The Italian Bug: A Flawed Procedure for Bi-Proportional Seat Allocation

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Abstract There is a serious technical flaw in the newly approved Italian electoral law. The flaw lies in the method used to allocate the Chamber of Deputies seats to parties (or coalitions) within multi-member regional constituencies. The procedure stated in the law could produce contradictory results: it could end up assigning a party more (or less) seats than it is entitled to receive on the basis of the same law. At least two types of paradoxes may occur. Although they have been utterly overlooked in the debate over electoral reform, they can be critical in practice when trying to determine the actual seat allocation. The failure of the current Italian electoral law was inherited from the previous one but the consequences are worse. Moreover, a correction mechanism introduced into the law at the last-minute does not prevent it from producing contradictory results. The paradoxes that undermine the Italian electoral law are pointed out and a solution is proposed. A broad conclusion is that a more extensive use of mathematics in the design and evaluation of electoral systems would help identify flaws and deliver more transparent, logical and fairer electoral laws.

Keywords: Bi-proportional allocation, Italian electoral system, proportional systems.

1. Introduction

On the 14th of December 2005 Italy endorsed a new law[1] for the election of representatives at the Chamber of Deputies and Senate. The new electoral law replaced a fairly recent hybrid system[2] with a proportional one, which includes a threshold for parties to be eligible to receive seats and a (potentially big) majority prize for the party (or coalition of parties) with the most votes. The debate over electoral reform questioned the opportunity of introducing

[1] The initial proposal for a reform of the electoral system was presented on 13th September 2005 by the Polo coalition in the lower house (Chamber of Deputies), approved with some modifications in October and ratified by the upper house (Senate) on the 14th December.

[2] In the mixed system (L. 4 agosto 1993), also know as "Legge Mattarella" or "Mattarellum", 75% of the seats were assigned on first-past-the-post rules and the remaining 25% on proportional basis. It was first introduced in 1993, at the time of a major turmoil in Italian political setting caused by Tangentopoli and the consequent collapse of the traditional parties.

such a radical change with general elections forthcoming in spring 2006 and mainly focused on its political effects in terms of coalition strategies and party fragmentation. The public and the media utterly overlooked a serious technical flaw that the new law inherits from the old one. The procedure adopted to transform votes into seats has a "bug" and one could end up with paradoxical results: the law might award a party more (or less) seats than those it is entitled to by the same law! This has considerable practical consequences on how to decide the actual seat allocation and casts doubt on the legitimacy of the electoral law itself. The flaw has to do with the fact that, for some voting outcomes, the procedure will get stuck when distributing the seats among parties/coalitions within the regional constituencies for the Chamber of Deputies. The purpose of this paper is not to discuss whether it was appropriate to change the Italian electoral system but to prove one of its shortcomings from a purely technical point of view. Whatever the opinion on the law and modifications it introduces, one would surely agree that it should guarantee a consistent and unique outcome in terms of seats and attempt, to the extent to which this is possible, to guarantee fairness (i.e., citizens' votes should have the same weight in determining the electoral outcome). The transformation of votes into seats is a mathematical problem and in order to satisfy basic requirements of logic, transparency and equity among citizens it should be consistently defined and correctly solved in all circumstances. Unfortunately, the system under exam fails to do so.

The new Italian electoral law for the election of representatives of the Chamber of Deputies (Ddl Camera 2620 13, 2005) allocates seats proportionally to the votes obtained by each party (and coalition of parties) at the national level and within multi-member regional constituencies. A majority prize is meant to ensure that the party or coalition with the greatest number of total votes wins a full majority of seats in the Chamber of Deputies (i.e., at least 340 seats) no matter how many votes the other parties receive[3] There is a single ballot and candidates are elected on the basis of regional "blocked" lists (citizens do not express their preference for a candidate but a vote for a party list). Moreover, a complex scheme of thresholds is adopted to select which parties and coalitions are eligible to compete in the seat allocation. The Italian Constitution sets the size of the Chamber of Deputies at 630 seats. There are 27 multi-member regional constituencies in total. The Constitution also establishes that the number of seats at stake in each regional constituency must be proportional to the number of its inhabitants, according to the latest population census. The only exception is the region of Valle d'Aosta which has a single-member dis-

[3]The seat bonus represents 54 per cent of the seats, no matter what the weight of the strongest party in terms of votes, and basically introduces a majority component which undermines the proportional principle attempted in the law.

trict. Finally, 12 seats are assigned to a constituency of Italian citizens resident abroad[4].

The electoral reform of 2005 is not the first attempt to modify the Italian mixed system which previously allocated 75% of the seats in single-member districts with first-past-the-post rules and the remaining 25% on a proportional basis. Several other proposals have been made in the last decade, but surprisingly enough they usually sought to abolish the proportional seats and introduce a fully first-past-the-post system. On the other hand, the new electoral law adopts a proportional logic, although mitigated by the majority prize.

2. Where the Italian System Fails

The Italian electoral law wishes to achieve a double proportionality: at the national level and within the regional constituencies. But the procedure implemented to achieve this is flawed and, for some voting outcomes, it will end up by awarding a party more (or less) seats within the regional constituencies than those the same party is entitled to at the national level. A similar flaw was identified by Balinski and Ramirez in the 1996 Mexican electoral law (Balinski and Ramírez González, 1997). In short, the new electoral law first allocates seats to parties at the national level and then assigns seats to the parties within each regional constituency. Both steps are carried out on a proportional basis, according to a method called *Hare* or *Largest Remainders*. A fundamental property of the method is that the number of seats is always equal to the exact share (quota) of seats a party should receive on a proportional basis, either rounded down or rounded up. There is an extensive literature on proportional electoral systems and the Largest Remainders method, for details see for example (Balinski, 2004; Balinski and Young, 1982; Grilli di Cortona et al., 1999).

Since the computation of the number of seats to each party or coalition at the national level is carried out first, the allocation of seats to parties within the regional constituencies is bound to satisfy two sub-totals: (a) the sum of the seats assigned to all parties within a given constituency must be equal to the number of seats actually at stake in the constituency and (b) the sum of the seats awarded to a given party in all constituencies must be equal to the number of seats it was awarded on the basis of the national computation.

The procedure adopted to allocate seats to parties within the regional constituencies starts by computing the exact number of seats due to each party one constituency at a time (starting from the smallest one). This number is equal to the size of the constituency multiplied by the percentage of ballots the given party has obtained. This "exact quota of seats" is not necessarily an integer

[4] Voting by Italian nationals resident abroad is governed by L. 27 dicembre 2001 n. 459 (known as "Legge Tremaglia") and by its implementing regulation (D.P.R. no.104 / 2003).

and usually carries a fractional part. Since a seat cannot be divided among different candidates, the law first assigns each party a number of seats equal to the exact quota rounded down. If there still remain seats to be assigned, these are awarded to parties in the order of the largest fractional remainders.

The problem with this procedure is it does not guarantee that, once all seats are assigned, the total amount awarded to each party is the same as the amount computed at the national level. Basically, by operating one constituency at a time, without worrying about the total amount of seats a party is entitled to at the national level, the sum constraint might not be satisfied. This is not a negligible defect and it has serious practical consequences: should such a paradoxical result occur, who will decide the final seat assignment? The size of the Chamber of Deputies cannot be changed. Some parties will gain more seats with the regional allocation but others will with the national one. The failure of the Italian electoral law could trigger a serious controversy between political parties on whether the result of the national allocation should prevail on the results of the regional allocations. Claims of the different political groups would presumably vary according to which case is the most advantageous for them.

3. Electoral Paradoxes

Small scale examples of the Italian electoral paradox have been already discussed in (Pennisi et al., 2005a) but more realistic examples can easily be produced. Consider the case of six political parties competing for the 617 seats at stake in the Italian Chamber of Deputies and the 26 regional constituencies[5]. Let the voting outcome be the one detailed in Table 1. In this example the number of votes is comparable with those expressed by Italian citizens in the last general elections held in 2001: there is, at the most, a 4.5 point difference between the share of party votes shown in the example and those obtained in 2001. Notice that party C is competing only in some constituencies. This was the case of the Northern League in the 2001 elections. Let the party with the greatest number of votes be the "majority list" and the quotient between the total number votes and the number of seats at stake (617) be called the fractional national coefficient. This number rounded downwards is called the national coefficient and represents the "cost" of a seat in terms of votes in the national contest.

[5]The actual size of the Chamber of Deputies is 630 seats, as established in the Italian Constitution (article 56), but 12 of them are reserved to the election of representatives of the Italians living abroad and 1 to the single/member district of Valle d'Aosta.

Table 1. Number of votes per party and constituency.

Constituency	Party A	Party B	Party C	Party D	Party E	Party F	Total votes	Seats at stake
Piemonte 1	400783	96054	73072	249544	237700	30383	1087536	24
Piemonte 2	285589	124753	194327	180623	132317	33655	951264	22
Lombardia 1	761543	249386	160114	321368	396133	48568	1937112	40
Lombardia 2	636642	192461	410789	192385	357854	97506	1887637	43
Lombardia 3	186671	78789	127148	170008	79445	1607	643668	15
Trentino Alto Adige	82778	55049	0	51517	67413	36885	293642	10
Veneto 1	329782	158402	169965	177987	251340	8289	1095765	29
Veneto 2	317041	89550	143286	131320	180218	44223	905638	20
Friuli Venezia Giulia	186371	96356	67321	66763	146959	1847	565617	13
Liguria	280399	84842	77958	249392	123161	43594	859346	17
Emilia Romagna	641699	307914	0	839563	469029	266707	2524912	43
Toscana	508202	350951	0	774294	360079	219358	2212884	38
Umbria	134556	134860	0	183441	113560	83630	650047	9
Marche	244594	179000	0	252977	183337	95155	955063	16
Lazio 1	564330	559634	0	473675	491237	173156	2262032	40
Lazio 2	307780	200949	0	183038	126949	84146	902862	15
Abruzzi	239180	157899	0	179005	131944	84537	792565	14
Molise	71393	59449	0	71950	59301	48741	310834	3
Campania 1	567890	239670	0	312391	197204	129222	1446377	33
Campania 2	463265	250225	0	212514	261133	101904	1289041	29
Puglia	681996	397202	0	341107	415243	150335	1985883	44
Basilicata	103244	72139	0	100214	101297	56334	433228	6
Calabria	270118	195793	0	223719	150159	97080	936869	22
Sicilia 1	478296	150643	0	190233	188381	88548	1096101	26
Sicilia 2	504881	221958	0	171734	271583	82452	1252608	28
Sardegna	301315	153912	0	209921	166320	126415	957883	18
TOTAL	9550338	4857840	1423980	6510683	5659296	2234277	30236414	617

The procedure adopted by the law first allocates the 617 seats to parties at the national level taking into account the majority prize, as follows[6]:

- assign to each party its exact quota of seats rounded down, i.e. divide the number of votes the party has obtained by the national coefficient and round this number down. Then count the number of seats that must still be awarded and assign an additional seat to those parties which have the greatest fractional remainders (this is a slight variant of the typical statement of the largest remainders method);

- check whether the majority list has achieved at least 340 seats;

[6]The whole setting is slightly more complicated because of the national and regional thresholds on the number of votes parties must obtain to compete in the electoral contest. For the sake of simplicity, suppose all parties in the example satisfy the thresholds. The technical flaw put forth in this paper is not related to the use of thresholds, although in paragraph 4 we do suggest that thresholds may play a role in making the paradoxes more likely to occur.

- if this is not the case, assign 340 seats to the majority list and calculate the majority electoral coefficient (i.e., the total majority list votes divided by 340 and rounded downwards) and the minority electoral coefficient (i.e., the sum of votes obtained by the other parties divided by 277 and rounded downwards). Distribute the 277 remaining seats among the minority parties with the method of largest remainders described above, but using the minority electoral coefficient.

The majority and minority coefficients represent the cost in terms of votes of a seat for the majority list and for all other parties, respectively. In a truly proportional electoral system the cost of a seat is the same for all parties, but here, the adoption of the 340-seat majority prize can introduce large differences: seats may cost much less for the majority list and much more for all other parties. In the example the cost of a seat for "minority" parties is 2.7 times larger that the cost of a seat for the majority list (approximately 28 thousand ballots are needed for the majority list to get a seat against about 75 thousand for the other parties!). The number of seats assigned to each party is given in Table 2.

Table 2. Seats awarded at the national level, taking into account the majority prize.

	Party A	**Party B**	**Party C**	**Party D**	**Party E**	**Party F**	**Total seats**
Seats awarded at the national level	340	65	19	87	76	30	617

At this point, seats must be allocated to the parties within the regional constituencies. The procedure consists of the following steps, for each constituency:

- divide the number of votes obtained by the majority list by the majority coefficient and for each other party, divide the number of votes by the minority coefficient; these indexes are the relative costs of a seat in the constituency.
- multiply the number of seats at stake in the constituency by each party's index and divide this product by the sum of all indexes to obtain the exact number of seats assigned to each party, and round this number down;
- assign the remaining seats at stake in the constituency on the basis of largest remainders.

Despite the convoluted formulation, this procedure is nothing more than a slight variant of the largest remainders method, applied at the constituency

level and taking into account two different totals (for majority and minority seats). The result is given in Table 3. The law's internal contradiction is clear: parties A, C and D end up with more seats than those they are entitled to at the national level while E and F have less. The only correct result is the number of seats assigned to party B.

In the bill for electoral reform presented in September 2005 the procedure ended at this point, totally neglecting the fact that a paradox - such as the one shown in the example - could occur. During parliamentary discussion the legislators must have realized, at the last moment, that something could go wrong. In a version of the law approved by the Chamber of Deputies in October a correction procedure was introduced. The idea underlying the correction mechanism is to re-balance the seat distribution through transfers of seats between parties with a surplus (in the example A, C and D) to parties with a deficit (E and F). Unfortunately, this mechanism is once again flawed!

The correction mechanism is executed whenever the sum of seats awarded to parties in the regional constituencies is not equal to the corresponding national seat allocation. It is applied starting from the party with the largest seat surplus, in decreasing order. Seats are transferred from the party with a surplus in those constituencies in which the party has obtained an additional seat thanks to its remainders, selecting the smallest remainders (the underlying idea is that seats are taken away from the party in those cases in which it was less entitled to them respect to other constituencies). The seats are transferred to a party with a seat deficit in the same constituency provided that such party has not already benefited from an additional seat on a remainder's basis and according to the largest unused remainder (the idea is to award the seat to the party which is next most entitled to it).

Although it is meant to correct the damage done, the mechanism does not always work because it operates only on seats rounded up, i.e. assigned to a given party thanks to its relatively "large" remainder. In other words the correction mechanism assumes that a paradox may occur, but only because a party has benefited too much from its exact quotas being rounded up. In Table 3 bold figures highlight exact quotas rounded upwards, i.e. cases in which an additional seat was awarded to the party during the regional allocation procedure thanks to the largest remainders. Note that party C is in surplus of seats although it has received only exact quotas rounded down. A double-star identifies cases in which a surplus party received an additional seat with relatively small remainders (the smallest among all remainders it has used) and a star identifies cases in which a deficit party has the largest unused remainders. These are the parties and constituencies involved in the seat transfers.

One can already notice that, despite transfer operations, the inconsistency between the sum of seats allocated to parties within the regional constituencies and the national allocation will still hold: party C will keep two extra

Table 3. Seats awarded to parties within the constituencies on the basis of largest remainders.

Constituency	Party A	Party B	Party C	Party D	Party E	Party F	Row sum
Piemonte 1	**15**	1	1	**4****	3	0*	24
Piemonte 2	**12**	**2**	3	**3**	2	0	22
Lombardia 1	25	3	2	4	**5**	**1**	40
Lombardia 2	**25**	**3**	6	**3**	5	1	43
Lombardia 3	**8**	1	2	**3**	1	0	15
Trentino Alto Adige	5	1	0	1	**2**	**1**	10
Veneto 1	**16****	**3**	3	3	4*	0	29
Veneto 2	12	1	2	**2**	2	**1**	20
Friuli Venezia Giulia	7	**2**	1	**1**	2	0	13
Liguria	**10****	1	1	3	**2**	0*	17
Emilia Romagna	20	**4**	0	10	**6**	3	43
Toscana	**17**	4	0	**10**	4	**3**	38
Umbria	**4**	1	0	**2**	1	**1**	9
Marche	**8**	2	0	**3**	2	1	16
Lazio 1	**19**	**7**	0	**6**	6	2	40
Lazio 2	**9**	2	0	**2**	1	**1**	15
Abruzzi	7	**2**	0	2	**2**	**1**	14
Molise	1	**1**	0	**1**	0	0	3
Campania 1	**21**	3	0	4	**3**	**2**	33
Campania 2	17	4	0	**3**	**4**	1	29
Puglia	25	**6**	0	**5**	**6**	2	44
Basilicata	**3**	**1**	0	**1**	1	0	6
Calabria	11	3	0	**4**	2	**2**	22
Sicilia 1	17	2	0	**3**	**3**	1	26
Sicilia 2	**18**	**3**	0	2	**4**	1	28
Sardegna	**10**	**2**	0	**3**	2	1	18
Column sum	**342**	**65**	**21**	**88**	**75**	**26**	**617**
Seats awarded at the national level	**340**	**65**	**19**	**87**	**76**	**30**	**617**
Surplus and/or deficit	**+2**	**0**	**+2**	**+1**	**-1**	**-4**	

seats and party F will lack them. There is no mention in the law on how to resolve such situations. In fact, the law states that when seat transfers within a same constituency are no longer possible (just as in the example), in order

to eliminate any other surplus, seats may be given to parties with a deficit in a different constituency, with the largest unused remainders first. However, it never acknowledges the fact that a surplus of seats may be due to the simple assignment of exact quotas rounded down and not to seats awarded according to largest remainders. Other realistic examples can be produced, as shown in (Pennisi et al., 2005b). There are at least two types of paradoxes undermining the Italian electoral law and for which its correction mechanism is not sufficient to repair:

- the *surplus paradox* for parties with exact regional quotas all rounded downwards: when the sum of the seats assigned to a party (or coalition of parties) in the constituencies is greater than the number of seats it is entitled to at national level and all its regional seats are the result of exact quotas rounded downwards;

- the *deficit paradox* for parties with exact regional quotas all rounded upwards: when the sum of the seats assigned to a party (or coalition) in the constituencies is smaller than the number of seats it is entitled to at the national level and it has already benefited from extra seats thanks to largest remainders in all constituencies where it has obtained votes.

The first type of paradox is shown in the example, the second is symmetric. In the second case the correction mechanism will get stuck because the law never considers the possibility that a lack of seats can occur although a party's exact quota of seats has already been rounded upwards in all constituencies (and therefore the party is never eligible to receive additional seats).

Moreover, applying the correction mechanism can cause a third type of paradox: the *constituency paradox*. In fact, given that seat transfers between parties in different constituencies are allowed, the total number of seats awarded in each constituency can end up being different from the number of seats actually at stake in the same constituency![7]

As mentioned earlier, the "bug" in the new Italian electoral law was inherited from the previous one. The proportional seat allocation of the hybrid system adopted for elections in 1994, 1996 and 2001 - also known as the Mattarellum system from the name of its maker - was in fact carried out with the same largest remainders procedure, applied first at the national level and then in the regional constituencies one at a time. There was no correction mechanism but while allocating additional seats to parties according to the Largest Remainders at the regional level, the number of seats each party was entitled to at the

[7] Actually, this occurred in the 2006 elections where 11 seats were assigned in Trentino Alto Adige - one more than the number of seats at stake in that constituency - and 2 seats were awarded in Molise instead of 3. Such a result is in clear contradiction with the Italian Constitution.

national level was considered an upper bound. Hence, the idea was to prevent a party from violating the constraint on the total number of seats it could receive by rounding its exact quota up only until it had not reached the number of seats already awarded at the national level. This does not prevent the paradoxes from occurring because, as we have stressed, even parties with exact quotas all rounded down can end up with a surplus and parties with exact quotas all rounded up can end up with a deficit.

In the new electoral law the technical flaw is even worse than in the old one, for several reasons. First of all, because it concerns the allocation of all 617 seats, harming the whole electoral outcome, while the Mattarellum system allocated only 25% of the seats on proportional basis. Secondly, because, in the case of coalitions competing in the electoral contest, the paradox occurring in trying to justify national and regional results may also occur in trying to re-allocate the seats awarded to the coalition among its member parties. Finally, the introduction of a correction mechanism while the bill was under exam in Parliament suggests that legislators saw a flaw in the procedure; the persisting failure of the correction mechanism proves they have not understood the real nature of the problem.

4. Tackling the Italian Electoral Problem

From a mathematical point of view, put aside the majority prize the electoral procedure adopted in the Italian case is meant to solve the following problem: find a matrix of nonnegative integers (the seats), whose row (the constituencies) and column (the political parties) sums are fixed and whose entries are "proportional" to a given matrix (the matrix of votes). This is the well-known bi-proportional allocation problem in integers which is in itself of great interest and has many applications, not only in the electoral field (for example Bacharach, 1970; Balinski, 1989a; Leti, 1970). Let M be a set of regional constituencies, N a set of political parties (or coalitions) and s a positive integer equal to the total amount of seats to be allocated (or house-size). The following notation is used:

v_{ij} the number of votes for party j in constituency i;
s_i the number of seats at stake in constituency i and such that $\sum_{i \in M} s_i = s$;
t_j the number of seats awarded to party j at the national level;
v the total number of votes.

Then v_{iN} and v_{Mj} are respectively the sum of the votes cast in constituency i (across all parties) and the sum of the votes cast for party j (across all constituencies):

$$v_{iN} = \sum_{j \in N} v_{ij}$$

$$v_{Mj} = \sum_{i \in M} v_{ij}$$

$$v_{MN} = \sum_{i \in M j \in N} v_{ij} = V$$

The bi-proportional allocation problem in integers is to find a matrix of seats s_{ij} for each constituency $i \in M$ and each party $j \in N$ such that the following constraints hold:

$$\begin{array}{ll} s_{MN} = s & \\ s_{iN} = s_i & \text{for every constituency } i \\ s_{Mj} = t_j & \text{for every party } j \\ s_{ij} \geq 0 & \text{for all } i, j \\ s_{ij} \text{ integer} & \text{for all } i, j \end{array} \tag{1}$$

Finally, one would like s_{ij} to be "*as proportional as possible*" to v_{ij} for all $i \in M$ and $j \in N$.

Let $q_{ij} = v_{ij} \frac{s_i}{v_{iN}}$ be the exact quota of seats for party j in constituency i. Now $q_{iN} = s_i$ and $q_{MN} = s$. Perfect proportionality is achieved by letting $s_{ij} = q_{ij}$. If there are no further constraints, this is the obvious solution to the problem, but s_{ij} must be integer and $s_{Mj} = t_j$ must hold as well. The idea underlying the Italian method is to consider the exact quotas each party is entitled to in the regional constituency and to round these numbers up or down (in the case the majority prize is assigned to some party, these quotas are not the exact ones but a modified version based on the majority or minority seats). Unfortunately, it is fairly easy to build realistic examples for which, however the rounding is carried out, it is impossible to satisfy both row and column constraints ($s_{iN} = s_i$ and $s_{Mj} = t_j$). The Italian electoral law adopts the method of largest remainders both at the national and regional level. Therefore, all resulting seat allocations comply with a property called quota satisfaction: i.e,

$$\left\lfloor v_{ij} \frac{s_i}{v_{iN}} \right\rfloor \leq s_{ij} \leq \left\lceil v_{ij} \frac{s_i}{v_{iN}} \right\rceil$$

holds for every party j and constituency i (at the regional level) but also:

$$\left\lfloor v_{Mj} \frac{s}{v} \right\rfloor \leq t_j \leq \left\lceil v_{Mj} \frac{s}{v} \right\rceil$$

holds for every party j at the national level.

The *surplus paradox* occurs if there is a party j such that:

$$\sum_{i \in M} \left\lfloor v_{ij} \frac{s_i}{v_{iN}} \right\rfloor > \left\lceil v_{Mj} \frac{s}{v} \right\rceil \tag{2}$$

The *deficit paradox* occurs if there is a party j such that:

$$\sum_{i \in M} \left\lceil v_{ij} \frac{s_i}{v_{iN}} \right\rceil < \left\lfloor v_{Mj} \frac{s}{v} \right\rfloor \tag{3}$$

For the paradoxes to occur there must be some kind of imbalance between vote/seat ratios at the national and regional level (or between the cost of a seat at the national and regional level), as shown below.

PROPOSITION 1 *If $s_i/v_{iN} = s/v$, for every $i \in M$ the two paradoxes cannot occur.*

Proof. If $s_i/v_{iN} = s/v$, for every $i \in M$, then for every party j:

$$\sum_{i \in M} \left\lfloor v_{ij} \frac{s_i}{v_{iN}} \right\rfloor = \left\lfloor v_{1j} \frac{s}{v} \right\rfloor + \left\lfloor v_{2j} \frac{s}{v} \right\rfloor + \ldots + \left\lfloor v_{Mj} \frac{s}{v} \right\rfloor \leq$$

$$\leq v_{1j} \frac{s}{v} + v_{2j} \frac{s}{v} + \ldots + v_{Mj} \frac{s}{v} = \frac{s}{v} \sum_{i \in M} v_{ij} \leq \left\lceil v_{Mj} \frac{s}{v} \right\rceil$$

which is the opposite of (2). The same can be shown for (3). ■

In other words discrepancy between national and regional coefficients is a necessary condition for the paradoxes, but it is not sufficient. Nevertheless, to get some intuition one may notice that when the regional seat apportionment plan is "perfect" - in the sense that the number of seats at stake in each constituency is perfectly proportional the corresponding regional population (fractional seats being allowed)- the "anti-paradox" condition $s_i/v_{iN} = s/v$ is equivalent to assuming the same rate of vote participation across the country.

PROPOSITION 2 *Let p be the total country population and p_i the population of the i-th constituency. Given a perfectly proportional seat apportionment plan, the condition $s_i/v_{iN} = s/v$ for every $i \in M$ is equivalent to the condition $v_{iN}/p_i = v/p$ for every $i \in M$.*

Proof. In a perfectly proportional seat apportionment plan $s_i/s = p_i/p$. If $s_i/v_{iN} = s/v$ holds for every $i \in M$, then $v_{iN} = s_i v/s = v p_i/p$ for every

$i \in M$ and vice-versa. ■

In real life a certain degree of discrepancy between the vote/seat ratios at the national and regional level is usual. Significant differences may be due to factors which are out of the legislator's control, such as different rates of absenteeism (citizens not voting at all) or protest (citizens casting invalid ballots) across the regions. As suggested in Proposition 2, they might also be due to the way the electoral law is put into practice, such as a "bad" regional apportionment plan (where the seats at stake in each regional constituency do not tend to reflect of the size of the constituency's population). Despite a "good" regional apportionment plan, there are at least two other features of the Italian electoral law that could be responsible for an imbalance between the national and regional ratios:

- The thresholds on the number of votes parties and coalitions must obtain to participate in the electoral contest. When a small party is cut out from the competition because of the threshold, its votes are deducted from the total constituency outcome in terms of votes. Parties running such a risk are typically groups of local interest which run only in very specific regions (at least in the Italian case);
- The majority prize. When the majority list wins 340 seats although it has obtained proportionally a much smaller amount of votes, the majority and minority coefficients tend to be very different, and different from the national vote/seat ratio.

Although the problem the Italian electoral law attempts to solve is not an easy one, a "sound" solution always exists as proved by Balinski and Demange (1989a and 1989b). The authors actually prove that a solution satisfying a number of basic properties (such as monotonicity, uniformity, relevance, exactness, etc.) can be found with an algorithm resembling the well-known out-of-kilter algorithm for minimum cost network flows.

5. Drawing Some Conclusions

The history of electoral systems is full of examples of paradoxes and failures - some of which have been used with bias for the purpose of political advantage. A mathematical approach to electoral systems can help identify such failures. In fact a more thorough use of mathematical tools to evaluate and design the many features that make an electoral system - from the design of electoral districts to the choice of a method to transform votes into seats - is fundamental (see also Balinski and Young, 1982; Grilli di Cortona et al., 2005). The paradoxes underlying the Italian law are not due to the fact that achieving double proportionality, at the national level and within regional constituencies,

is an unsolvable problem but to fact that the method adopted is not an appropriate one. In the case examined in this paper the legislators do not seem to have been aware of the underlying complexity of the problem they were facing and they have basically established a procedure which is too simple to address bi-proportional allocation in integers. Appropriate and correct procedures exist although they use somewhat sophisticated mathematics and might have to be carried out with the help of a computer program. This should not prevent electoral laws to adopt correct procedures: the University of Augsburg developed a Java-program for matrix apportionments using divisor methods and based on alternate scaling called BAZI, which has been adopted to shape the Zurich electoral law in 2003 (Pukelsheim, 2004). The idea of using a complex algorithm and a computer-aided solution to elect the representatives of Parliament opens to a number of questions. Surely such a fundamental law for democracy must be clear and transparent to all citizens and not only an optimum according to mathematicians. Moreover, the procedure must be replicable in all its steps and, above all, it must guarantee a unique solution.

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