Directional 3D Thinning Using 8 Subiterations

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Abstract. Thinning of a binary object is an iterative layer by layer erosion to extract an approximation to its skeleton. In order to provide topology preservation, different thinning techniques have been proposed. One of them is the directional (or border sequential) approach in which each iteration step is subdivided into subiterations where only border points of certain kind are deleted in each subiteration. There are six kinds of border points in 3D images, therefore, 6–subiteration parallel thinning algorithms were generally proposed. In this paper, we present two 8–subiteration algorithms for extracting "surface skeletons" and "curve skeletons", respectively. Both algorithms work in cubic grid for (26,6) images. Deletable points are given by templates that makes easy implementation possible.

1 Introduction

Thinning is a common pre–processing operation in pattern recognition. Its goal is to reduce binary objects to their "skeletons" in a topology–preserving way [4]. Border points of the binary object that satisfy certain topological and geometric constraints are deleted in iteration steps. Deletion means that 1's (object elements) are changed to 0's (background elements). The entire process is repeated until only the "skeleton" is left.

A thinning algorithm does *not* preserve topology if

- any object in the input picture is split (into two or more ones) or completely deleted,
- any cavity in the input picture is merged with the background or another cavity, or
- a cavity is created where there was none in the input picture.

There is an additional concept called hole in 3D pictures. A hole (that doughnuts have) is formed by 0's, but it is not a cavity [4,5]. Topology preservation implies that eliminating or creating any hole is not allowed.

A simple point is an object point whose deletion does not alter the topology of the picture [10]. Sequential thinning algorithms delete simple points which are not end points, since preserving end–points provides important information relative to the shape of the objects. Curve thinning preserves line–end points while surface thinning algorithms do not delete surface–end points.

Parallel thinning algorithms delete a set of simple points. A possible approach to preserve topology is to use directional strategy; each iteration step is composed

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of a number of parallel subiterations where only border points of certain kind can be deleted in each subiteration. There are six kinds of border points in 3D images on cubic grid, therefore, 6–subiteration parallel thinning algorithms were generally proposed [2,3,6,11,13,15] (with the exception of [12]).

In this paper, 8–subiteration directional algorithms are proposed for curve thinning and surface thinning. Our approach demonstrates a possible way for constructing non–conventional directional thinning algorithms.

2 Basic notions and results

Let p be a point in the 3D digital space $\mathbb{Z}^3$. Let us denote $N_j(p)$ (for $j = 6, 18, 26$) the set of points *j–adjacent* to point p (see Fig. 1 (a)). The sequence of distinct points $\langle x_0, x_1, \ldots, x_n \rangle$ is a *j–path* of length $n \geq 0$ from point x_0 to point x_n in a non–empty set of points X if each point of the sequence is in X and x_i is j–adjacent to x_{i-1} for each $1 \leq i \leq n$. (Note that a single point is a j–path of length 0.) Two points are *j–connected* in the set X if there is a j–path in X between them. A set of points X is *j–connected* in the set of points $Y \supseteq X$ if any two points in X are j–connected in Y.

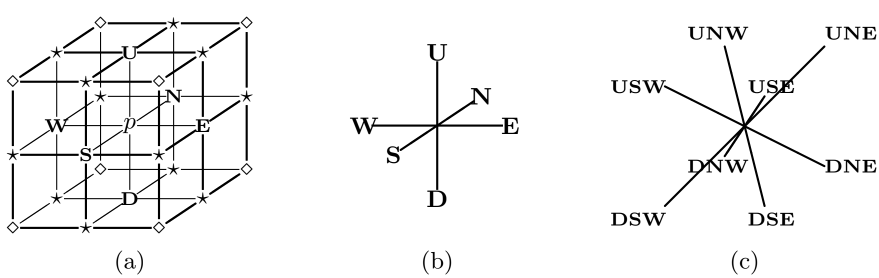

Fig. 1. The frequently used adjacencies in $\mathbb{Z}^3$ (a). The set $N_6(p)$ contains the central point p and points marked **U**, **N**, **E**, **S**, **W**, and **D**. The set of points $N_{18}(p)$ contains the set $N_6(p)$ and points marked "$\star$". The set of points $N_{26}(p)$ contains the set $N_{18}(p)$ and points marked "$\diamond$".
The usual 6 directions (b) corresponding to $N_6(p)$ and the proposed 8 deletion directions (c) corresponding to $N_{26}(p) \backslash N_{18}(p)$.

The *3D binary (m,n) digital picture* $\mathcal{P}$ is a quadruple $\mathcal{P} = (\mathbb{Z}^3, m, n, B)$ [4]. Each element of $\mathbb{Z}^3$ is called a *point* of $\mathcal{P}$. Each point in $B \subseteq \mathbb{Z}^3$ is called a *black point* and value 1 is assigned to it. Each point in $\mathbb{Z}^3 \backslash B$ is called a *white point* and value 0 is assigned to it. Adjacency m belongs to the black points and adjacency n belongs to the white points. A *black component* (or *object*) is a maximal m–connected set of points in B. A *white component* is a maximal n–connected set of points in $B \subseteq \mathbb{Z}^3$.

We are dealing with (26,6) pictures. It is assumed that any picture contains finitely many black points.

A black point is said to be *border point* if it is 6–adjacent to at least one white points. (Note that this definition is correct only for the special cases $m = 26$ and

$m = 18$.) A border point p is called **U**–*border point* if the point marked by **U** in Fig. 1 (a) is white. We can define **N**–, **E**–, **S**–, **W**–, and **D**–border points in the same way. The set $N_6(p)$ are subdivided into three kinds of *opposite pair of points* (**U**,**D**), (**N**,**S**), and (**E**,**W**).

A black point is called *simple point* if its deletion does not alter the topology of the picture. We make use the following result for (26,6) pictures:

Theorem 1. *Black point p is simple in picture $(\mathbb{Z}^3, 26, 6, B)$ if and only if all of the following two conditions hold* [9,14]*:*

1. *The set $(B \setminus \{p\}) \cap N_{26}(p)$ contains exactly one 26–component.*
2. *The set $(\mathbb{Z}^3 \setminus B) \cap N_6(p)$ is not empty and it is 6–connected in the set $(\mathbb{Z}^3 \setminus B) \cap N_{18}(p)$.*

We need to consider what is meant topology preservation when a number of black points are deleted simultaneously. Ma [7] and Kong [5] gave sufficient conditions for parallel reduction operations of 3D (26,6) pictures. Those require some additional concepts to be defined. Let $\mathcal{P}$ be a picture. The set $D = \{d_1, \ldots, d_k\}$ of black points is called a *simple set* of $\mathcal{P}$ if D can be arranged in a sequence $\langle d_{i_1}, \ldots, d_{i_k} \rangle$ in which each d_{i_j} is simple after $\{d_{i_1}, \ldots, d_{i_{j-1}}\}$ is deleted from $\mathcal{P}$, for $j = 1, \ldots, k$. (By definition, let the empty set be simple.) A *unit lattice square* is the set of four corners of a unit square embedded in $\mathbb{Z}^3$; a *unit lattice cube* is the set of eight corners of a unit cube embedded in $\mathbb{Z}^3$.

Theorem 2. *A 3D parallel reduction operation is topology preserving for (26,6) pictures if all of the following conditions hold* [5,7]*:*

1. *Only simple points are deleted.*
2. *If two black corners, p and q, of a unit lattice square are deleted, then the set $\{p, q\}$ is simple.*
3. *If three black corners, $p, q,$ and r, of a unit lattice square are deleted, then the set $\{p, q, r\}$ is simple.*
4. *If four black corners, $p, q, r,$ and s, of a unit lattice square are deleted, then the set $\{p, q, r, s\}$ is simple.*
5. *No black component contained in a unit lattice cube can be deleted completely.*

3 The new algorithms

Each 6–subiteration directional thinning algorithm uses the six deletion directions that can delete certain **U**–, **D**–, **N**–, **E**–, **S**–, and **W**–border points, respectively [2,3,6,11,13,15]. In our 8–subiteration approach, three kinds of border points can be deleted in each subiteration. The 8 directions are denoted by **USW**, **UWN**, **UNE**, **UES**, **DSW**, **DWN**, **DNE**, and **DES**. Note that each triple corresponds to the three points that are 6–adjacent to the point p and 18–adjacent to each others (see Fig. 1 (a)). The usual 6 and the proposed 8 deletion directions are illustrated in Fig. 1 (b) and Fig. 1 (c), respectively.

Suppose that the 3D (26,6) picture to be thinned contains finitely many back points. Therefore, it can be stored in a finite 3D binary array. (Outside

of this array every point is white.) Reduction operations associated with the 8 subiterations are called `deletion_from_USW`, ... , `deletion_from_DES`. We are now ready to present the 8–subiteration approach formally.
Input: binary array X that represents the picture to be thinned
Output: binary array Y that represents the thinned picture

```
8-subiteration_thinning(X,Y)
  begin
    Y = X;
    repeat
      Y = deletion_from_USW(Y);
      Y = deletion_from_DNE(Y);
      Y = deletion_from_USE(Y);
      Y = deletion_from_DNW(Y);
      Y = deletion_from_UNE(Y);
      Y = deletion_from_DSW(Y);
      Y = deletion_from_UNW(Y);
      Y = deletion_from_DSE(Y);
    until no points are deleted;
  end.
```

Note that choosing another order of the deletion directions yields another algorithm. The proposed order shows a kind of symmetry, therefore, the thinned objects are in their geometrically correct positions (i.e., the "middle" of the original objects).

Now, the successive parallel reduction operations are to be given. We propose a curve thinning algorithm and a surface thinning algorithm. The deletable points (that are simple points and not end points) are given by sets of $3 \times 3 \times 3$ matching templates. A black point is deletable if at least one template in the set of templates matches it.

We now give the characterizations of the curve–end points and the surface–end points.

Definition 3. *A black point p is a curve–end point in a picture $(\mathbb{Z}^3, 26, 6, B)$ if the set $(N_{26}(p) \cap B) \backslash \{p\}$ is singleton (i.e., p is 26–adjacent to exactly one black point).*

Definition 4. *A black point p is a surface–end point in a picture $(\mathbb{Z}^3, 26, 6, B)$ if the set $N_6(p) \backslash B$ contains at least one opposite pair of points (see Fig. 2). (Note that each curve–end point is a surface–end point.)*

Note that different surface–end point characterizations have been proposed by various authors [1,2,3,6,8,15].

A set of seven *base* templates $\mathcal{B}^c_{\mathbf{USW}}$ is given by Fig. 3. The set of templates $\mathcal{T}^c_{\mathbf{USW}}$ is assigned to the first subiteration of our *curve* thinning algorithm. It can be get from the base templates by the following rules:

1. if template $t \in \mathcal{B}^c_{\mathbf{USW}}$, then $t \in \mathcal{T}^c_{\mathbf{USW}}$,

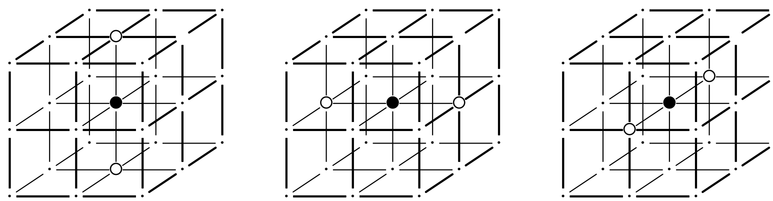

Fig. 2. The three configurations assigned to surface–end points. Every position marked "•" is a black point, every position marked "∘" is a white point, and every "·" ("don't care") may be either a black or a white point.

2. if template $t_1 \in \mathcal{T}^c_{\mathbf{USW}}$ and template t_2 is the reflected version of t_1 with respect to the three symmetry planes illustrated in Fig. 4, then $t_2 \in \mathcal{T}^c_{\mathbf{USW}}$.

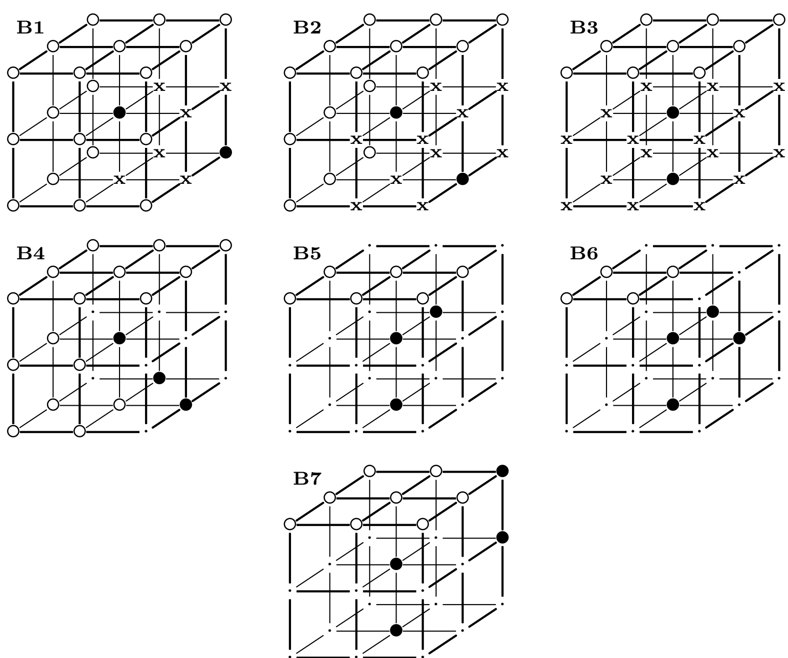

Fig. 3. The set of 7 base templates $\mathcal{B}^c_{\mathbf{USW}}$ assigned to the first subiteration of curve thinning. Notations: every position marked "•" matches a black point, every position marked "∘" matches a white point, every "·" ("don't care") matches either a black or a white point, at least one position marked "**x**" matches a black point.

It is easy to see that $\mathcal{T}^c_{\mathbf{USW}}$ contains 22 templates for deleting certain **U**–, **S**–, or **W**–border points that are not curve–end points. Note that template positions marked "**x**" are used for preserving curve–end points. These templates define the parallel reduction operation `deletion_from_USW`. The deletion rules assigned to the other seven subiterations can be derived from the appropriate rotations and reflections of the templates in $\mathcal{T}^c_{\mathbf{USW}}$.

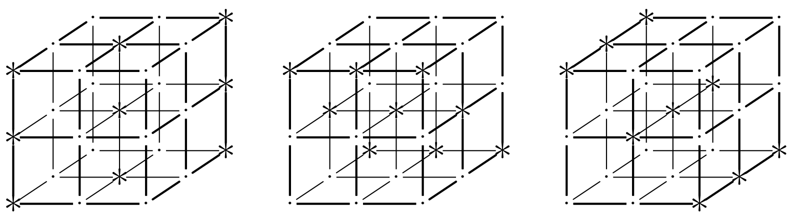

Fig. 4. The three symmetry planes for reflecting templates. Points belonging to the given planes are marked "$\ast$".

The set of four *base* templates $\mathcal{B}^s_{\mathbf{USW}}$ given by Fig. 5 belongs to the first subiteration of our *surface* thinning algorithm. The whole set of templates $\mathcal{T}^s_{\mathbf{USW}}$ can be derived in the same way as it was in the case of the curve thinning algorithm (with the help of the 3 symmetry planes). It is easy to see that $\mathcal{T}^s_{\mathbf{USW}}$ contains 15 templates for deleting certain **U**–, **S**–, or **W**–border points that are not surface–end points. Note that template positions marked "**x**" and "**y**" are used for preserving surface–end points. The sets of templates assigned to the other seven subiterations can be derived from the appropriate rotations and reflections of the templates in $\mathcal{T}^s_{\mathbf{USW}}$. Mention is to be taken that the set of templates $\mathcal{T}^s_{\mathbf{USW}}$ is derived from $\mathcal{T}^c_{\mathbf{USW}}$ by changing the end point criterion.

Both sets of templates ($\mathcal{T}^c_{\mathbf{USW}}$ and $\mathcal{T}^s_{\mathbf{USW}}$) were constructed for deleting some simple points which are neither end points nor extremities of curves/surfaces. Note that the same template is included in more sets of templates: for instance, template **B3** in $\mathcal{B}^c_{\mathbf{USW}}$ is in sets of templates assigned to deletion directions **UWN**, **UNE**, and **UES**, too.

4 Discussion

Different shapes of objects have been tested by the new algorithms. Here we present only three examples (see Figs. 6–7).

The behavior of the proposed algorithms has been also investigated. Here we present an example in Fig. 8 for thinning a "noise–free" solid doughnut of size $40 \times 40 \times 20$ and its noisy version. Noise was added to the boundary (i.e., some border points are deleted from the original noise–free object and some white points 26–adjacent to a border point are changed to black). Note that we consider a model of noise which consists of deleting and adding simple points. We can state that the proposed curve thinning algorithm is robust under noise. Unfortunately, the 8–subiteration surface point algorithm is rather sensitive to boundary noise, since a noisy boundary may contain a number of surface–end points to be preserved.

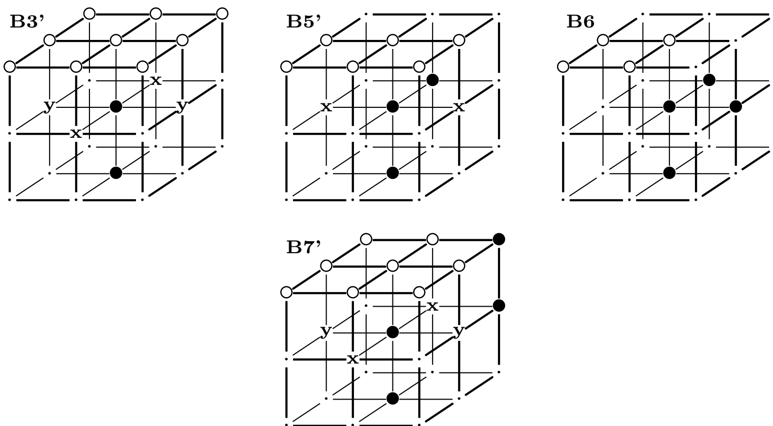

Fig. 5. The set of base templates $\mathcal{B}^s_{\mathbf{USW}}$ is derived from $\mathcal{B}^c_{\mathbf{USW}}$. It is assigned to the first subiteration of surface thinning. Notations: every position marked "•" matches a black point, every position marked "◦" matches a white point, every "·" ("don't care") matches either a black or a white point; at least one position marked "**x**" matches a black point, at least one position marked "**y**" matches a black point.

The proposed algorithms go around the object to be thinned according to the 8 deletion directions. One can say that the 8–subiteration approach seems to be slower than the 6–subiteration one. It is not necessarily true. Suppose that our sample object is a cube of size $16 \times 16 \times 16$. An iteration step of any 6–subiteration algorithm can generate a $14 \times 14 \times 14$ cube from the original one. A cube of size $8 \times 8 \times 8$ is created by the 8–subiteration algorithms. According to our experiments, less iteration steps are required by the proposed algorithms.

The proposed 8–subiteration algorithms are topology preserving for $(26, 6)$ images. It is sufficient to prove that reduction operation given by the set of templates $\mathcal{T}^c_{\mathbf{USW}}$ is topology preserving. If the first subiteration of the curve is topology preserving, then the other seven ones are topology preserving, too, since the applied rotations and reflections of the deletion templates do not alter the topological properties. Therefore, the entire curve thinning algorithm is topology preserving, since it is composed of topology preserving reductions. The surface thinning algorithm is topology preserving, too, since its subiterations can delete less kinds of border points.

Instead of using Theorem 2, we propose the following more general theorem that provides new sufficient conditions for 3D parallel reduction operations to preserve topology.

Theorem 5. *Let $\mathcal{T}$ be a parallel reduction operation on* $(26, 6)$ *pictures. Then $\mathcal{T}$ is topology preserving, if for all picture $\mathcal{P} = (\mathbb{Z}^3, 26, 6, B)$, all of the following conditions hold:*

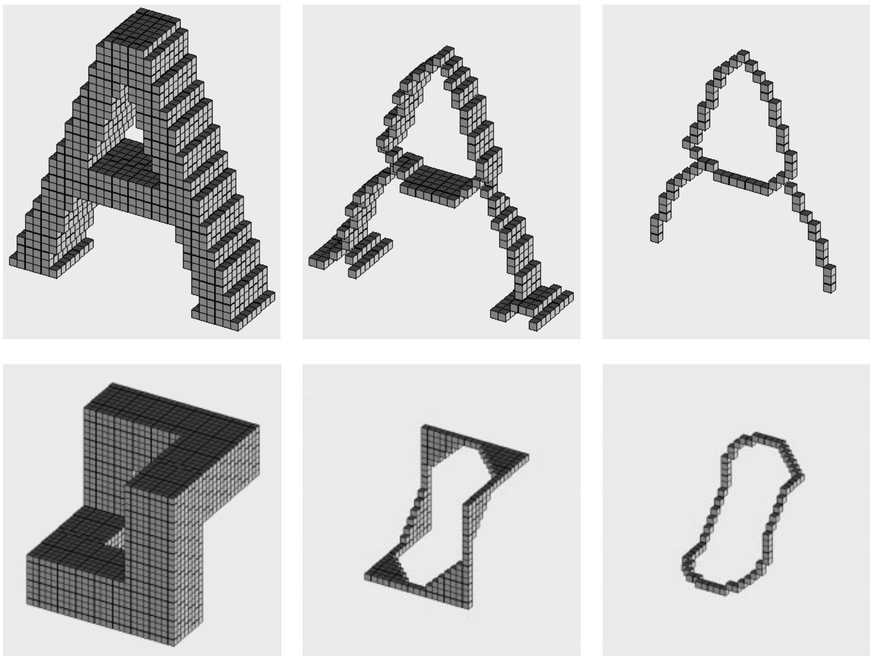

Fig. 6. Thinning of two synthetic objects (left); the results of surface thinning (centre); the results of curve thinning (right). (Cubes represents black points.)

1. *for all points $p \in B$ that are deleted by $\mathcal{T}$ and for all sets $Q \subseteq (N_{18}(p)\backslash\{p\}) \cap B$ that are deleted by $\mathcal{T}$, p is simple in the picture $(\mathbb{Z}^3, 26, 6, B\backslash Q)$; and*
2. *no black component contained in a unit lattice cube can be deleted completely by $\mathcal{T}$.*

Proof. (*sketch*) We show that condition 1 of Theorem 3 implies conditions 1–4 of Theorem 2.

It is obvious that the set $N_{18}(p)$ contains any unit lattice squares in which p is a corner. Point p is to be regarded as the last element of the simple sequence of corners while the preceding ones are in set Q. If $Q = \emptyset$, then we get Condition 1 of Theorem 2.

Condition 2 of Theorem 3 corresponds to condition 5 of Theorem 2. □

In order to prove both conditions of Theorem 3, we classify the elements of templates and state some properties of the set of templates $\mathcal{T}^c_{\mathbf{USW}}$.

The element in the very centre of a template is called *central*. A non–central template element is called *black* if it is marked "•". A non–central template element is called *white* if it is marked "○". Other non–central template element which is not white and not black, is called *potentially black*. A black or a potentially black non–central template element is called *non–white*.

Let us state some properties of the set of templates $\mathcal{T}^c_{\mathbf{USW}}$.

Fig. 7. Thinning of a ventricle extracted from a (greyscale) 3D MR brain study (left); the result of surface thinning (centre); the result of curve thinning (right).

Observation 6. Let us examine the configuration of $3 \times 3 \times 3$ points illustrated in Fig. 9. If point q is black or each point marked r is white, then black point p cannot be deleted by $\mathcal{T}^c_{\mathbf{USW}}$.

Observation 7. Let us examine the configurations of $3 \times 3 \times 3$ points illustrated in Fig. 10. If both points p and q are black and q can be deleted by $\mathcal{T}^c_{\mathbf{USW}}$, then point r is black and each point marked s is white.

Observation 8. Let us examine the configurations of $3 \times 3 \times 3$ points illustrated in Fig. 11. If both points p and q are black and q can be deleted by $\mathcal{T}^c_{\mathbf{USW}}$, then one of the following cases holds:

- point r_1 is black and both points marked s_1 are white and both points marked s_2 are white and both points marked s_3 are white;
- point r_2 is black and both points marked s_1 are white and both points marked s_4 are white and both points marked s_5 are white;
- point r_3 is black and each point marked s_i $(i = 1, \ldots, 5)$ is white;
- both points r_1 and r_2 are black and both points marked s_1 are white and both points marked s_2 are white;
- both points r_1 and r_2 are black and both points marked s_1 are white and both points marked s_4 are white.

Observation 9. Let us examine the configuration of $3 \times 3 \times 3$ points illustrated in Fig. 12. If black point p can be deleted by $\mathcal{T}^c_{\mathbf{USW}}$, then at least one point marked q is black.

Theorem 10. *Reduction operation given by the set of templates $\mathcal{T}^c_{\mathbf{USW}}$ is topology preserving.*

Proof. (*sketch*) It is easy to see that each template in $\mathcal{T}^c_{\mathbf{USW}}$ deletes only simple points of (26,6) pictures.

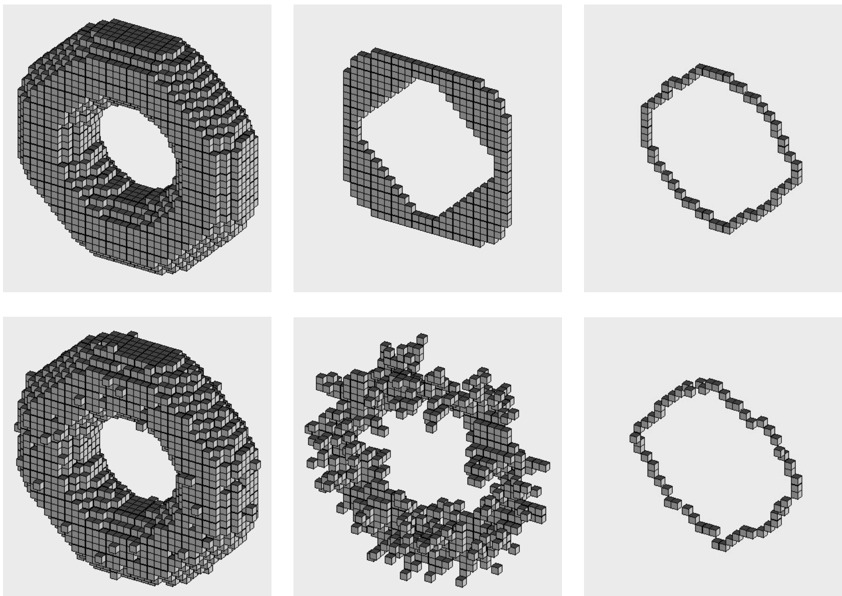

Fig. 8. Noise sensitiveness of the proposed algorithms. Thinning of a "noise–free" solid doughnut and its noisy version (left); the results of surface thinning (centre); the results of curve thinning (right).

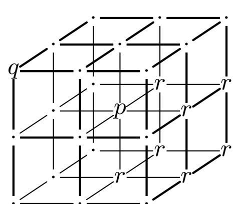

Fig. 9. Configuration assigned to Observation 1.

The first statement is to verify that there exists a 26–path between any two non–white positions (condition 1 of Theorem 1). It is sufficient to show that any potentially black position is 26–adjacent to a black position and any black position is 26–adjacent to another black position (if the template contains at least two black positions). It is obvious by careful examination of the templates in $\mathcal{T}^c_{\mathbf{USW}}$.

Let us examine condition 2 of Theorem 1. To prove it, it is sufficient to show for each template in $\mathcal{T}^c_{\mathbf{USW}}$

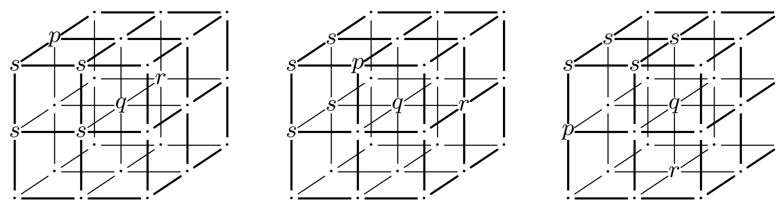

Fig. 10. Configurations assigned to Observation 2.

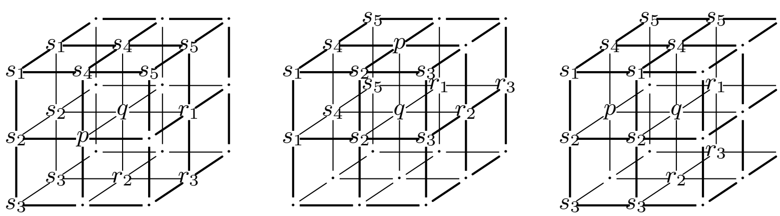

Fig. 11. Configurations assigned to Observation 3.

1. that there exists a white position 6–adjacent to the central position,
2. that for any two white positions 6–adjacent to the central position p are 6–connected in the set of white positions 18–adjacent to p,
3. and that for any potentially black position 6–adjacent to the central position p, there exists a 6–adjacent white 18–neighbour which is 6–adjacent to a white position 6–adjacent to p.

The three points are obvious by a careful examination of the templates in $\mathcal{T}^c_{\mathbf{USW}}$.

It can be stated that the value of any point coincides with a potentially black template position does not alter the simplicity of p. Therefore, it is sufficient to deal with sets Q containing points that can be deleted by $\mathcal{T}^c_{\mathbf{USW}}$ and coincide with black template positions. It is easy to see with the help of Observations 1–4 that deletion of such sets Q does not alter the simplicity of point p. Therefore, Condition 1 of Theorem 3 is satisfied.

Condition 2 of Theorem 3 can be seen with the help of Observation 1. □

Acknowledgements

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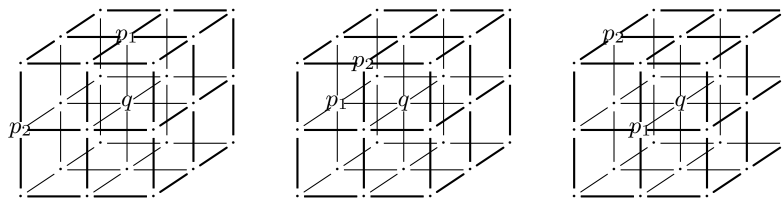

Fig. 12. Configurations assigned to Observation 4.

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