Smooth Interfaces for Spectral Element Approximations of Navier-Stokes Equations

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Abstract. A smoothing technique is developed to calculate the interface conditions of spectral element method for solving the incompressible Navier-Stokes equations. The first derivative of spectral element solution at the interface is calculated by using only the adjacent element information. Numerical simulations of an incompressible laminar fluid flow through a 2 : 1 planar contraction channel are presented for various Reynolds numbers.

Keywords: Spectral element method, A least square method, Navier-Stokes equations, Contraction channel flow

1 Introduction

Spectral element methods are high-order weighted residual techniques for the solution of partial differential equations typically in computational fluid dynamics [6]. Their success in the recent past in simulating complex flows derives from the flexibility of the method in representing accurately non-trivial geometries while preserving the good resolution properties of spectral method [1]. In the spectral element simulations, both the geometry and the solution are described through smooth functions so that the spectral element methods can obtain exponential accuracy by fully exploiting that regularity [6]. There are numerous fluid dynamics applications, however, where either very steep gradient of solutions or even discontinuous solutions are presented, e.g., a fluid through a channel with abrupt symmetrical contraction, interfaces in multiphase flows, or free surfaces in a die swell. A straightforward application of the spectral element methods in these situations may cause numerical instability as large errors induced by the discontinuous propagate in each element and eventually render the solution with oscillations everywhere. One reason for this instability phenomenon is that the spectral element method only enforces C^0 continuity at interfaces between each element. There have been methods proposed in which continuity of the first

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derivatives at the element interfaces is maintained [4], but they have the disadvantage that knowledge of a solution is required across the entire domain. When used on a parallel computer this translates into inter-processor communication which requires an extra amount of time.

In the current work we attempt to develop a spectral element scheme to approximate the interface conditions for the Gauss-Lobatto-Legendre polynomial approximations to the solutions of Navier-Stokes equations, and a smoothing technique to calculate the interface conditions at each element. The main idea presented here is to modify the Gauss-Lobatto-Legendre polynomial basis of the spectral element formulation by using a least square reconstruct procedure implemented on the first derivatives at interfaces of each element, in which the interface values can be calculated by using only the information on the adjacent elements. As a result the proposed interfacial smooth technique is implemented on the Navier-Stokes equations based on a channel flow with a symmetric abrupt contraction.

The paper is organized as follows: in Section 2 we introduce the basic idea upon which the spectral element formulation is based. In Section 3 we describe a smooth method to examine the accuracy between the numerical and analytical solution for the first derivative of trigonometric periodic function. Finally, the numerical simulation of the flow in a symmetric contraction channel is presented. The size of the salient corner vortex and the shape of stream function contours show a good agreement with the work of Dennis *et al.* [3] and Karageorghis *et al.* [5]. The separation length and the strength of the vortex increase as Re increases after $Re > 50$, and the downstream recirculations have been seen in the streamlines.

2 Spectral element approximation

The spectral element method is high-order weighted residual technique for the approximation of partial differential equations that combines the generality of finite element method with the accuracy of spectral method. In this section we will briefly describe the spectral element method based on a simple one-dimensional Poisson equation defined by

$$-u_{xx} = f, \quad x \in I = [a, b], \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$u(a) = u(b) = 0, \tag{2}$$

where f is a given function. Using the Galerkin technique, equations (1)-(2) can now be characterized by the following variational problem:

Find $u \in H_0^1(I)$, such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(I), \tag{3}$$

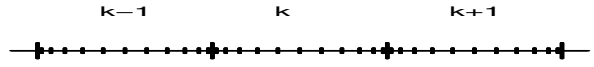

Fig. 1. Spectral element discretization in 1 D.

where

$$a(u,v) = \int_a^b u_x(x)v_x(x)dx, \quad (f,v) = \int_a^b f(x)v(x)dx,$$
$$H_0^1(I) = \{v | v \in L^2(I), v_x \in L^2(I), v(a) = v(b) = 0\},$$

$L^2(I)$ is the space of square integrable functions.

Following standard spectral element procedure, we begin, as usual, by introducing a family of partitions of I such that $I = \cup_{k=1}^{K} I^k,\ \forall k,\ l,\ k \neq l,\ I_k \cap I_l = \emptyset$, where K denotes the number of elements. In the development, we use N as the degree of the polynomial. Fig. 1 shows the spectral element discretization on one dimensional geometry. On elements $k-1$ and k the grid points are denoted $x_0^{k-1}, ..., x_N^{k-1}$ and $x_0^k, ..., x_N^k$, respectively. And then each element is mapped onto the parent element $\mathcal{I} = [-1, 1]$ by using the equation

$$x = \frac{L_k}{2}\xi^k + \frac{x_L^k + x_R^k}{2}, \tag{4}$$

where x_L^k and x_R^k denote the left and right coordinates of the elemental boundaries, L_k is the element length, and $\xi \in \mathcal{I}$. The interpolant of $u(x)$ in the kth element is then represented as

$$u^k(\xi) = \sum_{i=0}^{N} u_i^k h_i(\xi^k). \tag{5}$$

Here u_i^k are nodal values of u, and $h_i(\xi^k)$ are the basis functions corresponding to element k and node i, with property $h_i(\xi_j^k) = \delta_{ij}$. Expressions for these interpolants in terms of Chebyshev, Lengendre and other polynomials can be found in [2]. In this paper we choose the basis functions are the Gauss-Lobatto-Lengendre polynomials defined as

$$h_i(\xi) = -\frac{1}{N(N+1)L_N(\xi_i)} \frac{(1-\xi^2)L_N^{'}(\xi)}{\xi - \xi_i}, \quad \xi \in \mathcal{I}, \tag{6}$$

where $L_N(\xi)$ is the Legendre polynomial of degree N, $L_N^{'}(\xi) = dL_N(\xi)/d\xi$ and the ξ_i are the Gauss-Lobatto-Legendre collocation points. Furthermore, there also exists a unique set of positive real numbers, ρ_i, corresponding with ξ_i, $(0 \leq i \leq N)$, such that the integration rule

$$\int_{-1}^{1} \phi(\xi)d\xi = \sum_{i=0}^{N} \rho_i \phi(\xi_i), \tag{7}$$

is exact for all polynomials $\phi(x)$ of degree $\leq (2N-1)$ on the interval $[-1, 1]$. We follow the standard spectral element method [6], an expansion of the function $u(x)$ can be written in terms of elements as

$$u(x) = \sum_{k=1}^{K} u^k(x) = \sum_{k=1}^{K} \sum_{i=0}^{N} u_i^k h_i(x), \tag{8}$$

where u_i^k are the point values for element k, and x refers to the local coordinate. We require that the approximation function $u(x)$ is continuous through the interface of each element, i.e.,

$$u^k(x_N) = u^{k+1}(x_0), \quad 1 \leq k \leq K-1, \tag{9}$$

and also satisfies the essential boundary condition

$$u^1(x_0) = u^K(x_N) = 0. \tag{10}$$

Expansion (8) together with the boundary conditions (9) and (10) are now inserted into the weak formulation (3) and the discrete equations are then generated by choosing appropriate functions v which are unity at a point ξ_i and zero at all other Gauss-Lobatto-Legendre points. In matrix form the spectral element procedure for equation (3) can be written as

$$\mathbf{Au} = \mathbf{b}, \tag{11}$$

where $\mathbf{A}$ is the discrete Laplace operator and $\mathbf{b}$ is the right hand side vector with the boundary conditions.

It is clear that the approximation by spectral element discretization $u(x)$ defined by formulation (8) is only C^0 continuity and the first derivative may not continuous across each element. There has been a method proposed by Gottlieb *et al.* [4] in which continuity of the first derivatives at the sub-domain interfaces is maintained for the domain decomposition method. Following the same idea in [4], the element interfaces in the spectral element method can be calculated by enforcing C^1 continuity across each element interface, i.e., the first derivatives are calculated directly from formulation (8)

$$\frac{d}{dx}u(x) = \sum_{k=1}^{K} \frac{d}{dx}u^k(x) = \sum_{k=1}^{K} \sum_{i=0}^{N} u_i^k \frac{d}{dx}h_i(x), \tag{12}$$

and continuity of the first derivatives is enforced at each element interface

$$\frac{d}{dx}u^k(x_N) = \frac{d}{dx}u^{k+1}(x_0), \quad \forall k \in \{1, ..., K-1\}, \tag{13}$$

Equation (13) implies that

$$\sum_{i=0}^{N} u_i^k \frac{d}{dx}h_i(x_N) = \sum_{i=0}^{N} u_i^{k+1} \frac{d}{dx}h_i(x_0). \tag{14}$$

Although the continuity of the first derivative at each element interface is satisfied, the principal disadvantage of this enforcing method is that all of the elements are coupled together, which results in a relatively large amount of inter-processor communication and requires extra computer time to solve, in particular, in the parallel computational programs. The goal of this paper is to propose a simple modification for equation (14) so that the first derivative of the function $u(x)$ is almost continuous at each element interface but only using information from the adjacent elements in the approximation.

3 Smoothing interface method

In this section, we present a smoothing interface method (SIM) based on one-dimensional spectral element approximations. Let $u^{k-1}(x)$ and $u^k(x)$ denote the approximations to the unknown variable $u(x)$ on elements $k-1$ and k, respectively, and they can be represented as

$$u^{k-1}(x^{k-1}) = \sum_{i=0}^{N} u_i^{k-1} h_i(x^{k-1}), \quad u^k(x^k) = \sum_{i=0}^{N} u_i^k h_i(x^k), \tag{15}$$

where u_i^{k-1} and u_i^k are the values of $u(x)$ in the grid points x_i^{k-1} and x_i^k, respectively. Let $d^{k-1}(x)$ and $d^k(x)$ denote the first derivatives of $u^{k-1}(x)$ and $u^k(x)$, they can be written as

$$d^{k-1}(x) = \sum_{i=0}^{N} d_i^{k-1} h_i(x^{k-1}), \quad d^k(x) = \sum_{i=0}^{N} d_i^k h_i(x^k), \tag{16}$$

where d_i^{k-1}, d_i^k are the values of the first derivatives of the variable $u(x)$ at the local nodes x_i^{k-1} and x_i^k, respectively. The idea of the smoothing method is to find an alternative approach to calculate d_i^{k-1} and d_i^k by using the information from the adjacent elements $k-1$ and k only. We wish to determine the best values for d_i^k so that the deviations among d_i^k and $\frac{d}{dx}u(x_i^k)$ are minimized. It turns out that to find a functional Φ such that

$$\min \ \Phi(d_0^{k-1}, d_1^{k-1}, ..., d_N^{k-1}, d_0^k, ..., d_N^k, \lambda), \tag{17}$$

where the functional Φ is given by

$$\Phi = \int_{k-1} \left((d^{k-1}(x) - \frac{d}{dx} u^{k-1}(x) \right)^2 dx + \int_k \left(d^k(x) - \frac{d}{dx} u^k(x) \right)^2 dx$$

$$+ \ \lambda(d_N^{k-1} - d_0^k), \tag{18}$$

and λ is a parameter. If we substitute $d^k(x)$ as defined by equation (16) and $du^k(x)/dx$ as defined by equation (14) into equation (18), we get

$$\Phi = \int_{k-1} \left(\sum_{i=0}^{N} d_i^{k-1} h_i(x) - \sum_{i=0}^{N} u_i^{k-1} \frac{d}{dx} h_i(x) \right)^2 dx$$

$$+ \int_{k} \left(\sum_{i=0}^{N} d_i^{k} h_i(x) - \sum_{i=0}^{N} u_i^{k} \frac{d}{dx} h_i(x) \right)^2 dx + \lambda (d_N^{k-1} - d_0^k). \tag{19}$$

We observe that Φ is an ordinary function of the unknowns d_i^{k-1} and d_i^k as reflected in our notation. To minimize Φ, we need only to take its partial derivatives with respect to each unknown d_i^{k-1} (as well as d_i^k) and set to zero. This implies that, at the minimum, all the partial derivatives $\partial\Phi/\partial d_0^{k-1}, \ldots, \partial\Phi/\partial d_N^{k-1}$ and $\partial\Phi/\partial d_0^k, \ldots, \partial\Phi/\partial d_N^k$ vanish. Writing the equations for these gives $2(N+2)$ equations:

$$\frac{\partial \Phi}{\partial d_i^{k-1}} = 2 \int_{k-1} \left(\sum_{i=0}^{N} d_i^{k-1} h_i(x) - \sum_{i=0}^{N} u_i^{k-1} \frac{d}{dx} h_i(x) \right) h_i(x) dx = 0, \quad i = 0, ..., N-1,$$

$$\frac{\partial \Phi}{\partial d_N^{k-1}} = 2 \int_{k-1} \left(\sum_{i=0}^{N} d_i^{k-1} h_i(x) - \sum_{i=0}^{N} u_i^{k-1} \frac{d}{dx} h_i(x) \right) h_N(x) dx + \lambda = 0,$$

$$\frac{\partial \Phi}{\partial d_i{}^k} = 2 \int_{k} \left(\sum_{i=0}^{N} d_i^{k} h_i(x) - \sum_{i=0}^{N} u_i^{k} \frac{d}{dx} h_i(x) \right) h_i(x) dx = 0, \quad i = 1, ..., N,$$

$$\frac{\partial \Phi}{\partial d_0{}^k} = 2 \int_{k} \left(\sum_{i=0}^{N} d_i^{k} h_i(x) - \sum_{i=0}^{N} u_i^{k} \frac{d}{dx} h_i(x) \right) h_0(x) dx - \lambda = 0.$$

Solving these equations by using the integration formulation (7), we obtain

$$d_i^{k-1} = \frac{1}{J^{k-1}} \sum_{j=0}^{N} u_j^{k-1} \frac{d}{d\xi} h_j(\xi_i), \quad i = 0, ..., N-1, \tag{20}$$

$$d_i^{k} = \frac{1}{J^{k}} \sum_{j=0}^{N} u_j^{k} \frac{d}{d\xi} h_j(\xi_i), \quad i = 1, ..., N, \tag{21}$$

$$d_N^{k-1} = d_0^k = \frac{1}{J^{k-1} + J^k} \sum_{j=0}^{N} \left(u_j^{k-1} \frac{d}{d\xi} h_j(\xi_N) + u_j^k \frac{d}{d\xi} h_j(\xi_0) \right), \tag{22}$$

where J^{k-1} and J^k are the values of Jacobian that come from mapping subdomains Ω_{k-1} and Ω_k onto the parent element $\chi^2 = [-1, 1]$. With the above equations (20)-(22), the interface of elements can be calculated in the sense of

least-square approximations by using only information from the adjacent elements $k-1$ and k. It is important to realize that the deviations squared of the first derivatives should continually decrease as the degree of the polynomial is raised. If the approach is implemented on a parallel computer the only communication that is required is between adjacent processors.

Before leaving this section, we show some numerical experiments which illustrate the accuracy of the spectral element method with smoothing interface strategy. We computed the first derivatives of the test function

$$f(x) = |\sin(x)|, \quad -\pi \le x \le \pi. \tag{23}$$

Its first derivative is

$$\frac{d}{dx}f(x) = \begin{cases} -\cos(x), & -\pi \le x \le 0, \\ \cos(x), & 0 \le x \le \pi. \end{cases}$$

Clearly, the first derivative function $df(x)/dx$ is discontinuous at $x = 0$. How well does the smoothing interface method do on such a function? Fig. 2 shows the exact and numerical solutions for the first derivatives $df(x)/dx$ with different numbers of Gauss-Lobatto-Legendre collocation points as well as different numbers of the elements. As illustrated in Fig. 2, the numerical solutions agreed well with the exact solutions when the smoothing interface technique is used. It is also important to note that the smoothing solutions do not have any oscillations at $x = 0$ where the first derivative function $df(x)/dx$ is discontinuous. Table 1 shows that the numerical solutions calculated with and without smooth interface technique by using four elements and twenty-four collocation points on each element. The data show that both algorithms performed well for the discontinuous function $df(x)/dx$. It is quite self-evident that the smoothing solutions (SIM) are more accurate than the solutions without smoothing technique (SEM). This can be seen in Table 1 when $x = -\pi/2$ and $x = \pi/2$ the error for the SIM solutions is 10^{-13}, but it is only 10^{-6} for the SEM solutions.

Table 1. A comparison among the analytical, SIM and SEM solutions with 24 collocation points and 8 elements.

x	Analytical solution of $f'(x)$	SIM solution of $f'(x)$	SEM solution of $f'(x)$
$-\frac{3}{4}\pi$	$.70710670E+00$	$.70710670E+00$	$.70710680E+00$
$-\frac{1}{2}\pi$	$-.75497900E-07$	$-.75497890E-07$	$.36091110E-07$
$-\frac{1}{4}\pi$	$-.70710680E+00$	$-.70710680E+00$	$-.70710670E+00$
$\frac{1}{4}\pi$	$.70710680E+00$	$.70710680E+00$	$.70710690E+00$
$\frac{1}{2}\pi$	$.75497900E-07$	$.75497890E-07$	$.18708690E-06$
$\frac{3}{4}\pi$	$-.70710670E+00$	$-.70710670E+00$	$-.70710660E+00$

4 Numerical results

In this section, numerical results are presented for the incompressible Navier-Stokes flows in a 2 : 1 contraction channel by using the spectral element method

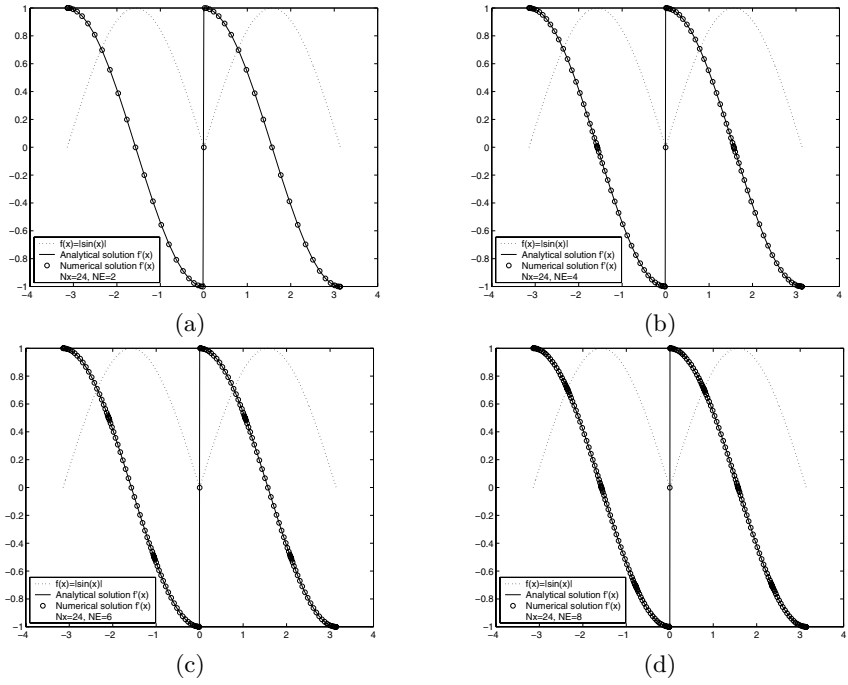

Fig. 2. Numerical solutions with the smoothing interface technique compare with the analytical solutions by using different numbers of collocation points (Nx) and elements (NE).

with smoothing interface technique. The incompressible Navier-Stokes equations are given by:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{Re} \triangle \mathbf{u} + \nabla p = \mathbf{f}, \tag{4.6}$$

$$-\nabla \cdot \mathbf{u} = 0, \tag{4.7}$$

where $\mathbf{u} = (u, v)$ is the velocity, p is the pressure, $\mathbf{f}$ is a given function, and $Re = \rho UL/\eta$ is Reynolds number defined the properties of the fluid.

4.1 Planar contraction flow

The 2 : 1 contraction channel flow geometry is shown in Fig. 3. Here the fluid enters upstream in the channel as a fully developed parabolic profile, and exits far downstream as a flat liquid sheet. The 2 : 1 planar contraction channel is chosen in order to compare with the results already published in Dennis *et al.* [3] and Karageorghis *et al.* [5]. In such a geometrical channel, the height of the inflow half channel is taken to be unit and the height of the outflow channel is $a = 1/2$, and the total length of the channel is 4.

Two non-uniform different meshes shown in Fig. 4 were used in the numerical simulations. Mesh 1 has three elements, twelve collocation points in the x direction and six collocation points in the y direction in each element, while there are

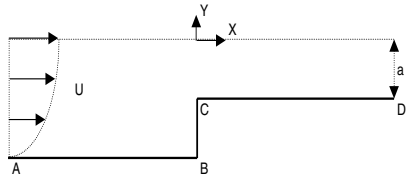

Fig. 3. The 2 : 1 planar contraction flow geometry.

three elements, sixteen and eight collocation points in the x and y directions on each element, respectively, in Mesh 2.

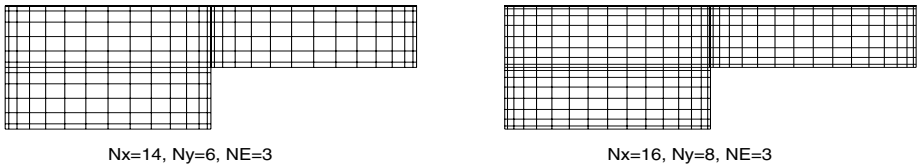

Fig. 4. Meshes for the 2 : 1 planar contraction channel problem.

Table 2 shows the length l_1 and the width l_2 of the salient corner vortex with different Reynolds numbers on Mesh 1 and Mesh 2. We can see that the size of corner vortex diminishes as Re increases from $Re = 0$ to $Re = 50$, and then begins to grow slowly with $Re > 50$. In Table 3, we compare the values of l_1 and l_2 for various Reynolds numbers on Mesh 2 with the results obtained by Dennis *et al.* [3] and Karageorghis *et al.* [5]. Table 3 shows that our numerical results are in good agreement with those results. Also, we find that l_1 grows more quickly than l_2 when Re increases from 50 to 200, which implies that the corner vortex grows in size along the upstream channel more quickly than up the wall at $x = 0$ as Re increases.

		$Re = 0$	$Re = 1$	$Re = 10$	$Re = 50$	$Re = 100$	$Re = 150$	$Re = 200$
l_1	Mesh 1	0.2548	0.2201	0.1477	0.1280	0.1379	0.1576	0.1870
	Mesh 2	0.2610	0.2399	0.1545	0.1077	0.1389	0.1467	0.1825
l_2	Mesh 1	0.3086	0.2793	0.1767	0.1215	0.1328	0.1474	0.1621
	Mesh 2	0.3181	0.2954	0.1706	0.1396	0.1298	0.1494	0.1706

Table 2. Values of l_1 and l_2 for various Re numbers on the Mesh 1 and Mesh 2.

Contours of the stream function for $Re = 0, 10, 50, 100, 150, 200$ on Mesh 2 are plotted in Fig. 5. The streamline plots give a qualitatively satisfactory to the flow solutions, we can see that the eddy of recirculation is happened for the Stokes flow at $Re = 0$, and as Re increases from zero, the length of corner vortex l_1 initially decreases until at $Re = 50$. For higher value Re number, the eddy length l_1 increases monotonically with Re. This phenomenon shows that the vortex develops as Re is increased.

		$Re=0$	$Re=1$	$Re=10$	$Re=50$	$Re=100$	$Re=150$	$Re=200$
l_1	(a)	0.2610	0.2399	0.1545	0.1077	0.1389	0.1467	0.1825
	(b)	0.268	0.243	0.153	0.128	0.138	–	–
	(c)	0.285	–	0.150	0.129	0.143	0.160	0.183
l_2	(a)	0.3181	0.2954	0.1706	0.1396	0.1298	–	–
	(b)	0.311	0.281	0.163	0.124	0.122	–	–

Table 3. The length and the width of the salient corner vortex for (a) smoothing interface spectral element method; (b) finite-difference scheme of Dennis *et al.* [3]; (c) spectral collocation method of Karageorghis *et al.* [5].

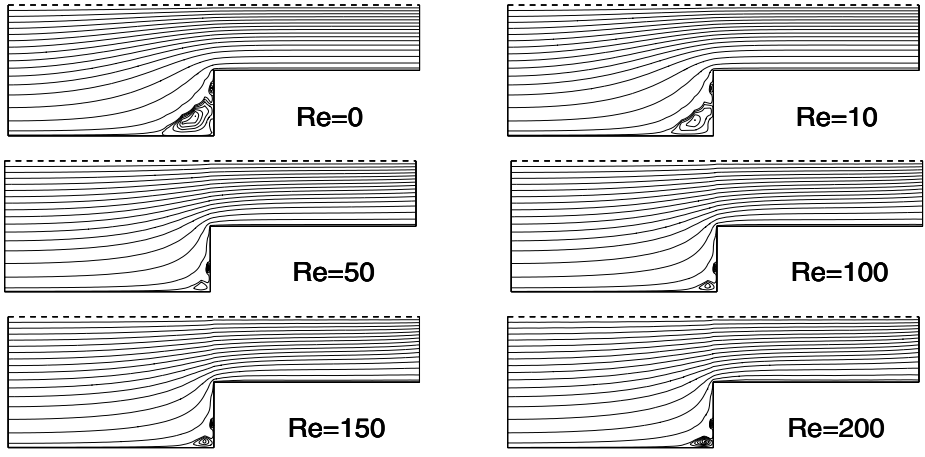

Fig. 5. Flow streamlines for various Re numbers on Mesh 2.

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