Choquet Integration and Correlation Matrices in Fuzzy Inference Systems

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A fuzzy rule inference system (FIS) of the type Takagi–Sugeno–Kang (TSK) [7, 8], is described by rules such as

$$R_i : \text{IF} x_1 \, is \, A_{i1} \text{ AND} \cdots \text{ AND} x_m \, is \, A_{im} \text{ THEN } y_i = c_{i0} + c_{i1}x_1 + \cdots + c_{im}x_m,$$

where $i = 1, ..., n$. The set of rules $\{R_1, R_2, \ldots, R_n\}$ forms the rule base. The consequent of each rule can be interpreted as a fuzzy singleton whose value is dependent on the system's inputs. Each triggered rule provides an activation level α, which is given by

$$\alpha_i = \otimes(\mu_{A_{i1}}, \ldots, \mu_{A_{im}}),$$

where $i = 1, ..., n$, $\otimes(\cdot)$ represents a t-norm and $\{\mu_{A_{ij}} | j = 1, ..., m\}$ represents the set of memberships for each rule's antecedent variables. The output for the TSK system is then obtained using an aggregation operator based on the standard weighted averaging

$$y(\underline{x}) = \frac{\sum_{i=1}^{n} \alpha_i y_i}{\sum_{j=1}^{n} \alpha_j} = \sum_{i=1}^{n} w_i y_i \quad \text{where} \quad w_i = \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j} \quad \text{and} \quad \sum_{i=1}^{n} w_i = 1,$$

where we consider the individual weights of the fuzzy rules, $w_i > 0$, to be the normalized activation levels α_i for $i = 1, \ldots, n$.

In this paper we propose an extension to the TSK fuzzy inference system based on Choquet interation [5, 6], called Choquet–TSK. In our model, a matrix of pairwise correlations among the activation levels of the FIS rules is explicitly used in the aggregation process, leading to attenuation effects when correlations are positive and emphasizing effects when correlations are negative.

Consider a finite set of interacting criteria $N = \{1, 2, \ldots, n\}$.

A *Choquet measure* [1] on the set N is a set function $\mu : \mathcal{P}(N) \longrightarrow [0, 1]$ satisfying

$$(1) \quad \mu(\emptyset) = 0, \ \mu(N) = 1, \quad (2) \quad S \subseteq T \subseteq N \Rightarrow \mu(S) \leq \mu(T). \qquad (1)$$

Given a Choquet measure μ we can define the *Choquet integral* [1–3] of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ with respect to μ as

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)}, \tag{2}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$. Also $A_{(i)} = \{(i), \ldots, (n)\}$ and $A_{(n+1)} = \emptyset$.

Notice that the Choquet integral with respect to an additive measure μ reduces to a weighted averaging operator, whose weights w_i are given by the $\mu(i)$ values,

$$\mu(A_{(i)}) = \mu((i)) + \mu((i+1)) + \cdots + \mu((n)),$$

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} = \sum_{i=1}^{n} \mu((i)) x_{(i)} = \sum_{i=1}^{n} w_i x_i\, . \tag{3}$$

Consider the pairwise correlation matrix $\mathbf{C} = [c_{ij}]$ among the various fuzzy rules

$$c_{ij} \in [-1, 1], \quad c_{ji} = c_{ij}, \quad i, j = 1, \ldots, n, \tag{4}$$

where for convenience reasons we take a null diagonal $c_{ii} = 0$ for $i = 1, \ldots, n$.

We consider $w_i > 0$ for $i = 1, \ldots, n$ individual weights of the fuzzy rules, normalized so that $\Sigma_{i=1}^{n} w_i = 1$.

Given a general pairwise correlation matrix $\mathbf{C} = [c_{ij}]$, we define a two-additive Choquet measure $\mu : 2^N \longrightarrow [0,1]$ in the following way: making use of the Möbius transform m of the measure μ, we define $m(i) = w_i/\mathcal{N}$ for each singlet $\{i\}$ and $m(ij) = -w_i c_{ij} w_j/\mathcal{N}$ for each doublet $\{i,j\}$, with null higher order terms. Then, we define the value of the two-additive measure μ on a coalition S as the sum of the singlets and doublets contained in the coalition S, as given by the Möbius transform m

$$\mu(S) = \sum_{\{i\} \subseteq S} w_i/\mathcal{N} + \sum_{\{i,j\} \subseteq S} (-w_i c_{ij} w_j)/\mathcal{N}, \tag{5}$$

where the normalization factor $\mathcal{N}$ is the sum of all singlets and doublets in the set N

$$\begin{aligned} \mathcal{N} &= \sum_{\{i\} \subseteq N} w_i + \sum_{\{i,j\} \subseteq N} (-w_i c_{ij} w_j) = 1 - \frac{1}{2} \sum_{i,j=1}^{n} w_i c_{ij} w_j \\ &= 1 - \frac{1}{2} \sum_{i=1}^{n} w_i c_i = 1 - c/2, \end{aligned} \tag{6}$$

where $c_i = \sum_{j=1}^{n} c_{ij} w_j$ and $c = \sum_{i=1}^{n} w_i c_i$ denote weighted averages of pairwise correlation values. In particular, we have

$$\mu(i) = w_i/\mathcal{N}, \quad i, j = 1, \ldots, n,$$

$$\mu(ij) = (w_i + w_j - w_i c_{ij} w_j)/\mathcal{N}. \tag{7}$$

The measure μ satisfies the boundary conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$, and is monotonic.

The graph interpretation of this definition, in which singlets correspond to nodes and doublets correspond to edges between nodes, is that the value of the two-additive measure μ on a coalition S is the sum of the nodes and edges contained in the subgraph associated with the coalition S.

Notice that the proposed Choquet–TSK integration model is an extension of the standard weighted averaging of the TSK FIS. If the matrix **C** is null (null pairwise correlations among the fuzzy rules, i.e., rules are really independent of each other) then the Choquet measure μ is additive and the Choquet integral coincides with the weighted arithmetic mean whose weights are w_i as in the standard TSK FIS.

The following example was adapted from the *Dinner for Two* example of MATLAB (www.matworks.com.). In this example we have two input variables and one output variable. The input variables *service* and *food* and the output variable *tip* are described by the following linguistic terms: Service = {poor, good, excellent}; Food= {rancid, delicious}; Tip= {cheap, average, generous}. The rules for this system are as follows:

R_1 : IF the *service* is poor OR the *food* is rancid THEN the *tip* is cheap.

R_2 : IF the *service* is good THEN the *tip* is average.

R_3 : IF the *service* is excellent OR the *food* is delicious THEN the *tip* is generous.

The results obtained for 50 randomly generated inputs for both TSK FIS and Choquet–TSK FIS gave a mean relative deviation of 2% between the values for both systems.

To observe how the results of the Choquet–TSK FIS can be affected by the correlation matrix values, some extreme values for this matrix where tested. For a **C** matrix with minus ones outside its main diagonal, the mean relative deviation was 6%, and for a **C** matrix with ones outside its main diagonal, the mean relative deviation was 9%. It should also be noted that, using a null **C** matrix returned, as expected, the same results as the usual TSK FIS.

References

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