On Intuitionistic Fuzzy Negations

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Summary. Up to now four intuitionistic fuzzy negations were constructed. In the present paper one new negation is described and its specific properties are discussed. The relations between it and the other negations are studied. The standard and modified Laws for Excluded Middle, the standard and modified De Morgan's Laws are checked for the new negation.

1 Introduction: on Some Previous Results

Variants of intuitionistic fuzzy implications are discussed in [3, 7–9, 11–13]. The implications from [7] are intuitionistic fuzzy versions of the fuzzy implications defined in [1]. In [8,9] the introduced implications are used as basis for obtaining of intuitionistic fuzzy negations. Here we will introduce for a new negation. Below we will study some properties of all negations and will show that they satisfy the properties of the intuitionistic negation.

Let x be a variable. Then its intuitionistic fuzzy truth-value is represented by the ordered couple

$$V(x) = \langle a, b \rangle, \tag{1}$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of nonvalidity of x. Any other formula is estimated by analogy. Obviously, when V is ordinary fuzzy truth-value estimation, for it $b = 1 - a$.

Everywhere below we shall assume that for the three variables x, y and z equalities: $V(x) = \langle a, b\rangle, V(y) = \langle c, d\rangle, V(z) = \langle e, f\rangle$ $(a, b, c, d, e, f, a + b, c + d, e + f \in [0, 1])$ hold.

For the needs of the discussion below we shall define the notion of intuitionistic fuzzy tautology (IFT, see, [1, 3]) by:

$$x \text{ is an IFT if and only if } a \geq b, \tag{2}$$

while x will be a (classical) *tautology* if $a = 1$ and $b = 0$.

In some definitions we shall use functions sg and $\overline{sg}$:

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad \overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases} \tag{3}$$

In ordinary intuitionistic fuzzy logic (see [1,3]) the negation of variable x is $N(x)$ such that

$$V(N(x)) = \langle b, a \rangle.$$

The operations "conjunction" (&) and "disjunction" ($\vee$) are defined (see [1,3] by:

$$V(x \,\&\, y) = \langle \min(a, c), \max(b, d) \rangle, \ \ V(x \vee y) = \langle \max(a, c), \min(b, d) \rangle. \tag{4}$$

In [11] (see also [15]) the explicit forms of all 23 implication are given and their 23 corresponding negations are obtained, using as a basis equality

$$N(x) = I(x, F), \text{ where } V(F) = \langle 0, 1 \rangle. \tag{5}$$

The negations, generated by the implication operations are given in Table 1.

For these negations and for their corresponding implications the following three properties are checked in [8,9,9]:

Property P1: $A \to \neg\neg A$,
Property P2: $\neg\neg A \to A$,
Property P3: $\neg\neg\neg A = \neg A$.

Obviously, negation $\neg_1$ is a classical negation (it satisfies simultaneously properties P1 and P2), while for the four other ones it is shown that they have intuitionistic behavior (they satisfy property P1 and do not satisfy property P2). All negations satisfy property P3.

In [10] the validity of the Law for Excluded Middle (LEM) in the following forms is studied:

$$\langle a, b \rangle \vee \neg \langle a, b \rangle = \langle 1, 0 \rangle \qquad \text{(tautology form)} \tag{6}$$

and

$$\langle a, b \rangle \vee \neg \langle a, b \rangle = \langle p, q \rangle, \qquad \text{(IFT form)} \tag{7}$$

and a Modified LEM in the forms:

$$\neg\neg \langle a, b \rangle \vee \neg \langle a, b \rangle = \langle 1, 0 \rangle \qquad \text{(tautology form)} \tag{8}$$

and

$$\neg\neg \langle a, b \rangle \vee \neg \langle a, b \rangle = \langle p, q \rangle, \qquad \text{(IFT form)} \tag{9}$$

where $1 \geq p \geq q \geq 0$.

Usually, De Morgan's Laws have the forms:

$$\neg x \wedge \neg y = \neg(x \vee y), \qquad \neg x \vee \neg y = \neg(x \wedge y). \tag{10}$$

Table 1. List of intuitionistic fuzzy negations

		Name	Form of negation
$\rightarrow_1$	$\neg_1$	Zadeh	$\langle b, a\rangle$
$\rightarrow_2$	$\neg_2$	Gaines-Rescher	$\langle 1-\text{sg}(a), \text{sg}(a)\rangle$
$\rightarrow_3$	$\neg_2$	Gödel	$\langle 1-\text{sg}(a), \text{sg}(a)\rangle$
$\rightarrow_4$	$\neg_1$	Kleene-Dienes	$\langle b, a\rangle$
$\rightarrow_5$	$\neg_1$	Lukasiewicz	$\langle b, a\rangle$
$\rightarrow_6$	$\neg_1$	Reichenbach	$\langle b, a\rangle$
$\rightarrow_7$	$\neg_1$	Willmott	$\langle b, a\rangle$
$\rightarrow_8$	$\neg_2$	Wu	$\langle 1-\text{sg}(a), \text{sg}(a).\text{sg}(1-b)\rangle$
$\rightarrow_9$	$\neg_3$	Klir and Yuan 1	$\langle b, a.b+a^2\rangle$
$\rightarrow_{10}$	$\neg_1$	Klir and Yuan 2	$\langle \text{sg}(1-a).b, \overline{sg}(1-a)+a.\text{sg}(1-a)\rangle$
$\rightarrow_{11}$	$\neg_2$	Atanassov 1	$\langle 1-\text{sg}(a), \text{sg}(a).\text{sg}(1-b)\rangle$
$\rightarrow_{12}$	$\neg_4$	Atanassov 2	$\langle b, 1-b\rangle$
$\rightarrow_{13}$	$\neg_1$	Atanassov and Kolev	$\langle b, a\rangle$
$\rightarrow_{14}$	$\neg_5$	Atanassov and Trifonov	$\langle 1-\text{sg}(a)-\overline{sg}(a).\text{sg}(1-b), \text{sg}(1-b)\rangle$
$\rightarrow_{15}$	$\neg_5$	Atanassov 3	$\langle 1-\text{sg}(\text{sg}(a)+\text{sg}(1-b)), 1-\overline{sg}(a).\overline{sg}(1-b)\rangle$
$\rightarrow_{16}$	$\neg_2$		$\langle 1-\text{sg}(a), \text{sg}(a)\rangle$
$\rightarrow_{17}$	$\neg_3$		$\langle b, a.b+a^2\rangle$
$\rightarrow_{18}$	$\neg_4$		$\langle b, 1-b\rangle$
$\rightarrow_{19}$	$\neg_5$		$\langle 1-\text{sg}(\text{sg}(a)+\text{sg}(1-b)), \text{sg}(1-b)\rangle$
$\rightarrow_{20}$	$\neg_2$		$\langle 1-\text{sg}(a), \text{sg}(a)\rangle$
$\rightarrow_{21}$	$\neg_3$		$\langle b, a.b+a^2\rangle$
$\rightarrow_{22}$	$\neg_4$		$\langle b, 1-b\rangle$
$\rightarrow_{23}$	$\neg_5$		$\langle 1-\text{sg}(\text{sg}(a)+\text{sg}(1-b)), \text{sg}(1-b)\rangle$

The above mentioned change of the LEM inspired the idea from [11] to study the validity of De Morgan's Laws that the classical negation $\neg$ (here it is negation $\neg_1$) satisfies. Really, easy it can be proved that the expressions

$$\neg_1(\neg_1 x \vee \neg_1 y) = x \wedge y, \qquad \neg_1(\neg_1 x \wedge \neg_1 y) = x \vee y \tag{11}$$

are IFTs, but the other negations do not satisfy these equalities. For them the following assertions are valid for every two propositional forms x and y:

$$\neg_i(\neg_i x \vee \neg_i y) = \neg_i\neg_i x \wedge \neg_i\neg_i y, \qquad \neg_i(\neg_i x \wedge \neg_i y) = \neg_i\neg_i x \vee \neg_i\neg_i y \tag{12}$$

for $i = 2, 4, 5$, while negation $\neg_3$ does not satisfy these equalities.

2 Main Results

Here we shall introduce a set of new negations. They are not connected with the previous ones. They will generalize the classical negation, but on the other hand, they will have some nonclassical properties. The set will have the form

$$\mathcal{N} = \{\neg^\varepsilon \mid 0 \le \varepsilon < 1\}. \tag{13}$$

Below we shall study some basic properties of an arbitrary element of $\mathcal{N}$.

Let everywhere below $0 \leq \varepsilon < 1$ be fixed. We define:

$$\neg^{\varepsilon}\langle a, b\rangle = \langle \min(1, b+\varepsilon), \max(0, a-\varepsilon)\rangle. \tag{14}$$

Obviously, $\neg_1 = \neg^0 \in \mathcal{N}$, i.e., $\mathcal{N}$ contains at least one element.

Figure 1 shows x and $\neg_1 x$, while Figs. 2 and 3 show y and $\neg^{\varepsilon} y$ and z and $\neg^{\varepsilon} z$, respectively.

We show that the couple $\langle \min(1, b+\varepsilon), \max(0, a-\varepsilon)\rangle$ is an intuitionistic fuzzy one. Indeed, if $a - \varepsilon \leq 0$, then

$$\min(1, b+\varepsilon) + \max(0, a-\varepsilon) = \min(1, b+\varepsilon) \leq 1. \tag{15}$$

If $a - \varepsilon > 0$, i.e., $a > \varepsilon$, then $b + \varepsilon < a + b \leq 1$ and

$$\min(1, b+\varepsilon) + \max(0, a-\varepsilon) = b + \varepsilon + a - \varepsilon = a + b \leq 1. \tag{16}$$

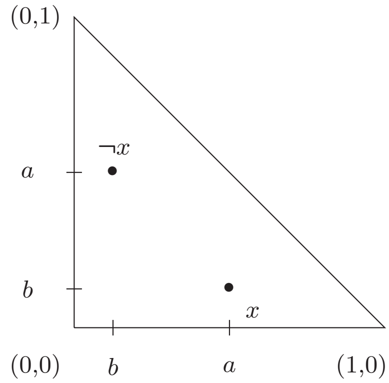

Fig. 1. Classical IFS Negation ($\neg_1$)

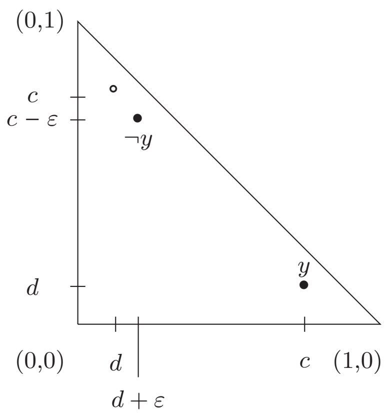

Fig. 2. ε-negation

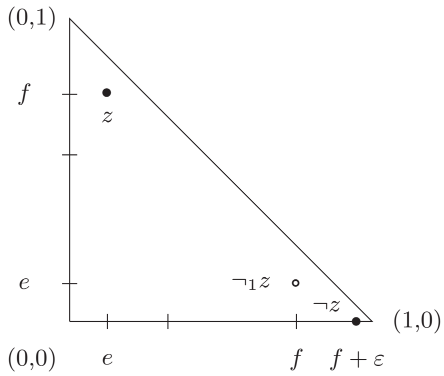

Fig. 3. ε-Negation, $e - \varepsilon < 1$

By analogy with above, we can construct two new implications, generated by the new negation. The first of them is based on $x \to y = \neg x \vee y$ or

$$V(x \to y) = \neg\langle a, b\rangle \vee \langle c, d\rangle \tag{17}$$

and has the form:

$$\begin{aligned}\langle a, b\rangle \to^{\varepsilon} \langle c, d\rangle &= \langle \max(c, \min(1, b + \varepsilon)), \min(d, \max(0, a - \varepsilon))\rangle \\ &= \langle \min(1, \max(c, b + \varepsilon)), \max(0, \min(d, a - \varepsilon))\rangle.\end{aligned} \tag{18}$$

Now, we see that

$$\langle a, b\rangle \to^{\varepsilon} \langle 0, 1\rangle = \langle \min(1, b + \varepsilon), \max(0, a - \varepsilon)\rangle, \tag{19}$$

i.e., the negation generated by implication $\to^{\varepsilon}$ coincides with negation $\neg^{\varepsilon}$.

The second implication that we can construct with negation $\neg^{\varepsilon}$ is based on $x \to y = \neg x \vee \neg\neg y$ or

$$V(x \to y) = \neg\langle a, b\rangle \vee \neg\neg\langle c, d\rangle \tag{20}$$

and has the form:

$$\begin{aligned}&\langle a, b\rangle \to^{\varepsilon} \langle c, d\rangle = \neg^{\varepsilon}\langle a, b\rangle \vee \neg^{\varepsilon}\neg^{\varepsilon}\langle c, d\rangle \\ &= \langle \min(1, b + \varepsilon), \max(0, a - \varepsilon)\rangle \vee \neg^{\varepsilon}\langle \min(1, d + \varepsilon), \max(0, c - \varepsilon)\rangle \\ &= \langle \min(1, b + \varepsilon), \max(0, a - \varepsilon)\rangle \vee \\ &\quad \langle \min(1, \max(0, c - \varepsilon) + \varepsilon), \max(0, \min(1, d + \varepsilon) - \varepsilon)\rangle \\ &= \langle \min(1, b + \varepsilon), \max(0, a - \varepsilon)\rangle \vee \langle \max(\varepsilon, c), \min(1 - \varepsilon, d))\rangle \\ &= \langle \max(\min(1, b + \varepsilon), \varepsilon, c)), \min(1 - \varepsilon, d, \max(0, a - \varepsilon)\rangle \\ &= \langle \min(\max(1, \varepsilon, c), \max(b + \varepsilon, \varepsilon, c)), \\ &\quad \max(\min(0, 1 - \varepsilon, d), \min(1 - \varepsilon, d, a - \varepsilon)\rangle \\ &= \langle \min(1, \max(b + \varepsilon, c)), \max(0, \min(a - \varepsilon, d)\rangle.\end{aligned} \tag{21}$$

Therefore, the two implications generated by negation $\neg^\varepsilon$ coincide.

By direct checking we see that the new negation is different than the other five and by this reason already we have six intuitionistic fuzzy negations. Also, we see that the new implication is different with the other 23 ones.

Now, we shall formulate similar assertions as in [8], but for the new negation.

Theorem 1. *Negation $\neg^\varepsilon$ satisfies Property 1 for its generated implication in as an IFT, but not as a tautology.*

Proof. Let x be a given propositional form.

$$\begin{aligned}\langle a,b\rangle \rightarrow^\varepsilon \neg^\varepsilon\neg^\varepsilon\langle a,b\rangle &= \langle a,b\rangle \rightarrow^\varepsilon \langle \max(\varepsilon,a),\min(1-\varepsilon,b)\rangle \\ &= \langle \min(1,\max(\varepsilon,a,b+\varepsilon)),\max(0,\min(1-\varepsilon,b,a-\varepsilon))\rangle \\ &= \langle \min(1,\max(a,b+\varepsilon)),\max(0,\min(b,a-\varepsilon))\rangle. \\ &= \langle \max(a,\min(1,b+\varepsilon)),\min(b,\max(1,a-\varepsilon))\rangle.\end{aligned} \tag{22}$$

Obviously, the latter expression cannot be a tautology. On the other hand

$$\max(a,\min(1,b+\varepsilon)) - \min(b,\max(1,a-\varepsilon)) \geq \min(1,b+\varepsilon) - b \geq 0, \tag{23}$$

i.e., Property 1 is an IFT. □

Theorem 2. *Negation $\neg^\varepsilon$ satisfies Property 2 for its generated implication as an IFT, but not as a tautology.*

Proof. Let x be a given propositional form.

$$\begin{aligned}&\neg^\varepsilon\neg^\varepsilon\langle a,b\rangle \rightarrow^\varepsilon \langle a,b\rangle = \langle \max(\varepsilon,a),\min(1-\varepsilon,b)\rangle \rightarrow^\varepsilon \langle a,b\rangle \\ &= \langle \min(1,\max(a,\min(1-\varepsilon,b)+\varepsilon),\max(0,\min(b,\max(\varepsilon,a)-\varepsilon))\rangle \\ &= \langle \min(1,\max(a,\min(1,b+\varepsilon))),\max(0,\min(b,\max(0,a-\varepsilon))\rangle \\ &= \langle \max(a,\min(1,b+\varepsilon)),\min(b,\max(0,a-\varepsilon))\rangle.\end{aligned} \tag{24}$$

Obviously, the latter expression cannot be a tautology. On the other hand

$$\max(a,\min(1,b+\varepsilon)) - \min(b,\max(0,a-\varepsilon)) \geq a - \max(0,a-\varepsilon) \geq 0, \tag{25}$$

i.e., Property 2 is an IFT. □

Therefore, we have constructed an example of a couple of a negation and an implication for which both properties P1 and P2 are IFTs for arbitrary propositional form x, but from this fact does not follow that x coincide with $\neg^\varepsilon\neg^\varepsilon x$. This is the third example for such a couple along with couples $(\neg_2, \rightarrow_{20})$ and $(\neg_5, \rightarrow_{23})$, described in [11].

Theorem 3. *Negation* $\neg^\varepsilon$ *satisfies Property 3.*

Proof. We shall use the above results:

$$\begin{aligned}\neg^\varepsilon\neg^\varepsilon\neg^\varepsilon\langle a,b\rangle &= \neg^\varepsilon\langle \max(\varepsilon,a),\min(1-\varepsilon,b)\rangle \\ &= \langle \min(1,\min(1-\varepsilon,b)+\varepsilon),\max(0,\max(\varepsilon,a)-\varepsilon)\rangle \\ &= \langle \min(1,\min(1,b+\varepsilon),\max(0,\max(0,a-\varepsilon)\rangle \\ &= \langle \min(1,b+\varepsilon),\max(0,a-\varepsilon)\rangle = \neg\langle a,b\rangle.\end{aligned} \tag{26}$$

Therefore Property 3 is valid. □

Now, we can classify each couple $(\neg,\rightarrow)$ as:

– Classical – it satisfies properties P1, P2, P3 and for each x: $V(x) = V(\neg\neg x)$;
– Intuitionistic – it satisfies properties P1, P3 and does not satisfy property P2;
– Nonstandard – it satisfies properties P1, P2, P3 and there is x: $V(x) \neq V(\neg\neg x)$.

Open problem 1 *Classify all different couples* $(\neg,\rightarrow)$ *to the three groups.*

Now we shall study the validity of the LEM and the De Morgan's Laws in the different forms, described above.

Theorem 4. *Negation* $\neg^\varepsilon$ *satisfies the LEM in its IFT form (7), but not in its tautological form (6).*

Theorem 5. *Negation* $\neg^\varepsilon$ *satisfies the Modified LEM in its IFT form (9), but not in its tautological form (8).*

Theorem 6. *Negation* $\neg^\varepsilon$*:*

(a) Does not satisfy the De Morgan's Laws in the form (10);
(b) Satisfies the De Morgan's Laws in the form (11);
(c) Satisfies the De Morgan's Laws in the form (12).

Finally, we shall study the relations between the different negations. By direct checks we can see the validity of the following Table 2.

The lack of relation between two implications is noted in Table 2 by " $*$ ".

The values from Table 3 are also interesting.

3 Conclusion: a New Argument that the Intuitionistic Fuzzy Sets Have Intuitionistic Nature

The above assertions show that all negations but the first one satisfy properties conditions of the intuitionistic logic not of the classical logic. A part

Table 2. List of the relations between the different intuitionistic fuzzy negations

	$\neg_1$	$\neg_2$	$\neg_3$	$\neg_4$	$\neg_5$	$\neg^{\varepsilon}$
$\neg_1$	$=$	$*$	$\leq$	$\geq$	$\geq$	$\leq$
$\neg_2$	$*$	$=$	$*$	$*$	$\geq$	$*$
$\neg_3$	$\geq$	$*$	$=$	$\geq$	$\geq$	$*$
$\neg_4$	$\leq$	$*$	$\leq$	$=$	$\geq$	$\leq$
$\neg_5$	$\leq$	$\leq$	$\leq$	$\leq$	$=$	$\leq$
$\neg^{\varepsilon}$	$\geq$	$*$	$*$	$\geq$	$\geq$	$=$

Table 3. List of the values of some special constants for the different intuitionistic fuzzy negations

$V(x)$	$\neg_1 V(x)$	$\neg_2 V(x)$	$\neg_3 V(x)$	$\neg_4 V(x)$	$\neg_5 V(x)$	$\neg^{\varepsilon} V(x)$
$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle \varepsilon,1-\varepsilon\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 1,0\rangle$	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle \varepsilon,0\rangle$

Table 4. List of the fuzzy negations, generated by intuitionistic fuzzy negations

Notation	Form of the intuitionistic fuzzy negation	Form of the fuzzy negation
$\neg_1$	$\langle b,a\rangle$	$1-a$
$\neg_2$	$\langle 1-\mathrm{sg}(a),\mathrm{sg}(a)\rangle$	$1-\mathrm{sg}(a)$
$\neg_3$	$\langle b,a.b+a^2\rangle$	$1-a$
$\neg_4$	$\langle b,1-b\rangle$	$1-a$
$\neg_5$	$\langle 1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)),\mathrm{sg}(1-b)\rangle$	$1-\mathrm{sg}(a)$
$\neg^{\varepsilon}$	$\langle \min(1,b+\varepsilon),\max(0,a-\varepsilon)\rangle$	$1-\max(0,a-\varepsilon)$

of these negations were generated by implications, that were generated by fuzzy implications. Now, let us return from the intuitionistic fuzzy negations to ordinary fuzzy negations. The result is shown on Table 4, where $b = 1 - a$.

Therefore, from the intuitionistic fuzzy negations we can generate fuzzy negations, so that two of them ($\neg_3$ and $\neg_4$) coincide with the standard fuzzy negation ($\neg_1$). Therefore, there are intuitionistic fuzzy negations that lose their properties when they are restricted to the ordinary fuzzy case. In other words, the construction of the intuitionistic fuzzy estimation

$$\begin{aligned}&\langle \text{degree of membership/validity},\\ &\ \text{degree of nonmembership/nonvalidity}\rangle\end{aligned} \tag{27}$$

that is specific for the intuitionistic fuzzy sets, is the reason for the intuitionistic behavior of these sets. Over them we can define intuitionistic as well as classical negations.

In the fuzzy case the negations $\neg_2$ and $\neg_5$ coincide, generating a fuzzy negation that satisfies Properties 1 and 3 and does not satisfy Property 2, i.e., it has intuitionistic character. As we see above, the new negation $\neg^{\varepsilon}$ has more strange behavior.

In [4] two other classes of negations are introduced. We will formulate the following interesting

Open problem 2 *What are the relation betweens these three sets of negations?*

References

1. Atanassov K (1988) Two variants of intuitonistc fuzzy propositional calculus. Preprint IM-MFAIS-5-88, Sofia
2. Atanassov K (1999) Intuitionistic Fuzzy Sets. Springer Physica, Heidelberg
3. Atanassov K (2001) Remarks on the conjunctions, disjunctions and implications of the intuitionistic fuzzy logic. Int. J. Uncertain. Fuzziness Knowl. Based Syst. 9(1), 55–65
4. Atanassov K (2004) On two sets of intuitionistic fuzzy negations. In: Issues in Intuitionistic Fuzzy Sets and Generalized Nets, Vol. 4(K. Atanassov, J. Kacprzyk and M. Krawczak, Eds.), Wydawnictwo WSISiZ, Warszawa, (in press)
5. Atanassov K (2005) Intuitionistic fuzzy implications and Modus Ponens. Notes on Intuitionistic Fuzzy Sets 11(1), 1–5
6. Atanassov K (2006) A new intuitionistic fuzzy implication from a modal type. Adv. Stud. Contemp. Math. 12(1), 117–122
7. Atanassov K (2006) On some intuitionistic fuzzy implications. Comptes Rendus de l'Academie bulgare des Sciences, Tome 59(1), 19–24
8. Atanassov K (2005) On some types of intuitionistic fuzzy negations. Notes Intuitionistic Fuzzy Sets 11(4), 170–172
9. Atanassov (2005) On some intuitionistic fuzzy negations. Notes Intuitionistic Fuzzy Sets 11(6), 13–20
10. Atanassov K (2006) On intuitionistic fuzzy negations and De Morgan's laws. In: Proceedings of IPMU'2006, Paris (in press)
11. Atanassov K (2006) On eight new intuitionistic fuzzy implications. In: Proceedings of IEEE Conf. Intelligent Systems'2006, London (in press)
12. Atanassov K, Kolev K. (2006) On an intuitionistic fuzzy implication from a possibilistic type. Adv. Stud. Contemp. Math. 12(1), 111–116
13. Atanassov K, Trifonov T. (2005) On a new intuitionistic fuzzy implication from Gödel's type. Proc. Jangjeon Math. Soci. 8(2), 147–152
14. Klir G, Bo Yuan (1995) Fuzzy Sets and Fuzzy Logic. Prentice Hall, New Jersey
15. Trifonov T, Atanassov (2006) On some intuitionistic properties of intuitionistic fuzzy implications and negations. In: the same proceedings