Interpolating Sporadic Data

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Abstract. We report here on the problem of estimating a smooth planar curve[a] $\gamma : [0, T] \to \mathbb{R}^2$ and its derivatives from an ordered sample of interpolation points $\{\gamma(t_0), \gamma(t_1), \ldots, \gamma(t_{i-1}), \gamma(t_i), \ldots, \gamma(t_{m-1}), \gamma(t_m)\}$, where $0 = t_0 < t_1 < \ldots < t_{i-1} < t_i < \ldots < t_{m-1} < t_m = T$, and the t_i are *not known precisely* for $0 < i < m$. Such situtation may appear while searching for the boundaries of planar objects or tracking the mass center of a rigid body with no times available. In this paper we assume that the distribution of t_i coincides with *more-or-less uniform sampling*. A fast algorithm, yielding *quartic convergence rate* based on 4-point piecewise-quadratic interpolation is analysed and tested. Our algorithm forms a substantial improvement (with respect to the speed of convergence) of piecewise 3-point quadratic Lagrange intepolation [19] and [20]. Some related work can be found in [7]. Our results may be of interest in computer vision and digital image processing [5], [8], [13], [14], [17] or [24], computer graphics [1], [4], [9], [10], [21] or [23], approximation and complexity theory [3], [6], [16], [22], [26] or [27], and digital and computational geometry [2] and [15].
Keywords: shape, image analysis and features, curve interpolation

1 Introduction

Let $\gamma : [0, T] \to \mathbb{R}^n$ be a smooth regular curve, namely γ is C^k for some $k \geq 1$ and $\dot{\gamma}(t) \neq \mathbf{0}$ for all $t \in [0, T]$ (with $0 < T < \infty$). Consider the problem of estimating γ from an ordered $m + 1$-tuple

$$\mathcal{Q} = (q_0, q_1, \ldots, q_m)$$

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of points in $\mathbb{R}^n$, where $q_i = \gamma(t_i)$, and $0 = t_0 < t_1 < \ldots < t_{i-1} < t_i < \ldots < t_{m-1} < t_m = T$. If the t_i are given then γ can be approximated in a variety of ways.

Example 1. Let γ be C^{r+2} where $r > 0$, and take m to be a multiple of r. Then $\mathcal{Q}$ gives $\frac{m}{r}$ subsets of $r+1$-tuples of the form

$$(q_0, q_1, \ldots, q_r)\ , \quad (q_r, q_{r+1}, \ldots, q_{2r})\ , \quad \ldots \quad , (q_{m-r}, q_{m-r+1}, \ldots, q_m)\ .$$

The jth $r+1$-tuple can be interpolated by a polynomial $\hat{\gamma}_j : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$ of degree r, and the track-sum $\hat{\gamma}$ of the $\hat{\gamma}_j$ is everywhere-continuous, and C^∞ except at $t_r, t_{2r}, \ldots, t_{m-r}$. Suppose that sampling is uniform i.e. $t_i = \frac{iT}{m}$ for $0 \leq i \leq m$. Then $\hat{\gamma}(t) = \gamma(t) + O(\frac{1}{m^{r+1}})$ for $t \in [0, T]$, and $\dot{\hat{\gamma}}(t) = \dot{\gamma}(t) + O(\frac{1}{m^r})$ for $t \neq t_r, t_{2r}, \ldots, t_{m-r}$. The error in length can be shown to be $O(\frac{1}{m^{r+1}})$ or $O(\frac{1}{m^{r+2}})$, accordingly as r is odd or even (see Theorem 1 in [19]).

In practice the t_i might not be given for $0 < i < m$.

Example 2. Let γ be C^4 curve in $\mathbb{R}^n$. For $0 \leq \varepsilon \leq 1$ the t_i are said to be *ε-uniformly sampled* when there is an order-preserving C^k reparameterization $\phi : [0, T] \to [0, T]$ such that

$$t_i = \phi(\frac{iT}{m}) + O(\frac{1}{m^{1+\varepsilon}})\ .$$

Although the set $\mathcal{Q}$ does not arise from perfectly uniform sampling, we can pretend that they do, and apply the method of Example 1. This is done in [19] and [20] with a view to estimating the length $d(\gamma)$ of γ. So far as γ and its derivatives are concerned the proof of Theorem 2 in [19] gives estimates of γ and $\dot{\gamma}$ with uniform $O(\frac{1}{m^{1+2\varepsilon}})$ and $O(\frac{1}{m^{2\varepsilon}})$ errors, respectively. The latter implies that $d(\hat{\gamma}) - d(\gamma) = O(\frac{1}{m^{4\varepsilon}})$ (see Theorem 2 in [19]). So when the distribution of the t_i is most nearly uniform ($\varepsilon = 1$) piecewise-quadratic Lagrange interpolation gives good estimates for γ, $\dot{\gamma}$, and $d(\gamma)$, namely cubic, quadratic and quartic, respectively. At the other extreme, where $\varepsilon = 0$, the methods of Example 1 have very little value. The extension of ε-uniform sampling for $\varepsilon > 1$ could also be considered. This case represents, however, a very small perturbation of uniform sampling (up to ϕ-order-preserving shift) which seems to be of less interest in applications. Nevertheless, by repeating the argument used in Theorem 2 (see [19]) it can be shown, that the case $\varepsilon > 1$ renders for γ, $\dot{\gamma}$, and $d(\gamma)$ estimation with piecewise-quadratic Lagrange interpolation the same results as for $\varepsilon = 1$.

A typical instance is shown in Figure 1, where ordinary Lagrange interpolation by piecewise-quadratics does not work well.

In general a less restrictive hypothesis than ε-uniformity is that the t_i should be sampled *more-or-less uniformly* in the following sense.

Definition 1. *Sampling is more-or-less uniform when there are constants* $0 < K_l < K_u$ *such that, for any sufficiently large integer* m, *and any* $1 \leq i \leq m$,

$$\frac{K_l}{m} \leq t_i - t_{i-1} \leq \frac{K_u}{m}\ .$$

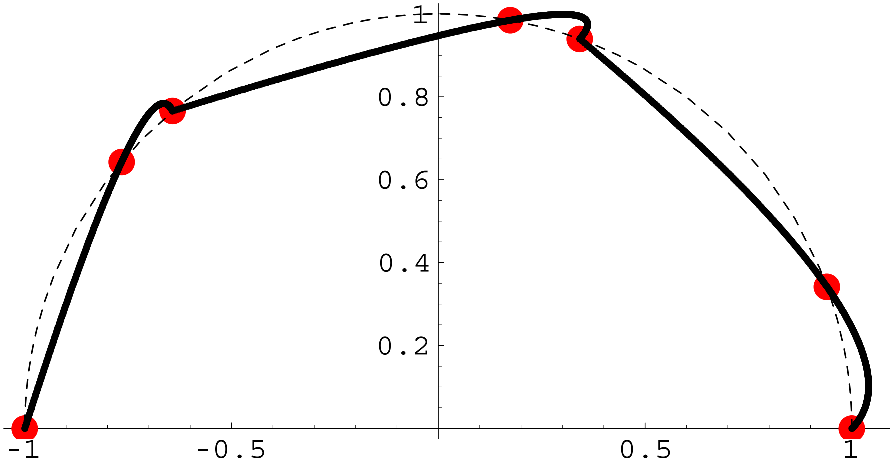

Fig. 1. 7 data points ($m = 6$), with 3 successive triples interpolated by piecewise-quadratics, giving length estimate $\pi + 0.0601035$ for the semicircle (shown dashed).

With more-or-less uniform sampling, increments between successive parameters are neither large nor small in proportion to $\frac{T}{m}$.

Example 3. For $0 < i < m$ set $t_i = \frac{(3i+(-1)^i)T}{3m}$. Then sampling is more-or-less uniform, with $K_l = \frac{T}{3}, K_u = \frac{5T}{3}$. Let $\gamma : [0, \pi] \to \mathbb{R}^2$ be the parameterization $\gamma(t) = (\cos t, \sin t)$ of the unit semicircle in the upper half-plane. When m is small the image of $\hat{\gamma}$ does not much resemble a semicircle, as in Figure 1, where $m = 6$.

Example 4. For $0 < i < m$ let t_i be random (according to some distribution) in the interval $[\frac{(3i-1)T}{3m}, \frac{(3i+1)T}{3m}]$. Then sampling is more-or-less uniform, with K_u, K_l as in Example 3.

Example 5. Choose $\theta > 0$ and $0 < L_l < L_u$. Set $s_0 = 0$. For $1 \leq i \leq m$ choose $\delta_i \in [\frac{L_l}{m}, \frac{L_u}{m}]$ independently from (say) the uniform distribution. Define $s_i = s_{i-1} + \delta_i$ for $i = 1, 2, \ldots, m$. The expectation of s_m is $\frac{L_u+L_l}{2}$ and the standard deviation $\frac{L_u-L_l}{2\sqrt{3m}}$. So if m is large $s_m \approx \frac{L_u+L_l}{2}$ with high probability. For $0 \leq i \leq m$, define $t_i = \frac{s_i T}{s_m}$. Set

$$K_l = \frac{2L_l T}{L_u + L_l} - \theta \, , \quad K_u = \frac{2L_u T}{L_u + L_l} + \theta \, .$$

Then with high probability for m large, the sampling $(t_0, t_1, t_2, \ldots, t_m)$ from $[0, T]$ is more-or-less uniform with constants K_l, K_u.

More-or-less uniform sampling is invariant with respect to reparameterizations, namely if $\phi : [0,T] \to [0,T]$ is an order-preserving C^1 diffeomorphism, and if $(t_0, t_1, \ldots, t_m)$ are sampled more-or-less uniformly, then so are $(\phi(t_0), \phi(t_1), \ldots, \phi(t_m))$. So reparameterizations lead to further examples from the ones already given. From now on take $n = 2$ and suppose that γ is C^4 and (without loss) paramerized by arc-length, namely $\|\dot{\gamma}\|$ is identically 1. The *curvature* of γ is defined as

$$k(t) = \det(M(t)) \ , \tag{1}$$

where $M(t)$ is the 2×2 matrix with columns $\dot{\gamma}(t), \ddot{\gamma}(t)$. Suppose that $k(t) \neq 0$ for all $t \in [0,T]$, namely γ is *strictly convex*. Let the t_i be sampled more-or-less uniformly. Then in section 2 we show how to carry out *piecewise 4-point quadratic interpolations* based on $\mathcal{Q}$. This approximation scheme is rather specialised, and much more elaborate than Lagrange interpolation. On the other hand it works well in cases such as in Figure 1. More precisely, from the proof of [18] Theorem 1, we obtain

Theorem 1. *Let γ be strictly convex and suppose that sampling is more-or-less uniform. Then we can estimate γ and $\dot{\gamma}$ from $\mathcal{Q}$ with $O(\frac{1}{m^4})$ and $O(\frac{1}{m^3})$ errors respectively.*

As a consequence of the last theorem we obtain $d(\hat{\gamma}) - d(\gamma) = O(\frac{1}{m^4})$ (see [18]). Applying piecewise 4-point quadratic interpolation to the data of Figure 1, gives a much more satisfactory estimate of the semicircle than Lagrange interpolation. This can be seen in Figure 2.

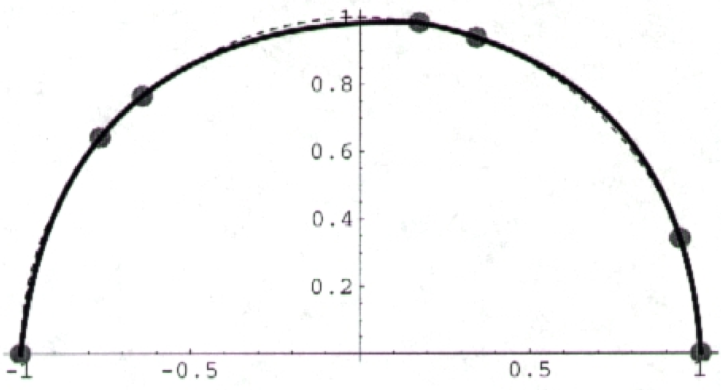

Fig. 2. Piecewise 4-point quadratic using 7 data points ($m = 6$) from a semicircle (shown dashed). Length estimate: $\pi - 0.00723637$.

The improvement is the result of a serious effort to estimate the parameters t_i from $\mathcal{Q}$. Although in practice it is difficult to discern a problem, our piecewise 4-point quadratic estimates $\tilde{\gamma}$ are usually not C^1. In theory, at least, this is a serious defect since γ is C^4. In section 3 we show how to refine the construction in section 2, replacing $\tilde{\gamma}$ by a C^1 curve with the same properties of approximation to γ.

2 Piecewise 4-Point Quadratics

Let $\mathcal{Q}$ be sampled more-or-less uniformly from γ, and suppose (without loss) that m is a positive integer multiple of 3. For each quadruple $(q_i, q_{i+1}, q_{i+2}, q_{i+3})$, where $0 \leq i \leq m-3$, define $a_0, a_1, a_2 \in \mathbb{R}^2$ and $Q^i(s) = a_0 + a_1 s + a_2 s^2$, by

$$Q^i(0) = q_i \ , \quad Q^i(1) = q_{i+1} \ , \quad Q^i(\alpha) = q_{i+2} \quad \text{and} \quad Q^i(\beta) = q_{i+3} \ .$$

Then $a_0 = q_i$, $a_2 = q_{i+1} - a_0 - a_1$, and we obtain two vector equations

$$\alpha a_1 + \alpha^2(p_1 - a_1) = p_\alpha \ , \quad \beta a_1 + \beta^2(p_1 - a_1) = p_\beta \ , \tag{2}$$

where $(p_1, p_\alpha, p_\beta) \equiv (q_{i+1} - q_i, q_{i+2} - q_i, q_{i+3} - q_i)$. Then (2) amounts to four quadratic scalar equations in four scalar unknowns $a_1 = (a_{11}, a_{12})$, α, β. Set

$$c = -\det(p_\alpha, p_\beta) \ , \quad d = -\det(p_\beta, p_1)/c \ , \quad e = -\det(p_\alpha, p_1)/c \ , \tag{3}$$

where $c, d, e \neq 0$ by strict convexity, and define

$$\rho_1 = \sqrt{e(1+d-e)/d} \ , \quad \rho_2 = \sqrt{d(1+d-e)/e} \ . \tag{4}$$

Then (2) has two solutions (see Appendix 1)

$$(\alpha_+, \beta_+) = \frac{(1+\rho_1, 1+\rho_2)}{e-d} \ , \quad (\alpha_-, \beta_-) = \frac{(1-\rho_1, 1-\rho_2)}{e-d} \tag{5}$$

provided ρ_1, ρ_2 are real and $e - d \neq 0$. In Appendix 1 it is also shown that these conditions hold, and in Appendix 2 it is proved that precisely one of (5) satisfies the additional constraint

$$1 < \alpha < \beta \ . \tag{6}$$

From now on, suppose[b] that $k(t) < 0$, for all $t \in [0, T]$. Define now

$$l(t) = \frac{\det(\frac{d\gamma}{dt}, \frac{d^3\gamma}{dt^3})}{k(t)}$$

and let $l = l(t_i)$. Then it is proved in [18] that

$$(\alpha_+, \beta_+) = \frac{((t_{i+2} - t_i)(1 + \frac{l(t_{i+2} - t_{i+1})}{6}), (t_{i+3} - t_i)(1 + \frac{l(t_{i+3} - t_{i+1})}{6}))}{t_{i+1} - t_i} + O(\frac{1}{m^2}) \ . \tag{7}$$

[b] The other case, where $k(t)$ is everywhere positive, is dealt with by considering the reversed curve $\gamma_r(t) = (\gamma_1(T-t), \gamma_2(T-t))$.

Note that here a third-order Taylor's expansion of γ is needed to justify the asymptotic behaviour of approximation results claimed in Theorem 1. Hence the assumption of $\gamma \in C^4$ is imposed. On the other hand, the justification of Appendices 1 and 2 requires only a second-order Taylor's expansion of γ and therefore a weaker restriction on smoothness of curve γ, namely $\gamma \in C^3$ is required. Set now $(\alpha, \beta) = (\alpha_+, \beta_+)$. Then, for $0 \leq s \leq \beta$, $Q^i(s) = q_i + a_1 s + a_2 s^2$, where

$$a_1 = \frac{p_\alpha - \alpha^2 p_1}{\alpha - \alpha^2} = \frac{p_\beta - \beta^2 p_1}{\beta - \beta^2}\,, \quad a_2 = \frac{\alpha p_1 - p_\alpha}{\alpha - \alpha^2} = \frac{\beta p_1 - p_\beta}{\beta - \beta^2}\,. \tag{8}$$

The quadratics Q^i, determined by $\mathcal{Q}$ and i, need to be reparameterized for comparison with the original curve γ. In doing so, let $\psi : [t_i, t_{i+3}] \to [0, \beta]$ be the cubic given by

$$\psi(t_i) = 0\,, \quad \psi(t_{i+1}) = 1\,, \quad \psi(t_{i+2}) = \alpha\,, \quad \psi(t_{i+3}) = \beta\,.$$

For m large ψ is an order-preserving diffeomorphism, and we define $\tilde{\gamma}_i = Q^i \circ \psi : [t_i, t_{i+3}] \to \mathbb{R}^2$. Then $\tilde{\gamma}_i$ is polynomial of degree at most 6. It turns out (as part of a difficult proof, given in [18]) that

$$\tilde{\gamma}_i(t) = \gamma(t) + O(\frac{1}{m^4}) \quad \text{and} \quad \dot{\tilde{\gamma}}_i(t) = \dot{\gamma}(t) + O(\frac{1}{m^3})\,, \quad \text{for} \quad t \in [t_i, t_{i+3}]\,.$$

Then the track-sum $\tilde{\gamma}$ of the arcs swept out by the Q^{3j} gives a $O(\frac{1}{m^4})$ uniformly accurate approximation of the image of γ. Although $\tilde{\gamma}$ is not C^1 at $t_3, t_6, \ldots, t_{m-3}$, the differences in left and right derivatives are $O(\frac{1}{m^3})$, and hardly discernible when m is large. In section 3 we show how to correct this minor defect.

The experiments verifying the rate of length estimation are discussed in [18].

3 C^1 Approximations

Instead of using 4-point quadratics as estimates of segments of γ, we can use them to estimate slopes of γ at $t_0, t_1, \ldots, t_m$. Except for $i = 0, m$ there is more than one choice of 4-point quadratic whose domain contains t_i. The choice does not appear to be critical, but for $0 < i < m$ we used estimates calculated from quadratics whose domain contained t_i in the interior.

It is then straightforward to produce a C^1 piecewise quadratic $\hat{\gamma}$ interpolating the given data points with the estimated slopes (for instance using the deCastlejau construction [3]). In practice $\hat{\gamma}$ seems slightly preferable to the already excellent estimate $\tilde{\gamma}$.

Example 6. Compared with the large discontinuities in derivatives at data points 3, 5 (from the right) in Figure 1, the tiny corner at the middle data point 4 in Figure 2 is only just discernible. The modification to a C^1 piecewise quadratic removes this blemish. Although there are only 7 sample points, and the t_i are unknown for $0 < i < 6$, the estimate $\hat{\gamma}$ shown in Figure 4 is difficult to distinguish from the underlying semicircle.

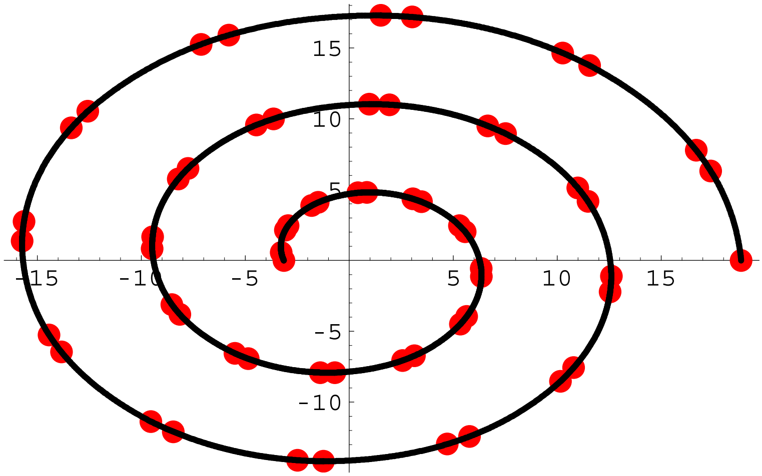

Fig. 3. A piecewise 4-point quadratic approximation to a spiral (singular point excluded), using the more-or-less uniform sampling of Example 3 and 61 data points ($m = 60$). True length: 173.608, estimate: 173.539, piecewise 3-point quadratic estimate: 181.311.

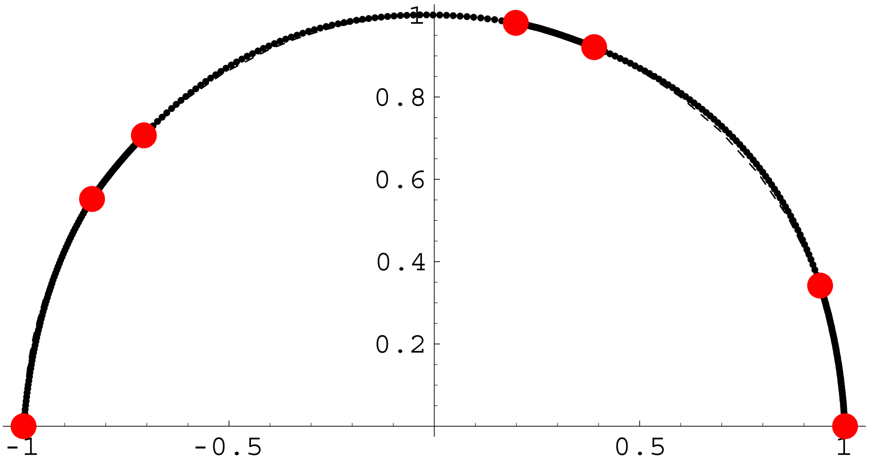

Fig. 4. C^1 piecewise-quadratic using 7 data points ($m = 6$) from a semicircle (shown dashed).

4 Concluding Remarks

Lagrange interpolation is reasonably effective for sporadic data when the t_i are distributed in an ε-uniform fashion and $\varepsilon \in (0,1]$. Better results are achieved with larger values of ε (see [19] and [20]). In general, the less restrictive condition, that the t_i be distributed more-or-less uniformly, is less straightforward, but (for strictly convex planar curves) the piecewise 4-point quadratic estimate of section 2 works well. The estimate $\tilde{\gamma}$ of section 2 is piecewise-polynomial, but not C^1. In section 3 we showed how to replace $\tilde{\gamma}$ by a piecewise-quadratic C^1 curve $\hat{\gamma}$.

There is also some analogous work for estimating lengths of digitized curves; indeed the analysis of digitized curves in $\mathbb{R}^2$ is one of the most intensively studied subjects in image data analysis. A digitized curve is the result of a process (such as contour tracing, 2D skeleton extraction, or 2D thinning) which maps a curve-like object (such as the boundary of a region) onto a computer-representable curve. As before, $\gamma : [0,T] \to \mathbb{R}^2$ is a strictly convex curve parameterized by arc-length. An analytical description of γ is not given, and numerical measurements of points on γ are corrupted by a process of *digitization*: γ is digitized within an orthogonal grid of points $(\frac{i}{m}, \frac{j}{m})$, where i, j are permitted to range over integer values, and m is a fixed positive integer called *the grid resolution*. Depending on the digitization model [11], γ is mapped onto a digital curve and approximated by a polygon whose length is an estimator for that of γ (see [2], [5], [8], [12], [13], [14] or [24]). We expect to revisit these issues in future.

Related work on interpolation, length estimation, noisy signal reconstruction and complexity involved can be found in [1], [4], [6], [7], [9], [10], [16], [17], [21], [22], [23], [25], [26] or [27].

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5 Appendix 1

In this Appendix we solve (2). Note that α (and β) cannot vanish as otherwise, by (2), the vector $p_\alpha = q_2 - q_0 = \mathbf{0}$ ($p_\beta = q_3 - q_0 = \mathbf{0}$) - *a contradiction* as interpolation points $\mathcal{Q}$ are assumed to be different. Similarly, as $q_2 \neq q_1$ and $q_3 \neq q_1$ we have $\alpha \neq 1$ and $\beta \neq 1$. Thus elimination of a_1 from (2) and further simplification yields

$$\alpha\beta(\alpha-\beta)p_1 = (\beta-\beta^2)p_\alpha - (\alpha-\alpha^2)p_\beta \, . \tag{9}$$

Consider now two vectors $p_\beta^\perp = (-p_{\beta 2}, p_{\beta 1})$ and $p_\alpha^\perp = (-p_{\alpha 2}, p_{\alpha 1})$, which are perpendicular to p_β and p_α, respectively. Taking the dot product of (9) first with $p_\beta^\perp$ and then with $p_\alpha^\perp$ results in

$$\alpha\beta(\alpha-\beta) < p_1|p_\beta^\perp > = (\beta-\beta^2) < p_\alpha|p_\beta^\perp > \, ,$$
$$\alpha\beta(\alpha-\beta) < p_1|p_\alpha^\perp > = -(\alpha-\alpha^2) < p_\beta|p_\alpha^\perp > \ .$$

Since α and β cannot vanish and $< p_\alpha|p_\beta^\perp > \neq 0$ and $< p_\beta|p_\alpha^\perp > \neq 0$ hold asymptotically (as γ is strictly convex) we obtain

$$\frac{\alpha(\alpha-\beta) < p_1|p_\beta^\perp >}{< p_\alpha|p_\beta^\perp >} = (1-\beta) \, , \quad \frac{\beta(\alpha-\beta) < p_1|p_\alpha^\perp >}{< p_\beta|p_\alpha^\perp >} = \alpha - 1 \, . \tag{10}$$

Note that by (3) and convexity of γ, $c \neq 0$ asymptotically. A simple verification shows:

$$c = - < p_\beta|p_\alpha^\perp > = < p_\alpha|p_\beta^\perp > \ . \tag{11}$$

Similarly

$$d = \frac{- < p_1|p_\beta^\perp >}{c} \, , \quad e = \frac{- < p_1|p_\alpha^\perp >}{c} \, .$$

The latter coupled with (11) yields

$$d = \frac{- < p_1|p_\beta^\perp >}{< p_\alpha|p_\beta^\perp >} \, , \quad e = \frac{< p_1|p_\alpha^\perp >}{< p_\beta|p_\alpha^\perp >}$$

which combined with (10) renders

$$\alpha(\alpha-\beta)d = \beta - 1 \, , \quad \beta(\alpha-\beta)e = \alpha - 1 \, . \tag{12}$$

The first equation of (12) yields

$$\alpha^2 d + 1 = \beta(1 + d\alpha) \, . \tag{13}$$

Note that $(1 + d\alpha) \neq 0$ as otherwise since $\alpha \neq 0$ we would have $d = -\alpha^{-1}$ and by (13) $\alpha^2 d + 1$ would vanish which combined with $d = -\alpha^{-1}$ would lead to $\alpha = 1$, *a contradiction*. Thus by (13)

$$\beta = \frac{\alpha^2 d + 1}{1 + d\alpha} \, . \tag{14}$$

Substituting (14) into the second equation of (12) yields

$$\frac{\alpha^2 d + 1}{1 + d\alpha}(1-\alpha)e = \alpha - 1$$

and taking into account that $\alpha \neq 1$ results in

$$(d^2 - de)\alpha^2 + 2d\alpha + 1 - e = 0 \,. \tag{15}$$

Assuming temporarily

$$\Delta = 4de(1 + d - e) > 0 \tag{16}$$

we arrive at (5). Having found $(\alpha_{\pm}, \beta_{\pm})$ the corresponding formulae (8) follow immediately. To show (16) recall that

$$d = \frac{\det(p_\beta, p_1)}{\det(p_\alpha, p_\beta)} \quad \text{and} \quad e = \frac{\det(p_\alpha, p_1)}{\det(p_\alpha, p_\beta)} \,. \tag{17}$$

As $\det(v, w) = \|v\|\|w\| sin(\sigma)$ (where σ is the oriented angle between v and w) for convex γ both $e < 0$ and $d < 0$ hold. Thus to justify (16) it is enough to show $1 + d - e > 0$. In fact, as γ is strictly convex all of above inequalities are separated from zero. The second-order Taylor's expansion of γ at $t = t_i$ yields

$$\gamma(t) = \gamma(t_i) + \dot{\gamma}(t_i)(t - t_i) + (1/2)\ddot{\gamma}(t_i)(t - t_i)^2 + O(\frac{1}{m^3})$$

as $0 < T < \infty$ and $\gamma \in C^4$ (in fact we need here only C^3). Thus taking into account that $\gamma(t_i) = q_i$, $\gamma(t_{i+1}) = q_{i+1}$, $\gamma(t_{i+2}) = q_{i+2}$, and $\gamma(t_{i+3}) = q_{i+3}$ we have

$$\begin{aligned}
p_1 &= \dot{\gamma}(t_i)(t_{i+1} - t_i) + (1/2)\ddot{\gamma}(t_i)(t_{i+1} - t_i)^2 + O(\frac{1}{m^3}) \,, \\
p_\alpha &= \dot{\gamma}(t_i)(t_{i+2} - t_i) + (1/2)\ddot{\gamma}(t_i)(t_{i+2} - t_i)^2 + O(\frac{1}{m^3}) \,, \\
p_\beta &= \dot{\gamma}(t_i)(t_{i+3} - t_i) + (1/2)\ddot{\gamma}(t_i)(t_{i+3} - t_i)^2 + O(\frac{1}{m^3}) \,.
\end{aligned} \tag{18}$$

Introducing $\gamma_2(t) = \dot{\gamma}(t_i)(t - t_i) + (1/2)\ddot{\gamma}(t_i)(t - t_i)^2$ and coupling it with (18) and more-or-less uniformity results in:

$$\begin{aligned}
\det(p_\beta, p_\alpha) &= \det(\gamma_2(t_{i+3}), \gamma_2(t_{i+2})) + O(\frac{1}{m^4}) \,, \\
\det(p_1, p_\beta) &= \det(\gamma_2(t_{i+1}), \gamma_2(t_{i+3})) + O(\frac{1}{m^4}) \,, \\
\det(p_\alpha, p_1) &= \det(\gamma_2(t_{i+2}), \gamma_2(t_{i+1})) + O(\frac{1}{m^4}) \,.
\end{aligned} \tag{19}$$

Set now $\mathcal{P}(c, d, e) = c(1 + d - e)$. Thus by (3) and (17) we have

$$\mathcal{P}(c, d, e) = \det(p_\beta, p_\alpha) + \det(p_1, p_\beta) + \det(p_\alpha, p_1) \,.$$

The latter combined with (1) and (19) yields

$$\mathcal{P}(c,d,e) = (1/2)k(t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})(t_{i+2}-t_{i+3}) + O(\frac{1}{m^4}) . \quad (20)$$

Similarly, by repeating the previous analysis we obtain

$$c = (1/2)k(t_i)(t_{i+3}-t_i)(t_{i+2}-t_i)(t_{i+2}-t_{i+3}) + O(\frac{1}{m^4}) . \quad (21)$$

Upon coupling (20) and (21) some factorization renders (note that curvature $k(t)$ is here bounded and separated from zero):

$$1+d-e = \frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+2}-t_i)} + O(\frac{1}{m}) . \quad (22)$$

As sampling is more-or-less uniform (see Definition 1) the latter amounts to

$$0 < \frac{K_l^2}{3K_u^2} \leq \frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+2}-t_i)} + O(\frac{1}{m}) .$$

Hence (16) follows.

6 Appendix 2

In this Appendix we show that one of the pairs $(\alpha_\pm, \beta_\pm)$ satisfies (6). More precisely, if curvature of curve γ satisfies $k(t) < 0$ then the pair (α_+, β_+) fulfills

$$1 < \alpha_+ < \beta_+ . \quad (23)$$

The opposite case involves the pair (α_-, β_-). It is sufficient (due to the analogous argument) to justify the first case only. To prove (23) we combine more-or-less uniformity, convexity of γ, with (3) and (18) to obtain for $\mathcal{R}(c,d) = cd$ which coincides with

$$\begin{aligned}\mathcal{R}(c,d) &= \det(\gamma_2(t_{i+1}), \gamma_2(t_{i+3})) + O(\frac{1}{m^4}) \\ &= (1/2)k(t_i)(t_{i+1}-t_i)(t_{i+3}-t_i)(t_{i+3}-t_{i+1}) + O(\frac{1}{m^4}) .\end{aligned}$$

Hence

$$d = \frac{-(t_{i+1}-t_i)(t_{i+3}-t_{i+1})}{(t_{i+2}-t_i)(t_{i+3}-t_{i+2})} + O(\frac{1}{m}) . \quad (24)$$

Similarly (taking also into account (24)) we arrive at

$$\begin{aligned}e &= \frac{-(t_{i+1}-t_i)(t_{i+2}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+3}-t_{i+2})} + O(\frac{1}{m}) , \\ \frac{e}{d} &= \frac{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}{(t_{i+3}-t_{i+1})(t_{i+3}-t_i)} + O(\frac{1}{m}) .\end{aligned}$$

Thus the latter combined with (22) yields

$$\frac{e}{d}(1+d-e) = \frac{(t_{i+2}-t_{i+1})^2}{(t_{i+3}-t_i)^2} + O(\frac{1}{m}) \; ,$$
$$e-d = 1 - \frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+2}-t_i)} + O(\frac{1}{m}) \; . \tag{25}$$

Hence Taylor's Theorem coupled with (4) and first equation from (25) renders

$$\rho_1 = \frac{(t_{i+2}-t_{i+1})}{(t_{i+3}-t_i)^2} + O(\frac{1}{m}) \; .$$

Finally, the second equation from (25) and further factorization in (5) results in

$$\alpha_+ = 1 + \frac{t_{i+2}-t_{i+1}}{t_{i+1}-t_i} + O(\frac{1}{m}) \; . \tag{26}$$

A similar analysis used to prove (26) shows that

$$\beta_+ = 1 + \frac{t_{i+3}-t_{i+1}}{t_{i+1}-t_i} + O(\frac{1}{m}) \; . \tag{27}$$

Because sampling is more-or-less uniform the formulae (26) and (27) guarantee that $1 < \alpha_+ < \beta_+$.