# Minimum Total Deviation Apportionments

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**Abstract** This note presents an algorithm for computing the minimum total deviation apportionment. Some properties of this apportionment are also explored. This particular apportionment arises from the jurisprudential concern that total deviation is the appropriate measure for the harm caused by malapportionment of the United States House of Representatives.

**Keywords:** Apportionment, House of Representatives, districting, one person - one vote.

## 1. Introduction

The goal of this paper is to revive interest in evaluating methods of apportionment based on the objective functions that they optimize rather than their intrinsic axiomatic properties. The latter approach is certainly the dominant one as evidenced by such texts as (Balinski and Young, 2001) and (Saari, 1994). Nevertheless there are circumstances for which this may not be the best approach. I have argued elsewhere (Edelman, to appear) that the case of the apportionment of the United States House of Representatives is exactly such a circumstance. Subsequent to the "one person, one vote" rulings of the mid-1960's, the United States Supreme Court has adopted the measure of *total deviation* to quantify the harm resulting from unequal voting district sizes. Once having established a measurement of the harm, the Court should require that any apportionment do what it can to mitigate that harm. This implies that any method of apportionment should look to minimize the total deviation.

As it happens there are two papers, both pre-dating "one person, one vote," investigating methods of apportionment that minimize total deviation. The first, by Burt and Harris (1963), argued in favor of apportioning the House of Representatives so as to minimize total deviation on equitable principles and presented an algorithm using dynamic programming to find such an apportionment. This paper has been cited a number of times in the literature.

A year later and in the same journal, Gilbert and Schatz (1964) published a response to Burt and Harris. Their rebuttal made three arguments: First, the equitable arguments in favor of minimizing total deviation were not convincing;

second, there may be many apportionments that minimize total deviation, and last, that the algorithm provided by Burt and Harris to produce the minimizing apportionment was unduly complicated. They provided quite an elegant algorithm, which I will present subsequently.

Oddly, Gilbert and Schatz's article seems to have escaped notice. As far as I know it has never been cited. Yet it contains some quite lovely ideas. A secondary purpose of this paper is to present the ideas of Gilbert and Schatz in a contemporary setting so they will get the attention that I think they deserve.

This paper is organized as follows: The next section presents the necessary background from the theory of apportionment. It is necessarily brief, and I will rely on the reader to have a basic familiarity with the techniques. Section 3 presents an algorithm to compute the minimum total deviation (mtd) apportionments. The method presented is due to Gilbert and Schatz (1964) although I have streamlined the presentation and proofs. The next three sections discuss technical issues associated with mtd apportionments. Section 4 confronts the problem of multiple mtd apportionments, Section 5 discusses bias and Section 6 examines the Alabama paradox. Section 7 is a brief conclusion.

## 2. Preliminaries

In this section I will introduce the necessary terminology. Since I am primarily interested in the apportionment of the United States House of Representatives, I will phrase the apportionment problem in terms of assigning seats to states. Assume that there are $s$ states and let $\mathbf{p} = (p_1, p_2, \ldots, p_s)$ be the state populations. For $h$ a positive integer we call $\mathbf{a} = (a_1, a_2, \ldots, a_s)$ an $h$-*apportionment* if $\sum a_i = h$. We will refer to $a_i$ as the number of seats that state $i$ receives. I will assume throughout that the state populations are generic in the sense that

$$\frac{p_i}{j} \neq \frac{p_k}{l}$$

for $1 \leq i, k \leq s$ and for all positive integers $1 \leq j, l \leq s$.

Given $\mathbf{p}$ and $h$-apportionment $\mathbf{a}$ let

1 $Max(\mathbf{p}, \mathbf{a}) = \max_i \frac{p_i}{a_i}$

2 $Min(\mathbf{p}, \mathbf{a}) = \min_i \frac{p_i}{a_i}$, and

3 $TD(\mathbf{p}, \mathbf{a}) = \max_{i,j}\{\frac{p_i}{a_i} - \frac{p_j}{a_j}\} = Max(\mathbf{p}, \mathbf{a}) - Min(\mathbf{p}, \mathbf{a})$.

Thus, $Max(\mathbf{p}, \mathbf{a})$ is largest population/seat ratio among the states, $Min(\mathbf{p}, \mathbf{a})$ is the smallest such value, and $TD(\mathbf{p}, \mathbf{a})$, the *total deviation* of the apportionment, is the gap between these two values.

Two methods of apportionment will be of particular importance in this paper. The first is the Adams method, which can be described in the following way (Balinski and Young, 2001, page 142): Given $s$ states with populations $\mathbf{p}$, and a house of size $h$, $h \geq s$, we let $Adams(\mathbf{p}, h)$ be the $h$-apportionment given recursively by

1 $Adams(\mathbf{p}, s) = (1, 1, \ldots, 1)$,

2 Let $\mathbf{A} = Adams(\mathbf{p}, h-1)$. If $t$ is the state so that $\frac{p_t}{a_t} = Max(\mathbf{p}, \mathbf{A})$ then define

$$Adams(\mathbf{p}, h)_i = \begin{cases} A_i + 1, & \text{if } i = t; \\ A_i, & \text{otherwise.} \end{cases}$$

Note that from the definition we have that $Max(\mathbf{p}, Adams(\mathbf{p}, h))$ is strictly decreasing as a function of $h$.

LEMMA 1 *The Adams apportionment* $Adams(\mathbf{p}, h)$ *minimizes* $Max(\mathbf{p}, \mathbf{a})$ *over all* $h$*-apportionments* $\mathbf{a}$.

**Proof.** This fact is noted in (Balinski and Young, 2001, page 104) without proof. For the sake of completeness I include one here. Let $Adams(\mathbf{p}, h) = \mathbf{A}$ and suppose there is an $h$-apportionment $\mathbf{a}$ so that

$$\frac{p_i}{a_i} = Max(\mathbf{p}, \mathbf{a}) < Max(\mathbf{p}, \mathbf{A}) = \frac{p_j}{A_j}.$$

It follows that $a_j > A_j$ and, since both $\mathbf{a}$ and $\mathbf{A}$ are $h$-apportionments, there must be some $k$ so that $a_k < A_k$. Since the Adams $h$-apportionment assigns more seats to state $k$ than $\mathbf{a}$ does, it follows that for some $h' < h$ we have $\frac{p_k}{a_k} = Max(\mathbf{p}, Adams(\mathbf{p}, h'))$. Since $Max(\mathbf{p}, Adams(\mathbf{p}, h))$is strictly decreasing as a function of $h$, $\frac{p_k}{a_k} > \frac{p_j}{A_j}$ which contradicts the assumption that $\frac{p_i}{a_i} = Max(\mathbf{p}, \mathbf{a})$. ■

If $h' > h$ and $\mathbf{a}$ and $\mathbf{a}'$ are $h$- and $h'$-apportionments, respectively, I will say that $\mathbf{a}'$ is an $h'$*-extension* of $\mathbf{a}$ if $\mathbf{a}' \geq \mathbf{a}$, i.e., $a'_k \geq a_k$ for all $1 \leq k \leq s$. The following lemma helps to illustrate this idea and will prove useful in the next section.

LEMMA 2 *Let* $\mathbf{A} = Adams(\mathbf{p}, h)$. *If* $\mathbf{a}$ *is an apportionment with*

$$Max(\mathbf{p}, \mathbf{a}) = Max(\mathbf{p}, \mathbf{A})$$

*then* $\mathbf{a}$ *is an* $h'$*-extension of* $\mathbf{A}$ *for some* $h' \geq h$.

**Proof.** Let $\frac{p_k}{A_k} = Max(\mathbf{p}, \mathbf{A})$. Because the Adams method always gives priority to the state with the largest ratio of population to seats, we know that $\frac{p_l}{A_l - 1} > \frac{p_k}{A_k}$ for all $l \neq k$, since the Adams method gave state $l$ its $A_l^{th}$ seat before $k$ gets its $A_k^{th}$. Thus, $Max(\mathbf{p}, \mathbf{a}) = Max(\mathbf{p}, \mathbf{A})$ implies that $a_l \geq A_l$ for all $l$ and thus $\mathbf{a}$ is an extension of $\mathbf{A}$. ■

If $\mathbf{a}$ is an $h$-apportionment, the $h'$-Jefferson extension of $\mathbf{a}$, $JExt(\mathbf{a}, h')$ is defined recursively by:

1 $JExt(\mathbf{a}, h) = \mathbf{a}$,

2 Let $\mathbf{J} = JExt(\mathbf{a}, h' - 1)$. If $t$ is the state so that

$$\frac{p_t}{J_t + 1} = Max_i\{\frac{p_i}{J_i + 1}\}$$

then define

$$JExt(\mathbf{p}, h')_i = \begin{cases} J_i + 1, & \text{if } i = t; \\ J_i, & \text{otherwise.} \end{cases}$$

It is clear that $Jeff(\mathbf{p}, h) = JExt(\mathbf{0}, h)$, where $\mathbf{0} = (0, \ldots, 0)$, is just the usual Jefferson $h$-apportionment (Balinski and Young, 2001, page 142).

LEMMA 3 *If* $\mathbf{a}$ *is an* $h$*-apportionment, and* $h' \geq h$*, then* $JExt(\mathbf{a}, h')$ *maximizes* $Min(\mathbf{p}, \mathbf{a}')$ *over all* $h'$*-extensions of* $\mathbf{a}$.

**Proof.** An essentially equivalent fact is stated in (Balinski and Young, 2001, page 104) without proof. For completeness I include one here.

Let $\mathbf{J} = JExt(\mathbf{a}, h')$. Suppose that $\mathbf{a}'$ is another $h'$-extension of $\mathbf{a}$ for which

$$\frac{p_i}{a'_i} = Min(\mathbf{p}, \mathbf{a}') > Min(\mathbf{p}, \mathbf{J}) = \frac{p_k}{J_k}.$$

It follows that $a'_k < J_k$, and since both $\mathbf{J}$ and $\mathbf{a}'$ are $h'$-extensions, there must be some $l$ so that $a'_l > J_l$. We also know that $a_k < J_k$ and so state $k$ received at least one more seat in $JExt$ than in $\mathbf{a}$. From the definition of $JExt$, then, we know that

$$\frac{p_k}{J_k} > \frac{p_l}{J_l + 1} \geq \frac{p_l}{a'_l} > \frac{p_i}{a'_i}$$

which is a contradiction. ■

There is but one last piece of notation required. Suppose that $\mathbf{a}$ is an $h$-apportionment and let $k$ be a state. By $\mathbf{a}|_k$ I mean the $(h - a_k)$-apportionment for the states with $k$ removed.

## 3. Minimum Total Deviation Apportionment

In this section I will present the algorithm for finding an apportionment that minimizes the total deviation function $TD(\mathbf{p}, \mathbf{a})$. This algorithm first appeared in (Gilbert and Schatz, 1964) and I have done little to improve it other than to update the terminology and streamline the proof. Their idea is quite clever and deserves to be more widely known. The key to the construction is to begin an apportionment using the Adams method, but extend it using the Jefferson extension. Since the Adams method minimizes $Max(\mathbf{p}, \mathbf{a})$ and the Jefferson extension maximizes $Min(\mathbf{p}, \mathbf{a})$ combining the two methods results in minimizing the gap $TD(\mathbf{p}, \mathbf{a})$.

Let $\mathbf{p}$ be the set of state populations as before, and suppose we want to find the $h'$-apportionment that minimizes the total deviation. For $h$, $h' \geq h \geq s$, let $\mathbf{A}^h = Adams(\mathbf{p}, h)$. If $k$ is the state so that

$$\frac{p_k}{A_k^h} = Max(\mathbf{p}, \mathbf{A}^h)$$

let $\mathbf{J}^h$ be the $h'$-apportionment obtained by taking $JExt(\mathbf{A}^h|_k, h' - A_k^h)$ and then assigning $A_k^h$ seats to state $k$ . That is, $\mathbf{J}^h$ is obtained by assigning the first $h$ seats using Adams method, setting aside the state which maximizes the population/seat ratio and then extending the rest of the apportionment using Jefferson's method. Thus, $\mathbf{J}^h$ is an $h'$-apportionment of $\mathbf{p}$ for every $h$, $h' \geq h \geq s$.

THEOREM 4 *The minimum of* $TD(\mathbf{p}, \mathbf{a})$ *over all* $h'$*-apportionments is equal to*

$$\min_{\{h \mid h' \geq h \geq s\}} TD(\mathbf{p}, \mathbf{J}^h).$$

*That is, the minimum of* $TD(\mathbf{p}, \mathbf{a})$ *over all* $h'$*-apportionments is achieved by one of the* $h'$*-apportionments in the set* $\{\mathbf{J}^h\}$.

**Proof.** Suppose that $\mathbf{a}'$ is an $h'$-apportionment that achieves the minimum of $TD(\mathbf{p}, \mathbf{a})$. Let

$$\frac{p_k}{a'_k} = Max(\mathbf{p}, \mathbf{a}').$$

Since the Adams $h'$-apportionment minimizes $Max(\mathbf{p}, \mathbf{a})$ over all $h'$-apportionments, it must be true that for some $h \leq h'$ we have

$$\frac{p_k}{a'_k} = Max(\mathbf{p}, Adams(\mathbf{p}, h)).$$

By Lemma 2 we know that $\mathbf{a}'$ is an $h'$-extension of $Adams(\mathbf{p}, h)$. Thus $\mathbf{a}'|_k$ is an extension of $Adams(\mathbf{p}, h)|_k$ and so, by Lemma 3, $Min(\mathbf{p}, \mathbf{a}') \leq Min(\mathbf{p}, \mathbf{J}^h)$. Thus,

$$TD(\mathbf{p}, \mathbf{a}') \geq TD(\mathbf{p}, \mathbf{J}^h)$$

and the theorem is proved. ■

This idea of starting an apportionment using one standard method and then extending it using a different one is interesting and understudied. It is also more subtle than it might seem at first glance. One might think that the above construction done in the opposite order, i.e., start an apportionment using the Jefferson method and then extend it using Adams, would result in the same outcome, but it need not. For while no extension of an apportionment can increase $Max(\mathbf{p}, \mathbf{a})$, most extensions will result in decreasing $Min(\mathbf{p}, \mathbf{a})$. Thus following Jefferson with Adams will almost surely result in losing control of $Min(\mathbf{p}, \mathbf{a})$ and no claim similar to Theorem 4 will be true.

## 4. A Multiplicity of Minima

An unfortunate aspect of total deviation is that minimizing apportionments need not be unique. This was observed first by Gilbert and Schatz (1964) who provided an example based on a modification of the 1960 US census data. The existence of such examples was, for them, a reason to disqualify minimizing total deviation as a means of choosing an apportionment. I have argued otherwise, elsewhere (Edelman, to appear). Nevertheless, this is an unusual aspect of total deviation which is worth considering further.

If the House of Representatives were apportioned using a total deviation minimizing method, then two of the twenty-two apportionments would not have been unique. In both 1810 and 1840 there were multiple apportionments that had the same minimum total deviation. Table 1 lists the 14 different apportionments for the House in 1810 which achieve the minimum.

**Table 1.** 1810 Minimum Total Deviation Apportionments

| State | Population | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| New York | 953043 | 26 | 26 | 26 | 26 | 26 | 26 | 27 | 27 | 27 | 27 | 25 | 25 | 25 | 25 |
| Virginia | 817615 | 22 | 22 | 22 | 23 | 23 | 23 | 22 | 22 | 22 | 23 | 22 | 23 | 23 | 23 |
| Pennsylvania | 809773 | 22 | 23 | 23 | 22 | 22 | 23 | 22 | 22 | 23 | 22 | 23 | 22 | 23 | 23 |
| Massachusetts | 700745 | 20 | 19 | 20 | 19 | 20 | 19 | 19 | 20 | 19 | 19 | 20 | 20 | 19 | 20 |
| North Carolina | 487971 | 14 | 14 | 13 | 14 | 13 | 13 | 14 | 13 | 13 | 13 | 14 | 14 | 14 | 13 |
| Kentucky | 374287 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| South Carolina | 336569 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| Maryland | 335946 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| Connecticut | 261818 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| Tennessee | 243913 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| New Jersey | 241222 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| Ohio | 230760 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| Vermont | 217895 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| New Hampshire | 214460 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| Georgia | 210346 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| Rhode Island | 76888 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| Delaware | 71004 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

It is interesting to note that the Hamilton, Webster, Dean, and Hill methods all give the same apportionment in this case and that apportionment, number 4 in Table 1, is a mtd apportionment. The population data for the 1840 census, which also has multiple *mtd* apportionments has a similar property; Hamilton, Webster, Dean and Hill all agree, although this time that apportionment is not *mtd*. This suggests that one might be able to say more about population data that produces multiple *mtd* apportionments. What can one say about situations for which the *mtd* apportionment is unique? To begin, observe that the genericity assumption that the population/seat ratios are all distinct does not exclude that the *differences* of the ratios are distinct. This opens the door for two apportionments to have the same total deviation by chance. For example, consider three states, $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, with populations $4704, 2076$ and 539, respectively. One can check that the minimum total deviation is achieved by two different apportionments, $(8, 3, 1)$ and $(7, 4, 1)$. In the first apportionment state $\mathbf{B}$ has the largest population/seat ratio of 692, $\mathbf{C}$ has the smallest at 539 for a total deviation of 153. In the second apportionment, state $\mathbf{A}$ is largest (672) and $\mathbf{B}$ is smallest (519) for the same total deviation.

I will call a set of populations *hyper-generic* if, not only are the population/seat ratios distinct (for all house sizes suitably small), but the differences of population/seat ratios are also distinct. For hyper-generic populations, two apportionments can have the same total deviation only if the states achieving the maximum and minimum population/seat ratios are the same in each apportionment and the differences occur in the distribution of seats among the other states. The data from the 1810 census illustrates this situation, where Ohio has the largest population/seat ratio and New Jersey the smallest. The variation in the apportionments comes from reallocating the seats among some of the other states in such a way that their population/seat ratios stay within the range established by Ohio and New Jersey. The question then becomes when such an internal reallocation is possible. There are a few situations in which we can say something concrete about whether a *mtd* apportionment is unique:

LEMMA 5 *If the state populations are hyper-generic, and the Adams apportionment minimizes total deviation, then it is the unique apportionment that minimizes total deviation.*

**Proof.** In order for there to be another *mtd* apportionment, one must be able to reallocate a seat from one state to another. But in the Adams apportionment reducing the number of seats to any state will increase its population/seat ratio above that of the current maximum and thus increase the total deviation. ■

In the following lemmas I will use the notation from Section 3. Recall that $\mathbf{J^h}$ is the $h'$-apportionment obtained from $Adams(\mathbf{p}, h)$ by taking $JExt(\mathbf{A}^h|_k, h'{-}A_k^h)$, where $k$ is the state with the largest population/seat ratio in $Adams(\mathbf{p}, h)$, and then assigning $A_k^h$ seats to state $k$.

LEMMA 6 *Suppose the state populations* $\mathbf{p}$ *are hyper-generic and* $\mathbf{J}^h$ *is a* mtd $h'$*-apportionment. This is the unique* mtd *apportionment if*

$$JExt(\mathbf{A}^h|_k, h' - A_k^h) = Jeff(\mathbf{p}|_k, h' - A_k^h).$$

*That is, if the Jefferson extension agrees with the actual Jefferson apportionment, then the* mtd *apportionment will be unique.*

**Proof.** It follows from Lemma 3 that any $h' - A_k^h$-apportionment $\mathbf{a}$ for the states $\mathbf{p}_k$ will have

$$Min(\mathbf{p}_k, \mathbf{a}) < Min(\mathbf{p}_k, Jeff(\mathbf{p}|_k, h' - A_k^h) = JExt(\mathbf{A}^h|_k, h' - A_k^h).$$

Thus $\mathbf{J}^h$ must be the unique *mtd* apportionment. ■

LEMMA 7 *Suppose that* $\mathbf{J}^h$ *is a* mtd $h'$*-apportionment, that it differs from* $Adams(\mathbf{p}, h)$ *in at least 2 states, and* $JExt(\mathbf{A}^h|_k, h' - A_k^h) \neq Jeff(\mathbf{p}|_k, h' - A_k^h)$. *Then the* mtd $h'$*-apportionment is not unique.*

**Proof.** That

$$JExt(\mathbf{A}^h|_k, h' - A_k^h) \neq Jeff(\mathbf{p}|_k, h' - A_k^h)$$

implies that there is some state $j \neq k$, with seat allocation $J_j^h$, so that

$$\frac{p_j}{J_j^h + 1} > Min(\mathbf{p}, \mathbf{J}^h).$$

Moreover, since there are at least two states on which $\mathbf{J}^h$ differs from $Adams(\mathbf{p}, h)$, there must be a state $i$ different from both $j$ and $k$ so that

$$\frac{p_i}{J_i^h - 1} < Max(\mathbf{p}, \mathbf{J}^h)$$

and hence the transfer of a seat from state $j$ to state $i$ will result in an $h'$-apportionment with the same total deviation as $\mathbf{J}^h$. ■

These three lemmas leave just a little uncertainty about the nature of the *mtd* apportionments that are not unique. The remaining case is if $\mathbf{J}^h$, the *mtd* apportionment, differs from $Adams(\mathbf{p}, h)$ in only one state. Such apportionments may or may not be unique depending on the existence of a second state to which a seat can be added without decreasing $Min(\mathbf{p}, \mathbf{J}^h)$. Either possibility can arise.

## 5. Bias

A traditional concern in apportionment is whether there is an inherent bias in the method with respect to the size of the state. It is well-established (Balinski and Young, 2001, Chapter 9) that among standard methods, Hamilton and Webster are unbiased while Adams is biased toward small states and Jefferson is biased toward large ones. What can one say about the bias inherent in the *mtd* apportionment?

As noted in the previous section, in 20 of the 22 apportionments of the United States, the *mtd* apportionment was the same as the Adams apportionment. Thus, one might conclude that the *mtd* apportionment has a bias to small states. The difficulty with this line of reasoning is that, as shown previously, there may be a multiplicity of *mtd* apportionments and in those situations, there may be little or no bias. For example, as previously noted, among the *mtd* apportionments for the 1810 census data is the apportionment that agrees with the Hamilton, Webster, Hill and Dean methods.

So, to prove anything conclusively we would need more detailed information on two aspects of *mtd* apportionments; first, how often are there multiple *mtd* apportionments, and second, can we choose among multiple *mtd* apportionments in such a way as to minimize the resulting bias overall. The results in the previous section are but a small step in the first direction. The second is totally unresearched.

## 6. Alabama Paradox

A method of apportionment is said to exhibit the Alabama paradox if an increase in the size of the house may result in the decrease in the number of seats allocated to a state. It is well-known that the divisor methods do not exhibit the paradox, while the Hamilton method does. In what way does the *mtd* apportionment behave?

Since *mtd* apportionments may not be unique one must be careful in how this problem is phrased. There is no question that if the *mtd* apportionments are chosen injudiciously the Alabama paradox may result. Consider the apportionment problem (taken from Figure 1) that consists of states New York, Virginia, Pennsylvania, New Jersey, and Ohio, with populations 953043, 817615, 809773, 241222, and 230760, respectively. One *mtd* 83-apportionment is 26, 22, 22, 7, and 6 seats for each state, respectively. A *mtd* 84-apportionment is 25, 23, 23, 7, and 6. So this pair of apportionments exhibits the Alabama paradox. On the other hand, 25, 23, 22, 7, and 6 is also a *mtd* 83-apportionment, which would show no Alabama paradox. It is also true that 26, 23, 22, 7, and 6 is a *mtd* 84-apportionment. So, by making an appropriate choice among the *mtd* apportionments one need not have the Alabama paradox manifest itself in this instance.

Can one always avoid the Alabama paradox in this way? I don't know. Balinski and Young (Balinski and Young, 2001, Proposition 3.9) assert that apportionments can exhibit the Alabama paradox, but they give no specific example. This leaves it unclear whether they were referring to the phenomenon just discussed or a more robust example in which the Alabama paradox is unavoidable.

## 7. Conclusion

What I have presented here is a method of apportionment designed to minimize total deviation, a particular measure of harm in malapportionments. It is the measure of harm that has been recognized by the United States Supreme Court. While this method has less desirable behavior than standard methods from an axiomatic point-of-view, that does not mean that it is inappropriate for certain purposes. And it certainly does not mean that it is not an interesting method worth studying further.

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