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Selfextensional Logics with Implication

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Abstract. The aim of this paper is to develop the theory of the selfextensional logics with an implication for which it holds the deduction-detachment theorem, as presented in [8], but avoiding the use of Gentzen-systems to prove the main results as much as possible.

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1. Introduction

Abstract Algebraic Logic (AAL) is the area of algebraic logic which studies the process of algebraization of the different logical systems. For information on AAL the reader is addressed to [10]. The concept of logic that is taken as primary in the AAL field is that of a consequence relation between sets of formulas and formulas which has the substitution-invariance property; informally speaking this means that if Γ is a set of formulas and φ is a formula that follows according to the logic from Γ, then for every pair (Δ, ψ) of the same form as (Γ, φ), ψ follows from Δ. A logic in this sense may have different replacement properties. The strongest one is shared by classical, intuitionistic and all the intermediate propositional logics. It says that if $\vdash_{\mathcal{S}}$ is the consequence relation of $\mathcal{S}$, for any set of formulas Γ, any formulas φ, ψ, δ and any variable p

$$\text{if } \Gamma, \varphi \vdash_{\mathcal{S}} \psi \text{ and } \Gamma, \psi \vdash_{\mathcal{S}} \varphi, \text{ then } \Gamma, \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi) \text{ and } \Gamma, \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi),$$

where $\delta(p/\varphi)$ and $\delta(p/\psi)$ are the formulas obtained by substituting φ for p and ψ for p in δ respectively. This strong replacement property can be seen as a formal counterpart of Frege's compositionality principle for truth. Logics satisfying this

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replacement property are called Fregean in [6]; the origin of the name comes from the studies by R. Suszko on his non-Fregean logic. Several important logics are not Fregean, for instance almost all the logics of the modal family. Many, like the so-called local consequence relation of the modal logic K, satisfy a weaker replacement property: for all formulas φ, ψ, δ,

$$\text{if } \varphi \vdash_{\mathcal{S}} \psi \text{ and } \psi \vdash_{\mathcal{S}} \varphi, \text{ then } \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi) \text{ and } \delta(p/\psi) \vdash_{\mathcal{S}} \delta(p/\varphi).$$

A logic is said to be *selfextensional* if it satisfies this weaker replacement property. In algebraic terms this means that the interderivability relation between formulas is a congruence relation of the formula algebra. R. Wójcicki coined the name in [17].

The class of protoalgebraic logics is the class of logics for which the theory of the algebraic-like semantics of its elements is the best understood in AAL. A logic is protoalgebraic if it has a generalizad implication, i.e. a set of formulas in two variables, which we denote as $\Rightarrow(p,q)$, with the generalized modus ponens rule (from p and $\Rightarrow(p,q)$ infer q) and such that for every $\varphi \in \Rightarrow(p,q)$, $\varphi(q/p)$ is a theorem. Roughly speaking protoalgebraic logics are the logics for which the semantics of logical matrices is well behaved from the point of view of universal algebra, in the sense that many of the results of universal algebra have counterparts of specific logical interest in the theory of logical matrices for protoalgebraic logics. A logical matrix is a pair $\langle \mathbf{A}, F\rangle$ where $\mathbf{A}$ is an algebra and F is a subset of the domain of $\mathbf{A}$; it is said to be a model of a logic $\mathcal{S}$ if $\mathbf{A}$ is of the type of $\mathcal{S}$ and F is closed under the interpretations in $\mathbf{A}$ of the inferences of $\mathcal{S}$, namely if for every set of formulas Γ, every formula φ and every interpretation v of the formulas in $\mathbf{A}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and the interpretations by v of the elements in Γ belong to F, then the interpretation by v of φ belongs to F. If this is the case it is said that F is an $\mathcal{S}$-filter of $\mathbf{A}$.

Several interesting logics are not protoalgebraic. For non protoalgebraic logics logical matrix semantics is not so well behaved. For instance, the class of algebras that the theory of logical matrices canonically associates with a non-protoalgebraic logic does not necessarily coincide with the class one would intuitively expect to be associated with it. An illustration of this phenomenon is found in the conjunction-disjunction fragment of classical logic. Here the expected class of algebras is the class of distributive lattices, but, as is shown in [11], this class is not the class of algebras the theory of matrices provides.

In [8] a general theory of the algebraization of logic is developed using generalized matrices (where they are called abstract logics) as possible models for logical systems. A generalized matrix is a pair $\langle \mathbf{A}, \mathcal{C}\rangle$ where $\mathbf{A}$ is an algebra and $\mathcal{C}$ the family of closed sets of some finitary closure operator on the domain A of $\mathbf{A}$. It is said to be a model of a logic $\mathcal{S}$ if $\mathbf{A}$ is of the type of $\mathcal{S}$ and $\mathcal{C}$ is a family of $\mathcal{S}$-filters of $\mathbf{A}$. Using generalized matrices, in [8] a canonical way is proposed to associate a class of algebras $\mathbf{Alg}\mathcal{S}$ with each logical system $\mathcal{S}$ that in the known non-protoalgebraic logics supplies the expected results and for protoalgebraic logics gives exactly the class of algebras the theory of logical matrices associates with them. In [8] several general results are proved that sustain the claim that the class of algebras $\mathbf{Alg}\mathcal{S}$ is the natural class of algebras that corresponds to a given deductive system $\mathcal{S}$, and a way to obtain $\mathbf{Alg}\mathcal{S}$ as the result of performing the Lindenbaum-Tarski method suitably generalized is given in [5] and [10].

Among the class of generalized matrices that are models of a given deductive system $\mathcal{S}$ we find the class of the full g-models of $\mathcal{S}$. A full g-model of $\mathcal{S}$ is a generalized matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ which from the logical point of view is equivalent to a generalized matrix of the form $\langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A} \rangle$, where $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is the set of all $\mathcal{S}$-filters of $\mathbf{A}$. The generalized matrices of this last form are called basic full g-models of $\mathcal{S}$ and the study of the class of full g-models can thus be reduced to their study. The class of full g-models of a deductive system was singled out in [8] as an important class and their systematic study was started.

Given a generalized matrix $\mathcal{A} = \langle \mathbf{A}, \mathcal{C} \rangle$, its "interderivability" relation is defined as follows: two elements are related if they belong to the same elements of $\mathcal{C}$. This relation is called the Frege relation of $\mathcal{A}$. Hence a logic $\mathcal{S}$ is selfextensional iff the Frege relation of the generalizad matrix $\langle \mathbf{Fm}, \mathbf{Th}\mathcal{S} \rangle$, where $\mathbf{Fm}$ is the algebra of formulas and $\mathbf{Th}\mathcal{S}$ the family of the theories of $\mathcal{S}$, is a congruence of $\mathbf{Fm}$. Among the selfextensional logics there is an important class introduced in [8], the class of fully selfextensional logics (note that there they are called strongly selfextensional). A logic $\mathcal{S}$ is fully selfextensional if for every full model $\langle \mathbf{A}, \mathcal{C} \rangle$ of $\mathcal{S}$ the Frege relation of $\langle \mathbf{A}, \mathcal{C} \rangle$ is a congruence of $\mathbf{A}$, which is equivalent to saying that for every algebra $\mathbf{A}$ the relation of belonging to the same $\mathcal{S}$-filters of $\mathbf{A}$ is a congruence. The class of fully selfextensional logics is included properly in the class of selfextensional logics as shown in [1].

The present paper studies a class of protoalgebraic selfextensional deductive systems using the tools of the semantics of generalized matrices. It is the class of selfextensional deductive systems $\mathcal{S}$ with a binary formula, or term, $p \Rightarrow q$ which has the deduction-detachment property, that is such that for every set of formulas Γ and all formulas φ, ψ,

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \psi.$$

Many deductive systems belong to this class, for instance the modal local consequence relations given by classes of Kripke frames in the standard language for many-modal logic. Hardly any of these are Fregean.

In [8] selfextensional deductive systems are studied using Gentzen systems as one of the main tools. The present paper develops part of the theory developed in [8] of the selfextensional deductive systems $\mathcal{S}$ with an implication $\Rightarrow$ with the deduction-detachment property without recourse to Gentzen systems. In this way we provide new and much simpler proofs of the following two results in [8].

1. For every deductive system $\mathcal{S}$ with an implication with the deduction-detachment property the class of algebras $\mathbf{Alg}\mathcal{S}$ is a variety (Theorems 4.27 of [8]).
2. Every selfextensional deductive system $\mathcal{S}$ with an implication with the deduction-detachment property is fully selfextensional (Theorems 4.31 and 4.46 of [8]).

In [13] it is proved that for every algebraic similarity type with a binary term $\wedge$ there is a dual isomorphism between the set of selfextensional deductive systems where $\wedge$ is a conjunction, ordered by the extension relation, and the set, ordered by inclusion, of all the subvarieties of the variety axiomatized by the semilattice equations $x \wedge x \approx x$, $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ and $x \wedge y \approx y \wedge x$. We prove the parallel result for selfextensional logics with an implication that has the deduction-detachment property, namely:

(3) for every algebraic similarity type and any of its binary terms $\Rightarrow$ there is a dual isomorphism between the set of selfextensional deductive systems where $\Rightarrow$ has the deduction-detachment property, ordered by the extension relation, and the set, ordered by inclusion, of all the subvarieties of the variety axiomatized by the Hilbert algebra equations H1-H4 below.

In our way to prove these results without recourse to Gentzen systems we characterize the selfextensional logics with a binary term $x \Rightarrow y$ that has the deduction-detachment property, as the logics $\mathcal{S}$ for which there is a class of algebras K such that the equations that define the Hilbert algebras

H1. $x \Rightarrow x \approx y \Rightarrow y$

H2. $(x \Rightarrow x) \Rightarrow x \approx x$

H3. $x \Rightarrow (y \Rightarrow z) \approx (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$

H4. $(x \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow y) \approx (y \Rightarrow x) \Rightarrow ((x \Rightarrow y) \Rightarrow x)$.

hold for the term $\Rightarrow$ in K and the following two conditions are satisfied:

1. $\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi$ iff $\forall \mathbf{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$
$$v(\varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi_n)\ldots)) = 1.$$
2. $\emptyset \vdash_{\mathcal{S}_{\mathsf{K}}} \varphi$ iff $\forall \mathbf{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}) v(\varphi) = 1$.

The deductive systems with these properties are called Hilbert-based in this paper.

At the end of Section 3 we give a characterization of the Hilbert-based deductive systems which are regularly algebraizable: they are the Fregean ones which are Hilbert-based. In Section 4 we obtain some results on these systems and a different proof of a result of Czelakowski and Pigozzi in [6]. Finally, in Section 5 we deal with Gentzen systems and we give a different, simpler proof of Proposition 4.47 (iii) and Proposition 4.44 in [8] using the results obtained in Section 3.

2. Preliminaries

In this section we survey the elements of AAL that will be used in the paper and we fix notation. For detailed expositions we address the reader to [2], [4], [8], [10] and [18].

Let $\mathcal{L}$ be an algebraic similarity type (or set of connectives) that we fix throughout this section. All algebras considered, etc., will be of this type. The set of all homomorphisms from an algebra $\mathbf{A}$ to an algebra $\mathbf{B}$ is denoted by $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$.

Let $\mathbf{Fm}$ be the absolutely free algebra of type $\mathcal{L}$ with a denumerable set Var of generators. The elements of Var will be called, as usual, propositional variables.

The algebra $\mathbf{Fm}$ is called the *formula algebra* of type $\mathcal{L}$ and the elements of its domain Fm are the *formulas* of type $\mathcal{L}$. A *deductive system* of type $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is a relation between sets of formulas and formulas such that

1. If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$.
2. If $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \psi$, then $\Delta \vdash_{\mathcal{S}} \varphi$.
3. If $\Gamma \vdash_{\mathcal{S}} \varphi$, then for any substitution σ, $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$, where a *substitution* is an homomorphism from the formula algebra $\mathbf{Fm}$ into itself.

From (1) and (2) it follows that:

(4) If $\Gamma \vdash_{\mathcal{S}} \varphi$ then for any ψ, $\Gamma \cup \{\psi\} \vdash_{\mathcal{S}} \varphi$.

The relation $\vdash_{\mathcal{S}}$ is called the *consequence relation* of $\mathcal{S}$.

A deductive system $\mathcal{S}$ is said to be *finitary* if for every set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\mathcal{S}} \varphi$ implies that $\Gamma' \vdash_{\mathcal{S}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$. All the deductive systems we deal with in the paper are finitary, so from now on *when we say 'deductive system' we understand finitary deductive system.* A *theory* of a deductive system $\mathcal{S}$, or $\mathcal{S}$-*theory* for short, is a set of formulas Γ that is closed under the consequence relation of $\mathcal{S}$, that is, for every formula φ, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\varphi \in \Gamma$. The set of $\mathcal{S}$-theories will be denoted by $\mathbf{Th}\mathcal{S}$.

A deductive system $\mathcal{S}$ is said to be *selfextensional* if its interderivability relation, denoted by $\dashv_{\mathcal{S}}\vdash$, is a congruence of the formula algebra, and it is said to be *Fregean* if for every set of formulas Γ, the interderivability relation modulo Γ, namely the relation defined by $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$ and $\Gamma, \psi \vdash_{\mathcal{S}} \varphi$, is a congruence of the formula algebra.

Given a deductive system $\mathcal{S}$ and an algebra $\mathbf{A}$ with universe A, a set $F \subseteq A$ is an $\mathcal{S}$-*filter* if for any homomorphism h from $\mathbf{Fm}$ into $\mathbf{A}$, any set of formulas Γ and any formula φ, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$. If the deductive system is finitary the condition can be replaced by the corresponding condition that requires in addition that Γ is finite. We denote the set of all $\mathcal{S}$-filters of an algebra $\mathbf{A}$ by $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. The set of all $\mathcal{S}$-filters of the formula algebra $\mathbf{Fm}$ is exactly the set $\mathbf{Th}\mathcal{S}$ of all the theories of $\mathcal{S}$. A *logical matrix*, abbreviatedly a matrix, is a pair $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is an algebra and F is a subset of the universe of $\mathbf{A}$. A matrix $\boldsymbol{\mathcal{M}} = \langle \mathbf{A}, F \rangle$ is a *(matrix) model* of a deductive system $\mathcal{S}$ if F is an $\mathcal{S}$-filter of $\mathbf{A}$. Therefore the matrix models of $\mathcal{S}$ on the formula algebra are the matrices of the form $\langle \mathbf{Fm}, T \rangle$ where T is an $\mathcal{S}$-theory.

A *finitary closed-set system* on a set A is a family $\mathcal{C}$ of subsets of A that contains A and is closed under arbitrary intersections and under unions of upward directed subfamilies with respect to the inclusion relation. If $\mathcal{C}$ is a finitary closed-set system on a set A the closure operator $\mathrm{Clo}_{\mathcal{C}}$ on A associated with $\mathcal{C}$ is the closure operator defined by

$$\mathrm{Clo}_{\mathcal{C}}(X) = \bigcap \{F \in \mathcal{C} : X \subseteq F\},$$

for each $X \subseteq A$. The closure operator $\mathrm{Clo}_{\mathcal{C}}$ is finitary in the following sense: if $a \in \mathrm{Clo}_{\mathcal{C}}(X)$, then there is a finite $Y \subseteq X$ such that $a \in \mathrm{Clo}_{\mathcal{C}}(Y)$. Moreover, given a finitary closure operator C on a set A, the family $\mathcal{C}_C$ of all C-closed subsets X of A, i.e. such that $C(X) = X$, is a finitary closed-set system. It is well known that $\mathrm{Clo}_{\mathcal{C}_C} = C$, and that if $\mathcal{C}$ is a finitary closed-set system, then $\mathcal{C}_{\mathrm{Clo}_\mathcal{C}} = \mathcal{C}$.

A *generalized matrix*, g-matrix for short, is a pair $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C}\rangle$ where $\mathbf{A}$ is an algebra and $\mathcal{C}$ is a finitary closed-set system on the universe A. Usually we will denote the closure operator determined by $\mathcal{C}$ on A by $\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}$. We will also refer to the closed-set system of a matrix $\boldsymbol{\mathcal{A}}$ by $\mathcal{C}_{\boldsymbol{\mathcal{A}}}$. Notice that for every finitary deductive system $\mathcal{S}$ the structure $\langle \mathbf{Fm}, \mathbf{Th}\mathcal{S}\rangle$ is a generalized matrix. Its associated closure operator can be identified with the consequence relation $\vdash_\mathcal{S}$. The finitarity of $\mathcal{S}$ is essential for obtaining that $\mathbf{Th}\mathcal{S}$ is closed under unions of upwards directed subfamilies (by the inclusion order). Generalized matrices are exactly the finitary abstract logics of the monograph [8].

A generalized matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C}\rangle$ is a *generalized model*, g-model for short, of a deductive system $\mathcal{S}$ if every element of $\mathcal{C}$ is an $\mathcal{S}$-filter, that is, if $\mathcal{C} \subseteq \mathrm{Fi}_\mathcal{S}\mathbf{A}$. The g-matrix $\langle \mathbf{Fm}, \mathbf{Th}\mathcal{S}\rangle$ is obviously a g-model of the deductive system $\mathcal{S}$.

Given a generalized matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C}\rangle$, its *Tarski congruence*, denoted by $\widetilde{\boldsymbol{\Omega}}_\mathbf{A}(\mathcal{C})$, is the greatest congruence of $\mathbf{A}$ compatible with every element of $\mathcal{C}$, that is, such that for every $F \in \mathcal{C}$ and every $a, b \in A$, if $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}_\mathbf{A}(\mathcal{C})$ and $a \in F$, then $b \in F$. Sometimes we will denote $\widetilde{\boldsymbol{\Omega}}_\mathbf{A}(\mathcal{C})$ by $\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}})$. A generalized matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C}\rangle$ is *reduced* if its Tarski congruence is the identity. The class of the algebraic reducts of the reduced g-matrix models of $\mathcal{S}$, denoted by $\mathbf{Alg}\mathcal{S}$, is the class of algebras that according to the general algebraic semantics for deductive systems developed in [8] deserves to be considered the canonical class of algeras of $\mathcal{S}$. This class turns out to have the following simpler description that is the best for working purposes in the present paper:

$$\mathbf{Alg}\mathcal{S} = \{\mathbf{A} : \langle \mathbf{A}, \mathrm{Fi}_\mathcal{S}\mathbf{A}\rangle \text{ is reduced}\}.$$

A *strict homomorphism* from a g-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C}\rangle$ to a g-matrix $\boldsymbol{\mathcal{B}} = \langle \mathbf{B}, \mathcal{D}\rangle$ is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ such that $\mathcal{C} = \{h^{-1}[F] : F \in \mathcal{D}\}$. Bijective strict homomorphisms are called *isomorphisms*, and surjective strict homomorphisms are called *bilogical morphisms* in [8]. If there is a strict surjective homomorphism from $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C}\rangle$ onto $\boldsymbol{\mathcal{B}} = \langle \mathbf{B}, \mathcal{D}\rangle$ we write $\boldsymbol{\mathcal{A}} \succeq \boldsymbol{\mathcal{B}}$. In other words this means that $\boldsymbol{\mathcal{B}}$ is a strict homomorphic image of $\boldsymbol{\mathcal{A}}$. The most typical surjective strict homomorphisms appear in the process of reducing a g-matrix. Given a g-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C}\rangle$, its *reduction* is the g-matrix $\boldsymbol{\mathcal{A}}^* = \langle \mathbf{A}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}), \mathcal{C}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}})\rangle$, where $\mathbf{A}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}})$ is the quotient algebra and $\mathcal{C}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) = \{F/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}}) : F \in \mathcal{C}\}$. The projection homomorphism $\pi : \mathbf{A} \to \mathbf{A}/\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}})$ is a surjective strict homomorphism from $\boldsymbol{\mathcal{A}}$ onto $\boldsymbol{\mathcal{A}}^*$. It is known ([8] Proposition 1.14) that if $\boldsymbol{\mathcal{A}} \succeq \boldsymbol{\mathcal{B}}$, then $\boldsymbol{\mathcal{A}}^*$ is isomorphic to $\boldsymbol{\mathcal{B}}^*$.

The notion of full g-model of a deductive system is one of the main notions introduced in [8]. A generalized matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C}\rangle$ is said to be a *basic full g-model* of a deductive system $\mathcal{S}$ if $\mathcal{C} = \mathrm{Fi}_\mathcal{S}\mathbf{A}$ and it is said to be a *full g-model* of $\mathcal{S}$ if there is a basic full g-model $\boldsymbol{\mathcal{B}}$ of $\mathcal{S}$ such that $\boldsymbol{\mathcal{A}} \succeq \boldsymbol{\mathcal{B}}$, that is, if one of its strict homomorphic images is a basic full g-model of $\mathcal{S}$. In [8] it is proved that if $\boldsymbol{\mathcal{A}} \succeq \boldsymbol{\mathcal{B}}$, then $\boldsymbol{\mathcal{A}}$ is a full g-model of $\mathcal{S}$ iff $\boldsymbol{\mathcal{B}}$ is so. Since the logical properties of the g-matrices are the properties which are preserved under strict homomorphisms, the class of full g-models is then the natural class of models one has to deal with. Moreover, it has many interesting properties that make it a very useful tool in the study of deductive systems, in particular for relating the algebraic treatment of a deductive system with the algebraic treatment of the several Gentzen calculi that define it. We will see this in the last section of the paper. Another important feature is that $\mathbf{Alg}\mathcal{S}$ is the class of the algebraic reducts of the reduced full g-models of $\mathcal{S}$.

Given a g-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C} \rangle$, its *Frege relation* $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{A}})$ is defined by

$$\langle a, b \rangle \in \boldsymbol{\Lambda}(\boldsymbol{\mathcal{A}}) \quad \text{iff} \quad \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(\{a\}) = \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(\{b\}).$$

It is easy to see that $\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{A}})$ is the largest congruence of $\mathbf{A}$ included in $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{A}})$. We will also denote the Frege relation of $\langle \mathbf{A}, \mathcal{C} \rangle$ by $\boldsymbol{\Lambda}_{\mathbf{A}}(\mathcal{C})$. For any deductive system $\mathcal{S}$, the interderivability relation ($\varphi \dashv_{\mathcal{S}} \vdash \psi$) is the Frege relation of the g-matrix $\langle \mathbf{Fm}, \mathbf{Th}\mathcal{S} \rangle$. Thus, a deductive system $\mathcal{S}$ is selfextensional iff the Frege relation of $\langle \mathbf{Fm}, \mathbf{Th}\mathcal{S} \rangle$ is a congruence. We denote the Frege relation of this g-matrix by $\boldsymbol{\Lambda}(\mathcal{S})$.

A deductive system $\mathcal{S}$ is said to be *fully selfextensional* when the Frege relation of every of its full g-models is a congruence. Thus, every fully selfextensional deductive system is selfextensional. The converse is not true as is shown in [1]. A deductive system $\mathcal{S}$ is said to be *fully Fregean*, if for every algebra $\mathbf{A}$ and every $\mathcal{S}$-filter F of $\mathbf{A}$, the Frege relation of the g-matrix $\langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A}^F \rangle$, where $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}^F = \{G \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A} : F \subseteq G\}$, is a congruence of $\mathbf{A}$. Clearly every fully Fregean deductive system is Fregean. In [1] it is shown that not every Fregean deductive system is fully Fregean.

Each deductive system has an associated variety, the variety $\mathsf{K}_{\mathcal{S}}$ generated by the algebra $\mathbf{Fm}/\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$, which is the free algebra over a denumerable set of generators of $\mathsf{K}_{\mathcal{S}}$ (see [8], [13]). The class $\mathsf{K}_{\mathcal{S}}$ is called the *intrinsic variety* of $\mathcal{S}$ in [13]. This variety plays an important role in the proof of the main theorems of the paper. The variety $\mathsf{K}_{\mathcal{S}}$ can be described as the variety whose valid equations are the equations $\varphi \approx \psi$ such that $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$. Thus

$$\mathsf{K}_{\mathcal{S}} \models \varphi \approx \psi \quad \text{iff} \quad \forall \delta \in Fm\; \forall p \in Var\; \delta(p/\varphi) \dashv_{\mathcal{S}} \vdash \delta(p/\psi).$$

In particular, if $\mathcal{S}$ is selfextensional,

$$\mathsf{K}_{\mathcal{S}} \models \varphi \approx \psi \quad \text{iff} \quad \varphi \dashv_{\mathcal{S}} \vdash \psi \quad \text{iff} \quad \langle \varphi, \psi \rangle \in \Lambda(\mathcal{S}).$$

The relation between the classes of algebras $\mathbf{Alg}\mathcal{S}$ and $\mathsf{K}_{\mathcal{S}}$ associated with a deductive system $\mathcal{S}$ is that of inclusion: $\mathbf{Alg}\mathcal{S} \subseteq \mathsf{K}_{\mathcal{S}}$. Moreover, $\mathsf{K}_{\mathcal{S}}$ is the variety generated by $\mathbf{Alg}\mathcal{S}$. Thus, when $\mathbf{Alg}\mathcal{S}$ is a variety, the two classes are equal. This is for instance the case for classical logic and for intuitionistic logic. But there are deductive systems $\mathcal{S}$ for which the inclusion is proper: for example the algebraizable logic BCK is such that $\mathbf{Alg}\mathrm{BCK} \subsetneq \mathsf{K}_{\mathrm{BCK}}$.

To conclude this section on preliminaries we recall the definitions of algebraizable deductive system and regularly algebraizable deductive system. A set of formulas $\Delta(p,q)$ in at most two variables is a *set of equivalence formulas* for a deductive system $\mathcal{S}$ if for every algebra $\mathbf{A}$ and every $\mathcal{S}$-filter F of $\mathbf{A}$, $\boldsymbol{\Omega}_{\mathbf{A}}(F) = \{\langle a,b\rangle \in A \times A : \Delta^{\mathbf{A}}(a,b) \subseteq F\}$. A set of equations $\tau(p)$ in at most one variable is a *set of defining equations* for a deductive system $\mathcal{S}$ for every algebra $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ the least $\mathcal{S}$-filter of $\mathbf{A}$ is the set of solutions in $\mathbf{A}$ of the equations in $\tau(p)$, that is the set $\{a \in A : \mathbf{A} \models \tau(p)[a]\}$. A deductive system $\mathcal{S}$ is *algebraizable* if it has a set of equivalence formulas and a set of defining equations. An algebraizable deductive system $\mathcal{S}$ is *regularly algebraizable* if for any set of equivalence formulas $\Delta(p,q)$ the G-rule holds, that is, $p, q \vdash_{\mathcal{S}} \Delta(p,q)$.

3. Hilbert-based deductive systems

Let $\mathcal{S}$ be a deductive system; we say that a binary term $\Rightarrow$ has *the deduction-detachment property*, or is a deduction-detachment term, if for every set of formulas Γ and every formulas φ, ψ,

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \psi.$$

A deductive system $\mathcal{S}$ is said to have the *uniterm deduction-detachement property* (u-DDP) relative to a binary term $\Rightarrow$ if the term $\Rightarrow$ has the deduction-detachment property, and it is said to have the *uniterm deduction-detachement property* if it has the uniterm deduction-detachement property relative to some binary term.

Notice that if $\mathcal{S}$ has the u-DDP relative to $\Rightarrow$ and relative to $\Rightarrow'$ then

$$p \Rightarrow q \dashv_{\mathcal{S}}\vdash p \Rightarrow' q.$$

Thus, if $\mathcal{S}$ is selfextensional, for all formulas φ, ψ, $\langle \varphi \Rightarrow \psi, \varphi \Rightarrow' \psi\rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$.

Definition 1. A class K of algebras is *Hilbert-based relative to a binary term* $\Rightarrow$ if the following equations are valid in K:

H1. $x \Rightarrow x \approx y \Rightarrow y$
H2. $(x \Rightarrow x) \Rightarrow x \approx x$
H3. $x \Rightarrow (y \Rightarrow z) \approx (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$
H4. $(x \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow y) \approx (y \Rightarrow x) \Rightarrow ((x \Rightarrow y) \Rightarrow x)$.

Thus K is Hilbert-based relative to a binary term $\Rightarrow$ if for every $\mathbf{A} \in \mathsf{K}$ the algebra $\langle A, \Rightarrow^{\mathbf{A}}\rangle$ is a Hilbert algebra. We will refer to the equations (H1)-(H4) as the *Hilbert equations*.

Definition 2. We say that a class of algebras is *Hilbert-based* if it is Hilbert-based relative to some binary term.

A class of algebras Q is said to be *pointed* if there is a term $\varphi(x_0, \ldots, x_n)$ with the property that $\varphi(x_0, \ldots, x_n) \approx \varphi(y_0, \ldots, y_n)$ is valid in Q for all variables $y_0, \ldots, y_n$. Thus for every $\mathbf{A} \in \mathsf{Q}$ and any two valuations v, v' on $\mathbf{A}$, $v(\varphi) = v'(\varphi)$. Such a term is called a constant term since it behaves like a constant. Once fixed we will usually refer to it by $\top$. Any Hilbert-based class of algebras K is pointed, because the term $x \Rightarrow x$ is a constant term, that is, for every algebra $\mathbf{A} \in \mathsf{K}$ and all $a, b \in A$, $a \Rightarrow a = b \Rightarrow b$. Let us denote the constant interpretation of $x \Rightarrow x$ in $\mathbf{A}$ by $1^{\mathbf{A}}$ or simply by 1. Given a Hilbert-based class of algebras K and an algebra $\mathbf{A} \in \mathsf{K}$ we define the relation $\leq^{\mathbf{A}}$ on A by

$$a \leq^{\mathbf{A}} b \quad \text{iff} \quad a \Rightarrow b = 1. \tag{1}$$

We will omit the superscript in $\leq^{\mathbf{A}}$ when no confusion is likely.

Definition 3. Given a Hilbert-based class of algebras K relative to $\Rightarrow$ and an algebra $\mathbf{A} \in \mathsf{K}$, a set $F \subseteq A$ is an $\Rightarrow$*-implicative filter* of $\mathbf{A}$ if

1. $1 \in F$
2. for all $a, b \in A$, if $a \Rightarrow b \in F$ and $a \in F$, then $b \in F$.

Definition 4. A deductive system $\mathcal{S}$ is *Hilbert-based* relative to a binary term $\Rightarrow$ and a class of algebras K which is Hilbert-based relative to $\Rightarrow$ if for all formulas $\varphi_0, \ldots, \varphi_n, \varphi$,

$$\begin{gathered}\varphi_0, \ldots, \varphi_n \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \forall \mathbf{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}), \\ v(\varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow (\varphi_n \Rightarrow \varphi) \ldots)) = 1\end{gathered} \tag{2}$$

and

$$\vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \forall \mathbf{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}), v(\varphi) = 1. \tag{3}$$

Property (2) is independent of the order in which the formulas $\varphi_0, \ldots, \varphi_n$ are taken because for any permutation π of $\{0, \ldots, n\}$, $v(\varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow (\varphi_n \Rightarrow \varphi) \ldots)) = 1$ iff $v(\varphi_{\pi(0)} \Rightarrow (\ldots \Rightarrow (\varphi_{\pi(n-1)} \Rightarrow (\varphi_{\pi(n)} \Rightarrow \varphi) \ldots)) = 1$. In the sequel when we say $\mathcal{S}$ is Hilbert-based relative to $\Rightarrow$ and K we assume that K is Hilbert-based relative to $\Rightarrow$.

We say that $\mathcal{S}$ is *Hilbert-based* if there is a binary term $\Rightarrow$ and a Hilbert-based class of algebras relative to $\Rightarrow$ such that $\mathcal{S}$ is Hilbert-based relative to them.

If $\mathcal{S}$ is Hilbert-based relative to $\Rightarrow$ and K then it is also Hilbert-based relative to $\Rightarrow$ and the variety generated by K. The remark in the next proposition together with Corollary 8 show that if $\mathcal{S}$ is Hilbert-based there is only one variety relative to which it is Hilbert-based. We can denote it by $\mathsf{V}(\mathcal{S})$.

Proposition 5. *If $\mathcal{S}$ is a Hilbert-based deductive system relative to K and $\Rightarrow$ and relative to K' and $\Rightarrow'$ then the varieties generated by K and by K' are the same.*

Proof. Assume that $\varphi \approx \psi$ is an equation valid in K. Then for every $\mathbf{A} \in \mathsf{K}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $v(\varphi) = v(\psi)$. Therefore $v(\varphi \Rightarrow \psi) = v(\psi \Rightarrow \varphi) = 1$. Hence, $\varphi \dashv_{\mathcal{S}}\vdash \psi$. Then for every $\mathbf{A} \in \mathsf{K}'$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $v(\varphi \Rightarrow' \psi) = v(\psi \Rightarrow' \varphi) = 1$. Therefore, $v(\varphi) = v(\psi)$. Hence, $\varphi \approx \psi$ is valid in K'. Analogously we obtain that the equations valid in K' are valid in K. $\square$

Remark 6. Condition (2) in the definition of Hilbert-based deductive system implies that if $\mathcal{S}$ is Hilbert-based relative to K, then $\varphi \dashv_{\mathcal{S}}\vdash \psi$ iff $\mathsf{K} \models \varphi \approx \psi$. Therefore, $\varphi \dashv_{\mathcal{S}}\vdash \psi$ iff $V(\mathcal{S}) \models \varphi \approx \psi$.

Notice that the definition of Hilbert-based deductive system implies that there cannot be two different deductive systems which are Hilbert-based relative to the same variety.

Proposition 7. *If $\mathcal{S}$ is Hilbert-based relative to $\Rightarrow$, then*

1. *$\mathcal{S}$ is selfextensional,*
2. *$\Rightarrow$ has the deduction-detachment property in $\mathcal{S}$,*
3. *the variety $V(\mathcal{S})$ is the intrinsic variety $\mathsf{K}_{\mathcal{S}}$, thus $\mathcal{S}$ is Hilbert-based relative to its intrinsic variety.*

Proof. Assume that $\mathcal{S}$ is Hilbert-based relative to $\Rightarrow$ and the variety K. (1) Let us see that $\boldsymbol{\Lambda}(\mathcal{S})$ is a congruence. If $\varphi \dashv_{\mathcal{S}}\vdash \psi$ then $V(\mathcal{S}) \models \varphi \approx \psi$, therefore for every formula δ and every variable p, $V(\mathcal{S}) \models \delta(p/\varphi) = \delta(p/\psi)$, which, by the above remark, implies that $\delta(p/\psi) \dashv_{\mathcal{S}}\vdash \delta(p/\psi)$. (2) Let us show that $\mathcal{S}$ has u-DDP relative to $\Rightarrow$. Assume that $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$. Let $\varphi_0, \dots, \varphi_{n-1} \in \Gamma$ such that $\varphi_0, \dots, \varphi_{n-1}, \varphi \vdash_{\mathcal{S}} \psi$ or $\varphi \vdash_{\mathcal{S}} \psi$. Therefore for every $\mathbf{A} \in \mathsf{K}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $v(\varphi_0 \Rightarrow (\dots \Rightarrow (\varphi \Rightarrow \psi)\dots)) = 1$ or for every $\mathbf{A} \in \mathsf{K}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $v(\varphi \Rightarrow \psi) = 1$. Hence, $\Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \psi$. On the other hand, if $\Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \psi$, it is also easy to seen that $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$. (3) From the definition of the intrinsic variety of $\mathcal{S}$ and the selfextensionality of $\mathcal{S}$ we have

$$\mathsf{K}_{\mathcal{S}} \models \varphi \approx \psi \quad \text{iff} \quad \varphi \dashv_{\mathcal{S}}\vdash \psi.$$

Therefore, by the above remark, $V(\mathcal{S}) = \mathsf{K}_{\mathcal{S}}$. □

Corollary 8. *If $\mathcal{S}$ is a Hilbert-based deductive system relative to $\Rightarrow$ and also relative to $\Rightarrow'$, then for every φ, ψ, $\varphi \Rightarrow \psi \dashv_{\mathcal{S}}\vdash \varphi \Rightarrow' \psi$, and $\langle \varphi \Rightarrow \psi, \psi \Rightarrow \varphi \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$.*

Proof. The first part follows immediately from the fact that, under the assumptions, by the above theorem both $\Rightarrow$ and $\Rightarrow'$ are deduction-detachment terms for $\mathcal{S}$. The second part follows from the selfextensionality of $\mathcal{S}$. □

Proposition 5 and Ccorollary 8 allow us to speak simply of Hilbert-based deductive systems when convenient.

Theorem 9. *A deductive system $\mathcal{S}$ is selfextensional and has the uniterm deduction-detachment property iff it is Hilbert-based.*

Proof. By the proposition above we have the implication from right to left. To prove the other implication assume that $\mathcal{S}$ is selfextensional and has u-DDP relative to $\Rightarrow$. Let us consider the algebra $\mathbf{Fm}/\boldsymbol{\Lambda}(\mathcal{S})$. It is not difficult to check that $\{\mathbf{Fm}/\boldsymbol{\Lambda}(\mathcal{S})\}$ is Hilbert-based relative to $\Rightarrow$. Moreover,

$$\begin{array}{rll} \varphi_0, \dots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi & \text{iff} & \vdash_{\mathcal{S}} \varphi_0 \Rightarrow (\dots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi)\dots) \\ & \text{iff} & \varphi \Rightarrow \varphi \dashv_{\mathcal{S}}\vdash \varphi_0 \Rightarrow (\dots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi)\dots) \\ & \text{iff} & \mathbf{Fm}/\boldsymbol{\Lambda}(\mathcal{S}) \models \varphi_0 \Rightarrow (\dots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi)\dots) \approx 1 \\ & \text{iff} & \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{Fm}/\boldsymbol{\Lambda}(\mathcal{S})), \\ & & v(\varphi_0 \Rightarrow (\dots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi)\dots)) = 1, \end{array}$$

and

$$\begin{array}{lll} \vdash_{\mathcal{S}} \varphi & \text{iff} & \varphi \Rightarrow \varphi \dashv_{\mathcal{S}}\vdash \varphi \\ & \text{iff} & \mathbf{Fm}/\mathbf{\Lambda}(\mathcal{S}) \models \varphi \approx 1 \\ & \text{iff} & \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{Fm}/\mathbf{\Lambda}(\mathcal{S})), v(\varphi) = 1. \end{array}$$

Thus $\mathcal{S}$ is Hilbert-based relative to the variety $\mathsf{K}_{\mathcal{S}} = \mathsf{V}(\mathbf{Fm}/\mathbf{\Lambda}(\mathcal{S}))$ and $\Rightarrow$. □

Let K be a Hilbert-based variety relative to $\Rightarrow$. We define the deductive system $\mathcal{S}^{\Rightarrow}_{\mathsf{K}}$ as follows:

$$\varphi_0, \ldots, \varphi_n \vdash_{\mathcal{S}^{\Rightarrow}_{\mathsf{K}}} \varphi \quad \text{iff} \quad \forall \mathbf{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$$

$$v(\varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow (\varphi_n \Rightarrow \varphi) \ldots)) = 1$$

and

$$\vdash_{\mathcal{S}^{\Rightarrow}_{\mathsf{K}}} \varphi \quad \text{iff} \quad \forall \mathbf{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})\ v(\varphi) = 1.$$

From the definitions it follows straightforwardly that:

Proposition 10. *For every Hilbert-based variety K relative to $\Rightarrow$ the deductive system $\mathcal{S}^{\Rightarrow}_{\mathsf{K}}$ is Hilbert-based relative to K and $\Rightarrow$ and $\mathsf{V}(\mathcal{S}^{\Rightarrow}_{\mathsf{K}}) = \mathsf{K}$.*

Let us fix a binary term $\Rightarrow$. From the results above it follows that there is a bijection between the Hilbert-based deductive systems relative to $\Rightarrow$ and the Hilbert-based varieties relative to $\Rightarrow$. This bijection is in fact a dual isomorphism when we order the deductive systems by extension and the varieties by the relation of being a subvariety.

A Hilbert-based deductive system $\mathcal{S}$ is determined exactly by its Frege relation, that is by the pairs of formulas $\langle \varphi, \psi \rangle$ which are interderivable in $\mathcal{S}$, and the extension relation between Hilbert-based deductive systems corresponds to the inclusion relation between their Frege relations.

Proposition 11. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two Hilbert-based deductive systems. Then*

$$\Lambda(\mathcal{S}) \subseteq \Lambda(\mathcal{S}') \quad \textit{iff} \quad \mathcal{S}' \textit{ is an extension of } \mathcal{S}.$$

Therefore, if $\Lambda(\mathcal{S}) = \Lambda(\mathcal{S}')$, then $\mathcal{S} = \mathcal{S}'$.

Proof. It is clear that if $\mathcal{S}'$ is an extension of $\mathcal{S}$ then $\Lambda(\mathcal{S}) \subseteq \Lambda(\mathcal{S}')$. Assume that $\Lambda(\mathcal{S}) \subseteq \Lambda(\mathcal{S}')$. Then

$$\begin{array}{lll} \varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi & \text{iff} & \vdash_{\mathcal{S}} \varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi) \ldots) \\ & \text{iff} & \varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi) \ldots) \dashv_{\mathcal{S}}\vdash \varphi \Rightarrow \varphi \\ & \text{then} & \varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi) \ldots) \dashv_{\mathcal{S}'}\vdash \varphi \Rightarrow \varphi \\ & \text{iff} & \vdash_{\mathcal{S}'} \varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow \varphi) \ldots). \\ & \text{iff} & \varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}'} \varphi \end{array}$$

and

$$\begin{array}{rll} \vdash_{\mathcal{S}} \varphi & \text{iff} & \varphi \dashv_{\mathcal{S}}\vdash \varphi \Rightarrow \varphi \\ & \text{then} & \varphi \dashv_{\mathcal{S}'}\vdash \varphi \Rightarrow \varphi \\ & \text{iff} & \vdash_{\mathcal{S}'} \varphi. \\ & \text{iff} & \vdash_{\mathcal{S}'} \varphi. \end{array}$$

Thus $\mathcal{S}'$ is an extension of $\mathcal{S}$. □

To state the theorem, given an algebraic similarity type $\mathcal{L}$ with a binary term $\Rightarrow$ let $\mathsf{K}^{\mathcal{L}}_{\Rightarrow}$ denote the variety axiomatized by the Hilbert equations (E1)-(E4).

Theorem 12. *For every algebraic similarity type and every one of its binary terms $\Rightarrow$ there is a dual isomorphism between the set of Hilbert-based deductive systems relative to $\Rightarrow$, ordered by extension, and the set of all subvarieties of the variety $\mathsf{K}^{\mathcal{L}}_{\Rightarrow}$, ordered by inclusion. The isomorphism is given by $\mathcal{S} \mapsto \mathsf{K}_{\mathcal{S}}$.*

Proof. Recall that for a selfextensional deductive system $\mathcal{S}$ the Frege relation determines exactly the equations that hold in the variety $\mathsf{K}_{\mathcal{S}}$, that is, $\langle \varphi, \psi \rangle \in \mathbf{\Lambda}(\mathcal{S})$ iff $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$. Thus if $\mathcal{S}$ and $\mathcal{S}'$ are Hilbert-based relative to $\Rightarrow$ and $\mathsf{K}_{\mathcal{S}} = \mathsf{K}_{\mathcal{S}'}$, then $\Lambda(\mathcal{S}) = \Lambda(\mathcal{S}')$. By Proposition 11, $\mathcal{S} = \mathcal{S}'$. Thus the function $\mathcal{S} \mapsto \mathsf{K}_{\mathcal{S}}$ is injective. Clearly it is onto since by Proposition 10 every Hilbert-based variety K defines a Hilbert-based deductive system whose class of algebras is K.

From Proposition 11 it follows that $\mathcal{S}$ is an extension of $\mathcal{S}'$ iff $\mathsf{K}_{\mathcal{S}}$ is a subvariety of $\mathsf{K}_{\mathcal{S}'}$. Therefore the function $\mathcal{S} \mapsto \mathsf{K}_{\mathcal{S}}$ is a dual isomorphism. □

We proceed to show that for any selfextensional deductive systems $\mathcal{S}$ with the uniterm deduction-detachment property, its class of algebras $\mathbf{Alg}\mathcal{S}$ is a variety, indeed we will show that it is the intrinsic variety of $\mathcal{S}$. This will give the following reformulation of the theorem above.

Theorem 13. *For every algebraic similarity type and every one of its binary terms $\Rightarrow$ the map $\mathcal{S} \mapsto \mathbf{Alg}\mathcal{S}$ is a dual isomorphism between the set of Hilbert-based deductive systems relative to $\Rightarrow$, ordered by extension, and the set of all subvarieties of the variety $\mathsf{K}^{\mathcal{L}}_{\Rightarrow}$, ordered by inclusion.*

Lemma 14. *Let $\mathcal{S}$ be a Hilbert-based deductive system relative to $\Rightarrow$. Then for every algebra $\mathbf{A} \in \mathsf{K}_{\mathcal{S}}$, the $\mathcal{S}$-filters of $\mathbf{A}$ are the implicative filters of $\mathbf{A}$.*

Proof. Let $\mathcal{S}$ be a deductive system which is Hilbert-based relative to $\Rightarrow$. Let $\mathbf{A} \in \mathsf{K}_{\mathcal{S}}$ and let F be an $\mathcal{S}$-filter of $\mathbf{A}$. Since $\vdash_{\mathcal{S}} p \Rightarrow p$ and $p, p \Rightarrow q \vdash_{\mathcal{S}} q$ it is clear that F is an implicative filter. Conversely, if F is an implicative filter of $\mathbf{A}$, assume that $\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi$ and that $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ is such that $v(\varphi_0), \ldots, v(\varphi_{n-1}) \in F$, then we have $v(\varphi_0 \Rightarrow (\ldots \Rightarrow (\varphi_{n-1} \Rightarrow \psi) \ldots)) = 1 \in F$. Thus, we conclude that $v(\psi) \in F$ as well. This shows that F is an $\mathcal{S}$-filter. □

Lemma 15. *Let $\mathcal{S}$ be a Hilbert-based deductive system. Then for every algebra $\mathbf{A} \in \mathsf{K}_{\mathcal{S}}$, the Frege relation of the g-matrix $\langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A} \rangle$ is the identity and therefore the matrix is reduced and has the congruence property.*

Proof. Let $a, b \in A$ be different elements. Consider the sets $F_a = \{c \in A : a \Rightarrow c = 1\}$ and $F_b = \{c \in A : b \Rightarrow c = 1\}$. It is easy to see that they are implicative filters, hence by Lemma 14 they belong to $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. Clearly, if $F_a = F_b$, then $a \Rightarrow b = b \Rightarrow a = 1$. Hence, $a = b$. Thus, if $a \neq b$, $a \notin F_b$ or $b \notin F_a$. Hence $\langle a, b\rangle \notin \mathbf{\Lambda}_{\mathbf{A}}(\mathrm{Fi}_{\mathcal{S}}\mathbf{A})$. This shows that the Frege relation of the g-matrix $\langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A}\rangle$ is the identity, which implies that $\langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A}\rangle$ is reduced and has the congruence property. □

Theorem 16. *If $\mathcal{S}$ is a Hilbert-based deductive system then*

1. $\mathbf{Alg}\mathcal{S} = \mathsf{K}_{\mathcal{S}} = \mathsf{V}(\mathcal{S})$.
2. $\mathbf{Alg}\mathcal{S}$ *is a variety.*
3. $\mathcal{S}$ *is Hilbert-based relative to* $\mathbf{Alg}\mathcal{S}$.

Proof. 1. We know that $\mathbf{Alg}\mathcal{S} \subseteq \mathsf{K}_{\mathcal{S}}$ always holds. By the previous lemma we obtain that $\mathsf{K}_{\mathcal{S}} \subseteq \mathbf{Alg}\mathcal{S}$. 2 follows from 1 because $\mathsf{K}_{\mathcal{S}}$ is a variety. 3 follows from 1 and item 3 in Proposition 7. □

In [8] it is proved that every selfextensional deductive system with the u-DDP is fully selfextensional. We are going to give a proof of this fact that does not make use of Gentzen systems.

Theorem 17. *Every selfextensional deductive system with the u-DDP is fully selfextensional.*

Proof. Let $\mathcal{S}$ be a selfextensional deductive system with the u-DDP relative to $\Rightarrow$. By Theorem 9 and the corollary to its proof, it is Hilbert-based relative to $\Rightarrow$ and $\mathsf{K}_{\mathcal{S}}$. By Theorem 16, $\mathsf{K}_{\mathcal{S}} = \mathbf{Alg}\mathcal{S}$. Thus, by Lemma 15, if $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$, $\mathbf{\Lambda}_{\mathbf{A}}(\mathrm{Fi}_{\mathcal{S}}\mathbf{A})$ is the identity relation on A; and therefore it is a congruence. If $\boldsymbol{\mathcal{A}}$ is a full g-model of $\mathcal{S}$, its reduction $\boldsymbol{\mathcal{A}}^*$ is of the form $\langle \mathbf{B}, \mathrm{Fi}_{\mathcal{S}}\mathbf{B}\rangle$ for some $\mathbf{B} \in \mathbf{Alg}\mathcal{S}$. By what we have just proved this g-matrix has the congruence property and by Proposition 2.40 in [8] this property is preserved by surjective strict homomorphisms (bilogical morphisms). Therefore, $\boldsymbol{\mathcal{A}}$ has the congruence property too. We can conclude that $\mathcal{S}$ is fully selfextensional. □

Given a pointed quasivariety variety Q with constant term $\top$ the $\top$-*assertional logic* of Q is the deductive system $\mathcal{S}^{ASL}\mathsf{Q} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}^{ASL}\mathsf{Q}}\rangle$ defined by

$$\Gamma \vdash_{\mathcal{S}^{ASL}\mathsf{Q}} \varphi \quad \textit{iff} \quad \forall \mathbf{A} \in \mathsf{V}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathsf{Q})(v[\Gamma] \subseteq \{1^{\mathbf{A}}\} \Longrightarrow v(\varphi) = 1^{\mathbf{A}}),$$

where $1^{\mathbf{A}}$ is the interpretation of the constant term $\top$ in $\mathbf{A}$. We will characterize the selfextensional deductive systems $\mathcal{S}$ with the deduction-detachment property such that $\mathcal{S}$ is equal to the $\top$-assertional logic of $\mathbf{Alg}\mathcal{S}$, where $\top$ is $x \Rightarrow x$ for the deduction-detachment term $\Rightarrow$ of $\mathcal{S}$.

A pointed quasivariety Q with constant term $\top$ is said to be *relatively point-regular* if for every $\mathbf{A} \in \mathsf{Q}$ and all the congruences θ, θ' of $\mathbf{A}$ such that $\mathbf{A}/\theta, \mathbf{A}/\theta' \in \mathsf{Q}$, $1^{\mathbf{A}}/\theta = 1^{\mathbf{A}}/\theta'$ implies $\theta = \theta'$.

The regularly algebraizable deductive systems are the assertional logics of the pointed quasivarieties that are reletively-point regular. If $\mathcal{S}$ is a regularly algebraizable deductive system, then all theorems of $\mathcal{S}$ are equivalent, so any one can be taken as the designated constant term $\top$ and $\mathcal{S}$ is the $\top$-assertional logic of $\mathbf{Alg}\mathcal{S}$.

Theorem 18 ([6] **Thm. 1.34).** *A deductive system $\mathcal{S}$ is regularly algebraizable iff $\mathbf{Alg}\mathcal{S}$ is a pointed and relatively-point regular quasivariety and $\mathcal{S} = \mathcal{S}^{ASL}\mathbf{Alg}\mathcal{S}$.*

Theorem 19. *Let $\mathcal{S}$ be a selfextensional deductive system with the uniterm deduction-detachment property. Then, $\mathcal{S}$ is regularly algebraizable iff $\mathcal{S} = \mathcal{S}^{ASL}\mathbf{Alg}\mathcal{S}$.*

Proof. Let $\mathcal{S}$ be a selfextensional deductive system with the uniterm deduction-detachment property for $\Rightarrow$. Then $\mathbf{Alg}\mathcal{S}$ is a pointed variety. By the theorem above, if $\mathcal{S}$ is regularly algebraizable, then $\mathcal{S} = \mathcal{S}^{ASL}\mathbf{Alg}\mathcal{S}$. Assume now that $\mathcal{S} = \mathcal{S}^{ASL}\mathbf{Alg}\mathcal{S}$. We show that $\mathbf{Alg}\mathcal{S}$ is point-regular. Let $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and let $\theta, \theta' \in \mathrm{Co}\mathbf{A}$ be such that $1/\theta = 1/\theta'$. Suppose that $\langle a, b\rangle \in \theta$. Then $\langle a \Rightarrow a, a \Rightarrow b\rangle \in \theta$, that is $\langle 1, a \Rightarrow b\rangle \in \theta$. Thus, Thus, $\langle 1, a \Rightarrow b\rangle \in \theta'$. Similarly, $\langle 1, b \Rightarrow a\rangle \in \theta'$. Hence, in $\mathbf{A}/\theta'$, $1 = a/\theta' \Rightarrow b/\theta'$ and $1 = b/\theta' \Rightarrow a/\theta'$. Since $\mathbf{Alg}\mathcal{S}$ is a variety, $\mathbf{A}/\theta' \in \mathbf{Alg}\mathcal{S}$. Therefore, $\langle A/\theta', \Rightarrow\rangle$ is a Hilbert algebra. Hence $a/\theta' = b/\theta'$. Thus, $\langle a, b\rangle \in \theta'$. By a similar argument we get the other inclusion. Now by the above theorem $\mathcal{S}$ is regularly algebraizable. □

In [8] (Thm. 3.18 and Prop. 3.20) it is shown that for any fully selfextensional deductive system $\mathcal{S}$, $\mathcal{S}$ is a Fregean, protoalgebraic deductive system with theorems iff $\mathcal{S}$ is regularly algebraizable. Thus, since every deductive system with the deduction-detachment property is protoalgebraic and has theorems, Theorem 17 implies that a selfextensional deductive system with the deduction-detachment property is Fregean iff it is regularly algebraizable. Moreover, Czelakowski and Pigozzi prove in [6] (Corollary 80) that if a deductive system is protoalgebraic and Fregean, then it is fully Fregean. Thus we have the equivalences below:

Theorem 20. *Let $\mathcal{S}$ be a selfextensional deductive system with the uniterm deduction-detachment property. The following statements are equivalent:*

1. *$\mathcal{S}$ is Fregean;*
2. *$\mathcal{S}$ is fully Fregean;*
3. *$\mathcal{S}$ is regularly algebraizable;*
4. *$\mathcal{S} = \mathcal{S}^{ASL}\mathbf{Alg}\mathcal{S}$.*

4. Fregean logics with a deduction-detachment theorem

We will obtain some results on Fregean logics with a uniterm deduction-detachment theorem using our results on selfextensional logics with a deduction-detachment theorem. In particular we give a different proof of the second part of Theorem 66 of Czelakowski and Pigozzi in [6].

Lemma 21 ([6]**).** *If $\mathcal{S}$ is a deductive system and $\Rightarrow$ is a deduction-detachment term for $\mathcal{S}$, then $\mathcal{S}$ is Fregean iff the set $\{p \Rightarrow q, q \Rightarrow p\}$ is an equivalence set of formulas for $\mathcal{S}$.*

If $\mathcal{S}$ is a selfextensional deductive system with a DDT-term $\Rightarrow$, then for all formulas $\varphi_0, \ldots, \varphi_n, \varphi$ and every permutation π of $\{0, \ldots, n\}$,

$$\varphi_0 \Rightarrow (\varphi_1 \Rightarrow (\ldots \Rightarrow (\varphi_n \Rightarrow \varphi) \ldots) \dashv_{\mathcal{S}}\vdash \varphi_{\pi(0)} \Rightarrow (\varphi_{\pi(1)} \Rightarrow (\ldots \Rightarrow (\varphi_{\pi(n)} \Rightarrow \varphi) \ldots).$$

In general, for every full model $\langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A} \rangle$ the analogous result holds, that is for every $a_0, \ldots, a_n, b \in A$ and every permutation π of $\{0, \ldots, n\}$, the sets $\mathrm{Clo}_{\mathrm{Fi}_{\mathcal{S}}\mathbf{A}}(a_0 \Rightarrow (a_1 \Rightarrow (\ldots \Rightarrow (a_n \Rightarrow b) \ldots))$ and $\mathrm{Clo}_{\mathrm{Fi}_{\mathcal{S}}\mathbf{A}}(a_{\pi(0)} \Rightarrow (a_{\pi(1)} \Rightarrow (\ldots \Rightarrow (a_{\pi(n)} \Rightarrow b) \ldots))$ are equal.

Given a sequence $\varphi_0, \ldots, \varphi_n$ of formulas and a formula ψ we introduce the notation $\overline{\varphi} \Rightarrow \psi$ to refer to the formula $\varphi_0 \Rightarrow (\varphi_1 \Rightarrow (\ldots \Rightarrow (\varphi_n \Rightarrow \psi) \ldots)$. Similarly, given a sequence $a_0, \ldots, a_n$ of elements of an algebra and an element b, $\overline{a} \Rightarrow b$ is the element $a_0 \Rightarrow (a_1 \Rightarrow (\ldots \Rightarrow (a_n \Rightarrow b) \ldots)$.

Proposition 22. *Let $\mathcal{S}$ be a selfextensional deductive system with a DDT-term $\Rightarrow$. $\mathcal{S}$ is Fregean iff for every n-ary connective $\star$, every k and every different variables $p_0, \ldots, p_k$, $q_0, \ldots, q_{n-1}$, $r_0, \ldots, r_{n-1}$ the quasiequations*

$$\Big(\bigwedge_{i<n} \overline{p} \Rightarrow (q_i \Rightarrow r_i) \approx \top \wedge \bigwedge_{i<n} \overline{p} \Rightarrow (r_i \Rightarrow q_i) \approx \top\Big) \longrightarrow \qquad (4)$$

$$\overline{p} \Rightarrow (\star(q_0, \ldots, q_{n-1}) \Rightarrow \star(r_0, \ldots, r_{n-1})) \approx \top$$

are valid in $\mathbf{Alg}\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a selfextensional deductive system with a DDT-term $\Rightarrow$. Then $\mathcal{S}$ is protoalgebraic. Suppose $\mathcal{S}$ is Fregean. By Theorem 20 it is fully Fregean. Let $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$. Then, $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A} \rangle$ is a Fregean g-matrix. Assume that $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ is such that for every $i < n$, $v(\overline{p} \Rightarrow (q_i \Rightarrow r_i)) = 1$ and $v(\overline{p} \Rightarrow (r_i \Rightarrow q_i)) = 1$. Then, letting $X = \{v(p_0), \ldots, v(p_k)\}$, for every $i < n$,

$$\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, v(q_i)) = \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, v(r_i)).$$

Hence, $\langle v(q_i), v(r_i) \rangle \in \boldsymbol{\Lambda}_{\mathbf{A}}(\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X))$. Therefore,

$$\langle \star(v(q_0), \ldots, v(q_{n-1})), \star(v(r_0), \ldots, v(r_{n-1})) \rangle \in \boldsymbol{\Lambda}_{\mathbf{A}}(\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X)).$$

Thus, since $\mathcal{S}$ is fully Fregean

$$\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, \star(v(q_0), \ldots, v(q_{n-1}))) = \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, \star(v(r_0), \ldots, v(r_{n-1}))).$$

Hence, $\star(v(r_0), \ldots, v(r_{n-1})) \in \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, \star(v(q_0), \ldots, v(q_{n-1})))$. Therefore,

$$v(\overline{p} \Rightarrow (\star(q_0, \ldots, q_{n-1}) \Rightarrow \star(q_0, \ldots, q_{n-1}))) \in \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(1).$$

This implies that $v(\overline{p} \Rightarrow (\star(q_0, \ldots, q_{n-1}) \Rightarrow \star(q_0, \ldots, q_{n-1}))) = 1$.

Suppose now that the quasiequations (4) of the statement of the proposition hold in $\mathbf{Alg}\mathcal{S}$. Let $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A} \rangle$ be a reduced full model of $\mathcal{S}$. Then $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$. Let X be a finite subset of A. We will show that $\boldsymbol{\Lambda}_{\boldsymbol{\mathcal{A}}}(X)$ is a congruence. Let $\star$ be a n-ary connective. Suppose for every $i < n$, $\langle a_i, b_i \rangle \in \boldsymbol{\Lambda}_{\boldsymbol{\mathcal{A}}}(X)$. Then, $\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, a_i) = \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, b_i)$, for every $i < n$. We can assume without losing generality that $1 \in X$. Thus, consider any sequence $\overline{X}$ of all the elements of X of length the cardinality of X, $\overline{X} \Rightarrow (a_i \Rightarrow b_i) \in \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(1)$ and $\overline{X} \Rightarrow (b_i \Rightarrow a_i) \in$

$\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(1)$. Thus, $\overline{X} \Rightarrow (a_i \Rightarrow b_i) = 1$ and $\overline{X} \Rightarrow (b_i \Rightarrow a_i) = 1$. Hence, using the quasiequations (4), $\overline{X} \Rightarrow (\star(a_0, \ldots, a_{n-1}) \Rightarrow \star(b_0, \ldots, b_{n-1})) = 1$, and similarly, $\overline{X} \Rightarrow (\star(b_0, \ldots, b_{n-1}) \Rightarrow \star(a_0, \ldots, a_{n-1})) = 1$. Hence, $\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, \star(a_0, \ldots, a_{n-1})) = \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(X, \star(b_0, \ldots, b_{n-1}))$. Thus, $\langle \star(a_0, \ldots, a_{n-1}), \star(b_0, \ldots, b_{n-1}) \rangle \in \boldsymbol{\Lambda}_{\boldsymbol{\mathcal{A}}}(X)$. □

Lemma 23. *Let* $\mathbf{A}$ *be an algebra and* $\Rightarrow$ *a binary term such that* $\langle A, \Rightarrow \rangle$ *is a Hilbert algebra. The quasiequations in* (4) *are valid in* $\mathbf{A}$ *iff for every* n*-ary connective* $\star$*, letting* X *be any sequence of all the elements of the set* $\{q_i \Rightarrow r_i, r_i \Rightarrow q_i : i < n\}$*, the equations*

$$\overline{X} \Rightarrow (\star(q_0, \ldots, q_{n-1}) \Rightarrow \star(r_0, \ldots, r_{n-1})) \approx 1 \tag{5}$$

are valid in $\mathbf{A}$.

Proof. Suppose that the quasiequations in (4) are valid in $\mathbf{A}$. Since $\langle A, \Rightarrow \rangle$ is a Hilbert algebra, the equations $\overline{X} \Rightarrow (q_i \Rightarrow r_i) \approx 1$ and $\overline{X} \Rightarrow (r_i \Rightarrow q_i) \approx 1$ are valid in $\mathbf{A}$. Hence, using the quasiequations (4), the equations

$$\overline{X} \Rightarrow (\star(q_0, \ldots, q_{n-1}) \Rightarrow \star(r_0, \ldots, r_{n-1})) \approx 1$$

and

$$\overline{X} \Rightarrow (\star(r_0, \ldots, r_{n-1}) \Rightarrow \star(q_0, \ldots, q_{n-1})) \approx 1$$

are valid in $\mathbf{A}$.

Suppose now that the equations

$$\overline{X} \Rightarrow (\star(q_0, \ldots, q_{n-1}) \Rightarrow \star(r_0, \ldots, r_{n-1})) \approx 1$$

are valid in $\mathbf{A}$. Then so are the equations

$$\overline{X} \Rightarrow (\star(r_0, \ldots, r_{n-1}) \Rightarrow \star(q_0, \ldots, q_{n-1})) \approx 1.$$

Let $p_0, \ldots, p_k, q_0, \ldots, q_{n-1}, r_0, \ldots, r_{n-1}$ be different variables. Let $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ be such that $v(\overline{p} \Rightarrow (q_i \Rightarrow r_i)) = 1$ and $v(\overline{p} \Rightarrow (r_i \Rightarrow q_i)) = 1$ for every $i < n$. From known facts on Hilbert algebras it follows that $v(\overline{p} \Rightarrow (\star(r_0, \ldots, r_{n-1}) \Rightarrow \star(q_0, \ldots, q_{n-1}))) = 1$. □

Corollary 24. *A Hilbert-based class* K *of algebras relative to* $\Rightarrow$ *is the variety* $\mathbf{Alg}\mathcal{S}$ *of a Fregean deductive system* $\mathcal{S}$ *with* $\Rightarrow$ *as binary term with the deduction-detachment property iff it is a subvariety of the variety axiomatized by the Hilbert equations and, for every connective* $\star$*, the equations*

$$\overline{X} \Rightarrow (\star(q_0, \ldots, q_{n-1}) \Rightarrow \star(r_0, \ldots, r_{n-1})) \approx 1 \tag{6}$$

where X *is a sequence of all the elements of the set* $\{q_i \Rightarrow r_i, r_i \Rightarrow q_i : i < n\}$.

Given an algebaric similarity type $\mathcal{L}$ and a binary term $\Rightarrow$, let $\mathsf{HI}^{\Rightarrow}_{\mathcal{L}}$ be the variery axiomatized by the Hilbert equations for $\Rightarrow$ and the above equations in (6). As a corollary we have:

Theorem 25. *For every algebraic similarity type* $\mathcal{L}$ *and every one of its binary terms* $\Rightarrow$ *there is a dual isomorphism between the set of Fregean Hilbert-based deductive systems relative to* $\Rightarrow$*, ordered by extension, and the set of all subvarieties of the variety* $\mathsf{HI}^{\Rightarrow}_{\mathcal{L}}$*. The isomorphism is given by* $\mathcal{S} \mapsto \mathbf{Alg}\mathcal{S}$*.*

5. Selfextensional logics with a deduction-detachment theorem and Gentzen calculi

Given a similarity type $\mathcal{L}$, in the present paper a *sequent* of type $\mathcal{L}$ will be a pair $\langle \Gamma, \varphi \rangle$ where Γ is a possibly empty finite set of formulas and φ is a formula. We will write $\Gamma \rhd \varphi$ instead of $\langle \Gamma, \varphi \rangle$.

A *Gentzen-style rule* is a pair $\langle X, \Gamma \rhd \varphi \rangle$ where X is a (possibly empty) finite set of sequents and $\Gamma \rhd \varphi$ is a sequent. A *substitution instance of a Gentzen-style rule* $\langle X, \Gamma \rhd \varphi \rangle$ is a Gentzen-style rule of the form $\langle \sigma[X], \sigma[\Gamma] \rhd \sigma(\varphi) \rangle$ for some substitution σ, where $\sigma[X] = \{\sigma[\Delta] \rhd \sigma(\psi) : \Delta \rhd \psi \in X\}$. A Gentzen-style rule $\langle X, \Gamma \rhd \varphi \rangle$ is *initial* if X is empty. We will use the standard fraction notation for Gentzen-style rules

$$\frac{\Gamma_0 \rhd \varphi_0, \ldots, \Gamma_{n-1} \rhd \varphi_{n-1}}{\Gamma \rhd \varphi}$$

For the purposes of this paper, a *Gentzen calculus* is a set of Gentzen-style rules. Just as for Hilbert style axiom systems there is the notion of proof from an arbitrary set of premises, given a Gentzen calculus **G** we can define the notion of proof from an arbitrary set of sequents in a similar way. A *proof* in a Gentzen calculus **G** from a set of sequents X is a finite succession of sequents each one of whose elements is a substitution instance of an initial rule of **G** or a sequent in X or is obtained by applying a substitution instance of a rule of **G** to previous elements in the sequence. A sequent $\Gamma \rhd \varphi$ is *derivable in* **G** *from* a set of sequents X if there is a proof in **G** from X whose last sequent is $\Gamma \rhd \varphi$; in this situation we write $X \vdash_{\mathbf{G}} \Gamma \rhd \varphi$. If $\Gamma \rhd \varphi$ is derivable from the emptyset of sequents it is said to be a *derivable* sequent of **G**. A rule $\langle X, \Gamma \rhd \varphi \rangle$ is a *derived rule* of a Gentzen calculus **G** if $X \vdash_{\mathbf{G}} \Gamma \rhd \varphi$. Notice that if a rule is a derived rule, so are all its substitution instances, and that, by the definition, every (primitive) rule of **G** is a derived rule.

A *Gentzen system* is a pair $\mathcal{G} = \langle \mathbf{Fm}, \vdash_{\mathcal{G}} \rangle$ where $\mathbf{Fm}$ is the algebra of formulas and $\vdash_{\mathcal{G}}$ is a finitary closure operator on the set of sequents that is *substitution-invariant*. This means, using the notation $X \vdash_{\mathcal{G}} \Gamma \rhd \varphi$, where X is any set of sequents, instead of the notation $\Gamma \rhd \varphi \in \vdash_{\mathcal{G}} (X)$ typical for closure operators, that if

$$\{\Gamma_i \rhd \psi_i : i < n\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi, \tag{7}$$

then for every substitution $\sigma \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{Fm})$

$$\{\sigma[\Gamma_i] \rhd \sigma(\psi_i) : i < n\} \vdash_{\mathcal{G}} \sigma[\Gamma] \rhd \sigma(\varphi). \tag{8}$$

We say that a Gentzen system $\mathcal{G} = \langle \mathbf{Fm}, \vdash_{\mathcal{G}} \rangle$ *satisfies a Gentzen-style rule*

$$\frac{\Gamma_i \rhd \psi_i : i < n}{\Gamma \rhd \varphi}$$

if $\{\Gamma_i \rhd \psi_i : i < n\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi$; in this situation we also say that the rule is a *sound rule* of $\mathcal{G}$. A Gentzen system $\mathcal{G} = \langle \mathbf{Fm}, \vdash_{\mathcal{G}} \rangle$ is said to be *structural* if it satisfies the structural rules of Weakening and Cut, and the Identity rule $\langle \emptyset, p \rhd p \rangle$ [1].

A Gentzen calculus $\mathbf{G}$ determines the Gentzen system $\mathcal{G}_{\mathbf{G}} = \langle \mathbf{Fm}, \vdash_{\mathbf{G}} \rangle$. If $\mathbf{G}$ has the structural rules (either as primitive or derived), the Gentzen system $\mathcal{G}_{\mathbf{G}}$ is structural.

Every structural Gentzen system $\mathcal{G}$ defines a deductive system $\mathcal{S}_{\mathcal{G}}$ as follows

$$\Gamma \vdash_{\mathcal{S}_{\mathcal{G}}} \varphi \text{ iff there is a finite } \Delta \subseteq \Gamma \text{ such that } \emptyset \vdash_{\mathcal{G}} \Delta \rhd \varphi.$$

We will say that a Gentzen system $\mathcal{G}$ is *adequate* for a deductive system $\mathcal{S}$ if $\mathcal{S} = \mathcal{S}_{\mathcal{G}}$.

Generalized matrices can be used as models of Gentzen-style rules, Gentzen calculi and Gentzen systems. The double nature of g-matrices as models of both deductive systems and Gentzen systems allows us to study in a natural way the connections between the algebraic theory of deductive systems and the algebraic theory of Gentzen systems. We explore some of these connections here for selfextensional logics with a deduction-detachment term.

A g-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C} \rangle$ is said to be a *model of a Gentzen-style rule*

$$\frac{\{\Gamma_i \rhd \psi_i : i < n\}}{\Gamma \rhd \varphi}$$

if for every homomorphism $h \in \text{Hom}(\mathbf{Fm}, \mathbf{A})$, $h(\varphi) \in \text{Clo}_{\mathcal{C}}(h[\Gamma])$ whenever for all $i < n$ $h(\varphi_i) \in \text{Clo}_{\mathcal{C}}(h[\Gamma_i])$. It is a *model of a Gentzen calculus* if it is a model of all its rules, and it is a *model of a Gentzen system* if it is a model of all its sound rules. The following observations follow immediately from the definitions:

1. if a g-matrix is a model of a Gentzen-style rule, it is also a model of all its substitution instances,
2. if a g-matrix is a model of a Gentzen calculus, then it is a model of the Gentzen system that it defines,
3. if a g-matrix is a model of a Gentzen system $\mathcal{G}$, it is a g-model of the associated deductive system $\mathcal{S}_{\mathcal{G}}$.

The *congruence rules* for an n-ary connective $\star$ are the Gentzen-style rules of the form

$$\frac{\{\varphi_i \rhd \psi_i, \ \ \psi_i \rhd \varphi_i : i \leq n\}}{\star(\varphi_0 \dots \varphi_{n-1}) \rhd \star(\psi_0 \dots \psi_{n-1})}$$

We say that a Gentzen calculus *has the congruence rules* if the congruence rules of every connective are derived rules. A Gentzen system *has the congruence rules* if it satisfies the congruence rules of every connective.

Let $\mathcal{S}$ be from now on a selfextensional logic with the deduction-detachment property for $\Rightarrow$. Recall that then on every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ the operation $\Rightarrow^{\mathbf{A}}$ defines

[1] We do not need to consider the other structural rules - exchange and contraction - because we consider sets of premises in our sequents and not successions.

by condition (1) an order that we denote by $\leq^{\mathbf{A}}$. We say that a Gentzen calculus $\mathbf{G}$ adequate for $\mathcal{S}$ is $\mathbf{Alg}\mathcal{S}$-*order-sound* if whenever

$$\{\varphi_i \rhd \psi_i : I \in I\} \vdash_{\mathbf{G}} \varphi \rhd \psi,$$

then for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and every valuation $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$

$$\text{if for all } i \in I,\ v(\varphi_i) \leq^{\mathbf{A}} v(\psi_i), \text{ then } v(\varphi) \leq^{\mathbf{A}} v(\psi).$$

We say that it is $\mathbf{Alg}\mathcal{S}$-*order-complete* if the converse of the main implication in the above statement holds.

Lemma 26. *If* $\mathbf{G}$ *is a Gentzen calculus adequate for* $\mathcal{S}$ *which is* $\mathbf{Alg}\mathcal{S}$*-order-complete, then it has the congruence rules.*

Proof. Let $\star$ be an n-ary connective. Let $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$. Assume that for φ_i, ψ_i with $i < n$, $v(\varphi_i) \leq^{\mathbf{A}} v(\psi_i)$ and $v(\psi_i) \leq^{\mathbf{A}} v(\varphi_i)$. Thus, $v(\varphi_i) = v(\psi_i)$. Therefore, $v(\star\,\varphi_0 \ldots \varphi_{n-1}) = v(\star\,\psi_0 \ldots \psi_{n-1})$. By $\mathbf{Alg}\mathcal{S}$-order-completeness it follows that the congruence rules for $\star$ are derived rules of $\mathbf{G}$. □

We say that a Gentzen calculus $\mathbf{G}$ *has the DDT rules* if the rules of the forms

$$\frac{\Gamma, \varphi \rhd \psi}{\Gamma \rhd \varphi \Rightarrow \psi} \qquad \frac{\Gamma \rhd \varphi \Rightarrow \psi}{\Gamma, \varphi \rhd \psi}$$

are derived rules.

Remark 27. If $\mathbf{G}$ has the DDT rules then for every sequent $\Gamma \rhd \varphi$,

$$\Gamma \rhd \varphi \dashv_{\mathbf{G}}\vdash \top \rhd \overline{\overline{\Gamma}} \Rightarrow \varphi.$$

The remark implies the lemma below.

Lemma 28. *Let* $\mathbf{G}$ *be a Gentzen calculus with the DDT rules, then for every family of sequents* $\{\Gamma_i \rhd \varphi_i : i \in I\}$ *and every sequent* $\Gamma \rhd \varphi$ *the following statements are equivalent:*

1. $\{\Gamma_i \rhd \varphi_i : i \in I\} \vdash_{\mathbf{G}} \Gamma \rhd \varphi$
2. $\{\top \rhd \overline{\overline{\Gamma_i}} \Rightarrow \varphi_i : i \in I\} \vdash_{\mathbf{G}} \top \rhd \overline{\overline{\Gamma}} \Rightarrow \varphi.$

Lemma 29. *If* $\mathbf{G}$ *is a structural Gentzen calculus adequate for* $\mathcal{S}$ *which is* $\mathbf{Alg}\mathcal{S}$*-order-sound, has the DDT rules and has the congruence rules, then it is* $\mathbf{Alg}\mathcal{S}$*-order-complete.*

Proof. Assume that the family of sequents with elements $\varphi_i \rhd \psi_i$ with $i \in I$ and $\varphi \rhd \psi$ is such that for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and every valuation $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, if for all $i \in I$, $v(\varphi_i) \leq^{\mathbf{A}} v(\psi_i)$, then $v(\varphi) \leq^{\mathbf{A}} v(\psi)$. Then, setting $\top := p \Rightarrow p$ for some fixed variable p,

$$\{\varphi_i \Rightarrow \psi_i \approx \top : i \in I\} \models_{\mathbf{Alg}\mathcal{S}} \varphi \Rightarrow \psi \approx \top.$$

Thus by completeness of the quasiequational logic of $\mathbf{Alg}\mathcal{S}$ and the fact that $\mathbf{Alg}\mathcal{S}$ is a variety, there is a proof of the equation $\varphi \Rightarrow \psi \approx \top$ from the equations in $\{\varphi_i \Rightarrow \psi_i \approx \top : i \in I\}$ and the equations which are valid in $\mathbf{Alg}\mathcal{S}$, which are the equations $\delta \approx \varepsilon$ such that $\delta \dashv\vdash_{\mathcal{S}} \varepsilon$. An easy inductive argument will show that for every equation $\gamma \approx \delta$ in such a proof

$$\{\varphi_i \rhd \psi_i : i \in I\} \vdash_{\mathbf{G}} \gamma \rhd \delta, \delta \rhd \gamma.$$

If $\gamma \approx \delta$ is $\varphi_i \Rightarrow \psi_i \approx \top$ with $i \in I$, then the above remark and the fact that (using Identity, the DDT rules and Weakening) $\vdash_{\mathbf{G}} \varphi_i \Rightarrow \psi_i \rhd \top$ gives the result. If $\gamma \approx \delta$ is valid in $\mathbf{Alg}\mathcal{S}$, then $\gamma \dashv\vdash_{\mathcal{S}} \delta$; therefore the sequents $\gamma \rhd \delta, \delta \rhd \gamma$ are derivable in $\mathbf{G}$ and we have the result. Now if $\gamma \approx \delta$ follows by symmetry of the equality from previous equations in the proof, then it is clear. If it follows by transitivity of equality, then Cut gives the desired result. Finally, if it follows by replacement from previous equations in the proof, applying the congruence rules and Cut we obtain the result. Hence, by the induccion principle we obtain that $\top \rhd \varphi \Rightarrow \psi$ is derivable in $\mathbf{G}$ from $\{\varphi_i \rhd \psi_i : i \in I\}$. Thus $\varphi \rhd \psi$ is also derivable from this set. □

Corollary 30. *If* $\mathbf{G}$ *is a structural Gentzen calculus with the DDT rules which is adequate for the deductive system* $\mathcal{S}$ *and which is* $\mathbf{Alg}\mathcal{S}$*-order-sound, then* $\mathbf{G}$ *is* $\mathbf{Alg}\mathcal{S}$*-order-complete iff it has the congruence rules.*

From the corollary follows that there is always a structural Gentzen calculus which is adequate for the deductive system $\mathcal{S}$ and is $\mathbf{Alg}\mathcal{S}$-order-sound and $\mathbf{Alg}\mathcal{S}$-order-complete. It is the Gentzen calculus $\mathbf{G}_{\mathcal{S}}$ defined by the following rules:

1. the structural rules of identity, weakening and cut,
2. the congruence rules for the connectives,
3. the DDT rules
4. for every finite Γ and every φ such that $\Gamma \vdash_{\mathcal{S}} \varphi$, the initial rule

$$\frac{}{\Gamma \rhd \varphi}$$

That this calculus is $\mathbf{Alg}\mathcal{S}$-order-sound follows easily from Lemma 28 and the fact that $\mathbf{Alg}\mathcal{S}$ is Hilbert-based with respect to $\Rightarrow$. The $\mathbf{Alg}\mathcal{S}$-order-completeness follows from the corollary. We will denote by $\mathcal{G}(\mathcal{S})$ the Gentzen system of the calculus $\mathbf{G}_{\mathcal{S}}$.

If we have a nice Hilbert style axiomatization of $\mathcal{S}$ we can consider the Gentzen calculus like the one described above except that, instead of the rules in (4), it has the Gentzen-style rules that naturally correspond to the axioms and rules of the Hilbert style axiomatization.

Remark 31. From Remark 27 it follows that any two structural Gentzen calculi adequate for $\mathcal{S}$ and with the DDT rules which are $\mathbf{Alg}\mathcal{S}$-order-sound and $\mathbf{Alg}\mathcal{S}$-order-complete define the same Gentzen system.

We say that a Gentzen system adequate for a deductive system $\mathcal{S}$ with theorems is *fully-adequate* if the g-matrix models of the Gentzen system are the full g-models of $\mathcal{S}$. This notion is introduced in [8] under the name 'strongly adequate'. We give a different, simpler proof of Proposition 4.47 (iii) in [8]. We will use the lemma below.

Lemma 32. *Let $\mathcal{S}$ be a Hilbert-based deductive system. If $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C} \rangle$ is g-matrix model of $\mathcal{S}$ such that its Frege relation is the identity, then $\mathcal{C} = \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ and $\mathbf{A} \in \mathsf{K}_{\mathcal{S}}$*

Proof. Since the Frege relation of $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C} \rangle$ is the identity, this g-matrix is reduced, so $\mathbf{A} \in \mathbf{Alg}\mathcal{S} \subseteq \mathsf{K}_{\mathcal{S}}$. To prove that $\mathcal{C} = \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ assume that $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. By Lemma 14, F is an implicative filter. We show that $F = \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(F)$. If $b \in \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(F)$, then let $a_0, \dots, a_n \in F$ such that $a \in \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(\{a_0, \dots, a_n\})$. Since $\boldsymbol{\mathcal{A}}$ is a model of the DDT rules, $\overline{\overline{a}} \Rightarrow b \in \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(\top)$. Therefore, $\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(\top) = \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(\overline{\overline{a}} \Rightarrow b)$. Thus, $\overline{\overline{a}} \Rightarrow b = \top$. Since $\top \in F$ because it is an implicative filter, $\overline{\overline{a}} \Rightarrow b \in F$. Hence, since $a_0, \dots, a_n \in F$ and F is an implicative filter, $b \in F$. □

Theorem 33. *Let $\mathcal{S}$ be a deductive system with the deduction-detachment property. Then, $\mathcal{S}$ is selfextensional iff the Gentzen system $\mathcal{G}(\mathcal{S})$ is fully adequate for $\mathcal{S}$.*

Proof. If the Gentzen system $\mathcal{G}(\mathcal{S})$ for $\mathcal{S}$ is fully adequate, then the full model $\langle \mathbf{Fm}, \mathbf{Th}\mathcal{S} \rangle$ is a model of $\mathcal{G}(\mathcal{S})$, thus of the congruence rules. This implies that $\mathcal{S}$ is selfextensional. Assume now that $\mathcal{S}$ is selfextensional. Then, by Theorem 17, $\mathcal{S}$ is fully selfextensional. Thus, if $\langle \mathbf{A}, \mathcal{C} \rangle$ is a full g-model of $\mathcal{S}$, then it is a model of the congruence rules of every connective and of the sequents $\Gamma \rhd \varphi$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$. Clearly it is also a model of the structural rules. Moreover, by Theorem 2.48 of [8] $\langle \mathbf{A}, \mathcal{C} \rangle$ has the deduction-detachment property. Thus it is a model of the DDT rules. Therefore, it is a model of the Gentzen system $\mathcal{G}(\mathcal{S})$. To finish the proof it is enough to show that if $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \mathcal{C} \rangle$ is a reduced g-matrix model of $\mathcal{G}(\mathcal{S})$, then $\mathcal{C} = \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. If $\boldsymbol{\mathcal{A}}$ is a reduced g-matrix model of $\mathcal{G}(\mathcal{S})$ it is a model of $\mathcal{S}$ and of the congruence rules. This implies that its Frege relation is a congruence; therefore since $\boldsymbol{\mathcal{A}}$ is reduced, its Frege relation is the identity. Moreover, $\boldsymbol{\mathcal{A}}$ is a model of the DDT rules. Lemma 32 implies that $\mathcal{C} = \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. □

The notion of $\mathbf{Alg}\mathcal{S}$-order-sound and $\mathbf{Alg}\mathcal{S}$-order-complete structural Gentzen system is strongly related to the notion of algebraizable Gentzen system. We recall this notion here.

A *structural translation t of sequents into equations* is a mapping that maps every sequent to a finite set of equations and satisfies the following structurality property: for every sequent $\Gamma \rhd \varphi$ and every substitution σ,

if $t(\Gamma \rhd \varphi) = \{\varepsilon_i \approx \delta_i : i < n\}$, then $t(\sigma[\Gamma] \rhd \sigma(\varphi)) = \{\sigma(\varepsilon_i) \approx \sigma(\delta_i) : i < n\}$.

A *structural translation s from equations into sequents* is a mapping that maps every equation to a finite set of sequents and has the corresponding structurality property, that is, for every equation $\varepsilon \approx \delta$ and every substitution σ,

if $s(\varepsilon \approx \delta) = \{\Gamma_i \rhd \varphi_i : i < n\}$, then $s(\sigma(\varepsilon) \approx \sigma(\delta)) = \{\sigma[\Gamma_i] \rhd \sigma[\varphi_i] : i < n\}$.

If t is a translation of sequents into equations and X is a set of sequents, the set of equations $t(X)$ is defined by

$$t(X) = \bigcup \{t(\Gamma \rhd \varphi) : \Gamma \rhd \varphi \in X\}.$$

If s is a translation from equations into sequents and E is a set of equations, the set of sequents $s(E)$ is defined by

$$s(E) = \bigcup\{s(\varphi \approx \psi) : \varphi \approx \psi \in E\}.$$

A Gentzen system $\mathcal{G}$ is said to be *algebraizable* if there is a class of algebras K and a structural translation t from sequents into equations and a structural translation s from equations into sequents such that the following two conditions hold

$$\{\Gamma_i \rhd \varphi_i : i \in I\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi \quad \text{iff} \quad t(\{\Gamma_i \rhd \varphi_i : i \in I\}) \models_{\mathsf{K}} t(\Gamma \rhd \varphi). \tag{9}$$

$$\varphi \approx \psi \models_{\mathsf{K}} t(s(\varphi \approx \psi)) \quad \text{and} \quad t(s(\varphi \approx \psi)) \models_{\mathsf{K}} \varphi \approx \psi, \tag{10}$$

where $\models_{\mathsf{K}}$ denotes the equational consequence defined by the class of algebras K as follows:

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad \forall \mathbf{A} \in \mathsf{K} \text{ and } \forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}),$$

$$\text{if } \forall i \in I \ h(\varphi_i) = h(\psi_i), \text{ then } h(\varphi) = h(\psi).$$

When to the right of " $\models_{\mathsf{K}}$ " there is a set it means that every element of the set follows from the equations in the set to the left of " $\models_{\mathsf{K}}$ ".

From [16] it follows that if a Gentzen system $\mathcal{G}$ is algebraizable, there is always a quasivariety K, which is unique, such that (9) and (10) are satisfied for some structural translations t and s. This quasivariety is called the *equivalent algebraic semantics* of $\mathcal{G}$.

The notion of algebraizable Gentzen system was introduced in [15] and its theory developed in [16]. The theory of algebraizable Gentzen systems is an extension to these objects of the theory of algebraizable deductive systems developed by W. Blok and D. Pigozzi in [2]. It is also a particular case of the notion of equivalence between Gentzen systems introduced and studied in [16].

Let us fix a selfextensional deductive system $\mathcal{S}$ with the deduction-detachment property for $\Rightarrow$. We define the structural translation t from sequents to equations and the structural translation sq from equations to sequents respectively by

$$t(\Gamma \rhd \varphi) = \top \approx \overline{\overline{\Gamma}} \Rightarrow \varphi \qquad sq(\varphi \approx \psi) = \{\varphi \rhd \psi, \psi \rhd \varphi\}.$$

Theorem 34. *For every selfextensional logic $\mathcal{S}$ with the deduction-detachment property for $\Rightarrow$, the Gentzen system defined by the Gentzen calculus $\mathbf{G}_{\mathcal{S}}$ is algebraizable with equivalent algebraic semantics $\mathbf{Alg}\mathcal{S}$ and translations t and sq.*

Proof. By the results above $\mathbf{G}_{\mathcal{S}}$ is $\mathbf{Alg}\mathcal{S}$-order-sound and $\mathbf{Alg}\mathcal{S}$-order-complete, thus Lemma 28 gives condition (7) of the definition of algebraizable Gentzen system holds. Condition (8) holds because in any Hilbert algebra, $a = b$ iff $a \Rightarrow b = 1$ and $b \Rightarrow a = 1$. $\square$

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