The Sunfish Against the Octopus: Opposing Compactness to Gerrymandering

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Abstract Gerrymandering - the artful and partisan manipulation of electoral districts - is a well known pathology of electoral systems, especially majoritarian ones. In this paper, we try to give theoretical and experimental answers to the following questions: 1) How much biased can the assignment of seats be under the effect of gerrymandering? 2) How effective is compactness as a remedy against gerrymandering? Accordingly, the paper is divided into two parts. In the first one, a highly stylized combinatorial model of gerrymandering is studied; in the second one, a more realistic multiobjective graph-partitioning model is adopted and local search techniques are exploited in order to find satisfactory district designs. In a nutshell, our results for the theoretical model mean that gerrymandering is as bad as one can think of and that compactness is as good as one can think of. These conclusions are confirmed to a large extent by the experimental results obtained with the latter model on some medium-large real-life test problems.

Keywords: Gerrymandering, partition, graph coloring.

1. Introduction

Gerrymandering - the partisan manipulation of electoral district boundaries - has plagued modern democracies since their early times. Far from being defeated, it keeps displaying its perverse effects even at present (Balinski, 2004). It was only with the rise of the electronic computer that researchers started thinking about neutral and rational procedures for political districting. Its nature as a multicriteria decision problem was soon recognized. Suppose that the territory is subdivided into elementary administrative units (counties, townships, wards,..). The most commonly adopted districting criteria are the following: integrity (no unit may be split between two or more districts); contiguity (the units within the same district should be geographically contiguous);

population equality (the district populations should be equal or nearly equal, especially in majoritarian systems); compactness (each district should be compact, that is, “closely and neatly packed together” (Oxford Dictionary)); conformity to administrative boundaries (the electoral district boundaries should not cross other administrative boundaries, such as those of regions, provinces, local or minority communities). Among these criteria, compactness stands as a powerful weapon against gerrymandering, since it bans indented or elongated districts: a sunfish-shaped district is deemed to be compact, while an octopus-shaped or an eel-shaped one is not.

The present paper deals with the following two basic problems:

1) How bad can gerrymandering be?

2) How effective is compactness in preventing gerrymandering?

We shall give both theoretical and experimental answers to these two problems. Accordingly, our paper is divided into two parts. In the first one, an idealized combinatorial model is investigated; in the second part, a more realistic and flexible multicriteria graph-theoretic model is adopted, and computational results are presented for some medium to large real-life test problems.

2. A Combinatorial Gerrymandering Model

As a motivation for the present section we mention a striking artificial example of gerrymandering given by Dixon and Plischke (1950). Suppose that only two parties P and C compete under a first-past-the-post system and that, as in Figure 1, the territory is divided into elementary units having the same population with an homogeneous electoral behavior, that is, the whole population of an elementary unit votes for the same party. If the district map of Figure 1 (a) is adopted, party C wins in 8 districts out of 9; however, if the alternative district map of Figure 1(b) is adopted, party C wins only in 2 districts out of 9, so the outcome is drastically reversed.

A careful look at Figure 1 gives us a clue about an effective strategy for maximizing the number of districts won by either party: the districts should be designed so that every win should be close and every loss should be sweeping.

In this section we shall consider an idealized graph-theoretic formulation that captures the essence of the artificial example by Dixon and Plischke. Given a territory composed by territorial units, define the following integers:

- n is the number of territorial units;
- p is the number of districts;

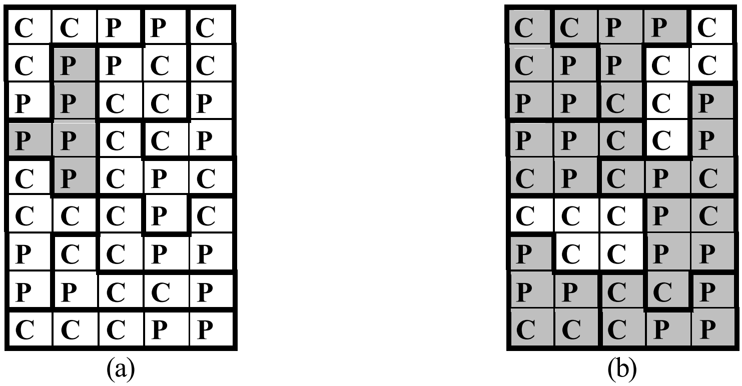


Fig. 1. Example by Dixon and Plischke: (a) Party P wins 1 seat and party C wins 8; (b) Party P wins 7 seat and party C wins 2.

- s is the common district size (number of territorial units in each district).

Clearly, the three parameters n, p, s must satisfy the relation $n = ps$.

We model the territory as an undirected graph $G = (V, E)$ with $|V| = n$, where the vertices represent territorial units and the edges represent adjacency between territorial units.

A *connected partition* of G is a partition of its set of vertices V such that each component induces a connected subgraph of G.

A *district design* is a connected partition of the graph into p components or *districts* of the same size. Notice that this definition takes into account the criteria of integrity, contiguity and population equality. If at least one such partition exists, the graph is said to be p-equipartitionable. Checking such property is not easy: in fact, Frieze and Dyer (1985) proved its NP-completeness even for bipartite graphs. We assume that G is p-equipartitionable.

A *vote outcome* is a bicoloring of the vertices that assigns to each vertex either the color blue or the color red: this means that all voters in the corresponding unit vote for the same party, *blue* or *red*, respectively. A vote outcome is *balanced* if the number of blue vertices is equal to the number of red ones.

A balanced vote outcome corresponds to a situation in which the electoral population is perfectly split between two parties.

From now on, except for the last section, we shall consider only balanced vote outcomes. We shall also make the following assumptions on the integers n, s, and p:

- n is even: this is a necessary condition for the existence of balanced vote outcomes;
- s is odd and greater or equal to 3: this assumption forbids trivial cases and ties between the two parties;
- p is even: this follows from the relation $n = ps$.

If in a district D the number of blue vertices is greater than the number of red ones, we will say that D is a *blue district*. In a similar way we can define a *red district*. We will denote by Π the set of all district designs and by Ω the set of all possible balanced vote outcomes.

We define an *electoral competition* to be a pair (ω, π) such that $\omega \in \Omega$ and $\pi \in \Pi$. The functions $b(\omega, \pi)$ and $r(\omega, \pi)$, compute the number of blue and red districts, respectively, resulting from the electoral competition (ω, π). Let

$$B(G) = \max_{\omega \in \Omega, \pi \in \Pi} b(\omega, \pi)$$

be the maximum number of blue districts for all the electoral competitions $(\omega, \pi) \in \Omega \times \Pi$. In a similar way we can define $R(G)$ with respect to $r(\omega, \pi)$.

PROPERTY 1 *Since, for any bicoloring, it is possible to switch the colors of the vertices so that the red vertices become the blue vertices and viceversa, any property related to the blue party that does not explicitly depend on any given bicoloring must hold for the red party also. In particular we have that* $B(G) = R(G)$.

By this property we can define the function

$$W(G) = B(G) = R(G).$$

Moreover the results that we will provide for the blue party hold also for the red one.

Given an electoral competition $(\omega, \pi) \in \Omega \times \Pi$, for any district k, $k = 1, ..., p$, let

- b_k = number of blue vertices in district k,
- r_k = number of red vertices in district k.

PROPERTY 2 *Given a p-equipartitionable graph G, for any* $(\omega, \pi) \in \Omega \times \Pi$ *the following inequality holds:*

$$b(\omega, \pi) \leq \lfloor n/(s+1) \rfloor.$$

Proof. Given an electoral competition $(\omega, \pi) \in \Omega \times \Pi$, for each district k let b_k and r_k be defined as above. Since ω is balanced, we may assume:

$$\sum_{k=1,\dots,p} (b_k - r_k) = 0.$$

Hence:

$$0 = \sum_{k=1,\dots,p} (b_k - r_k) = \sum_{k:b_k > r_k} (b_k - r_k) + \sum_{k:b_k < r_k} (b_k - r_k)$$
$$\geq b(\omega, \pi) - s(p - b(\omega, \pi)) = (s+1)b(\omega, \pi) - sp$$

Since $n = ps$ and $b(\omega, \pi)$ is a natural number we obtain:

$$b(\omega, \pi) \leq \lfloor n/(s+1) \rfloor.$$

■

COROLLARY 1 *If G is p-equipartitionable, then* $W(G) = \lfloor n/(s+1) \rfloor$.

Proof. Let $\pi \in \Pi$ be any district design. It is possible to color the vertices of the graph G in such a way that $\lfloor n/(s+1) \rfloor$ districts have at least $(s+1)/2$ blue vertices. In fact, in any balanced vote outcome, the number of blue vertices is $n/2$ and:

$$\frac{s+1}{2} \left\lfloor \frac{n}{s+1} \right\rfloor \leq \frac{n}{2}.$$

Since a district with $(s+1)/2$ blue vertices is blue, we obtain a vote outcome with at least $\lfloor n/(s+1) \rfloor$ blue districts. But, by Proposition 2, this is an upper bound for the number of blue districts, hence $W(G) = \lfloor n/(s+1) \rfloor$. ■

COROLLARY 2 *If G is p-equipartitionable, and* $p = q(s+1) + r$ *with* $1 \leq r \leq s+1$ *then* $W(G) = qs + r - 1$ [1].

Proof. From Corollary 1 we have:

$$W(G) = \left\lfloor \frac{n}{s+1} \right\rfloor = qs + \left\lfloor \frac{rs}{s+1} \right\rfloor.$$

[1]Notice that q and r might not coincide with the quotient and the remainder, respectively, of the division of p by $s+1$.

Since $r \leq s+1$,

$$\left\lfloor \frac{rs}{s+1} \right\rfloor = \left\lfloor r - \frac{r}{s+1} \right\rfloor = r-1,$$

hence

$$W(G) = qs + r - 1.$$

■

Given a bicoloring $\omega \in \Omega$ and a partition $\pi \in \Pi$, we say that a district is (*blue*) *edgy* if it contains $(s+1)/2$ blue vertices and $(s-1)/2$ red vertices, while we will say that a district is (*blue*) *sweeping* if all its vertices are blue. Moreover we say that a district design π is (*blue*) *extremal* if the number of blue districts $b(\omega, \pi)$ is equal to its upper bound $\lfloor n/(s+1) \rfloor$. Similar concepts can be introduced for the red party.

REMARK 3 *If $p \leq s+1$, each blue extremal partition has $p-1$ blue districts and one red district.*

We are especially interested in the following optimization problem:

$$GAP(G) = \max_{\omega \in \Omega}(\max_{\pi \in \Pi} b(\omega, \pi) - \min_{\pi \in \Pi} b(\omega, \pi)).$$

For a given graph G the function $GAP(G)$ is a measure of the maximum bias of an electoral outcome in terms of number of seats in single member majority districts.

PROPOSITION 4 *$GAP(G) \leq 2W(G) - p = 2\lfloor \frac{n}{s+1} \rfloor - p$.*

Proof. Since $b(\omega, \pi) + r(\omega, \pi) = p$, we have

$$GAP(G) = \max_{\omega \in \Omega}(\max_{\pi \in \Pi} b(\omega, \pi) + \max_{\pi \in \Pi} r(\omega, \pi)) - p \leq \quad (1)$$

$$\max_{\omega \in \Omega} \max_{\pi \in \Pi} b(\omega, \pi) + \max_{\omega \in \Omega} \max_{\pi \in \Pi} r(\omega, \pi) - p = 2W(G) - p.$$

■

For a given p-equipartitionable graph G we are interested in finding, if it exists, a bicoloring $\omega^* \in \Omega$ such that there are both a blue extremal partition and a red extremal one, both w.r.t. ω^*. If such a bicoloring exists, we will say that G is *two-faced* and there exist two partitions $\pi_b, \pi_r \in \Pi$ such that:

$$b(\omega^*, \pi_b) = r(\omega^*, \pi_r) = W(G) = \lfloor n/(s+1) \rfloor.$$

COROLLARY 5 *We have*

$$GAP(G) = 2W(G) - p \tag{2}$$

if and only if G is two-faced.

Proof. Follows from (1). ■

Two-faced graphs are those for which gerrymandering exhibits its worst case bias. There is an absolute threshold for the largest number of seats that a party can obtain when the vote outcome is balanced. In two-faced graphs, for a suitable balanced vote, both parties can achieve this threshold by artful gerrymandering.

3. Theoretical Results on Grid Graphs

The main result of this section is that, under the above assumptions on n, s, and p, any grid graph with an even number of vertices is two-faced.

Let G be a grid graph with M rows and N columns, and $n = MN$. Since we assume that n is even, at least one between M or N must be even. In the following we assume, without loss of generality, that M is even.

Even grids feature one simple property which is crucial for the development of the results to follow: they are hamiltonian (see Figure 2). On the one hand, this property implies that even grids are p-equipartitionable, since obviously a cycle of length $n = ps$ can always be partitioned into p paths of length s (remember that p-equipartitionability is NP-complete for general graphs). On the other hand, in an even cycle there are only s partitions into subpaths of the same size s. Each of them results from cutting p equidistant edges of the cycle, and thus it can be easily obtained from the others by a suitable rotation of the cuts along the cycle. If one can show that there exists one such partition satisfying certain properties, then this is sufficient to establish the existence in an even grid of a district map satisfying the same properties. This tool will be often exploited in our constructions.

We start from the case $p = s + 1$, where a blue extremal partition has exactly s edgy districts and one sweeping district. In fact, by Corollary 2 with $q = 0$ and $r = s + 1$, the upper bound on the number of blue districts is s. These districts must be edgy since the number of blue vertices in G is $s(s + 1)/2$. It follows that the remaining district is red sweeping. We will show how to construct such an extremal partition on a hamiltonian cycle H of G. We suppose that the vertices of H are consecutively numbered from 1 to n along the cycle (traversed clockwise).

A *boa* is a path with $(s + 1)(s - 1)/2$ vertices that can be partitioned into $(s+1)/2$ components having $(s-1)/2$ consecutive blue vertices and $(s-1)/2$

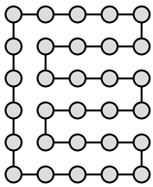


Fig. 2. Hamiltonian cycle in a grid graph with an even number of rows.

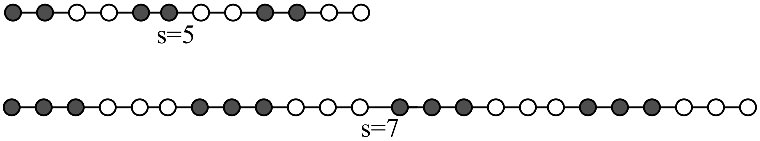


Fig. 3. Examples of boas.

consecutive red vertices each. Boas have the following nice property: if one cuts the s-th, the $2s$-th, ... , the $((s-1)s/2)$-th edge from left to right, one obtains $(s-1)/2$ red edgy districts and the remaining $(s-1)/2$ nodes are blue; a symmetrical property holds when one interchanges the two colors “red” and “blue”, as well as “right” and “left”.

In Figure 3 the boas for $s = 5$ and $s = 7$ are shown. Here, as in all black and white figures in the sequel, blue vertices are displayed in white and red vertices in black.

In Figure 4 we consider the case $s = 5$ and we show how to use two boas in order to find a bicoloring of H for which there are both a blue extremal partition and a red extremal one. One obtains such bicoloring by splitting H into four consecutive subpaths that are colored in the following way:

- the first subpath P_1 extends from vertex 1 to vertex $(s+1)/2$ and all its vertices are red;
- the second subpath P_2 is a boa starting from the red vertex $(s+1)/2+1$ and ending at the blue vertex $s(s+1)/2$;
- the third subpath P_3 extends from vertex $s(s+1)/2+1$ to vertex $(s+1)(s+1)/2$ and all its vertices are blue;

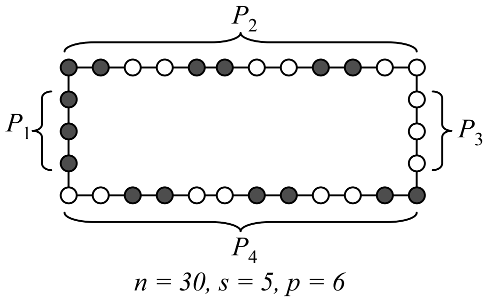


Fig. 4. Bicoloring for the case $p = s + 1$.

- the fourth subpath P_4 is a boa starting from the red vertex $(s+1)(s+1)/2+1$ and ending at the blue vertex $s(s+1)$.

It is easy to verify that the number of blue vertices is equal to the number of red ones. Since H is a cycle, one can obtain an arbitrary partition into p connected components by cutting p edges. In Figure 5 the two extremal partitions are shown for the case $s = 5$. If the cut edges are $(s, s+1), (2s, 2s+1), ..., (s^2, s^2+1), ((s+1)s, 1)$ the district containing vertices from 1 to s is red sweeping and all the other ones are blue edgy (Figure 5 (a)). Thus the partition is blue extremal. By shifting each cut to its next edge (clockwise) $(s+1)/2$ times, we obtain a blue sweeping district from vertex $s(s+1)/2+1$ to vertex $s(s+1)/2+s$ and all the other districts are red edgy. So the partition is red extremal (Figure 5 (b)).

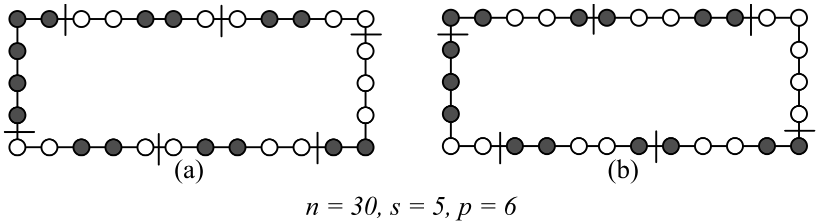


Fig. 5. Partitions for the case $p = s + 1$.

Let us consider now the case $p < s+1$. Since p is even and positive we can suppose $p = (s+1) - 2k$ for some k such that $1 \leq k \leq (s-1)/2$. As shown in Figure 6 for the case $s = 5$ and $k = 1$, starting from the bicoloring of the case $p = s + 1$ we delete from the subpath P_2 the last ks vertices and from the subpath P_4 the first ks vertices. We obtain a cycle with $s(s+1) - 2ks$ vertices where the number of blue vertices is equal to the number of red ones. If one cuts the edges as above, starting from $(s, s+1)$, the district containing vertices from 1 to s is red sweeping and all the other ones are blue edgy except the one containing the subpath P_3, which is not edgy because it contains $(s+1)/2 + k$ blue vertices and $(s-1)/2 - k$ red vertices. The obtained partition is blue extremal. By shifting the cuts as for the case $p = s + 1$, the resulting partition is red extremal. In fact, in the district containing the subpath P_3, the blue party wins since there are $s - k$ blue vertices and k red vertices, while all the other districts are red edgy.

n = 20, s = 5, p = 4, k = 1

Fig. 6. Bicoloring and Partitions for the case $p < s + 1$.

Finally suppose that $p > s + 1$.

PROPOSITION 6 *Under the above assumptions on M, N, p and s, G can be decomposed into p grid subgraphs having s vertices each.*

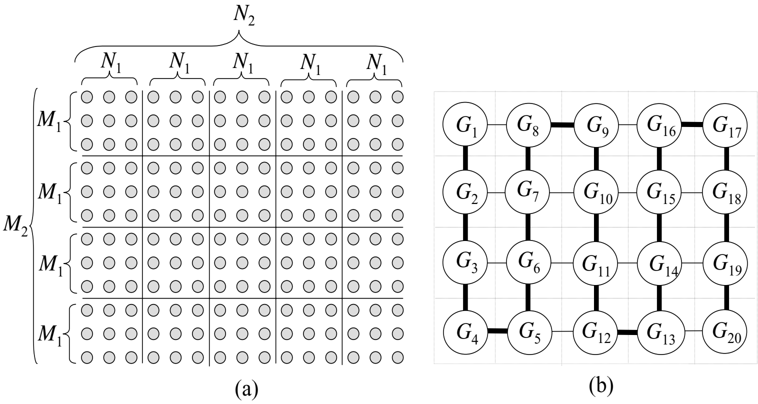
Proof. Since $MN = ps$ there exist four natural numbers M_1, M_2, N_1 and N_2 such that:

$$M = M_1 M_2,\ N = N_1 N_2,\ M_1 N_1 = s,\ M_2 N_2 = p.$$

As shown in Figure 7 (a), by partitioning the columns of G into N_2 components having N_1 columns each and the rows of G into M_2 components having M_1 columns each, one can decompose G into p grid subgraphs having M_1 rows and N_1 columns each. Notice that, since s is odd, also M_1 and N_1 are odd; hence, since M is even, also M_2 is even. ■

As in Corollary 2, we suppose that $p = q(s+1) + r$, with $q \geq 1$ and $1 \leq r \leq s + 1$. Notice that, since $s + 1$ and p are even, also r must be even.

We represent the decomposition given in Proposition 6 by a grid graph $\overline{G}$, with M_2 rows and N_2 columns, whose vertices V_k, $k = 1, ..., p$, correspond to the grid subgraphs and there is an edge connecting the vertices V_k and V_j if



$s = 9, p = 20, M = 12, N = 15, M_1 = 3, M_2 = 4, N_1 = 3, N_2 = 5$

Fig. 7. Decomposition of G into p grid subgraphs. The hamiltonian path is marked bold.

some vertex of the grid corresponding to V_k is adjacent to some vertex of the grid corresponding to V_j (see Figure 7 (b)). Let us consider the hamiltonian path $\overline{P} = (V_1, V_2, ..., V_p)$ of $\overline{G}$ and partition it into q subpaths having $s+1$ vertices each and one subpath having r vertices. Let P_j be the j-th subpath of $\overline{P}$.

LEMMA 7 *For each $j = 1, ..., q+1$, and for each column c of $\overline{G}$, the number of vertices of P_j in column c is even.*

Proof. The proof is based on the fact that the number of rows of $\overline{G}$, M_2, and the number of vertices in each subpath P_j, $s+1$ or r, are even. Let c_1 be the smallest numbered column whose intersection with some of the subpaths P_j is odd. Then c_1 must intersect in an odd number of nodes an even positive number of subpaths P_j. But then the smallest numbered such subpath, by the minimality assumption on c_1, would contain an odd number of nodes, a contradiction.

■

As shown in Figure 8, the subpaths P_j, $j = 1, ..., q+1$, define in G a decomposition into $q+1$ connected subgraphs $H_1, ..., H_{q+1}$.

PROPOSITION 8 *For each $j = 1, ..., q+1$, H_j is hamiltonian.*

Proof. As shown in Figure 8, each H_j can be decomposed into at most three grid subgraphs which, by Lemma 7, have an even number of rows. Hence it is possible to find a hamiltonian cycle of H_j as in the graph of Figure 9. ■

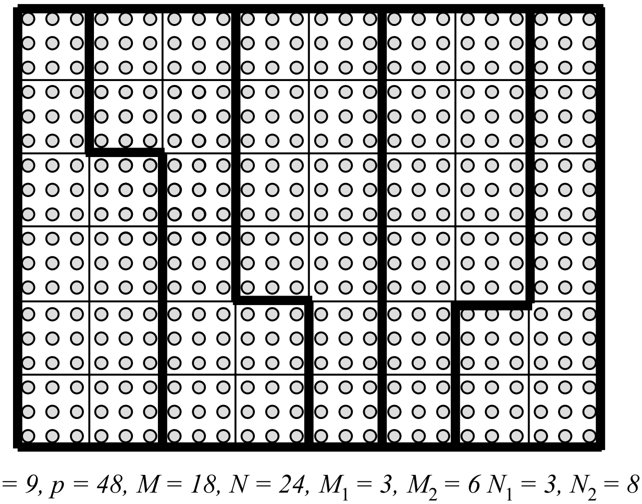


Fig. 8. Decomposition of G into $q + 1$ hamiltonian subgraphs.

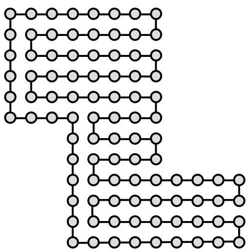


Fig. 9. Hamiltonian cycle in a H_j subgraph of G.

Since H_j, $j = 1, ..., q + 1$ is hamiltonian, then, as shown before, it is two-faced and so it is possible to find a bicoloring such that there exist a blue extremal partition and a red extremal one. By using the blue extremal partitions of the subgraphs H_j, one can obtain a partition of G having $qs + r - 1$ blue districts. In fact, by Corollary 2, in each of the q subgraphs having $s(s + 1)$ vertices, there are s blue districts and in the subgraph having rs vertices there are $r - 1$ blue districts. But, again by Corollary 2, $qs + r - 1$ is an upper bound on $W(G)$, hence the partition of G is blue extremal. The same arguments can be used for obtaining a red extremal partition. Then G is two-faced.

By the constructions shown for the cases $p = s + 1$ and $p < s + 1$ and the decomposition found for the case $p > s + 1$, the following theorem holds.

THEOREM 9 *Under the above assumptions on p and s, any grid graph with ps vertices is two-faced.*

COROLLARY 10 *If $G(s+1, s)$ is a grid graph with $s+1$ rows and s columns, then*

$$lim_{odd\ s \to \infty} \frac{GAP(G(s+1,s))}{s+1} = 1.$$

Proof. After Theorems 5 and 9, one has

$$\frac{GAP(G(s+1,s))}{s+1} = \frac{2W(G(s+1,s)) - s - 1}{s+1} = \frac{2s - s - 1}{s+1} = \frac{s-1}{s+1}.$$

When s odd $\to \infty$, the thesis follows. ■

Corollary 10 is stunning: it means that, for certain infinite families of grids, as the number and size of the districts grow, vicious gerrymandering can make the percentages of blue districts and red ones both arbitrarily close to 1 even under the assumptions that the vote outcome is the same and that the blue party and the red one get the same total number of votes.

In conclusion, we have shown that for all even grids one can construct Dixon-Plischke-like examples where gerrymandering can heavily reverse the electoral result in terms of Parliament seats.

Our final result shows that for some highly symmetric colorings, on the one hand, there are blue and red extremal district designs; on the other hand, the most compact design, namely, the partition of the grid into square subgrids, yields the same number of blue and red districts.

To address the question we introduce the notion of *skew-symmetric* coloring.

Let φ be the mapping of the grid onto itself that maps node (i, j) into $(M + 1 - i, N + 1 - j)$. Notice that φ is the product of two reflections, the first one around the y-axis, the second one around the x-axis. Since M is even, φ fixes no point of G. A coloring $\omega \in \Omega$ is *skew-symmetric* if (i, j) and $\varphi(i, j)$ have opposite colors.

If a grid is skew-symmetrically colored, then $\varphi(G)$ is isomorphic to G, the colors of its vertices being interchanged (in fact φ is an automorphism of the grid). In other words, up to the labels of the vertices, the effect of φ on G reduces to switching the colors of its vertices.

THEOREM 11 *Let G be an $M \times N$ grid having ps vertices with $p \leq s+1$ and p even. One can always find a blue- and a red- extremal partition with respect to some skew-symmetric bicoloring of G.*

Proof. (Sketch). We can divide the grid into two equally sized parts, say L and R, of $\frac{ps}{2}$ vertices each, in such a way that: (i) $(i,j) \in L$ if and only if $\varphi(i,j) \in R$; (ii) both L and R induce subgraphs containing hamiltonian paths.

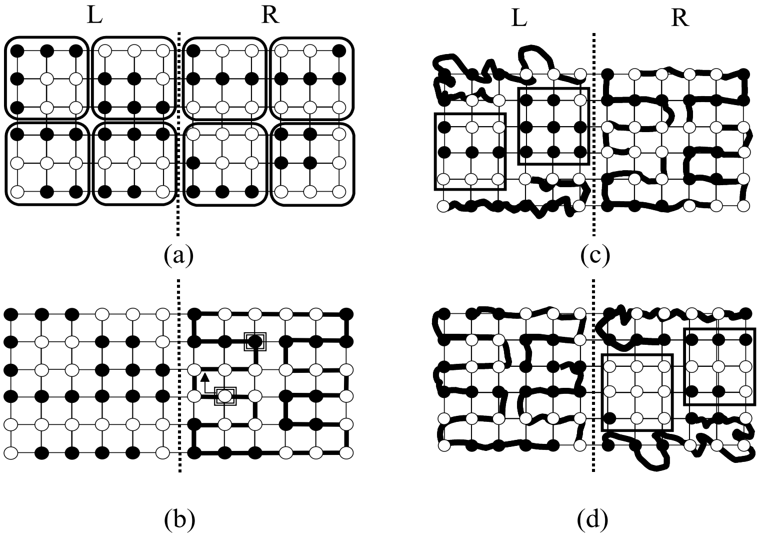


Fig. 10. **(a):** The most compact and equitable partition of a 6×12 skew-symmetrically colored grid. **(b):** The hamiltonian cycle from which the two extremal partitions in (c) and (d) are generated. Starting from the framed blue (white) vertex, and cutting the 9th, 18th, 27th and 36th edges of the cycle (clockwise) the right hand side of the partition in (c) is generated (the left hand side of the partition in (d) can be obtained by symmetry). Similarly, the right hand side of the partition in (d) (and, by symmetry, the left hand side of the partition in (c)) is generated by starting from the framed red (black) vertex. **(c),(d):** Red and blue extremal partitions.

Let us consider the subgraph G_R induced by R. We can define a coloring of G_R and two connected partitions π'_R and π''_R into $p/2$ components such that: π'_R is a partition all whose districts are red edgy, π''_R is a partition all whose districts but one are blue edgy, the exceptional one being red (see Figure 10). Using φ we extend the coloring of G_R to the entire grid. By construction this coloring is skew-symmetric. Moreover, if C is any component of either π'_R or π''_R, $\varphi(C)$ is a connected component of G_L (the graph induced by L), isomorphic to C but with colors interchanged. It follows that if π'_L and π''_L are the partitions of G_L corresponding via φ to π'_R and π''_R, respectively, then $\pi'_R \cup \pi''_L$ and $\pi''_R \cup \pi'_L$ are extremal partitions of G. ■

However, skew-symmetric colorings give rise not only to maximally biased designs, but also to minimally biased compact designs (see Figure 10).

THEOREM 12 *Let G be a skew-symmetrically colored $M \times N$ grid. Suppose that G can be divided into squares of sides of length $\sqrt{s}$ and let π be the partition formed by such squares. Then, in π, the number of red district equals the number of blue districts .*

Theorem 11 shows that even highly symmetrical vote outcomes can be manipulated in a partisan way. Nevertheless, in view of Theorem 12, compactness can be considered (at least within the frame of our idealized model) as an effective remedy against gerrymandering.

4. Experimental Results on Real-Life Test Problems

In this section we provide a multiobjective graph partitioning model for political districting and we study gerrymandering from an experimental point of view on real-life data. Starting from the graph-theoretic model described in Section 2, here we relax some of the previous assumptions in order to adhere to reality as much as possible. In both models the territory is represented by a graph and one looks for a connected partition of the graph in order to enforce the integrity and contiguity requirements. In the previous sections the underlying graph was assumed to be a rectangular grid, while here it may be, more generally, an arbitrary planar graph. In the former model nodes were unweighted and a vote outcome was but a node bicoloring; here nodes are weighted both by their populations and their votes. Whereas the stylized previous model is more amenable to theoretical investigation, the one we shall study in this section is more flexible and offers a more accurate description of real-life political districting. It is no coincidence that variants of it have been considered by several Authors (Bussamra et al., 1996, Garfinkel and Nemhauser, 1970, Merhotra et al., 1998, Nygreen, 1988, Ricca and Simeone, 2005). In spite of their differences, both models lead, in different ways, to the same conclusions: gerrymandering can drastically reverse the final outcome of an election, and compactness does provide an effective protection. Thus the experimental results of the present section corroborate and validate the theoretical results obtained so far.

Remember that our aim is to investigate both how bad gerrymandering can be and, simultaneously, to determine if there exist effective weapons against it, such as compactness or other districting criteria, which can be adopted in order to avoid this practice. Thus, in our experiments we undertake a multicriteria approach and develop an optimization model in which different objective functions - measuring different criteria - are considered one at a time.

4.1 The Model and the Database

As before, n denotes the total number of territorial units in the territory, $n = |V|$, and p, $1 \leq p \leq n$, is a positive integer denoting the number of districts. Let p_i, $\forall i \in V$, be positive integral node-weights, representing territorial unit populations and d_{ij}, $\forall i, j \in V$, be positive real distances defined for each unit pair (i, j). For each territorial unit, the list of all those administrative areas (regions, provinces,...) that contain the unit is known. Finally, with reference to political elections in Italy, for each territorial unit we introduce two positive integral node weights, vo_i and vp_i, $\forall i \in V$, representing the number of votes obtained in unit i by the Olive Tree and by the Pole of Liberties, respectively[2].

The general graph partitioning problem can be formulated as follows:
Given a graph G, partition its set of nodes into p subsets (districts) such that the subgraph induced by each subset is connected and a given function of the partition is optimized.

In the sequel, we use the term “district design” as a synonym of “connected partition into p components”. Actually, we are no longer imposing the further restriction that the districts be equally sized since in real-life cases this requirement is too strict and we can only try to get close to the ideal case as much as possible by optimizing a suitable objective function.

In our experiments we used data of three Italian Regions, namely, Piedmont, Latium and Abruzzi, whose townships are taken to be the territorial units. The weights p_i associated to territorial units correspond to the Italian population from 1991 Census, and we considered the real road distances between pairs of territorial units. In this application we considered the Italian (majoritarian) vote distribution of Political Elections of 1996.

4.2 Districting Criteria and Local Search Algorithm

In our real life model we considered several of the most commonly adopted districting criteria discussed in Section 1. In particular, integrity and contiguity are automatically guaranteed by the graph-theoretic model in which each elementary territorial unit is represented by a vertex of the graph. The remaining criteria of population equality (PE), compactness (C) and conformity to admin-

[2]In this application we consider the Italian (majoritarian) vote distribution of Political Elections of 1996. The Olive Tree and Pole of Liberties parties were the center-right and center-left coalitions, respectively, which were in competition at that time.

istrative boundaries (AC) are measured by proper indicators to be optimized. To this purpose, we adopted some indicators already used in similar applications (Grilli di Cortona et al., 1999). Actually, they measure non-population equality, non-compactness and non-administrative conformity, therefore they must be minimized. Firstly, an ideal situation of perfect population equality, perfect compactness and perfect administrative conformity is defined and a proper index is chosen so that its value is equal to 0 when the ideal situation is met, while in the other cases it provides a measure of the corresponding error. These indexes are generally normalized in order to be independent of scaling factors. Therefore, they can be read as percentages.

Let $C_1, C_2, \ldots, C_p \subset V$ be the subsets of nodes of the p districts of a given district design. Let $P_k = \sum_{i \in C_k} p_i$, $k = 1, 2, \ldots, p$, be the population of district C_k. Then, the population equality index for the district design is given by

$$PE = \frac{\sum_k |P_k - \bar{P}|}{p\bar{P}} \tag{3}$$

where $\bar{P} = \frac{\sum_k P_k}{p}$ is the average district population. This is the average deviation of the population of each district from $\bar{P}$, divided by the normalization factor $\bar{P}$.

On the basis of the distances d_{ij}, for each pair of vertices $i, j \in V$, we define a global compactness index given by the sum of compactness indices computed over each district separately. For a given district C it can be briefly described as follows. Let d_{ij} be the distance between unit i and unit j. For each unit compute its eccentricity

$$d(i) = \max_{j \in C} d_{ij}$$

and set

$$\delta = d(s) = \min_{i \in C} d(i)$$

By definition, s is the center of district C and the compactness in district C is measured by:

$$C = \frac{\sum_{i \in C} p_i}{\sum_{j \in D} p_j} \tag{4}$$

where $D = \{j \in V : d_{js} \leq \delta\}$.

The compactness index (4) is a measure of the deviation of the districts from the ideal situation in which they all have a regular, “round” shape.

The administrative conformity index adopted in this application is defined on the basis of the discrepancies between the already existing administrative district maps (of different type) and the electoral district design. For a given district and a given type of administrative boundaries it is basically computed as a measure of those units which produce discrepancies. The global index, which varies between 0 and 1, is obtained by averaging over all types of administrative boundaries and over all the electoral districts. A detailed description of the index is reported in (Grilli di Cortona et al., 1999).

We considered these three indexes as objective functions in our optimization model, both separately and combined together into a single objective function given by a convex combination of them.

Notice that the population equality index defined for the real-life application can be considered as the counterpart of the principle of equal size districts stated in the combinatorial model of Section 2. In our graph-theoretic model a vertex corresponds to an elementary unit of the territory. In general - as in our case - territorial units are given by townships and it is not guaranteed that they have the same size. Thus, the requirement of Section 2 which forces each district to have exactly the same number of units as each other here does not work. Actually, the *one-man-one-vote* principle addressed by that assumption here must be necessarily pursued through population equality, regarded not as a hard constraint, but as a criterion to be fulfilled as much as possible.

On the other hand, the idea of compact districts sketched in Section 3 (see Figure 10) perfectly matches the principle embedded in our compactness index (4).

The additional administrative conformity index was considered in our experiments since it is generally included among the commonly and widely accepted political districting criteria. The experimental results related to it add some more information to our knowledge and can be useful for evaluating the actual relevance of this criterion in a districting procedure.

Since we are interested in studying how far gerrymandering can be pushed, we must also consider partisan criteria. Here we are obliged to adhere to reality and, in our case, we refer to the real vote distribution of Italian political elections of 1996. With respect to this vote, for any given district design, we are able to compute how many seats are assigned to the Pole and to the Olive party, respectively. The idea is that both Pole and Olive would like to win the election. To this purpose, if they each had the opportunity of designing their own political districts, they would try to find the district design that makes them win as many seats as possible (gerrymandering). Actually, for a given district design, the number of seats assigned to a party can be considered as a measure of the *utility* of the district design for that party. For a given district, let ρ be the ratio between the number of votes for the Pole and those for the

Olive. Then, the utility of this single district for the Pole can be measured through the following step function:

$$h(\rho) = \begin{cases} 0, & \text{if} \quad \rho < 1 \\ 1, & \text{if} \quad \rho \geq 1, \end{cases} \tag{5}$$

and the sum of such district utilities for the Pole over all the districts provides a partisan index for the Pole. Similarly, we can compute a partisan index for the Olive party. These two indexes can be adopted as objective functions in our experimental analysis when we study how far the Pole and the Olive party can manipulate the district design, respectively.

In our experiments we used the Old Bachelor Acceptance metaheuristic (Hu et al., 1995) in order to find solutions that minimize the six different objectives. This metaheuristic has shown to perform well when applied to territorial political districting problems. For details, see (Ricca and Simeone, 2005).

We notice here that local search techniques are particularly suitable also for the design of partisan districts. Actually, starting from an initial district design, they work by performing small perturbations of the current solution. At each step a node belonging to the boundary of a district migrates towards an adjacent district. Thus, two consecutive district designs differ just for one node in only two districts and it is hard to distinguish between them. Migration by migration, it is possible to obtain a district design which favors a given party (its utility is maximized) and such that the initial given district design is modified as little as possible. However, when applying local search techniques, (5) is not sufficiently sensitive to the migration of a vertex from one district to another. This explains why we chose to replace the step function (5) by a smoother objective function. For a given party, say the Pole, in each district we compute the following district-utility *logistic* function for that party

$$g(\rho) = \frac{c}{1 + exp(b\ (1 - \rho))},$$

where c and b are suitably chosen in order to get the desired shape of the utility function. The idea is that the district-utility increases rapidly when ρ is near 1 (see Figure 11).

The aim of our experimental study is twofold. On the one hand, we want to test if our four objective functions, given by PE, C and AC, and their convex combination, are good weapons against gerrymandering. On the other hand, starting from a given district design we try to manipulate it as much as possible in order to maximize the objective function given by the utilities of the Pole and the Olive party, respectively. The underlying idea is that gerrymandering can be investigated experimentally in order to identify worst case configurations and the corresponding upper bounds over the maximum number of seats that a party can get. From our previous experimental works, we already know

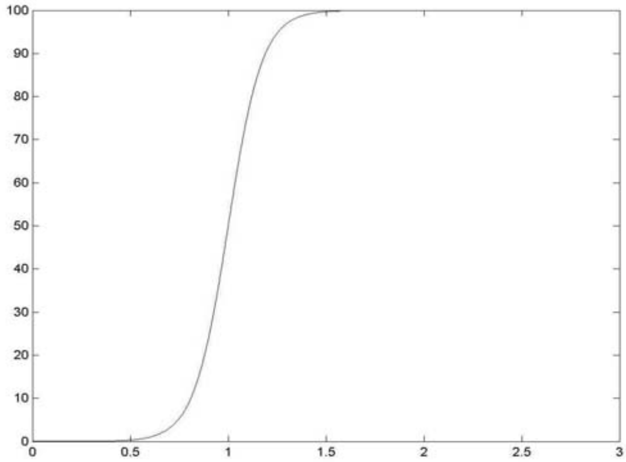


Fig. 11. District-utility logistic function for $c = 100$ and $b = 11, 51$.

the good performance of PE, C and AC as objective functions in non-partisan districting problems. However, in this paper the results referring to the manipulation of the districts are new. Moreover, as we will see in the next section, our experimental results show that our neutral objectives can be used as an alarm signal for gerrymandering, since they tend to deteriorate when gerrymandering is practiced.

It is clear that our experimental results cannot be compared to the exact bounds given in Section 2. However, we believe that these results are of interest on their own because they represent the real-life counterpart of our theoretical results of the previous sections. Therefore, such results are not a mere mathematical curiosity, but they capture the gist of the real threat posed by gerrymandering. As we will see in Section 4.3, there are regions in which, for suitable district designs, the Pole or the Olive party gets the total number of seats.

4.3 Experimental Plan and Results

Table 1 shows the main characteristics of the graphs representing the territories of three Italian regions considered in our experimental plan.

As before, here PE means "Population Equality", C means "Compactness" and AC means "Administrative Conformity", while MT refers to the "Mixed Target" which is defined as the following convex combination of PE, C and

Table 1. Graphs of the Italian Regions

Region	N. of Nodes	N. of Edges	Density	N. of Districts
Piedmont	1208	3527	2.92	28
Latium	374	1006	2.69	19
Abruzzi	305	847	2.78	11

AC:

$$0.5PE + 0.3C + 0.2AC.$$

For each region we performed six different runs of Old Bachelor Acceptance metaheuristic with PE, C, AC, MT and the two utility functions as objectives, respectively. Following (Ricca and Simeone, 2005), we implemented a randomized version of this metaheuristic, that is, starting from an initial solution, at each iteration Old Bachelor Acceptance chooses a random solution in the neighborhood of the current one. Notice that randomization is a useful tool for the diversification of the search: it is used to avoid cycling and explore a large amount of different solutions. When the objectives are PE, C, or AC, the initial solution is generated randomly. After a spanning tree T of G is randomly generated, $p - 1$ randomly chosen edges of T are cut in order to get p subtrees whose node-sets correspond to the p initial districts. For the MT criterion we preferred to start from the district map generated by the ADEN heuristic in (Grilli di Cortona et al., 1999).

The optimal solutions found in the previous four runs were adopted as possible initial solutions for the case in which the objective is to maximize the utility of a given party. The idea was that starting from an already optimized set of districts could make it more difficult to manipulate the given district design in favor of one of the two parties. However, also the Institutional district design of the Italian Political Elections of 1996 was considered as possible initial solution. Among the results obtained w.r.t. these 5 different initial district designs, we selected the worst observed case.

Tables 2-4 show our experimental results on the three different graphs. The last row of Tables 2-4 refers to the values of the six objectives computed for the Institutional district design adopted in Italy for the Political Elections of 1996. This row was included in order to favor the comparison between our - neutral and partisan - district designs and the one that was actually adopted in 1996.

On the basis of our experiments, we can state the following conclusions:

1. Given a vote distribution, gerrymandering is able to dramatically reverse the electoral outcome (see, the fifth and the sixth row of each table).

Table 2. Piedmont

District Design	PE	C	AC	MT	Pole seats	Olive seats
Min PE	**0.075**	0.911	0.577	0.426	10	18
Min C	0.771	**0.531**	0.347	0.614	11	17
Min AC	0.940	0.643	**0.113**	0.686	12	16
Min MT	0.094	0.762	0.288	**0.334**	11	17
Max Pole	1.052	0.777	0.454	0.850	**21**	7
Max Olive	1.364	0.593	0.263	0.913	3	**25**
Institutional	0.105	0.859	0.143	0.339	11	17

Table 3. Latium

District Design	PE	C	AC	MT	Pole seats	Olive seats
Min PE	**0.046**	0.778	0.523	0.361	13	6
Min C	1.226	**0.166**	0.143	0.692	12	7
Min AC	1.072	0.620	**0.050**	0.732	13	6
Min MT	0.050	0.502	0.270	**0.230**	10	9
Max Pole	1.512	0.321	0.061	0.864	**19**	0
Max Olive	1.299	0.277	0.131	0.759	3	**16**
Institutional	0.060	0.683	0.202	0.275	10	9

Table 4. Abruzzi

District Design	PE	C	AC	MT	Pole seats	Olive seats
Min PE	**0.040**	0.744	0.508	0.345	4	7
Min C	0.668	**0.390**	0.288	0.508	4	7
Min AC	0.894	0.539	**0.056**	0.620	4	7
Min MT	0.113	0.442	0.263	**0.242**	4	7
Max Pole	1.217	0.425	0.320	0.800	**10**	1
Max Olive	1.129	0.473	0.328	0.772	1	**10**
Institutional	0.078	0.633	0.215	0.272	5	6

2. The districting bias produced by gerrymandering algorithms implies the deterioration of the values of all the traditional PD criteria.

3. It turns out that there is a substantial stability of the number of seats attributed to the Pole and to the Olive when the criteria of Population Equality, Compactness, Administrative Conformity and the Mixed one are optimized.

4. Compactness is a good shield against the practice of gerrymandering. On the other hand, in view of 3, and since gerrymandering deteriorates *all* the districting criteria, satisfying the other criteria helps in preventing gerrymandering. This is why the use of more than one traditional PD criterion is generally recommended.

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