Trends and Applications in Constructive Approximation
(Eds.) M.G. de Bruin, D.H. Mache & J. Szabados
International Series of Numerical Mathematics, Vol. 151, 13–26

On Some Classes of Diffusion Equations and Related Approximation Problems

Francesco Altomare and Ioan Rasa

Abstract. Of concern is a class of second-order differential operators on the unit interval. The C_0-semigroup generated by them is approximated by iterates of positive linear operators that are introduced here as a modification of Bernstein operators. Finally, the corresponding stochastic differential equations are also investigated, leading, in particular to the evaluation of the asymptotic behaviour of the semigroup.

1. Introduction

In this paper we study the positive semigroups $(T(t))_{t\geq 0}$ generated by the differential operators

$$Wu(x) = x(1-x)u''(x) + (a+1-(a+b+2)x)u'(x)$$

defined on suitable domains of $C[0,1]$ which incorporate several boundary conditions.

In the spirit of a general approach introduced by the first author ([1], see also, e.g., [4], [5], [7], [14], [23]–[25] and the references given there) we show that the semigroups can be approximated by iterates of some positive linear operators that are introduced here, perhaps, for the first time.

These operators are a simple modification of the classical Bernstein operators and, as them, are of interpolatory type.

We also investigate some shape preserving properties of these operators that, in turn, imply similar ones for the semigroup $(T(t))_{t\geq 0}$.

By following a recent approach due to the second author ([24]–[25]) we also consider the solutions $(Y_t)_{t\geq 0}$ of the stochastic equations associated with W and which are formally related to the semigroups by the formula

$$T(t)f(x) = E^x f(Y_t) \quad (0 \leq x \leq 1,\ t \geq 0).$$

For suitable values of a and b we get information about $(T(t))_{t\geq 0}$ and $(Y_t)_{t\geq 0}$, in particular about their asymptotic behaviour. Analogous results have been obtained in [25] considering the operators described in [16].

We finally point out that the generation properties of the operators W have been also investigated in $L^1[0,1]$ (see [7]). In the spirit of [24], Section 7, W may be viewed also as generator of a C_0-semigroup on $L^2[0,1]$; details will appear elsewhere.

2. The semigroup

Let $C[0,1]$ be the space of all real-valued continuous functions, endowed with the supremum norm and the usual order.

For $a, b \in \mathbb{R}$ consider the differential operator

$$Wu(x) = x(1-x)u''(x) + (a+1-(a+b+2)x)u'(x)\ ,\ 0 < x < 1,\ u \in C^2(0,1).$$

The corresponding diffusion equation, i.e.,

$$\begin{gathered} u_t(t,x) = x(1-x)u_{xx}(t,x) + (a+1-(a+b+2)x)u_x(t,x) \\ (0 < x < 1,\ t \geq 0) \end{gathered} \tag{1}$$

occurs in some stochastic model from genetics discussed in [13] (see also [9] and [27]).

Usually the above diffusion equation is coupled with some initial boundary conditions.

Let

$$\begin{aligned} D_V(W) &= \{u \in C[0,1] \cap C^2(0,1) &&: \lim_{x\to 0,1} Wu(x) = 0\}, \\ D_M(W) &= \{u \in C[0,1] \cap C^2(0,1) &&: Wu \in C[0,1]\}, \\ D_{VM}(W) &= \{u \in C[0,1] \cap C^2(0,1) &&: \lim_{x\to 0} Wu(x) = 0, \lim_{x\to 1} Wu(x) \in \mathbb{R}\}, \\ D_{MV}(W) &= \{u \in C[0,1] \cap C^2(0,1) &&: \lim_{x\to 1} Wu(x) = 0, \lim_{x\to 0} Wu(x) \in \mathbb{R}\} \end{aligned}$$

and set

$$D(W) = \begin{cases} D_V(W), & \text{if} \quad a, b < 0; \\ D_M(W), & \text{if} \quad a, b \geq 0; \\ D_{VM}(W), & \text{if} \quad a < 0, b \geq 0; \\ D_{MV}(W), & \text{if} \quad a \geq 0, b < 0. \end{cases}$$

For every $u \in D(W), Wu$ can be continuously extended to $[0,1]$.

We shall continue to denote by Wu this extension and so we obtain a linear operator $W : D(W) \longrightarrow C[0,1]$.

As a particular case of the results of [7], we have:

Theorem 2.1. *In each of the following cases*

1) $a \geq 0$, 2) $b \geq 0$, 3) $a, b \leq -1$, 4) $-1 < a, b < 0$,

$(W, D(W))$ is the infinitesimal generator of a strongly continuous positive semigroup $(T(t))_{t\geq 0}$ on $C[0,1]$. Moreover, $T(t)1 = 1, t \geq 0$, i.e., $(T(t))$ is a contraction semigroup.

For $u \in C[0,1] \cap C^2(0,1)$ define the boundary conditions:

$$N_a \begin{cases} u \in C^1[0, \frac{1}{2}]\ ,\ u'(0) = 0, \lim\limits_{x\to 0} xu''(x) = 0 & \text{if } a < -1; \\ \lim\limits_{x\to 0} xu''(x) = 0 & \text{if } a = -1; \\ u \in C^1[0, \frac{1}{2}], \lim\limits_{x\to 0} xu''(x) = 0 & \text{if } a \geq 0. \end{cases}$$

$$N_b \begin{cases} u \in C^1[\frac{1}{2}, 1], u'(1) = 0, \lim\limits_{x\to 1} (1-x)u''(x) = 0 & \text{if } b < -1; \\ \lim\limits_{x\to 1} (1-x)u''(x) = 0 & \text{if } b = -1; \\ u \in C^1[\frac{1}{2}, 1], \lim\limits_{x\to 1} (1-x)u''(x) = 0 & \text{if } b \geq 0. \end{cases}$$

As a particular case of [7], Theorem 2.3, we get

Theorem 2.2. *Let $a, b \in (-\infty, -1] \cup [0, +\infty)$ and $u \in C[0,1] \cap C^2(0,1)$. Then*

(i) *$u \in D(W)$ if and only if u satisfies N_a and N_b.*

(ii) *$C^2[0,1] \cap D(W)$ is a core of $(W, D(W))$.*

3. Approximation of the semigroup by modified Bernstein operators

In this section we shall introduce a modification of Bernstein operators and we shall approximate the semigroup considered in the previous section by suitable iterates of these modified operators.

From now on we shall assume that $a, b \in [-1, +\infty)$. For every $n \geq M_0 := \max\{a+1, b+1\}$ we shall consider the following positive linear operator $L_n : C[0,1] \longrightarrow C[0,1]$ defined by

$$L_n f(x) := \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} f\Big(\Big(1 - \frac{a+b+2}{2n}\Big)\frac{h}{n} + \frac{a+1}{2n}\Big) \tag{2}$$

for every $f \in C[0,1]$ and $x \in [0,1]$.

Note that if we consider the auxiliary function

$$v(x) = a + 1 - (a+b+2)x \qquad (0 \leq x \leq 1) \tag{3}$$

then

$$L_n f = B_n\Big(f \circ \Big(e_1 + \frac{v}{2n}\Big)\Big) \qquad (f \in C[0,1]) \tag{4}$$

where B_n denotes the nth Bernstein operator and $e_1(x) := x \quad (0 \leq x < 1)$.

From formula (4) and from well-known properties of Bernstein operators it is possible to obtain the approximation properties of the sequence $(L_n)_{n\geq M_0}$.

As usual we set $e_j(x) := x^j \quad (0 \leq x \leq 1),\ j = 0, 1, \ldots$.

Then for every $n \geq M_0$,

$$L_n e_0 = e_0, \tag{5}$$

$$L_n e_1 = e_1 + \frac{v}{2n}, \tag{6}$$

$$\begin{aligned} L_n e_2 = &\Big(1 - \frac{a+b+2}{2n}\Big)^2 \Big(e_2 + \frac{e_1 - e_2}{n}\Big) \\ &+ \frac{a+1}{n}\Big(1 - \frac{a+b+2}{2n}\Big)e_1 + \frac{(a+1)^2}{4n^2} e_0. \end{aligned} \tag{7}$$

We are now in the position to state the following result.

Theorem 3.1. *For every $f \in C[0,1]$,*

$$\lim_{n\to\infty} L_n f = f \quad \textit{uniformly on} \quad [0,1].$$

Moreover, for every $x \in [0,1]$ and $n \geq M_0$,

$$|L_n f(x) - f(x)| \leq \omega_1\Big(f, \frac{|v(x)|}{2n}\Big) + M\omega_2\Big(f, \Big(1 - \frac{a+b+2}{2n}\Big)\sqrt{\frac{x(1-x)}{n}}\Big),$$

where ω_1 and ω_2 denote the ordinary first and second moduli of smoothness and M is a suitable constant independent of f, n and x.

Proof. The first statement follows from formulae (5)–(7) and the Korovkin theorem. As regards the subsequent estimate, taking formula (5.2.43) of [4] into account, we get

$$\begin{aligned} &|L_n f(x) - f(x)| \\ &\leq |B_n\Big(f \circ \Big(e_1 + \frac{v}{2n}\Big)\Big)(x) - f\Big(x + \frac{v(x)}{2n}\Big)| + |f\Big(x + \frac{v(x)}{2n}\Big) - f(x)| \\ &\leq M\omega_2\Big(f \circ \Big(e_1 + \frac{v}{2n}\Big), \sqrt{\frac{x(1-x)}{n}}\Big) + \omega_1\Big(f, \frac{|v(x)|}{2n}\Big) \\ &\leq M\omega_2\Big(f, \Big(1 - \frac{a+b+2}{2n}\Big)\sqrt{\frac{x(1-x)}{n}}\Big) + \omega_1\Big(f, \frac{|v(x)|}{2n}\Big). \end{aligned}$$ □

We are now going to show some shape preserving properties of the operators L_n. As usual, for $0 < \alpha \leq 1$ and $M \geq 0$ we set

$$\mathrm{Lip}(\alpha, M) := \{f \in C[0,1] : |f(x) - f(y)| \leq M|x-y|^\alpha \quad \text{for every} \quad x, y \in [0,1]\}.$$

Proposition 3.2. *The following statements hold true:*

(i) *Each operator L_n maps increasing continuous functions into increasing continuous functions and convex continuous functions into convex continuous functions.*

(ii) *For every $n \geq M_0,\ 0 < \alpha \leq 1,\ M \geq 0$,*

$$L_n(\mathrm{Lip}(\alpha, M)) \subset \mathrm{Lip}\Big(\alpha, M\Big(1 - \frac{a+b+2}{2n}\Big)^\alpha\Big).$$

(iii) *For every* $n \geq M_0,\ f \in C[0,1]$ *and* $0 \leq \varepsilon \leq 1/2$,

$$\omega_1(L_n f, \varepsilon) \leq 2\omega_1\Big(f, \Big(1 - \frac{a+b+2}{2n}\Big)\varepsilon\Big)$$

and

$$\omega_2(L_n f, \varepsilon) \leq 3\omega_2\Big(f, \Big(1 - \frac{a+b+2}{2n}\Big)\varepsilon\Big).$$

Proof. Since the function v is affine and increasing statement (i) can be easily proved by using (4) together with the property of Bernstein operators of leaving invariant the cone of continuous increasing functions as well as the cone of convex continuous functions.

As regards statement (ii), if $f \in \mathrm{Lip}(\alpha, M)$ then

$$f \circ \Big(e_1 + \frac{v}{2n}\Big) \in \mathrm{Lip}\Big(\alpha, M\Big(1 - \frac{a+b+2}{2n}\Big)^{\alpha}\Big)$$

and then the result follows because $\mathrm{Lip}\Big(\alpha, M\Big(1 - \frac{a+b+2}{2n}\Big)^{\alpha}\Big)$ is invariant under the Bernstein operator B_n. Finally, the inequalities

$$\omega_1(B_n\varphi, \varepsilon) \leq 2\omega_1(\varphi, \varepsilon) \quad \text{and} \quad \omega_2(B_n\varphi, \varepsilon) \leq 3\omega_2(\varphi, \varepsilon) \quad (\varphi \in C[0,1], \varepsilon > 0)$$

obtained, respectively, in [6] and [22], imply statement (iii) because

$$\omega_i\Big(f \circ \Big(e_1 + \frac{v}{2n}\Big), \varepsilon\Big) \leq \omega_i\Big(f, \Big(1 - \frac{a+b+2}{n}\Big)\varepsilon\Big),\ i = 1, 2. \qquad \square$$

The sequence $(L_n)_{n \geq M_0}$ verifies the following asymptotic formula.

Theorem 3.3. *For every* $f \in C^2[0,1]$,

$$\lim_{n\to\infty} n(L_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x) + \frac{v(x)}{2} f'(x)$$

uniformly with respect to $x \in [0,1]$.

Proof. Since

$$L_n f(x) - f(x) = \Big(B_n\Big(f \circ \Big(e_1 + \frac{v}{2n}\Big)\Big)(x) - B_n f(x)\Big) + (B_n f(x) - f(x)),$$

we have to determine only the limit of the expression

$$n\Big(B_n\Big(f \circ \Big(e_1 + \frac{v}{2n}\Big)\Big)(x) - B_n f(x)\Big),$$

because, as it is well known, $n(B_n f(x) - f(x)) \longrightarrow \dfrac{x(1-x)}{2} f''(x)$ uniformly with respect to $x \in [0,1]$.

Let $p_{nk}(x) = \binom{n}{k}x^k(1-x)^{n-k}$. The above-mentioned expression becomes successively

$$\begin{aligned}
&n\sum_{k=0}^{n} p_{nk}(x)\Big(f\Big(\frac{k}{n}+\frac{1}{2n}v\Big(\frac{k}{n}\Big)\Big)-f\Big(\frac{k}{n}\Big)\Big)\\
&= n\sum_{k=0}^{n} p_{nk}(x)\frac{1}{2n}v\Big(\frac{k}{n}\Big)f'(c_{nk})\\
&= \frac{1}{2}\sum_{k=0}^{n} p_{nk}(x)v\Big(\frac{k}{n}\Big)f'\Big(\frac{k}{n}\Big)+\frac{1}{2}\sum_{k=0}^{n} p_{nk}(x)v\Big(\frac{k}{n}\Big)\Big(f'(c_{nk})-f'\Big(\frac{k}{n}\Big)\Big)\\
&= \frac{1}{2}\sum_{k=0}^{n} p_{nk}(x)v\Big(\frac{k}{n}\Big)f'\Big(\frac{k}{n}\Big)+\frac{1}{2}\sum_{k=0}^{n} p_{nk}(x)v\Big(\frac{k}{n}\Big)f''(d_{nk})\Big(c_{nk}-\frac{k}{n}\Big),
\end{aligned}$$

where c_{nk} and d_{nk} are between $\frac{k}{n}$ and $\frac{k}{n}+\frac{1}{2n}v\left(\frac{k}{n}\right)$.

As $n\to\infty$, the uniform limit of the first sum is $\frac{1}{2}v(x)f'(x)$, while the uniform limit of the second sum is zero. So the theorem is proved. $\square$

By using Theorem 3.3 we can quickly proceed to give a representation of the semigroup studied in Section 2 in terms of iterates of the operators L_n. However this representation will be proved only for the particular cases $a\ge 0$, $b\ge 0$ or $a=-1$ and $b\ge 0$ or $a\ge 0$ and $b=-1$. In the remaining cases, the problem of constructing a suitable approximation process whose iterates approximate the semigroup remains open.

We finally point out that, if $a=b=-1$, then $L_n = B_n$ and the corresponding result about the representation of the semigroup is well known (see, e.g., [4, Ch. VI]).
When $a\ge 0$ and $b\ge 0$, the semigroup can be also represented by iterates of the operators introduced in [7, Theorem 4.6] or in [17] (see [23]).

According to the previous section we set

$$D(W)=\begin{cases} D_M(W) & \text{if } \ a\ge 0, b\ge 0,\\ D_{VM}(W) & \text{if } \ a=-1, b\ge 0,\\ D_{MV}(W) & \text{if } \ a\ge 0, b=-1\end{cases}$$

and denote by $(T(t))_{t\ge 0}$ the semigroup generated by $(W, D(W))$.

Theorem 3.4. *In each of the above-mentioned cases, for every $f\in C[0,1]$ and $t\ge 0$,*

$$T(t)f=\lim_{n\to\infty} L_n^{k(n)}f \quad \textit{uniformly on} \quad [0,1]$$

where $(k(n))_{n\ge 1}$ is an arbitrary sequence of positive integers such that $\frac{k(n)}{n}\to 2t$ and $L_n^{k(n)}$ denotes the iterate of L_n of order $k(n)$.

Proof. From Theorem 3.3 it follows that

$$\lim_{n\to\infty} n(L_n u - u) = \frac{1}{2} W u \quad \text{in} \quad C[0,1]$$

for every $u \in C^2[0,1] \subset D(W)$.

Since $C^2[0,1]$ is a core for $(W, D(W))$ (see Theorem 2.2) and $||L_n^p|| \le 1$ for every $n \ge M_0$ and $p \ge 1$, then, by a result of Trotter [27], denoting by $(S(t))_{t\ge 0}$ the semigroup generated by $(\frac{1}{2}W, D(W))$, for every $f \in C[0,1]$ and $t \ge 0$ and for every sequence $(k(n))_{n\ge 1}$ of positive integers such that $\dfrac{k(n)}{n} \to t$ we have

$$S(t)f = \lim_{n\to\infty} L_n^{k(n)} f \quad \text{in} \quad C[0,1].$$

Now the result obviously follows since $T(t) = S(2t)$ for every $t \ge 0$. □

Taking Theorem 3.4 and Proposition 3.2 into account, we can easily derive the following properties of the semigroup $(T(t))_{t\ge 0}$. These properties can be immediately translated into the corresponding ones for the solutions $u(t,x)$ of the diffusion equation (1) coupled with an initial condition $u_0 \in D(W)$ and which is given by

$$u(t,x) = T(t)u_0(x) \quad (0 \le x \le 1,\ t \ge 0).$$

Corollary 3.5. *In each of the three cases considered in Theorem* 3.4*, the following statements hold true:*

(i) *Each operator* $T(t) \quad (t \ge 0)$ *maps increasing continuous functions into increasing continuous functions as well as convex continuous functions into convex continuous functions.*

(ii) *For every* $0 < \alpha \le 1,\ M \ge 0$ *and* $t \ge 0$

$$T(t)(\mathrm{Lip}(\alpha, M)) \subset \mathrm{Lip}(\alpha, M\exp(-(a+b+2)\alpha t)).$$

Proof. We need only to prove (ii). Let $f \in \mathrm{Lip}(\alpha, M)$ and $t \ge 0$. Consider a sequence $(k(n))_{n\ge 1}$ of positive integers such that $\frac{k(n)}{n} \to 2t$. Replacing, if necessary, f by f/M, we can always assume that $M = 1$. Then for every $n \ge 1$

$$L_n^{k(n)} f \in \mathrm{Lip}\Big(\alpha, \Big(1 - \frac{a+b+2}{2n}\Big)^{\alpha k(n)}\Big).$$

Passing to the limit as $n \to \infty$ we get that $T(t)f \in \mathrm{Lip}(\alpha, \exp(-(a+b+2)\alpha t))$ by virtue of Theorem 3.4. □

The limit behaviour of the semigroup, i.e., the limit $\lim\limits_{t\to\infty} T(t)$, will be studied in the next section.

4. The stochastic equation

Consider the stochastic equation associated to W:

$$dY_t = \sqrt{2Y_t(1-Y_t)}dB_t + (a+1-(a+b+2)Y_t)dt\ ,\ t \geq 0,$$

with $Y_0 = x \in (0,1)$. (see, e.g., [10], [11], [12], [15], [21], [26]).

Feller's test is applicable; we omit the proof and give only the final result:

Theorem 4.1. *Let ζ be the lifetime of the solution $(Y_t)_{t\geq 0}$.*

(i) *If $a \geq 0$ and $b \geq 0$, then $P(\zeta = \infty) = 1$ and* $P(\inf_{0\leq t<\infty} Y_t = 0) = P(\sup_{0\leq t<\infty} Y_t = 1) = 1.$

(ii) *If $a \geq 0$ and $b < 0$, we have $P(\zeta < \infty) = 1$ and* $P(\inf_{0\leq t<\zeta} Y_t > 0) = P(\lim_{t\to\zeta} Y_t = 1) = 1.$

(iii) *For $a < 0$ and $b \geq 0$, $P(\zeta < \infty) = 1$ and* $P(\lim_{t\to\zeta} Y_t = 0) = P(\sup_{0\leq t<\zeta} Y_t < 1) = 1.$

(iv) *For $a < 0$ and $b < 0$, $E\zeta < \infty$ and* $P(\lim_{t\to\zeta} Y_t = 1) = 1 - P(\lim_{t\to\zeta} Y_t = 0) = \varphi(x)$, *where*

$$\varphi(x) = \frac{\int\limits_0^x u^{-a-1}(1-u)^{-b-1}du}{\int\limits_0^1 u^{-a-1}(1-u)^{-b-1}du}.$$

For $a \geq 0$ and $b \geq 0$ it is possible to apply results from [18], [19] in order to prove that the semigroup $(T(t))$ is compact and $||T(t) - T|| \longrightarrow 0$ exponentially, as $t \to \infty$, where

$$Tf(x) = \frac{\int_0^1 u^a(1-u)^b f(u)du}{\int_0^1 u^a(1-u)^b du}\ ,\ f \in C[0,1],\ x \in [0,1].$$

(See also [24], (10.8).)

The kernel of the integral representation of $T(t)$ for $a \geq 0$ and $b \geq 0$ (see [14], [8], and [18], Theorem 4.4) is

$$p(t,x,y) = \sum_{n=0}^{\infty} e^{-n(n+a+b+1)t} J_n^{(a,b)}(x) J_n^{(a,b)}(y) y^a (1-y)^b,$$

where $J_n^{(a,b)}(x)$ are the Jacobi polynomials orthonormal on the interval $[0,1]$ with weight $x^a(1-x)^b$.

The asymptotic behavior of the semigroup $(T(t))$ in the remaining cases is suggested by Theorem 4.1 (see also ([24], (10.10)) and described in

Theorem 4.2. *For all* $f \in C[0,1]$,

1. $\lim_{t\to\infty} T(t)f = f(0)$ *if* $a < 0,\ b \geq 0$;
2. $\lim_{t\to\infty} T(t)f = f(1)$ *if* $a \geq 0,\ b < 0$;
3. $\lim_{t\to\infty} T(t)f = f(0)(1-\varphi) + f(1)\varphi$ *if* $a, b \leq -1$ *or* $-1 < a, b < 0$, *where the function* φ *is defined in Theorem* 4.1.

Proof. 1. Let $v(x) = x^{-a},\ x \in [0,1]$. Then $v \in C[0,1] \cap C^2(0,1)$ and $Wv = a(b+1)v$. We deduce that $v \in D_{VM}(W) = D(W)$ and

$$T(t)v = e^{a(b+1)t}v,\ t \geq 0.$$

Now let $T : C[0,1] \to C[0,1],\ Tf(x) = f(0),\ f \in C[0,1],\ x \in [0,1]$.

Then $\lim_{t\to\infty} T(t)1 = T1$ and $\lim_{t\to\infty} T(t)v = Tv$.

An application of Theorem 3.4.3 [4] shows that

$$\lim_{t\to\infty} T(t)f = Tf,\ f \in C[0,1],$$

which is the first statement of the theorem.

2. The proof of the second statement is similar.

3. Let $w(x) = x^{-a}(1-x)^{-b},\ x \in [0,1]$. Then $w \in C[0,1] \cap C^2(0,1)$ and $Ww = (a+b)w$. It follows that $w \in D_V(W) = D(W)$ and

$$T(t)w = e^{(a+b)t}w,\ t \geq 0.$$

On the other hand, $\varphi \in C[0,1] \cap C^2(0,1)$ and $W\varphi = 0$, which means that $T(t)\varphi = \varphi,\ t \geq 0$.

Let $T : C[0,1] \to C[0,1],\ Tf = f(0)(1-\varphi) + f(1)\varphi,\ f \in C[0,1]$. For $t \to \infty$ we have $T(t)1 \to T1,\ T(t)\varphi \to T\varphi,\ T(t)w \to Tw$. To conclude the proof it suffices to apply Theorem 3.4.3 [4]. □

From the above proof we can deduce also quantitative versions of Theorem 4.2:

1. Let $a < 0,\ b \geq 0$. If $f \in C[0,1]$ and
$$|f(x) - f(0)| \leq C_f x^{-a},\ x \in [0,1] \tag{8}$$
for some constant C_f, then
$$|T(t)f(x) - f(0)| \leq C_f e^{a(b+1)t} x^{-a},\ x \in [0,1],\ t \geq 0.$$
2. Let $a \geq 0,\ b < 0$, and $f \in C[0,1]$ with
$$|f(x) - f(1)| \leq K_f (1-x)^{-b},\ x \in [0,1]. \tag{9}$$
Then we have
$$|T(t)f(x) - f(1)| \leq K_f e^{(a+1)bt}(1-x)^{-b},\ x \in [0,1],\ t \geq 0.$$

3. Let $a, b \leq -1$ or $-1 < a,\ b < 0$. Then φ satisfies both (8) and (9). If a function $f \in C[0,1]$ also satisfies both (8) and (9), then for $x \in [0,1],\ t \geq 0$,

$$\begin{aligned} &|T(t)f(x) - f(0)(1-\varphi(x)) - f(1)\varphi(x)| \\ &\leq (C_f K_\varphi + C_\varphi K_f) e^{(a+b)t} x^{-a}(1-x)^{-b}. \end{aligned} \tag{10}$$

4. Let $a = b = -1$; then $\varphi(x) = x$. Let $(T(t))$ be the corresponding semigroup. Consider also the semigroup$(S(t))$ associated with the classical Bernstein operators (see [4], Theorem 6.3.5). The generator of $(S(t))$ is $\frac{1}{2}W$, which means that $S(t) = T(t/2)$. From (10) we get

$$|S(t)f(x) - f(0)(1-x) - f(1)x| \leq (C_f + K_f)e^{-t}x(1-x). \tag{11}$$

Another proof of (11) can be obtained using the approximation of $S(t)$ by iterates of classical Bernstein operators.

5. Explicit solutions

We shall use Lamperti's method [12, pp. 294–295] and the method of Doss-Sussmann [12, pp. 295–296], [26, pp. 382–383] in order to get information about the solution Y_t.

By using Lamperti's method in our setting, take $U = \sqrt{2} arcsin \sqrt{Y}$. Itô's formula yields

$$dU = dB + \frac{a - b + (a+b+1)\cos(\sqrt{2}U)}{\sqrt{2}\sin(\sqrt{2}U)} dt,$$

with $U_0 = \sqrt{2}\arcsin\sqrt{x}$.

So the generator of the process $(U_t)_{t \geq 0}$ is

$$\frac{1}{2}\frac{d^2}{du^2} + \frac{\sqrt{2}}{2}\left(\frac{a-b}{\sin(\sqrt{2}u)} + (a+b+1)\cot(\sqrt{2}u)\right)\frac{d}{du}.$$

In particular, for $b = a$ the generator of U becomes

$$\frac{1}{2}\frac{d^2}{du^2} + \frac{\sqrt{2}}{2}(2a+1)\cot(\sqrt{2}u)\frac{d}{du},$$

which means that U is a *Legendre process* [26, p. 357].

So we have

Theorem 5.1. $Y_t = \sin^2 \frac{U_t}{\sqrt{2}}$. *For $b = a$, U is a Legendre process.*

The Doss-Sussmann method yields

Theorem 5.2. $Y_t = \sin^2\left(\frac{B_t}{\sqrt{2}} + \arcsin\sqrt{X_t}\right)$, *where* X_t *is the solution of the equation*

$$\frac{dX}{dt} = \frac{\sqrt{X(1-X)}(a-b+(a+b+1)\cos(B\sqrt{2}+2\arcsin\sqrt{X}))}{\sin(B\sqrt{2}+2\arcsin\sqrt{X})},$$

with $X_0 = x$.

When $a = b = -\frac{1}{2}$, the following explicit solution can be obtained by using either Theorem 5.1 or Theorem 5.2:

Corollary 5.3. *For* $a = b = -\dfrac{1}{2}$ *we have*

$$Y_t = \sin^2\left(\frac{B_t}{\sqrt{2}} + \arcsin\sqrt{x}\right),\ 0 \le t < \zeta.$$

Similar solutions for corresponding problems are given in [15], Section 4.4, and [21], Chapter 5.

We conclude with two examples.

I. Let $a = b = -\frac{1}{2}$, i.e.,

$$Wu(x) = x(1-x)u''(x) + (\frac{1}{2} - x)u'(x),$$

with $D(W) = D_V(W)$ described in Section 2.
By using the expression of Y_t given in Corollary 5.3 it can be seen that $Wu = \lim_{n\to\infty} n(M_n u - u)$ for every $u \in C^2[0,1]$, where

$$M_n f(x) := Ef(Y_{1/n}) = (W_{1/4n}(f\circ g))(g^{-1}(x)),$$

$g(x) = \sin^2(x)$ and $(W_t)_{t>0}$ are the classical Gauss-Weierstrass convolution operators (see [4], (5.2.78)).
Now let $f_n(x) = (\arcsin\sqrt{x})^n$, $0 \le x \le 1,\ n \ge 1$.
According to Corollary 5.3,

$$\begin{aligned} u_n(t,x) := Ef_n(Y_t) &= E\Big(\Big(\frac{B_t}{\sqrt{2}} + \arcsin\sqrt{x}\Big)^n\Big) \\ &= \sum_{j=0}^{[n/2]} \binom{n}{2j} \frac{(2j)!}{j!} \Big(\frac{t}{4}\Big)^j (\arcsin\sqrt{x})^{n-2j}. \end{aligned}$$

Consequently, the function $u_n(t,x)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} u_n(t,x) = Wu_n(t,x) \\ u_n(0,x) = f_n(x) \end{cases}$$

for $t \ge 0,\ 0 \le x \le 1$.

Since f_1 and $u_1(t,\cdot)$ are in $D(W)$, we get

$$T(t)f_1(x) = u_1(t,x) = f_1(x).$$

Thus $T(t)f_1 = f_1$; this is also a consequence of the fact that $Wf_1 = 0$.

II. Let now $a \geq 0,\ b \geq 0$. Then

$$WJ_n^{(a,b)} = -n(n+a+b+1)J_n^{(a,b)}\ ,\ n \geq 0,$$

and

$$T(t)J_n^{(a,b)} = e^{-n(n+a+b+1)t}J_n^{(a,b)}\ ,\ n \geq 0,\ t \geq 0.$$

Since 0 and 1 are entrance boundaries, the diffusion (Y_t) extends to a continuous Feller process on $[0,1]$ (see [11], Theorem 23.13). We have also

$$EJ_n^{(a,b)}(Y_t) = e^{-n(n+a+b+1)t}J_n^{(a,b)}(x).$$

Now it is possible to compute the moments of Y_t. For example,

$$EY_t = \frac{a+1}{a+b+2} + \frac{1}{a+b+2}e^{-(a+b+2)t}j_1(x)$$

and

$$\begin{aligned} EY_t^2 = &\frac{(a+1)(a+2)}{(a+b+2)(a+b+3)} \\ &+ \frac{2(a+2)}{(a+b+2)(a+b+4)}\ e^{-(a+b+2)t}j_1(x) \\ &+ \frac{1}{(a+b+3)(a+b+4)}\ e^{-2(a+b+3)t}j_2(x), \end{aligned}$$

where

$$j_1(x) = (a+b+2)x - (a+1)$$

and

$$j_2(x) = (a+b+3)(a+b+4)x^2 - 2(a+2)(a+b+3)x + (a+1)(a+2)$$

differ from $J_1^{(a,b)}(x)$, respectively $J_2^{(a,b)}(x)$, only by some constant factors.

Taking into account the asymptotic behavior of the semigroup $(T(t))$ we get also

$$\lim_{t\to\infty} E(Y_t^n) = \frac{(a+1)(a+2)\ldots(a+n)}{(a+b+2)(a+b+3)\ldots(a+b+n+1)}\ .$$

The same results can be achieved by using the fact that the probability density of Y_t is the function $p(t,x,\cdot)$ from Section 4.

Moreover,

$$\begin{aligned} EY_t^{-a} &= e^{a(b+1)t}x^{-a}, \\ E(1-Y_t)^{-b} &= e^{(a+1)bt}(1-x)^{-b}, \\ EY_t^{-a}(1-Y_t)^{-b} &= e^{(a+b)t}x^{-a}(1-x)^{-b}. \end{aligned}$$

Acknowledgement

Part of this work has been developed during a stay at both Bommerholz-Witten and Oberwolfach in the framework of the RiP programme of the Mathematisches Forschungsinstitut.

We would like to express our thanks to Professor D.Mache and to MFO for the stimulating atmosphere and the warm hospitality.

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Francesco Altomare
University of Bari
Department of Mathematics
Via E.Orabona, 4
I-70125 Bari (Italy)
e-mail: `altomare@dm.uniba.it`

Ioan Rasa
Technical University of Cluj-Napoca
Department of Mathematics
Str.C. Daicoviciu, 15
R-3400 Cluj-Napoca (Romania)
e-mail: `Ioan.Rasa@math.utcluj.ro`