Chapter 4

Uniqueness, Generation of Contraction Semigroups, Error Estimates

In all this section we consider the "quadratic" approximation scheme (2.0.3b), (2.0.4) for 2-curves of maximal slope and we identify the "weak" topology σ with the "strong" one induced by the distance d as in Remark 2.1.1: thus we are assuming that

$$\begin{gathered} p = 2, \quad (\mathscr{S}, d) \text{ is a complete metric space and} \\ \phi : \mathscr{S} \to (-\infty, +\infty] \text{ is a proper, coercive (2.4.10), l.s.c. functional,} \end{gathered} \tag{4.0.1}$$

but *we are not imposing any compactness assumptions on the sublevels of* ϕ. Existence, uniqueness and semigroup properties for minimizing movement $u \in MM(\Phi; u_0)$ (and not simply the generalized ones, recall Definition 2.0.6) are well known in the case of lower semicontinuous *convex functionals in Hilbert spaces* [28]. In this framework the resolvent operator in $J_\tau[\cdot]$ (3.1.2) is single valued and *non expansive*, i.e.

$$d(J_\tau[u], J_\tau[v]) \le d(u, v) \quad \forall\, u, v \in \mathscr{S},\ \tau > 0; \tag{4.0.2}$$

this property is a key ingredient, as in the celebrated Crandall-Ligget generation Theorem [46], to prove the uniform convergence of the exponential formula (cf. (2.0.9))

$$u(t) = \lim_{n\to\infty} (J_{t/n})^n[u_0], \qquad d\Big(u(t), (J_{t/n})^n[u_0]\Big) \le \frac{2|\partial\phi|(u_0)\, t}{\sqrt{n}}, \tag{4.0.3}$$

and therefore to define a contraction semigroup on $\overline{D(\phi)}$. Being generated by a convex functional, this semigroup exhibits a nice regularizing effect [27], since

$u(t) \in D(|\partial\phi|)$ whenever $t > 0$ even if the starting value u_0 simply belongs to $\overline{D(\phi)}$. Moreover the function u can be characterized as the unique solution of the *evolution variational inequality*

$$\langle \tfrac{d}{dt}u(t), u(t) - v\rangle + \phi(u(t)) \le \phi(v) \quad \forall\, v \in D(\phi), \tag{4.0.4}$$

$\langle\cdot,\cdot\rangle$ being the scalar product in $\mathscr{S}$.

More recently, optimal *a priori* and *a posteriori* error estimates have also been derived [18, 115, 102]: the original $O(\tau^{1/2}) = O(1/\sqrt{n})$ order of convergence established by Crandall and Ligget for $u_0 \in D(|\partial\phi|)$ and a uniform partition (2.0.8), has been improved to

$$d\Big(u(t), (J_{t/n})^n[u_0]\Big) \le \frac{|\partial\phi|(u_0)\, t}{n\sqrt{2}} \tag{4.0.5}$$

and extended to the general scheme (2.0.4), (2.0.7)

$$d^2(\overline{U}_{\boldsymbol\tau}(t), u(t)) \le |\boldsymbol\tau|\Big(\phi(u_0) - \inf_{\mathscr{S}} \phi\Big), \quad d^2(\overline{U}_{\boldsymbol\tau}(t), u(t)) \le |\boldsymbol\tau|^2\, \frac{|\partial\phi|^2(u_0)}{2}, \tag{4.0.6}$$

thus establishing an optimal error estimate of the same order $O(|\boldsymbol\tau|)$ of the Euler method in a smooth and finite dimensional setting.

Similar results for gradient flows of convex functionals in general (non Hilbertian) Banach spaces are still completely open: at least heuristically, this fact suggests that some structural property of the distance should play a crucial role, besides the convexity of the functional ϕ.

A first step in this direction has been obtained by U. Mayer [96] (see also [85]), who considered gradient flows of geodesically convex functionals on *non-positively curved metric spaces*: these are length spaces (i.e. each couple of points v_0, v_1 can be connected through a minimal geodesic) where the distance maps

$$v \mapsto \tfrac{1}{2}d^2(v, w) \text{ are 1-convex along geodesics} \quad \forall\, w \in \mathscr{S}. \tag{4.0.7}$$

This property was introduced by Aleksandrov on the basis of the analogous inequality satisfied in Euclidean spaces (2.4.4) and in Riemannian manifolds of non-positive sectional curvature [84, §2.3]; it allows to prove (4.0.2), and to obtain the generation formula (4.0.3) by following the same Crandall-Liggett arguments. Observe that Mayer's assumptions yield in particular that the variational functional defined by (2.0.3b)

$$\begin{aligned}&v \mapsto \Phi(\tau, w; v) = \frac{1}{2\tau}d^2(v, w) + \phi(v)\\ &\text{is } (\tau^{-1} + \lambda)\text{-}\textit{convex along geodesics} \quad \forall\, w \in \mathscr{S}.\end{aligned} \tag{4.0.8}$$

These assumptions, though quite general, do not cover the case of the metric space of probability measures endowed with the L^2-Wasserstein distance: we will show in Section 7.3 that, in fact, the distance of this space satisfies the opposite inequality, thus providing a positively curved space, as formally suggested also by [107]. Example 7.3.3 will also show that the squared L^2-Wasserstein distance does not satisfy any λ-convexity properties, even for negative choice of $\lambda \in \mathbb{R}$.

Our idea is to concentrate our attention directly on the functional $\Phi(\tau, w; \cdot)$ and to allow more flexibility in the choice of the connecting curves, along which it has to satisfies the convexity assumption (4.0.8): we formalize this requirement in the following assumption:

Assumption 4.0.1 **($(\tau^{-1} + \lambda)$-convexity of $\Phi(\tau, u; \cdot)$).** *We suppose that for every choice of w, v_0, and v_1 in $D(\phi)$ there exists a curve $\gamma = \gamma_t$, $t \in [0,1]$, with $\gamma_0 = v_0, \gamma_1 = v_1$ such that*

$$v \mapsto \Phi(\tau, w; v) \text{ is } \Big(\frac{1}{\tau} + \lambda\Big)\text{-convex on } \gamma \text{ for each } 0 < \tau < \frac{1}{\lambda^-}, \tag{4.0.9}$$

i.e. the map $\Phi(\tau, w; \gamma_t)$ satisfies the inequality

$$\Phi(\tau, w; \gamma_t) \le (1-t)\Phi(\tau, w; v_0) + t\Phi(\tau, w; v_1) - \frac{1+\lambda\tau}{2\tau} t(1-t) d^2(v_0, v_1). \tag{4.0.10}$$

Remark 4.0.2. Of course, Assumption 4.0.1 covers the case of a (geodesically) λ-convex functional on a nonpositively curved metric space considered by [96], in particular the case of a (geodesically) λ-convex functional in a Riemannian manifold of nonpositive sectional curvature or in a Hilbert space.

Remark 4.0.3. Assumption 4.0.1 is *stronger* than 2.4.5, since this last one is a particular case of (4.0.1) when the "base point" w coincides with v_0.

We collect the main results in this case

Theorem 4.0.4 (Generation and main properties of the evolution semigroup). *Let us assume that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold for some $\lambda \in \mathbb{R}$.*

i) Convergence and exponential formula: *for each $u_0 \in \overline{D(\phi)}$ there exists a unique element $u = S[u_0]$ in $MM(\Phi; u_0)$ which therefore can be expressed through the exponential formula*

$$u(t) = S[u_0](t) = \lim_{n\to\infty} (J_{t/n})^n [u_0]. \tag{4.0.11}$$

ii) Regularizing effect: *u is a locally Lipschitz curve of maximal slope with $u(t) \in D(|\partial\phi|) \subset D(\phi)$ for $t > 0$; in particular, if $\lambda \ge 0$, the following a priori bounds hold:*

$$\begin{aligned} \phi(u(t)) \le \phi_t(u_0) \le \phi(v) + \frac{1}{2t} d^2(v, u_0) \quad & \forall\, v \in D(\phi), \\ |\partial\phi|^2(u(t)) \le |\partial\phi|^2(v) + \frac{1}{t^2} d^2(v, u_0) \quad & \forall\, v \in D(|\partial\phi|). \end{aligned} \tag{4.0.12}$$

iii) Uniqueness and evolution variational inequalities: *u is the unique solution of the evolution variational inequality*

$$\frac{1}{2}\frac{d}{dt}d^2(u(t),v)+\frac{1}{2}\lambda d^2(u(t),v)+\phi(u(t))\le\phi(v)\quad \mathscr{L}^1\text{-a.e. } t>0,\ \forall v\in D(\phi), \tag{4.0.13}$$

among all the locally absolutely continuous curves such that $\lim_{t\downarrow 0} u(t)=u_0$ *in* $\mathscr{S}$.

iv) Contraction semigroup: *The map* $t\mapsto S[u_0](t)$ *is a* λ*-contracting semigroup i.e.*

$$d\big(S[u_0](t),S[v_0](t)\big)\le e^{-\lambda t}d(u_0,v_0)\quad \forall\, u_0,v_0\in\overline{D(\phi)}. \tag{4.0.14}$$

v) Optimal a priori estimate: *if* $u_0\in D(\phi)$ *and* $\lambda=0$ *then*

$$d^2\Big(S[u_0](t),(J_{t/n})^n[u_0]\Big)\le\frac{t}{n}\Big(\phi(u_0)-\phi_{t/n}(u_0)\Big)\le\frac{t^2}{2n^2}|\partial\phi|^2(u_0). \tag{4.0.15}$$

Remark 4.0.5. Let us collect some comments about this result:

(a) The regularizing effect provided by (4.0.12) is stronger than the analogous property proved in Theorem 2.4.15 for λ-convex function, since in this case we simply need $u_0\in\overline{D(\phi)}$ instead of $u_0\in D(\phi)$. Inequality (4.0.12) also implies a faster decay of $|\partial\phi|(u(t))$ as $t\uparrow+\infty$.

(b) Since for differentiable curves u in a Hilbert space $\mathscr{S}=\mathscr{H}$

$$\langle\frac{d}{dt}u(t),u(t)-v\rangle=\frac{1}{2}\frac{d}{dt}|u(t)-v|^2=\frac{1}{2}\frac{d}{dt}d^2(u(t),v)\quad \forall\, v\in\mathscr{H},$$

the variational inequality formulation (4.0.13) is formally equivalent to (4.0.4) (in the case $\lambda=0$), but it does not require neither the existence of the pointwise derivative of u nor a vectorial structure. A similar idea was introduced by P. Bénilan [22] for the definition of the integral solutions of evolution equations governed by m-accretive operators in Banach spaces. The integral formulation corresponds to condier (4.0.13) in the weaker distributional sense:

$$\frac{1}{2}d^2(u(t),v)-\frac{1}{2}d^2(u(s),v)\le\int_s^t\Big(\phi(v)-\phi(u(r))-\frac{\lambda}{2}d^2(u(r),v)\Big)\,dr, \tag{4.0.16}$$

for every $v\in D(\phi)$ and $0<s<t$; in this way, one can simply require that u is a continuous curve with $\phi\circ u\in L^1_{loc}(0,+\infty)$, thus avoiding any *a priori* regularity assumption on the evolution curve. It would not be difficult to show that there exists at most one integral solution with prescribed initial datum and that this formulation is equivalent to (4.0.13).

(c) The semigroup S satisfies the contracting property (4.0.14) (e.g. for $\lambda = 0$) even if at the discrete level the resolvent operator does not satisfy in general the analogous property (4.0.2).

(d) In the case $\lambda > 0$ (4.0.14) provides another estimates of the exponential decay of the solution u to the unique minimum point $\bar{u}$ of ϕ (cf. (2.4.12)), as already discussed in Theorem 2.4.14, i.e.

$$d(u(t), \bar{u}) \leq e^{-\lambda t} d(u_0, \bar{u}) \quad \forall\, t \geq 0. \tag{4.0.17}$$

(e) The estimates (4.0.15) are exactly the same of the Hilbert framework: in fact the first one is even slightly better than the previously known results, since it exhibits an order of convergence $o(\sqrt{1/n})$ instead of $O(\sqrt{1/n})$ for $u_0 \in D(\phi)$ and it shows that the error is related to the speed of convergence of the Moreau-Yosida approximation ϕ_τ to ϕ as $\tau \downarrow 0$. Starting from this formula, it would not be difficult to relate the order of convergence to the regularity of u_0, measured in suitable (nonlinear) interpolation classes between $D(\phi)$ and $D(|\partial\phi|)$ (see e.g. [29], [19]).

In the limiting case $\lambda = 0$ the exponential decay does not occur, in general, but we can still prove some weaker results on the asymptotic behaviour of u, which are easy consequences of (4.0.12) and of (4.0.13).

Corollary 4.0.6. *Suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold with* $\lambda = 0$*, and that* $\bar{u}$ *is a minimum point for* ϕ*. Then the solution* $u = S[u_0]$ *provided by Theorem* 4.0.4 *satisfies*

$$\begin{gathered} |\partial\phi|(u(t)) \leq \frac{d(u_0, \bar{u})}{t}, \qquad \phi(u(t)) - \phi(\bar{u}) \leq \frac{d^2(u_0, \bar{u})}{2t}, \\ \textit{the map} \quad t \mapsto d(u(t), \bar{u}) \quad \textit{is not increasing.} \end{gathered} \tag{4.0.18}$$

In particular, if the sublevels of ϕ *are compact, then* $u(t) \stackrel{d}{\to} u_\infty$ *as* $t \to \infty$ *and* u_∞ *is a minimum point for* ϕ*.*

General a priori and a posteriori error estimates. (4.0.15) is a particular case of the general error estimates which can also be proved for non uniform partitions; quite surprisingly, they reproduce exactly the same structure of the Hilbertian setting and can be derived by a preliminary *a posteriori error analysis* (we refer to [102] for a detailed account of the various contributions to the subject of the *a priori and a posteriori* error estimates in the Hilbert case).

As we have already seen in (4.0.15), for each estimate the order of convergence depends on the regularity of the initial datum: the best one is obtained if $u_0 \in D(|\partial\phi|)$, whereas an intermediate order $O(\sqrt{|\boldsymbol{\tau}|})$ can be proved if $u_0 \in D(\phi)$; simple linear examples show that these bounds are optimal.

We first present the most interesting result for $\lambda = 0$ and then we will show how the various constants are affected by different values of λ.

Theorem 4.0.7 (The case $\lambda = 0$). *Suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold with* $\lambda = 0$, *let* $u \in MM(\Phi; u_0)$ *be the unique solution of the equation* (4.0.13) *and let* $\overline{U}_{\boldsymbol{\tau}}$ *be a discrete solution associated to the partition* $\mathcal{P}_{\boldsymbol{\tau}}$ (2.0.1). *If* $u_0 \in D(\phi)$ *and* $T = t^N_{\boldsymbol{\tau}} \in \mathcal{P}_{\boldsymbol{\tau}}$

$$d^2(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \le d^2(U^0_{\boldsymbol{\tau}}, u_0) + \sum_{n=1}^{N-1} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}}, \tag{4.0.19}$$

where

$$\mathscr{E}^n_{\boldsymbol{\tau}} := \frac{\phi(U^{n-1}_{\boldsymbol{\tau}}) - \phi(U^n_{\boldsymbol{\tau}})}{\tau_n} - \frac{d^2(U^{n-1}_{\boldsymbol{\tau}}, U^n_{\boldsymbol{\tau}})}{\tau_n^2} \tag{4.0.20}$$

and

$$\sum_{n=1}^{N} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}} \le |\boldsymbol{\tau}| \Big(\phi(U^0_{\boldsymbol{\tau}}) - \phi_T(U^0_{\boldsymbol{\tau}})\Big); \tag{4.0.21}$$

if $U^0_{\boldsymbol{\tau}} \equiv u_0$ *we have*

$$d^2(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \le |\boldsymbol{\tau}| \Big(\phi(u_0) - \phi_T(u_0)\Big) \le |\boldsymbol{\tau}| \Big(\phi(u_0) - \inf_{\mathscr{S}} \phi\Big) \quad \forall T > 0. \tag{4.0.22}$$

If $U^0_{\boldsymbol{\tau}} \in D(|\partial\phi|)$ *we have*

$$\sum_{n=1}^{N} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}} \le \frac{|\boldsymbol{\tau}|^2}{2} |\partial\phi|^2(U^0_{\boldsymbol{\tau}}); \tag{4.0.23}$$

if $U^0_{\boldsymbol{\tau}} \equiv u_0$ *we have*

$$d^2(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \le \frac{|\boldsymbol{\tau}|^2}{2} |\partial\phi|^2(u_0) \quad \forall T > 0. \tag{4.0.24}$$

Remark 4.0.8. (4.0.21) is slightly worse than (4.0.15), which in the case of a uniform mesh and $u_0 \in D(\phi)$ provides an $o(\sqrt{|\boldsymbol{\tau}|})$ estimates, instead of $O(\sqrt{|\boldsymbol{\tau}|})$: this fact depends on a finer cancellation effect which seems to be related to the choice of uniform step sizes.

In the case $\lambda \ne 0$ the error $d(\overline{U}_{\boldsymbol{\tau}}(T), u(T))$ should be affected by an exponential factor $e^{-\lambda T}$, corresponding to (4.0.14) or $e^{-\lambda_{\boldsymbol{\tau}} T}$, where

$$\lambda_{\boldsymbol{\tau}} := \frac{\log(1 + \lambda|\boldsymbol{\tau}|)}{|\boldsymbol{\tau}|} \quad \text{as for the discrete bounds of Lemma 3.4.1;} \tag{4.0.25}$$

the involved constants could also be perturbed by the presence of λ: here the main technical difficulty is to obtain estimates which exhibit the right coefficient of the exponential grow (or decay) and constants which reduce to the optimal ones (4.0.22), (4.0.24) when $\lambda = 0$.

We limit us to detail the *a priori* bounds of the error: we adopt the convention to denote by $c = c(\lambda, |\boldsymbol{\tau}|, T)$ the constants which depend only on the parameters λ, $|\boldsymbol{\tau}|$, T, exhibit at most a polynomial (in fact linear or quadratic) growth with respect to T, and are asymptotic to 1 as $\lambda \to 0$.

Theorem 4.0.9 (The case $\lambda < 0$). *Suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *holds for* $\lambda < 0$, *let* $u \in MM(\Phi; u_0)$ *be the unique solution of the equation* (4.0.13) *and let* $\overline{U}_{\boldsymbol{\tau}}$ *be the discrete solution associated to the partition* $\mathcal{P}_{\boldsymbol{\tau}}$ *in* (2.0.1) *with* $|\boldsymbol{\tau}| < (-\lambda)^{-1}$. *If* $U^0_{\boldsymbol{\tau}} = u_0 \in D(\phi)$ *we have*

$$d^2(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \le c\,|\boldsymbol{\tau}|\Big(\phi(u_0) - \inf_{\mathscr{S}} \phi\Big) e^{-2\lambda T}, \qquad c := \Big(1 + \sqrt{\tfrac{4}{3}|\lambda|\,|\boldsymbol{\tau}|}\Big)^2. \tag{4.0.26}$$

If $U^0_{\boldsymbol{\tau}} = u_0 \in D(|\partial\phi|)$, $\lambda_{\boldsymbol{\tau}}$ *is defined as in* (4.0.25), *and* $T_{\boldsymbol{\tau}} = \min\big\{t^k_{\boldsymbol{\tau}} \in \mathcal{P}_{\boldsymbol{\tau}} : t^k_{\boldsymbol{\tau}} \ge T\big\}$, *we have*

$$d(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \le c\,\frac{|\boldsymbol{\tau}|}{\sqrt{2}}|\partial\phi|(u_0)\, e^{-\lambda_{\boldsymbol{\tau}} T}, \qquad c := \frac{1 + 2|\lambda|\,T_{\boldsymbol{\tau}}}{1 + \lambda|\boldsymbol{\tau}|}. \tag{4.0.27}$$

We recall that in the case $\lambda > 0$ the function ϕ is bounded from below.

Theorem 4.0.10 (The case $\lambda > 0$). *Suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold for* $\lambda > 0$, *let* $u \in MM(\Phi; u_0)$ *be the unique solution of the equation* (4.0.13), *let* $\overline{U}_{\boldsymbol{\tau}}$ *be a discrete solution associated to the partition* $\mathcal{P}_{\boldsymbol{\tau}}$ (2.0.1), *and let* $\lambda_{\boldsymbol{\tau}}$ *be defined as in* (4.0.25). *If* $U^0_{\boldsymbol{\tau}} = u_0 \in D(\phi)$ *and* $T_{\boldsymbol{\tau}} \in \mathcal{P}_{\boldsymbol{\tau}}$ *is defined as in the above Theorem, we have*

$$\begin{aligned} d^2(\overline{U}_{\boldsymbol{\tau}}(T), u(T))) &\le c\,|\boldsymbol{\tau}|\Big(\phi(u_0) - \inf_{\mathscr{S}} \phi\Big) e^{-2\lambda_{\boldsymbol{\tau}} T}, \\ c &:= (1 + \lambda|\boldsymbol{\tau}|)\big(1 + \sqrt{2\lambda T_{\boldsymbol{\tau}}}\big)^4. \end{aligned} \tag{4.0.28}$$

If $U^0_{\boldsymbol{\tau}} = u_0 \in D(|\partial\phi|)$ *we have*

$$d^2(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \le c\,\frac{|\boldsymbol{\tau}|^2}{2}|\partial\phi|^2(u_0)\, e^{-2\lambda_{\boldsymbol{\tau}} T}, \qquad c := 1 + 2\lambda\, T_{\boldsymbol{\tau}}. \tag{4.0.29}$$

We split the proof of the previous theorems in many steps:

4.1.1: discrete variational inequalities. First of all we derive the variational evolution inequalities (4.1.3), which are the discrete counterparts of (4.0.13). They provide a crucial property satisfied by the discrete solutions and are a simple consequence of the convexity assumption 4.0.1; all the subsequent estimates can be deduced from this fundamental point.

4.1.2: Cauchy-type estimates. Here we introduce a general way to pass from a discrete variational inequality to a continuous one, though affected by a perturbation term; the main technical difficulty is the lackness of an underlying linear structure, which prevents an easy interpolation of the discrete values in the ambient space $\mathscr{S}$. We circumvent this fact by considering affine interpolations of the values of the functions instead of trying to interpolate their arguments (see also [101] for a similar approach). Once continuous versions of the evolution variational inequalities are at our disposal, it will not be difficult to derive Cauchy-type estimates, by also applying a Gronwall lemma in the case $\lambda \neq 0$.

4.2: convergence. This section is devoted to control the perturbation terms in the previously derived estimates, in order to prove the convergence of the scheme. We first consider the easier case $u_0 \in D(\phi)$ and then we extend the results to a general $u_0 \in \overline{D(\phi)}$.

4.3: regularizing effect and semigroup generation. Here we show that the unique element $u \in MM(\Phi; u_0)$ exhibits the regularizing effect (4.0.12) and then derives the differential characterization (4.0.13) which also yield the λ-contracting semigroup property (4.0.14).

4.4: optimal error estimates. Finally, we refine the error estimates which have been derived in the first section, and we prove Theorems 4.0.7, 4.0.9, 4.0.10, and the related estimate (4.0.15). For the ease of the reader, the main ideas are first presented in the case $\lambda = 0$; the more technical results for $\lambda \neq 0$ are discussed in Section 4.4.2

4.1 Cauchy-type estimates for discrete solutions

4.1.1 Discrete variational inequalities

Let us first state an auxiliary lemma:

Lemma 4.1.1. *Let us suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold for some* $\lambda \in \mathbb{R}$, *and let* $0 < \tau < \frac{1}{\lambda^-}$. *If* $u \in \overline{D(\phi)}$ *and* (v_n) *is a sequence in* $D(\phi)$ *satisfying*

$$\limsup_{n\to\infty} \Phi(\tau, u; v_n) \leq \phi_\tau(u), \tag{4.1.1}$$

then (v_n) *converges to* $v \in D(\phi)$ *and* $v = u_\tau = J_\tau[u]$ *is the unique element of* $J_\tau[u]$.

Proof. Being $u \in \overline{D(\phi)}$, we can find a sequence $(u_n) \subset D(\phi)$ converging to u such that

$$\limsup_{n\to\infty} \Phi(\tau, u_n; v_n) = \limsup_{n\to\infty} \Phi(\tau, u; v_n) \leq \phi_\tau(u).$$

We argue as in the proof of Lemma 2.4.8: observe that, being ϕ_τ continuous (cf. Lemma 3.1.2) and $\phi_\tau(u) < +\infty$, (4.1.1) yields

$$\begin{aligned}\Phi(\tau, u_n; v_n) &= \phi_\tau(u_n) + \Big(\phi_\tau(u) - \phi_\tau(u_n)\Big) + \Big(\Phi(\tau, u_n; v_n) - \phi_\tau(u)\Big) \\ &= \phi_\tau(u_n) + \omega_n \quad \text{with} \quad \limsup_{n\to\infty} \omega_n \leq 0.\end{aligned}$$

We apply the convexity property (4.0.10) with $w := u_n, v_0 := v_n, v_1 := v_m$ at $t = 1/2$ to find $v_{n,m}$ such that

$$\phi_\tau(u_n) \leq \Phi(\tau, u_n; v_{n,m}) \leq \phi_\tau(u_n) + \frac{\omega_n + \omega_m}{2} - \frac{1 + \lambda\tau}{8\tau} d^2(v_n, v_m).$$

Since $1 + \lambda\tau > 0$ this implies that

$$\limsup_{n,m\to\infty} d^2(v_n, v_m) \le \frac{4\tau}{1+\lambda\tau} \limsup_{n,m\to\infty} \big(\omega_n + \omega_m\big) = 0,$$

therefore (v_n) is a Cauchy sequence and the lower semicontinuity of ϕ gives that $\Phi(\tau, u; v) = \phi_\tau(u)$, i.e. $v \in J_\tau[u]$. The same argument also shows that v is the unique element of $J_\tau[u]$. □

The following result is a significant improvement of Theorem 3.1.6:

Theorem 4.1.2 (Variational inequalities for u_τ). *Let us suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *holds for some* $\lambda \in \mathbb{R}$*.*

(i) *If* $u \in \overline{D(\phi)}$ *and* $\lambda\tau > -1$ *then the minimum problem* (2.0.5) *has a unique solution* $u_\tau = J_\tau[u]$*. The map* $u \in \overline{D(\phi)} \mapsto J_\tau[u]$ *is continuous.*

(ii) *If* $u \in \overline{D(\phi)}$ *and* $u_\tau = J_\tau[u]$*, for each* $v \in D(\phi)$ *we have*

$$\frac{1}{2\tau} d^2(u_\tau, v) - \frac{1}{2\tau} d^2(u, v) + \frac{1}{2}\lambda d^2(u_\tau, v) \le \phi(v) - \phi_\tau(u). \tag{4.1.2}$$

Proof. (i) In order to show the existence of a minimum point $u_\tau \in J_\tau[u]$ we simply apply the previous Lemma 4.1.1 by choosing an arbitrary minimizing sequence, thus satisfying (4.1.1).

The continuity of J_τ follows by the same argument; simply take a sequence $(u_n) \subset \overline{D(\phi)}$ converging to u and observe that $v_n := J_\tau[u_n]$ is bounded in $\mathscr{S}$ and satisfies

$$\limsup_{n\to\infty} \Phi(\tau, u; v_n) = \limsup_{n\to\infty} \Phi(\tau, u_n; v_n) = \lim_{n\to\infty} \phi_\tau(u_n) = \phi_\tau(u).$$

(ii) Since the map J_τ is continuous, by a standard approximation argument we can suppose $u \in D(\phi)$. We apply (4.0.10) again with $w := u$, $v_0 := u_\tau$ and $v_1 := v$, obtaining a family $v_t \in D(\phi)$, $t \in (0, 1)$, such that

$$\Phi(\tau, u; u_\tau) \le \Phi(\tau, u; v_t) \le (1-t)\Phi(\tau, u; u_\tau) + t\Phi(\tau, u; v) - \frac{1+\lambda\tau}{2\tau} t(1-t) d^2(u_\tau, v).$$

Subtracting $\Phi(\tau, u; u_\tau)$ by each term of the inequality, dividing by t, and passing to the limit as $t \downarrow 0$ we get

$$0 \le -\Phi(\tau, u; u_\tau) + \Phi(\tau, u; v) - \frac{1+\lambda\tau}{2\tau} d^2(u_\tau, v)$$

which is equivalent to (4.1.2) since $\phi_\tau(u) = \Phi(\tau, u; u_\tau)$. □

Corollary 4.1.3 (Variational inequalities for discrete solutions). *Under the same assumptions of the previous Lemma, every discrete solution* $\{U^n_{\boldsymbol\tau}\}_{n=0}^{+\infty}$ *with* $U^0_{\boldsymbol\tau} \in \overline{D(\phi)}$ *satisfies*

$$\begin{aligned}\frac{1}{2\tau_n}\Big(d^2(U^n_{\boldsymbol\tau},V) - d^2(U^{n-1}_{\boldsymbol\tau},V)\Big) + \frac{1}{2}\lambda\, d^2(U^n_{\boldsymbol\tau},V)\\ \le \phi(V) - \phi(U^n_{\boldsymbol\tau}) - \frac{1}{2\tau_n} d^2(U^n_{\boldsymbol\tau},U^{n-1}_{\boldsymbol\tau}) \quad \forall\, V \in D(\phi),\ n \ge 1.\end{aligned} \tag{4.1.3}$$

4.1.2 Piecewise affine interpolation and comparison results

Now we formalize a general way to write a discrete difference inequality as a continuous one: first of all, let us introduce the "delayed" piecewise constant function $\underline{U}_{\boldsymbol\tau}$

$$\underline{U}_{\boldsymbol\tau}(t) \equiv U^{n-1}_{\boldsymbol\tau} \quad \text{if } t \in (t^{n-1}_{\boldsymbol\tau}, t^n_{\boldsymbol\tau}],$$

and the interpolating functions

$$\ell_{\boldsymbol\tau}(t) := \frac{t - t^{n-1}_{\boldsymbol\tau}}{\tau_n}, \quad 1 - \ell_{\boldsymbol\tau}(t) = \frac{t^n_{\boldsymbol\tau} - t}{\tau_n} \quad \text{if } t \in (t^{n-1}_{\boldsymbol\tau}, t^n_{\boldsymbol\tau}]. \tag{4.1.4}$$

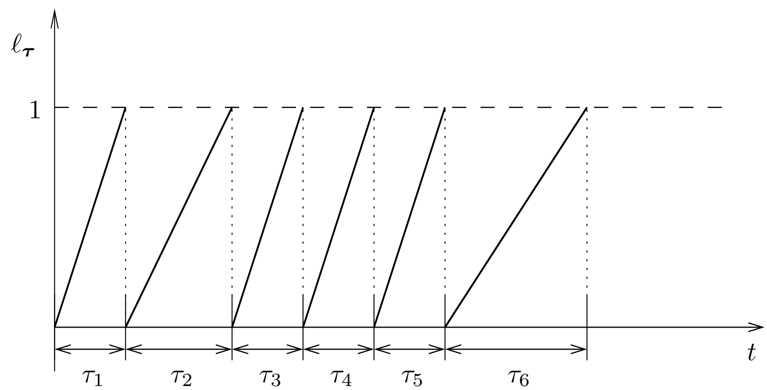

Figure 4.1: The interpolating functions $\ell_{\boldsymbol\tau}$.

If $\zeta : \mathscr{S} \to (-\infty, +\infty]$ is a function which is finite on the discrete solution $\{U^n_{\boldsymbol\tau}\}_{n=0}^{+\infty}$, we can define its *affine interpolation* as

$$\begin{aligned}\zeta_{\boldsymbol\tau}(t) :=&(1-\ell_{\boldsymbol\tau}(t))\zeta(\underline{U}_{\boldsymbol\tau}(t)) + \ell_{\boldsymbol\tau}(t)\zeta(\overline{U}_{\boldsymbol\tau}(t))\\ =&(1-\ell_{\boldsymbol\tau}(t))\zeta(U^{n-1}_{\boldsymbol\tau}) + \ell_{\boldsymbol\tau}(t)\zeta(U^n_{\boldsymbol\tau}) \quad \text{if } t \in (t^{n-1}_{\boldsymbol\tau}, t^n_{\boldsymbol\tau}].\end{aligned} \tag{4.1.5}$$

In other words, $\zeta_{\boldsymbol\tau}$ is the continuous piecewise affine function which interpolates the values $\zeta(U^n_{\boldsymbol\tau})$ at the nodes $t^n_{\boldsymbol\tau}$ of the partition $\mathcal{P}_{\boldsymbol\tau}$. In this way, for $V \in \mathscr{S}$, we can consider the functions

$$d^2_{\boldsymbol{\tau}}(t;V) := (1-\ell_{\boldsymbol{\tau}}(t))d^2(U^{n-1}_{\boldsymbol{\tau}},V) + \ell_{\boldsymbol{\tau}}(t)d^2(U^n_{\boldsymbol{\tau}},V) \qquad t\in(t^{n-1}_{\boldsymbol{\tau}},t^n_{\boldsymbol{\tau}}], \tag{4.1.6}$$

$$\varphi_{\boldsymbol{\tau}}(t) := (1-\ell_{\boldsymbol{\tau}}(t))\phi(U^{n-1}_{\boldsymbol{\tau}}) + \ell_{\boldsymbol{\tau}}(t)\phi(U^n_{\boldsymbol{\tau}}) \qquad t\in(t^{n-1}_{\boldsymbol{\tau}},t^n_{\boldsymbol{\tau}}]. \tag{4.1.7}$$

The main idea here is to "interpolate a function" instead of evaluating it on a (more difficult) interpolation of the arguments (see also [101] for another application of this technique); of course, for convex functional in Euclidean space these two approaches are slightly different but in our metric framework the first one is particularly convenient.

Finally, to every discrete solution $\{U^n_{\boldsymbol{\tau}}\}_{n=0}^{+\infty}\subset D(\phi)$ defined as before we associate the "squared discrete derivative"

$$D^n_{\boldsymbol{\tau}} := \frac{d^2(U^{n-1}_{\boldsymbol{\tau}},U^n_{\boldsymbol{\tau}})}{\tau_n^2}, \quad n=1,\cdots, \tag{4.1.8}$$

and the residual function $\mathscr{R}_{\boldsymbol{\tau}}$, defined for $t\in(t^{n-1}_{\boldsymbol{\tau}},t^n_{\boldsymbol{\tau}}]$ by

$$\mathscr{R}_{\boldsymbol{\tau}}(t) := 2(1-\ell_{\boldsymbol{\tau}}(t))\Big(\phi(U^{n-1}_{\boldsymbol{\tau}})-\phi(U^n_{\boldsymbol{\tau}})-\frac{\tau_n}{2}D^n_{\boldsymbol{\tau}}\Big)-\ell_{\boldsymbol{\tau}}(t)\tau_nD^n_{\boldsymbol{\tau}} \tag{4.1.9a}$$

$$=2(1-\ell_{\boldsymbol{\tau}}(t))\tau_n\mathscr{E}^n_{\boldsymbol{\tau}} + \big(1-2\ell_{\boldsymbol{\tau}}(t)\big)\tau_nD^n_{\boldsymbol{\tau}}. \tag{4.1.9b}$$

Observe that (3.1.20) yields

$$\begin{aligned}(1+\lambda\tau_n)|\partial\phi|^2(U^n_{\boldsymbol{\tau}}) &\le (1+\lambda\tau_n)D^n_{\boldsymbol{\tau}} \le \frac{2}{\tau_n}\Big(\phi(U^{n-1}_{\boldsymbol{\tau}})-\phi(U^n_{\boldsymbol{\tau}})-\frac{\tau_n}{2}D^n_{\boldsymbol{\tau}}\Big)\\ &\le \frac{1}{1+\lambda\tau_n}|\partial\phi|^2(U^{n-1}_{\boldsymbol{\tau}}) \le \frac{1}{1+\lambda\tau_n}D^{n-1}_{\boldsymbol{\tau}},\end{aligned} \tag{4.1.10}$$

so that, if $U^{n-1}_{\boldsymbol{\tau}}\in D(|\partial\phi|)$ then (4.1.9a) yields

$$\mathscr{R}_{\boldsymbol{\tau}}(t) \le \tau_n\frac{1-\ell_{\boldsymbol{\tau}}(t)}{1+\lambda\tau_n}|\partial\phi|^2(U^{n-1}_{\boldsymbol{\tau}})-\ell_{\boldsymbol{\tau}}(t)\tau_nD^n_{\boldsymbol{\tau}} \qquad t\in(t^{n-1}_{\boldsymbol{\tau}},t^n_{\boldsymbol{\tau}}]. \tag{4.1.11}$$

Theorem 4.1.4. *Let us suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold for* $\lambda\in\mathbb{R}$, *and* $U^0_{\boldsymbol{\tau}}\in D(\phi)$. *The interpolated functions* $d_{\boldsymbol{\tau}}$, $\varphi_{\boldsymbol{\tau}}$ *defined as in* (4.1.6), (4.1.7) *starting from the discrete solution* $\{U^n_{\boldsymbol{\tau}}\}_{n=0}^{+\infty}$ *satisfy the following system of variational inequalities almost everywhere in* $(0,+\infty)$*:*

$$\frac12\frac{d}{dt}d^2_{\boldsymbol{\tau}}(t;V)+\frac{\lambda}{2}d^2(\overline{U}_{\boldsymbol{\tau}}(t),V)+\varphi_{\boldsymbol{\tau}}(t)-\phi(V)\le\frac12\mathscr{R}_{\boldsymbol{\tau}}(t) \quad \forall V\in D(\phi). \tag{4.1.12}$$

Proof. If $t \in (t_{\boldsymbol\tau}^{n-1}, t_{\boldsymbol\tau}^n]$, using (4.1.3) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}d_{\boldsymbol\tau}^2(t;V) + \frac{1}{2}\lambda d^2(\overline{U}_{\boldsymbol\tau}(t),V) + \varphi_{\boldsymbol\tau}(t) - \phi(V)\\
&= \frac{1}{2\tau_n}\Big(d^2(U_{\boldsymbol\tau}^n,V) - d^2(U_{\boldsymbol\tau}^{n-1},V)\Big) + \frac{1}{2}\lambda d^2(U_{\boldsymbol\tau}^n,V) + \varphi_{\boldsymbol\tau}(t) - \phi(V)\\
&\le -\frac{1}{2\tau_n}d^2(U_{\boldsymbol\tau}^n,U_{\boldsymbol\tau}^{n-1}) + \phi(V) - \phi(U_{\boldsymbol\tau}^n) + \varphi_{\boldsymbol\tau}(t) - \phi(V)\\
&= -\frac{1}{2\tau_n}d^2(U_{\boldsymbol\tau}^n,U_{\boldsymbol\tau}^{n-1}) + (1-\ell_{\boldsymbol\tau}(t))\Big(\phi(U_{\boldsymbol\tau}^{n-1}) - \phi(U_{\boldsymbol\tau}^n)\Big)\\
&= (1-\ell_{\boldsymbol\tau}(t))\Big(\phi(U_{\boldsymbol\tau}^{n-1}) - \phi(U_{\boldsymbol\tau}^n) - \frac{1}{2\tau_n}d^2(U_{\boldsymbol\tau}^n,U_{\boldsymbol\tau}^{n-1})\Big) - \ell_{\boldsymbol\tau}(t)\frac{1}{2\tau_n}d^2(U_{\boldsymbol\tau}^n,U_{\boldsymbol\tau}^{n-1}).
\end{aligned}
$$

Recalling the Definition (4.1.9a) of $\mathscr{R}_{\boldsymbol\tau}(t)$ we conclude. □

Comparison between discrete solutions for $\lambda = 0$. In the next Corollary we are finally able to compare two discrete solutions.

Corollary 4.1.5 (Comparison for $\lambda = 0$). *Under the same assumptions of Theorem 4.1.4, let us suppose that $\lambda = 0$ and let $\{U_{\boldsymbol\eta}^m\}_{m=0}^{+\infty}$, $U_{\boldsymbol\eta}^0 \in D(\phi)$, be another discrete solution associated to the admissible partition*

$$\mathcal{P}_{\boldsymbol\eta} := \Big\{0 = t_{\boldsymbol\eta}^0 < t_{\boldsymbol\eta}^1 < \ldots < t_{\boldsymbol\eta}^m, \ldots\Big\}, \quad \eta_m = t_{\boldsymbol\eta}^m - t_{\boldsymbol\eta}^{m-1}. \tag{4.1.13}$$

The continuous and piecewise affine function

$$d_{\boldsymbol\tau\boldsymbol\eta}^2(t,s) := (1-\ell_{\boldsymbol\eta}(s))d_{\boldsymbol\tau}^2(t;\underline{U}_{\boldsymbol\eta}(s)) + \ell_{\boldsymbol\eta}(s)d_{\boldsymbol\tau}^2(t;\overline{U}_{\boldsymbol\eta}(s)) \quad t,s \ge 0 \tag{4.1.14}$$

satisfies the differential inequality

$$\frac{d}{dt}d_{\boldsymbol\tau\boldsymbol\eta}^2(t,t) \le \mathscr{R}_{\boldsymbol\tau}(t) + \mathscr{R}_{\boldsymbol\eta}(t) \quad \forall\, t \in (0,+\infty) \setminus (\mathcal{P}_{\boldsymbol\tau} \cup \mathcal{P}_{\boldsymbol\eta}) \tag{4.1.15}$$

and therefore the integral bound

$$d_{\boldsymbol\tau\boldsymbol\eta}^2(T,T) \le d^2(U_{\boldsymbol\tau}^0, U_{\boldsymbol\eta}^0) + \int_0^T \Big(\mathscr{R}_{\boldsymbol\tau}(t) + \mathscr{R}_{\boldsymbol\eta}(t)\Big)\,dt. \tag{4.1.16}$$

Proof. Defining the function $\varphi_{\boldsymbol\eta}(s)$ as in (4.1.5) by

$$\varphi_{\boldsymbol\eta}(s) := (1-\ell_{\boldsymbol\eta}(s))\phi(\underline{U}_{\boldsymbol\eta}(s)) + \ell_{\boldsymbol\eta}(s)\phi(\overline{U}_{\boldsymbol\eta}(s)), \tag{4.1.17}$$

a convex combination of (4.1.12) for $V := \underline{U}_{\boldsymbol\eta}(s)$ and $V := \overline{U}_{\boldsymbol\eta}(s)$ yields

$$\frac{1}{2}\frac{\partial}{\partial t}d_{\boldsymbol\tau\boldsymbol\eta}^2(t,s) + \varphi_{\boldsymbol\tau}(t) - \varphi_{\boldsymbol\eta}(s) \le \frac{1}{2}\mathscr{R}_{\boldsymbol\tau}(t) \quad \forall\, s > 0,\ t \in (0,+\infty) \setminus \mathcal{P}_{\boldsymbol\tau}.$$

Analogously, writing (4.1.12) for the function $d^2_{\boldsymbol\eta}$ defined as in (4.1.6)

$$d^2_{\boldsymbol\eta}(s;V) := (1-\ell_{\boldsymbol\eta}(s))d^2(\underline{U}_{\boldsymbol\eta}(s),V) + \ell_{\boldsymbol\eta}(s)d^2(\overline{U}_{\boldsymbol\eta}(s),V),$$

and reversing the roles of $\boldsymbol\eta$ and $\boldsymbol\tau$ we obtain

$$\frac{1}{2}\frac{\partial}{\partial s}d^2_{\boldsymbol{\eta\tau}}(s,t) + \varphi_{\boldsymbol\eta}(s) - \varphi_{\boldsymbol\tau}(t) \le \frac{1}{2}\mathscr{R}_{\boldsymbol\eta}(s) \quad \forall\, t>0,\ s\in(0,+\infty)\setminus\mathcal{P}_{\boldsymbol\eta},$$

where

$$d^2_{\boldsymbol{\eta\tau}}(s,t) := (1-\ell_{\boldsymbol\tau}(t))d^2_{\boldsymbol\eta}(s;U^{n-1}_{\boldsymbol\tau}) + \ell_{\boldsymbol\tau}(t)d^2_{\boldsymbol\eta}(s;U^n_{\boldsymbol\tau}) \quad \text{for } t\in(t^{n-1}_{\boldsymbol\tau},t^n_{\boldsymbol\tau}]. \tag{4.1.18}$$

Summing up the two contributions we find

$$\frac{\partial}{\partial t}d^2_{\boldsymbol{\tau\eta}}(t,s) + \frac{\partial}{\partial s}d^2_{\boldsymbol{\eta\tau}}(s,t) \le \mathscr{R}_{\boldsymbol\tau}(t) + \mathscr{R}_{\boldsymbol\eta}(s) \quad \forall\, s,t\in(0,+\infty)\setminus(\mathcal{P}_{\boldsymbol\tau}\cup\mathcal{P}_{\boldsymbol\eta}).$$

Finally, by the symmetry property

$$d^2_{\boldsymbol{\tau\eta}}(t,s) = d^2_{\boldsymbol{\eta\tau}}(s,t), \tag{4.1.19}$$

evaluating the previous inequality for $s=t$ we end up with (4.1.15). □

Comparison between discrete solutions for $\lambda\neq 0$. If $\lambda\neq 0$ we need to rewrite (4.1.12) in a more convenient form; let us first observe that the concavity of the square root provides the inequalities for $V\in\mathscr{S}$

$$(1-\ell_{\boldsymbol\tau}(t))d(\underline{U}_{\boldsymbol\tau}(t),V) + \ell_{\boldsymbol\tau}(t)d(\overline{U}_{\boldsymbol\tau}(t),V) \le d_{\boldsymbol\tau}(t,V) \quad \forall\, t>0, \tag{4.1.20}$$

$$(1-\ell_{\boldsymbol\eta}(s))d_{\boldsymbol\tau}(t,\underline{U}_{\boldsymbol\eta}(s)) + \ell_{\boldsymbol\eta}(s)d_{\boldsymbol\tau}(t,\overline{U}_{\boldsymbol\eta}(s)) \le d_{\boldsymbol{\tau\eta}}(t,s) \quad \forall\, t,s>0. \tag{4.1.21}$$

Lemma 4.1.6. *Under the same assumptions of Theorem* 4.1.4, *for a discrete solution* $\{U^n_{\boldsymbol\tau}\}_{n=0}^{+\infty}$ *with* $U^0_{\boldsymbol\tau}\in D(\phi)$ *let us define*

$$\mathscr{D}_{\boldsymbol\tau}(t) := \big(1-\ell_{\boldsymbol\tau}(t)\big)d(\overline{U}_{\boldsymbol\tau}(t),\underline{U}_{\boldsymbol\tau}(t)) = \tau_n\big(1-\ell_{\boldsymbol\tau}(t)\big)\sqrt{D^n_{\boldsymbol\tau}}, \quad t\in(t^{n-1}_{\boldsymbol\tau},t^n_{\boldsymbol\tau}]. \tag{4.1.22}$$

Then for every element $V\in D(\phi)$ *the interpolated functions* $d_{\boldsymbol\tau},\varphi_{\boldsymbol\tau}$ *defined by* (4.1.6) *and* (4.1.7) *satisfy the following system of variational inequalities almost everywhere in* $(0,+\infty)$*:*

$$\frac{d}{dt}\frac{1}{2}d^2_{\boldsymbol\tau}(t;V) + \frac{\lambda}{2}d^2_{\boldsymbol\tau}(t;V) - |\lambda|\mathscr{D}_{\boldsymbol\tau}(t)d_{\boldsymbol\tau}(t;V) + \varphi_{\boldsymbol\tau}(t) - \phi(V) \le \frac{1}{2}\mathscr{R}_{\boldsymbol\tau}(t) + \frac{\lambda^-}{2}\mathscr{D}^2_{\boldsymbol\tau}(t), \tag{4.1.23}$$

where $\lambda^- = \max(-\lambda,0)$.

Proof. If $\lambda\ge 0$ the inequality (4.1.23) is an immediate consequence of (4.1.12) and

$$-2d_{\boldsymbol\tau}(t;V)\mathscr{D}_{\boldsymbol\tau}(t) \le d^2(\overline{U}_{\boldsymbol\tau}(t),V) - d^2_{\boldsymbol\tau}(t;V)$$

which, in turn, follows by the triangle inequality. If $\lambda < 0$ it follows by (4.1.12) and

$$d^2(\overline{U}_{\boldsymbol{\tau}}(t), V) - d^2_{\boldsymbol{\tau}}(t; V) \leq 2d_{\boldsymbol{\tau}}(t; V)\mathscr{D}_{\boldsymbol{\tau}}(t) + \mathscr{D}^2_{\boldsymbol{\tau}}(t). \tag{4.1.24}$$

Let us prove (4.1.24). Suppose $t \in (t^{n-1}_{\boldsymbol{\tau}}, t^n_{\boldsymbol{\tau}}]$ and $d^2(\overline{U}_{\boldsymbol{\tau}}(t), V) \geq d^2_{\boldsymbol{\tau}}(t; V)$, otherwise (4.1.24) is obvious; the elementary identity $a^2 - b^2 = 2b(a - b) + (a - b)^2$ yields

$$\begin{aligned} d^2(\overline{U}_{\boldsymbol{\tau}}(t), V) - d^2_{\boldsymbol{\tau}}(t; V) &= 2d_{\boldsymbol{\tau}}(t; V)\big(d(\overline{U}_{\boldsymbol{\tau}}(t), V) - d_{\boldsymbol{\tau}}(t; V)\big) \\ &+ \big(d(\overline{U}_{\boldsymbol{\tau}}(t), V) - d_{\boldsymbol{\tau}}(t; V)\big)^2. \end{aligned}$$

On the other hand the concavity inequality (4.1.20) gives

$$\begin{aligned} d(\overline{U}_{\boldsymbol{\tau}}(t), V) - d_{\boldsymbol{\tau}}(t; V) &\leq d(\overline{U}_{\boldsymbol{\tau}}(t), V) - (1 - \ell_{\boldsymbol{\tau}}(t))d(\underline{U}_{\boldsymbol{\tau}}(t), V) \\ &- \ell_{\boldsymbol{\tau}}(t)d(\overline{U}_{\boldsymbol{\tau}}(t), V) \leq \mathscr{D}_{\boldsymbol{\tau}}(t). \end{aligned}$$

These two inequalities imply (4.1.24). □

Corollary 4.1.7 (Comparison for $\lambda \neq 0$). *Under the same assumption of the previous Lemma, let $\mathcal{P}_{\boldsymbol{\tau}}, \mathcal{P}_{\boldsymbol{\eta}}$ be two admissible partitions; the "error" function $d_{\boldsymbol{\tau\eta}}(t, s)$ defined by* (4.1.14) *satisfies the differential inequality*

$$\begin{aligned} \frac{d}{dt}d^2_{\boldsymbol{\tau\eta}}(t, t) + 2\lambda d^2_{\boldsymbol{\tau\eta}}(t, t) \leq & 2|\lambda|\big(\mathscr{D}_{\boldsymbol{\tau}}(t) + \mathscr{D}_{\boldsymbol{\eta}}(t)\big)d_{\boldsymbol{\tau\eta}}(t, t) \\ & + (\mathscr{R}_{\boldsymbol{\tau}}(t) + \mathscr{R}_{\boldsymbol{\eta}}(t)) + \lambda^-\big(\mathscr{D}^2_{\boldsymbol{\tau}}(t) + \mathscr{D}^2_{\boldsymbol{\eta}}(t)\big), \end{aligned} \tag{4.1.25}$$

and therefore the Gronwall-like estimate

$$\begin{aligned} e^{\lambda T}d_{\boldsymbol{\tau\eta}}(T, T) \leq & \left(d^2(U^0_{\boldsymbol{\tau}}, V^0_{\boldsymbol{\eta}}) + \mathsf{R}_{\boldsymbol{\tau}}(T) + \mathsf{R}_{\boldsymbol{\eta}}(T) + \int_0^T e^{2\lambda t}\lambda^-\big(\mathscr{D}^2_{\boldsymbol{\tau}}(t) + \mathscr{D}^2_{\boldsymbol{\eta}}(t)\big)\,dt\right)^{1/2} \\ & + 2\int_0^T |\lambda|e^{\lambda t}\big(\mathscr{D}_{\boldsymbol{\tau}}(t) + \mathscr{D}_{\boldsymbol{\eta}}(t)\big)\,dt, \end{aligned} \tag{4.1.26}$$

where $\mathsf{R}_{\boldsymbol{\tau}}$ (and analogously $\mathsf{R}_{\boldsymbol{\eta}}$) are defined by

$$\mathsf{R}_{\boldsymbol{\tau}}(T) := \sup_{t\in[0,T]} \int_0^t e^{2\lambda r}\mathscr{R}_{\boldsymbol{\tau}}(r)\,dr \leq \int_0^T e^{2\lambda r}\Big(\mathscr{R}_{\boldsymbol{\tau}}(r)\Big)^+ dr \quad \forall T > 0. \tag{4.1.27}$$

Proof. Starting from the inequality (4.1.23) we easily obtain (4.1.25) by arguing as in Corollary 4.1.5 and by using (4.1.21). Inequality (4.1.26) is a direct consequence of (4.1.25) and of the following version of the Gronwall Lemma [18]. □

Lemma 4.1.8 (A version of Gronwall Lemma). *Let $x : [0, +\infty) \to \mathbb{R}$ be a locally absolutely continuous function, let a, $b \in L^1_{\rm loc}([0, +\infty))$ be given functions satisfying, for $\lambda \in \mathbb{R}$,*

$$\frac{d}{dt}x^2(t) + 2\lambda x^2(t) \leq a(t) + 2b(t)x(t) \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t > 0. \tag{4.1.28}$$

Then for every $T > 0$ we have

$$e^{\lambda T}|x(T)| \le \Big(x^2(0) + \sup_{t\in[0,T]}\int_0^t e^{2\lambda s}a(s)\,ds\Big)^{1/2} + 2\int_0^T e^{\lambda t}|b(t)|\,dt. \tag{4.1.29}$$

Proof. Multiplying (4.1.28) by $e^{2\lambda t}$ we obtain

$$\frac{d}{dt}\big(e^{\lambda t}x(t)\big)^2 \le e^{2\lambda t}a(t) + 2e^{\lambda t}b(t)\big(e^{\lambda t}x(t)\big) \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0, \tag{4.1.30}$$

therefore it is sufficient to prove (4.1.29) for $\lambda = 0$.

Introducing the functions

$$\begin{aligned} X(T) &:= \sup_{t\in(0,T)}|x(t)|, \quad A(T) := \sup_{t\in(0,T)}\int_0^t a(s)\,ds, \\ B(T) &:= \int_0^T |b(s)|\,ds, \end{aligned} \tag{4.1.31}$$

and integrating the equation we obtain

$$x^2(t) \le x^2(0) + \int_0^t a(s)\,ds + 2B(t)X(t) \quad \forall\, t > 0. \tag{4.1.32}$$

Therefore, taking the supremum w.r.t. $t \in [0,T]$ we get

$$X^2(T) \le x^2(0) + A(T) + 2B(T)X(T), \tag{4.1.33}$$

and adding $B^2(T)$ to both sides gives

$$X(T) \le B(T) + \sqrt{B^2(T) + x^2(0) + A(T)} \le 2B(T) + \sqrt{x^2(0) + A(T)}.$$

Recalling (4.1.31) we obtain (4.1.29). □

4.2 Convergence of discrete solutions

4.2.1 Convergence when the initial datum $u_0 \in D(\phi)$

The previous Corollaries 4.1.5, 4.1.7 show the importance to obtain a priori bounds of the integral of $\mathscr{R}_{\boldsymbol{\tau}}, \mathscr{D}_{\boldsymbol{\tau}}$, and $\mathscr{D}^2_{\boldsymbol{\tau}}$. In this section we mainly focus our attention on the convergence of the discrete solutions, by quickly deriving rough estimates of these integrals and we postpone a finer analysis of the error to Section 4.4. It is not restrictive to assume $\lambda \le 0$.

Lemma 4.2.1. *Let us suppose that the convexity Assumption* 4.0.1 *holds with* $\lambda \le 0$, *let* $\mathscr{R}_{\boldsymbol\tau}, \mathscr{D}_{\boldsymbol\tau}$ *be the residual terms associated to a discrete solution* $\{U^n_{\boldsymbol\tau}\}_{n=0}^{+\infty}$ *defined as in* (4.1.9a), (4.1.22), *and let us choose* T *in the interval* $I^N_{\boldsymbol\tau} = (t^{N-1}_{\boldsymbol\tau}, t^N_{\boldsymbol\tau}]$. *Then*

$$\int_0^T e^{2\lambda t}\Big([\mathscr{R}_{\boldsymbol\tau}(t)]^+ - \lambda\mathscr{D}^2_{\boldsymbol\tau}(t)\Big)\,dt \le |\boldsymbol\tau|\Big(\phi(U^0_{\boldsymbol\tau}) - \phi(U^N_{\boldsymbol\tau})\Big), \tag{4.2.1}$$

$$\Big(\int_0^T |\lambda| e^{\lambda t}\mathscr{D}_{\boldsymbol\tau}(t)\,dt\Big)^2 \le \frac12\int_0^T |\lambda| e^{2\lambda t}\mathscr{D}^2_{\boldsymbol\tau}(t)\,dt \tag{4.2.2}$$

$$\le \frac{|\lambda|\,|\boldsymbol\tau|^2}{3}\Big(\phi(U^0_{\boldsymbol\tau}) - \phi(U^N_{\boldsymbol\tau})\Big). \tag{4.2.3}$$

Proof. First of all we observe that

$$\int_{I^n_{\boldsymbol\tau}} \big[\mathscr{R}_{\boldsymbol\tau}(t)\big]^+\,dt \le \tau_n\Big(\phi(U^{n-1}_{\boldsymbol\tau}) - \phi(U^n_{\boldsymbol\tau}) - \frac{d^2(U^n_{\boldsymbol\tau}, U^{n-1}_{\boldsymbol\tau})}{2\tau_n}\Big), \tag{4.2.4}$$

which is a direct consequence of (4.1.9a) and

$$\phi(U^{n-1}_{\boldsymbol\tau}) - \phi(U^n_{\boldsymbol\tau}) - \frac{d^2(U^n_{\boldsymbol\tau}, U^{n-1}_{\boldsymbol\tau})}{2\tau_n} \ge 0, \qquad \int_{I^n_{\boldsymbol\tau}} (1-\ell_{\boldsymbol\tau}(t))\,dt = \int_{I^n_{\boldsymbol\tau}} \ell_{\boldsymbol\tau}(t)\,dt = \frac12.$$

Since

$$\int_{I^n_{\boldsymbol\tau}} (1-\ell_{\boldsymbol\tau}(t))^2\,dt = \frac13\tau_n,$$

and

$$\int_{I^n_{\boldsymbol\tau}} |\lambda|\mathscr{D}^2_{\boldsymbol\tau}(t)\,dt \le \frac13|\lambda|\tau_n d^2(U^n_{\boldsymbol\tau}, U^{n-1}_{\boldsymbol\tau}) \le \frac13 d^2(U^n_{\boldsymbol\tau}, U^{n-1}_{\boldsymbol\tau}), \tag{4.2.5}$$

from (4.2.4) we get

$$\int_{I^n_{\boldsymbol\tau}} e^{2\lambda t}\Big[\big(\mathscr{R}_{\boldsymbol\tau}(t)\big)^+ + |\lambda|\mathscr{D}^2_{\boldsymbol\tau}(t)\Big]\,dt \le \tau_n\big(\phi(U^{n-1}_{\boldsymbol\tau}) - \phi(U^n_{\boldsymbol\tau})\big) \tag{4.2.6}$$

which yields (4.2.1). Starting from (4.2.5) and recalling (3.2.8) we obtain

$$\int_0^T |\lambda|\mathscr{D}^2_{\boldsymbol\tau}(t)\,dt \le \frac23|\lambda|\,|\boldsymbol\tau|^2(\phi(U^0_{\boldsymbol\tau}) - \phi(U^N_{\boldsymbol\tau})),$$

so that

$$\Big(\int_0^T |\lambda| e^{\lambda t}\mathscr{D}_{\boldsymbol\tau}(t)\,dt\Big)^2 \le \int_0^T |\lambda| e^{2\lambda t}\,dt\int_0^T |\lambda|\mathscr{D}^2_{\boldsymbol\tau}(t)\,dt \le \frac{|\lambda|\,|\boldsymbol\tau|^2}{3}\Big(\phi(U^0_{\boldsymbol\tau}) - \phi(U^N_{\boldsymbol\tau})\Big),$$

which yields (4.2.2) and (4.2.3). □

Theorem 4.2.2. *Suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold for* $\lambda \in \mathbb{R}$ *and*

$$\lim_{|\boldsymbol{\tau}|\downarrow 0} d(U^0_{\boldsymbol{\tau}}, u_0) = 0, \quad \sup_{\boldsymbol{\tau}} \phi(U^0_{\boldsymbol{\tau}}) = S < +\infty. \tag{4.2.7}$$

Then the family $\{\overline{U}_{\boldsymbol{\tau}}\}_{\boldsymbol{\tau}}$ *of the discrete solutions generated by* $U^0_{\boldsymbol{\tau}}$ *is convergent to a function* u *as* $|\boldsymbol{\tau}| \downarrow 0$, *uniformly in each bounded interval* $[0,T]$; *in particular* u *is the unique element of* $MM(\Phi; u_0)$.

Proof. We fix a time $t \in [0,T]$ and we prove that $\{\overline{U}_{\boldsymbol{\tau}}(t)\}_{\boldsymbol{\tau}}$ is a Cauchy family as $|\boldsymbol{\tau}|$ goes to 0. We already know from the a priori estimates of Lemma 3.2.2 that there exists a constant C dependent on S, T, λ but independent of $\boldsymbol{\tau}$ such that

$$d^2(\overline{U}_{\boldsymbol{\tau}}(t), \underline{U}_{\boldsymbol{\tau}}(t)) \le C\,|\boldsymbol{\tau}|, \quad \phi(U^0_{\boldsymbol{\tau}}) - \phi(U^n_{\boldsymbol{\tau}}) \le C \quad 1 \le n \le N, \tag{4.2.8}$$

for the integer N such that the interval $I^N_{\boldsymbol{\tau}}$ contains T. Moreover, choosing two partitions $\mathcal{P}_{\boldsymbol{\tau}}, \mathcal{P}_{\boldsymbol{\eta}}$ as in Corollary 4.1.7, by (4.1.14) we have

$$\begin{aligned} d^2(\overline{U}_{\boldsymbol{\tau}}(t), \overline{U}_{\boldsymbol{\eta}}(t)) &\le 3d^2_{\boldsymbol{\tau\eta}}(t,t) + 3d^2(\underline{U}_{\boldsymbol{\tau}}(t), \overline{U}_{\boldsymbol{\tau}}(t)) + 3d^2(\underline{U}_{\boldsymbol{\eta}}(t), \overline{U}_{\boldsymbol{\eta}}(t)) \\ &\le 3d^2_{\boldsymbol{\tau\eta}}(t,t) + 3C(|\boldsymbol{\tau}| + |\boldsymbol{\eta}|), \end{aligned}$$

therefore we simply have to show that $\lim_{|\boldsymbol{\tau}|,|\boldsymbol{\eta}|\downarrow 0} d_{\boldsymbol{\tau\eta}}(t,t) = 0$. By (4.1.26), (4.2.1), and (4.2.3) we obtain

$$e^{2\lambda t} d^2_{\boldsymbol{\tau\eta}}(t,t) \le 2d^2(U^0_{\boldsymbol{\tau}}, U^0_{\boldsymbol{\eta}}) + 2C\big(|\boldsymbol{\tau}| + |\boldsymbol{\eta}|\big) + 2|\lambda| C(|\boldsymbol{\tau}|^2 + |\boldsymbol{\eta}|^2), \tag{4.2.9}$$

and this conclude the proof of the convergence; since the constant C in the bound (4.2.9) is independent of t, the convergence is also uniform in $[0,T]$.
Finally, it is easy to check that the limit does not depend on the particular family of initial data $(U^0_{\boldsymbol{\tau}})$ satisfying (4.2.7): if $(V^0_{\boldsymbol{\tau}})$ is another sequence approximating u_0, we can apply the same convergence result to a third family $(W^0_{\boldsymbol{\tau}})$ which coincides with the previous ones along two different subsequences of step sizes $\boldsymbol{\tau}_n$, $\boldsymbol{\tau}'_n$ with $|\boldsymbol{\tau}_n|, |\boldsymbol{\tau}'_n| \downarrow 0$ as $n \to \infty$. □

Corollary 4.2.3. *Under the same assumption of the previous Theorem, let* $u = MM(\Phi; u_0)$ *and let* $\overline{U}_{\boldsymbol{\tau}}$ *be the discrete solution associated to the partition* $\mathcal{P}_{\boldsymbol{\tau}}$. *Then if* $T \in \mathcal{P}_{\boldsymbol{\tau}}$ *and* $\lambda = 0$ *we have*

$$d^2(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \le d^2(U^0_{\boldsymbol{\tau}}, u_0) + \int_0^T \mathscr{R}_{\boldsymbol{\tau}}(t)\,dt, \tag{4.2.10}$$

whereas for $\lambda \ne 0$ *we have*

$$\begin{aligned} e^{\lambda T} d(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \le &\left(d^2(U^0_{\boldsymbol{\tau}}, u_0) + \mathsf{R}_{\boldsymbol{\tau}}(T) + \int_0^T e^{2\lambda t} \lambda^- \mathscr{D}^2_{\boldsymbol{\tau}}(t)\,dt \right)^{1/2} \\ &+ 2\int_0^T |\lambda| e^{\lambda t} \mathscr{D}_{\boldsymbol{\tau}}(t)\,dt, \end{aligned} \tag{4.2.11}$$

where $\mathsf{R}_{\boldsymbol{\tau}}$ *is defined by* (4.1.27).

Proof. We simply pass to the limit as $|\boldsymbol{\eta}| \downarrow 0$ in (4.1.16) or (4.1.26), observing that the integrals of $\big(\mathscr{R}_{\boldsymbol{\eta}}\big)^+, \mathscr{D}_{\boldsymbol{\eta}}, \mathscr{D}^2_{\boldsymbol{\eta}}$ are infinitesimal by the estimates of Lemma 4.2.1; on the other hand, by (4.2.8) we have for $T \in \mathcal{P}_{\boldsymbol{\tau}}$

$$\lim_{|\boldsymbol{\eta}|\downarrow 0} d_{\boldsymbol{\tau\eta}}(T,T) = d_{\boldsymbol{\tau}}(T,u(T)), \quad \text{and} \quad d_{\boldsymbol{\tau}}(T,u(T)) = d(\overline{U}_{\boldsymbol{\tau}}(T), u(T)). \qquad \square$$

4.2.2 Convergence when the initial datum $u_0 \in \overline{D(\phi)}$.

Now we conclude the proof of (4.0.11) in the statement of Theorem 4.0.4 when the starting point belongs to the closure in $\mathscr{S}$ of the proper domain of ϕ: in this case, it is more difficult to exhibit an explicit order of convergence for the approximate solutions and we have to take care of the loss of regularity of the initial datum.

Let us start with a comparison result between two discrete solutions related to the same partition $\mathcal{P}_{\boldsymbol{\tau}}$:

Lemma 4.2.4. *Let $\overline{U}_{\boldsymbol{\tau}}, \overline{V}_{\boldsymbol{\tau}}$ be discrete solutions associated to the same choice of step size $\boldsymbol{\tau}$ and to the initial values $U^0_{\boldsymbol{\tau}} \in \overline{D(\phi)}, V^0_{\boldsymbol{\tau}} \in D(\phi)$ respectively. If $T \in I^N_{\boldsymbol{\tau}} = (t^{N-1}_{\boldsymbol{\tau}}, t^N_{\boldsymbol{\tau}}]$, and $\lambda_{\boldsymbol{\tau}}$ is defined in (4.0.25), then for $-1 < \lambda|\boldsymbol{\tau}| \le 0$ we have*

$$\begin{aligned} e^{2\lambda_{\boldsymbol{\tau}}(T+|\boldsymbol{\tau}|)} d^2(\overline{U}_{\boldsymbol{\tau}}(T), \overline{V}_{\boldsymbol{\tau}}(T)) &\le e^{2\lambda_{\boldsymbol{\tau}} t^N_{\boldsymbol{\tau}}} d^2(U^N_{\boldsymbol{\tau}}, V^N_{\boldsymbol{\tau}}) \\ &\le d^2(U^0_{\boldsymbol{\tau}}, V^0_{\boldsymbol{\tau}}) + 2|\boldsymbol{\tau}|\Big(\phi(V^0_{\boldsymbol{\tau}}) - \phi(V^N_{\boldsymbol{\tau}})\Big). \end{aligned} \tag{4.2.12}$$

Proof. Choosing $V := V^{n-1}_{\boldsymbol{\tau}}$ in (4.1.3) and multiplying the inequality by $2\tau_n$ we obtain

$$\begin{aligned} &d^2(U^n_{\boldsymbol{\tau}}, V^{n-1}_{\boldsymbol{\tau}}) - d^2(U^{n-1}_{\boldsymbol{\tau}}, V^{n-1}_{\boldsymbol{\tau}}) \le 2\tau_n\phi(V^{n-1}_{\boldsymbol{\tau}}) - 2\tau_n\phi(U^n_{\boldsymbol{\tau}}) - d^2(U^n_{\boldsymbol{\tau}}, U^{n-1}_{\boldsymbol{\tau}}) \\ &\quad - \lambda\tau_n d^2(U^n_{\boldsymbol{\tau}}, V^{n-1}_{\boldsymbol{\tau}}). \end{aligned}$$

Analogously, we choose $V := U^n_{\boldsymbol{\tau}}$ in the discrete inequality (4.1.3) written for the discrete solution $\{V^n_{\boldsymbol{\tau}}\}_{n=0}^{+\infty}$ obtaining

$$(1+\lambda\tau_n)d^2(V^n_{\boldsymbol{\tau}}, U^n_{\boldsymbol{\tau}}) - d^2(V^{n-1}_{\boldsymbol{\tau}}, U^n_{\boldsymbol{\tau}}) \le 2\tau_n\phi(U^n_{\boldsymbol{\tau}}) - 2\tau_n\phi(V^n_{\boldsymbol{\tau}}) - d^2(V^n_{\boldsymbol{\tau}}, V^{n-1}_{\boldsymbol{\tau}}).$$

Recalling the elementary inequality $(a+b)^2 \le \varepsilon^{-1}a^2 + (1-\varepsilon)^{-1}b^2$, $0 < \varepsilon < 1$, choosing $\varepsilon := -\lambda\tau_n$ we get

$$-\lambda\tau_n d^2(U^n_{\boldsymbol{\tau}}, V^{n-1}_{\boldsymbol{\tau}}) \le d^2(U^n_{\boldsymbol{\tau}}, U^{n-1}_{\boldsymbol{\tau}}) - \frac{\lambda\tau_n}{1+\lambda\tau_n} d^2(U^{n-1}_{\boldsymbol{\tau}}, V^{n-1}_{\boldsymbol{\tau}});$$

summing up the previous inequalities we obtain

$$(1+\lambda\tau_n)d^2(V^n_{\boldsymbol{\tau}}, U^n_{\boldsymbol{\tau}}) - \frac{1}{1+\lambda\tau_n} d^2(U^{n-1}_{\boldsymbol{\tau}}, V^{n-1}_{\boldsymbol{\tau}}) \le 2\tau_n\Big(\phi(V^{n-1}_{\boldsymbol{\tau}}) - \phi(V^n_{\boldsymbol{\tau}})\Big).$$

Multiplying the inequality by $e^{\lambda_{\boldsymbol{\tau}}(2t^{n-1}_{\boldsymbol{\tau}} + \tau_n)} < 1$ and recalling that $\phi(V^{n-1}_{\boldsymbol{\tau}}) \ge \phi(V^n_{\boldsymbol{\tau}})$, we get by (3.4.10)

$$e^{2\lambda_{\boldsymbol{\tau}} t^n_{\boldsymbol{\tau}}} d^2(V^n_{\boldsymbol{\tau}}, U^n_{\boldsymbol{\tau}}) \le e^{2\lambda_{\boldsymbol{\tau}} t^{n-1}_{\boldsymbol{\tau}}} d^2(V^{n-1}_{\boldsymbol{\tau}}, U^{n-1}_{\boldsymbol{\tau}}) + 2\tau_n\Big(\phi(V^{n-1}_{\boldsymbol{\tau}}) - \phi(V^n_{\boldsymbol{\tau}})\Big).$$

Summing these inequalities from $n = 1$ to N we get (4.2.12). □

The following Corollary extends the previous Theorem 4.2.2 and concludes the *proof* of the convergence part of Theorem 4.0.4:

Corollary 4.2.5. *Suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold for* $\lambda \in \mathbb{R}$ *and*

$$U^0_{\boldsymbol\tau} \in \overline{D(\phi)}, \qquad \lim_{|\boldsymbol\tau|\downarrow 0} d(U^0_{\boldsymbol\tau}, u_0) = 0. \tag{4.2.13}$$

The family $\{\overline{U}_{\boldsymbol\tau}\}_{\boldsymbol\tau}$ *of the discrete solutions generated by* $U^0_{\boldsymbol\tau}$ *is convergent to the function* $u = S[u_0]$ *as* $|\boldsymbol\tau| \downarrow 0$ *defined by Corollary* 4.3.3*, uniformly in each bounded interval* $[0, T]$*; in particular* u *is the unique element of* $MM(\Phi; u_0)$.

Proof. It is not restrictive to assume $\lambda \le 0$. Let $\overline{U}_{\boldsymbol\tau}, \overline{U}_{\boldsymbol\eta}$ be two discrete solutions corresponding to the admissible partitions $\mathcal{P}_{\boldsymbol\tau}, \mathcal{P}_{\boldsymbol\eta}$, let us choose an arbitrary initial datum $v_0 \in D(\phi)$, and let us introduce the correspondent discrete solutions $\overline{V}_{\boldsymbol\tau}, \overline{V}_{\boldsymbol\eta}$ associated to the same partitions $\mathcal{P}_{\boldsymbol\tau}, \mathcal{P}_{\boldsymbol\eta}$ with $V^0_{\boldsymbol\tau} = V^0_{\boldsymbol\eta} = v_0$.

Applying the previous Lemma 4.2.4 we get

$$\begin{aligned} d(\overline{U}_{\boldsymbol\tau}(t), \overline{U}_{\boldsymbol\eta}(t)) &\le d(\overline{U}_{\boldsymbol\tau}(t), \overline{V}_{\boldsymbol\tau}(t)) + d(\overline{V}_{\boldsymbol\tau}(t), \overline{V}_{\boldsymbol\eta}(t)) + d(\overline{V}_{\boldsymbol\eta}(t), \overline{U}_{\boldsymbol\eta}(t)) \\ &\le e^{-\lambda_{\boldsymbol\tau}(t+|\boldsymbol\tau|)} \Big[d^2(v_0, U^0_{\boldsymbol\tau}) + 2|\boldsymbol\tau| [\phi(v_0) - \phi(\overline{V}_{\boldsymbol\tau}(t))] \Big]^{1/2} \\ &\quad + e^{-\lambda_{\boldsymbol\eta}(t+|\boldsymbol\eta|)} \Big[d^2(v_0, U^0_{\boldsymbol\eta}) + 2|\boldsymbol\tau| [\phi(v_0) - \phi(\overline{V}_{\boldsymbol\eta}(t))] \Big]^{1/2} + d(\overline{V}_{\boldsymbol\tau}(t), \overline{V}_{\boldsymbol\eta}(t)). \end{aligned}$$

Since $v_0 \in D(\phi)$, passing to the limit as $|\boldsymbol\tau|, |\boldsymbol\eta| \downarrow 0$ and applying Theorem 4.2.2, we get

$$\limsup_{|\boldsymbol\tau|,|\boldsymbol\eta|\downarrow 0} d(\overline{U}_{\boldsymbol\tau}(t), \overline{U}_{\boldsymbol\eta}(t)) \le 2 e^{-\lambda t} d(u_0, v_0) \quad \forall\, v_0 \in D(\phi).$$

Since $u_0 \in \overline{D(\phi)}$, taking the infimum with respect to v_0 we conclude. □

4.3 Regularizing effect, uniqueness and the semigroup property

The λ-contractivity property is an immediate consequence of Lemma 4.2.4:

Proposition 4.3.1. *Suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold,* $\lambda \in \mathbb{R}$*. If* $u_0, v_0 \in \overline{D(\phi)}$ *and* $u = MM(u_0; \Phi), v = MM(v_0; \Phi)$*, then*

$$d(u(t), v(t)) \le e^{-\lambda t} d(u_0, v_0). \tag{4.3.1}$$

Proof. If $v_0 \in D(\phi)$, we can simply pass to the limit as $|\boldsymbol\tau| \downarrow 0$ in (4.2.12), choosing e.g. $U^0_{\boldsymbol\tau} = u_0, V^0_{\boldsymbol\tau} = v_0$.

When $v_0 \in \overline{D(\phi)} \setminus D(\phi)$, we consider an auxiliary initial datum $w_0 \in D(\phi)$ and the Minimizing Movement $w = MM(w_0; \Phi)$, obtaining by the triangular inequality

$$d(u(t), v(t)) \le d(u(t), w(t)) + d(w(t), v(t)) \le e^{-\lambda t}\big(d(u_0, w_0) + d(w_0, v_0)\big).$$

(4.3.1) follows now by taking the infimum of the right hand member of the previous inequality w.r.t. $w_0 \in D(\phi)$. □

Theorem 4.3.2. *Suppose that* (4.0.1) *and the convexity Assumption* 4.0.1 *hold,* $\lambda \in \mathbb{R}$. *If* $u \in MM(u_0; \Phi)$ *then* u *satisfies* (4.0.13). *In particular, setting*

$$\iota(T) := \int_0^T e^{\lambda t}\, dt = \frac{e^{\lambda T} - 1}{T}, \tag{4.3.2}$$

we have

$$\phi(u(T)) \le \frac{1}{\iota(T)} \int_0^T \phi(u(t))\, e^{\lambda t} dt \le \phi_{\iota(T)}(u_0), \tag{4.3.3}$$

and, if $\lambda \ge 0$,

$$\begin{aligned} |\partial\phi|(u(T)) &\le \frac{1}{T} d(u_0, u(T)), \\ |\partial\phi|^2(u(T)) &\le |\partial\phi|^2(V) + \frac{1}{T^2} d^2(V, u_0) \quad \forall\, V \in D(|\partial\phi|). \end{aligned} \tag{4.3.4}$$

Proof. By a simple approximation argument via the λ-contraction property of Proposition 4.3.1 and the lower semicontinuity of ϕ, it is not restrictive to assume $u_0 \in D(\phi)$. In this case, we already know from Theorem 2.4.15 that u is locally Lipschitz in $(0, +\infty)$. Keeping the same notation of Section 4.1.2, observe that

$$\lim_{|\boldsymbol{\tau}| \downarrow 0} d_{\boldsymbol{\tau}}(t, V) = d(u(t), V), \quad \lim_{|\boldsymbol{\tau}| \downarrow 0} \varphi_{\boldsymbol{\tau}}(t) = \phi(u(t)) \quad \forall\, t \ge 0,\ V \in \mathscr{S}.$$

Integrating (4.1.12) from S to T and passing to the limit as $|\boldsymbol{\tau}| \downarrow 0$ gives

$$\frac{1}{2} d^2(u(T), V) - \frac{1}{2} d^2(u(S), V) + \int_S^T \Big(\phi(u(t)) + \frac{\lambda}{2} d^2(u(t), V)\Big)\, dt \le (T - S)\phi(V) \tag{4.3.5}$$

which easily yields (4.0.13). Moreover, multiplying (4.0.13) by $e^{\lambda t}$ and integrating from 0 to T, since $t \mapsto \phi(u(t))$ is decreasing we have

$$\iota(T)\phi(u(T)) \le \int_0^T \phi(u(t))\, e^{\lambda t} dt \le \iota(T)\phi(V) + \frac{1}{2} d^2(u_0, V) - \frac{e^{\lambda T}}{2} d^2(u(T), V)$$

for any $V \in D(\phi)$. Taking the infimum w.r.t. V we get (4.3.3). Finally, if $\lambda = 0$, multiplying (2.4.26) by t and integrating in time we get

$$\begin{aligned} \frac{T^2}{2}|\partial\phi|^2(u(T)) &\leq \int_0^T t|\partial\phi|^2(u(t))\,dt \leq -\int_0^T t\big(\phi(u(t)\big)'\,dt \\ &= \int_0^T \phi(u(t))\,dt - T\phi(u(T)) \\ &\leq T\phi(V) + \frac{1}{2}d^2(u_0, V) - T\phi(u(T)) - \frac{1}{2}d^2(u(T), V). \end{aligned}$$

Choosing $V := u(T)$ yields the first estimate of (4.3.4); on the other hand, if $V \in D(|\partial\phi|)$ the right hand side of the last formula can be bounded by

$$T|\partial\phi|(V)d(V, u(T)) - \frac{1}{2}d^2(u(T), V) + \frac{1}{2}d^2(u_0, V) \leq \frac{T^2}{2}|\partial\phi|^2(V) + \frac{1}{2}d^2(u_0, V),$$

which gives the second inequality of (4.3.4). □

Corollary 4.3.3. *The λ-contractive map $u_0 \mapsto S[u_0](t)$, $S[u_0]$ being the Minimizing movement $MM(u_0; \Phi)$, provides the unique solution of the evolution variational inequality* (4.0.13), *and it satisfies the semigroup property $S[u_0](t+s) = S[S[u_0](t)](s)$ for every choice of $t, s \geq 0$.*

Proof. Let us first observe that if u is a continuous solution of the system (4.0.13), then an integration from $t-h$ to t gives for every $v \in D(\phi)$

$$\frac{1}{2}d^2(u(t), v) + \frac{1}{2}d^2(u(t-h), v) + \int_{t-h}^t \Big(\frac{\lambda}{2}d^2(u(r), v) + \phi(u(r))\Big)\,dr \leq h\phi(v).$$

Dividing by h and passing to the limit as $h \downarrow 0$, the lower semicontinuity of ϕ and Fatou's Lemma yield

$$\begin{aligned} \limsup_{h\downarrow 0} h^{-1}\Big(\frac{1}{2}d^2(u(t), v) - \frac{1}{2}d^2(u(t-h), v)\Big) \\ + \frac{\lambda}{2}d^2(u(t), v) + \phi(u(t)) \leq \phi(v) \quad \forall\, t > 0. \end{aligned} \tag{4.3.6}$$

By the same argument we also get the analogous pointwise estimate for the right derivative

$$\begin{aligned} \limsup_{h\downarrow 0} h^{-1}\Big(\frac{1}{2}d^2(u(t+h), v) - \frac{1}{2}d^2(u(t), v)\Big) \\ + \frac{\lambda}{2}d^2(u(t), v) + \phi(u(t)) \leq \phi(v) \quad \forall\, t > 0. \end{aligned} \tag{4.3.7}$$

Let now $u, w \in AC_{\rm loc}(0, +\infty; \mathscr{S})$ be two curves valued in $D(\phi)$ which satisfy the system (4.0.13) and take (by continuity as $t \downarrow 0$) the initial values $u_0, w_0 \in \overline{D(\phi)}$.

Choosing $v := w(t)$ in (4.3.6), $v := u(t)$ in the analogous inequality (4.3.7) written for the function w, and applying the next lemma we find that

$$\frac{d}{dt} d^2(u(t), w(t)) + 2\lambda d^2(u(t), w(t)) \le 0 \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0,$$

i.e.

$$\frac{d}{dt} e^{2\lambda t} d^2(u(t), w(t)) \le 0, \quad d^2(u(t), w(t)) \le e^{-2\lambda t} d^2(u_0, w_0) \quad \forall\, t > 0.$$

In particular, if $u_0 = w_0$ the functions u, w coincides and therefore the system (4.0.13) has at most one solution for a given initial datum u_0.

Since the curve $u(t) := S[u_0](t)$, defined as the value at t of $u \in MM(u_0; \Phi)$ for $u_0 \in D(\phi)$, solves (4.0.13), we obtain that u is the unique solution of (4.0.13). The semigroup property follows easily be the uniqueness for solutions of (4.0.13). □

The following elementary lemma is stated just for convenience for functions in the unit interval $(0, 1)$.

Lemma 4.3.4. *Let $d(s, t) : (0, 1)^2 \to \mathbb{R}$ be a map satisfying*

$$|d(s, t) - d(s', t)| \le |v(s) - v(s')|, \quad |d(s, t) - d(s, t')| \le |v(t) - v(t')|$$

for any s, t, s', $t' \in (0, 1)$, for some locally absolutely continuous map $v : (0, 1) \to \mathbb{R}$ and let $\delta(t) := d(t, t)$. Then δ is locally absolutely continuous in $(0, 1)$ and

$$\frac{d}{dt}\delta(t) \le \limsup_{h\downarrow 0} \frac{d(t,t) - d(t-h,t)}{h} + \limsup_{h\downarrow 0} \frac{d(t,t+h) - d(t,t)}{h} \quad \mathscr{L}^1\text{-a.e. in } (0,1)$$

Proof. Since $|\delta(s) - \delta(t)| \le 2|v(s) - v(t)|$ the function δ is locally absolutely continuous. We fix a nonnegative function $\zeta \in C_c^\infty(0, 1)$ and $h > 0$ such that $\pm h + \operatorname{supp}\zeta \subset (0, 1)$. We have then

$$\begin{aligned} -\int_0^1 \delta(t) \frac{\zeta(t+h) - \zeta(t)}{h}\, dt &= \int_0^1 \zeta(t) \frac{d(t,t) - d(t-h, t-h)}{h}\, dt \\ = \int_0^1 \zeta(t) \frac{d(t,t) - d(t-h,t)}{h}\, dt &+ \int_0^1 \zeta(t+h) \frac{d(t,t+h) - d(t,t)}{h}\, dt, \end{aligned}$$

where the last equality follows by adding and subtracting $d(t - h, t)$ and then making a change of variables in the last integral. Since

$$h^{-1}\,|d(t,t) - d(t-h,t)| \le h^{-1}\,|v(t) - v(t-h)| \to |v'(t)| \quad \text{in } L^1_{\rm loc}(0,1) \text{ as } h \downarrow 0$$

and an analogous inequality holds for the other difference quotient, we can apply (an extended version of) Fatou's Lemma and pass to the (superior) limit in the integrals as $h \downarrow 0$; denoting by a and b the two upper derivatives in the statement of the Lemma we get $-\int \delta\zeta'\, dt \le \int (a + b)\zeta\, dt$, whence the inequality between distributions follows. □

4.4 Optimal error estimates

4.4.1 The case $\lambda = 0$

In this section we mainly focus our attention on the case $\lambda = 0$ and we postpone the analysis of the other situation to Section 4.4.2.

Lemma 4.4.1. *Let us suppose that the convexity Assumption* 4.0.1 *holds for* $\lambda = 0$, *let* $\mathscr{R}_{\boldsymbol{\tau}}, \mathscr{E}^n_{\boldsymbol{\tau}}$ *be defined as in* (4.1.9a) *and* (4.0.20), *let* $I^n_{\boldsymbol{\tau}} := (t^{n-1}_{\boldsymbol{\tau}}, t^n_{\boldsymbol{\tau}}]$, *and let us define*

$$\mathscr{I}_{\boldsymbol{\tau}}(T) := \int_{t^{N-1}_{\boldsymbol{\tau}}}^T \mathscr{R}_{\boldsymbol{\tau}}(t)\,dt \quad \text{for } T \in I^N_{\boldsymbol{\tau}} = (t^{N-1}_{\boldsymbol{\tau}}, t^N_{\boldsymbol{\tau}}]. \tag{4.4.1}$$

Then

$$\int_{I^n_{\boldsymbol{\tau}}} \mathscr{R}_{\boldsymbol{\tau}}(t)\,dt = \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}}, \tag{4.4.2}$$

$$\mathscr{I}_{\boldsymbol{\tau}}(T) \le \tau_N \Big(\phi(U^{N-1}_{\boldsymbol{\tau}}) - \phi(U^N_{\boldsymbol{\tau}}) - \frac{1}{2}\tau_N D^N_{\boldsymbol{\tau}}\Big) \le \frac{1}{2}\tau_N^2 |\partial\phi|^2(U^{N-1}_{\boldsymbol{\tau}}), \tag{4.4.3}$$

$$\mathscr{E}^n_{\boldsymbol{\tau}} \le \frac{1}{2}\Big(|\partial\phi|^2(U^{n-1}_{\boldsymbol{\tau}}) - D^n_{\boldsymbol{\tau}}\Big) \le \frac{1}{2}\Big(|\partial\phi|^2(U^{n-1}_{\boldsymbol{\tau}}) - |\partial\phi|^2(U^n_{\boldsymbol{\tau}})\Big), \tag{4.4.4}$$

$$\int_0^T \mathscr{R}_{\boldsymbol{\tau}}(t)\,dt \le \sum_{n=1}^{N-1} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}} + \mathscr{I}_{\boldsymbol{\tau}}(T). \tag{4.4.5}$$

Proof. (4.4.2) follows directly from (4.1.9b) since

$$\int_{I^n_{\boldsymbol{\tau}}} (1 - \ell_{\boldsymbol{\tau}}(t))\,dt = \int_{I^n_{\boldsymbol{\tau}}} \ell_{\boldsymbol{\tau}}(t)\,dt = \frac{1}{2}, \quad \int_{I^n_{\boldsymbol{\tau}}} (1 - 2\ell_{\boldsymbol{\tau}}(t))\,dt = 0. \tag{4.4.6}$$

(4.2.4) and (4.1.10) yield (4.4.3) and (4.4.4); finally, (4.4.7) is a direct consequence of (4.4.2) and (4.4.1). □

Corollary 4.4.2. *Under the same assumption of the previous lemma, let us suppose that* $\lambda = 0$ *and* $U^0_{\boldsymbol{\tau}} \in D(\phi)$; *then we have*

$$\sum_{n=1}^{N-1} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}} + \mathscr{I}_{\boldsymbol{\tau}}(T) \le |\boldsymbol{\tau}| \Big\{\phi(U^0_{\boldsymbol{\tau}}) - \phi_T(U^0_{\boldsymbol{\tau}})\Big\} \le |\boldsymbol{\tau}| \Big\{\phi(U^0_{\boldsymbol{\tau}}) - \inf_{\mathscr{S}} \phi\Big\}, \tag{4.4.7}$$

and, if $U^0_{\boldsymbol{\tau}} \in D(|\partial\phi|)$,

$$\sum_{n=1}^{N-1} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}} + \mathscr{I}_{\boldsymbol{\tau}}(T) \le \frac{1}{2}|\boldsymbol{\tau}|^2\, |\partial\phi|^2(U^0_{\boldsymbol{\tau}}). \tag{4.4.8}$$

Moreover, when the partition $\mathcal{P}_{\boldsymbol{\tau}}$ *is* uniform *(i.e.* $\tau_n \equiv \tau = |\boldsymbol{\tau}|$ *is independent of* n, *cf. Remark* 2.0.3*), then the following sharper estimate holds, too:*

$$\int_0^T \mathscr{R}_{\boldsymbol{\tau}}(t)\,dt \le \sum_{n=1}^{N-1} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}} + \mathscr{I}_{\boldsymbol{\tau}}(T) \le \tau \Big\{\phi(U^0_{\boldsymbol{\tau}}) - \phi_\tau(U^0_{\boldsymbol{\tau}})\Big\} \le \frac{\tau^2}{2}\, |\partial\phi|^2(U^0_{\boldsymbol{\tau}}). \tag{4.4.9}$$

Proof. Since $\mathscr{E}^n_{\boldsymbol{\tau}} \geq 0$ by (4.1.10), we easily have

$$\begin{aligned}\sum_{n=1}^{N-1} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}} &\leq |\boldsymbol{\tau}| \sum_{n=1}^{N-1} \Big(\big(\phi(U^{n-1}_{\boldsymbol{\tau}}) - \phi(U^n_{\boldsymbol{\tau}})\big) - \tau_n D^n_{\boldsymbol{\tau}} \Big) \\ &\leq |\boldsymbol{\tau}| \sum_{n=1}^{N-1} \Big(\phi(U^{n-1}_{\boldsymbol{\tau}}) - \phi(U^n_{\boldsymbol{\tau}}) \Big) - |\boldsymbol{\tau}| \sum_{n=1}^{N-1} \tau_n D^n_{\boldsymbol{\tau}} \\ &= |\boldsymbol{\tau}| \Big\{ \phi(U^0_{\boldsymbol{\tau}}) - \phi(U^{N-1}_{\boldsymbol{\tau}}) - |\boldsymbol{\tau}| \sum_{n=1}^{N-1} \tau_n D^n_{\boldsymbol{\tau}} \Big\}.\end{aligned}$$

Summing up the contribution of $\mathscr{I}_{\boldsymbol{\tau}}(T)$ and recalling that

$$\sum_{n=1}^{N} \tau_n D^n_{\boldsymbol{\tau}} = \sum_{n=1}^{N} \frac{d^2(U^n_{\boldsymbol{\tau}}, U^{n-1}_{\boldsymbol{\tau}})}{\tau_n} \geq \frac{1}{T} d^2(U^0_{\boldsymbol{\tau}}, U^N_{\boldsymbol{\tau}}), \tag{4.4.10}$$

we obtain (4.4.7).

Since $n \mapsto |\partial\phi|^2(U^n_{\boldsymbol{\tau}})$ is decreasing, too, if $U^0_{\boldsymbol{\tau}} \in D(|\partial\phi|)$ then (4.4.4) yields

$$\begin{aligned}\sum_{n=1}^{N} \tau_n^2 \mathscr{E}^n_{\boldsymbol{\tau}} &\leq \frac{|\boldsymbol{\tau}|^2}{2} \sum_{n=1}^{N-1} \Big(|\partial\phi|^2(U^{n-1}_{\boldsymbol{\tau}}) - |\partial\phi|^2(U^{n-1}_{\boldsymbol{\tau}}) \Big) \\ &\leq \frac{|\boldsymbol{\tau}|^2}{2} \Big(|\partial\phi|^2(U^0_{\boldsymbol{\tau}}) - |\partial\phi|^2(U^{N-1}_{\boldsymbol{\tau}}) \Big) \leq \frac{|\boldsymbol{\tau}|^2}{2} |\partial\phi|^2(U^0_{\boldsymbol{\tau}}) - \mathscr{I}_{\boldsymbol{\tau}}(T),\end{aligned}$$

which proves (4.4.8). When $\tau_n \equiv \tau$ we can use a different estimate for $\mathscr{E}^n_{\boldsymbol{\tau}}$ which comes from (4.1.10)

$$\begin{aligned}\mathscr{E}^n_{\boldsymbol{\tau}} &\leq \tau^{-1} \Big(\big(\phi(U^{n-1}_{\boldsymbol{\tau}}) - \phi_\tau(U^{n-1}_{\boldsymbol{\tau}})\big) - \frac{1}{2}\tau_n D^n_{\boldsymbol{\tau}} \Big) \\ &\leq \tau^{-1} \Big(\big(\phi(U^{n-1}_{\boldsymbol{\tau}}) - \phi_\tau(U^{n-1}_{\boldsymbol{\tau}})\big) - \big(\phi(U^n_{\boldsymbol{\tau}}) - \phi_\tau(U^n_{\boldsymbol{\tau}})\big) \Big),\end{aligned} \tag{4.4.11}$$

thus obtaining

$$\begin{aligned}\sum_{n=1}^{N-1} \tau^2 \mathscr{E}^n_{\boldsymbol{\tau}} &\leq \tau \sum_{n=1}^{N-1} \Big(\big(\phi(U^{n-1}_{\boldsymbol{\tau}}) - \phi_\tau(U^{n-1}_{\boldsymbol{\tau}})\big) - \big(\phi(U^n_{\boldsymbol{\tau}}) - \phi_\tau(U^n_{\boldsymbol{\tau}})\big) \Big) \\ &\leq \tau\big(\phi(U^0_{\boldsymbol{\tau}}) - \phi_\tau(U^0_{\boldsymbol{\tau}})\big) - \tau\big(\phi(U^{N-1}_{\boldsymbol{\tau}}) - \phi_\tau(U^{N-1}_{\boldsymbol{\tau}})\big)\Big) \\ &\leq \tau\big(\phi(U^0_{\boldsymbol{\tau}}) - \phi_\tau(U^0_{\boldsymbol{\tau}})\big) - \mathscr{I}_{\boldsymbol{\tau}}(T),\end{aligned}$$

which proves (4.4.9). □

Corollary 4.4.3. *Suppose that the convexity Assumption* 4.0.1 *holds with* $\lambda \geq 0$. *Then the estimate* (4.0.15) *of Theorem* 4.0.4 *and all the estimates of Theorem* 4.0.7 *hold.*

Proof. We simply apply (4.2.10) and the results of the previous corollary. Observe that when $T = t^N_{\boldsymbol{\tau}} \in \mathcal{P}_{\boldsymbol{\tau}}$ then $\mathscr{I}_{\boldsymbol{\tau}}(T) = 0$, so that we have (4.0.19) without any correction term. □

4.4.2 The case $\lambda \neq 0$

First of all, let us observe that the first estimate (4.0.26) of Theorem 4.0.9 follows directly from Corollary 4.2.3 and (4.2.1), (4.2.3).

In order to get the other error bounds, we need refined estimates of the integral terms in the right-hand side of (4.2.11). Since $\lambda_{\boldsymbol{\tau}} \leq \lambda$, by replacing λ by $\lambda_{\boldsymbol{\tau}}$ in the left-hand side of the differential inequality (4.1.25), we easily get bounds analogous to (4.1.26) and (4.2.11) where the coefficient $\lambda_{\boldsymbol{\tau}}$ occours in each exponential term, thus obtaining for $U^0_{\boldsymbol{\tau}} = u_0$

$$e^{\lambda_{\boldsymbol{\tau}} T} d(\overline{U}_{\boldsymbol{\tau}}(T), u(T)) \leq \left(\mathsf{R}_{\boldsymbol{\tau}}(T) + \lambda^- \int_0^T e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{D}^2_{\boldsymbol{\tau}}(t)\, dt\right)^{1/2} + 2\int_0^T |\lambda| e^{\lambda_{\boldsymbol{\tau}} t} \mathscr{D}_{\boldsymbol{\tau}}(t)\, dt. \tag{4.4.12}$$

Let us observe that if $T \in (t^{N-1}_{\boldsymbol{\tau}}, t^N_{\boldsymbol{\tau}}]$ for some $N \in \mathbb{N}$,

$$\mathsf{R}_{\boldsymbol{\tau}}(T) = \sup_{t \in [0,T]} \int_0^t e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{R}_{\boldsymbol{\tau}}(r)\, dr \tag{4.4.13a}$$

$$\leq \sup_{1 \leq M \leq N} \Big(\int_0^{t^{M-1}_{\boldsymbol{\tau}}} e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{R}_{\boldsymbol{\tau}}(r)\, dr + \int_{I^M_{\boldsymbol{\tau}}} e^{2\lambda_{\boldsymbol{\tau}} t} \big[\mathscr{R}_{\boldsymbol{\tau}}(r)\big]^+ dr \Big) \tag{4.4.13b}$$

$$\leq \sup_{1 \leq M \leq N} \Big(\sum_{n=1}^{M-1} \int_{I^n_{\boldsymbol{\tau}}} e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{R}_{\boldsymbol{\tau}}(r)\, dr + \int_{I^M_{\boldsymbol{\tau}}} e^{2\lambda_{\boldsymbol{\tau}} t} \big[\mathscr{R}_{\boldsymbol{\tau}}(r)\big]^+ dr \Big), \tag{4.4.13c}$$

and, recalling (4.1.11), the integral of the positive part of $\mathscr{R}_{\boldsymbol{\tau}}$ can be bounded by

$$\int_{I^M_{\boldsymbol{\tau}}} e^{2\lambda_{\boldsymbol{\tau}} t} \big[\mathscr{R}_{\boldsymbol{\tau}}(r)\big]^+ dr \leq \tau^2_M \frac{\max\big[e^{2\lambda_{\boldsymbol{\tau}} t^{M-1}_{\boldsymbol{\tau}}}, e^{2\lambda_{\boldsymbol{\tau}} t^M_{\boldsymbol{\tau}}}\big]}{2(1+\lambda\tau_M)} |\partial\phi|^2(U^{M-1}_{\boldsymbol{\tau}}). \tag{4.4.14}$$

The next two lemmas provide the estimates of the other integral in the right-hand side of (4.4.13b) and of the integrals involving $\mathscr{D}_{\boldsymbol{\tau}}$ in (4.4.12). Combining these results with (4.4.12) we complete the proof of Theorems 4.0.9 and 4.0.10.

Proposition 4.4.4. *Suppose that $\lambda < 0$ and $U^0_{\boldsymbol{\tau}} \in D(\partial\phi)$; then for $T > 0$ we have*

$$\mathsf{R}_{\boldsymbol{\tau}}(T) \leq \frac{|\boldsymbol{\tau}|^2}{2(1+\lambda|\boldsymbol{\tau}|)} |\partial\phi|^2(U^0_{\boldsymbol{\tau}}), \tag{4.4.15}$$

and, recalling that $T_{\boldsymbol{\tau}} := \min\big\{t^k_{\boldsymbol{\tau}} \in \mathcal{P}_{\boldsymbol{\tau}} : t^k_{\boldsymbol{\tau}} \geq T\big\}$,

$$|\lambda| \int_0^T e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{D}^2_{\boldsymbol{\tau}}(t)\, dt \leq |\boldsymbol{\tau}|^2 \frac{|\lambda|\, T_{\boldsymbol{\tau}}}{3(1+\lambda|\boldsymbol{\tau}|)^2} |\partial\phi|^2(U^0_{\boldsymbol{\tau}}),$$
$$2|\lambda| \int_0^T e^{\lambda_{\boldsymbol{\tau}} t} \mathscr{D}_{\boldsymbol{\tau}}(t)\, dt \leq |\boldsymbol{\tau}| \frac{|\lambda|\, T_{\boldsymbol{\tau}}}{1+\lambda|\boldsymbol{\tau}|} |\partial\phi|(U^0_{\boldsymbol{\tau}}). \tag{4.4.16}$$

Proof. Let us suppose that $T \in I_{\boldsymbol{\tau}}^N = (t_{\boldsymbol{\tau}}^{N-1}, t_{\boldsymbol{\tau}}^N]$ so that $T_{\boldsymbol{\tau}} = t_{\boldsymbol{\tau}}^N$, and $1 \le M \le N$. Since

$$\int_{I_{\boldsymbol{\tau}}^n} e^{2\lambda_{\boldsymbol{\tau}} t}(1 - \ell_{\boldsymbol{\tau}}(t))\, dt \le \frac{1}{2}\tau_n e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}}, \tag{4.4.17}$$

$$\int_{I_{\boldsymbol{\tau}}^n} e^{2\lambda t}\ell_{\boldsymbol{\tau}}(t)\, dt \ge \frac{1}{2}\tau_n e^{\lambda(t_{\boldsymbol{\tau}}^{n-1}+t_{\boldsymbol{\tau}}^n)} = \frac{1}{2}\tau_n e^{2\lambda t_{\boldsymbol{\tau}}^{n-1}}\, e^{\lambda\tau_n}, \tag{4.4.18}$$

recalling (4.1.11) and (3.4.10) we get

$$\begin{aligned}\int_{I_{\boldsymbol{\tau}}^n} e^{2\lambda_{\boldsymbol{\tau}} t}\mathscr{R}_{\boldsymbol{\tau}}(t)\, dt &\le \frac{\tau_n^2}{2(1+\lambda\tau_n)} e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}} \Big\{ |\partial\phi|^2(U_{\boldsymbol{\tau}}^{n-1}) - (1+\lambda\tau_n)e^{\lambda_{\boldsymbol{\tau}}\tau_n}|\partial\phi|^2(U_{\boldsymbol{\tau}}^n)\Big\} \\ &\le \frac{\tau_n^2}{2(1+\lambda\tau_n)}\Big(e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}}|\partial\phi|^2(U_{\boldsymbol{\tau}}^{n-1}) - e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^n}|\partial\phi|^2(U_{\boldsymbol{\tau}}^n)\Big).\end{aligned}$$

Since the map $n \mapsto e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^n}|\partial\phi|^2(U_{\boldsymbol{\tau}}^n)$ is decreasing, we get

$$\sum_{n=1}^{M-1}\int_{I_{\boldsymbol{\tau}}^n} e^{2\lambda_{\boldsymbol{\tau}} t}\mathscr{R}_{\boldsymbol{\tau}}(t)\, dt \le \frac{|\boldsymbol{\tau}|^2}{2(1+\lambda|\boldsymbol{\tau}|)}\Big(|\partial\phi|^2(U_{\boldsymbol{\tau}}^0) - e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{M-1}}|\partial\phi|^2(U_{\boldsymbol{\tau}}^{M-1})\Big).$$

Taking into account (4.4.14) we obtain (4.4.15). Finally, we easily have

$$\begin{aligned}|\lambda|\int_{I_{\boldsymbol{\tau}}^n} e^{2\lambda_{\boldsymbol{\tau}} t}\mathscr{D}_{\boldsymbol{\tau}}^2(t)\, dt &\le \frac{|\lambda|\tau_n}{3} e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}} d^2(U_{\boldsymbol{\tau}}^n, U_{\boldsymbol{\tau}}^{n-1}) \\ &\le \frac{|\lambda|\tau_n^3}{3(1+\lambda\tau_n)^2} e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}} |\partial\phi|^2(U_{\boldsymbol{\tau}}^{n-1}) \le \frac{|\lambda||\boldsymbol{\tau}|^2\tau_n}{3(1+\lambda|\boldsymbol{\tau}|)}|\partial\phi|^2(U_{\boldsymbol{\tau}}^0),\end{aligned}$$

and

$$\begin{aligned}2|\lambda|\int_{I_{\boldsymbol{\tau}}^n} e^{\lambda_{\boldsymbol{\tau}} t}\mathscr{D}_{\boldsymbol{\tau}}(t)\, dt &\le |\lambda|\tau_n e^{\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}} d(U_{\boldsymbol{\tau}}^n, U_{\boldsymbol{\tau}}^{n-1}) \\ &\le \frac{|\lambda|\tau_n^2}{1+\lambda\tau_n} e^{\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}} |\partial\phi|(U_{\boldsymbol{\tau}}^{n-1}) \le \frac{|\lambda||\boldsymbol{\tau}|\tau_n}{1+\lambda|\boldsymbol{\tau}|}|\partial\phi|(U_{\boldsymbol{\tau}}^0).\end{aligned}$$

Summing up all the contribution from $n = 1$ to N we obtain (4.4.16). □

Proposition 4.4.5. *Assume that* $\lambda > 0$, $\inf_{\mathscr{S}} \phi = 0$, $U_{\boldsymbol{\tau}}^0 \in D(\phi)$, *and* $T_{\boldsymbol{\tau}}$ *is defined as in the above proposition. We have*

$$\mathsf{R}_{\boldsymbol{\tau}}(T) \le \int_0^T e^{2\lambda_{\boldsymbol{\tau}} t}\Big(\mathscr{R}_{\boldsymbol{\tau}}(t)\Big)^+ dt \le |\boldsymbol{\tau}|(1+\lambda|\boldsymbol{\tau}|)(1+\lambda T_{\boldsymbol{\tau}})\phi(U_{\boldsymbol{\tau}}^0), \tag{4.4.19}$$

$$\int_0^T e^{\lambda_{\boldsymbol{\tau}} t}\mathscr{D}_{\boldsymbol{\tau}}(t)\, dt \le |\boldsymbol{\tau}|\Big(2T_{\boldsymbol{\tau}}(1+\lambda T_{\boldsymbol{\tau}})\phi(U_{\boldsymbol{\tau}}^0)\Big)^{1/2}. \tag{4.4.20}$$

Moreover, if $U^0_{\boldsymbol{\tau}} \in D(|\partial\phi|)$ *then*

$$\mathsf{R}_{\boldsymbol{\tau}}(T) \le \frac{1}{2}|\boldsymbol{\tau}|^2(1+\lambda T_{\boldsymbol{\tau}})\,|\partial\phi|^2(U^0_{\boldsymbol{\tau}}), \tag{4.4.21}$$

$$2\int_0^T e^{\lambda_{\boldsymbol{\tau}} t}\mathscr{D}_{\boldsymbol{\tau}}(t)\,dt \le T_{\boldsymbol{\tau}}|\boldsymbol{\tau}||\partial\phi|(U^0_{\boldsymbol{\tau}}). \tag{4.4.22}$$

Proof. As before suppose that $T \in I^N_{\boldsymbol{\tau}} = (t^{N-1}_{\boldsymbol{\tau}}, t^N_{\boldsymbol{\tau}}]$. Since Lemma 2.4.13 yields

$$D^n_{\boldsymbol{\tau}} \ge |\partial\phi|^2(U^n_{\boldsymbol{\tau}}) \ge 2\lambda\phi(U^n_{\boldsymbol{\tau}}),$$

by (4.1.9a) and recalling (3.4.10) and (3.4.9), we get

$$\begin{aligned}\int_{I^n_{\boldsymbol{\tau}}} e^{2\lambda_{\boldsymbol{\tau}} t}\Big(\mathscr{R}_{\boldsymbol{\tau}}(t)\Big)^+\,dt &\le \tau_n e^{2\lambda_{\boldsymbol{\tau}} t^n_{\boldsymbol{\tau}}}\Big(\phi(U^{n-1}_{\boldsymbol{\tau}}) - (1+\lambda\tau_n)\phi(U^n_{\boldsymbol{\tau}})\Big)\\
&\le \tau_n e^{2\lambda_{\boldsymbol{\tau}} t^n_{\boldsymbol{\tau}}}(1+\lambda\tau_n)\Big(\frac{\lambda\tau_n}{(1+\lambda\tau_n)^2}\phi(U^{n-1}_{\boldsymbol{\tau}}) + \frac{1}{(1+\lambda\tau_n)^2}\phi(U^{n-1}_{\boldsymbol{\tau}}) - \phi(U^n_{\boldsymbol{\tau}})\Big)\\
&\le |\boldsymbol{\tau}|(1+\lambda|\boldsymbol{\tau}|)\Big(\lambda\tau_n e^{2\lambda_{\boldsymbol{\tau}} t^{n-1}_{\boldsymbol{\tau}}}\phi(U^{n-1}_{\boldsymbol{\tau}}) + e^{2\lambda_{\boldsymbol{\tau}} t^{n-1}_{\boldsymbol{\tau}}}\phi(U^{n-1}_{\boldsymbol{\tau}}) - e^{2\lambda_{\boldsymbol{\tau}} t^n_{\boldsymbol{\tau}}}\phi(U^n_{\boldsymbol{\tau}})\Big)\\
&\le |\boldsymbol{\tau}|(1+\lambda|\boldsymbol{\tau}|)\Big(\lambda\tau_n\phi(U^0_{\boldsymbol{\tau}}) + e^{2\lambda_{\boldsymbol{\tau}} t^{n-1}_{\boldsymbol{\tau}}}\phi(U^{n-1}_{\boldsymbol{\tau}}) - e^{2\lambda_{\boldsymbol{\tau}} t^n_{\boldsymbol{\tau}}}\phi(U^n_{\boldsymbol{\tau}})\Big).\end{aligned}$$

Summing up for $n = 1$ to N we obtain

$$\int_0^T e^{2\lambda_{\boldsymbol{\tau}} t}\mathscr{R}_{\boldsymbol{\tau}}(t)\,dt \le |\boldsymbol{\tau}|(1+\lambda|\boldsymbol{\tau}|)(1+\lambda T_{\boldsymbol{\tau}})\phi(U^0_{\boldsymbol{\tau}}).$$

Moreover,

$$2\int_0^T e^{\lambda_{\boldsymbol{\tau}} t}\mathscr{D}_{\boldsymbol{\tau}}(t)\,dt \le \sum_{n=1}^N \tau_n^2 e^{\lambda_{\boldsymbol{\tau}} t^n_{\boldsymbol{\tau}}}\sqrt{D^n_{\boldsymbol{\tau}}} \le \sqrt{2T_{\boldsymbol{\tau}}}\,|\boldsymbol{\tau}|\Big(\sum_{n=1}^N \tau_n e^{2\lambda_{\boldsymbol{\tau}} t^n_{\boldsymbol{\tau}}}\frac{D^n_{\boldsymbol{\tau}}}{2}\Big)^{1/2}$$

and

$$(1+\lambda\tau_n)\tau_n\frac{D^n_{\boldsymbol{\tau}}}{2} \le \Big(\phi(U^{n-1}_{\boldsymbol{\tau}}) - (1+\lambda\tau_n)\phi(U^n_{\boldsymbol{\tau}})\Big). \tag{4.4.23}$$

Arguing as before, we find

$$2\int_0^T e^{\lambda_{\boldsymbol{\tau}} t}\mathscr{D}_{\boldsymbol{\tau}}(t)\,dt \le |\boldsymbol{\tau}|\Big(2T_{\boldsymbol{\tau}}(1+\lambda T_{\boldsymbol{\tau}})\phi(U^0_{\boldsymbol{\tau}})\Big)^{1/2}.$$

Finally, if $U^0_{\boldsymbol{\tau}} \in D(|\partial\phi|)$, we first observe that

$$\int_{I^n_{\boldsymbol{\tau}}} e^{2\lambda_{\boldsymbol{\tau}} t}(1 - 2\ell_{\boldsymbol{\tau}}(t))\,dt \le 0, \tag{4.4.24}$$

so that by (4.1.9b) we have

$$\int_{I_{\boldsymbol{\tau}}^n} e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{R}_{\boldsymbol{\tau}}(t)\, dt \le \tau_n^2 e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^n} \mathscr{E}_{\boldsymbol{\tau}}^n. \tag{4.4.25}$$

Since

$$\begin{aligned} 2\mathscr{E}_{\boldsymbol{\tau}}^n &\le \frac{1}{1+\lambda\tau_n} |\partial\phi|^2(U_{\boldsymbol{\tau}}^{n-1}) - |\partial\phi|^2(U_{\boldsymbol{\tau}}^n) \\ &\le \frac{\lambda\tau_n}{(1+\lambda\tau_n)^2} |\partial\phi|^2(U_{\boldsymbol{\tau}}^{n-1}) + \frac{1}{(1+\lambda\tau_n)^2} |\partial\phi|^2(U_{\boldsymbol{\tau}}^{n-1}) - |\partial\phi|^2(U_{\boldsymbol{\tau}}^n) \end{aligned}$$

we obtain

$$\begin{aligned} \int_{I_{\boldsymbol{\tau}}^n} e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{R}_{\boldsymbol{\tau}}(t)\, dt \quad &\le \quad \frac{1}{2}\tau_n^2 \Big(e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}} |\partial\phi|^2(U_{\boldsymbol{\tau}}^{n-1}) - e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^n} |\partial\phi|^2(U_{\boldsymbol{\tau}}^n) \Big) \\ &+ \quad \frac{\lambda}{2}\tau_n^3 e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{n-1}} |\partial\phi|^2(U_{\boldsymbol{\tau}}^{n-1}). \end{aligned}$$

Summing up from $n = 1$ to $M - 1$ and adding the contribution of the integral in the last interval $I_{\boldsymbol{\tau}}^M$ as in (4.4.14), by a repeated application of (4.1.10) we find

$$\begin{aligned} &\sum_{n=1}^{M-1} \int_{I_{\boldsymbol{\tau}}^n} e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{R}_{\boldsymbol{\tau}}(t)\, dt + \int_{I_{\boldsymbol{\tau}}^M} e^{2\lambda_{\boldsymbol{\tau}} t} \Big(\mathscr{R}_{\boldsymbol{\tau}}(t) \Big)^+ dt \\ &\quad \le \frac{|\boldsymbol{\tau}|^2}{2} \Big(|\partial\phi|^2(U_{\boldsymbol{\tau}}^0) - e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{M-1}} |\partial\phi|^2(U_{\boldsymbol{\tau}}^{M-1}) \Big) + \frac{\lambda|\boldsymbol{\tau}|^2 t_{\boldsymbol{\tau}}^{M-1}}{2} |\partial\phi|^2(U_{\boldsymbol{\tau}}^0) \\ &\qquad + \tau_M^2 \frac{e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^{M-1}}}{2} |\partial\phi|^2(U_{\boldsymbol{\tau}}^{M-1}) + \tau_M^2 \frac{\lambda\tau_M e^{2\lambda_{\boldsymbol{\tau}} t_{\boldsymbol{\tau}}^M}}{(1+\lambda_{\boldsymbol{\tau}}\tau_M)^2} |\partial\phi|(U_{\boldsymbol{\tau}}^{M-1}) \\ &\quad \le \frac{|\boldsymbol{\tau}|^2}{2} |\partial\phi|^2(U_{\boldsymbol{\tau}}^0)(1 + \lambda t_{\boldsymbol{\tau}}^M), \end{aligned}$$

which yields (4.4.21). Analogously,

$$2\int_0^T e^{2\lambda_{\boldsymbol{\tau}} t} \mathscr{D}_{\boldsymbol{\tau}}(t)\, dt \le T_{\boldsymbol{\tau}} |\partial\phi|(U_{\boldsymbol{\tau}}^0),$$

which concludes the proof. □

Part II

Gradient Flow in the Space of Probability Measures