Comparison of Electoral Systems: Simulative and Game Theoretic Approaches

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Abstract Simulation may be a useful tool to address some basic problems concerning the choice of the electoral system. A case study is analyzed as an example. The utility of including power indices is discussed. A simulation program is illustrated.

Keywords: Simulation, Electoral Systems, Weighted Majority Games.

1. Introduction

Simulation may be very useful in the comparative assessment of electoral systems; actually, it is difficult to imagine a field where the simulative approach may be more effective. There are two reasons for that. The first is that the real world feature that must be simulated is very simple - a set of preferences. The assessment of the relative performance of electoral systems requires a set of preferences, but is entirely *downstream* of the reasons that produced a set or another one. A 'virtual' case of a society that uses perfect proportionality and where there are some major parties and a cohort of minor ones provides nearly the same information offered by an analogous real-world case (pre-reform Italy, in this paper).

The second reason is even more compelling, and possibly less obvious, at least for non-social scientists. While the virtual *set* of preferences is nearly as informative as a real one, the *single* virtual subject is *identical* to a real one. According to the basic theorems of choice (Arrow's and May's), and more generally to the basic individualistic paradigm of social sciences, no preference must be privileged. Hence, there is no reason to ask *why* a given subject provided a given choice. The entire process of evaluating the result of his/her and others' combined choices is again downstream. In other terms, in this field the virtual subjects include *all* the relevant features of the real ones.

In this paper we argue in favor of the simulative approach for the evaluation of electoral systems. In Section 2 we present a simple empirical rule that allows choosing among electoral systems, in the line discussed in (Fragnelli et al., 2005). As we will see, the procedure requires the decision-maker to explicit its preferences; consequently, the choice rule is not affected by the theorem of Arrow (and cannot aspire to pinpoint the system *objectively* preferable). Section 3 contains an example obtained employing ALEX4, the improved version of a new and more powerful simulation program, announced (as ALEX3), but not used in (Fragnelli et al., 2005)[1]. Section 4 extends the discussion to power matters, proposing tools and methodologies for defining better indices. Conclusions are in Section 5. Technicalities are in appendices.

2. The Choice of the Optimal Electoral System

The choice of the best electoral system affects a lot of facets of the political process. (Ortona, 2002) provides a list with some twenty items, arguably not complete. Fortunately, however, there is a general agreement that the efficiency in representing electors' will (*representativeness,* R) and the effect on the efficiency of the resulting government (*governability,* G) are of paramount relevance [2], [3]. There are at least two good reasons to privilege R and G.

First, to summon the representatives and to form a government are the basic aims of a Parliament (bar, obviously, to make laws). Possible pitfalls of other dimensions may be managed in other moments of the political process, but this is not the case for representativeness and governability, if we admit the sovereignty of the voters in choosing their representatives and that of the representatives in choosing the government. In addition, it is sensible to think that other dimensions are lexicographic with respect to them [4]. If this is so, the results obtained with reference to R and G will keep their validity irrespective of the dimensions judged relevant.

R and G may be evaluated through the assessment of plausible (albeit arbitrary) numerical indicators. The ones used in our simulations are briefly described in Appendix B (for further details, see (Bissey et al., 2004)). We will label them r and g, respectively [5]. The range of both is the interval 0-1.

[1] The most relevant additional features are the consideration of new electoral systems (including the Single Transferable Vote) and the possibility to define individually the virtual voters.

[2] See (Bowler et al., 2005).

[3] A more detailed characterization of both R and G and of the related trade-off (R is likely to increase with the number of parties and G to decrease) is provided in Appendix B, through the definition of the indices employed to assess them. For a broader discussion, see (Ortona, 1998) and (Bissey et al., 2004).

[4] Note however that the method outlined here may be extended to further dimensions, provided that suitable indices are available.

[5] A slightly different version of these indices has been employed also in (Ortona, 1998).

Results for different electoral systems, *referring to a single case*, may consequently be computed out. There are three possibilities. First, the values of both r and g of a given system may be greater than those of *all* other systems considered; we define that system *dominant*, and it is obviously the best. Unfortunately, this system is very likely not to exist, given the trade-off between the two dimensions. Second, the values of both r and g for a given system may be lower than those of *another* one. We define that system *dominated*, and it may safely be excluded: no need to consider system X, if system Y, better on both dimensions, is available. Third, systems may be neither dominant nor dominated, i.e. all of them are Pareto optimal, like (usually) plurality voting and proportional representation in real world. We label these systems *alternative*. Obviously, the rule we look for is useful only if it allows choosing among alternative systems. Note that there may be at most one (strongly) dominant system, while the dominated systems can be more than one.

In principle, to compare different electoral systems, we need voting results for different systems: a majoritarian vote, a proportional vote, a list of voters' ordered preferences for Condorcet voting, and so on. It is usually impossible to collect these data from real world. But given a set of virtual electors, each with her/his preferences, it is perfectly possible to produce them. Given the votes, every system considered will produce a potential Parliament, and each Parliament will have a pair of values of r and g. If a system will result as dominant, it is the good one; but, as we noticed, this result is very unlikely, given the trade-off between the two dimensions. Apparently, what we need to compare them is a social utility function (*SUF*) - admittedly a quite formidable requirement, to say little. Actually, we may be satisfied with something less.

Let us admit the *SUF* for representativeness and governability to be a typical Cobb-Douglas function in g and r, $U = Kg^a r^b$, where K is a suitable constant. We choose this form not only for its simplicity and versatility, but also for the meaning of a and b, the partial elasticity of U with reference to g and r, respectively; as we will see, this provides a meaningful characterization of the choice rule. Now consider two non-dominated systems, X and Y. We may write that:

$$X \succ Y \iff Kg_X^a r_X^b > Kg_Y^a r_Y^b \tag{1}$$

where $X \succ Y$ means that system X is preferred to system Y.

Let $p = \dfrac{a}{b}$. It is easy to obtain that condition (1) reduces to:

$$p > \frac{\ln \frac{r_Y}{r_X}}{\ln \frac{g_X}{g_Y}} \tag{2}$$

supposing that X refers to the system with the higher value of g.

REMARK 1 *The comparison of systems is strongly influenced by the actual scaling of the indices with respect to each other. This inconvenience is reduced* *by the choice of a multiplicative utility function. Suppose that a decision-maker thinks that g should be given more (less) relevance, and that the increase (decrease) of relevance may be established through the attachment to g of a multiplicative constant > 1 (< 1). This procedure leaves the choice rule unaffected.*

It is important to notice that the ratio of the elasticities, p, can be seen as the *price* in terms of a relative decrease of r that the community accepts to pay for a given relative increase of g. If, for instance, we have $p = 2$, it is worthwhile to accept a 2% reduction of r to gain a 1% increase of g (but for the approximation due to the use of differentials). In general, if an increase in g is valued more than the same increase in r, then $p > 1$, and vice versa [6].

The only a priori information we need to assess the fulfillment of the condition, is the value of p. We argue that this parameter may actually be provided by the political system. Several procedures may be adopted, as discussed in Ortona (2005).

Equation (2) allows for binary comparisons of non-dominated electoral systems, and hence for finding out the Condorcet winner. The winner is the best system [7].

Alternatively, we may trace indifference curves and pick the system that lies on the higher curve. This procedure allows for a graphical individuation of the preferred system. For details, see (Fragnelli et al., 2005).

3. An Example

In this section we provide an example that mirrors the actual Italian case.

The input is a representative survey of electoral preferences of Italian citizens collected by the Osservatorio del Nord Ovest of the Università di Torino in the first quarter of 2004. The simulation program described in Appendix A provides the data of Table 1 and Figure 1.

The choice set may be considerably reduced through the exclusion of systems that are *dominated* or *weakly dominated*. This criterion leaves us with the ten systems labeled 2, 3, 4, 6, 8, 11, 17, 18, 20, 21.

An elicitation procedure implemented with 80 students at the Laboratorio di Economia Sperimentale e Simulativa of the Universita' del Piemonte Orientale and described in detail in Ortona, 2005, provided the value 0.696 for p (with 0.402 standard deviation). However, *each* participant to the experiment provided his/her value; given these values, it was tedious but simple to apply the choice method described in this paper to the ten systems above *and to each participant*. It is not inappropriate to state that participants *voted* their

[6] For the proof see (Fragnelli et al., 2005).

[7] A Condorcet cycle may result only by chance, and may be ruled out simply by adding a further figure while rounding the results - or by tossing a coin.

Table 1. A simulation of an Italian-like case.

	System	r	g	share of seats of the governing coalition	parties of the governing coalition
1	Borda	0.66	0.275	0.55	2
2	Run-off plurality	0.66	0.300	0.60	2
3	Plurality	0.74	0.233	0.70	3
4	Mixed-sc. (a)	0.85	0.207	0.61	3
5	Mixed (a)	0.82	0.207	0.62	3
6	Prop. (1 district)	1.00	0.104	0.52	5
7	Threshold Prop. (b)	0.87	0.170	0.51	3
8	Condorcet	0.70	0.295	0.59	2
9	Prop. Hare (c)	0.92	0.135	0.54	4
10	Prop. Imperiali (c)	0.88	0.087	0.52	6
11	Prop. Sainte-Lague (c)	0.94	0.135	0.54	4
12	Prop. D'Hondt (c)	0.84	0.180	0.54	3
13	STV N.B. (c)	0.94	0.106	0.53	5
14	STV Droop (c)	0.95	0.108	0.54	5
15	STV Hare (c)	0.91	0.108	0.54	5
16	Prop. Hare (d)	0.99	0.106	0.53	5
17	Prop. Imperiali (d)	0.99	0.106	0.53	5
18	Prop. Sainte-Lague (d)	0.98	0.108	0.54	5
19	Prop. D'Hondt (d)	0.96	0.104	0.52	5
20	Mixed-sc (d)	0.91	0.177	0.53	3
21	Mixed (d)	0.87	0.190	0.57	3
22	Threshold prop. (b, d)	0.96	0.106	0.53	5

(a) 25 seats assigned through one-district proportionality, 75 through plurality.
'sc' (after the Italian word 'scorporo') means that votes used for the proportional share are not considered for the assignment of the plurality seats.
(b) Threshold 5%.
(c) Ten ten-seat districts.
(d) Five twenty-seat districts. The program ran out of memory for STV.
Simulations were performed with 100 seats.
• prop. = pure proportionality
• STV = single transferable vote

preferred electoral system. Condorcet got 46 votes, pure proportionality 17, mixed (5 districts) 12, and mixed (ten districts) 5. The Condorcet winner is Condorcet; a result hardly unexpected for a theorist, but not that easy to detect from data, as Condorcet ranks second in g but only second to last in r.

4. The Role of Power

We think that representativeness and governability should take into account more than the distribution of seats w.r.t. the distribution of votes. Mathematics

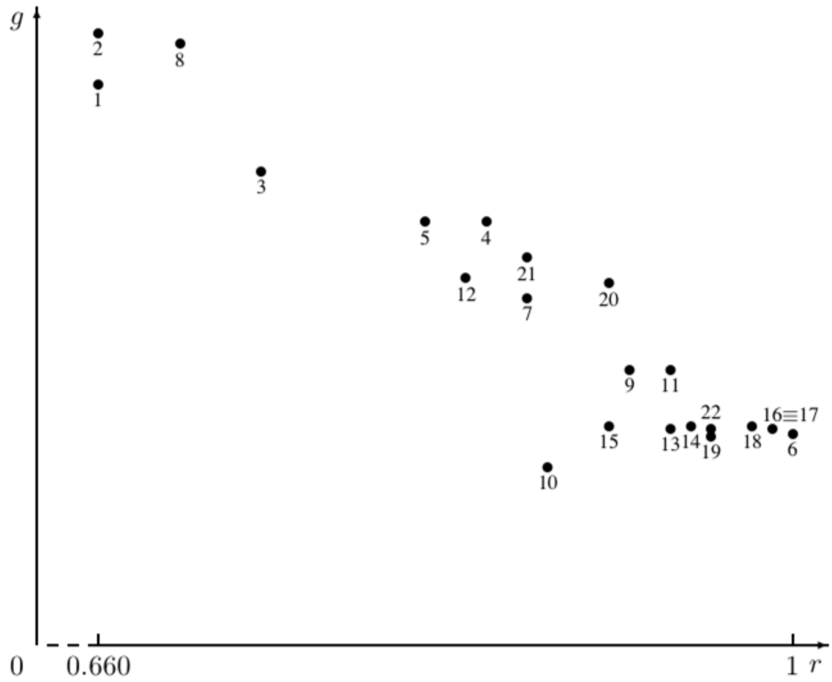

Fig. 1. The voting systems of Table 1 in the $r - g$ space.

offers a lot of distances or norms in order to measure the distance of the distribution of voters, v_i and the distribution of seats according a system h, s_i^h; among them the most widely used are:

- Norm 1: $d_1^h = \sum_{i \in N} |v_i - s_i^h|$
- Norm 2: $d_2^h = \sqrt{\sum_{i \in N} (v_i - s_i^h)^2}$
- Norm ∞: $d_\infty^h = \max_{i \in N} |v_i - s_i^h|$

where N is the set of parties.

This approach may be largely far from our needs, as shown in the following example.

EXAMPLE 2 [8] *Suppose that there are four parties P_A, P_B, P_C and P_D; the preferences of the voters are respectively 40, 25, 20 and 15 per cent and the*

[8]Taken from (Fragnelli et al., 2005).

majority quota is 50 per cent; suppose also that the parliament consists of 4 seats and that two voting systems generate the two partitions of seats (2,1,1,0) and (1,1,1,1). We start by computing the distances of the two partitions from the distribution of voters:

	(2,1,1,0)	*(1,1,1,1)*
d_1	*0.4*	*0.4*
d_2	*0.01* $\sqrt{350}$	*0.01* $\sqrt{350}$
d_∞	*0.15*	*0.15*

The two voting systems seem to be equivalent. $\diamondsuit$

In order to avoid these unlikely situations we can relate the indices not directly to the distributions of votes and seats, but to the *power* of the parties.

The elusive notion of power has a lot to do with the choice of the electoral system; and both with governability and with representativeness. If we stick to the microcosm notion of representativeness, we should want a distribution of power similar to that of preferences, while the governability is normally supposed to be enhanced if the power is highly concentrated. To find out the 'right' distribution of power is a formidable task, and we will not deal with it. More modestly, we argue that in order to tackle that problem it is necessary to be able to compare the distribution of power with that of preferences; and again simulation is highly useful, for the same reasons that we discussed above - the non-availability of reliable real world data.

So, the new problem we have to face is to determine the distribution of the power of the parties.

Game theory is a natural habitat for the problem of evaluating the power of the parties in a voting situation. Since the pivotal paper of Shapley (1953) different indices were introduced, with the aim of assigning to each agent a number that represents his/her relevance in a multiagent situation. It may be useful to recall some basic notions. A *cooperative game with transferable utility (TU-game)* is a pair $G = (N, v)$, where N is the set of players (the agents) and v is the characteristic function that assigns to each subset of players $S \subseteq N$, called *coalition*, a real number that can be considered as its worth independently from the behavior of the other players. A game is said to be *simple* if $v(S) \in \{0, 1\}$, i.e. the worth of a coalition may be only 0 or 1; a game is said to be *monotonic* if $S \subset T$ implies $v(S) \leq v(T)$, i.e. if a coalition is enlarged then its worth cannot decrease.

In particular we are interested in the *weighted majority games*, simple monotonic games that are widely used in voting situations. Suppose that each player $i \in N$ is associated with a non negative real number, the *weight* w_i, and suppose that if some players join to form a coalition S the weight of the coalition is the sum of the weights of the players, i.e. the weights are additive; if the weight of a coalition is strictly larger than a given positive real number q, the so-called *quota*, the coalition is said to be *winning*, and it is said to be *losing* otherwise. Formally we define the characteristic function w of a weighted majority game as:

$$w(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i > q \\ 0 & \text{if } \sum_{i \in S} w_i \leq q \end{cases} \qquad \forall\, S \subseteq N$$

Usually such a situation is summarized by the $(n+1)$-upla $(q; w_1, ..., w_n)$.

As a consequence we can say that if $v(S) = 1$ then S is a winning coalition and if $v(S) = 0$ then S is a losing coalition. A winning coalition is called *minimal* if all its subcoalitions are losing.

The weighted majority games associated to the distributions of voters and of seats, according to a given electoral system, allow us evaluating the importance of each party with respect to a suitable power index. Game theory dealt with this problem from the beginning of its history. Many different power indices were proposed, each of them emphasizing different properties of the underlying situation. In this paper we consider the Shapley-Shubik index, the normalized Banzhaf-Coleman index, the Deegan-Packel index and the Holler (or Public goods) index.

The *Shapley-Shubik index* (Shapley and Shubik, 1954), ϕ, is the natural extension of the Shapley value (Shapley, 1953) to simple games. Let Π be the set of all the permutations of the players and for each $\pi \in \Pi$ let $P(i, \pi)$ be the set of players that precede player i in π; the Shapley value is the average marginal contribution of each player w.r.t. the possible permutations [9]:

$$\phi_i = \frac{1}{|N|!} \sum_{\pi \in \Pi} [v(P(i,\pi) \cup \{i\}) - v(P(i,\pi)] \qquad \forall\, i \in N$$

The *normalized Banzhaf-Coleman index* (Banzhaf, 1965 and Coleman, 1971), β, is similar to the Shapley-Shubik index, but it considers the marginal contributions of a player to all possible coalitions, without considering the order of the players. Let us introduce $\beta_i^* = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N, S \ni i} [v(S) - v(S \setminus \{i\})], i \in N$. By normalization we get:

$$\beta_i = \frac{\beta_i^*}{\sum_{j \in N} \beta_j^*} \qquad \forall\, i \in N$$

The *Deegan-Packel index* (Deegan and Packel, 1978), δ, considers only the minimal winning coalitions; the power is firstly equally divided among minimal winning coalitions and then the power of each is equally divided among

[9] We denote by $|A|$ the cardinality of the set A

its members:

$$\delta_i = \sum_{S_k \in W, S_k \ni i} \frac{1}{|W|} \frac{1}{|S_k|} \qquad \forall\, i \in N$$

The *Holler index*, or *Public Goods index* (Holler, 1982 and Holler and Packel, 1983), H, considers the number of minimal winning coalitions which player i belongs to, $c_i, i \in N$; then by normalization we get:

$$H_i = \frac{c_i}{\sum_{j \in N} c_j} \qquad \forall\, i \in N$$

The different indices take into account various aspects of the coalition formation process, so that the power of a given party may assume different values. In particular the power could be concentrated in few large parties or spread on many of them.

EXAMPLE 3 *Referring to Example 2, we can define the majority games $w(v)$ on voters, $w(s^1)$ on the first parliament and $w(s^2)$ on the second parliament*

game	*1*	*2*	*3*	*4*	*12*	*13*	*14*	*23*	*24*	*34*	*123*	*124*	*134*	*234*	*N*
$w(v)$	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1
$w(s^1)$	0	0	0	0	1	1	0	0	0	0	1	1	1	0	1
$w(s^2)$	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1

whose corresponding indices are:

$game$	ϕ	β	δ	H
$w(v)$	$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$	$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$	$\left(\frac{9}{24}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24}\right)$	$\left(\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right)$
$w(s^1)$	$\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0\right)$	$\left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right)$
$w(s^2)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

Finally, for each index, we compute the distances between the power w.r.t. the voters and to each parliament:

	(2,1,1,0)				*(1,1,1,1)*			
	ϕ	β	δ	H	ϕ	β	δ	H
d_1	**0.333**	**0.333**	0.417	0.444	0.500	0.500	**0.250**	**0.167**
d_2	**0.236**	**0.200**	0.250	0.281	0.289	0.289	**0.144**	**0.096**
d_∞	**0.167**	**0.167**	0.222	0.208	0.250	0.250	**0.125**	**0.083**

where numbers in bold indicate the 'best' voting system according to each index.
The distances on the power indices allow us to distinguish the two systems. In particular the indices of Shapley-Shubik and Banzhaf-Coleman favor the first parliament, while the second parliament is preferable according to the indices of Deegan-Packel and Holler. $\diamondsuit$

Another measure may be obtained referring to the distribution of voters, v, to the assignment of seats according to an electoral system h, s^h, and to the power of the parties related to the votes and to the seats, φ and φ^h respectively (see Gambarelli and Biella, 1992). The resulting distance Δ is:

$$\Delta = \max_{i \in N} \left| |v_i - s_i^h| - |\varphi - \varphi^h| \right|$$

EXAMPLE 4 *Referring again to Example 2, we obtain the following distances:*

	Δ			
Voting system	$\varphi = \phi$	$\varphi = \beta$	$\varphi = \delta$	$\varphi = H$
s^1	**0.067**	**0.033**	**0.058**	**0.072**
s^2	0.100	0.100	**0.058**	**0.072**

where again numbers in bold indicate the 'best' voting system according to each index.
Also using this measure the Shapley-Shubik and Banzhaf-Coleman favor the first parliament, while the two parliaments are equivalent according to the indices of Deegan-Packel and Holler. ♢

REMARK 5 *The first two indices and the last two have similar behavior; this depends on the matter that the first two take into account all the coalitions, while the last two consider only minimal winning coalitions.*

The main conclusions of this section are the following. First, the indication of the example - were it for real - would be precious. Yet the starting point are, by necessity, the data on votes. If votes are those actually cast in a, say, plurality election, they are useless to compare the distribution of power with that of *preferences*. Arguably, the distribution of votes may be assumed as a proxy to that of preferences only in proportional systems with large districts (and, we must add, with low running costs). The same conclusions of Section 3 apply. Real data cannot provide useful information; the simulation does. To accumulate experimental (i.e. simulative) evidence would probably provide relevant suggestions for real world analysis and policing.

Second, in our opinion, a better definition of both representativeness and governability should rest on the notion of power. Game theory is a very useful tool for this aim. In particular the index of concentration (Gini, 1914) applied to the distribution of power may be exploited to define a representativeness index, while for the governability index two approaches seem promising: the coalitional value (Owen, 1977), that takes into account the role of the *a priori* agreements of the parties and the propensity to disruption (Gately, 1974) that measures the relative gain of the players when leaving the grand coalition.

5. Concluding Remarks

We argued that the simulation approach to the evaluation of electoral systems is very powerful. To add evidence, we suggest a (very partial) list of problems that could profitably be tackled this way. What is the difference *in results* between Borda and Condorcet? When do pure proportionality and single transferable vote provide analogous results? What is the actual effect of district magnitude on proportionality? How do indices of proportionality perform? Are Condorcet cycles really a problem?

It is not difficult to add others, so we will not pursue this point further. Instead, we argue that experimental results may be improved if the simulation programs are further elaborated. We suggest that main methodological improvements should regard the possibility of including and managing survey data, the addition of further indices, mostly but not only with reference to power issues, and obviously the addition of further electoral systems. However, to our opinion the main methodological challenge is the addition of new evaluation *dimensions*, and consequently indicators. Obviously, this requires that they may be quantified, and consensus on what we desire about.

To conclude, simulation is very useful to analyze the performance of the electoral systems including random elements, e.g. the absence of some members of the parliament in a voting session, or to study the possibility of manipulating the elections, e.g. via merging or splitting of the parties in order to profit of suitable features of the system.

Appendix A - The Simulation Program

Given the utility and the versatility of the simulative approach for the analysis of electoral systems, it is quite surprising that it is so little employed in the political science literature. A survey is in (Fragnelli et al., 2005); there are some, but not that many, suggestive case studies, but very few papers address the matter we are dealing with here, to compare electoral systems, after some pioneering papers (see Mueller, 1989; Merrill, 1984 and Merrill, 1985). (Gambarelli and Biella, 1992) analyze the effect in Italy of a change to a number of electoral systems, and (Christensen, 2003), compares six majoritarian systems, but without reference to a Parliament. Consequently, it is not surprising that the simulation programs so far available (like those developed by Accuratedemocracy, www.accuratedemocracy.org) are of limited use for purposes of the kind suggested here.

The simulations produced in this paper have been carried out with a specific program, ALEX4; its number refers to the version currently in use. ALEX4 is written in Java, and it is the heir of a program originally written in Visual Basic, *g&r* (for Governability and Representativeness), dating back to 1998 (Ortona, 1998; Trinchero, 1998). ALEX4 is a cosmetic improvement of ALEX3, which is described in detail in (Bissey et al., 2004); hence here we provide only some basic hints. All the versions of ALEX have been written by Marie-Edith Bissey at the Universita' del Piemonte Orientale.

The user is requested to provide some basic inputs, namely, the size of the Parliament, the number of voters, the number of parties (i.e. the number of candidates in every constituency for non-proportional systems), the share of votes of the parties, the concentration of the parties across the constituencies, the probability that second and further preferred parties are the closest to the first, to the second etc., the probability that third and further preferred parties are the closest to the second, to the third etc., and the probability that second and further preferred candidates are the closest to the first, the second, etc. The first two probabilities are employed to generate a complete set of preferences for parties, for each voter; the last one to generate a complete set of preferences for candidates, to be employed for single transferable vote. The program produces the Parliaments for (up to now) sixteen systems, namely one-district proportionality; one-district threshold proportionality [10]; Hare, D'Hondt, Imperiali and Sainte-Lague multi-district proportionality; N.B., Droop and Hare multi-district single transferable vote; two mixed-member systems [11]; plurality; run-off majority; Condorcet; Borda; and VAP, a suggested new system described in detail in (Ortona, 2004). Approval voting is not included (but it will be in further versions) because previous experiments (with $g\&r$) indicated that it is commonly dominated by other systems. Finally, the program computes the index of representativeness and the index of governability (the user is requested to define the governing coalition). Both indices are described in Appendix B.

Appendix B - The Indices Employed

Index of Representativeness, *r*

An index of representativeness suitable to compare electoral systems cannot be based on the difference between the share of votes and that of seats, albeit all the indices of *proportionality* commonly employed, like Gallagher's [12], are constructed this way. The obvious and compelling reason is that the voting behavior is affected by the electoral system itself. Instead, our index is based on the difference between *votes cast in a nation-wide proportional district and*

[10] The threshold may be fixed by the user.
[11] With and without the exclusion of votes employed in the plurality election from the proportional election. The share of seats assigned through proportionality may be fixed by the user.
[12] Introduced by (Gallagher, 1991).

seats assigned by a given electoral system. The formula is:

$$r_h = 1 - \frac{\sum_{i \in N} |S_i^h - S_i^{PP}|}{\sum_{i \in N} |S_i^u - S_i^{PP}|}$$

where N is the set of parties, S_i^h is the number of seats of party i with system h, S_i^{PP} is the number of seats of party i with the perfect proportional system and S_i^u is the total number of seats for the relative majority party under system h and it is 0 otherwise.

The index reads as follows. For the sum at the numerator, we assume that the representativeness R is maximal under perfect proportionality rule (PP). Hence the loss of representativeness incurred by party i is the (absolute) difference between the seats it would get under PP and those actually obtained. Summing this loss across all the parties we obtain the total loss of R. The sum at the denominator is introduced to normalize this value. It is the maximum possible loss of R. This maximum is obtained when 'winner takes all' in a very strict sense, that is when the relative majority party, according to the selected system, takes all the seats instead of just its quota. The ratio of the sums is a loss of representativeness index, normalized in the range 0-1; subtracting it from 1 we transform it into a representativeness index.

EXAMPLE 6 *Suppose three parties, L, C and R, in a parliament of 100 seats. Under PP they obtain 49, 31 and 20 seats respectively, under majority* (M) *90, 10 and 0, and under some other system* (S) *30, 55 and 15. So* $r_M = 1 - \frac{41+21+20}{51+31+20} = 0.196$ *and* $r_S = 1 - \frac{19+24+5}{49+45+20} = 0.579$ *(obviously* $r_{PP} = 1 - \frac{0}{51+31+20} = 1$*)* ♢

As this index is not that easy to grasp, in the example described in Section 3 above we employed a simpler one, which is 1 minus the ratio between the total number of seats assigned in excess to the proportional share and the total number of seats. In the previous example the value of this second index is 0.59 for M and 0.89 for S (and 1 for PP). For more realistic cases, however, the two indices are strongly correlated; for data of Section 3 the correlation index is 0.963.

Index of Governability, *g*

According to the mainstream doctrine, governability is inversely related to the number of parties that take part in the governing majority. Our index is based on this assumption. It depends on the number of parties of the governing coalition that may destroy the majority if they withdraw, m, and on the share of seats of the majority, f. m is more important, so we add (lexicographically) the f-component to the m-component. Hence the index is made by the sum of two terms, the first related to m, g_m, and the second related to f, g_f. Thus,

$g = g_m + g_f$. The range of the second term is the difference between successive values of the first: the term in m defines a lower and an upper bound, and the term in f specifies the value of the index between them.

The range defined for g_m is simply $\frac{1}{m}$ (upper bound) and $\frac{1}{m+1}$ (lower bound). For instance, if the government is supported by just one party, g is in between 0.5 and 1; if it supported by two parties, then g is in between 0.333 and 0.5, and so on. The number of seats of the majority coalition specifies the value of g in the given range. The amount g_f to be added to the lower bound depends from the lead of the majority coalition, according to the proportion $\frac{g_f}{\frac{1}{m} - \frac{1}{m+1}} = \frac{f - \frac{T}{2}}{T - \frac{T}{2}}$. In sum, the formula for g is:

$$g = g_m + g_f = \frac{1}{m+1} + \frac{1}{m(m+1)} \frac{f - \frac{T}{2}}{\frac{T}{2}}$$

For instance, if there are 100 seats and the governing majority is made up of one party with 59 members, we have $g_f = \frac{9}{50}\frac{1}{2} = 0.09$. This value must be added to 0.5, to give $g = 0.59$.

The maximum value of g is 1, when a party has all the seats; the lowest tends to zero as the number of parties increases, thus justifying the claim that the range of g is in between 0 and 1.

Again, in Section 3 we employed a simpler index, based on the same theoretical assumptions, i.e. the ratio between the share of seats and the number of parties of the governing coalition. In the example, the value of this index[13] is again 0.59; and this index is strongly correlated with the previous one; for data of Section 3 the correlation index is 0.994.

Appendix C - A Short Description of the Electoral Systems

This appendix is taken from the *readme* file of ALEX4 package. Many systems allow for variants; the definitions provided here refer to those adopted in ALEX4. For a description of how the systems are implemented, see the *Final Note* of this paper, and the *readme* file quoted. For an easy-to-read, more detailed description of the systems, see (Farrell, 2001).

- *Plurality* In each district, the winner is the candidate with most votes.

[13] There is a reason for dissatisfaction with indices of governability based on the number of parties, which to our opinion is why this kind of indices perform quite poorly when applied to real cases, and more generally why the governability is not that greater in majoritarian systems (see Lijphart, 1999). The reason is that a party may be and may be not be a single subject. At one extreme it is, but at the other it is a set of independent decision-makers. ALEX5, the next version of the program ,will take into account this feature through the addition of a suitable parameter.

- *One-district Proportionality* The seats in the Parliament are distributed according to the shares of votes in the population, rounded to the closer integer.

- *Threshold Proportionality* All the parties who have a share of votes in the population smaller than the established threshold are excluded from the Parliament. The seats are distributed proportionally among the remaining parties.

- *Run-off Plurality* In each district all parties but the two with the most votes are excluded. The second round is carried out with these two parties only and the one with the most votes wins. If after the first round the first party has at least 50% of the votes, it wins the seat without the need of a second round.

- *Mixed* Part of the parliament is elected with the Plurality System, and the rest is elected using the Proportional System.

- *Mixed with 'Scorporo'* As for the previous system, but the votes used to elect the Plurality share are lost for the Proportional share.

- *Borda count* This system uses the electors' complete preference ordering. Each elector gives points to each party, from 0 for the most preferred party to $N-1$ for the least preferred party, where N is the total number of parties. In each district, the winner is the party with fewer points.

- *Condorcet winner* In each district, the Condorcet winner is the party that beats all the others when taken in pairs.

- *Multi-district Proportionality* The method is the same as in one-district proportionality. In this case, however, the rounding procedure is relevant. We employed four: D'Hondt, Hare, Imperiali and Sainte-Lague.

- *Single Transferable Vote* The seats for each party, in each plurinominal district, are assigned according to a quota value. If some seats are not assigned with this method, the votes unused by the elected candidates are transferred to the next candidates in the elector's preference ordering, and the candidates with the highest number of votes (obtained + transferred) are elected. The quota value may be computed according to three different procedures: N.B., Droop, and Hare.

Final Note

If you are interested, you may download and use the simulation program ALEX4. There will be no charge, but you will be asked to observe some gentleperson-agreement conditions - basically, no liability for possible mistakes and quotation of the source of the program. Please contact Guido Ortona for further details or for downloading instructions.

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