Drawing Outer-Planar Graphs in $O(n \log n)$ Area

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Abstract. In this paper, we study drawings of outer-planar graphs in various models. We show that $O(n \log n)$ area can be achieved for such drawings if edges are allowed to have bends or if vertices may be represented by boxes. The question of straight-line grid-drawings of outer-planar graphs in $o(n^2)$ area remains open.

1 Introduction

A planar graph is a graph that can be drawn without crossing. Fáry, Stein and Wagner [Fár48,Ste51,Wag36] proved independently that every planar graph has a drawing such that all edges are drawn as straight-line segments. De Fraysseix, Pach and Pollack [FPP90], and independently Schnyder [Sch90] established that in fact $O(n^2)$ area suffices for a straight-line drawing of an n-vertex planar graph, with vertices placed at grid points. This is asymptotically optimal, since there are planar graphs that need $\Omega(n^2)$ area [FPP88].

A number of other graph drawing models (e.g., poly-line drawings, orthogonal drawings, visibility representations) exist for planar graphs. See Section 2 for precise definitions. In all these models, $O(n^2)$ area can be achieved for planar graphs, see for example [Kan96,FKK97,Wis85]. On the other hand, $\Omega(n^2)$ area is needed, even in these models, for the graph in [FPP88].

Our paper was motivated by the question whether an area of $o(n^2)$ is possible, at least in some models, for subclasses of planar graphs such as trees, series-parallel graphs, and outer-planar graphs. Every tree has a straight-line drawing in $O(n \log n)$ area [Shi76], and in $O(n)$ area if the maximum degree is asymptotically smaller than n [GGT96]. On the other hand, one can easily construct a series-parallel graph that requires $\Omega(n^2)$ area if the planar embedding must be preserved (see also Fig. 11). The question of whether every series-parallel graph can be drawn in $o(n^2)$ area when the planar embedding may be changed appears to be open. See [DBETT98] for many other upper and lower bounds regarding drawings of trees and series-parallel graphs under special restrictions.

Not many drawing results are known for outer-planar graphs. Leiserson [Lei80] showed that outer-planar graphs have an orthogonal point-drawing in $O(n)$ area

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if the maximum degree is at most 4. By splitting vertices of higher degrees, and combining them in the resulting drawing after replacing each grid-line with Δ grid-lines (where Δ is the maximum degree), one can obtain orthogonal box-drawings of area $O(\Delta^2 n)$. However, this is an improvement over the $O(n^2)$ bound only if Δ is small.

In another result, any n points in general position can be used for a straight-line drawing of any outer-planar graph [CU96]. However, $O(n^2)$ area is needed to create n grid points in general positions, so this result does not lead to smaller area bounds either.

In this paper, we provide the following results:

- Every outer-planar graph has an orthogonal box-drawing with area $O(n \log n)$.
- By converting this drawing, it follows easily that every outer-planar graph has a poly-line grid drawing in area $O(n \log n)$.
- Every outer-planar graph has a visibility representation in $O(n \log n)$ area; however, not all vertices are necessarily drawn on the outer-face.

The question whether $o(n^2)$ area is possible for straight-line drawing remains open. In fact, not even any lower bounds better than the obvious $\Omega(n)$ appear to be known; we provide some partial results for lower bounds as well.

2 Definitions

Let $G = (V, E)$ be a graph with $n = n(G) = |V|$ vertices. We assume that G is *simple*, i.e., it has no loops and multiple edges. Throughout this paper, we will assume that G is planar, i.e., that G can be drawn without crossing. Such a planar drawing can be characterized by the cyclic order of edges around each vertex. A planar drawing splits the plane into connected pieces; the unbounded piece is called the *outer-face*, all other pieces are called *interior faces*.

An *outer-planar graph* is a planar graph that can be drawn such that all vertices are incident to the outer-face. Throughout this paper we will assume that G is outer-planar; in fact, we will assume that it is *maximal outer-planar*, i.e., no edge can be added to G without destroying simplicity or outer-planarity. (If G is not maximal outer-planar, simply add edges until it is, draw the resulting graph, and delete the extra edges from the drawing.) A maximal outer-planar graph consists of an n-cycle with chords and every interior face is a triangle.

We use the following drawing models (see also Fig. 1):

- *Straight-line drawings*: Vertices are points, all edges are straight-line segments.
- *Poly-line drawings*: Vertices are points, all edges are contiguous sequences of straight-line segments.
- *Orthogonal point-drawings*: A poly-line drawing where all edge segments are horizontal or vertical. Such drawings exist only if the maximum degree is at most 4.
- *Orthogonal box-drawings*: Vertices are (axis-parallel) boxes, edges are contiguous sequences of horizontal or vertical segments.[1]

[1] In this paper, the term "box" is referring to an axis-parallel box.

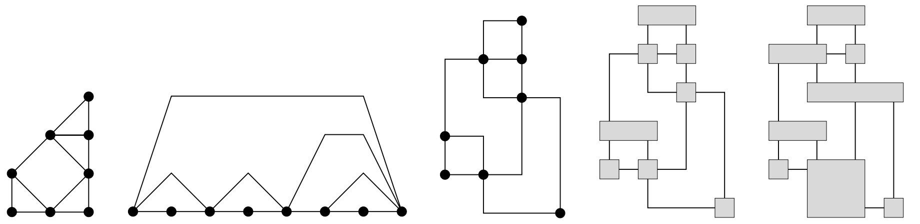


Fig. 1. The same graph in a straight-line drawing, a poly-line drawing, an orthogonal point-drawing, an orthogonal box-drawing, and a visibility representation.

– *Visibility representations*: Vertices are boxes, edges are horizontal or vertical segments.

For a planar graph, such drawings should be planar, i.e., have no crossing. We also assume that all defining features have integral coordinates; in particular points of vertices and transition-points (*bends*) in the routes of edges have integral coordinates, and boxes of vertices have integral corner points. We allow boxes to be degenerate, i.e., to be line segments or points.

The *width* of a box is the number of vertical grid lines (*columns*) that are occupied by it. The *height* of a drawing is the number of horizontal grid lines (*rows*) that are occupied by it. A drawing whose minimum enclosing box has width w and height h is called an $w \times h$-drawing, and has *area* $w \cdot h$.

3 Orthogonal Box-Drawings

Let $G = (V, E)$ be a maximal outer-planar graph. In this section, we show how to construct an orthogonal box-drawing of G in a grid of area $O(n \log n)$. For the remainder of this section, "drawing" will refer to "orthogonal box-drawing".

We use induction on the number of vertices with the following induction hypothesis (see also Fig. 2): Let (u, v) be an edge on the outer-face of G, with v after u in clockwise order on the outer-face. Then G has a drawing such that

- u occupies the top right corner of the drawing,
- v occupies the bottom right corner of the drawing,
- vertices have height 1, i.e., each vertex intersects only one row,[2]
- edges that attach horizontally at an endpoint have no bend.

In the base case ($n = 2$), simply place u atop v; see Fig. 2. The conditions of the induction hypothesis are clearly satisfied.

For $n \geq 3$, let w be the third vertex on the interior face incident to (u, v). Let G_1 be the graph induced by the vertices between w and u in clockwise order on the outer-face (we call this the *graph attached to edge* (u, w); see also Fig. 3). Let G_2 be the subgraph attached to (w, v). Note that G_1 or G_2 may contain only one edge. We will for now assume that $n(G_1) \leq n(G_2)$; the other case will be treated later.

[2] In our drawings, we "fatten" vertices for better visibility.

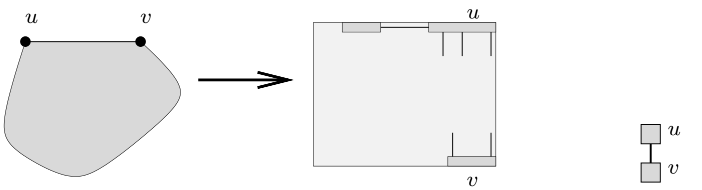


Fig. 2. Illustration of the induction hypothesis, and the base case $n = 2$.

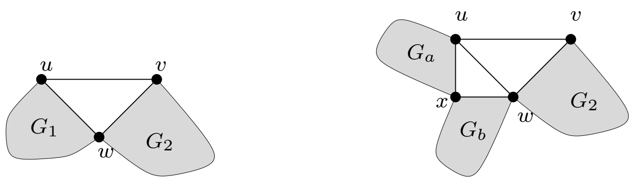


Fig. 3. Breaking the graph into parts (and breaking G_1 further in Case 2).

We have two sub-cases:

Case 1: If G_1 has only two vertices u and w, then recursively draw G_2 with respect to edge (w, v). Place u to the right of w, extend v, and draw the new edge as straight line. See Fig. 4. One can easily verify the induction hypothesis.

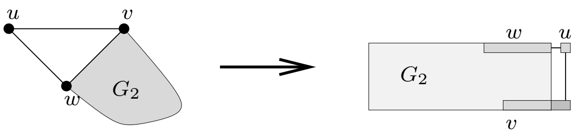


Fig. 4. The case when G_1 consists of only one edge.

Case 2: If G_1 contains at least three vertices, then let x be the third vertex on the face other than $\{u, v, w\}$ incident to (u, w). Let G_a and G_b be the two subgraphs attached to edges (u, x) and (x, w); see also Fig. 3. Recursively draw G_a, G_b and G_2, with respect to edges (u, x), (x, w) and (w, v), respectively, and let Γ_a, Γ_b and Γ_2 be the corresponding drawings.

We now modify Γ_a to achieve an additional property. By the induction hypothesis, vertex x occupies the bottom right corner of Γ_a. We say that x *spans the bottom of* Γ_a if additionally x occupies the bottom left corner, i.e., if x occupies the complete bottom row of Γ_a. As we show now, we can always modify Γ_a by adding at most one row so that this holds. For if x does not occupy the bottom left corner of Γ_a, then add one more row below Γ_a, move x into it and expand x so that it occupies the whole width of the drawing. Now re-route the incident edges of x. All edges that attach vertically at x can simply be extended.

If an edge (z, x) attaches horizontally at x, then (by the induction hypothesis) this edge has no bend. Since x only intersects one row by induction hypothesis, this implies that z occupies part of the last row of Γ_a. Therefore, we can route edge (z, x) vertically from z to x, which is now one row below. See also Fig. 5.

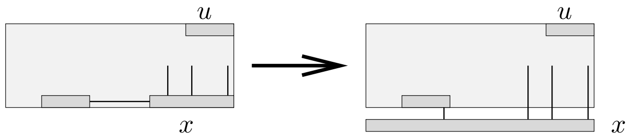


Fig. 5. Modifying Γ_a into Γ'_a.

Let Γ'_a be the drawing that we obtain from Γ_a by making x span the bottom. In a similar fashion, we modify Γ_b such that x spans the top, i.e., x occupies both top corners. (By induction hypothesis x already occupies the top right corner of Γ_b.) Call this new drawing Γ'_b.

Now we obtain a drawing of G by placing Γ_2, Γ'_b and Γ'_a next to each other. See Fig. 6 for an illustration of the following process. Stretch Γ'_a or Γ'_b, if necessary, until both drawings have the same height. (Note that any orthogonal box-drawing can be stretched by inserting empty rows.) Next, stretch Γ_2, if necessary, such that it is at least two units taller than Γ'_a and Γ'_b. Now place Γ_2, then Γ'_b (turned upside down) and then Γ'_a such that their tops are aligned.

By the placement, the two boxes of w are adjacent in the same row and can simply be rejoined. Since Γ'_a and Γ'_b have the same height, and since x spans both drawings Γ'_a and Γ'_b, the boxes of x are also adjacent in the same row and can be rejoined. Since Γ_2 is at least two units taller than Γ'_a, there are two empty rows underneath Γ'_a and Γ'_b. We use these to extend v to the right bottom corner and to route edge (w, u) with two bends. Edge (u, v) is a vertical line in the rightmost column. One easily verifies the induction hypothesis.

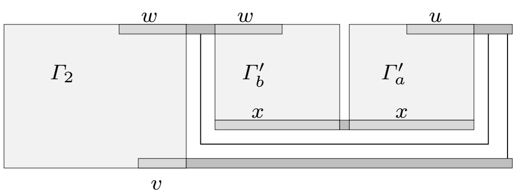


Fig. 6. Combining Γ'_a, Γ'_b and Γ_2.

Thus we are done with the case $n(G_1) \leq n(G_2)$. The case $n(G_1) > n(G_2)$ is symmetric; we do not give the details here and refer to Fig. 7.

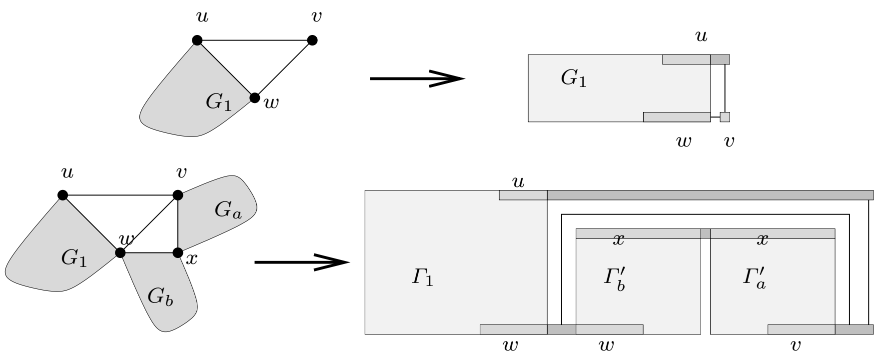


Fig. 7. The construction if $n(G_1) > n(G_2)$.

Theorem 1. *Every outer-planar graph has an orthogonal box-drawing such that the following holds:*

- *The height is at most* $3 \log n - 1$*,*
- *the width is at most* $\frac{5}{2}n - 4$*,*
- *the planar embedding is respected,*
- *the total number of bends is at most* $n - 2$*,*
- *every edge has at most two bends, every edge on the outer-face has no bends.*

Proof. The construction is given above. The bounds on the width, height and number of bends are easily verified in the base case and in Case 1 of the recursion, so we will only focus on Case 2. Let $G_1, G_2, G_a, G_b, \Gamma_2, \Gamma_a, \Gamma_b, \Gamma'_a$ and Γ'_b be defined as before.

Since $n(G_1) \leq n(G_2)$, we have $n(G_a) \leq \frac{n}{2}$. Therefore by induction Γ_a has height at most $3 \log n(G_a) - 1 \leq 3 \log n - 4$. We add one grid line to obtain Γ'_a, so Γ'_a has height at most $3 \log n - 3$. Similarly Γ'_b has height at most $3 \log n - 3$. Finally, Γ_2 has height at most $3 \log n - 1$ by induction. When stretching the drawings to achieve suitable heights, we will not exceed these bounds. Finally, during the merging process (Fig.6), we use two more grid lines near Γ'_a and Γ'_b, resulting in a total height of at most $3 \log n - 1$.

The width of the drawing is the combined width of Γ_2, Γ_a and Γ_b, with three more columns added. Since $n(G_2) + n(G_a) + n(G_b) = n + 2$, and using induction, the width is at most

$$3 + \frac{5}{2}n(G_2) - 4 + \frac{5}{2}n(G_a) - 4 + \frac{5}{2}n(G_b) - 4 = 3 + \frac{5}{2}(n+2) - 12 = \frac{5}{2}n - 4.$$

Similarly, the total number of bends is at most

$$2 + n(G_2) - 2 + n(G_a) - 2 + n(G_b) - 2 = 2 + (n+2) - 6 = n - 2.$$

Every edge receives at most 2 bends, if it is edge (w, u). Note in particular that this can be the case only for a chord; any edge on the outer-face of G is drawn as a straight line.

4 Poly-line Drawings

From any orthogonal box-drawing, we can easily obtain a poly-line drawing with asymptotically the same area as follows: Add empty grid-lines until every segment of every edge has length at least 2. Now for every vertex v, create a point for v at an arbitrary grid point inside the box of v. For each incident edge e of v, re-route e to end at this point by placing a new bend (if needed) at the grid-line incident to the box of v. (Such a grid-line must exist, and does not contain any other vertex, since e had length at least 2 by assumption.) See Fig. 8 for an illustration.

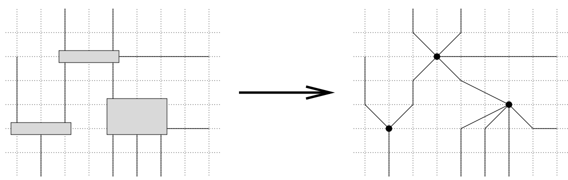


Fig. 8. Converting to a point-drawing.

If we start with the orthogonal box-drawing obtained in Section 3, then by analyzing the construction more carefully, we can improve this general construction as follows:

- We only need to add $\log n$ new rows, since only in the base case and in the construction in Fig. 5 vertical edge segments of length 1 can be created. More precisely, we can show by induction on the number of vertices similarly as before that there exists an orthogonal box-drawing of height $4 \log n - 1$ and width $\frac{5}{2}n - 4$ such that all vertical edges segments have length at least 2. This is obtained by adding one extra row with the base case, and one extra row whenever we need to modify the drawing to make a vertex span the bottom or top of a drawing.
- We need not add new columns, since every vertex has height 1, and therefore horizontal edge segments can simply be extended without adding a bend.
- We can achieve at most three bends per edge with a good choice of grid points for vertices. Note that every edge obtains at most one extra bend at every endpoint, so we only have to be careful for edges that had two bends in the orthogonal drawing. By construction, this happens only for an edge (w, u) added during Case 2 of the recursive construction.
  If (w, u) is such an edge, with say w as the left endpoint (see also Fig. 6), then place the grid point for w at the column where edge (w, u) attaches to w. Note that every vertex is at most once the left endpoint of an edge with two bends, so this is well-defined. Edge (w, u) now does not receive an extra bend at w, and hence has at most three bends as well.

Combining the bounds of Theorem 1 with the above observations, we obtain the following results:

Theorem 2. *Every outer-planar graph has a poly-line grid-drawing in a* $(\frac{5}{2}n - 4) \times (4 \log n - 1)$*-grid such that all vertices are on the outer-face. Every edge has at most 3 bends.*

For an outer-planar graph G with maximum degree $\Delta \leq 4$, we can similarly obtain an orthogonal point-drawing with area $O(n \log n)$, by starting with an orthogonal box-drawing, adding rows and columns to make vertex boxes sufficiently big, and then replacing each box with a subgraph to connect all edges to a point. However, note that this is not an improvement over the known $O(n)$ area bound for orthogonal point-drawings of outer-planar graphs with maximum degree 4 [Lei80].

5 Visibility Representations

In this section, we show that by changing the planar embedding we can avoid the bends in Fig. 6 and obtain a visibility representation.

The construction is very similar to the one in Section 3. No bends are added in either the base case or Case 1 of the recursion. The only change to the construction thus occurs in Case 2 of the recursion.

Define $G_a, G_b, G_2, \Gamma_a, \Gamma_b$ and Γ_2 as in Case 2 of Section 3. Recall that u occupies the top right and x occupies the bottom right corner of Γ_a. We showed that by adding one row, we can modify Γ_a such that x spans the bottom of the drawing. By applying this modification twice and adding two rows, we can obtain a drawing such that x spans the bottom of the drawing and u spans the top of the drawing. Call the resulting drawing Γ''_a.

Similarly we can modify Γ_b into Γ''_b by adding at most two rows such that now w spans the bottom and x spans the top of the drawing.

Now we combine Γ_2, Γ''_b and Γ''_a similar as before, but route (u, w) with a horizontal segment (see Fig. 9). Note that this removes x from the outer-face of the drawing. We save the two columns needed for edge (w, u) before, and hence have a smaller width. Using induction, one can show that the height is at most $3 \log n - 1$ as before, and the width is at most $\frac{3}{2}n - 1$.

Theorem 3. *Every outer-planar graph has a visibility representation in a* $(\frac{3}{2}n - 2) \times (3 \log n - 1)$*-grid.*

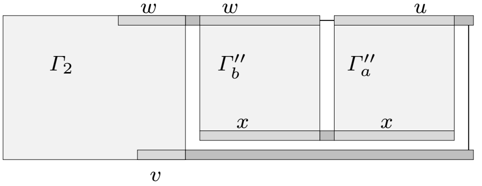


Fig. 9. Combining without bends.

6 Towards Lower Bounds

6.1 The Snowflake Graph

We have shown that $O(n \log n)$ drawings are possible in a variety of drawing models. We suspect that this is best-possible, in particular for the graph shown in Fig. 10 (and referred to as the *snowflake graph*).

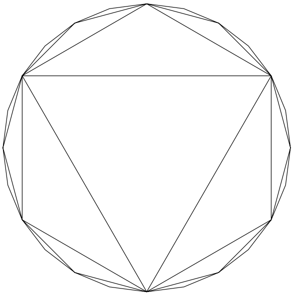


Fig. 10. The snowflake graph.

Conjecture 1. Any poly-line drawing of the snowflake graph has $\Omega(n \log n)$ area.

We have not been able to prove this conjecture, but believe it to be true based on the following observations:

- It appears to be the case that in any drawing of this graph, at least one dimension is $\Omega(n)$. Is this true, and how can this be shown?
- It also appears to be the case that in any drawing of this graph, both dimensions are $\Omega(\log n)$. Is this true, and does the fact that any edge separator (i.e., any set of edges whose removal splits the graph into two graphs that are a constant fraction smaller) has $\Omega(\log n)$ size help in proving this?

Note that if this conjecture is true, then the lower bound would hold orthogonal box-drawings as well by the results in Section 4, and therefore also for visibility representations. Of course, this conjecture would also imply an $\Omega(n \log n)$ lower bound for straight-line drawings.

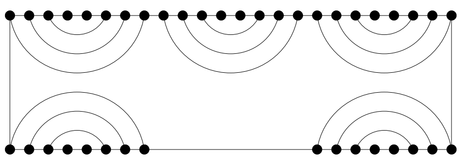
6.2 A More Restrictive Model

As a second contributions to lower bounds, we would like to give an $\Omega(n^2)$ lower bound on the area in a special model. One possible criticism of the orthogonal box-drawings obtained with our algorithm is that although all vertices are technically on the outer-face, they are not clearly visible as being on the outer-face. For example, vertex x in Fig. 6 appears removed from the outer-face, because it is far from the boundary of the enclosing rectangle.

Another model for drawing outer-planar graphs would thus be to require all vertices are placed on the boundary of the minimum enclosing rectangle (respectively touch it if they are boxes). It is known that trees may require $\Omega(n \log n)$ are in this model [Ull83, p.83ff]. We show now that for outer-planar graphs, $\Omega(n^2)$ area is required in this model.

Theorem 4. *There exists an outer-planar graph G such that any poly-line drawing Γ of G with all vertices on the boundary of the minimum enclosing rectangle of the graph has area $\Omega(n^2)$.*

Proof. Let G be the outer-planar graph illustrated in Fig. 6.2. It consists of 5 groups of $n/5$ vertices, connected with chords between them.



Now assume we have a drawing Γ such that all vertices are on the boundary $\mathcal{B}$ of the minimum enclosing rectangle. By planarity, and since the outer-face is a cycle, the order of vertices along $\mathcal{B}$ must be the same as the order around the outer-face. Therefore, at least one of the five groups of the graph entirely on one side of $\mathcal{B}$, say the top. Consider the subgraph induced by this group of vertices. We can duplicate its drawing and flip it upside down to obtain a drawing of asymptotically the same area. This drawing is a drawing of a multi-graph well-known to need $\Omega(n^2)$ area in any drawing that preserves this planar embedding (see also Fig. 11). So the area of the original drawing must have been $\Omega(n^2)$ as well.

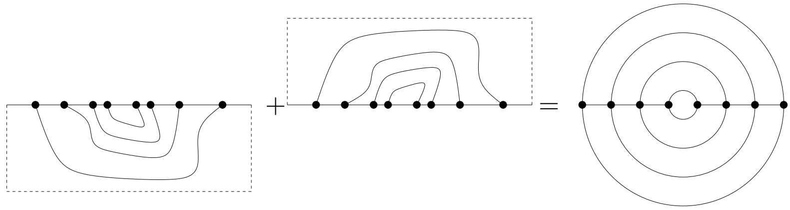


Fig. 11. Any poly-line drawing must have $\Omega(n^2)$ area.

We note here without proof that the same graph, with a similar proof, also yields a lower bound of $\Omega(n^2)$ for various other models of "being visible on the outer-face", such as "having a horizontal or vertical segment that reaches to the boundary of the enclosing rectangle", or even "having an escape hatch" (see [Lei80]).

7 Conclusion

In this paper, we studied planar drawings of outer-planar graphs. Using a simple recursive algorithm, we achieved an orthogonal box-drawing in $O(n \log n)$ area. This can be converted into a poly-line drawing in $O(n \log n)$ area. With a slight variation, we can also obtain visibility representations, at the cost of not having all vertices drawn on the outer-face. Some partial results towards lower bounds were provided as well.

Many open problems remain:

- What is the correct area bound for straight-line drawings of outer-planar graph? Can we achieve $o(n^2)$ for all outer-planar graphs?
  Note that $o(n^2)$ can be achieved for all outer-planar graphs that are balanced in some sense. The adjacencies between the interior faces of an outer-planar graph form a tree. It is relatively straightforward to show that if this "dual tree" has diameter d, then the graph has a straight-line drawing in a $d \times n$-grid (Erik Demaine and the author, private communication).
  Unfortunately, there are outer-planar graphs for which the dual tree is a path, and which therefore $d = n - 3$. While these graphs can be drawn in $o(n^2)$ area (in fact, in $O(n)$ area), it is not clear how to handle outer-planar graphs for which the dual tree has diameter $\Omega(n)$, but is not a path.
- We gave visibility representations of area $O(n \log n)$, but at the cost of not respecting the planar embedding. Can we obtain visibility representations of area $O(n \log n)$ even with all vertices drawn on the outer-face?
- As illustrated in the drawing of the snowflake graph in Figure 12, our drawings may have small area, but are somewhat disappointing in other aspects. In particular, the aspect ratio of the drawing is $\Omega(n/ \log n)$, and the aspect ratio of some boxes may be $\Omega(n)$.

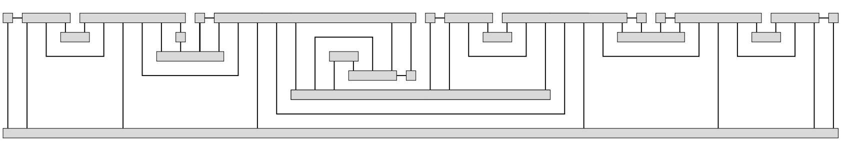


Fig. 12. The drawing of the snowflake graph created with our algorithm.

  Is it true that any poly-line drawing of aspect ratio $O(1)$ requires $\Omega(n^2)$ area? (We believe this to be true for the snowflake graph.) And what can be said about drawings where vertex boxes may not have arbitrary aspect ratios, e.g., orthogonal box-drawings in the Kandinsky-model [FK96] or the PG-model [BMT98]?
- What can be said about series-parallel graphs? Does every series-parallel graph have a drawing in $o(n^2)$ area if we are allowed to change the planar embedding? (This is not possible if all edges are required to be drawn "upward", see [BCDB$^+$94].)

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