Critical Curves and Surfaces for Euclidean Reconstruction

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Abstract. The problem of recovering scene structure and camera motion from images has a number of inherent ambiguities. In this paper, configurations of points and cameras are analyzed for which the image points alone are insufficient to recover the scene geometry uniquely. Such configurations are said to be critical. For two views, it is well-known that a configuration is critical only if the two camera centres and all points lie on a ruled quadric. However, this is only a necessary condition. We give a complete characterization of the critical surfaces for two calibrated cameras and any number of points. Both algebraic and geometric characterizations of such surfaces are given. The existence of critical sets for n-view projective reconstruction has recently been reported in the literature. We show that there are critical sets for n-view Euclidean reconstruction as well. For example, it is shown that for any placement of three calibrated cameras, there always exists a critical set consisting of any number of points on a fourth-degree curve.

1 Introduction

Early in the twentieth century it was noticed that for two cameras and any number of scene points all lying on a particular surface, the solution to the reconstruction problem is not unique. There are several ambiguous configurations of cameras and scene points, which produce identical images. In the German photogrammetry community, such surfaces were termed 'gefährliche Örter' [7], which is translated as critical surfaces. In this paper, such critical configurations are analyzed for two and more calibrated cameras.

For two views it is well-known that a configuration is critical only if all points and the two camera centres lie on a ruled quadric, that is, a hyperboloid of one sheet, or one of its degenerate versions [8]. In [9], it was proven that six points and any number of cameras lying on a ruled quadric are critical. This was shown to be the dual to the two-view case [5] in the Carlsson duality sense [2]. Recently, the existence of critical configurations with more than two views and arbitrarily many points has been reported. In [4], it was shown that for any three cameras, there exists a critical fourth-degree curve containing the camera centres. The extension to arbitrarily many views was done in [6] where it was shown that

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cameras centres and scene points lying on the intersection of two quadrics are critical for projective reconstruction. However, critical configurations for more than two *calibrated* views have previously not been investigated in the published literature (except in the case of lines [1]).

In this paper, we focus on critical configurations for Euclidean reconstruction, starting with the calibrated two-view case. The most complete investigation of this is Maybank's book [8], to which the present paper owes much. A basic result of [8] is that a critical surface is necessarily a ruled, rectangular quadric, but the question of sufficiency is not substantially considered there. In the present paper, we obtain a necessary and sufficient condition for a configuration to be critical. The key is a restriction on the possible positions of the camera centres on the critical quadric. The condition is related to an algebraic constraint obtained by Maybank, but formulated here in a simple geometric form. The two-view analysis is used as a basis for exploring Euclidean multiview critical configurations. Previously unknown examples of critical configurations for more than two views are exhibited.

2 Background

A camera is represented by a 3×4 matrix of rank 3, the camera matrix $\mathtt{P}$. Provided the camera centre (which is the generator of the right null-space of $\mathtt{P}$) is a finite point, the camera matrix may be decomposed as $\mathtt{P} = \mathtt{K}[\,\mathtt{R} \,|\, \mathbf{t}\,]$, where $\mathtt{R}$ is a rotation matrix and $\mathtt{K}$ is an upper-triangular matrix, the calibration matrix. If $\mathtt{K}$ is known, then the camera is called *calibrated*. Matrix $\mathtt{K}$ represents a coordinate transformation in the image, this transformation may be undone by multiplying image coordinates by $\mathtt{K}^{-1}$. The camera matrix may therefore be assumed to be of the form $\mathtt{P} = [\,\mathtt{R} \,|\, \mathbf{t}\,]$, which will be referred to as a *calibrated camera*.

Consider a configuration of $n \geq 2$ cameras and a set of points. Denote the camera matrices by $\mathtt{P}^i$ for $i = 0, ..., n-1$ and the set of points with $\mathbf{P}_j$. The set $\{\mathtt{P}^i, \mathbf{P}_j\}$ is said to be *critical* if there exists an inequivalent configuration of cameras $\mathtt{Q}^i$ and points $\mathbf{Q}_j$ such that $\mathtt{P}^i\mathbf{P}_j = \mathtt{Q}^i\mathbf{Q}_j$ for all i and j. Two configurations are considered to be equivalent if the essential matrices for all pairs of views are the same. [1] The alternative configuration $\{\mathtt{Q}^i, \mathbf{Q}_j\}$ is called a *conjugate configuration*. In a critical configuration it is not possible to recover the cameras and the scene points unambiguously from the image points alone, as there are two alternative solutions. Note that when considering critical configurations for calibrated cameras all camera matrices $\mathtt{P}^i$ and $\mathtt{Q}^i$ are required to be of the form $[\mathtt{R}|\mathbf{t}]$, which is quite a restrictive condition. Consequently critical configurations for calibrated cameras may be expected to be a subset of uncalibrated critical configurations.

For two calibrated cameras, the relative pose is encapsulated by the essential matrix. We define an *essential matrix* to be a 3×3 matrix writable as a product of

[1] This definition excludes the trivial ambiguity that arises from points lying on a line containing all camera centres, as such points may vary along this base line. In addition, it excludes the two-view "twisted-pair ambiguity" discussed later – such an ambiguity being deemed in some sense trivial also.

a rotation and a skew-symmetric matrix. Thus, $\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times$, where $\mathtt{R}$ is a rotation, and $[\mathbf{t}]_\times$ denotes the matrix such that $[\mathbf{t}]_\times \mathbf{v} = \mathbf{t} \times \mathbf{v}$ for any 3-vector $\mathbf{v}$.

3 Critical Sets for Euclidean Reconstruction

The basic equation for the shape of a critical set in the calibrated (Euclidean) case is not very different from that for the uncalibrated (projective) case. Following the derivation in [4] one arrives at the following result.

Theorem 1. *Let* $(\mathtt{P}^0, \mathtt{P}^1)$ *and* $(\mathtt{Q}^0, \mathtt{Q}^1)$ *be two pairs of calibrated cameras, and let* $\mathtt{E}_\mathtt{P}$ *and* $\mathtt{E}_\mathtt{Q}$ *be the corresponding essential matrices for the two camera pairs. The surfaces* $S_\mathtt{P}$ *and* $S_\mathtt{Q}$ *defined by*

$$S_\mathtt{P} = \mathtt{P}^{0\top}\mathtt{E}_\mathtt{Q}\mathtt{P}^1 + \mathtt{P}^{1\top}\mathtt{E}_\mathtt{Q}{}^\top\mathtt{P}^0 \quad \textit{and} \quad S_\mathtt{Q} = \mathtt{Q}^{0\top}\mathtt{E}_\mathtt{P}\mathtt{Q}^1 + \mathtt{Q}^{1\top}\mathtt{E}_\mathtt{P}{}^\top\mathtt{Q}^0 \tag{1}$$

are ruled quadric surfaces[2]*, which are critical for reconstruction. In particular:*

1. *If* $\mathbf{P}$ *is a point on* $S_\mathtt{P}$*, then there exists a point* $\mathbf{Q}$ *on* $S_\mathtt{Q}$ *such that* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for* $i = 0, 1$.
2. *Conversely, if* $\mathbf{P}$ *and* $\mathbf{Q}$ *are points such that such that* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for* $i = 0, 1$*, then* $\mathbf{P}$ *lies on* $S_\mathtt{P}$ *and* $\mathbf{Q}$ *lies on* $S_\mathtt{Q}$.

This leads us to the following definition.

Definition 1. *A triple* $(S, \mathtt{P}^0, \mathtt{P}^1)$ *where* S *is a symmetric* 4×4 *matrix representing a quadric, and* $\mathtt{P}^0$ *and* $\mathtt{P}^1$ *are calibrated camera matrices representing cameras with centres lying on the quadric* S *is called* critical for Euclidean reconstruction *if there exists an essential matrix* $\mathtt{E}$ *such that*

$$S = \mathtt{P}^{0\top}\mathtt{E}\mathtt{P}^1 + \mathtt{P}^{1\top}\mathtt{E}^\top\mathtt{P}^0. \tag{2}$$

The twisted-pair ambiguity. In Definition 1, matrix $\mathtt{E}$ is the essential matrix for a conjugate camera pair $(\mathtt{Q}^0, \mathtt{Q}^1)$. Note however that (unlike in the projective case), the essential matrix $\mathtt{E}$ does not determine the two camera matrices $(\mathtt{Q}^0, \mathtt{Q}^1)$ uniquely even up to a similarity, because of the “twisted-pair ambiguity.” Thus, (apart from sign and scale ambiguities) there are two essentially different ways of decomposing $\mathtt{E}$, namely $\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times = \mathtt{R}'[\mathbf{t}]_\times$, involving different rotation matrices $\mathtt{R}$ and $\mathtt{R}'$ (see [8,3] for more details). Note that the vector $\mathbf{t}$ is the same in both cases, since it is the generator of the null-space of $\mathtt{E}$.

Normalized camera matrices. As shown in [4], for a configuration of points and cameras to be critical it is sufficient to consider only the positions of the cameras, and not their orientation[3]. In particular, in investigating whether a configuration is critical, we may assume that the two cameras have the form $[\,\mathtt{I} \mid -\mathbf{t}_i\,]$, where $\mathbf{t}_i$ is an inhomogeneous 3-vector representing the location of

[2] A quadric S is defined by the set of points $\mathbf{P} \in \mathcal{P}^3$ such that $\mathbf{P}^\top S\mathbf{P} = 0$.

[3] Proved in [4] for the uncalibrated case, but easily extended to calibrated cameras.

the camera centre. Camera matrices in this form will be referred to as *normalized*. The form of a critical quadric is particularly simple for normalized camera matrices:

$$S = \begin{bmatrix} \mathtt{E}^{01} + \mathtt{E}^{10} & -\mathtt{E}^{10}\mathbf{t}_0 - \mathtt{E}^{01}\mathbf{t}_1 \\ -\mathbf{t}_0{}^\top \mathtt{E}^{01} - \mathbf{t}_1{}^\top \mathtt{E}^{10} & 2\mathbf{t}_0{}^\top \mathtt{E}^{01}\mathbf{t}_1 \end{bmatrix}. \tag{3}$$

4 Rectangular Quadrics

We now define *rectangular quadric* surfaces, which will turn out to be the critical surfaces for calibrated cameras.

Definition 2. *A quadric represented by a symmetric matrix S is called a* rectangular *quadric if the upper left hand 3×3 block $\mathtt{M}$ of S may be written as $\mathtt{E}+\mathtt{E}^\top$ for some essential matrix $\mathtt{E}$.*

In [8] rectangular quadrics are characterized by different algebraic conditions on $\mathtt{M}$, which are next seen to be equivalent to Definition 2.

Proposition 1. *Let $\mathtt{M}$ be a 3×3 symmetric matrix. The following conditions are equivalent.*

1. $\mathtt{M} = \mathtt{E} + \mathtt{E}^\top$ *for some essential matrix* $\mathtt{E}$.
2. $\mathtt{M} = \mathbf{m}\mathbf{n}^\top + \mathbf{n}\mathbf{m}^\top - 2\mathbf{m}\mathbf{n}^\top\mathtt{I}$ *for two 3-vectors $\mathbf{m}$ and $\mathbf{n}$ called the* principal points *on* $\mathtt{M}$.
3. *The eigenvalues of $\mathtt{M}$ are of the form λ_1, λ_2 and $\lambda_1+\lambda_2$, where $\lambda_1\lambda_2 \le 0$.*

Proof. $1 \to 2$ Suppose $\mathtt{M} = \mathtt{E} + \mathtt{E}^\top$. Let $\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times$. Without substantially altering the problem, $\mathtt{E}$ may be replaced by an essential matrix $\mathtt{UEU}^\top$, where $\mathtt{U}$ is a rotation. Using this observation, one may without loss of generality assume that the rotation axis of $\mathtt{R}$ is the vector $\mathbf{n} = (0,0,1)^\top$. Thus, $\mathtt{R}$ is a rotation about the Z-axis. Let $\mathbf{t} = (x,y,z)^\top$. Then

$$\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times = \begin{bmatrix} c & -s & \\ s & c & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = \begin{bmatrix} -sz & -cz & cy+sx \\ cz & -sz & sy-cx \\ -y & x & 0 \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin\theta$ and θ is the angle of rotation. So

$$\mathtt{E}+\mathtt{E}^\top = \begin{bmatrix} -2sz & 0 & sx+(c-1)y \\ 0 & -2sz & sy-(c-1)x \\ sx+(c-1)y & sy-(c-1)x & 0 \end{bmatrix}. \tag{4}$$

Now, setting $\mathbf{m} = (sx+(c-1)y,\ sy-(c-1)x,\ sz)^\top$ and $\mathbf{n} = (0,0,1)^\top$, it is easily verified that $\mathbf{m}\mathbf{n}^\top + \mathbf{n}\mathbf{m}^\top - 2\mathbf{m}^\top\mathbf{n}\mathtt{I} = \mathtt{E}+\mathtt{E}^\top$ as required. In addition, apart from scaling, or swapping $\mathbf{m}$ and $\mathbf{n}$, the choice of $\mathbf{m}$ and $\mathbf{n}$ is unique.

$2 \to 1$ Suppose $\mathtt{M} = \mathbf{m}\mathbf{n}^\top + \mathbf{n}\mathbf{m}^\top - 2\mathbf{m}^\top\mathbf{n}\mathtt{I}$. Once again, one may rotate coordinates to ensure that $\mathbf{n} = (0,0,1)^\top$, and that furthermore $\mathbf{m}$ lies in the XZ-plane, and so $\mathbf{m}$ is of the form $\mathbf{m} = (2p,0,q)^\top$. Then

$$\mathtt{M} = \mathbf{m}\mathbf{n}^\top + \mathbf{n}\mathbf{m}^\top - 2\mathbf{m}^\top\mathbf{n}\mathtt{I} = 2\begin{bmatrix} -q & & p \\ & -q & 0 \\ p & 0 & 0 \end{bmatrix} . \tag{5}$$

Now, it is easily verified that if

$$\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -q & -p \\ q & 0 & -p \\ p & p & 0 \end{bmatrix} = \begin{bmatrix} -q & 0 & p \\ 0 & -q & -p \\ p & p & 0 \end{bmatrix} \tag{6}$$

then $\mathtt{E} + \mathtt{E}^\top$ equals (5) as required.

$2 \longleftrightarrow 3$. See [8] for a proof. ■

Note that matrix M, the upper-left 3×3 block of S represents the conic in which the quadric S meets the plane at infinity. The principal points $\mathbf{m}$ and $\mathbf{n}$ lie on this conic, since $\mathbf{m}^\top \mathtt{M}\mathbf{m} = \mathbf{n}^\top \mathtt{M}\mathbf{n} = 0$. Points on M may also be thought of as representing asymptotic directions of the quadric. When S is a ruled quadric, points on M are the direction vectors of the generators (straight lines) on S.

Restriction. Henceforth in this paper, we will avoid having to deal with special cases by assuming that the matrix M is non-singular, i.e., S meets the plane at infinity in a non-degenerate conic. In terms of the representation of M by principal points, this corresponds to an assumption that $\mathbf{m}$ and $\mathbf{n}$ represent neither collinear, nor orthogonal directions. Equivalently, in (5) neither p nor q is zero. It is shown in [8] that a rectangular quadric with two equal principal points is a circular cylinder, which can not be a critical surface.

Proposition 1 gives *algebraic* conditions for a quadric to be rectangular. Various equivalent geometric conditions for a quadric to be rectangular are given in the appendix. These help to provide geometric intuition.

5 Standard Position for a Rectangular Quadric

The definition of a rectangular quadric given in Definition 2 specifies only the form of the top-left block of the matrix S. In other words whether a quadric is rectangular or not depends only on its intersection with the plane at infinity. However, if M is non-singular (which we are now assuming), then by a translation of coordinates, S may be transformed to a block-diagonal matrix of the form

$$S = \begin{bmatrix} \mathtt{M} & \mathbf{0} \\ \mathbf{0}^\top & d \end{bmatrix} . \tag{7}$$

Note that this quadric is symmetric about the origin, in that if $(\mathbf{X}^\top, k)^\top$ lies on the quadric S, then so does $(-\mathbf{X}^\top, k)^\top$. We may also assume, as before that the two principal points of the quadric are $\mathbf{n} = (0, 0, 1)^\top$ and $\mathbf{m} = (2p, 0, q)^\top$. In this case, the matrix representing the quadric is of the form (7), where M has the form given by (5). Such a quadric is said to be in *standard position.*

Symmetry of a rectangular quadric. A quadric in standard position has a rotational symmetry about the the Y-axis. Namely, if $\mathbf{X} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z})^\top$ lies on the quadric, then so does $(-\mathrm{X}, \mathrm{Y}, -\mathrm{Z})^\top$. There is a further symmetry mapping

$\mathbf{X}$ to $(\mathrm{X}, -\mathrm{Y}, \mathrm{Z})^\top$, as well as a symmetry swapping the two principal points, but we will not be so concerned with these. Let $\mathtt{R}_{sym}$ represent this rotation of the quadric about the Y-axis. As a matrix, $\mathtt{R}_{sym}$ has the form $\mathrm{diag}(-1, 1, -1)$. For a quadric not in standard position, we still use $\mathtt{R}_{sym}$ to represent this rotational symmetry of the quadric, though it will be an arbitrary rotation matrix.

Different choices of $\mathtt{E}$**.** Let S be a rectangular quadric and $\mathtt{M}$ the upper left-hand block. By definition, $\mathtt{M}$ can be written as $\mathtt{M} = \mathtt{E} + \mathtt{E}^\top$ for some essential matrix. The decomposition (6) involved a choice of $\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times$ for which $\mathtt{R}$ was a rotation through 90 degrees about the axis $\mathbf{n} = (0, 0, 1)^\top$. It will next be shown that this decomposition of the quadric is not unique. The following result enumerates all possible ways of expressing $\mathtt{M}$ as $\mathtt{E} + \mathtt{E}^\top$.

Terminology: By left and right *epipoles* of an essential matrix $\mathtt{E}^{01}$ are meant the vectors $\mathbf{e}_0$ and $\mathbf{e}_1$ such that $\mathtt{E}^{01\top}\mathbf{e}_0 = \mathtt{E}^{01}\mathbf{e}_1 = \mathbf{0}$.

Theorem 2. *Let* $\mathtt{M}$ *be a non-singular matrix* $\mathbf{m}\mathbf{n}^\top + \mathbf{n}\mathbf{m}^\top - 2\mathbf{m}^\top\mathbf{n}\mathtt{I}$*, where* $\mathbf{m}$ *and* $\mathbf{n}$ *are two principal points. Let* $\mathtt{E}$ *be an essential matrix such that* $\mathtt{M} = \mathtt{E} + \mathtt{E}^\top$*.*

1. *The left and right epipoles* $\mathbf{e}_0$ *and* $\mathbf{e}_1$ *of* $\mathtt{E}$ *as well as the two principal points* $\mathbf{m}$ *and* $\mathbf{n}$ *lie on the conic* $\mathtt{M}$*.*
2. *If* $\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times = \mathtt{R}'[\mathbf{t}]_\times$ *are the two distinct ways of decomposing* $\mathtt{E}$ *into a rotation and skew-symmetric matrix, then the rotation axes of* $\mathtt{R}$ *and* $\mathtt{R}'$ *are the two principal points of* $\mathtt{M}$*.*
3. *For every point* $\mathbf{x}$ *lying on the conic* $\mathtt{M}$*, with the exception of the two principal points, there exists a unique* $\mathtt{E}$ *such that* $\mathtt{M} = \mathtt{E} + \mathtt{E}^\top$ *and* $\mathtt{E}\mathbf{x} = 0$*.*
4. *If* $\mathbf{e}_0$ *is one of the epipoles of* $\mathtt{E}$*, then the other one,* $\mathbf{e}_1$ *is the point* $\mathtt{R}_{sym}\mathbf{e}_0$ *obtained by rotating* $\mathbf{e}_0$ *about the symmetry axis of the quadric.*
5. *For any* $\mathtt{E}$ *satisfying* $\mathtt{M} = \mathtt{E} + \mathtt{E}^\top$*, the relation* $\mathtt{R}_{sym}\mathtt{E}\mathtt{R}_{sym} = \mathtt{E}^\top$ *holds.*

Proof. **Part 1.** A point $\mathbf{x}$ lies on the conic $\mathtt{M}$ if and only if $\mathbf{x}^\top\mathtt{M}\mathbf{x} = 0$. The fact that the two principal points $\mathbf{m}$ and $\mathbf{n}$ lie on $\mathtt{M}$ is easily verified. Similarly, if $\mathtt{E}\mathbf{e} = 0$, then $\mathbf{e}^\top(\mathtt{E} + \mathtt{E}^\top)\mathbf{e} = 0$, so $\mathbf{e}$ lies on $\mathtt{M}$.

Part 2. We may without loss of generalization assume that $\mathtt{R}$ is a rotation about the Z-axis. If for some $\mathbf{m}$ and $\mathbf{n}$ one has $\mathtt{E} + \mathtt{E}^\top = \mathbf{m}\mathbf{n}^\top + \mathbf{n}\mathbf{m}^\top - 2\mathbf{m}^\top\mathbf{n}\mathtt{I}$, then $\mathtt{E} + \mathtt{E}^\top - \mathrm{tr}(\mathtt{E} + \mathtt{E}^\top)/2 = \mathbf{m}\mathbf{n}^\top + \mathbf{n}\mathbf{m}^\top$. However, the form of $\mathtt{E} + \mathtt{E}^\top$ is given by (4), from which it follows that $\mathtt{E} + \mathtt{E}^\top - \mathrm{tr}(\mathtt{E} + \mathtt{E}^\top)/2$ has an upper left-hand 2×2 block of zeros. From this it easily follows that either $\mathbf{m}$ or $\mathbf{n}$ is $(0, 0, 1)^\top$ and hence equal to the the rotation axis of $\mathtt{R}$. The axis of the other rotation matrix $\mathtt{R}'$ is distinct from that of $\mathtt{R}$, and by the same argument must therefore be the other of the two principal points.

Part 3. Let $\mathtt{M}$ be in standard position, with principal points given by $(0, 0, 1)^\top$ and $(2p, 0, 1)^\top$. In this case, $\mathtt{M}$ has the form given by (5) with $q = 1$. In any decomposition of $\mathtt{M}$, the rotation axis is one of these two principal points, and so we may assume $\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times$ where $\mathtt{R}$ is a rotation about the Z axis. Let the rotation angle be θ and $\mathbf{t} = (x, y, z)^\top$, then $\mathtt{E}$ is given by (4). Equating (5) with (4) the unique solution is $\mathbf{t} = (p(c + 1), -ps, 1)^\top$, provided that $s \neq 0$.

Thus, with $\mathtt{M}$ given, $\mathtt{E}$ is uniquely determined by its rotation angle,

$$\mathtt{E} = \mathtt{R}[\mathbf{t}]_\times = \begin{bmatrix} -s & -c & ps \\ c & -s & -p(c+1) \\ ps & p(c+1) & 0 \end{bmatrix} . \tag{8}$$

Now, as a conic (5), with $q = 1$, $\mathtt{M}$ may be written as $\mathrm{X}^2 + \mathrm{Y}^2 - 2p\mathrm{XZ} = 0$. Dehomogenizing by setting $\mathrm{Z} = 1$ and completing the square, this becomes $(\mathrm{X} - p)^2 + \mathrm{Y}^2 = p^2$, which is a circle centred at $(p, 0)$ of radius p. As $c = \cos(\theta)$ and $s = \sin(\theta)$ vary with θ, the epipole $\mathbf{t} = (p(c + 1), -ps, 1)^\top$ traces out this circle. Thus, the points $\mathbf{x}$ on $\mathtt{M}$ are in one-to-one correspondence with the rotation angle, and so $\mathtt{E}$ is uniquely determined by $\mathbf{x}$.

Part 4. Simply observe that the left epipole of $\mathtt{E}$ in (8) is $(-p(c+1),\ -ps,\ -1)^\top$ which is the rotation of the right epipole about the Y-axis.

Part 5. The identity $\mathtt{R}_{sym}\mathtt{E}\mathtt{R}_{sym} = \mathtt{E}^\top$ is verified by direct computation using (8) and $\mathtt{R}_{sym} = \mathrm{diag}(-1, 1, -1)$ for the symmetry of a quadric in standard position. ■

6 Characterization of Critical Surfaces

We are now ready to determine the critical surfaces for calibrated reconstruction.

Theorem 3. *If $(S, \mathtt{P}^0, \mathtt{P}^1)$ is a critical configuration, then S is a ruled rectangular quadric. Further, the camera centres $\mathbf{t}_0$ and $\mathbf{t}_1$ of $\mathtt{P}^0$ and $\mathtt{P}^1$ satisfy the condition that $\mathtt{R}_{sym}\mathbf{t}_0$ and $\mathbf{t}_1$ lie on a common generator of the quadric.*

Conversely, if $\mathtt{R}_{sym}\mathbf{t}_0$ and $\mathbf{t}_1$ lie on a common generator, then the configuration is critical, provided that the generator does not pass through a principal point of the quadric.

That the quadric is rectangular follows directly from (3) and the definition of a rectangular quadric in the case where the two cameras are normalized. For general cameras of the form $\mathtt{R}_0[\mathtt{I}|\mathbf{t}_0]$ and $\mathtt{R}_1[\mathtt{I}|\mathbf{t}_1]$, the upper left-hand block of (2) is of the form $\mathtt{M} = \mathtt{R}_0{}^\top\mathtt{E}\mathtt{R}_1 + \mathtt{R}_1{}^\top\mathtt{E}^\top\mathtt{R}_0$. However, if $\mathtt{E}$ is an essential matrix, then so is $\mathtt{R}_0{}^\top\mathtt{E}\mathtt{R}_1$, and so S is a rectangular quadric.

The necessary condition on the camera centres is completed by the following lemma, which specifies the relationship between the camera centres and the essential matrix more precisely.

Lemma 1. *If S is a critical quadric for normalized camera matrices with centres at $\mathbf{t}_0$ and $\mathbf{t}_1$, and $\mathtt{E}^{01}$ is the essential matrix satisfying (3), then $\mathtt{R}_{sym}\mathbf{t}_0$ and $\mathbf{t}_1$ lie on a generator of S with direction vector given by $\mathbf{e}_1$, the right epipole of $\mathtt{E}^{01}$.*

Proof. We may assume that S is in standard position. From this and (3) it follows that $\mathtt{E}^{10}\mathbf{t}_0 + \mathtt{E}^{01}\mathbf{t}_1 = 0$. Multiplying on the left by $\mathbf{e}_1$ gives $\mathbf{e}_1{}^\top\mathtt{E}^{01}\mathbf{t}_1 = 0$.

Two things need to be proved: (i) For all α, point $\mathbf{t}_1 + \alpha\mathbf{e}_1$ lies on S, and (ii) $\mathtt{R}_{sym}\mathbf{t}_0 = \mathbf{t}_1 + \alpha\mathbf{e}_1$. Since the quadric has the diagonal block form $S = \mathrm{diag}(\mathtt{E}^{01} + \mathtt{E}^{10}, d)$, the first point is proved by showing that

$$\mathbf{t}_1{}^\top\mathtt{M}\mathbf{t}_1 + 2\alpha\mathbf{e}_1{}^\top\mathtt{M}\mathbf{t}_1 + \alpha^2\mathbf{e}_1{}^\top\mathtt{M}\mathbf{e}_1 + d = 0.$$

However, $\mathbf{e}_1{}^\top \mathtt{M}\mathbf{e}_1 = 0$, because $\mathbf{e}_1$ is on $\mathtt{M}$ and $\mathbf{t}_1{}^\top \mathtt{M}\mathbf{t}_1 + d = 0$, because $\mathbf{t}_1$ is on S. The remaining term reduces to $2\alpha\mathbf{e}_1\mathtt{E}^{01}\mathbf{t}_1$, which is zero, as was just shown.

Now to the second point. Showing that $\mathtt{R}_{sym}\mathbf{t}_0 = \mathbf{t}_1 + \alpha\mathbf{e}_1$ is equivalent to showing that $\mathtt{E}^{01}(\mathtt{R}_{sym}\mathbf{t}_0 - \mathbf{t}_1) = 0$. However,

$$\begin{aligned}\mathtt{E}^{01}(\mathtt{R}_{sym}\mathbf{t}_0 - \mathbf{t}_1) &= \mathtt{E}^{01}\mathtt{R}_{sym}\mathbf{t}_0 + \mathtt{E}^{10}\mathbf{t}_0 \quad \text{since} \quad \mathtt{E}^{10}\mathbf{t}_0 + \mathtt{E}^{01}\mathbf{t}_1 = 0 \\ &= (\mathtt{R}_{sym} + \mathtt{I})\mathtt{E}^{10}\mathbf{t}_0 \quad \text{since} \quad \mathtt{E}^{10}\mathtt{R}_{sym} = \mathtt{R}_{sym}\mathtt{E}^{01}.\end{aligned}$$

It has been shown that $\mathtt{e}_1{}^\top \mathtt{E}^{01}\mathbf{t}_1 = 0$, and trivially $\mathtt{e}_0{}^\top \mathtt{E}^{01}\mathbf{t}_1 = 0$. Thus $\mathtt{E}^{01}\mathbf{t}_1$ is perpendicular to both $\mathbf{e}_0$ and $\mathbf{e}_1$. Consequently $\mathtt{E}^{01}\mathbf{t}_1 = k(\mathbf{e}_0 \times \mathbf{e}_1)$. In turn,

$$\mathtt{R}_{sym}(\mathbf{e}_0 \times \mathbf{e}_1) = (\mathtt{R}_{sym}\mathbf{e}_0) \times (\mathtt{R}_{sym}\mathbf{e}_1) = \mathbf{e}_1 \times \mathbf{e}_0 = -(\mathbf{e}_0 \times \mathbf{e}_1).$$

So $(\mathtt{R}_{sym} + \mathtt{I})\mathtt{E}^{01}\mathbf{t}_1 = k(\mathtt{R}_{sym} + \mathtt{I})(\mathbf{e}_0 \times \mathbf{e}_1) = 0$, as required. ■

Converse. Suppose that S is in standard position, and that $\mathbf{t}_1$ and $\mathtt{R}_{sym}\mathbf{t}_0$ lie on a common generator, whose direction vector we denote by $\mathbf{e}_1$. According to hypothesis, $\mathbf{e}_1$ is not coincident with one of the principal points of the quadric. In this case, according to Theorem 2, there exists an essential matrix $\mathtt{E}^{10}$ such that $\mathtt{E}^{01}\mathbf{e}_1 = \mathbf{0}$ and $\mathtt{M} = \mathtt{E}^{01} + \mathtt{E}^{10}$. According to Theorem 2 again, $\mathbf{e}_0 = \mathtt{R}_{sym}\mathbf{e}_1$ is the other epipole of $\mathtt{E}^{10}$, satisfying $\mathbf{e}_0{}^\top \mathtt{E}^{01} = \mathbf{0}$.

Our goal is to demonstrate that (3) holds for this choice of $\mathtt{E}^{10}$. Since S is assumed to be in standard position, $S = \mathrm{diag}(\mathtt{M}, d)$, it suffices to prove that $\mathtt{E}^{10}\mathbf{t}_0 + \mathtt{E}^{01}\mathbf{t}_1 = \mathbf{0}$ and $2\mathbf{t}_0{}^\top \mathtt{E}^{01}\mathbf{t}_1 = d$. Let $\mathbf{w} = \mathtt{E}^{10}\mathbf{t}_0 + \mathtt{E}^{01}\mathbf{t}_1$. Then

$$\mathtt{R}_{sym}\mathbf{w} = \mathtt{R}_{sym}\mathtt{E}^{10}\mathbf{t}_0 + \mathtt{R}_{sym}\mathtt{E}^{01}\mathbf{t}_1 = \mathtt{E}^{01}\mathtt{R}_{sym}\mathbf{t}_0 + \mathtt{E}^{10}\mathtt{R}_{sym}\mathbf{t}_1 = \mathtt{E}^{01}\mathbf{t}_1 + \mathtt{E}^{10}\mathbf{t}_0 = \mathbf{w}.$$

On the other hand, by assumption $\mathbf{t}_1$ lies on a generator with direction vector $\mathbf{e}_1$. Mimicking part of the proof of Lemma 1 leads to the conclusion that $\mathbf{e}_1{}^\top \mathtt{E}^{01}\mathbf{t}_1 = 0$, and hence $\mathbf{e}_1{}^\top \mathbf{w} = 0$. Similarly $\mathbf{e}_0{}^\top \mathbf{w} = 0$. Consequently, up to scale, $\mathbf{w} = \mathbf{e}_1 \times \mathbf{e}_0$. As in the proof of Lemma 1, it follows that $\mathtt{R}_{sym}\mathbf{w} = -\mathbf{w}$, and so $\mathbf{w} = \mathtt{E}^{10}\mathbf{t}_0 + \mathtt{E}^{01}\mathbf{t}_1 = 0$ as required.

Finally, since $\mathbf{t}_0$ lies on S, it follows that $d = -\mathbf{t}_0{}^\top(\mathtt{E}^{01} + \mathtt{E}^{10})\mathbf{t}_0$. Using $\mathtt{E}^{10}\mathbf{t}_0 + \mathtt{E}^{01}\mathbf{t}_1 = 0$, it follows that $d = 2\mathbf{t}_0{}^\top \mathtt{E}^{01}\mathbf{t}_1$, and the proof is complete. ■

Number of conjugate configurations. This theorem gives us insight into how many conjugate configurations (that is, different essential matrices $\mathtt{E}_\mathtt{Q}$) exist for a given critical configuration. It was shown that the essential matrix $\mathtt{E} = \mathtt{E}_\mathtt{Q}$ is uniquely determined by the quadric S and the vanishing point of the generator containing $\mathbf{t}_1$ and $\mathtt{R}_{sym}\mathbf{t}_0$. The only possibility for there to exist two distinct essential matrices $\mathtt{E}_\mathtt{Q}$ is if $\mathbf{t}_1 = \mathtt{R}_{sym}\mathbf{t}_0$, in which case each of the generators through $\mathbf{t}_1$ leads to a different essential matrix $\mathtt{E}_\mathtt{Q}$.

7 A Condition for Ambiguity in 3 Views and More

Since Euclidean ambiguities are special cases of projective ambiguities, it is useful to have a (nearly) necessary and sufficient condition for ambiguity. Such a condition is given by the following theorem, which is a restatement of Theorem 1 and Corollary 1 of [4], in slightly simpler form.

Theorem 4. *Let* $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ *and* $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ *be two triplets of camera matrices. For each of the pairs (i,j)=(0,1),(0,2) and (1,2), let* $S^{ij}_{\mathtt{P}}$ *and* $S^{ij}_{\mathtt{Q}}$ *be the ruled quadric critical surfaces defined in (1) for camera pairs* $(\mathtt{P}^i, \mathtt{P}^j)$ *and* $(\mathtt{Q}^i, \mathtt{Q}^j)$*, respectively.*

(i) If there exist points $\mathbf{P}$ *and* $\mathbf{Q}$ *such that* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for all* $i = 0, 1, 2$ *then* $\mathbf{P}$ *must lie on the intersection* $S^{01}_{\mathtt{P}} \cap S^{02}_{\mathtt{P}} \cap S^{12}_{\mathtt{P}}$ *and* $\mathbf{Q}$ *must lie on* $S^{01}_{\mathtt{Q}} \cap S^{02}_{\mathtt{Q}} \cap S^{12}_{\mathtt{Q}}$*.*

(ii) Conversely, if $\mathbf{P}$ *is a point lying on the intersection of quadrics* $S^{01}_{\mathtt{P}} \cap S^{02}_{\mathtt{P}} \cap S^{12}_{\mathtt{P}}$*, but not satisfying the condition*

$$\begin{bmatrix} (e^{10}_{\mathtt{Q}} \times e^{20}_{\mathtt{Q}})^\top \mathtt{P}^0 \\ (e^{21}_{\mathtt{Q}} \times e^{01}_{\mathtt{Q}})^\top \mathtt{P}^1 \\ (e^{02}_{\mathtt{Q}} \times e^{12}_{\mathtt{Q}})^\top \mathtt{P}^2 \end{bmatrix} \mathbf{P} = 0, \tag{9}$$

where each $e^{ij}_{\mathtt{Q}}$ *is an epipole (the image of the camera centre of* $\mathtt{Q}^i$ *in the image formed by* $\mathtt{Q}^j$*), then there exists a point* $\mathbf{Q}$ *lying on* $S^{01}_{\mathtt{Q}} \cap S^{02}_{\mathtt{Q}} \cap S^{12}_{\mathtt{Q}}$ *such that* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for all* $i = 0, 1, 2$*.*

If a point $\mathbf{P}$ happens to satisfy the condition (9) then there may or may not be a conjugate point $\mathbf{Q}$. In a reasonable sense, most points lying on the intersection $S^{01}_{\mathtt{P}} \cap S^{02}_{\mathtt{P}} \cap S^{12}_{\mathtt{P}}$ are critical. Notice, however that if the three cameras $\mathtt{Q}^i$ are collinear, then each of the vector products $(e^{ij}_{\mathtt{Q}} \times e^{kj}_{\mathtt{Q}})$ vanishes, and so condition (9) is satisfied for all $\mathbf{P}$. In this case we can make no conclusion regarding the existence of a conjugate point $\mathbf{Q}$. However, if the three cameras $\mathtt{Q}$ are not collinear then we may say more.

Proposition 2. *Given the assumptions of Theorem 4, suppose further that the three cameras* $\mathtt{Q}^i$ *are distinct and non-collinear. Then any point satisfying the condition (9) must lie on the intersection of quadrics* $S^{01}_{\mathtt{P}} \cap S^{02}_{\mathtt{P}} \cap S^{12}_{\mathtt{P}}$*.*

Proof. Let i, j and k represent the three indices 0, 1 and 2 in some permuted order, i.e. $i \neq j \neq k$. If the three cameras are non-collinear, then for each j the cross product $e^{ij}_{\mathtt{Q}} \times e^{kj}_{\mathtt{Q}}$ is non-vanishing. Let $\mathbf{P}$ be a point satisfying (9). Then $(e^{ij}_{\mathtt{Q}} \times e^{kj}_{\mathtt{Q}})^\top(\mathtt{P}^j\mathbf{P}) = 0$, which implies that $\mathtt{P}^j\mathbf{P}$ lies in the span of $e^{ij}_{\mathtt{Q}}$ and $e^{kj}_{\mathtt{Q}}$, and so we write $\mathtt{P}^j\mathbf{P} = \alpha_{ij}e^{ij}_{\mathtt{Q}} + \alpha_{kj}e^{kj}_{\mathtt{Q}}$ for some constants α_{ij} and α_{kj}. Now, $\mathbf{P}$ lies on $S^{ij}_{\mathtt{P}}$ if and only if $\mathbf{P}^\top(\mathtt{P}^{i\top}\mathtt{F}^{ij}_{\mathtt{Q}}\mathtt{P}^j)\mathbf{P} = 0$. Substituting for $\mathtt{P}^i\mathbf{P}$ and $\mathtt{P}^j\mathbf{P}$ gives

$$\mathbf{P}^\top(\mathtt{P}^{i\top}\mathtt{F}^{ij}_{\mathtt{Q}}\mathtt{P}^j)\mathbf{P} = (\alpha_{ji}e^{ji}_{\mathtt{Q}} + \alpha_{ki}e^{ki}_{\mathtt{Q}})^\top\mathtt{F}^{ij}_{\mathtt{Q}}(\alpha_{ij}e^{ij}_{\mathtt{Q}} + \alpha_{kj}e^{kj}_{\mathtt{Q}}) = (\alpha_{ki}e^{ki}_{\mathtt{Q}})^\top\mathtt{F}^{ij}_{\mathtt{Q}}(\alpha_{kj}e^{kj}_{\mathtt{Q}}).$$

The last equality holds, because $e^{ji\top}_{\mathtt{Q}}\mathtt{F}^{ij}_{\mathtt{Q}} = \mathtt{F}^{ij}_{\mathtt{Q}}e^{ij}_{\mathtt{Q}} = 0$. Finally, $e^{ki\top}_{\mathtt{Q}}\mathtt{F}^{ij}_{\mathtt{Q}}e^{kj}_{\mathtt{Q}} = 0$, since $e^{ki}_{\mathtt{Q}}$ and $e^{kj}_{\mathtt{Q}}$ are a matching point pair in images i and j, corresponding to the camera centre of $\mathtt{Q}^k$. Thus, $\mathbf{P}^\top(\mathtt{P}^{i\top}\mathtt{F}^{ij}_{\mathtt{Q}}\mathtt{P}^j)\mathbf{P} = 0$ and so $\mathbf{P}$ lies on $S^{ij}_{\mathtt{P}}$. ■

The points $\mathbf{P}$ that satisfy (9) must be either a single point, a line or a plane lying in the intersection of the three quadrics $S^{ij}_{\mathtt{P}}$. If this quadric intersection does not contain a complete line or a plane, then the latter two cases are not possible. In addition, it may be shown by continuity that if (9) defines a single point, then this point must be critical (a conjugate $\mathbf{Q}$ exists) unless it is an isolated single point in the intersection of the $S_{\mathtt{P}}^{ij}$. We may therefore state a general ambiguity result:

Theorem 5. *Let* $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ *and* $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ *be two triplets of camera matrices, with cameras* $\mathtt{Q}^i$ *non-collinear. Then for any point* $\mathbf{P}$ *in the intersection* $S_{\mathtt{P}}^{01} \cap S_{\mathtt{P}}^{02} \cap S_{\mathtt{P}}^{12}$ *there exists a conjugate point* $\mathbf{Q}$ *satisfying* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for all* i*, with the possible exception of*

1. *A single isolated point* $\mathbf{P}$ *in* $S_{\mathtt{P}}^{01} \cap S_{\mathtt{P}}^{02} \cap S_{\mathtt{P}}^{12}$*, or*
2. *Points* $\mathbf{P}$ *on a single line or plane contained in the intersection* $S_{\mathtt{P}}^{01} \cap S_{\mathtt{P}}^{02} \cap S_{\mathtt{P}}^{12}$*.*

This theorem simplifies the search for critical configurations, since it is not necessary to worry about the points. It is sufficient to find sets of cameras that define quadric intersections of interest. If we are searching for critical calibrated configurations, then the two sets of cameras must of course be calibrated.

The question arises as to whether the exceptional conditions of Theorem 5 really occur (the points that are non-critical). It was shown in [9] that if the three quadrics intersect in 8 points then indeed one of these points (the exceptional point identified in Theorem 5) does not have a conjugate. For the case where the three quadrics intersect in a line, an example is given later in which the points on the line in fact do not have conjugates.

8 Euclidean Ambiguities in 3 Views or More

As seen in the previous section, calibrated critical configurations involving three views and seven points abound. It is natural to ask if calibrated critical configurations exist involving more than two views and infinite numbers of points. In the projective case, it has been shown that elliptic quartics (a fourth-degree curve given as the intersection of two quadrics) are critical for projective reconstruction [6]. The calibration information restricts the class of critical sets to a class which is strictly smaller than in the projective case. Still, we will show that for any three cameras, there exists an elliptic quartic through the three camera centres such that the points on the quartic and the three cameras form a critical configuration. First some properties of pencils of rectangular quadrics are given.

Lemma 2. *Let* S_1 *and* S_2 *be two rectangular quadrics with principal points* $(\mathbf{m}_1, \mathbf{n}_1)$ *and* $(\mathbf{m}_2, \mathbf{n}_2)$*, respectively.*

(i) There exists in general a third rectangular quadric in the pencil $\alpha S_1 + \beta S_2$ *.*
(ii) All the quadrics in the pencil $\alpha S_1 + \beta S_2$ *are rectangular if and only if (a) one of the principal points* $(\mathbf{m}_1, \mathbf{n}_1)$ *coincides with one of* $(\mathbf{m}_2, \mathbf{n}_2)$*, or (b) all four principal points are collinear.*

Proof. From Proposition 1 it follows that $\mathrm{tr}(M)/2$ is an eigenvalue of M. Thus, a necessary constraint for a rectangular quadric is that $\det[M - \frac{\mathrm{tr}(M)}{2} I] = 0$, which is also sufficient (provided the product of the two other eigenvalues is positive

- otherwise the principal points will be complex). Applying the constraint to $\alpha S_1 + \beta S_2$ yields

$$\alpha^2\beta \det\left[\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{m}_1\right] \det\left[\mathbf{m}_1\ \mathbf{m}_2\ \mathbf{n}_1\right] + \alpha\beta^2 \det\left[\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{m}_2\right] \det\left[\mathbf{m}_1\ \mathbf{m}_2\ \mathbf{n}_2\right] = 0.$$

This is a homogeneous polynomial constraint in (α, β) where two solutions are $(1, 0)$ and $(0, 1)$. Since it is a cubic constraint, there is always a third solution which proves (i). All quadrics in the pencil are rectangular if and only if the two coefficients of the polynomial vanish. It follows that (1) either $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m}_1)$ or $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{n}_1)$ are collinear and (2) either $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{m}_2)$ or $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{n}_2)$ are collinear, which occurs exactly in the two cases given by (ii) above. ■

Based on the observations in Theorem 3 and Lemma 2 we are now ready to prove the following result on critical configurations for calibrated cameras.

Theorem 6. *Given three calibrated cameras* $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$*, then there exists an elliptic quartic curve (given as the intersection of two quadrics) which contains the three camera centres and such that the points lying on the quartic curve and the three cameras constitute a critical configuration.*

Proof. According to Theorem 4, we need to find a triplet of conjugate cameras $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ which are calibrated and where the corresponding three quadrics $S^{01}_{\mathtt{P}}$,$S^{02}_{\mathtt{P}}$ and $S^{12}_{\mathtt{P}}$ are linearly dependent. Without loss of generality we can assume normalized cameras and that the two camera centres of $\mathtt{P}^0$ and $\mathtt{P}^1$ are given by $\mathbf{t}_0 = (0, 0, 0)^\top$ and $\mathbf{t}_1 = (0, 1, 0)^\top$, respectively.

An explicit solution to the problem will given, but first we will describe how the solution was discovered. Start with three general camera matrices $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ with $\mathtt{Q}^0 = \left[I \,|\, 0\right]$ and $\mathtt{Q}^i = \mathtt{R}^i_{\mathtt{Q}} \left[I \,|\, -\mathbf{t}_{\mathtt{Q},i}\right]$ for $i = 1, 2$. According to Theorem 2, the rotation axes of $\mathtt{R}^1_{\mathtt{Q}}$, $\mathtt{R}^2_{\mathtt{Q}}$ and $\mathtt{R}^{1\top}_{\mathtt{Q}}\mathtt{R}^2_{\mathtt{Q}}$ coincide with one of the two principal points in the quadrics $S^{01}_{\mathtt{P}}$,$S^{02}_{\mathtt{P}}$ and $S^{12}_{\mathtt{P}}$, respectively. At the same time, the pencil should be rectangular. By choosing a fix rotation axis, denoted by $\mathbf{m}$, for $\mathtt{R}^1_{\mathtt{Q}}$ and $\mathtt{R}^2_{\mathtt{Q}}$, implies that the rotation axis of $\mathtt{R}^{1\top}_{\mathtt{Q}}\mathtt{R}^2_{\mathtt{Q}}$ will also be $\mathbf{m}$. Furthermore, one of the principal points for $S^{01}_{\mathtt{P}}$,$S^{02}_{\mathtt{P}}$ and $S^{12}_{\mathtt{P}}$ will be $\mathbf{m}$ and hence the pencil spanned by $S^{01}_{\mathtt{P}}$ and $S^{02}_{\mathtt{P}}$ is rectangular according to Lemma 2. So ensuring these constraints is sufficient in order to generate a pencil of rectangular quadrics.

Now let $\mathbf{m} = (1, 0, 0)^\top$. Denote the camera centre coordinates of $\mathtt{P}^2$ with $\mathbf{t}_2 = (x, y, z)^\top$ and let $\mu = \sqrt{2(y - 1/2)^2 + 2(z + 1/2)^2}$ and $\nu = y^2 - y + z^2$. Straightforward calculations show that

$$\mathtt{Q}^1 = \begin{bmatrix} 1 & 0 & 0 & x(-2y+1) \\ 0 & 0 & 1 & -\nu \\ 0 & -1 & 0 & \nu \end{bmatrix}, \quad \mathtt{Q}^2 = \begin{bmatrix} \mu & 0 & 0 & \mu x(z - y - \mu + 1) \\ 0 & z - y + 1 & y + z & -(y+z)\nu \\ 0 & -z - y & z - y + 1 & \nu(y - z + \mu - 1) \end{bmatrix} \tag{10}$$

generates a pencil spanned by

$$S^{01}_{\mathtt{P}} = \begin{bmatrix} 0 & -2\nu & 0 & \nu \\ -2\nu & 2x(2y-1) & 0 & x(-2y+1) \\ 0 & 0 & 2x(2y-1) & 0 \\ \nu & x(-2y+1) & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$S_{\mathtt{P}}^{02} = \begin{bmatrix} 0 & -2\nu\xi & 0 & \nu(\nu+y\mu) \\ -2\nu\xi & 2x(y+z)\xi & 0 & -x(y+z)\xi \\ 0 & 0 & 2x(y+z)\xi & x\nu(-2y-\mu+1) \\ \nu(\nu+y\mu) & -x(y+z)\xi & x\nu(-2y-\mu+1) & 0 \end{bmatrix},$$

where $\xi = y - z + \mu - 1$ and $S_{\mathtt{P}}^{12} = \alpha S_{\mathtt{P}}^{01} + \beta S_{\mathtt{P}}^{02}$ for some $(\alpha, \beta) \in \mathcal{P}^1$. The pencil contains the three cameras centres $\mathbf{t}_0$, $\mathbf{t}_1$ and $\mathbf{t}_2$ and the whole intersection curve of the pencil (which is an elliptic quartic) is critical as the exception condition (9) contains in general only a single point, cf. Proposition 2 and Theorem 5. The above solution breaks down, when for example $\mu = 0$. By interchanging the roles of two cameras, say $\mathtt{P}^0$ and $\mathtt{P}^1$, will then generally produce a valid solution. ■

As the proof is constructive, it is easy to generate examples.

Example 1. Let $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ be three normalized cameras with centres $\mathbf{t}_0 = (0,0,0)^\top$, $\mathbf{t}_1 = (0,1,0)^\top$ and $\mathbf{t}_2 = (1,1,3)^\top$ lying on a pencil spanned by

$$S_1 = \begin{bmatrix} 0 & 18 & 0 & -9 \\ 18 & -2 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ -9 & 1 & 0 & 0 \end{bmatrix} \text{ and } S_2 = \begin{bmatrix} 0 & 0 & 0 & 18 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -9 \\ 18 & -1 & -9 & 0 \end{bmatrix}. \tag{11}$$

Let $g(t)$ be the solution to the quadratic equation $(2t+1)X^2 + (-18t+9)X - t - t^2 + 2t^3 = 0$, then the intersection curve can be written in homogeneous form $\mathbf{P}(t) = (g(t), t(2t+1), g(t)(2t+1), 2t+1)^\top$. Further, according to (10), the conjugate cameras are

$$\mathtt{Q}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathtt{Q}^1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -9 \\ 0 & -1 & 0 & 9 \end{bmatrix} \text{ and } \mathtt{Q}^2 = \begin{bmatrix} 5 & 0 & 0 & -10 \\ 0 & 3 & 4 & -36 \\ 0 & -4 & 3 & 18 \end{bmatrix}.$$

The corresponding quadrics $S_{\mathtt{P}}^{01}$, $S_{\mathtt{P}}^{02}$ and $S_{\mathtt{P}}^{12}$ lie in the pencil $\alpha S_1 + \beta S_2$. The elliptic quartic given by quadrics $S_{\mathtt{Q}}^{01}$ and $S_{\mathtt{Q}}^{02}$ can be parametrized by $\mathbf{Q}(t) = (2(t+5)(t-1), 2(t+5)(18t-9-(2t+1)g(t)), 2(t+5)(t-1)(2t-1), 2t^2+17t-10-(2t+1)g(t))^\top$. Finally, one verifies that $\mathtt{P}^i\mathbf{P}(t) = \mathtt{Q}^i\mathbf{Q}(t)$ (up to scale) for $i = 0, 1, 2$ and all t. Thus, the configuration is indeed critical.

Next, we wish to study critical configurations of $n > 3$ calibrated views.

Theorem 7. *A configuration of $n \geq 3$ calibrated cameras $\mathtt{P}^i$, $i = 0, ..., n-1$ and points $\mathbf{P}_j$ is critical if the set of cameras $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^k)$ and points $\mathbf{P}_j$ is critical with respect to some conjugate cameras $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^k)$ for $k = 2, ..., n-1$.*

Proof. We prove the result for 4 views. The general result for n views follows by induction. The three cameras $\mathtt{P}^0$,$\mathtt{P}^1$ and $\mathtt{P}^2$ along with the points form a critical set, and hence a conjugate configuration exists. Similarly a second conjugate configuration exists for the cameras $\mathtt{P}^0$,$\mathtt{P}^1$ and $\mathtt{P}^3$ and the points. The goal is to show that these two conjugate configurations are consistent.

By assumption, the conjugate pair $(\mathtt{Q}^0, \mathtt{Q}^1)$ is the same for both triplets $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ and $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^3)$. Denote the conjugate points by $\mathbf{Q}_j$ and $\mathbf{Q}'_j$ in the first and second triplet, respectively. Consider the way the conjugate points $\mathbf{Q}_j$ are obtained in the first triplet. From the image points in the first two views, one can determine the position of the conjugate points by triangulation using $\mathtt{Q}^0$ and $\mathtt{Q}^1$. However, and this is the main point, the third camera is not used in this construction. It follows that $\mathbf{Q}_j = \mathbf{Q}'_j$ and the theorem is proved. ■

Example 2. Consider again the pencil spanned by S_1 and S_2 in (11). Are there any additional camera positions in Example 1 for which the configuration remains critical? Yes, the following camera pair does not break the ambiguity:

$$\mathtt{P}^3 = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 9/5 \\ 0 & 0 & 1 & -78/5 \end{bmatrix} \quad \text{and} \quad \mathtt{Q}^3 = \begin{bmatrix} 5 & 0 & 0 & -45 \\ 0 & -4 & -3 & 27 \\ 0 & 3 & -4 & 81 \end{bmatrix},$$

How were these cameras discovered? Well, if a camera $\mathtt{P}^3$ is to be critical, Theorem 7 says that we only need to show that $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^3)$ and the points on the quartic curve are critical. The way to do that is by means of Theorem 4. Thus, the constraints that have to be satisfied are (i) the camera centre of $\mathtt{P}^3$ lies on both S_1 and S_2 and (ii) the quadrics $S^{ij}_{\mathtt{P}}$ in (1) for pairs $(\mathtt{P}^0, \mathtt{P}^3)$ and $(\mathtt{P}^1, \mathtt{P}^3)$, respectively, lie in the pencil $\alpha S_1 + \beta S_2$. Again, without loss of generality, one can assume that $\mathtt{P}^3$ is normalized. The only valid solution to this system of polynomial equations is the one given above.

In the uncalibrated case, n cameras with centres and points lying on an elliptic quartic are critical [6]. The previous example shows that this is not true in the calibrated case. One might suspect that there are only critical configurations with a finite number of cameras.

Example 3. Consider the pencil $\alpha S_1 + \beta S_2$ where

$$S_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}.$$

The intersection curve splits up into a line and a twisted cubic, where the line is the X-axis and the points on the twisted cubic can be parametrized by $\mathbf{P}(\theta) = (2\theta(2\theta^2 - 2\theta + 1), \theta^2(2\theta - 1), (2\theta - 1)(-\theta + 1), 2\theta^2 - 2\theta + 1)^\top$. Let

$$\mathtt{P}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathtt{P}^1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathtt{P}^2 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

A conjugate configuration is given by

$$\mathtt{Q}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathtt{Q}^1 = \begin{bmatrix} 2\sqrt{2} & 0 & 0 & 2(\sqrt{2}-1) \\ 0 & 2 & 2 & -1 \\ 0 & -2 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \mathtt{Q}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

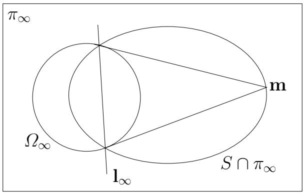

Fig. 1. Intersection of a rectangular quadric S with the plane at infinity π_∞ with principal point $\mathbf{m}$.

as the corresponding quadrics $S^{01}_{\mathtt{P}}$, $S^{02}_{\mathtt{P}}$ and $S^{12}_{\mathtt{P}}$ lie in the pencil $\alpha S_1 + \beta S_2$. However, the whole intersection curve is not critical. The exception condition in (9) consists of the X-axis and by inspection, one finds that there are no conjugate points for the X-axis. Thus, the points on the twisted cubic and the cameras $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ form a critical configuration. This can also be verified by direct computations: The conjugate points to the twisted cubic is a conic curve, which can be parametrized by $\mathbf{Q}(\theta) = (-4\theta^2 + 4\theta - 2, \theta(-2\theta+1), (2\theta-1)(\theta-1), 4\theta^2 - 4\theta + 2)^\top$ and $\mathtt{P}^i\mathbf{P}(\theta) = \mathtt{Q}^i\mathbf{Q}(\theta)$ for $i = 0, 1, 2$ and all θ.

Are there any additional camera positions for which the configuration remains critical? Yes, for any camera $\mathtt{P}$ lying on the twisted cubic there is a conjugate calibrated camera $\mathtt{Q}$,

$$\mathtt{P}(\eta) = \begin{bmatrix} 1\,0\,0 & -2\eta \\ 0\,1\,0 & \frac{\eta^2(-2\eta+1)}{2\eta^2-2\eta+1} \\ 0\,0\,1 & \frac{\eta(2\eta-1)(\eta-1)}{2\eta^2-2\eta+1} \end{bmatrix} \quad \text{and} \quad \mathtt{Q}(\eta) = \begin{bmatrix} \xi & 0 & 0 & \xi(-\xi+1) \\ 0 & -\eta+1 & \eta & -\eta^2 \\ 0 & -\eta & -\eta+1 & \eta(\eta-1) \end{bmatrix},$$

where $\xi = \sqrt{2(\eta - 1/2)^2 + 1/2}$. Notice that the camera centres in the conjugate configuration lie on a conic. In order to verify that the configuration is indeed critical, it is enough to check that the corresponding critical quadrics for pairs $(\mathtt{P}^0, \mathtt{P}(\eta))$ and $(\mathtt{P}^1, \mathtt{P}(\eta))$ lie in the pencil $\alpha S_1 + \beta S_2$ or, alternatively, that $\mathtt{P}(\eta)\mathbf{P}(\theta) = \mathtt{Q}(\eta)\mathbf{Q}(\theta)$ for all η, θ.

9 Conclusions

In this paper we have given a complete characterization of critical surfaces for two calibrated cameras. We have shown several new results on critical configurations for multiple views. For example, for any placement of three calibrated cameras there exists a critical elliptic quartic curve. Further, the existence of critical configurations containing arbitrarily many points and cameras have been shown, even though they are less frequent than in the uncalibrated case.

Appendix

Geometric interpretation. The definition of a rectangular quadric and its properties as stated in Proposition 1 are purely based on algebraic concepts. We will now give a more geometrically oriented characterization.

Proposition 3. *A principal point* **m** *of a rectangular quadric is contained in the intersection of the quadric with the plane at infinity with the following property: The tangents from* **m** *to the absolute conic meet the absolute conic at points lying on the quadric.*

In [8], this was used for defining a principal point. See Figure 1 for an illustration. This is still quite abstract. Before we give another interpretation, we need two simple facts about the absolute conic. For justification, refer to [10].

Proposition 4. *A planar conic is a circle if and only if it meets the plane at infinity at two (imaginary) points lying on the absolute conic.*

Now consider any line in space meeting the plane at infinity at a point **m**. The *polar* of **m** with respect to the absolute conic is the line $\mathbf{l}_\infty$ joining the two points of tangency from **m** to the absolute conic (see Figure 1). This line is the vanishing line of a plane perpendicular to the line first mentioned.

Proposition 5. *A plane and a line are perpendicular if and only if they meet the plane at infinity in a polar line-point pair with respect to the absolute conic.*

Now, refer to Figure 1. Let π be a plane that vanishes at the line $\mathbf{l}_\infty$ on π_∞. This plane meets the quadric S in a conic curve. At the plane at infinity π_∞, the quadric S, the absolute conic Ω_∞ and the plane π all meet. According to Proposition 4, this means that π and the quadric S meet in a circle.

The point **m** is the polar of the line $\mathbf{l}_\infty$ with respect to the absolute conic, and hence represents the vanishing direction perpendicular to the plane π. If **m** is a principal point of the quadric S, then it lies on S.

Proposition 6. *A quadric S is rectangular if there exists a plane that meets the quadric in a circle and such that the perpendicular direction to the plane is asymptotic to the quadric.*

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