Power Indices Taking into Account Agents' Preferences

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Abstract A set of new power indices is introduced extending Banzhaf power index and taking into account agents' preferences to coalesce. An axiomatic characterization of intensity functions representing a desire of agents to coalesce is given. A set of axioms for new power indices is presented and discussed. An example of use of these indices for Russian parliament is given.

Keywords: Ordinal power index, Cardinal power index, intensity function, consistency of factions.

1. Introduction

Power indices have become a very powerful instrument for study of electoral bodies and an institutional balance of power in these bodies (Brams, 1975; Felsenthal and Machover, 1998; Grofman and Scarrow, 1979; Herne and Nurmi, 1993; Laruelle and Valenciano, 2001; Leech, 2004). One of the main shortcomings mentioned almost in all publications on power indices is the fact that well-known indices do not take into account the preferences of agents (Felsenthal and Machover, 1998; Steunenberg et al., 1999). Indeed, in construction of those indices, e.g., Shapley-Shubik or Banzhaf power indices (Banzhaf, 1965; Shapley and Shubik, 1954), all agents are assumed to be able to coalesce. Moreover, none of those indices evaluates to which extent the agents are free in their wishes to create a coalition, how intensive are the connections inside one or another coalition[1].

Until recently the only index taking into account preferences of voters was that of Shapley – Owen (Shapley and Owen, 1989). However, the application

[1]First study on the coalition formation taking into account preferences of agents to coalesce was Dreze and Greenberg (1980). However, the problem of power distribution among agents in that study had not been considered.

of it to real data reveals some serious problems. They have been discussed in Barr and Passarelli (2004). That is why several attempts have been made to construct power indices do take into account preferences of voters to coalesce (Napel and Widgren, 2004, 2005).

In this article we try to construct another approach to define such indices. Consider an example. Let three parties A, B and C with 50, 49 and 1 sets, respectively, are presented in a parliament, and the voting rule is the simple majority one, i.e., 51 votes for. Then winning coalitions are $A+B$, $A+C$, $A+B+C$ and A is pivotal in all coalitions, B is pivotal in the first coalition and C is pivotal in the second one. (Normalized) Banzhaf power index[2] for these parties is equal to

$$\beta(A) = 3/5, \beta(B) = \beta(C) = 1/5.$$

Assume now that parties A and B never coalesce in pairwise coalition, i.e., coalition $A+B$ is impossible. Let us, however, assume that the coalition $A+B+C$ can be implemented, i.e., in the presence of 'moderator' C parties A and B can coalesce. Then the winning coalitions are $A+C$ and $A+B+C$, and A is pivotal in both coalitions while C is in one; B is pivotal in none of the winning coalitions. In this case $\beta(A) = 2/3$, $\beta(C) = 1/3$ and $\beta(B) = 0$, i.e., although B has almost half of the seats in the parliament, its power is equal to 0.

If A and B never coalesce even in the presence of a moderator C, then the only winning coalition is $A+C$, in which both parties are pivotal. Then, $\beta(A) = \beta(C) = 1/2$. Such situations are met in real political systems. For instance, Russian Communist Party in the second parliament (1997-2000) had had about 35% of seats, however, its power during that period was always almost equal to 0 (Aleskerov et al., 2003).

We introduce here two new types of indices based on the idea similar to Banzhaf power index, however, taking into account agents' preferences to coalesce. In the first type the information is used about agents' preferences over other agents. These preferences are assumed to be linear orders. Since these preferences may not be symmetric, the desire of agent 1 to coalesce with agent 2 can be different than the desire of agent 2 to coalesce with agent 1. These indices take into account in a different way such asymmetry of preferences. In the second type of power index the information about the intensity of prefer-

[2] Banzhaf power index is evaluated as

$$\beta(i) = \frac{b_i}{\sum_j b_j},$$

b_i is the number of winning coalitions in which agent i is pivotal, i.e., if agent i expels from the coalition it becomes a loosing one (Banzhaf, 1965). This form of Banzhaf index is called the normalized one.

ences is taken into account as well, i.e., we extend the former type of power index to cardinal information about agents' preferences.

The structure of the paper is as follows. Section 2 gives main notions. In Section 3 we define and discuss 'ordinal' power indices. In Section 4 cardinal indices are introduced. In Section 5 we evaluate power distribution of groups and factions in the Russian Parlament in 2000-2003 using some of new indices. Section 6 and 7 provides some axioms for the indices introduced. Section 8 cocludes.

2. Main Notions

The set of agents is denoted as N, $N = \{1, ..., n\}$, $n > 1$. A coalition ω is a subset of N, $\omega \subseteq N$. We consider the situation when the decision of a body is made by voting procedure; agents who do not vote 'yes' vote against it, i.e., the abstention is not allowed.

Each agent $i \in N$ has a predefined number of votes, $v_i > 0, i = 1, \ldots, n$. It is assumed that a quota q is predetermined and as a decision making rule the voting with quota is used, i.e., the decision is made if the number of votes for it is not less than q,

$$\sum_i v_i \geq q.$$

The model describes a voting by simple and qualified majority, voting with veto (as in the Security Council of UN), etc.

A coalition ω is called winning if the sum of votes in the coalition is no less than q. An agent i is called pivotal in a coalition ω if the coalition $\omega \backslash \{i\}$ is a loosing one.

For such voting rule the set of all winning coalitions Ω possesses the following properties:

$$\begin{array}{rcl} \emptyset & \notin & \Omega, \\ N & \in & \Omega, \\ \omega & \in & \Omega,\ \omega' \supseteq \omega \Longrightarrow \omega' \in \Omega. \end{array}$$

Sometimes, one additional condition is applied as well

$$\omega \in \Omega \Longrightarrow N \backslash \omega \notin \Omega,$$

implying $q \geq \lceil \frac{n}{2} \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Next we introduce two types of indices, ordinal and cardinal. Both types are constructed on the following basis: the intensity of connection $f(i, \omega)$ of the agent with other members of ω is defined. Then for such agent i the value χ_i is evaluated as

$$\chi_i = \sum_{\omega} f(i, \omega),$$

i.e., the sum of intensities of connections of i over those coalitions in which i is pivotal.

Naturally, other functions instead of summation can be considered.

Then the power indices are constructed as

$$\alpha(i) = \frac{\chi_i}{\sum_j \chi_j}.$$

The very idea of the index α is the same as for Banzhaf index, with the difference that in Banzhaf index we evaluate the number of coalitions in which i is pivotal, i.e., in the definition of Banzhaf index χ_i is equal to 1, on the contrary, in our case χ_i is defined by the value of intensity function.

The main question is how to construct the intensity functions $f(i, \omega)$. Below we give two ways how to construct those functions.

Each agent i is assumed to have a linear order[3] P_i revealing her preferences over other agents in the sense that i prefers to coalesce with agent j rather than with agent k if P_i contains the pair (j, k). Obviously, P_i is defined on the Cartesian product $(N \backslash \{i\}) \times (N \backslash \{i\})$.

Since P_i is a linear order, the rank p_{ij} of the agent j in P_i can be defined. We assume that $p_{ij} = |N| - 1$ for the most preferable agent j in P_i.

The value p_{ij} shows how many agents less preferable than j are in P_i. For instance, if $N = \{A, B, C, D\}$ and $P_A : B \succ C \succ D$, then $p_{AB} = 3, p_{AC} = 2$ and $p_{AD} = 1$.

Using these ranks, one can construct different intensity functions.

A second way of construction of $f(i, \omega)$ is based on the idea that the values p_{ij} of connection of i with j are predetermined somehow. In general, it is not assumed $p_{ij} = p_{ji}$. Then the intensity function can be constructed as above.

Below we give six different ways how to construct $f(i, \omega)$ in ordinal case and sixteen ways of construction of cardinal function $f(i, \omega)$.

3. Ordinal Indices

For each coalition ω and each agent i construct now an intensity $f(i, \omega)$ of connections in this coalition. In other words, f is a function which maps $N \times \Omega$ $(= (2^N \backslash \{\emptyset\})$ into $\mathcal{R}^1$, $f : N \times \Omega \to \mathcal{R}^1$. This very value is evaluated using the ranks of members of coalition. Several different ways to evaluate f using different information about agents' preferences are provided:

a) *Intensity of i's preferences.*

[3] i.e. irreflexive, transitive and connected binary relation. We often denote it as $\succ$.

In this form only preferences of i's agent over other agents are evaluated, i.e.,

$$f^+(i,\omega) = \sum_{j\in\omega} \frac{p_{ij}}{|\omega|};$$

b) *Intensity of preferences for i.* In this case we consider the sum of ranks of i given by other members of coalition ω.

$$f^-(i,\omega) = \sum_{j\in\omega} \frac{p_{ji}}{|\omega|};$$

c) *Average intensity with respect to i's agent*

$$f(i,\omega) = \frac{f^+(i,\omega) + f^-(i,\omega)}{2};$$

d) *Total positive average intensity.*

Consider any coalition ω of size $k \leq n$. Without loss of generality one can put $\omega = \{1, \ldots, k\}$. Then consider $f^+(i,\omega)$ for each i and construct

$$f^+(\omega) = \frac{\sum\limits_{i\in\omega} f^+(i,\omega)}{|\omega|};$$

e) *Total negative average intensity* is defined similarly by the formula

$$f^-(\omega) = \frac{\sum\limits_{i\in\omega} f^-(i,\omega)}{|\omega|};$$

f) *Total average intensity* is defined as

$$f(\omega) = \frac{\sum\limits_{i\in\omega} f(i,\omega)}{|\omega|}.$$

It is worth emphasizing here that the intensities d) – f) do not depend on agent i, i.e., for any agent i in the following calculation of power indices we assume that for any i in the coalition ω the corresponding intensity is the same.

Consider now several examples.

Example 1. Let $n = 3, N = \{A, B, C\}, v(A) = v(B) = v(C) = 33, q = 50$. Consider two preference profiles given in Tables 1 and 2.

For both preference profiles there are three winning coalitions in which agents are pivotal. These coalitions are $A + B, A + C$ and $B + C$.

Table 1. First preference profile

P_A	P_B	P_C
C	C	A
B	A	B

Table 2. Second preference profile

P_A	P_B	P_C
B	C	A
C	A	B

Let us calculate the functions f as above for each agent in each winning coalition. The preferences from Tables 1 and 2 can be re-written in the matrix form as

$$\|p_{ij}\| = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

$$\|p_{ij}\| = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

Now, for the profile given in Table 1 one can calculate the values of intensities a)–f) obtained by each agent i in each winning coalition ω. These values for the first preference profile are given in Table 3 and for the second one – in Table 4.

Using these intensity functions one can define now the corresponding power indices $\alpha(i)$. Let i be a pivotal agent in a winning coalition ω. Denote by χ_i the number equal to the value of the intensity function for a given coalition ω and agent i. Then the power index is defined as follows

$$\alpha(i) = \frac{\sum\limits_{\substack{\omega \\ i \text{ is pivotal in } \omega}} \chi_i}{\sum\limits_{j \in N} \sum\limits_{\substack{\omega \\ j \text{ is pivotal in } \omega}} \chi_j}$$

Table 3. Intensity values for the first preference profile

	$f^+(i,\omega)$			$f^-(i,\omega)$			$f(i,\omega)$		
	A	B	C	A	B	C	A	B	C
$A+B$	1/2	1/2	-	1/2	1/2	-	1/2	1/2	-
$A+C$	1	-	1	1	-	1	1	-	1
$B+C$	-	1	1/2	-	1/2	1	-	3/4	3/4
	$f^+(i,\omega)$			$f^-(i,\omega)$			$f(i,\omega)$		
	A	B	C	A	B	C	A	B	C
$A+B$	1/2	1/2	-	1/2	1/2	-	1/2	1/2	-
$A+C$	1	-	1	1	-	1	1	-	1
$B+C$	-	3/4	3/4	-	3/4	3/4	-	3/4	3/4

Table 4. Intensity values for the second preference profile

	$f^+(i,\omega)$			$f^-(i,\omega)$			$f(i,\omega)$		
	A	B	C	A	B	C	A	B	C
$A+B$	1	1/2	-	1/2	1	-	3/4	3/4	-
$A+C$	1/2	-	1	1	-	1/2	3/4	-	3/4
$B+C$	-	1	1/2	-	1/2	1	-	3/4	3/4
	$f^+(i,\omega)$			$f^-(i,\omega)$			$f(i,\omega)$		
	A	B	C	A	B	C	A	B	C
$A+B$	3/4	3/4	-	3/4	3/4	-	3/4	3/4	-
$A+C$	3/4	-	3/4	3/4	-	3/4	3/4	-	3/4
$B+C$	-	3/4	3/4	-	3/4	3/4	-	3/4	3/4

Table 5. Power indices values

	First profile (Table 1)			Second profile (Table 2)		
	A	B	C	A	B	C
α_1	$1/3$	$1/3$	$1/3$	$1/3$	$1/3$	$1/3$
α_2	$1/3$	$2/9$	$4/9$	$1/3$	$1/3$	$1/3$
α_3	$1/3$	$5/18$	$7/18$	$1/3$	$1/3$	$1/3$
α_4	$1/3$	$5/18$	$7/18$	$1/3$	$1/3$	$1/3$
α_5	$1/3$	$5/18$	$7/18$	$1/3$	$1/3$	$1/3$
β	$1/3$	$5/18$	$7/18$	$1/3$	$1/3$	$1/3$

As we already mentioned this index is similar to the Banzhaf index. The difference is that in the Banzhaf index χ_i is equal to 1, in the case under study χ_i represents some intensity value.

The indices $\alpha(i)$ will be denoted by $\alpha_1(i), \ldots, \alpha_6(i)$.

Let us evaluate now the values $\alpha_1(\cdot) - \alpha_6(i)$ for all agents for the preference profile from Table 1.

The agent A (as well as agents B and C) is pivotal in two coalitions; the sum of the values $f^+(i,\omega)$ for each i is equal to 3/2. Then

$$\alpha_1 = \frac{3/2}{3/2 + 3/2 + 3/2} = \frac{1}{3} = \alpha_1(B) = \alpha_1(C).$$

The value $\alpha_2(\cdot)$ is evaluated differently. The sum of values $f^-(i,\omega)$ from Table 3 for all i and ω is equal to 9/2. However, for A $\sum\limits_{\omega} f(A,\omega) = 3/2$, $\sum\limits_{\omega} f(B,\omega) = 1$ and $\sum\limits_{\omega} f(C,\omega) = 2$. Then $\alpha_2(A) = \frac{3}{9} = \frac{1}{3}$; $\alpha_2(B) = \frac{2}{9}$ and $\alpha_2(C) = \frac{4}{9}$.

The values of the indices $\alpha_1(\cdot)$–$\alpha_6(i)$ for both preference profiles are given in Table 5 as well as the values of Banzhaf index β.

Consider now another example.

Example 2. Let $N = \{A, B, C, D, E\}$, each agent has one vote, $q = 3$ and the preferences of agents are given in Table 6. The values of indices $\alpha_2(\cdot)$–$\alpha_4(i)$ are given in Table 7.

Note that α_1 is equal to the Banzhaf index, which for this case gives $\forall i \in N$ $\beta(i) = 1/5$.

Example 3. Consider the case when 3 parties A, B and C have 50, 49 and 1 seats, respectively. Assume that decision making rule is simple majority, i.e. 51 votes. Then the winning coalitions are A+B, A+C and A+B+C. Note that

Table 6. Preferences of agents for $N = \{A, B, C, D, E\}$

P_A	P_B	P_C	P_D	P_E	rank
B	A	D	A	B	4
C	C	A	B	A	3
D	D	B	C	D	2
E	E	E	E	C	1

Table 7. The values of the indices $\alpha_2 - \alpha_4$ for Example 2.

	A	B	C	D	E
α_2	0.28	0.26	0.18	0.2	0.008
α_3	0.24	0.23	0.19	0.2	0.14
α_4	0.22	0.21	0.2	0.2	0.17

A is pivotal in all three coalitions, B and C are pivotal in one coalition each. Then $\beta(A) = 3/5$, $\beta(B) = \beta(C) = 1/5$.

Consider now the case with the preferences of agents given below: $P_A; C \succ B$; $P_B : C \succ A$ and $P_C : A \succ B$.

Then the values of α_1 and α_2 (constructed by $f^+(i, \omega)$ and $f^-(i, \omega)$) are as follows

$$\begin{array}{lll} \alpha_1(A) = 5/12, & \alpha_1(B) = 1/4, & \alpha_1(C) = 1/3, \\ \alpha_2(A) = 5/12 & \alpha_2(B) = 7/36 & \alpha_2(C) = 7/18. \end{array}$$

Consider another preference profile: $P'_A : C \succ B$, $P'_B : C \succ A$ and $P'_C : B \succ A$, i.e., the only agent C changes her preferences. Then one can easily evaluate $\alpha'_1(A) = 5/11$, $\alpha'_1(B) = 3/11$, $\alpha'_1(C) = 3/11$, $\alpha'_2(A) = 10/33$, $\alpha'_2(B) = 3/11$, $\alpha'_2(C) = 14/33$.

4. Cardinal Indices

Assume now that the desire of party i to coalesce with party j is given as real number p_{ij}, $\sum_j p_{ij} = 1, i, j = 1, \ldots, n$. In general, it is not assumed that $p_{ij} = p_{ji}$.

One can call the value p_{ij} as an intensity of connection of i with j. It may be interpreted as, for instance, a probability for i to form a coalition with j.

We define now several intensity functions

a) average intensity of i's connection with other members of coalition ω

$$f^+(i,\omega) = \frac{\sum\limits_{j\in\omega} p_{ij}}{|\omega|};$$

b) average intensity of connection of other members of coalition with i

$$f^-(i,\omega) = \frac{\sum\limits_{j\in\omega} p_{ji}}{|\omega|};$$

c) average intensity for i

$$f(i,\omega) = \frac{1}{2}\left(f^+(i,\omega) + f^-(i,\omega)\right);$$

d) average positive intensity in ω

$$f^+(i,\omega) = \frac{\sum\limits_{i\in\omega} f^+(i,\omega)}{|\omega|},$$

e) average negative intensity in ω

$$f^-(i,\omega) = \frac{\sum\limits_{i\in\omega} f^-(i,\omega)}{|\omega|},$$

f) average intensity in ω

$$f(\omega) = \frac{\sum\limits_{i\in\omega} f(i,\omega)}{|\omega|},$$

In contrast to ordinal case now we can introduce several new intensity functions:

g) minimal intensity of i's connections

$$f^+_{\min}(i,\omega) = \min_j p_{ij};$$

h) maximal intensity of i's connections

$$f^+_{\max}(i,\omega) = \max_j p_{ij};$$

i) maximal fluctuation of i's connections

$$f_{mf}(i,\omega) = \frac{1}{2}\left(\min_j p_{ij} + \max_j p_{ij}\right);$$

j) minimal intensity of connections of other agents in ω with i

$$f^{-}_{\min}(i,\omega) = \min_j p_{ji}$$

k) maximal intensity of connections of other agents ω in with i

$$f^{-}_{\max}(i,\omega) = \max_j p_{ji}$$

l) s-mean intensity of i's connections with other agents in ω

$$f^{+}_{sm}(i,\omega) = \frac{1}{|\omega|}\sqrt[s]{\sum_j p^s_{ij}};$$

m) s-mean intensity of connections of other agents ω in with i

$$f^{+}_{sm}(i,\omega) = \frac{1}{|\omega|}\sqrt[s]{\sum_j p^s_{ji}};$$

n) max min intensity

$$f_{\max\ \min}(\omega) = \max_i \min_j p_{ij};$$

o) min max intensity

$$f_{\min\ \max}(\omega) = \min_i \max_j p_{ji};$$

p) maximal fluctuation

$$f_{mf}(\omega) = \frac{1}{2}\left(f_{\max\ \min}(\omega) + f_{\min\ \max}(\omega)\right).$$

Note that the intensity functions in the cases d)–f), n)–p) do not depend on agent herself but only on coalition ω.

Now the corresponding power indices can be defined as above, i.e.,

$$\alpha^{\text{card}}(i) = \frac{\sum\limits_{\substack{\omega \text{ is winning} \\ i \text{ is pivotal in } \omega}} \chi_i}{\sum\limits_{j\in N} \sum\limits_{\substack{\omega \text{ is winning} \\ j \text{ is pivotal in } \omega}} \chi_i(\omega)},$$

where χ_i is one of the above intensity functions.

Example 4. Let $N = \{A, B, C, D\}$, each voter has only one vote, the quota is equal to $q = 3$, and the matrix $\|p_{ij}\|$ is given in Table 8. In Table 9 the power indices are given for the cases a), b), e), h).

Table 8. Matrix $\|p_{ij}\|$ for Example 3

	A	B	C	D
A		0.7	0.2	0.1
B	0.3		0.5	0.2
C	0.1	0.7		0.2
D	0.7	0.2	0.1	

Table 9. Some cardinal indices for Example 3

	A	B	C	D
$\alpha_{a)}$	0.25	0.25	0.25	0.25
$\alpha_{b)}$	0.27	0.40	0.20	0.13
$\alpha_{c)}$	0.25	0.27	0.24	0.23
$\alpha_{h)}$	0.25	0.25	0.25	0.25

5. Evaluation for Russian Parliament

We will study now a distribution of power among factions in the third Russian Parliament (1999-2003) using these new indices. The matrix $\|p_{ij}\|$ is constructed using the consistency index; the latter (the index of consistency of positions of two groups) is constructed as

$$c(q_1, q_2) = 1 - \frac{|q_1 - q_2|}{\max(q_1, 1 - q_1, q_2, 1 - q_2)},$$

where q_1 and q_2 be the share of “ay” votes in two groups of MPs (Aleserkov et al., 2003).

We consider the value of consistency index as the value of intensity of connections between agents i and j. Then we are in cardinal framework, and one can use one of the indices introduced in the previous section.

On Fig. 1 the values of $\alpha_{a)}$ index are given for the Russian Parliament from 2000 to 2003 on the monthly basis. It can be readily seen that index α gives lower values for Communist Party (sometimes up to 3%) and higher values for Edinstvo (up to 1%). It is interesting to note that Liberal-Democrats (Jirinovski’s Party) had had almost equal values by both indices, which corresponds to the well-known flexibility of that party position.

Let us note that different ways to use the index α are possible. For instance, following the approach from Aleskerov et al. (2003), we may assume that if the consistency value for two factions is less than some threshold value δ, then parties do not coalesce, i.e., $p_{ij} = 0$.

Figs. 2 and 3 give power distribution for the factions in the Russian Parliament for the same period calculated on the basis of factions coordinates on a political map. On that map each faction at each period is characterized by two coordinates – the level to which extent it is liberal or state oriented and the level of support of the president (pro-reforms or anti-reforms) (Aleskerov et al., 2005).

Having these two coordinates, we calculate the distance on the map between the positions of two factions. Then it is possible to construct a measure τ_{ij} – intensity of connections among factions i and j – as

$$\tau_{ij} = \frac{1}{\sqrt{2}} \left(\frac{1+\sqrt{2}}{1+d_{ij}} - 1 \right),$$

where d_{ij} is the Euclidean distance between positions of factions i and j on political map.

It can be easily seen that $\tau_{ij} = 0$ if $d_{ij} = \sqrt{2}$ (the maximal distance on the map), and $\tau_{ij} = 1$ if $d_{ij} = 0$ (i.e., when positions of two factions coincide). Using the values τ_{ij} for two factions and consider them as a measure of preference to coalesce, one can calculate the cardinal indices introduced above, in particular, the index for the case a). These very evaluations are given on Figs. 2 and 3 for five main parties in Russian parliament during the period 2000-2003.

6. Axiomatic Construction of a Cardinal Intensity Function

Now we will try to axiomatize a construction of cardinal intensity function.

First, we define an intensity function depending on intensities p_{ij} of connections of i with other members of coalition ω, i.e., if $\omega = \{1, \ldots, m\}$, $m \leq n$,

$$f(i, \omega) = f_i \left(p_{11}, \ldots, p_{1m}, p_{21}, \ldots, p_{2m}, \ldots, p_{i1}, \ldots, p_{im}, \ldots, p_{mm} \right).$$

As it is seen, the intensity function for i depends not only of i's connections with other members of coalition, but depends also of connections of other members among themselves. We can consider, for instance, the case when the intensity of agent i to join a coalition ω depends on the average intensity of connections between members of ω, say, the intensity can be low if that average intensity is below some threshold.

However, we will restrict this function in a way which is similar to independence of irrelevant alternatives (Arrow, 1963): $f(i, \omega)$ will depend on connections of agent i with other members of coalition ω only, i.e.,

$$f(i, \omega) = f_i \left(p_{i1}, \ldots, p_{im} \right).$$

For the sake of simplicity we put $p_{ij}^{\omega} \geq 0$ for all i, j and $\forall i \sum\limits_{j \in \omega} p_{ij}^{\omega} = 1$.

I would like to emphasize that in this formulation the sum of p_{ij}^{ω} is equal to 1 in each ω, i.e., now connections are defined by $2^N - 1$ matrices $\|p_{ij}^{\omega}\|$ for each coalition ω.

Consider several axioms which reasonable function $f(i, \omega)$ should satisfy to.

Axiom 1. For any m – tuple of values $(p_{i1}, \ldots, p_{im})$ there exist a function $f(i, \omega)$ such that $0 \leq f(i, \omega) \leq 1$, f is continuous differentiable function of each of its arguments.

Axiom 2. If $p_{ij} = 0$ for any j, then $f(i, \omega) = 0$.

Axiom 3. (Monotonicity). A value of $f(i, \omega)$ increases if any value p_{ij} increases, and a value of $f(i, \omega)$ decreases if p_{ij} decreases. Moreover, equal changes in intensities p_{ij} lead to equal changes of $f(i, \omega)$. This means that

$$\frac{\partial f_i}{\partial p_{ij}} = \mu_i \quad \text{for any } j,$$

and

$$\frac{\partial f_i}{\partial p_{lj}} = 0 \quad \text{for any } l \neq i.$$

Then the following theorem holds

Theorem. An intensity function $f(i, \omega)$ satisfies Axioms 1–3 iff it is represented in the form

$$f(i, \omega) = \frac{\sum\limits_{j} p_{ij}}{|\omega|}.$$

Proof is a re-formulation of the proof of the theorem from Intriligator (1973) given in the framework of probabilistic social choice and hence is omitted.

An axiomatic characterization of other types of intensity functions is still an open problem.

7. Axioms for Power Indices

We introduce several axioms, which any reasonable power index should satisfy to.

First, we call a voting situation a four-tuple $[N, q, v, \vec{P}]$, where N is a set of agents, $|N| = n$, $n > 1$, q is a quota, $v = (v_1, \ldots, v_n)$ is a set of votes which agents possess, $\vec{P}$ is a preference profile, where each agent $i \in N$ has a preference (linear order) P_i over $N \backslash \{i\}$ or preference matrix $\|p_{ij}\|$.

Axiom 1. Under a given quota rule for any agent $i \in N$ there exists a preference profile $\vec{P}$ such that $\alpha(i) > 0$.

In words, for no agent it is known in advance, independently of agents' preferences, that her power is equal to 0.

Axiom 2. Consider two voting situations $[N, q, v, \vec{P}]$ and $[N, q, v', \vec{P}]$. Let $\exists A \in N$ such that $v'(A) \geq v(A)$, and $\forall B \in N$, $v'(B) = v(B)$. Then, $\alpha'(A) \geq \alpha(A)$.

Assume that for a given distribution of votes and a given preference profile we evaluate power distribution among agents. Then we increase the number of votes for a given agent A, keeping the votes of other agents unchanged. Then Axiom 2 states that voting power of A in new situation should not be less than before.

Axiom 3. (Symmetry) Let η be a one-to-one correspondence of N to N. Then

$$\eta(\alpha_1, \ldots, \alpha_n) = (\alpha_{\eta(1)}, \ldots, \alpha_{\eta(n)}).$$

Axiom 3 states that power of agents does not depend of their names, i.e., the procedure of evaluation of power distribution must treat agents in a similar way.

Axiom 4. Let $i \in N$ be pivotal in no winning coalition ω. Then, $\alpha(i) = 0$.

It is usual axiom in voting power models (in fact, in game – theoretic models, see Shapley and Shubik (1954)): a dummy player has power equal to 0.

Axiom 5'. First Monotonicity Axiom (FMA). Consider two voting situations $[N, q, v, \vec{P}]$ and $[N, q, v, \vec{P'}]$. Let for some i and any $k \neq i$ $P_k = P'_k$ holds. Let additionally for some $p'_{ij} > p_{ij}$ holds. Then, $\alpha'(j) \geq \alpha(j)$.

This axiom can be explained in a simple way: all preferences except i's are the same in two profiles; in i'th preference the evaluation of j is higher in new profile than in the old one. Then in new voting situation (with $\vec{P'}$) the power of j should not be less than before.

Axiom 5''. Second Monotonicity Axiom (SMA). Consider two voting situations $[N, q, v, \vec{P}]$ and $[N, q, v, \vec{P'}]$. Let for two agents i and j $\alpha(i) \geq \alpha(j)$ holds, where $\alpha(i)$ is the voting power of i in the first voting situation. Let $\vec{P'}$ is such that for any $k \neq l$ $P_k = P'_k$ holds, and in the preferences of l's agent

$$p'_{li} - p'_{lj} > p_{li} - p_{lj}$$

holds.

Then $\alpha'(i) \geq \alpha(j)$ (weak version of SMA) or $\alpha'(i) > \alpha(j)$ (strong version of SMA), where $\alpha'(i)$ is the voting power of i with respect to second voting situation.

In words, assume that the power of i is not less than the power of j with respect to first voting situation. Let $\vec{P'}$ is such that for any agent but ℓ her new preferences coincide with old ones, and in l's preference the relative position of i with respect to j is higher in P'_l than in P_l. Then the voting power of i

should be not less than that of j in new voting situation (weak version) or even must be greater than that of j (strong version).

Axiom 6. Let $\vec{P}'$ be an intensity matrix such that $p'_{ij} = kp_{ij}$ for every $i, j = 1, \ldots, n$. Then $\alpha'(i) = \alpha(i)$ where α' is the power vector obtained from $\vec{P}'$.

Axiom 6 deals with cardinal power indices. It says that voting power of agents does not change under the transformation of scale of intensities in the form

$$p'_{ij} = kp_{ij},$$

i.e., when intensities multiply to the same constant k.

It is possible to formulate axioms similar to those from Section 5 and prove a theorem similar to the given above but for α–indices. However, it will be interesting to analyze how the axioms from this Section provide an axiomatic characterization of α indices.

8. Concluding Remarks

We have considered three ways to construct power indices taking into account voters' preferences to coalesce. The first one is based on the consistency index showing to which extent two groups of voters (party factions) vote in a similar way. The values of consistency index define the possibility of these groups to coalesce. Then the Banzhaf index is defined on the set of admissible coalitions only.

The second way uses the functions defining the intensity of factions to coalesce being based on the intensity to coalesce of individual faction. We have defined six ordinal intensity indices and sixteen cardinal ones. For a simplest cardinal intensity index the corresponding axioms are introduced and the characterization theorem is proved.

Then the power index is defined in a way similar to Banzhaf index – instead of calculating number of coalitions in which faction is pivotal we calculate an intensity of faction to coalesce in the coalitions in which it is pivotal.

Finally, we define an intensity function as a function of distance using the coordinates of factions on the political map. The latter is constructed using data of real voting in a parliament (see, for instance, Aleskerov et al. (2005)).

Then using this intensity of faction one can calculate one of the power indices defined above for a cardinal case.

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