Apportionment: Uni- and Bi-Dimensional

Michel Balinski
Laboratoire d'Économétrie, École Polytechnique

Abstract This paper characterizes divisor methods for vector and matrix apportion problems with very simple properties. For the vector problem—a vector gives the votes of parties or the populations of states, a single number the size of the house—they are shown to be the only methods that are *coherent with* the definition of the corresponding divisor method when applied to only two states or parties. For the matrix problem—rows correspond to districts, columns to parties, entries to votes for party-lists, and the number of seats due to each row (or district) and each column (or party) is known—one extra property is necessary. The method must be *proportional*: it must give identical answers to a problem obtained by re-scaling any rows and/or any columns of the matrix of votes.

Keywords: Apportionment, divisor method, coherence, biproportional apportionment, rounding, justified rounding.

1. Introduction

"Bi-dimensional" (or "matrix") apportionment is now a recognized system for designating winners in an election system. It is the law of the land in the Swiss canton and the city of Zürich (Pukelsheim and Schuhmacher, 2004), and it may well become so in the Faroe Islands (Zachariassen and Zachariassen, 2005). Developed, justified, explained and applied in a series of papers and a book (Balinski and Demange, 1989a,b; Balinski and Rachev, 1997; Balinski and Ramírez, 1997,1999a; Balinski, 2002, 2004) it may also be viewed as a simple and direct extension of the more familiar "uni-dimensional" (or "vector") apportionment problem. That is what this paper aims to do.

2. Vector Apportionment: a Primer

A *vector (or uni-dimensional) apportionment problem* is a pair (v, h), where $v = (v_i) > 0$ for $i = 1, \ldots, m$ are the populations of m regions (or the votes of m parties) and h is the number of seats in an assembly to be distributed "proportionally" among them. An *apportionment* is a vector $a = (a_1, \ldots, a_m)$, where $a_i \geq 0$ is integer valued and $\sum_i a_i = h$. Vector apportionment is the classical problem of allocating seats to regions or states when v is the vector of their populations, or of allocating seats to political parties when v is the vector of their votes: by what method should a solution be chosen from among the many possible apportionments?

In general, a *(vector) method of apportionment* Φ selects a nonempty subset of apportionments $\Phi(v,h)$ for any problem (v,h).

A *divisor criterion* is any real valued function d on the nonnegative integers $k \geq 0$ that satisfies $k \leq d(k) \leq k+1$ and for which there are no two integers $p > 0$ and $q \geq 0$ where $d(p) = p$ and $d(q) = q+1$. In effect, a divisor criterion is simply a point on each closed interval $[k, k+1]$ for $k \geq 0$ and integer, with the stipulation that if in some interval the point is at the lower (the upper) end then in no other interval can it be at the upper (the lower) end. Suppose that a real number x is in the interval $[a, a+1]$, a an integer. Then a *d-rounding* $[x]_d$ of $x > 0$ is a if $x < d(a)$ and $a+1$ if $x > d(a)$; if $x = d(a)$ then $[x]_d$ is either a or $a+1$ (so in fact $[x]_d$ is a set that is usually single valued). A d-rounding of 0 is always 0: $[0]_d = 0$. The $d(a)$ are thresholds in the intervals $[a, a+1]$: below the threshold x is rounded-down to a, above it is rounded-up to $a+1$, at the threshold it is either rounded-up or -down.

A *divisor method based on* d is the set of apportionments

$$\Phi_d(v,h) = \left\{ a = (a_i) : a_i = [\lambda v_i]_d \text{ for } \lambda \text{ chosen so that } \sum_i a_i = h \right\}. \quad (1)$$

If, contrary to the definition, $d(p) = p, d(q) = q+1$ for some integers $p > 0, q \geq 0$, then $(p-1, q+1) \in \Phi_d\big((p,q), p+q\big)$, showing that although a perfect apportionment exists it may not be chosen, and explaining the exclusion. Note also that $d(0) = 0$ implies $[\lambda v_i]_d \geq 1$ for every $\lambda > 0$ and $v_i > 0$.

If $a \in \Phi_d$ then $d(a_i - 1) \leq \lambda v_i \leq d(a_i)$ for all i, implying

$$\Phi_d(v,h) = \left\{ a = (a_i) : \min_{a_i > 0} \frac{v_i}{d(a_i - 1)} \geq \max_{a_j \geq 0} \frac{v_j}{d(a_j)}, \sum_i a_i = h \right\}, \quad (2)$$

where $v_j/0 = \infty$ and $d(-1) = 0$. Consequently, a divisor method may also be described recursively as follows. $\Phi_d(v,0) = 0$ and suppose $a \in \Phi_d(v,h)$. Then

$$\bar{a} \in \Phi_d(v, h+1) \text{ where } \bar{a} = a, \text{ except } \bar{a}_l = a_l + 1 \text{ for } \frac{v_l}{d(a_l)} = \max_i \frac{v_i}{d(a_i)}. \quad (3)$$

This description implies that $a \in \Phi_d(v,h)$ if a solves

$$\max_a \min_i \frac{v_i}{d(a_i - 1)} \text{ when } \sum_i a_i = h \text{ and } a_i \geq 0 \text{ integer}. \quad (4)$$

From the recursive definition it also follows that an apportionment a of the divisor method Φ_d is a solution to

$$\max_a \sum_i \sum_{k=0}^{a_i-1} \frac{v_i}{d(k)} \text{ when } \sum_i a_i = h \text{ and } a_i \geq 0 \text{ integer,} \tag{5}$$

(assuming $h \geq m$ if $d(0) = 0$). This is easy to see because the solution is "greedy": at each allocation of an "extra" seat give to the integer variable k the value that maximizes $v_i/d(k)$ over i in conformity with the recursive procedure. Precisely the same argument shows that an apportionment a of the divisor method Φ_d is also a solution to

$$\max_a \prod_i \prod_{k=0}^{a_i-1} \frac{v_i}{d(k)} \text{ when } \sum_i a_i = h \text{ and } a_i \geq 0 \text{ integer.} \tag{6}$$

There are many other "objective functions" that are optimized by the apportionments of one or another of the divisor methods.

The *parametric (divisor) method* Φ_δ based on δ, for $0 \leq \delta \leq 1$, is the divisor method Φ_d based on d where $d(k) = k + \delta$ for all integer $k \geq 0$. Adams's method is the parametric method based on $\delta = 0$; Condorcet's method is the parametric method based on $\delta = \frac{2}{5}$; Webster's or Sainte-Laguë's is based on $\delta = \frac{1}{2}$; and Jefferson's or D'Hondt's is based on $\delta = 1$.

Letting $\bar{v} = \sum v_i$, an apportionment a of the parametric method Φ_δ is a solution to (see Balinski and Ramírez (1999b)):

$$\min_a \sum_i v_i \left(\frac{a_i + \delta - \frac{1}{2}}{v_i} - \frac{h}{\bar{v}} \right)^2 \text{ when } \sum_i a_i = h \text{ and } a_i \geq 0 \text{ integer.} \tag{7}$$

Notice that solutions to the optimization problems (4), (5) and (6) do not change when v is replaced by λv for any $\lambda > 0$.

There are an infinite number of divisor methods for vector problems, and they can yield very different apportionments. They have been characterized by a set of properties so desirable in the context of apportionment that it is fair to say they are the *only* acceptable methods (Balinski and Young, 1982). The most important of these properties—coherence—stems from a very simple idea.

Suppose that the h seats have been apportioned among the several regions in a manner that is "fair". Any subset of the regions could reasonably ask the question: Do our shares represent a "fair" division among us considered as a *separate* group? Suppose that they believed that a different division of their pooled shares would be more fair. In that case it would be possible to substitute this different division for their initial apportionment to obtain a new apportionment among all that is “fairer” for the regions of the subset, and the same for all others. How could one then affirm that the initial division was fair to all? A rule that did this would surely be judged to be “incoherent”! The idea is captured in the slogan: “Any part of a fair division must be fair.”

To be precise, consider an apportionment chosen by a method—a “global” apportionment—sum the seats it assigns to any subset of the regions, and consider the apportionment(s) obtained by applying the same method to redistribute this sum among the members of the subgroup (each of the latter set is a *local apportionment*). The method is *coherent* when two properties hold: (i) the shares assigned to each of the regions of the subset by the original (global) apportionment is a local apportionment and (ii) if there is another local apportionment among the regions of the subgroup, then another (global) apportionment of the method to all the regions is found as follows: substitute the shares in the local apportionment for what those regions have in the original apportionment.

For example, imagine an apportionment of the seats in the U.S. House of Representatives: if the method that is used yields an apportionment that gives 29 of the country’s congressional seats to New York and 53 to California (as it did in accordance with the 2000 census), then surely the same method should divide the sum of 82 seats between New York and California in the same way. If the method *also* gives rise to a local apportionment that assigned 28 seats to New York and 54 to California, then replacing their shares in the initial apportionment with these should yield a second apportionment that belongs to the method. There would then be two possible apportionments of the House (a theoretical possibility unlikely to occur in practice). The concept of a coherent method[1] is quite general and is germane to many problems of fair division (Balinski, 2005), but it first arose in the context of the apportionment problem (Balinski and Young, 1982) where it is particularly important.

3. From Divisions Between Two to Divisions Among All in Uni-Dimensional Apportionment

Given any apportionment problem (v, h), consider an apportionment obtained by a coherent rule. By definition, every pair of regions must share the seats they receive together h' in accordance with the rule applied to those regions when they are to be allocated h' seats. This immediately implies that knowing how to divide any number of seats between any *two* regions (meaning two regions having any number of inhabitants) suffices to completely determine a coherent rule. Deciding how to divide seats between only two regions is obviously an easier task than deciding how to divide seats among an arbitrary

[1]Coherence was earlier called “uniformity” and also “consistency.”

number: this idea is pursued here to show how simple it is to extend vector apportionment to matrix apportionment.

Let Φ^2 be a method for dividing any number of seats between two regions (or parties), and Φ^2_d be the divisor method based on d for dividing any number of seats between two regions (or parties). The methods Φ^2_d enjoy two evident properties that will shortly be used. First, they are "monotone": if $(a_1, a_2) \in \Phi^2_d\big((v_1, v_2), h\big)$ and $(a'_1, a'_2) \in \Phi^2_d\big((v_1, v_2), h'\big)$ for $h' > h$ then $a'_1 \geq a_1$ and $a'_2 \geq a_2$. Second, if $\{(a_1, a_2), (a'_1, a'_2)\} \subseteq \Phi^2_d\big((v_1, v_2), h\big)$ then $|a_1 - a'_1| \leq 1$ and $|a_2 - a'_2| \leq 1$: two apportionments for a same h can differ by at most 1.

Given any two apportionments a, b of a problem (v, h), consider $a - b$. It is a vector of integers summing to 0: so a may be obtained from b in a sequence of changes each of which transfers one seat from one party (or region) i where $b_i > a_i$ to another j where $a_j < b_j$: "local" change always involves just two parties (or regions).

A single property suffices to determine a divisor method.

PROPERTY 3 *A method Φ for vector problems is said to be* ***coherent with*** Φ^2 *if for every pair i, j*

$$a \in \Phi(v, h) \textit{ implies } (a_i, a_j) \in \Phi^2\big((v_i, v_j), a_i + a_j\big). \textit{ Moreover,}$$

$$(b_i, b_j) \in \Phi^2\big((v_i, v_j), a_i + a_j\big) \textit{ implies } a' \in \Phi(v, h),$$
$$\textit{where } a' = a \textit{ except } a'_i = b_i, a'_j = b_j.$$

The really essential—and at first blush surprising—point about coherence is that Φ treats *every pair i, j* exactly as does Φ^2. In addition, if a change could be made that agrees with Φ^2 then it would yield another apportionment of Φ.

THEOREM 1 *The unique method Φ for vector problems that is coherent with Φ^2_d is the divisor method Φ_d.*

Proof. The condition is necessary, since Φ_d is obviously coherent with Φ^2_d.

To see that the condition is sufficient, suppose Φ is any method that is coherent with Φ^2_d and that $a \in \Phi(v, h)$. It is shown that $a \in \Phi_d(v, h)$. Choose $\lambda > 0$ such that $\sum_i [\lambda v_i]_d = h$, and let $b_i = [\lambda v_i]_d$, so $b \in \Phi_d(v, h)$, implying, in particular, that $(b_i, b_j) \in \Phi^2_d\big((v_i, v_j), b_i + b_j\big)$ for every pair i, j. The coherence of Φ with Φ^2_d implies that $(a_i, a_j) \in \Phi^2_d\big((v_i, v_j), a_i + a_j\big)$ for every pair i, j. Suppose $b \neq a$. Then there exists a pair i, j for which $a_i > b_i$ and $a_j < b_j$. But this is impossible unless $a_i + a_j = b_i + b_j$ so $a_i = b_i + 1$, $a_j = b_j - 1$ and $\{(a_i, a_j), (b_i, b_j)\} \subseteq \Phi^2_d\big((v_1, v_2), a_i + a_j\big)$. The coherence of Φ_d with Φ^2_d implies that substituting (a_i, a_j) for (b_i, b_j) in b yields an apportionment that belongs to $\Phi_d(v, h)$ which agrees with more components of a than did b. Repeating the same argument until a is obtained shows that $a \in \Phi_d(v, h)$, and so completes the proof.

So, if one wishes to verify that $a \in \Phi_d(v,h)$ it suffices to check that $(a_i,a_j) \in \Phi_d^2\big((v_i,v_j),a_i+a_j\big)$ for every pair i,j: "local" conditions determine the solution, just as in well-behaved optimization problems no possible local "improvement" implies the solution in hand is an optimum. The analogy with optimization carries further: the λ found in (1) may be viewed as a kind of "dual" variable which, once known, makes it possible to solve the problem by assigning the obvious value to each a_i *independently*, namely, by taking it to be a d-rounding of λv_i.

The parametric methods Φ_δ, $0 \le \delta \le 1$, each have a particularly simple closed-form formula for calculating how two regions divide any number of seats between them. The simplest—and most natural—of all is Webster's or Sainte-Laguë's: the proportional share of each is rounded to the closest integer. For a positive real number x, suppose $x = n + r$, n integer and $0 \le r < 1$. Define $[x]_\delta = n$ if $r \le \delta$, and $[x]_\delta = n+1$ if $r \ge \delta$ (so $[n+\delta]_\delta = n$ or $n+1$). It is a simple exercise to show:

LEMMA 2 *Given a two-region problem* $\big((v_1,v_2),h\big)$, *let* $\bar{v}_k = v_k/(v_1+v_2)$, $k = 1,2$. *Then* $a_k = \big[\bar{v}_k(h+2\delta-1)\big]_\delta$, $k = 1,2$ *(and if* $\bar{v}_k(h+2\delta-1)$ *has a remainder of exactly* δ *then one of the* a*'s is rounded-up, the other is rounded-down).*

4. Matrix Apportionment: a Primer

A *matrix (or two-dimensional) apportionment problem* is a triple (v,r,c), where $v = (v_{ij}) \ge 0$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$ is a nonnegative matrix with no row or column of 0's, and $r = (r_1,\ldots,r_m) > 0$ and $c = (c_1,\ldots,c_n) > 0$ are vectors of integers whose sums are equal, $\sum r_i = \sum c_j = h$. An *apportionment* is a matrix $a = (a_{ij})$, where $a_{ij} \ge 0$ is integer valued, $\sum_j a_{ij} = r_i$ for all i and $\sum_i a_{ij} = c_j$ for all j. Matrix apportionment is a more recent problem where v_{ij} is the vote of party i's list in region j, r_i is the number of seats deserved by each party i (on the basis of the total vote of all of its lists $\sum_j v_{ij}$, for example) and c_j is the number of seats assigned to each region j (typically on the basis of its population): which one of the many possible apportionments a should be chosen?

In general, a *(matrix) method of apportionment* Φ selects a nonempty subset of apportionments $\Phi(v,r,c)$ for any problem (v,r,c).

When $v_{ij} = 0$ (or, more generally, when v_{ij} is less than some preset positive threshold) imposes that $a_{ij} = 0$, it may be that no apportionment exists. But this is rather unlikely. The example of figure 1 is typical of the only situations when none exists. The subset of regions (or columns) J that consists of the 4th through the 7th regions are to receive together a total of 8 seats (in general, $c(J) = \sum_J c_j$ seats). The subset of parties (or rows) I_J each of which received some votes (or more than the threshold of votes) from at least one of the regions

	1st	2nd	3rd	4th	5th	6th	7th	seats
Party 1	+	+	+	+	+	+	+	2
Party 2	+	+	+	+	+	+	+	5
Party 3	+	+	+	0	0	0	0	4
Party 4	+	+	+	0	0	0	0	6
Seats	4	2	3	1	1	2	4	

Fig. 1. Example of votes that allows no feasible apportionment. ($+$ means any number of votes, 0 means no votes or too few to permit a seat.)

of J, namely, the parties 1 and 2, deserve a total of 7 seats (in general, $I_J = \{i : v_{ij} = + \text{ for some } j \in J\}$ having a total of $r(I_J) = \sum_{I_K} r_i$ seats). Thus the regions J are to have 8 seats but they can only fill them from candidates of the parties I_J who deserve 7 seats: clearly, there can be no apportionment in this case. If regions assigned 8 seats give all of their votes to parties that in total only deserve 7 seats, the total turnout in those regions must be abnormally low. There is a symmetric explanation. The set I consisting of parties 3 and 4 deserve $r(I) = 10$ seats; the set J_I consisting of those regions who gave some votes (or more than the threshold) to at least one of the lists of parties I, namely, regions 1, 2, and 3, are to receive $c(J_I) = 9$ seats. This is again clearly impossible since it asks that parties deserving 10 seats get them all from regions having only 9 seats, but also unlikely for in this case the turnout in the regions J_I must be abnormally high. Yet, as the following theorem shows, this is the only situation that can deny the existence of apportionments (its proof is easily deduced from duality in linear programming or from the min-cut, max-flow theorem of network flows).

THEOREM 3 *There exist apportionments if and only if* $c(K) \leq r(I_K)$ *for every subset of the regions (or columns)* K.

A problem that has apportionments will be said to be *feasible*.

A *(matrix) divisor method based on* d is, for any feasible problem, the set of apportionments:

$$\Phi_d(v, r, c) = \tag{8}$$
$$\{a = (a_{ij}) : a_{ij} = [\lambda_i v_{ij} \mu_j]_d \text{ for } \lambda, \mu \text{ such that } \sum_j a_{ij} = r_i \text{ and } \sum_i a_{ij} = c_j\}.$$

Note, again, that $d(0) = 0$ implies $[\lambda_i v_{ij} \mu_j]_d \geq 1$ for every $\lambda_i v_{ij} \mu_j > 0$. This means that $\Phi_d(v, r, c)$ may be empty when $d(0) = 0$ despite the fact that the problem is feasible. In order for $\Phi_d(v, r, c)$ to be nonempty when $d(0) = 0$ there must exist an apportionment a that satisfies the row- and column- equations and also $a_{ij} \geq 1$ when $v_{ij} > 0$ and $a_{ij} = 0$ when $v_{ij} = 0$: call such problems *super-feasible*.

Properties of matrix methods of apportionment similar to those appealed to for vector methods of apportionment characterize the matrix divisor methods (Balinski and Demange, 1989a).

THEOREM 4 *(Balinski and Demange, 1989b). For any feasible matrix problem (v, r, c) there exist multipliers λ, μ and an $a \in \Phi_d(v, r, c)$ with $a_{ij} = [\lambda_i v_{ij} \mu_j]_d$ when $d(0) > 0$; and when $d(0) = 0$ the same is true if the problem is super-feasible. The multipliers are not unique, and there may be several apportionments in Φ_d; however, if there is more than one apportionment in Φ_d, all of them are obtained with a same set of multipliers.*

5. From Divisions Between Two to Divisions Among All in Bi-Dimensional Apportionment

One property sufficed to determine a divisor method for uni-dimensional problems. For bi-dimensional problems two properties suffice.

Given an m by n matrix v, an m-vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ and an n-vector $\mu = (\mu_1, \ldots, \mu_n)$, let $\lambda \circ v \circ \mu = (\lambda_i v_{ij} \mu_j)$; that is, the matrix obtained from v by multiplying its ith row by λ_i and its jth column by μ_j, for all i, j.

An essential property in vector apportionment is that a method (*any* method) should yield the same solutions to (v, h) and to $(\lambda v, h)$ for any scalar $\lambda > 0$: that is, how votes are scaled should make no difference. It came for free in the uni-dimensional case, but must be called upon in the bi-dimensional case. Since a party i (or row) deserves a fixed number of seats r_i, rescaling by multiplying its votes by $\lambda_i > 0$ should (as in the vector problem) change nothing; symmetrically, since a region j is assigned a fixed number of seats c_j, rescaling its votes by $\mu_j > 0$ should (as in the vector problem) change nothing as well.

PROPERTY 4 *A method Φ for matrix problems is said to be **proportional** if*

$$\Phi(v, r, c) = \Phi(\lambda \circ v \circ \mu, r, c) \text{ for every real } \lambda, \mu > 0.$$

Given any two apportionments a, b of a problem (v, r, c), consider $a - b$. It is a matrix of integers each of whose rows and columns sums to 0: so a may be obtained from b in a sequence of changes each of which transfers 1 seat from one to another entry of the matrix within a *simple cycle* C

$$\begin{array}{ccccccccccc}
i(1)j(1) & & i(2)j(2) & & \cdots & & i(k-1)j(k-1) & & i(k)j(k) & & i(1)j(1) \\
\downarrow & \nearrow & \downarrow & \nearrow & & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
i(1)j(2) & & i(2)j(3) & & \cdots & & i(k-1)j(k) & & i(k)j(1) & &
\end{array} \qquad (9)$$

for which $b_{i(s)j(s)} > a_{i(s)j(s)}$ and $b_{i(s)j(s+1)} < a_{i(s)j(s+1)}$ for $s = 1, \ldots, k$, where $k + 1$ is taken as 1, the indices $i(s)$ are different and so are the indices $j(s)$. The change decreases the $b_{i(s)j(s)}$ by 1 and increases the $b_{i(s)j(s+1)}$

by 1 in the cycle: "local" change always involves party-regions (i,j) that form a cycle. To simplify the description below, rename the "even" entries $(i(s),j(s))=(s,s)$ and the "odd" entries $(i(s),j(s+1))=(s,s+1)$ and $(i(k),j(1))=(k,1)$.

PROPERTY 5 *A method Φ for matrix problems is said to be* ***coherent with*** Φ^2 *if for any problem (v,r,c) there exists an equivalent problem (v',r,c), $v'=\lambda\circ v\circ\mu$, for which*

$$a\in\Phi \text{ implies } (a_{kl},a_{st})\in\Phi^2\big((v'_{kl},v'_{st}),a_{kl}+a_{st}\big),$$

for every pair of indices $(k,l),(s,t)$. Moreover, suppose that for some simple cycle C as in (9) there is a b for which

$$(b_{ss},b_{ss+1})\in\Phi^2\big((v'_{ss},v'_{ss+1}),a_{ss}+a_{ss+1}\big) \text{ for all } s(\text{mod } k), \text{ and}$$

$$(b_{s-1s},b_{ss})\in\Phi^2\big((v'_{s-1s},v'_{ss}),a_{s-1s}+a_{ss}\big) \text{ for all } s(\text{mod } k).$$

Then

$$a'\in\Phi(v',r,c), \text{ where } a'=a \text{ except } a'_{ij}=b_{ij} \text{ for } (i,j)\in C.$$

Again, the really essential—and at first blush surprising—point about coherence is that Φ treats *every pair* $(k,l),(s,t)$ exactly as does Φ^2 relative to the equivalent problem (v',r,c). In addition, if a change could be made that agrees with Φ^2 with respect to the equivalent problem, then that would yield another apportionment of Φ. But a change in a matrix apportionment implies at least a change in a simple cycle C.

THEOREM 5 *The unique proportional method for matrix problems Φ that is coherent with Φ_d^2 is the divisor method Φ_d.*

Proof. The conditions are necessary since Φ_d is obviously proportional and coherent with Φ_d^2.

To see that the conditions are sufficient, suppose Φ is any proportional method that is coherent with Φ_d^2 and that $a\in\Phi(v,r,c)$. It will be shown that $a\in\Phi_d(v,r,c)$.

There exist $\lambda^b>0,\ \mu^b>0$ so that $b=(b_{ij})\in\Phi_d(v,r,c)$, where $b_{ij}\in[\lambda_i^b v_{ij}\mu_j^b]_d$.

$a\in\Phi(v,r,c)$ and Φ coherent with Φ_d^2 implies there exist $\lambda^a>0,\ \mu^a>0$ so that $a_{ij}\in[\lambda_i^a v_{ij}\mu_j^a]_d$.

Suppose $a\neq b$. Then for some (i,j), $a_{ij}<b_{ij}$, and (i,j) belongs to a simple cycle C as in (8). Simplifying the notation again, let $\{(1,1),(2,2),\dots,(s,s)\}$ be the even entries and $\{(1,2),\dots,(s-1,s),(s,1)\}$ be the odd entries, so that $a_{ii}<b_{ii},\ a_{ii+1}>b_{ii+1}$ for $i=1,\dots,s(\text{mod } s)$. Multiplying λ^a by

λ_1^b/λ_1^a and dividing μ^a by the same amount changes nothing, so it may be assumed that $\lambda_1^a = \lambda_1^b$. Now notice that in general, if x, y are reals, $a \in [x]_d$ and $b \in [y]_d$, then $a > b$ implies $x \geq y$ and $a > b + 1$ implies $x > y$.

Begin the cycle C at $(1, 1)$ and follow it with the indices increasing. $\lambda_1^a = \lambda_1^b$ and $a_{11} < b_{11}$ implies $\mu_1^a \leq \mu_1^b$ (with strict inequality if $a_{11} + 1 < b_{11}$). Also, $\lambda_1^a = \lambda_1^b$ and $a_{12} > b_{12}$ implies $\mu_2^a \geq \mu_2^b$ (with strict inequality if $a_{12} > b_{12} + 1$). But $\mu_2^a \geq \mu_2^b$ and $a_{22} < b_{22}$ implies $\lambda_2^a \leq \lambda_2^b$ (with strict inequality if either $a_{22} + 1 < b_{22}$ or $\mu_2^a > \mu_2^b$). Continuing around the cycle, $\mu_s^a \geq \mu_s^b$ and $a_{ss} < b_{ss}$ implies $\lambda_s^a \leq \lambda_s^b$ (with strict inequality if either $a_{ss} + 1 < b_{ss}$ or $\mu_s^a > \mu_s^b$). But this means that $\lambda_s^a \leq \lambda_s^b$, $\mu_1^a \leq \mu_1^b$ and $a_{s1} > b_{s1}$, a contradiction *unless* the following holds: $\lambda_i^a = \lambda_i^b$, $\mu_i^a = \mu_i^b$ for $i = 1, \ldots, s$ *and* the differences between the values of the a's and b's in the cycle C are all exactly 1. But in this case there is a massive "tie". Defining $b' = b$, except that every b-entry in the cycle C is replaced by the corresponding value of a, another apportionment $b' \in \Phi_d(v, r, c)$ is obtained. Repeating the same argument until a is obtained shows that $a \in \Phi_d(v, r, c)$, and so completes the proof.

The analogy with optimization may be carried further here as well: the vectors λ, μ found in (7) may be thought of as "dual" variables which, once known, make it possible to solve the problem by assigning the obvious (or "greedy") value independently to each a_{ij}, namely, by taking it to be a d-rounding of $\lambda_i v_{ij} \mu_j$.

There can be no multipliers λ, μ that yield different solutions, showing that uni- and bi-proportional apportionments are, in essence, the same problem, and that a matrix apportionment treats every pair $(i, j), (k, l)$ fairly.

Acknowledgements

I am indebted to my friend and colleague, Friedrich Pukelsheim, for incisive constructive comments and corrections.

References

M. Balinski (2004), *Le suffrage universel inachevé*, Éditions Belin, Paris.

M. Balinski (2002), "Une 'dose' de proportionnelle: le système électorale mexicain," *Pour la science*, 58-59.

M. Balinski (2005), What is just? *American Mathematical Monthly*, **112**, 502-511.

M. Balinski and G. Demange (1989a), An axiomatic approach to proportionality between matrices, *Mathematics of Operations Research* **14**, 700-719.

M. Balinski and G. Demange (1989b), Algorithms for proportional matrices in reals and integers, *Mathematical Programming* **45**, 193-210.

M. Balinski and S. Rachev (1997), Rounding proportions : methods of rounding, *The Mathematical Scientist* **22**, 1-26.

M. Balinski and V. Ramírez (1997), Mexican electoral law: 1996 version, *Electoral Studies* **16**, 329-340.

M. Balinski and V. Ramírez (1999a), Mexico's 1997 apportionment defies its electoral law, *Electoral Studies* **18**, 117-124.

M. Balinski and V. Ramírez (1999b), Parametric methods of apportionment, rounding and production, *Mathematical Social Sciences* **37**, 107-122.

M.L. Balinski and H.P. Young (1982), *Fair Representation: Meeting the Ideal of One Man, One Vote*, Yale University Press, New Haven; 2nd ed., Brookings Institution Press, Washington, D.C., 2001.

F. Pukelsheim and C. Schuhmacher (2004), Das neue Zürcher Zuteilungsverfahren für Parlamentswahlen, *Aktuelle Juristichische Praxis* **5**, 505-522.

P. Zachariassen and Martin Zachariassen (2005), A comparison of electoral formulae for the Faroese Palriament (The Løgting), Technical report, Náttúruvísindadeildin, Føroya, 1 August 2005.