$$U_i' = A_i U_i + B_i V_i, \quad V_i' = C_i U_i - A_i^* V_i,$$

where U_1, V_1 are $n \times r$ matrices and U_2, V_2 are $n \times n$ matrices with U_2 nonsingular. Define $W = V_2 U_2^{-1}$. Then if $A_1 = A_2$ and $B_1 = B_2$,

$$[U_1^* W U_1 - U_1^* V_1]' = U_1^* (C_2 - C_1) U_1 - [V_1 - W U_1]^* B_2 [V_1 - W U_1].$$

The general result can be found in [88, p. 354].

Calvin Ahlbrandt has pointed out that prior to Weierstrass it was thought that, as for point functions, if an admissible arc satisfied the Euler equation, the strengthened Legendre condition and the strengthened Jacobi condition (condition (3) in Theorem 4.4), then it would provide a local minimum. This was true for weak local minimums, but not for strong local minimums. Thus the theory of the second variation was discredited as having the analogous utility as the second derivative for point functions.

Theorem 4.4 gives immediately a comparison theorem. If $B_1(x) \geq B(x)$ and $C_1(x) \geq C(x)$, and J_1 is the functional corresponding to (4.9), then $J_1(\eta, a, b) \geq J(\eta, a, b)$ so that disconjugacy of (4.5) implies disconjugacy of

$$U' = AU + B_1 V, \quad V' = C_1 U - A^* V.$$

For (4.6), J is given by

$$J(\eta, a, b) = J(y, a, b) = \int_a^b [ry''^2 - py'^2 + qy^2]dx$$

over those sufficiently smooth y satisfying $y(a) = y'(a) = y(b) = y'(b) = 0$. Hence the comparison reads $r_1(x) \geq r(x), p_1(x) \leq p(x), q_1(x) \geq q(x)$ and disconjugacy of (4.6) implies disconjugacy of

$$(r_1 y'')'' + (p_1 y')' + q_1(x) y = 0.$$

Similar comparisons are immediate for the $2n$th-order symmetric differential expression $l[y] = \sum_{i=0}^{n} (p_i y^{(i)})^{(i)}$.

It was Morse [78] who gave the first generalizations of the Sturm theorems of separation and comparison to self-adjoint second-order linear differential systems. Morse proved the number of points on an interval (a, b) which are conjugate to the point a is the same as the number of negative eigenvalues of a quadratic form defined on a certain finite-dimensional space. This quadratic form is constructed from the form (4.9) of Theorem 4.4. This development can be found in [79] and [86]. In particular, it establishes a comparison between conjugate points for two systems of the form (4.5).

A solution of the problem of extending Sturm's separation Theorem C may be stated as follows [86, p. 307]. If for (4.5) there are q points conjugate to a on $(a, b]$, then for any conjoined basis of (4.5) there are at most $q+n$ points conjugate to a on $(a, b]$ and at least $q - n$ points conjugate to a on $(a, b]$. Thus if we take U, V to be the solution of (4.7) with initial conditions $U(a) = 0$, $V(a) = I$, and suppose $\det U(x)$ is zero exactly $n+1$ times in $(a, b]$, then for any other conjoined solution U_1, V_1, $\det U_1(x) = 0$ at least once. For $n = 1$ this is Sturm's theorem. Note also if $\det U(x) = 0$ infinitely many times on $[a, \infty)$, then $\det U_1(x) = 0$ infinitely many times on $[a, \infty)$.

5. Parameter dependent boundary conditions and indefinite weights

A large class of physical problems have the eigenparameter in the boundary conditions. Examples are vibration problems under various loads such as a vibrating string with a tip mass or heat conduction through a liquid solid interface. See [34] for a list of references. With the boundary condition at one endpoint containing the eigenparameter, the eigenvalue problem on $[a, b]$ takes the form of (1.3) with boundary conditions

$$y(a)\cos\alpha - (py')(a)\sin\alpha = 0, \tag{5.1}$$

$$[\beta_1\lambda + \beta_1']y(b) = [\beta_2\lambda + \beta_2'](py')(b). \tag{5.2}$$

It was noted independently by several authors, Binding, Browne, and Seddighi [14], Harrington [42], and Linden [72] that there is a skip in the counting of the zeros of the eigenfunction compared to the index of the eigenvalue. The development in [14] is the most comprehensive and also shows how the eigenvalues of (5.1)-(5.2) interlace with those of a standard Sturm-Liouville problem. We quote here Linden's theorem.

Theorem 5.1 (Linden, 1991). *For the eigenvalue problem* (1.3), (5.1), *and* (5.2), *suppose that* $\beta_1'\beta_2 - \beta_2'\beta_1 > 0$. *Then there is a countable sequence* $\lambda_1 < \lambda_2 < \cdots$ *of real simple eigenvalues with* $\lambda_k \to \infty$ *for* $k \to \infty$. *Let* y_k *denote the eigenfunction corresponding to the eigenvalue* λ_k. *If* $\beta_2' = 0$, *then* y_k *has exactly* $(k-1)$ *zeros in* (a,b). *If* $\beta_2' \neq 0$, *then for* $\lambda_k < -\beta_2/\beta_2'$, y_k *has exactly* $(k-1)$ *zeros in* (a,b), *and for* $\lambda_k \geq -\beta_2/\beta_2'$, y_k *has exactly* $(k-2)$ *zeros in* (a,b).

In the case of a parameter in the boundary condition at both endpoints there is in general a skip of two zeros in the indexing of the eigenfunctions [14, p. 65]. The case of the eigenparameter occurring rationally in the boundary conditions has been considered by Binding [12].

Everitt, Kwong, and Zettl [32] considered (1.3) with the separated boundary conditions

$$y(a)\cos\alpha - (py')(a)\sin\alpha = 0, \quad y(b)\cos\beta + (py')(b)\sin\beta = 0, \tag{5.3}$$

where the conditions on p and w were relaxed to

$$p(x),\ w(x) \geq 0, \quad \int_a^b w(x)\,dx > 0.$$

Under these conditions they were able to prove that there is a sequence $\lambda_0 < \lambda_1 < \cdots$ of simple eigenvalues tending to infinity with associated eigenfunctions $\psi_0, \psi_1, \ldots$, where each ψ_n has only a finite number m_n of zeros in the open interval (a,b) and such that

(i) $m_{n+1} = m_n + 1$,
(ii) Given any integer $r \geq 0$ there exist p, q, and w such that $m_0 = r$ and so $m_n = m_0 + n = n + r$ for n=1,2, ...

Of course $m_0 = 0$ in the standard case where $p(x)$, $w(x) > 0$. Property (i) may also be deduced from Theorem IV of [90] (see also Section 6 of [90]).

We turn now to the case where w may change sign. This occurs in some physical problems, e.g., the equation

$$-((1-x^2)y')' = \lambda x y, \quad -1 < x < 1,$$

occurs in electron transport theory. We associate with the differential expression $L[y] = -(py')' + qy$ and boundary conditions (5.3) the quadratic forms

$$Q[y,y] \;=\; \langle L[y], y\rangle \;=\; |y(a)|^2 \cot\alpha + |y(b)|^2 \cot\beta + \int_a^b [p|y'|^2 + q|y|^2]dx \quad (5.4)$$

and

$$W[y,y] = \int_a^b w|y|^2 dx. \quad (5.5)$$

Then the equation (1.3) is called *left definite* (polar by Hilbert and his school) if $Q[y,y] > 0$ for all $y \neq 0$ in the domain of Q which consists of all absolutely continuous y such that $\int_a^b p|y'|^2 dx < \infty$. It is called *right definite* if $W[y,y] > 0$ for all $y \neq 0$ such that $\int_a^b |w||y|^2 dx < \infty$. It is called *indefinite* (non-definite by Richardson) if $\int_a^b w_+ dx > 0$ and $\int_a^b w_- dx > 0$ where $w_+ = \max\{w, 0\}$, $w_- = \max\{-w, 0\}$. In his survey article, Mingarelli [76] attributes the first investigations of the general indefinite case to Haupt [47] and Richardson [89]. The indefinite equations have been studied in Krein and Pontrjagin spaces where the indefinite metric is given by $\int_a^b w|y|^2 dx$, but more for questions of completeness of eigenfunction expansions and operator theory. The indefinite problems may have complex eigenvalues, but can have only finitely many.

An early result (see Mingarelli [76]) of Haupt [47] and Richardson [90] is that in the indefinite case there exists an integer $n_R > 0$ such that for each $n > n_R$ there are at least two solutions of (1.3) and (5.3) having exactly n zeros in (a,b) while for $n < n_R$ there are no real solutions having n zeros in (a,b). Furthermore there exists a possibly different integer $n_H \geq n_R$ such that for each $n \geq n_H$ there are precisely two solutions of (1.3) and (5.3) having exactly n zeros in (a,b). It has been shown by Mingarelli that both cases $n_R = n_H$ and $n_R < n_H$ may occur.

However, in the left definite indefinite case things are more orderly and we quote the following result from Ince [53, p. 237]. If in (1.3) and (5.3), $q(x) \geq 0, 0 \leq \alpha,\ \beta \leq \pi/2$, and the problem is indefinite, then there are eigenvalues

$$\cdots < \lambda_1^- < \lambda_0^- < 0 < \lambda_0^+ < \lambda_1^+ < \cdots$$

with corresponding eigenfunctions y_n^-, y_n^+ such that both y_n^-, y_n^+ have exactly n zeros in $a < x < b$ for $n = 0, 1, \ldots$.

Further work on left-definite and indefinite problems may be found in Binding and Browne [13], Binding and Volkmer [15], and Kong, Wu, and Zettl [60]. In [15] and in Möller [77] the coefficient p is also allowed to change sign. Again the eigenvalues are unbounded above and below.

It is clear that the work of Sturm on oscillation theory has had an enduring impact in mathematics. We have only discussed a few ways in which the theory has been extended. It has been necessary to omit many important topics such as the theory of principal solutions and the renormalization theory of Gesztesy, Simon and Teschl [35] (for the latter see the text of B. Simon in the present volume). Important work on the constants of oscillation theory (as in Hille's 1948 theorem) has been done by O. Došlý [26] and others. We have just touched on the Riccati equations which arise in diverse applications and are a research area by themselves, see Reid [87]. Oscillation theory is a subject in its own right, and theorems such as Theorem 4.4 show it can be pursued independently. In his remark "Le principe sur lequel reposent les théorèmes que je développe, n'a jamais, si je ne me trompe, été employé dans l'analyse, et il ne me paraît pas susceptible de s'étendre à d'autres équations différentielles", Sturm [94] was too pessimistic that his methods could not be applied to other differential equations.

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