ADVANCES IN *SOFT COMPUTING*

Bernd Reusch
Editor

International Conference
9th Fuzzy Days in Dortmund, Germany,
September 18–20, 2006 Proceedings

Bernd Reusch (Ed.)

Computational Intelligence, Theory and Applications

Advances in Soft Computing

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Preface

For the 9th time since 1991 we invite researchers to participate in the Dortmund Fuzzy-Days. I am very glad that our conference has established itself as an international forum for the discussion of new results in the filed of Computational Intelligence. Again all papers had to undergo a thorough review: each one was judged by five referees to guarantee a solid quality of the programme.

From the beginning of the Fuzzy-Days on Lotfi A. Zadeh felt associated with the conference. I would like to express my gratitude for his encouragement and support and I am particularly glad that he once again delivers a keynote speech. Much to my pleasure Ewa Orlowska, Radko Mesiar together with Vilém Novák, Ernesto Damiani together with Tharam Dillon and Nik Kasabov have also agreed to present new results of their work as keynote speakers.

Many thanks go to my friends Janusz Kacprzyk and Enric Trillas who together with Lotfi Zadeh again served as honorary chairmen.

Due to my retirement in 2006, these are the last Dortmund Fuzzy Days in the form we had developed over the years. At this point I have to leave open, whether we find another forum or not.

I wish to thank all participants of the Dortmund Fuzzy-Days for their commitment to the conference and the organisers, namely Mrs Ulrike Lippe, for the excellent job they did. Last but not least, I am obliged to the German research council for their valuable financial support.

September 2006 *Bernd Reusch*

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From Search Engines to Question-Answering Systems: The Problems of World Knowledge, Relevance, Deduction, and Precisiation

Lotfi A. Zadeh*

Summary. Existing search engines, with Google at the top, have many truly remarkable capabilities. Furthermore, constant progress is being made in improving their performance. But what is not widely recognized is that there is a basic capability which existing search engines do not have: deduction capability – the capability to synthesize an answer to a query by drawing on bodies of information which reside in various parts of the knowledge base. By definition, a question-answering system, or a Q/A system for short, is a system which has deduction capability. Can a search engine be upgraded to a question-answering system through the use of existing tools – tools which are based on bivalent logic and probability theory? A view which is articulated in the following is that the answer is: no.

The first obstacle is world knowledge – the knowledge which humans acquire through experience, communication, and education. Simple examples are: "Icy roads are slippery," "Princeton usually means Princeton University," "Paris is the capital of France," and "There are no honest politicians." World knowledge plays a central role in search, assessment of relevance and deduction. The problem with world knowledge is that it is, for the most part, perception-based. Perceptions – and especially perceptions of probabilities – are intrinsically imprecise, reflecting the fact that human sensory organs, and ultimately the brain, have a bounded ability to resolve detail and store information. Imprecision of perceptions stands in the way of using conventional techniques – techniques which are based on bivalent logic and probability theory – to deal with perception-based information. A further complication is that much of world knowledge is negative knowledge in the sense that it relates to what is impossible and/or nonexistent. For example, "A person cannot have two fathers," and "Netherlands has no mountains."

The second obstacle centers on the concept of relevance. There is an extensive literature on relevance, and every search engine deals with relevance in its own way, some at a high level of sophistication. But what is quite obvious is that the problem of assessment of relevance is quite complex and far from solution.

There are two kinds of relevance (a) question relevance, and (b) topic relevance. Both are matters of degree. For example, on a very basic level, if the question is q: "Number of cars in California?" and the available information is p: "Population of California is 37,000,000," then what is the degree of relevance of p to q? Another example: To what degree is a paper entitled "A New Approach to Natural Language Understanding" of relevance to the topic of machine translation.

Basically, there are two ways of approaching assessment of relevance (a) semantic, and (b) statistical. To illustrate, in the number of cars example, relevance of p to q is a matter of semantics and world knowledge. In existing search engines, relevance is largely a matter of statistics, involving counts of links and words, with little if any consideration of semantics. Assessment of semantic relevance presents difficult problems whose solutions lie beyond the reach of bivalent logic and probability theory. What should be noted is that assessment of topic relevance is more amendable to the use of statistical techniques, which explains why existing search engines are much better at assessment of topic relevance than question relevance.

The third obstacle is deduction from perception-based information. As a basic example, assume that the question is q: What is the average height of Swedes?, and the available information is p: Most adult Swedes are tall. Another example is: Usually Robert returns from work at about 6 p.m. What is the probability that Robert is at home at 6:15 p.m.? Neither bivalent logic nor probability theory provide effective tools for dealing with problems of this type. The difficulty is centered on deduction from premises which are both uncertain and imprecise.

Underlying the problems of world knowledge, relevance, and deduction is a very basic problem – the problem of natural language understanding. Much of world knowledge and web knowledge is expressed in a natural language. A natural language is basically a system for describing perceptions. Since perceptions are intrinsically imprecise, so are natural languages.

A prerequisite to mechanization of question-answering is mechanization of natural language understanding, and a prerequisite to mechanization of natural language understanding is precisiation of meaning of concepts and proposition drawn from a natural language. To deal effectively with world knowledge, relevance, deduction, and precisiation, new tools are needed. The principal new tools are: precisiated natural language (PNL); protoform theory (PFT), and the generalized theory of uncertainty (GTU). These tools are drawn from fuzzy logic – a logic in which everything is, or is allowed to be, a matter of degree.

The centerpiece of the new tools is the concept of a generalized constraint. The importance of the concept of a generalized constraint derives from the fact that in PNL and GTU it serves as a basis for generalizing the universally accepted view that information is statistical in nature. More specifically, the point of departure in PNL and GTU is the fundamental premise that, in general, information is representable as a system of generalized constraints,

with statistical information constituting a special case. This, much more general, view of information is needed to deal effectively with world knowledge, relevance, deduction, precisiation, and related problems.

In summary, the principal objectives of this paper are (a) to make a case for the view that a quantum jump in search engine IQ cannot be achieved through the use of methods based on bivalent logic and probability theory; and (b) to introduce and outline a collection of nonstandard concepts, ideas, and tools which are needed to achieve a quantum jump in search engine IQ.

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A Fuzzy Approach to Optimal R&D Project Portfolio Selection

Christer Carlsson, Robert Fullér, and Péter Majlender

1 Introduction

A major advance in the development of strategic investment selection tools came with the application of options reasoning to the fields of Research and Development (R&D). By *real options* we understand the opportunity to invest in and thus support a project opportunity that essentially involves acquisition or building of real assets. In every step of the investment program, when making the appropriate entry (or exit) decisions, we also have to take into consideration that the underlying projects can open or close the possibility for further options (which might be more profitable). Defining phases and actively scheduling and managing investment activities, we can collect information to decide whether we are ready to go ahead with the investment or not.

Formulating from this point of view, we seek to correct the deficiencies of traditional investment valuation methods by incorporating the managerial flexibility that can (and usually does) bring significant value to projects. From our experience, we found that the main issue in the options approach to strategic project valuation is the correct characterization of the nonstatistical imprecision that we encounter when judging or estimating future cash flows. Working out schemes for phasing and scheduling systems of interrelated projects, we will develop a basic model for valuing options on R&D investment opportunities, when future revenues and expected costs are estimated by trapezoidal possibility distributions. Furthermore, drawing on our results, we shall present a fuzzy mixed integer programming model for the R&D optimal project portfolio selection problem.

The *real options valuation methods* were first tried and implemented as tools for working with very large industrial investments, also called as *giga-investments*. They presented a unique source of income for corporations through capturing significant market share from their rivals. However, those opportunities were often left abandoned due to the huge risks and uncertainties: there was fear that capital invested in very large projects, with an expected life time of more than a decade is not very productive and that their overall activity is not very profitable.[1]

[1] The Waeno project; Tekes 40470/00.

Investment opportunities of R&D types compete for major portions of the risk-taking capital, and as their outcome is particularly uncertain, compromises have to be made on their productivity. The short-term productivity may not be high, although the overall return of the investment program can be forecasted as very good. Another way of motivating an R&D investment is to point to strategic advantages, which would not be possible without the knowledge that the investment yields. Thus, R&D projects do offer some indirect (intangible) returns as well.

Our experience shows that R&D investments made in the paper and pulp industry face fierce competition and scenarios of slow growth (2–3% p.a.) in their key market segments. However, this environment does not prevent other more effective competitors to gain footholds in their main markets.

There are other issues. Global financial markets make sure that capital cannot be used nonproductively, as its owners are offered other opportunities, and the capital will move (often quite fast) to capture these opportunities. The capital market has learned "the American way," i.e., there is a shareholder dominance among the actors, which has often brought short-term shareholder return to the forefront as a key indicator of success, profitability, and productivity. There are also lessons learned from the Japanese industry, which point to the importance of *immaterial investments*. They show that investments in buildings, production, and supporting technologies become enhanced with immaterial investments, and that these are even more important for further investments and gradually growing maintenance investments.

The core products and services created by R&D investments are enhanced with life-time services, with gradually more advanced maintenance and financial add-in services. These features make it difficult to actually assess the productivity and profitability of the original R&D project, especially if the products and services are repositioned to serve other (e.g., emerging) markets. New technology and enhanced technological innovations have been changing the life cycle of R&D investments. The challenge is to find the right time and the right innovation to modify the life cycle in an optimal way. Technology providers are actively involved throughout the life cycle of R&D projects, which actually changes the way we assess the profitability and the productivity of such investments.

R&D projects, and in particular, portfolios of R&D projects generate commitments, which possess:

1. Long life cycles (taking into account their possible impacts on other investments)
2. Uncertain (i.e., *vague*), sometimes overly optimistic or pessimistic future cash flow estimates
3. Uncertain (i.e., *biased*), sometimes questionable profitability estimates
4. Imprecise assessments of future effects on productivity, market positions, competitive advantages, and shareholder value
5. The ability to generate series of further investments

Jensen and Warren [14] propose to use options theory to value R&D in the telecom service sector. The reasons are rather similar to those we identified above: research managers are under pressure to explain the value of R&D programs to the senior management, and at the same time they need to evaluate individual projects to make management decisions on their own R&D portfolio. The research in real options theory has evolved from general presentations of flexibility of investments in industrial cases to more theoretical contributions, which resulted in the application of real option valuation methods to industrial R&D projects. The term real option was introduced by Kester [15] and Myers [20] in 1984. The option to postpone an investment opportunity was discussed by McDonald and Siegel [22]. Pakes [23] considered patents as options. Siegel et al. [24] discussed the option valuation of offshore oil properties. Majd and Pindyck [21] analyzed the optimal time and computed the option value of building operations in investment decisions. A fundamental book on managerial flexibility and strategy in resource allocation, written by Trigeorgis [25], presented a theory of real options. Abel et al. [1] discussed a theory of option valuation of real capital and investments. Faulkner [13] discussed the application of real options to the valuation of R&D projects at Kodak. Kulatilaka et al. [16] discussed a capability-based real options approach to managing information technology investments.

The use of fuzzy sets to work with real options is a novel approach, which has not been considered and analyzed widely so far. One of the first results to apply fuzzy mathematics in finance was presented by Buckley [4], where he worked out how to use fuzzy sets to formulate the concepts of future value, present value, and internal rate of return. Carlsson and Fullér [5] also dealt with fuzzy internal rate of return in the context of investment decisions to control paper mills in the industry. Later, Carlsson and Fullér [6] developed a method for managing capital budgeting problems with fuzzy cash flows. However, there are a growing number of papers in the intersection of the disciplines of *real options* and *fuzzy sets*. In one of the first papers on developing the fuzzy Black–Scholes model, Carlsson and Fullér [7] presented a fuzzy real option valuation method. Muzzioli and Torricelli [19] used fuzzy sets to frame the binomial option pricing model. Carlsson and Fullér [8] analyzed the optimal timing of investment opportunities with fuzzy real options. Carlsson et al. [10,12] developed and tested a method for project selection with optimal timing and scheduling by using the methodology of fuzzy real options. Majlender [18] presented a comprehensive overview of the development of investment valuation methods in a possibilistic environment.

2 Real Options for R&D Portfolios

The options approach to R&D project valuation seeks to correct the deficiencies of traditional methods of valuation that are based on the methodologies of net present valuation (NPV) and discounted cash flow (DCF) analyses,

through the recognition of managerial flexibility and interaction with the underlying investment opportunities. This uncertainty can bring significant value to a project.

Real options in option thinking are based on the same principles as financial options. In real options, the options involve "real" (i.e., productive) assets as opposed to financial ones, where the options relate to some financial instruments [2]. To have a "real option" means to have the possibility for a certain period of time to either choose for or against something, without binding ourselves up front. The value of a real option is computed by [17]

$$\mathrm{ROV} = S_0 e^{-\delta T} N(d_1) - X e^{-r_f T} N(d_2),$$

where

$$d_1 = \frac{\ln(S_0/X) + (r_f - \delta + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln(S_0/X) + (r_f - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and where S_0 is the present value of expected cash flows, X is the nominal value of fixed costs, δ is the value lost over the duration of the option, r_f is the annualized continuously compounded rate on a safe asset, T is the time to maturity of the option in years, and σ stands for the uncertainty of the expected cash flows potentially involved in S_0, and finally $N(d)$ denotes the probability that a random draw from a standard normal distribution will be less than d.

The main question that a firm must answer for a deferrable investment opportunity is the following.

2.1 How Long Should We Postpone the Investment up to T Time Periods?

To answer this question, Benaroch and Kauffman [3] suggested the following decision rule for an optimal investment strategy.

Where the maximum deferral time is T, make the investment (i.e., exercise the real option) at time t^*, $0 \le t^* \le T$, for which the value of the option C_{t^*} is positive and attends its maximum value. That is,

$$C_{t^*} = \max_{t=0,1,\ldots,T} \{V_t e^{-\delta t} N(d_1) - X e^{-r_f t} N(d_2)\} > 0, \tag{1}$$

where

$$V_t = \mathrm{PV}(\mathrm{cf}_0, \mathrm{cf}_1, \ldots, \mathrm{cf}_T; r) - \mathrm{PV}(\mathrm{cf}_0, \mathrm{cf}_1, \ldots, \mathrm{cf}_{t-1}; r) = \mathrm{PV}(\mathrm{cf}_t, \ldots, \mathrm{cf}_T; r)$$
$$= \sum_{j=0}^{T} \frac{\mathrm{cf}_j}{(1+r)^j} - \sum_{j=0}^{t-1} \frac{\mathrm{cf}_j}{(1+r)^j} = \sum_{j=t}^{T} \frac{\mathrm{cf}_j}{(1+r)^j},$$

and where cf_t denotes the expected cash flows at time t, $t = 0, 1, \ldots, T$, and r is the project-specific risk-adjusted discount rate.

Of course, this decision rule has to be reapplied each time when new information arrives during the deferral period to see how the optimal investment strategy changes in light of the new information. From a real option perspective, it can be worthwhile to undertake R&D investments with a negative net present value (NPV), when early investment can provide information about future benefits or losses of the whole investment program.

3 A Hybrid Approach to Real Option Valuation

A fuzzy set $\tilde{A}$ on the real line $\mathbb{R}$ is called a trapezoidal fuzzy number with core $[a, b]$, left width $\alpha \geq 0$ and right width $\beta \geq 0$, if its membership function is of the following form:

$$\tilde{A}(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha < t < a, \\ 1 & \text{if } a \leq t \leq b, \\ 1 - \dfrac{t-b}{\beta} & \text{if } b < t < b + \beta, \\ 0 & \text{otherwise,} \end{cases}$$

and we use the notation $\tilde{A} = (a, b, \alpha, \beta)$.

Usually, the present value of the expected cash flows cannot be characterized by a single number. However, they can be estimated by a trapezoidal possibility distribution of the form

$$\tilde{S}_0 = (a, b, \alpha, \beta).$$

That is, the most possible values of the present value of the expected cash flows lie in the interval $[a, b]$ (which is the core of the trapezoidal fuzzy number $\tilde{S}_0$), and $(b+\beta)$ is the upward potential and $(a-\alpha)$ is the downward potential for the present value of the expected cash flows. In a similar manner, we can estimate the nominal value of the expected costs by using a trapezoidal possibility distribution of the form

$$\tilde{X} = (a', b', \alpha', \beta').$$

That is, the most possible values of the expected costs lie in the interval $[a', b']$ (which is the core of the trapezoidal fuzzy number $\tilde{X}$), and $(b' + \beta')$ is the upward potential and $(a' - \alpha')$ is the downward potential for expected costs.

In 2003, Carlsson and Fullér [11] suggested the use of the following fuzzy-probabilistic formula for computing fuzzy real option values

$$\tilde{C}_0 = \tilde{S}_0 e^{-\delta T} N(d_1) - \tilde{X} e^{-r_f T} N(d_2), \tag{2}$$

where

$$d_1 = \frac{\ln(E(\tilde{S}_0)/E(\tilde{X})) + (r_f - \delta + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln(E(\tilde{S}_0)/E(\tilde{X})) + (r_f - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and where $E(\tilde{S}_0)$ denotes the possibilistic mean value of the present value of expected cash flows, $E(\tilde{X})$ stands for the possibilistic mean value of the expected costs and $\sigma = \sigma(\tilde{S}_0)$ is the possibilistic variance of the present value of the expected cash flows [9]. Based on (2), Carlsson and Fullér [11] derived a similar formula to (1) for the optimal investment strategy in a possibilistic setting.

4 A Possibilistic Approach to R&D Portfolio Selection

Facing a set of project opportunities of R&D type, the company is usually able to estimate the expected investment costs of the projects with a high degree of certainty. Thus, in the following we will assume that $\tilde{X} = X \in \mathbb{R}$ is a crisp number. However, the cash flows received from the projects do involve uncertainty, and they are modeled by trapezoidal possibility distributions. Let us fix a particular project of length L and maximal deferral time T with cash flows

$$\tilde{\text{cf}}_i = (A_i, B_i, \Phi_i, \Psi_i), \quad i = 0, 1, \ldots, L.$$

Now, instead of the *absolute* values of the cash flows, we shall consider their fuzzy returns on investment (FROI) by computing the return that we receive on investment X in year i of the project as

$$\text{FROI}_i = \tilde{R}_i = \frac{\tilde{\text{cf}}_i}{X} = \left(\frac{A_i}{X}, \frac{B_i}{X}, \frac{\Phi_i}{X}, \frac{\Psi_i}{X}\right) = (a_i, b_i, \alpha_i, \beta_i).$$

For example, let $\tilde{\text{cf}}_i = (0.9, 8.4, 3.9, 5.6)$ and $X = 6$. Then

$$\tilde{R}_i = (15\%, 140\%, 65\%, 93\%)$$

with possibilistic mean value

$$E(\tilde{R}_i) = \frac{a_i + b_i}{2} + \frac{\beta_i - \alpha_i}{6} = \frac{15 + 140}{2} + \frac{93 - 65}{6} = 82.17\%,$$

and (possibilistic) standard deviation

$$\sigma(\tilde{R}_i) = \sqrt{\left(\frac{b_i - a_i}{2} + \frac{\alpha_i + \beta_i}{6}\right)^2 + \frac{(\alpha_i + \beta_i)^2}{72}}$$
$$= \sqrt{\left(\frac{140 - 15}{2} + \frac{65 + 93}{6}\right)^2 + \frac{(65 + 93)^2}{72}} = 90.76\%.$$

We compute the fuzzy net present value of the project by

$$\text{FNPV} = \left[\sum_{i=0}^{T} \frac{\tilde{R}_i}{(1+r)^i} - 1\right] \times X.$$

If a project with rates of return on investment

$$\{\tilde{R}_0, \tilde{R}_1, \ldots, \tilde{R}_L\}$$

can be postponed by a *maximum* of T years, then we will define the value of its possibilistic deferral flexibility by

$$\mathcal{F} = (1+\sigma(\tilde{R}_0)) \times (1+\sigma(\tilde{R}_1)) \times \cdots \times (1+\sigma(\tilde{R}_{T-1})) \times \text{FNPV},$$

where $1 \leq T \leq L$. If a project cannot be postponed then its possibilistic flexibility equals to its fuzzy net present value. That is, if $T = 0$ then $\mathcal{F} =$ FNPV.

The basic optimal R&D project portfolio selection problem can be formulated as the following fuzzy mixed integer programming problem

$$\begin{aligned} \text{maximize } & \mathcal{F} = \sum_{i=1}^{N} u_i \mathcal{F}_i \\ \text{subject to } & \sum_{i=1}^{N} u_i X_i + \sum_{i=1}^{N} (1-u_i) c_i \leq B \\ & u_i \in \{0,1\},\ i = 1, \ldots N, \end{aligned} \tag{3}$$

where N is the number of R&D projects; B is the whole investment budget; u_i is the decision variable associated with project i, which takes value one if the project i starts now (i.e., at time zero) and takes value zero if it is postponed and going to start at a later time; c_i denotes the cost of the postponement of project i (i.e., the capital expenditure required to keep the associated real option alive); X_i and $\mathcal{F}_i$ stand for the investment cost and the possibilistic deferral flexibility of project i, respectively, $i = 1, \ldots, N$.

In our approach to fuzzy mathematical programming problem (3), we have used the following defuzzifier operator for $\mathcal{F}$

$$\nu(\mathcal{F}) = \big(E(\mathcal{F}) - \tau \times \sigma(\mathcal{F})\big) \times X,$$

where $0 \leq \tau \leq 1$ denotes the decision maker's risk aversion.

Since R&D projects are characterized by the long planning horizon and very large uncertainty, the value of managerial flexibility can be substantial. Therefore, the fuzzy real options model is quite practical and useful. The standard work in the field use probability theory to account for the uncertainties involved in future cash flow estimates. This may be defended for financial options, for which we can assume the existence of an efficient market with

numerous players and numerous stocks for trading, which in turn justifies the assumption of the validity of the laws of large numbers and thus the use of statistical methods. The situation for real options associated with an investment opportunity of R&D type is quite different. The option to postpone an R&D project does have consequences, which differs from efficient markets, as the number of players producing the consequences is quite small. The imprecision we encounter when judging or estimating future cash flows is nonstochastic by nature, and the use of probability theory can give us a misleading level of precision and a notion that the consequences are somehow repetitive. This is not the case, since in our case the uncertainty is genuine, i.e., we simply do not know the exact level of future cash flows. Without introducing fuzzy real option models, it would not be possible to formulate this genuine uncertainty.

The proposed model that incorporates subjective judgments as well as statistical uncertainties can give investors a better understanding of the problem when making R&D investment decisions.

5 Summary

Multinational enterprises with large R&D departments often face the difficulty of selecting an appropriate portfolio of research projects. The cost of developing a new product or technology is low as compared to the cost of its introduction to the global market. The NPV rule and other discounted cash flow techniques for making R&D investment decisions seem to be inappropriate for selecting a portfolio of R&D projects, as they favor short-term projects in relatively certain markets over long-term and relatively uncertain markets. Since many new products are identified as failures during the R&D stages, the possibility of refraining from market introduction can add a significant value to the NPV of the R&D project. Therefore R&D investments can be interpreted as the price of an option on major follow-on investments.

In our *OptionsPort* project,[2] we represented the optimal R&D portfolio selection problem by a fuzzy 0–1 mathematical programming problem, where the optimal solution(s) defined the optimal portfolio(s) of R&D projects with the biggest (aggregate) possibilistic flexibility value.

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Choquet Integration and Correlation Matrices in Fuzzy Inference Systems

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A fuzzy rule inference system (FIS) of the type Takagi–Sugeno–Kang (TSK) [7, 8], is described by rules such as

$$R_i : \text{IF} x_1 \, is \, A_{i1} \text{ AND} \cdots \text{ AND} x_m \, is \, A_{im} \text{ THEN } y_i = c_{i0} + c_{i1}x_1 + \cdots + c_{im}x_m,$$

where $i = 1, ..., n$. The set of rules $\{R_1, R_2, \ldots, R_n\}$ forms the rule base. The consequent of each rule can be interpreted as a fuzzy singleton whose value is dependent on the system's inputs. Each triggered rule provides an activation level α, which is given by

$$\alpha_i = \otimes(\mu_{A_{i1}}, \ldots, \mu_{A_{im}}),$$

where $i = 1, ..., n$, $\otimes(\cdot)$ represents a t-norm and $\{\mu_{A_{ij}} | j = 1, ..., m\}$ represents the set of memberships for each rule's antecedent variables. The output for the TSK system is then obtained using an aggregation operator based on the standard weighted averaging

$$y(\underline{x}) = \frac{\sum_{i=1}^{n} \alpha_i y_i}{\sum_{j=1}^{n} \alpha_j} = \sum_{i=1}^{n} w_i y_i \quad \text{where} \quad w_i = \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j} \quad \text{and} \quad \sum_{i=1}^{n} w_i = 1,$$

where we consider the individual weights of the fuzzy rules, $w_i > 0$, to be the normalized activation levels α_i for $i = 1, \ldots, n$.

In this paper we propose an extension to the TSK fuzzy inference system based on Choquet interation [5, 6], called Choquet–TSK. In our model, a matrix of pairwise correlations among the activation levels of the FIS rules is explicitly used in the aggregation process, leading to attenuation effects when correlations are positive and emphasizing effects when correlations are negative.

Consider a finite set of interacting criteria $N = \{1, 2, \ldots, n\}$.

A *Choquet measure* [1] on the set N is a set function $\mu : \mathcal{P}(N) \longrightarrow [0, 1]$ satisfying

$$(1) \quad \mu(\emptyset) = 0, \ \mu(N) = 1, \quad (2) \quad S \subseteq T \subseteq N \Rightarrow \mu(S) \leq \mu(T). \qquad (1)$$

Given a Choquet measure μ we can define the *Choquet integral* [1–3] of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ with respect to μ as

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)}, \tag{2}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$. Also $A_{(i)} = \{(i), \ldots, (n)\}$ and $A_{(n+1)} = \emptyset$.

Notice that the Choquet integral with respect to an additive measure μ reduces to a weighted averaging operator, whose weights w_i are given by the $\mu(i)$ values,

$$\mu(A_{(i)}) = \mu((i)) + \mu((i+1)) + \cdots + \mu((n)),$$

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} = \sum_{i=1}^{n} \mu((i)) x_{(i)} = \sum_{i=1}^{n} w_i x_i\,. \tag{3}$$

Consider the pairwise correlation matrix $\mathbf{C} = [c_{ij}]$ among the various fuzzy rules

$$c_{ij} \in [-1,1], \quad c_{ji} = c_{ij}, \quad i,j = 1, \ldots, n, \tag{4}$$

where for convenience reasons we take a null diagonal $c_{ii} = 0$ for $i = 1, \ldots, n$.

We consider $w_i > 0$ for $i = 1, \ldots, n$ individual weights of the fuzzy rules, normalized so that $\Sigma_{i=1}^{n} w_i = 1$.

Given a general pairwise correlation matrix $\mathbf{C} = [c_{ij}]$, we define a two-additive Choquet measure $\mu : 2^N \longrightarrow [0,1]$ in the following way: making use of the Möbius transform m of the measure μ, we define $m(i) = w_i/\mathcal{N}$ for each singlet $\{i\}$ and $m(ij) = -w_i c_{ij} w_j/\mathcal{N}$ for each doublet $\{i,j\}$, with null higher order terms. Then, we define the value of the two-additive measure μ on a coalition S as the sum of the singlets and doublets contained in the coalition S, as given by the Möbius transform m

$$\mu(S) = \sum_{\{i\} \subseteq S} w_i/\mathcal{N} + \sum_{\{i,j\} \subseteq S} (-w_i c_{ij} w_j)/\mathcal{N}, \tag{5}$$

where the normalization factor $\mathcal{N}$ is the sum of all singlets and doublets in the set N

$$\begin{aligned} \mathcal{N} &= \sum_{\{i\} \subseteq N} w_i + \sum_{\{i,j\} \subseteq N} (-w_i c_{ij} w_j) = 1 - \frac{1}{2} \sum_{i,j=1}^{n} w_i c_{ij} w_j \\ &= 1 - \frac{1}{2} \sum_{i=1}^{n} w_i c_i = 1 - c/2, \end{aligned} \tag{6}$$

where $c_i = \sum_{j=1}^{n} c_{ij} w_j$ and $c = \sum_{i=1}^{n} w_i c_i$ denote weighted averages of pairwise correlation values. In particular, we have

$$\mu(i) = w_i/\mathcal{N}, \quad i, j = 1, \ldots, n,$$

$$\mu(ij) = (w_i + w_j - w_i c_{ij} w_j)/\mathcal{N}. \tag{7}$$

The measure μ satisfies the boundary conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$, and is monotonic.

The graph interpretation of this definition, in which singlets correspond to nodes and doublets correspond to edges between nodes, is that the value of the two-additive measure μ on a coalition S is the sum of the nodes and edges contained in the subgraph associated with the coalition S.

Notice that the proposed Choquet–TSK integration model is an extension of the standard weighted averaging of the TSK FIS. If the matrix $\mathbf{C}$ is null (null pairwise correlations among the fuzzy rules, i.e., rules are really independent of each other) then the Choquet measure μ is additive and the Choquet integral coincides with the weighted arithmetic mean whose weights are w_i as in the standard TSK FIS.

The following example was adapted from the *Dinner for Two* example of MATLAB (www.matworks.com.). In this example we have two input variables and one output variable. The input variables *service* and *food* and the output variable *tip* are described by the following linguistic terms: Service = {poor, good, excellent}; Food= {rancid, delicious}; Tip= {cheap, average, generous}. The rules for this system are as follows:

R_1 : IF the *service* is poor OR the *food* is rancid THEN the *tip* is cheap.

R_2 : IF the *service* is good THEN the *tip* is average.

R_3 : IF the *service* is excellent OR the *food* is delicious THEN the *tip* is generous.

The results obtained for 50 randomly generated inputs for both TSK FIS and Choquet–TSK FIS gave a mean relative deviation of 2% between the values for both systems.

To observe how the results of the Choquet–TSK FIS can be affected by the correlation matrix values, some extreme values for this matrix where tested. For a $\mathbf{C}$ matrix with minus ones outside its main diagonal, the mean relative deviation was 6%, and for a $\mathbf{C}$ matrix with ones outside its main diagonal, the mean relative deviation was 9%. It should also be noted that, using a null $\mathbf{C}$ matrix returned, as expected, the same results as the usual TSK FIS.

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Linguistic Summarization of Some Static and Dynamic Features of Consensus Reaching

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Summary. Consensus reaching has been widely recognized as an important component of the decision-making process. In previous works Fedrizzi and Kacprzyk introduced a new concept of consensus referring to the idea of fuzzy majority and based on the Zadeh's calculus of linguistically quantified propositions. Basically, the (degree of) consensus was meant as the degree to which Q_1 (e.g., most) of the I (e.g., important) individuals agree as to Q_2 (e.g., almost all) of B (e.g., relevant) options. The approach was extended in further works by Fedrizzi, Kacprzyk, Nurmi, and Zadrożny. Recently Kacprzyk and Zadrożny proposed to apply linguistic summaries in the sense of Yager to support the consensus reaching process. For instance, "most individuals definitely preferring option o_1 over option o_2 also definitely prefer option o_5 to option o_7," "almost all options dominating option o_3 in the opinion of expert e_2 also dominate option o_6 in the opinion of expert e_4," etc. In the present paper, we extend this idea and propose to take into account dynamic features of the consensus reaching process while constructing the linguistic summaries. Basically, linguistic summaries are meant as a concise description of the current status in the group of individuals in terms of their preferences. These descriptions may concern particular individuals, the whole group, or particular options. Moreover we propose here to take into account also how the preferences are evolving over time. For instance, "individual e_2 is very flexible with respect to his or her preferences between options o_3 and o_5," etc. Such an information might be even more useful for the running, moderating, etc. of a consensus reaching process than a static description.

Key words: Consensus reaching, Fuzzy majority, Fuzzy preferences, Fuzzy quantifiers, Linguistic data summary, Linguistic summarization of times series.

1 Introduction

This paper deals with an aspect of consensus reaching processes in a fuzzy environment, i.e., under fuzzy preferences and a fuzzy majority. We assume that there is a set of individuals (experts, decision-makers, ...) and a set of options (alternatives, variants, decisions, issues, ...). The individuals provide their testimonies concerning alternatives in question which are assumed to be

fuzzy preference relations. Normally, the individuals initially disagree in their testimonies, i.e., they are far from "consensus." Then, assuming that the individuals are seriously committed to reaching consensus, they are expected to update step-by-step their testimonies via an exchange of information, rational argument (e.g., by a moderator), etc., and hopefully to finally attain a "consensus."

Traditionally, consensus is meant as a full and unanimous agreement, i.e., that the testimonies of all the individuals should be the same at consensus. Unfortunately, this is utopian in practice. This has implied a need of some reconsideration of the very essence of "consensus" exemplified by a citation from Lower and Laddaga [31]: "... It can correctly be said that there is a consensus among biologists that Darwinian natural selection is an important cause of evolution though there is currently no consensus concerning Gould's hypothesis of speciation. This means that there is a widespread agreement among biologists concerning the first matter but disagreement concerning the second ..." (cf. also [30]). The above given quotation suggests that a *fuzzy majority* is appropriate, and that it makes sense to speak about a consensus to a degree.

A "soft" degree of consensus meant as the degree to which, say, "*most* of the *relevant* individuals agree as to *almost all* of the *relevant* issues (aspects, etc.) was proposed by Kacprzyk [9], Kacprzyk and Fedrizzi [10–15], and Fedrizzi and Kacprzyk [2,3] (see also Kacprzyk et al. [16–18]), and then by Kacprzyk and Zadrożny [22,24,25,27]. Fuzzy logic with linguistic quantifiers (cf. [35]) was employed.

Consensus is normally reached via a consensus reaching process run by a moderator (cf. [5,29]). Thus some tools may be helpful, notably linguistic summaries as proposed by Kacprzyk and Zadrożny (c.f., e.g., [28]).

In this paper we will add some other analytic tools, notably some linguistic assessment of how opinions of the individuals evolve over time.

In Sect. 2 we will show how to derive soft degrees of consensus under fuzzy preferences and a fuzzy majority, in Sect. 3 the idea and basic aspects of linguistic data summaries are presented, and some methods of their derivation are presented; moreover, it is shown how they can be used to support the running of a consensus reaching process. In Sect. 4 dynamic aspects of consensus reaching process are dealt with.

2 Degrees of Consensus under Fuzzy Preferences and a Fuzzy Majority

We operate in the following basic setting. We have a set of $n \geq 2$ options (alternatives, variants, issues, ...), $O = \{o_1, \ldots, o_n\}$, and a set of $m \geq 2$ individuals, $E = \{e_1, \ldots, e_m\}$. Moreover B is a fuzzy set of *relevant* options and I is a fuzzy set of *important* individuals. Each individual $e_k \in E$ provides his or her testimony as to the options in O, assumed to be an individual fuzzy preference relation in $O \times O$.

An *individual fuzzy preference relation* of individual e_k, $R^k = [r_{ij}^k]$, is given as

$$\mu_{R^k}(o_i, o_j) = \begin{cases} 1 & \text{if } o_i \text{ is definitely preferred to } o_j, \\ c \in (0.5, 1) & \text{if } o_i \text{ is slightly preferred to } o_j, \\ 0.5 & \text{in the case of indifference,} \\ d \in (0, 0.5) & \text{if } o_j \text{ is slightly preferred to } o_i, \\ 0 & \text{if } o_j \text{ is definitely preferred to } o_i. \end{cases} \tag{1}$$

The "soft" degree of consensus is derived in three steps:

1. for each pair of individuals we derive a degree of agreement as to their preferences between *all* the pairs of options,
2. we aggregate these degrees to obtain a degree of agreement of each pair of individuals as to their preferences between Q_1 (a linguistic quantifier as, e.g., "most," "almost all," "much more than 50%," ...) pairs of *relevant* options B, and
3. we aggregate these degrees to obtain a degree of agreement of Q_2 (a linguistic quantifier similar to Q_1) pairs of *important* individuals I as to their preferences between Q_1 pairs of *relevant* options B, and this is meant to be the *degree of consensus* sought.

The point of departure is clearly a degree to which each pair of individuals agrees as to the preference between a particular pair of options. In the most basic case, we may define the degree of strict agreement between individuals e_k and e_l as to their preferences between options o_i and o_j

$$v_{ij}(k, l) = \begin{cases} 1 \text{ if } r_{ij}^k = r_{ij}^l, \\ 0 \text{ otherwise,} \end{cases} \tag{2}$$

where here and later on in this section, if not otherwise specified, $k = 1, \ldots, m-1$; $l = k+1, \ldots, m$; $i = 1, \ldots, n-1$; $j = i+1, \ldots, n$.

The relevance of a pair of options, $(o_i, o_j) \in O \times O$, may be defined, say, as $b_{ij}^B = \frac{1}{2}[\mu_B(o_i) + \mu_B(o_j)]$, which is clearly the most straightforward choice; evidently, $b_{ij}^B = b_{ji}^B$, and b_{ii}^B do not matter; for each i, j.

The importance, $b_{k,l}^I$, of a pair of individuals, (e_k, e_l), may be defined as $b_{k,l}^I = \frac{1}{2}[\mu_I(e_k) + \mu_I(e_l)]$.

The degree of agreement between individuals e_k and e_l as to their preferences between *all* the pairs of *relevant* options is

$$v_B(k, l) = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [v_{ij}(k, l) \wedge b_{ij}^B]}{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij}^B}. \tag{3}$$

The degree of agreement between individuals e_k and e_l as to their preferences between Q_1 *relevant* pairs of options is

$$v_{Q_1}^B(k, l) = \mu_{Q_1}[v_B(k, l)]. \tag{4}$$

In turn, the degree of agreement of *all* the pairs of *important* individuals as to their preferences between Q_1 pairs of *relevant* options is

$$v_{Q_1}^{I,B} = \frac{2}{m(m-1)} \frac{\sum_{k=1}^{m-1} \sum_{l=k+1}^{m} [v_{Q_1}^{B}(k,l) \wedge b_{k,l}^{I}]}{\sum_{k=1}^{m-1} \sum_{l=k+1}^{m} b_{k,l}^{I}}. \tag{5}$$

Finally, the degree of agreement of Q_2 pairs of *important* individuals as to their preferences between Q_1 pairs of *relevant* options, called the *degree of* $Q1/Q2/I/B$*-consensus*, is

$$con(Q_1, Q_2, I, B) = \mu_{Q_2}(v_{Q_1}^{I,B}). \tag{6}$$

For some extensions, see, e.g. [6–8].

It is worth noticing that the required consensus does not have to concern directly the preference relations expressed by the individuals. Usually the ultimate goal of the decision-making session is the *choice* of the *best* option(s). Obviously an agreement between the individuals on the level of their preference relations makes this choice easier [32]. However if the individuals do not agree with respect to *all* or *most* pairs of options they still may be fairly in agreement as to which option is the best. Thus it may be advantageous to consider the agreement also on the level of the *choice sets* of options, i.e., the sets of options that should be selected as the best taking into account each individual preference relation separately. Such a multilevel measuring of the consensus was proposed by Kacprzyk and Zadrożny in [23, 24, 26, 27], see also [22] for a discussion of *linguistic choice rules*.

3 A Consensus Reaching Process and Linguistic Data Summarization

We assume the following setting of the consensus reaching process (cf. [4,5,29, 36]). We have a set of individuals and a distinguished person, a *moderator* who is responsible for running the consensus reaching session. The individual fuzzy preference relations may initially differ to a large extent, i.e., the group may be far from consensus. A moderator stimulates an exchange of information, rational argument, discussion, creative thinking, clarification of positions, etc. If the individuals are rationally committed to consensus, a change of testimonies usually occurs, and they get closer to consensus. It is assumed that some individuals, even if they are still convinced they are right with their original preferences, they can accept a consensual preferences established by the group provided their arguments has been heard and discussed. Thus, their acceptance of consensus may be effectively treated as a change of their preferences. This is repeated until the group gets sufficiently close to consensus.

Among some means for supporting consensus reaching, linguistic summaries of what happens to the preferences, relations between options, etc. may be useful.

A linguistic summary is meant as a natural language like sentence that subsumes the very essence (from a certain point of view) of a set of data. This set is assumed to be numeric and is usually large, not comprehensible in its original form by the human being. The original Yager's approach to the linguistic summaries (cf. Yager [33], Kacprzyk and Yager [20], Kacprzyk et al. [21], and Kacprzyk and Zadrożny [28]) may be expressed as follows:

- $Y = \{y_1, \ldots, y_n\}$ is a set of objects
- $A = \{A_1, \ldots, A_m\}$ is a set of attributes characterizing objects from Y, $A_j(y_i)$ denotes a value of attribute A_j for object y_i

A linguistic summary of set Y consists of:

- A summarizer S, i.e., an attribute together with a linguistic term (label) defined on the domain of attribute A_j
- A quantity in agreement Q, i.e., a linguistic quantifier (e.g., most)
- Truth (validity) T of the summary, i.e., a number from the interval $[0, 1]$ assessing the truth (validity) of the summary (e.g., 0.7); usually, only summaries with a high value of T are interesting
- Optionally, a qualifier P, i.e., another attribute together with a linguistic term (label) defined on the domain of attribute A_k determining a (fuzzy) subset of Y

Note that for brevity we will often identify summarizers and qualifiers with the linguistic terms they contain.

Basically, the core of a linguistic summary is a *linguistically quantified proposition* in the sense of Zadeh [35]. A linguistically quantified proposition, of type I may be written as:

$$Qy\text{s are } S \tag{7}$$

and the one of type II may be written as

$$QPy\text{s are } S. \tag{8}$$

Then, the component of a linguistic summary, T, i.e., its truth (validity), directly corresponds to the truth value of (7) or (8). This may be calculated by using either original Zadeh's calculus of linguistically quantified statements (cf. [35]), or other interpretations of linguistic quantifiers, including Yager's OWA operators [34].

Using Zadeh's [35] fuzzy logic-based calculus of linguistically quantified propositions, a (proportional, nondecreasing) linguistic quantifier Q is assumed to be a fuzzy set in the interval $[0, 1]$ as, e.g.,

$$\mu_Q(x) = \begin{cases} 1 & \text{for } x \leq 0.8, \\ 2x - 0.6 & \text{for } 0.3 < x < 0.8, \\ 0 & \text{for } x \geq 0.3. \end{cases} \tag{9}$$

Then, the truth values (from $[0,1]$) of (7) and (8) are calculated, respectively, as

$$\text{truth}(Qy\text{s are } S) = \mu_Q\left(\frac{1}{n}\sum_{i=1}^{n}\mu_S(y_i)\right), \tag{10}$$

$$\text{truth}(QPy\text{s are } S) = \mu_Q\left(\frac{\sum_{i=1}^{n}(\mu_P(y_i)\wedge\mu_S(y_i))}{\sum_{i=1}^{n}\mu_P(y_i)}\right), \tag{11}$$

where "$\wedge$" is the minimum operation, i.e., $a \wedge b = \min(a,b)$, which can be replaced by, e.g., a t-norm.

Linguistic summaries may, in a convenient way, describe the current state of agreement in the group and serve as guiding indicators for a further discussion, if needed. They may point out how far the group is from consensus, what are main obstacles in reaching consensus, which preference matrix may be a candidate for a consensual one, etc. As shown by Kacprzyk and Zadrożny [27] the very definition of consensus (6) may be interpreted as a kind of a linguistic summary.

In [27] other types of linguistic summaries have also been proposed. The summarizers S and qualifiers P refer to features of either individuals or options and linguistic terms expressing degrees of preferences, importance of individuals, and relevance of options.

First, the summarized objects Y may be identified with the individuals E, and their attributes A are preference degrees for particular pairs of options as well as their importance degrees. Then, the summaries of the following type may be useful and helpful for running a consensus reaching session:

Most individuals *definitely* prefer o_1 to o_2, *moderately* prefer o_3 to $o_4, \ldots$

Most individuals *definitely* preferring o_1 to o_3 also *definitely* prefer o_2 to o_4.

The summaries concerning the *choice sets* of particular individuals may be exemplified by:

Most individuals choose options $o_1, o_3, \ldots$

Most individuals reject options $o_1, o_4, \ldots$

The summaries of the latter type may help exclude some options from a further consideration and thus better focus the discussion.

Second, the summarized objects Y may be identified with the options O, and their attributes A are preference degrees over other options as expressed by particular individuals as well as their relevance degrees. Then, for instance, the following summaries may be useful:

Most options are dominated by option o_2 in opinion of individual e_3.

Most options are dominated by option o_2 in opinion of individual $e_2, e_4, \ldots$

Most options are preferred to option o_1 in opinion of individual e_3 also preferred to option o_2 in opinion of individual e_4.

Thus, summarized are here the preference matrices of the individuals at a given point in time, i.e., at a given iteration (stage) of the discussion. Quite clearly, preferences evolve over time, and an interesting problem is to how to linguistically assess those changes. This is dealt with in the next section.

4 A Dynamic View of Consensus Reaching

Let us now consider some linguistic summaries that take into account dynamic, temporal aspects of the consensus reaching process, and notably express how the preferences evolve over time. These may be exemplified by a summary

$$\text{Individual } e_k \text{ is } \textit{very flexible} \text{ with respect to his/her preferences between options } o_i \text{ and } o_j. \tag{12}$$

In such a summary we do not capture an "up" and "down" direction of change, and what really matters is an absolute change of a preference degree r_{ij}^k. Thus the time series data considered here represent the cumulative changes up to certain point of time (i.e., iteration of discussion) of an expert's preferences regarding a pair of options. This may be expressed as the sum of absolute values of all changes that have occurred until a given point of time t_s, i.e.

$$\text{change}(t_s) = \sum_{q=1}^{s} \left| r_{ij}^k(t_q) - r_{ij}^k(t_{q-1}) \right|, \tag{13}$$

where $r_{ij}^k(t)$ denotes the r_{ij}^k element of the preference matrix of the kth expert at time (iteration) t. Clearly, the maximal value of change in a single step in comparison to the previous one equals 1.

As the consensus reaching sessions, and thus a "time series of preference changes," are usually very short, then we assume that only one trend, concerning the whole time span is observed. Such a trend is meant to describe a *flexibility* of an expert's opinion: an expert is identified as either flexible or inflexible (stubborn). To extract it we use a simple least squares approximation (LSA) of function (13) by a linear function.

In the summaries we use the slope of the line obtained via LSA to characterize the trend. However it might be impractical to use the actual precise value of the slope. Instead we use a *fuzzy granulation* in order to meet the users' needs and a task specificity. The user may construct a scale of *linguistic terms* corresponding to various slopes of the line identifying the trend as, e.g.,:

- very flexible,
- flexible,
- moderately flexible,

– inflexible, and
– very inflexible (stubborn).

Figure 1 illustrates the lines corresponding to the particular linguistic terms. In fact, each term represents a fuzzy granule of directions.

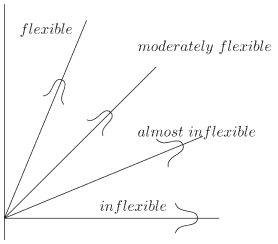

Fig. 1. A visual representation of slope granules defining the dynamics of change

The truth value of the summary (12) is computed via the following three steps:

1. Compute the values of a cumulative change for all time points until the current one – we treat them as a time series
2. Find the slope of the LSA line (α)
3. The truth value of the summary is equal to $\mu_S(\alpha)$, where S is the fuzzy set representing the summarizer of the summary; in case of (12) it is the fuzzy set representing the linguistic term *very flexible*

5 Concluding Remarks

We have proposed to use linguistic summaries to evaluate trends of how the individuals' preferences evolve over time. Starting from a "soft" definition of a consensus degree, we have shown that these summaries provide much of information that can help run a consensus reaching session aimed at reaching a possibly good agreement among a group of individuals concerning their preferences.

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Consistency for Nonadditive Measures: Analytical and Algebraic Methods

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Summary. We consider the problem of choosing an alternative in a set $\mathbf{A} = \{A_1, A_2, \ldots, A_m\}$ of alternatives, given a set $D = \{d_1, d_2, \ldots, d_h\}$ of decision makers and a set $\Omega = \{O_1, O_2, \ldots, O_n\}$ of objectives. We assume that any decision maker d_k assigns to any pair (alternative A_i, objective O_j) a number a_{ijk} that measures to what extent A_i satisfies O_j.

We assume that Ω is a subset of a universal set U and, for every alternative A_i and decision maker d_k, the function m_{ik} that associates a_{ijk} to O_j is a fuzzy measure. We propose to aggregate the scores a_{ijk} by means of a t-conorm $\oplus_\lambda$ of a family Φ_λ of t-conorms such that every m_{ik} is a $\oplus_\lambda$-decomposable measure.

We consider also some algebraic and geometric representations of the Archimedean fuzzy unions and their additive generators in terms of the theory of hypergroups.

By considering the O_j as events, we propose also to assign the scores a_{ijk} in such a way that for some λ the assessment is consistent and to aggregate such evaluations with the correspondent t-conorm $\oplus_\lambda$.

Finally we generalize the previous procedure by considering fuzzy measures of type 2, having as a range a set of fuzzy numbers with the interval $[0, 1]$ as support.

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Key words: Fuzzy measures, Multicriteria and multiperson decision making, Fuzzy models, Hypergroups.

1 Choosing Among Several Alternatives

The problem of choosing an alternative in a set $\mathbf{A} = \{A_1, A_2, \ldots, A_m\}$ of alternatives is considered, given a set $D = \{d_1, d_2, \ldots, d_h\}$ of decision makers and a set $\Omega = \{O_1, O_2, \ldots, O_n\}$ of objectives.

Let $I = \{1, 2, \ldots, m\}$, $J = \{1, 2, \ldots, n\}$, and $H = \{1, 2, \ldots, h\}$ be finite sets. Every decision maker $d_k, k \in H$, assigns to any pair (A_i, O_j) a number a_{ijk} that measures the grade in which the alternative A_i satisfies the objective O_j.

It is assumed also that, for any $j \in J$, weights ω_j are associated to objectives O_j, with the condition:

$$\omega_1 + \omega_2 + \cdots + \omega_n = 1.$$

A classical *two phases choice algorithm* is:

Phase F1. Aggregate the set of the measures assigned by all the decision makers to an alternative A_i with respect to an objective O_j, by considering the *arithmetic mean* b_{ij} of the numbers a_{ijk} with i and j fixed. Then obtain a matrix $b_{ij}, i \in I, j \in J$ such that b_{ij} is the *global grade* in which A_i satisfies the objective O_j.

Phase F2. For every alternative A_i as *global utility* is assumed a *weighted mean* of the numbers $b_{ij}, j \in J$ with the weights ω_j. The alternatives are ordered with respect to their utilities: the preferred alternative has the greatest utility.

Let us call the algorithm above "the fair criterion". We now describe a new algorithm where the two phases are exchanged and aggregate the decision makers scores with criteria of different types, e.g., decision makers bargaining possibly by utilizing cooperative games and in particular their characteristic functions.

Precisely we consider also the "the bargaining criterion" with the following phases:

Phase B1. For every alternative A_i as *global utility with respect to the decision maker* d_k is assumed a *weighted mean* u_{ik} of the numbers $a_{ijk}, j \in J$ with the weights ω_j.

Phase B2. The decision makers agree with the criteria to aggregate, for every alternative A_j, the utilities u_{ik}.

In any case, in this paper we introduce a double generalization. We assume the objectives are subsets of a universe set U.

As a first generalization we consider the scores $a_{ijk}, j \in J$ or the aggregate scores $b_{ij}, j \in J$ are the values of a fuzzy measure m_i over the set Ω of the objectives and we aggregate such values with a t-conorm $\oplus$ such that m_i is decomposable with respect to $\oplus$.

As a further generalization we consider fuzzy measures of type 2 by replacing the scores a_{ijk} with fuzzy scores a^*_{ijk} that are fuzzy numbers with support in the interval $[0, 1]$.

2 Fuzzy Measures on the Set of Objectives

Let us recall some definition and results [1,6,8].

Definition 1. *Let U be a universal set and $\mathcal{F}$ a family of subsets of U containing $\emptyset, U$. A fuzzy measure on $(U, \mathcal{F})$ is a function*

$$h : \mathcal{F} \to [0,1],$$

such that:

FM1 $h(\emptyset) = 0, \quad h(U) = 1$;
FM2 $\forall A, B \in \mathcal{F}, A \subseteq B \Rightarrow h(A) \leq h(B)$.

Definition 2. *A fuzzy measure bel on* $(U, \mathcal{F})$ *is said to be a belief measure if:*

BF1 $\mathcal{F}$ *is an algebra of subsets of* U;
BF2 for every positive integer n *and for every collection* $\{A_1, A_2, \cdots, A_n\}$ *of elements of* $\mathcal{F}$ *we have:*

$$bel(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \sum_i bel(A_i) - \sum_{i<j} bel(A_i \cap A_j) + \cdots + (-1)^{n+1} bel(A_1 \cap A_2 \cap \cdots \cap A_n). \quad (1)$$

In particular a belief measure *bel* has the superadditive property:

$$\forall A, B \in \mathcal{F}, \quad A \cap B = \emptyset \Rightarrow bel(A \cup B) \geq bel(A) + bel(B). \quad (2)$$

For further results see [8].

Definition 3. *A fuzzy measure pl on* $(U, \mathcal{F})$ *is said to be a plausibility measure if:*

PL1 $\mathcal{F}$ *is an algebra of subsets of* U;
PL2 for every positive integer n *and for every collection* $\{A_1, A_2, \ldots, A_n\}$ *of elements of* $\mathcal{F}$ *we have:*

$$pl(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \sum_i pl(A_i) - \sum_{i<j} pl(A_i \cup A_j) + \cdots + (-1)^{n+1} pl(A_1 \cup A_2 \cup \cdots \cup A_n). \quad (3)$$

In particular a plausibility measure *pl* has the subadditive property:

$$\forall A, B \in \mathcal{F}, \quad A \cap B = \emptyset \Rightarrow pl(A \cup B) \leq pl(A) + pl(B). \quad (4)$$

For every fuzzy measure $h : \mathcal{F} \to [0,1]$ on $(U, \mathcal{F})$ with $\mathcal{F}$ an algebra of subsets of U, the function $h^* : \mathcal{F} \to [0,1]$ such that $\forall A \in \mathcal{F}, h^*(A) = 1 - h(\overline{A})$ is also a fuzzy measure on $(U, \mathcal{F})$, called *the associate measure* to h.

It is easy to prove the following.

Theorem 1. *Let* $\mathcal{F}$ *be an algebra of subsets of the universal set* U. *The function* $h : \mathcal{F} \to [0,1]$ *is a belief measure if and only if its associate measure* h^* *is a plausibility measure. Moreover* h *is a finitely additive probability if and only if* $h = h^*$.

Definition 4. *A basic probability assignment bp on* $(U, \mathcal{F})$*, with* $\mathcal{F}$ *an algebra of subsets of* U*, is a function* $bp : \mathcal{F} \to [0,1]$ *such that* $bp(\emptyset) = 0$ *and*

$$\sum_{A \in \mathcal{F}} bp(A) = 1. \tag{5}$$

From a basic probability assignment bp we obtain a belief measure bel and the associate plausibility measure $pl = bel^*$ by the formulas:

$$bel(A) = \sum_{B:B \subseteq A} bp(B); \tag{6}$$

$$pl(A) = \sum_{B:B \cap A \neq \emptyset} bp(B). \tag{7}$$

In particular, by the previous formulas, we have that bel (and then also pl) are probability measures if and only if the set $\{A \in \mathcal{F} : bp(A) > 0\}$ is contained in the set $\mathcal{C}$ of the atoms of the algebra $\mathcal{F}$.

In this paper we assume that the set $\Omega = \{O_1, O_2, ..., O_n\}$ of the objectives is a family of subsets of the universal set U and $\mathcal{F}$ is an algebra that includes Ω.

If we introduce, for every alternative A_i, a basic probability assignment bp_i, we can obtain the scores of the objectives by the formulas (6) or (7), according to whether we wish have belief or plausibility measures.

3 Aggregation of Fuzzy Measures with Respect to a t-Conorm

Suppose that the scores of the objectives $O_j, j \in J$ with respect to an alternative A_i are the numbers $b_{ij}, j \in J$, belonging to the interval $[0,1]$. Our aim is to give a criterion to aggregate these scores and obtain a global score $c_i \in [0,1]$. Then we have to consider an operation $\oplus$ on $[0,1]$ such that:

$$c_i = b_{i1} \oplus b_{i2} \oplus \cdots \oplus b_{in}.$$

The expected properties of such an operation lead us to the concept of the t-conorm.

Definition 5. *A t-conorm or fuzzy union is an operation* $\oplus$ *on* $[0,1]$ *with the following properties:*

FU1 $\forall a \in [0,1], \quad a \oplus 0 = a;$ *(boundary conditions)*
FU2 $\forall a, b, c \in [0,1], \quad (a \oplus b) \oplus c = a \oplus (b \oplus c);$ *(associativity)*
FU3 $\forall a, b \in [0,1], \quad a \oplus b = b \oplus a;$ *(commutativity)*
FU4 $\forall a, b, c \in [0,1], \quad b \leq c \Rightarrow a \oplus b \leq a \oplus c.$ *(monotonicity)*

The t-conorm $\oplus$ is *strictly monotonic* if it is strictly increasing in the open interval $(0,1)^2$ with respect to every variable.

A fuzzy union $\oplus$ is *Archimedean* if it has the following properties:

FU5 the function $\oplus : (a,b) \in [0,1]^2 \to a \oplus b$ is continuous; (continuity)
FU6 $\forall a \in (0,1), \quad a \oplus a > a$. (superidempotence)

A fuzzy union $\oplus$ *strictly monotonic* and *Archimedean* is said to be *strictly Archimedean.*

A characterization of the Archimedean t-conorms is given in the following theorem (see [9]).

Theorem 2. *A binary operation $\oplus$ on $[0,1]$ is an archimedean t-conorm if and only if there exists a strictly increasing and continuous function*

$$g : [0,1] \to [0,+\infty], \quad \textit{with} \quad g(0) = 0,$$

such that

$$\forall a,b \in [0,1], \quad a \oplus b = g^{-1}[min(g(a) + g(b),\ g(1))].$$

Moreover $\oplus$ is strict if and only if $g(1) = +\infty$.

The function g, *called* an additive generator *of $\oplus$, is unique up to a positive constant factor.*

Definition 6. *Let $\mathcal{F}$ be an algebra of subsets of a universal set U and $\oplus$ a t-conorm. A fuzzy measure m on $(U,\mathcal{F})$ is said to be a $\oplus$-decomposable measure if:*

$$\forall A,B \in \mathcal{F}, A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) \oplus m(B).$$

We have the following classification theorems (see [15]).

Theorem 3. *Let $\oplus$ be a strictly Archimedean t-conorm with the additive generator g. A fuzzy measure m on $(U,\mathcal{F})$ is a $\oplus$-decomposable measure if and only if the function:*

$$g \circ m : A \in \mathcal{F} \to g(m(A)) \in [0,+\infty]$$

is an infinite additive measure with $(g \circ m)(U) = +\infty$.

Theorem 4. *Let $\oplus$ be a nonstrictly Archimedean t-conorm with the additive generator g. A A fuzzy measure m on $(U,\mathcal{F})$ is a $\oplus$-decomposable measure if and only if the function $g \circ m$ satisfy the following property:*

$$\forall A,B \in \mathcal{F}, A \cap B = \emptyset, (g \circ m)(A \cup B) = min[(g \circ m)(A) + (g \circ m)(B), g(1)].$$

By Theorem 4 it follows

Corollary 1. *For every $A,B \in \mathcal{F}, A \cap B = \emptyset$*

$$(g \circ m)(A \cup B) < g(1) \Rightarrow (g \circ m)(A \cup B) = (g \circ m)(A) + (g \circ m)(B).$$

4 Algebraic and Geometric Representations by Means of Hypergroups

4.1 Basic Definitions

We show that a fundamental tool to the representation and the treatment of the uncertainty is the *theory of the algebraic hyperstructures* that consider multi-valued operations, i.e., the case in which an operation has a nonempty set of possible results, in general greater to 1.

The theory started in 1934 with a paper by Marty [10], but its present development begins in 1978 with the First International Congress on Algebraic Hyperstructures and Applications (AHA), and the book *Join geometries* by Prenowitz and Jantosciak [12].

For further developments and results see [3] and [4].

We recall some basic definitions.

Definition 7. *A hypergroupoid, or hyperstructure with a hyperoperation, is a pair* (H, σ)*, where* H*, called the support, is a nonempty set and*

$$\sigma : H \times H \to \wp^*(H) = \wp(H) - \{\emptyset\},$$

called the hyperoperation, is a function that associates to any ordered pair (a, b) *of elements of* H *a nonempty subset of* H*, denoted* $a\sigma b$*.*

The elements of H *are called points and any singleton* $\{a\}, a \in H$*, is identified with the point* a*. So, for every* $a, b \in H$*, if* $a\sigma b$ *is a singleton, the hyperoperation reduces to an operation on* H*.*

For every $A, B \in \wp^*(H)$ *we assume:*

$$A\sigma B = \{a\sigma b : a \in A, b \in B\}.$$

Definition 8. *A hypergroupoid* (H, σ) *is said to be:*

- *a semihypergroup, if for every* $a, b, c \in H, a\sigma(b\sigma c) = (a\sigma b)\sigma c$ *(associative property);*
- *a weak semihypergroup, if for every* $a, b, c \in H, a\sigma(b\sigma c) \cap (a\sigma b)\sigma c \neq \emptyset$ *(weak associative property);*
- *a quasihypergroup if for every* $a \in H, a\sigma H = H = H\sigma a$ *(reproducibility property);*
- *an hypergroup if it is both a semihypergroup and a quasihypergroup;*
- *a commutative hypergroupoid if for every* $a, b \in H, a\sigma b = b\sigma a$ *(commutative property); and*
- *a weak commutative hypergroupoid if for every* $a, b \in H, a\sigma b \cap b\sigma a \neq \emptyset$*. (weak commutative property).*

The most important case is the one in which (H, σ) is a hypergroup. For the geometric applications we have to consider the following definitions.

Definition 9. *A hypergroup (H, σ) is said to be a geometric hypergroup if it is commutative and*

$$\forall a \in H, a\sigma a = \{a\} \quad (idempotence).$$

The hyperoperation σ is usually called a hypermultiplication. If (H, σ) is a commutative hypergroup, we define also the division / as the function that to any pair $(a, b) \in H^2$ associates the set $a/b = \{x \in H : a \in x\sigma b\}$.

The division is an hyperoperation on H if and only if $\forall a, b \in H, a/b \neq \emptyset$. For further results and details see [3, 4, 12].

Definition 10. *A commutative hypergroup (H, σ) is said to be a join space if the following incidence property holds:*

$$\forall a, b, c, d \in H, \quad a/b \cap c/d \neq \emptyset \Rightarrow a\sigma d \cap b\sigma c \neq \emptyset.$$

Prenowitz and Jantosciak [12] proved that the Euclidean Spaces and the Convex Sets are particular join spaces. The incidence property is a different formulation of the Pasch axiom (see, e.g., [2]).

Also the Projective Spaces are join spaces and the incidence axiom is reduced to the Veblen–Young axiom (see, e.g., [2]).

4.2 Fuzzy Sets and Coherent Probability Assessments as Hypergroup

In order to deal with the uncertainty it is very important to recall that a fuzzy set with universal set U is a particular hypergroup with support U. Actually the following theorem holds (see [4, 11]).

Theorem 5. *Let $\varphi : U \to [0, 1]$ be a fuzzy set. For every $a, b \in U$, let*

$$a\sigma b = \{x \in U : min\{\varphi(a), \varphi(b)\} \leq \varphi(x) \leq max\{\varphi(a), \varphi(b)\}\}.$$

The pair (H, σ) is a commutative hypergroup, called the hypergroup associated to the fuzzy set φ. Precisely it is a geometric hypergroup and a join space.

Also the set of coherent subjective probability assessments is a hypergroup.

Let $\Gamma = \{E_1, E_2, \ldots, E_n\}$ be a finite set of events and U the Euclidean space E^n having such events as axes. If K is an atom associated to Γ, we call *the representative* of K on U the point $P(K) \in U$ such that, for every E_i, its projection $P_i(K)$ on the axis E_i is 1 if $K \subseteq E_i$ and 0 if $K \subseteq \overline{E}_i$, the contrary of E_i. Let Δ be the set of all the atoms associated to Γ.

As shown by de Finetti [5], a point $p = (p_1, p_2, \ldots, p_n) \in U$ is a coherent probability assessment on Γ, with $p_i = p(E_i)$, if and only if $p \in conv(\Delta)$, with $conv(\Delta)$ the convex set generated by Δ.

We can easily prove the following theorem (see [11]).

Theorem 6. *Let S be the subset of the Euclidean space E^n of all the coherent probability assessments on a set Γ of n events. The application σ that to any pair (a, b) of elements of S associates the close segment with extremes a and b is an hyperoperation on S. The pair (S, σ) is a commutative hypergroup, that we call the hypergroup of the coherent probability assessments on Γ. Precisely it is a geometric hypergroup and a join space.*

4.3 Geometric Spaces of the Fuzzy Unions and Additive Generators

The Space of the *L*-Fuzzy Sets

Let L be a totally ordered set, U the universal set, and Φ the set of the *L-fuzzy sets* with universe U. For every $\alpha, \beta \in \Phi$ we put

$$\alpha\sigma\beta = \{\varphi \in \Phi : \forall x \in U, min\{\alpha(x), \beta(x)\} \leq \varphi(x) \leq max\{\alpha(x), \beta(x)\}\}.$$

We have $\{\alpha, \beta\} \subseteq \alpha\sigma\beta$ and then (Φ, σ) is a commutative quasihypergroup. Let $\sigma(\alpha, \beta, \gamma)$ be the set

$$\{\varphi \in \Phi : \forall x \in U, min\{\alpha(x), \beta(x), \gamma(x)\} \leq \varphi(x) \leq max\{\alpha(x), \beta(x), \gamma(x)\}\}.$$

We can prove that:

- $\forall \alpha, \beta, \gamma \in \Phi, (\alpha\sigma\beta)\sigma\gamma = \sigma(\alpha, \beta, \gamma) = \alpha\sigma(\beta\sigma\gamma)$;
- $\forall \alpha \in \Phi, \alpha\sigma\alpha = \{\alpha\}$.

Then we have the following theorem.

Theorem 7. *The pair (Φ, σ) is a geometric hypergroup.*

We denote with $\mathcal{B}$ the set of all the hyperproducts $\alpha\sigma\beta$ with $\alpha, \beta \in \Phi$. The elements of $\mathcal{B}$ are called *blocks* of Φ and the pair $(\Phi, \mathcal{B})$ the *geometric space associated* to Φ.

The Space of the Fuzzy Unions

Let T be the set of the t-conorms. We define on T the hyperoperation τ such that, $\forall a, b \in T$, $a\tau b$ is the set:

$$\{c \in T : \forall x, y \in [0, 1], min\{a(x, y), b(x, y)\} \leq c(x, y) \leq max\{a(x, y), b(x, y)\}\}.$$

By Theorem 7 we have:

Corollary 2. *The pair (T, τ) is a geometric hypergroup, called the hypergroup of the fuzzy unions.*

The Space of the Additive Generators

Let AG be the set of the additive generators of Archimedean t-conorms. We define on AG the hyperoperation μ such that $\forall f, g \in AG$:

$$f\mu g = \{h \in AG : \forall x \in [0,1], min\{f(x), g(x)\} \leq h(x) \leq max\{f(x), g(x)\}.$$

By Theorem 7 we have:

Corollary 3. *The pair (AG, μ) is a geometric hypergroup, called the hypergroup of the additive generators.*

5 Assessment of Consistent Measures to the Objectives

5.1 Conditions for a Consistent Assessment

The *atoms* with respect to the set $\Omega = \{O_1, O_2, \ldots, O_n\}$ of objectives are the logical possible intersections $I_1 \cap I_2 \cap \cdots \cap I_n$, with $I_j, j \in \{1, 2, \ldots, n\}$, element of the set $\{O_j, \overline{O_j}\}$. We can always assume that the objectives O_j are subsets of a universal set U, and the contrary $\overline{O_j}$ of O_j is the complement of O_j with respect to U. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_s\}$ be the set of atoms.

Suppose $\oplus$ is a nonstrictly Archimedean t-conorm and g is an additive generator of $\oplus$. Assume m is a fuzzy $\oplus$-decomposable measure.

By Corollary 1, an assessment of evaluations

$$m(\Omega) = \{m(O_1), m(O_2), \ldots, m(O_n)\} \quad with \quad m(O_j) < 1, \forall j \in J$$

over the objectives is consistent if and only if there exist measures

$$m(C_r), \quad r = 1, 2, \ldots, s$$

of the atoms such that:

$$a_{j1}g(m(C_1)) + a_{j2}g(m(C_2)) + \cdots + a_{js}g(m(C_s)) = g(m(O_j), j = 1, 2, \cdots, n, \tag{8}$$

$$g(m(C_1)) + g(m(C_2)) + \cdots + g(m(C_s)) \geq g(1), \tag{9}$$

with

$$a_{jr} = \begin{cases} 0, \text{ if } C_r \subseteq \overline{O_j}; \\ 1, \text{ if } C_r \subseteq O_j. \end{cases}$$

We put

$$x_r = g(m(C_r)), \quad b_j = m(O_j), \quad b_0 = g(1) = g(m(U))$$

and

$$x_0 = \sum_{r=1}^{s} g(m(C_r)) - g(1).$$

Then (8) and (9) with the positivity constraints reduce to the system:

$$\begin{aligned} &\sum_{r=1}^{s} a_{jr}x_r = b_j, \quad j = 1, 2, \ldots, J, \\ &\sum_{r=1}^{s} x_r - x_0 = b_0, \\ &x_0 \geq 0, x_r \geq 0, \quad r = 1, 2, \ldots, s. \end{aligned} \tag{10}$$

If we wish to have the solutions of the system (10) such that (9) is "nearest to the equality" we have to consider the *linear programming problem*:

minimize the objective function

$$y = x_0$$

with the *system of constraints* given by (10).

If we have a family $\oplus_\lambda, \lambda \in \Lambda$, of nonstrictly Archimedean t-conorms and g_λ is the correspondent family of additive generators we have:

$$\forall i \in J, b_j = b_j(\lambda) = g_\lambda(m(O_j)), \quad b_0 = b_0(\lambda) = g_\lambda(m(U)).$$

Then we can find the subset Δ of Λ such that the assessment of valuations m on Ω is consistent with $\oplus_\lambda$ if and only if $\lambda \in \Delta$.

5.2 The Case of Sugeno Measures

For $\lambda > -1$, the function $U_\lambda(a, b) = min(a + b + \lambda ab, 1)$ with $a, b \in [0, 1]$, defines a nonstrictly Archimedean t-conorm U_λ with additive generator

$$g_\lambda(x) = \frac{ln(1 + \lambda x)}{x}.$$

In particular:

- For $\lambda = 0$, we have the *bounded sum* $U_0(a, b) = min(a + b, 1)$, with generator $g_0(x) = x$
- For $\lambda \to -1$, we have the strictly Archimedean *sum-product* $U_{-1}(a, b) = a + b - ab$, with generator $g_{-1}(x) = -ln(1 - x)$
- For $\lambda \to +\infty$ we have the *drastic fuzzy union*

For further details see [13–15].

The U_λ-decomposable measures are called λ-additive measures (see [15]).

Banon [1] and Berres proved that λ-additive measures are *plausibility measures* if $-1 < \lambda < 0$, and *belief measures* if $\lambda > 0$.

For $\lambda = 0$ the λ-additive measures are *probability measures.*

6 An Algorithm to Compare the Alternatives

We are now able to exhibit the following algorithm to compare the alternatives $A_i, i \in I$.

1. Any decision maker d_k assigns the score a_{ijk} to the alternative A_i with respect to the objective O_j.
2. By considering a mean or an agreement among the decision makers, we obtain, for every alternative A_i and objective O_j an overall score b_{ij}.
3. We assign a class of nonstrict Archimedean t-conorms $\oplus_\lambda, \lambda \in \Lambda$ and we calculate the corresponding class of additive generators $g_\lambda, \lambda \in \Lambda$.
4. For every $i \in I$ we find the set Δ_i of all the $\lambda \in \Lambda$ such that the set of evaluations $b_{ij}, j \in J$ is consistent, i.e., there exists a fuzzy $\oplus_\lambda$-decomposable measure m_i such that $b_{ij} = m_i(O_j), \forall j \in J$.
5. Put $\Delta = \cap_{i \in I} \Delta_i$. If $\Delta = \emptyset$ we return to step 1 in order to updating scores or to 3 to consider a different class of nonstrict Archimedean t-conorms. If $\Delta \neq \emptyset$ we go to step 6.
6. For every $i \in I$ we aggregate the evaluations $b_{ij}, j \in J$ with respect to a chosen $\oplus_\lambda$ with $\lambda \in \Delta$ and we obtain the overall score $c_i = b_{i1} \oplus_\lambda b_{i2} \oplus_\lambda ... \oplus_\lambda b_{in}$ of the alternative A_i.
7. We order alternatives by considering as the preferable the alternative A_i with the maximum score c_i.

7 An Extension to the Fuzzy Measures of Type 2

We assume that, because of some elements of uncertainty are involved into the assessment of scores or the scores are values of linguistic variables, the real numbers a_{ijk} are replaced by triangular fuzzy numbers a^*_{ijk} with support in the interval $[0, 1]$ and core a_{ijk} (see, e.g., [7]).

For every pair $\alpha = (a, c, b)$ and $\beta = (a', c', b')$ of triangular fuzzy numbers with $\{a, a', b, b'\} \subseteq [0, 1]$ and for every t-conorm $\oplus$ we put:

$$\alpha \oplus \beta = (a \oplus a', c \oplus c', b \oplus b').$$

The algorithm of Sect. 6 is modified as follows.

1. Any decision maker d_k assigns as a score a triangular fuzzy number a^*_{ijk} with support contained in $[0, 1]$ and with core a_{ijk} to the alternative A_i with respect to the objective O_j.
2. By considering a mean or an agreement among the decision makers, we obtain, for every alternative A_i and objective O_j as overall score with respect to all the decision makers a triangular fuzzy number b^*_{ij} with support contained in $[0, 1]$ and with core b_{ij}.
3. We assign a class of nonstrict Archimedean t-conorms $\oplus_\lambda, \lambda \in \Lambda$ and we calculate the corresponding class of additive generators $g_\lambda, \lambda \in \Lambda$.

4. For every $i \in I$ we find the set Δ_i of all the $\lambda \in \Lambda$ such that the set of evaluations $\{b_{ij}, j \in J\}$ is consistent, i.e., there exists a fuzzy measure m_i $\oplus_\lambda$-decomposable such that $b_{ij} = m_i(O_j), \forall j \in J$.
5. Put $\Delta = \cap_{i \in I} \Delta_i$. If $\Delta = \emptyset$ we return to step 1 in order to updating scores or to 3 to consider a different class of nonstrict Archimedean t-conorms. If $\Delta \neq \emptyset$ we go to step 6.
6. For every $i \in I$ we aggregate the fuzzy evaluations $b^*_{ij}, j \in J$ with respect to a chosen $\oplus_\lambda$ with $\lambda \in \Delta$ and we obtain the overall score $c^*_i = b^*_{i1} \oplus_\lambda b^*_{i2} \oplus_\lambda ... \oplus_\lambda b^*_{in}$ of the alternative A_i.
7. We order alternatives by considering some criteria to have a preference relation $\preceq$ among triangular fuzzy numbers.

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Neuro-Fuzzy Kolmogorov's Network with a Modified Perceptron Learning Rule for Classification Problems

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Summary. A novel *Neuro-Fuzzy Kolmogorov's Network* (NFKN) is considered. The NFKN is based on the famous Kolmogorov's superposition theorem (KST) and is the development of the previously proposed Fuzzy Kolmogorov's Network (FKN). Modifications of the FKN architecture include multiple outputs as required for classification problems with more than two classes, as well as the possibility of defining different number of membership functions at each input. A new learning algorithm, based on the modified perceptron learning rule and designed for classification problems, is proposed. The validity of theoretical results and the advantages of the new NFKN are confirmed by experiments in data classification and visualization.

Key words: Kolmogorov's superposition theorem, Neo-fuzzy neuron, Membership function, Fuzzy inference, Classification, Perception learning rule, Batch training.

1 Introduction

A universal approximator called *Fuzzy Kolmogorov's Network* (FKN) with simple structure based on the Kolmogorov's Superposition Theorem [5], and its training procedure with high rate of convergence were proposed in [6, 7]. It was demonstrated that the FKN can be successfully used for time series prediction problems, such as the Mackey–Glass time series prediction and electric load forecasting, as well as for data classification like separation of two intertwined spirals and solving the XOR problem. However, the FKN training algorithm may require a large number of computations in the problems of high dimension, because it is based on the least squares technique, which requires inversion of matrices. This problem is alleviated in the *Neuro-Fuzzy Kolmogorov's Network* (NFKN) [1, 8], which is trained with a hybrid algorithm, where the least squares method is used only for the output layer, and the hidden layer is trained with a gradient descent-based procedure.

Although the FKN [6, 7] and especially the NFKN with the hybrid algorithm [1,8] demonstrated very promising results in classification problems (e.g. the NFKN was demonstrated to solve the N-parity problem for N=18 after

only two training epochs in [8]), their training algorithms are based on the quadratic error function and thus are better suited for regression rather than for classification problems, because minimization of the sum of squared errors does not necessarily lead to the reduction in the number of misclassifications.

In this paper, we propose an efficient and computationally simple learning algorithm, whose complexity depends linearly on both the dimension of the input space and the number of neurons. The proposed algorithm is a batch multioutput modification of the perceptron learning rule [9] with improved convergence, and is designed for classification problems. We use the modified NFKN architecture [8], which is suitable for classification problems with large number of inputs and two or more classes. The efficiency of the new algorithm is confirmed by experiments, in which improved accuracy and the reduction in network size are achieved.

2 Network Architecture

The original FKN architecture [6, 7] is comprised of two layers of neo-fuzzy neurons (NFNs) [11] and is described by the following equations:

$$\hat{f}(x_1,\dots,x_d)=\sum_{l=1}^{n} f_l^{[2]}(o^{[1,l]}),\quad o^{[1,l]}=\sum_{i=1}^{d} f_i^{[1,l]}(x_i),\quad l=1,\dots,n, \tag{1}$$

where n is the number of hidden layer neurons, $f_l^{[2]}(o^{[1,l]})$ is the lth nonlinear synapse in the output layer, $o^{[1,l]}$ is the output of the lth NFN in the hidden layer, $f_i^{[1,l]}(x_i)$ is the ith nonlinear synapse of the lth NFN in the hidden layer.

The universality of internal functions in the KST [4] suggests that we can introduce an extended version of the FKN, called NFKN, with Q outputs [8], having the same hidden layer for all the output neurons:

$$\hat{f}_q(x_1,\dots,x_d)=\sum_{l=1}^{n} f_l^{[2,q]}(o^{[1,l]}),\quad o^{[1,l]}=\sum_{i=1}^{d} f_i^{[1,l]}(x_i), \tag{2}$$
$$l=1,\dots,n, q=1,\dots,Q,$$

where Q is the number of output layer neurons, $f_l^{[2,q]}(o^{[1,l]})$ is the lth nonlinear synapse of the qth NFN in the output layer.

The equations for the hidden and output layer synapses are

$$f_i^{[1,l]}(x_i)=\sum_{h=1}^{m_{1,i}} \mu_{i,h}^{[1]}(x_i) w_{i,h}^{[1,l]},\quad f_l^{[2,q]}(o^{[1,l]})=\sum_{j=1}^{m_{2,l}} \mu_{l,j}^{[2]}(o^{[1,l]}) w_{l,j}^{[2,q]}, \tag{3}$$
$$l=1,\dots,n, i=1,\dots,d, q=1,\dots,Q,$$

where $m_{1,i}$ and $m_{2,l}$ are the number of membership functions (MFs) per input in the hidden and output layers, respectively, $\mu_{i,h}^{[1]}(x_i)$ and $\mu_{l,j}^{[2]}(o^{[1,l]})$ are the

MFs, $w_{i,h}^{[1,l]}$ and $w_{l,j}^{[2,q]}$ are the tunable weights. We assume that the MFs are fixed, triangular (piecewise-linear), and equidistantly spaced over the range of each NFN input. The parameters (centers) of the MFs are not tuned.

As in the FKN, the MFs in the NFKN at each input in the hidden and output layers are shared between all neurons. However, in the NFKN architecture we allow for different number of MFs at each input. This property is essential for the processing of data sets with mixed numerical and categorical inputs, such that each category value of a categorical input corresponds to one MF and is encoded with a numerical value corresponding to the center of that MF. This is a more parsimonious and convenient approach than conventional binary coding of categories, because we do not have to introduce additional inputs to the classifier. When a missing input is encountered, no membership function for that input is activated, and the corresponding input synapse produces zero value. The NFKN architecture is shown in Fig. 1.

The outputs of the NFKN are computed via the following two-stage fuzzy inference procedure:

$$\hat{y}_q = \sum_{l=1}^{n} \sum_{j=1}^{m_{2,l}} \mu_{l,j}^{[2]} \left[\sum_{i=1}^{d} \sum_{h=1}^{m_{1,i}} \mu_{i,h}^{[1]}(x_i) w_{i,h}^{[1,l]} \right] w_{l,j}^{[2,q]}, \quad q = 1, \ldots, Q. \tag{4}$$

The description 4 corresponds to the following two-level fuzzy rule base:

$$\begin{gathered} \text{IF} x_i \text{ IS } X_{i,h} \text{ THEN } o^{[1,1]} = w_{i,h}^{[1,1]} d \text{ AND} \ldots \text{AND } o^{1,n} = w_{i,h}^{[1,n]} d, \\ i = 1, \ldots, d, h = 1, \ldots, m_{1,i}, \end{gathered} \tag{5}$$

$$\begin{gathered} \text{IF } o^{[1,l]} \text{ IS } O_{l,j} \text{ THEN } \hat{y}_1 = w_{l,j}^{[2,1]} n \text{ AND} \ldots \text{AND } \hat{y}_Q = w_{l,j}^{[2,Q]} n, \\ l = 1, \ldots, n, j = 1, \ldots, m_{2,l}, \end{gathered} \tag{6}$$

where $X_{i,h}$ and $O_{l,j}$ are the antecedent fuzzy sets in the first and second level rules, respectively.

Total number of rules is

$$N_R^{NFKN} = \sum_{i=1}^{d} m_{1,i} + \sum_{l=1}^{n} m_{2,l}, \tag{7}$$

i.e., it depends *linearly* on the number of inputs d. Straightforward grid-partitioning approach would produce $\prod_{i=1}^{d} m_{1,i}$ fuzzy rules, leading to combinatorial explosion and being practically not feasible for $d > 4$.

3 Learning Algorithm

The number of tunable weights in an NFKN is $S = S_1 + S_2$, where $S_1 = \sum_{i=1}^{d} m_{1,i} \cdot n$ is the number of parameters in the hidden layer, and $S_2 = \sum_{l=1}^{n} m_{2,l} \cdot Q$

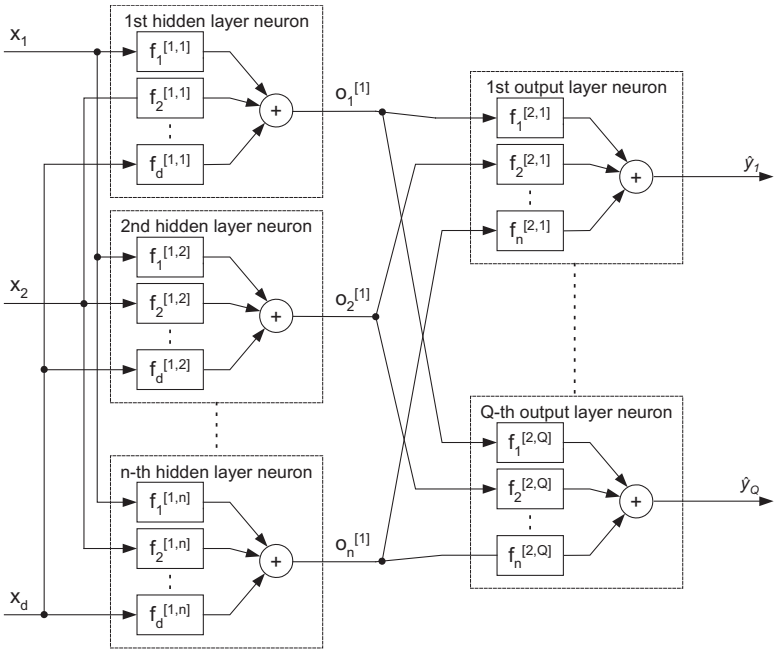

Fig. 1. NFKN with d inputs, n hidden layer neurons, and Q output layer neurons

is the number of parameters in the output layer. The weights of the NFKN here are determined by means of a batch-training algorithm as described below.

A training set containing N samples is used. The minimized error function is a batch Q-output modification of the function investigated in [9]:

$$E(t) = \sum_{q=1}^{Q}\sum_{k=1}^{N}\left[\,|\hat{y}_q(t,k)| - y_q(k)\hat{y}_q(t,k)\right] \tag{8}$$

$$= \sum_{q=1}^{Q}\sum_{k=1}^{N}\left[(sign\,\hat{y}_q(t,k) - y_q(k))\hat{y}_q(t,k)\right] = Tr\left[(sign\hat{Y}(t) - Y)^T\hat{Y}(t)\right],$$

$$Y = (Y_1, Y_2, \ldots, Y_Q), \quad \hat{Y}(t) = (\hat{Y}_1(t), \hat{Y}_2(t), \ldots, \hat{Y}_Q(t)),$$

$$Y_q = [y_q(1), \ldots, y_q(N)]^T, \quad \hat{Y}_q(t) = [\hat{y}_q(t,1), \ldots, \hat{y}_q(t,N)]^T, \quad q = 1, \ldots, Q,$$

where Y is the matrix $(N \times Q)$ of target values, $\hat{Y}(t)$ is the matrix $(N \times Q)$ of network outputs at epoch t. Target values in Y are encoded as –1 or +1.

It is only the sign of the output rather than the output value itself that is important for the determination of classification error, and this is explicitly reflected in (8). So the error function (8) would be a potentially better choice for classification problems compared to the sum of squared errors as in [6, 7] or [1, 8].

The error function (8) can be minimized through recursive gradient-based optimization of the synaptic weights. To derive such a procedure for the output layer in a compact matrix–vector notation, re–write (4) as follows:

$$\hat{y} = {W^{[2]}}^T \varphi^{[2]}(o^{[1]}), \quad W^{[2]} = \begin{bmatrix} w_{1,1}^{[2,1]}, w_{1,2}^{[2,1]}, \ldots, w_{n,m_{2,n}}^{[2,1]} \\ \vdots \\ w_{1,1}^{[2,Q]}, w_{1,2}^{[2,Q]}, \ldots, w_{n,m_{2,n}}^{[2,Q]} \end{bmatrix}^T, \tag{9}$$
$$\varphi^{[2]}(o^{[1]}) = \left[\mu_{1,1}^{[2]}(o^{[1,1]}), \mu_{1,2}^{[2]}(o^{[1,1]}), \ldots, \mu_{n,m_{2,n}}^{[2]}(o^{[1,n]})\right]^T,$$

where $W^{[2]}$ is the matrix $(\sum_{l=1}^{n} m_{2,l} \times Q)$ of the output layer weights.

The update procedure for the output layer weights will be

$$W^{[2]}(t+1) = W^{[2]}(t) - \gamma \frac{\partial E(t)/\partial W^{[2]}}{\left\|\Phi^{[2]}\right\|} = W^{[2]}(t) + \gamma \frac{{\Phi^{[2]}}^T (Y - sign\hat{Y}(t))}{\left\|\Phi^{[2]}\right\|},$$
$$\Phi^{[2]} = \left[\varphi^{[2]}(o^{[1]}(1)), \ldots, \varphi^{[2]}(o^{[1]}(N))\right]^T, \left\|\Phi^{[2]}\right\| = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{\sum_{l=1}^{n} m_{2,l}} (\Phi^{[2]})_{i,j}^2}, \tag{10}$$

where γ is the learning rate, and $\Phi^{[2]}$ is the regressor matrix $(N \times \sum_{l=1}^{n} m_{2,l})$ for the linear output layer. The norm $\left\|\Phi^{[2]}\right\|$ in the denominator of (10) is present to speed up convergence.

We can derive a similar gradient descent-based learning rule for the hidden layer as well. To do this, let us introduce the vector $(S_1 \times 1)$ of the hidden layer weights $W^{[1]} = \left[w_{1,1}^{[1,1]}, w_{1,2}^{[1,1]}, \ldots, w_{d,m_{1,d}}^{[1,1]}, \ldots, w_{d,m_{1,d}}^{[1,n]}\right]^T$ and the Jacobian matrix $\Phi^{[1]} = \left[\varphi^{[1]}(x(1),1), \ldots, \varphi^{[1]}(x(N),1), \ldots, \varphi^{[1]}(x(1),Q), \ldots, \varphi^{[1]}(x(N),Q)\right]^T$ of size $(NQ \times S_1)$, where

$$\Phi^{[1]} = \left[\varphi^{[1]}(x(1),1), \ldots, \varphi^{[1]}(x(N),1), \ldots, \varphi^{[1]}(x(1),Q), \ldots, \varphi^{[1]}(x(N),Q)\right]^T,$$
$$\varphi^{[1]}(x,q) = \left[\varphi_{1,1}^{[1,1]}(x_1,q), \varphi_{1,2}^{[1,1]}(x_1,q), \ldots, \varphi_{d,m_{1,d}}^{[1,1]}(x_d,q), \ldots, \varphi_{d,m_{1,d}}^{[1,n]}(x_d,q)\right]^T,$$
$$\varphi_{i,h}^{[1,l]}(x_i,q) = a_l^{[2,q]}(o^{[1,l]}) \cdot \mu_{i,h}^{[1]}(x_i), i = 1, \ldots, d,$$
$$h = 1, \ldots, m_{1,i},\ l = 1, \ldots, n,\ q = 1, \ldots, Q, \tag{11}$$

and $a_l^{[2,q]}(o^{[1,l]})$ are determined as in [8]:

$$a_l^{[2,q]}(o^{[1,l]}) = \frac{w_{l,p+1}^{[2,q]} - w_{l,p}^{[2,q]}}{c_{l,p+1}^{[2]} - c_{l,p}^{[2]}}, \tag{12}$$

where $w_{l,p}^{[2,q]}$ and $c_{l,p}^{[2]}$ are the weight and center of the pth MF in the lth synapse of the qth output layer neuron, respectively. The MFs in an NFN are always chosen such that only two adjacent MFs p and $p+1$ fire at a time [11].

Now we can obtain the gradient-based update procedure for the hidden layer weights:

$$\begin{gathered} W^{[1]}(t+1) = W^{[1]}(t) - \gamma \frac{\nabla_{W^{[1]}} E(t)}{\left\|\Phi^{[1]}\right\|} = W^{[1]}(t) + \gamma \frac{\Phi^{[1]^T}(\overrightarrow{Y} - sign\overrightarrow{\hat{Y}}(t))}{\left\|\Phi^{[1]}\right\|}, \\ \overrightarrow{Y} = (Y_1^T, Y_2^T, \ldots, Y_Q^T)^T, \quad \overrightarrow{\hat{Y}}(t) = (\hat{Y}_1^T(t), \hat{Y}_2^T(t), \ldots, \hat{Y}_Q^T(t))^T, \\ \left\|\Phi^{[1]}\right\| = \sqrt{\sum_{i=1}^{NQ} \sum_{j=1}^{S_1} (\Phi^{[1]})_{i,j}^2}, \end{gathered} \tag{13}$$

where $\overrightarrow{Y}$ and $\overrightarrow{\hat{Y}}(t)$ are vectors $(NQ \times 1)$.

Since no matrix inversions are involved, the considered training algorithm is less computationally intensive that both the original algorithm for the FKN [6, 7] and its hybrid modifications [1, 8]. The number of computations in (10) for both layers at each epoch depends linearly on both the dimension of the input space d and the number of neurons n.

For the processing of very large data sets when the storage of matrices for the complete data set is impossible because of memory limitations, the most memory-consuming calculations of the gradient $\nabla_{W^{[1]}} E(t) = -\Phi^{[1]} T(\overrightarrow{Y} - sign\overrightarrow{\hat{Y}}(t))$ and the norm $\left\|\Phi^{[1]}\right\|$ can be performed cumulatively sample by sample.

4 Experiments

To verify the theoretical results and compare the performance of the proposed network to the known approaches, we carried out experiments using the data from the well-known UCI repository [10]: Iris, Wisconsin Breast Cancer, Australian Credit, and German Credit. The parameters of the data sets are listed in Table 1. Note that two data sets, "Australian" and "German", have several categorical inputs, and the "Wisconsin" data set has 16 samples with missing values.

Table 1. Data sets used in experiments

Data set	Number of samples	Samples with missing values	Numerical attributes	Categorical attributes	Classes
Iris	150	0	4	0	3
Wisconsin	699	16	9	0	2
Australian	690	0	6	8	2
German	1000	0	7	13	2

The results of the experiments with the NFKN and the proposed training algorithm are summarized in Table 2. The column "neurons" describes the NFKN architectures: the numbers separated by "+" indicate the number of the hidden and output neurons, respectively. The column "weights" shows the number of tunable parameters. The next column shows the average number of epochs required for the learning algorithm to converge. The last two columns show the classification error rates. All the results in Table 2 were obtained with the learning rate $\gamma = 0.05$ in the procedures (10) and (13).

The most important advantages of the NFKN classifier are its simple architecture, which is not affected by the curse of dimensionality, and fast training procedures providing at the same time high accuracy of classification. All the results are at the level of accuracy achieved with the best classification techniques, e.g. the support vector machines [2]. For comparison, the best results obtained with the NFKN and the hybrid training algorithm from [8] are shown in Table 3.

Note that for the "Wisconsin" and "Iris" data the accuracy on the checking set in Tables 2 and 3 is the same, and for the "Australian" and "German" data sets the new training algorithm yields better results, which can be seen from the comparison of checking set errors in Tables 2 and 3. In addition, for the "Wisconsin" and "German" data best results are achieved with fewer neurons and weights.

Since all the best results from Table 2 are achieved with only two neurons in the hidden layer, visualization of multidimensional data in the two-dimensional space, formed by the outputs of the hidden layer neurons, is possible (an example is shown in Fig. 2).

Table 2. Results of experiments for NFKN with the proposed modified perceptron learning rule (tenfold cross-validation)

Data set	Neurons	Weights	Epochs	Training set errors(%)	Checking set errors(%)
Iris	2+3	54	9.9	1.33	4
Wisconsin	2+1	78	9.1	2.23	3.01
Australian	2+1	116	10.8	11.18	13.48
German	2+1	160	11.6	21.04	24.1

Table 3. Results of experiments for NFKN with the hybrid training algorithm (tenfold cross-validation)

Data set	Neurons	Weights	Epochs	Training set errors(%)	Checking set errors(%)
Iris	2+3	54	4	1.04	4
Wisconsin	4+1	176	11.9	0.54	3.01
Australian	2+1	116	11.7	10.42	14.2
German	3+1	240	29	14.32	24.8

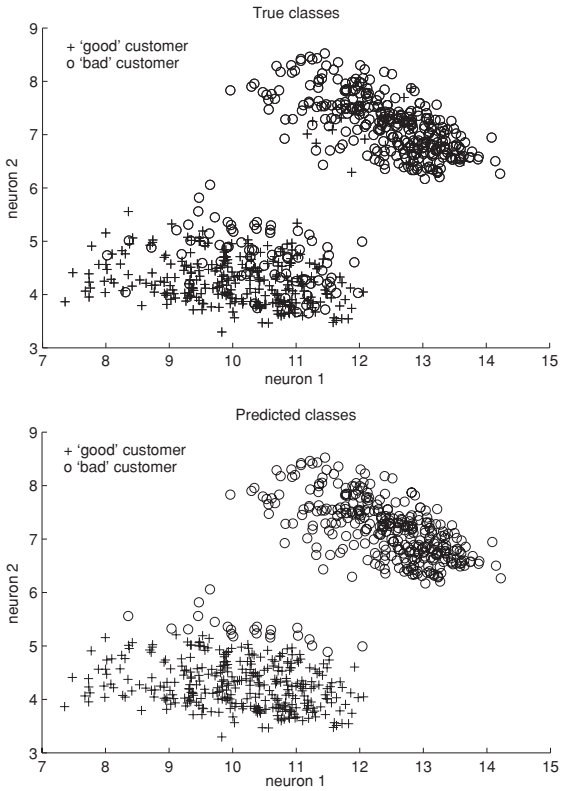

Fig. 2. Australian credit data projected by the hidden layer of the NFKN trained with the proposed algorithm (10), (13)

5 Conclusion

A simple and practical approach to the construction of neuro-fuzzy classifiers was considered. The described NFKN architecture is not affected by the curse of dimensionality because of its two-level structure according to the KST, and is suitable for classification problems with multiple classes and both continuous and discrete (categorical) input variables.

The use of the neo-fuzzy neurons enabled us to develop fast and simple training procedures for both the hidden and output layer parameters, based on the perceptron learning rule. A new batch multioutput modification of this learning rule was proposed.

We expect that the NFKN with the new learning algorithm can find applications in decision support and data mining [3], where classification and visualization of high-dimensional data are the key problems.

An important issue that needs to be further investigated is the interpretability improvement of the fuzzy rules in the NFKN, because the two-level rule base (5), (6) lacks transparency and differs from the rule format used in most of fuzzy modeling approaches.

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A Self-Tuning Controller for Teleoperation System using Evolutionary Learning Algorithms in Neural Networks

Habib Allah Talavatifard, Kamran Razi, and Mohammad Bagher Menhaj

Summary. Using an evolutionary learning algorithm, a self-tuning neural network based controller for teleoperation system has been presented. The proposed control scheme comprises a PID controller, and a neural network which updates the PID controller gains in both master and slave sides assuming that the environmental conditions are given. To do so, this technique uses the least information possible (master and slave positions), thus no force feedback signal is required. To meet the desired solution, a well-defined fitness function is introduced. The proposed controller is implemented in a one degree-of-freedom teleoperatory system. The simulation results showed that the designed controller provides satisfactory telepresence in different environmental impedances.

Key words: Teleoperation system, Neural networks, Evolutionary computation, Evolutionary programming, Self-tuning controller.

1 Introduction

Teleoperation systems have been widely used in areas such as hazardous environments and surgical operations. The term – teleoperation system – refers to a robotic system in which the master controller, typically a human interface, and the slave robot, which manipulate the remote task, are at different locations. Like any other robotic system, teleoperatory systems have two main objectives: accuracy and safety. These systems have also limited flexibility in adapting environment changes in the presence of the frequently experienced stability problems.

To have a performance index for these systems, *telepresence* is introduced. Telepresence is a measure of how accurate an operator working with a teleoperatory system feels the remote task. Ideal telepresence means the operator has a sense of performing the task directly. Figure 1 shows a general structure for teleoperation systems.

To achieve a high degree of telepresence, several methods have been proposed. Brooks [1] suggested response requirements for teleoperation systems,

Fig. 1. A teleoperatory system general structure

which resulted in finding an appropriate bandwidth for position and force signals. The force signal was a measure of environment stiffness. Lawrence [2] used four-channel architecture for the system and introduced mathematical definition for ideal transparency. Hannaford [3,4] suggested a popular two-port hybrid network model for teleoperation system which is now frequently used. Using this model, we can easily calculate the transferred impedance to the operator. Figure 2 shows a two port hybrid network. Transferred impedance to the operator with respects to environment impedance is derived in 1.

$$\begin{bmatrix} F_h(s) \\ X_m(s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \times \begin{bmatrix} X_s(s) \\ -F_e(s) \end{bmatrix}$$
$$Z_t = (H_{11} - H_{12}Z_e)(H_{21} - H_{22}Z_e)^{-1} \tag{1}$$

If the hybrid parameters are not functions of Z_h and Z_e, impedance matching is obtained if and only if these conditions are satisfied:

$$\begin{cases} H_{22} = H_{11} = 0, \\ H_{21}H_{12} = -1 \end{cases} \tag{2}$$

To achieve the ideal telepresence, Lawrence [2] proposed a time invariant controller using force, position, and velocity signals at both sides. Using four different signals means that additional measurement noise can permeate into the system and possibly decrease the system stability. However, this paper try to suggest that a self-tuned controller is able to satisfy conditions stated in 2 knowing just positions, and their derivatives.

Cavusoglu et al. [5] introduced a task based optimization framework instead of seeking a generic ideal teleoperation. They proposed a fidelity measure which quantifies the teleoperation system's ability to transmit changes in the compliance of the environment. Niemeyer et al. [6] suggested a hybrid control by employing an adaptive scheme for slave free motion. Their control scheme enables system to attain a stable interaction with any passive environment. Alternatively, Hashtrudi Zaad et al. [7] proposed another adaptive method that generates the force signal through an estimation of environment impedance. In order to prove their method's efficiency, they presented some numerical solutions and left further experiments to future. Hannaford [8] studied teleoperation system's stability in terms of time domain passivity and designed a passivity controller to achieve a stable condition.

Two significant notions must be reminded about above methods: first, some methods are model-based [2, 7] and hence sensitive to model variations, therefore the resulting controller shows small stability margins and lacks

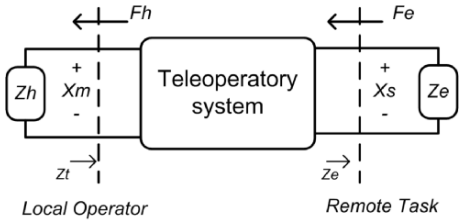

Fig. 2. A two port hybrid network: X_m, X_s, F_h and F_e are the laplace transforms of the master and slave positions, the force applied by the operator in the master side, and the force exerted by the slave on the remote environment, respectively

robustness. Besides, passivity is a conservative index and can degrade telepresence. As a result, all above methods are reduced to a stability-performance trade-offs [2]. Second, methods that use force signal [2], add additional noise to system and increase its cost dramatically. In fact, a control architecture without a force feedback signal is much more desirable today.

Traditional control methods such as PID controllers can be used to control slave robot to follow the motion of the master. These methods are easy to implement and inexpensive but they have one big deficiency: they cannot track the environment changes; hence, they fail to achieve a good sense of telepresence. Anyway, there is an important point to cite here, for a time invariant environment the performance of a well-tuned PID controller (in both master and slave) is quite acceptable [9].

To provide a PID controller which is able to adapt itself to model changes, we use a self-tuning control scheme. Our design includes an artificial neural network updating the PID controller gains according to model changes. In the training stage, neural network learns the nonlinear mapping from environment impedances to proper PID gains. This mapping contains the system model implicitly, and the goal is ideal transparency as described above.

Neural Networks are commonly used due to their intrinsic robustness and model independence. Since in most teleoperation applications it is not possible to predict the environment model, a neural network based controller is an apt choice.

Choosing a particular neural network for an application concerns two considerations: first, the neural network architecture, and second, its learning algorithm. Taking into account that our goal is to learn a nonlinear mapping, a feed-forward multilayer architecture seems to be suitable. Besides, although gradient based learning algorithms are common (Back-Propagation for example), they cannot be applied to this problem because they do need the error in network outputs as a supervisory signal and here, network target outputs – the PID gains – are not known.

Evolutionary computation is a powerful model-free method in optimization problems. It can be used in our problem to adjust network parameters [10] due to the fact that it does not require any error signal [11]. In this method, instead of error, a carefully selected fitness function will be used as a measure of how well the neural network has learned its task.

2 Neural Controller and Evolutionary Programming

As mentioned above, traditional PID controller performs satisfactorily if there are no changes in environment impedance. It is apparent that in terms of any environmental changes an adaptive controller is necessary [7, 9, 12]. Recently designed adaptive controllers for teleoperation system are model-based: [2, 7], and lack a key element: robustness. Providing both adaptively and robustness, neural network-based controllers can be excellent alternatives in this problem. With the force sensors eliminated, the neural network should be able to approximate external impedance knowing barely master and slave positions. However, first and even higher order position derivatives should also appear in the inputs to allow the neural network to track the environment impedance dynamical behavior. These derivatives can be substituted by position values in past sampling times.

There exist two ways for constructing a controller: first, the neural network directly produces output control signal for both master and slave. In this method an evolutionary real-time algorithm was in charge of learning process [9]. Second, the neural network is used to adjust PID gains in a classical control scheme.

Since a traditional PID controller performs satisfactorily if the environmental conditions are known [9], a PID controller which is being tuned by a neural network according to environmental changes can lead to very good results. In this way, the simple but efficient classical control framework is allowed to adapt itself to environmental changes. Use of the second method reduces the neural network complexity which is crucial for quick and efficient learning, and the neural network does not need to select control framework by itself. Combining this control scheme with an offline and "in depth" evolutionary learning process can boost the system ability to handle any possible impedance.

To update neural network weights in training phase, we need an error independent learning algorithm in the view of the fact that the neural network target output, and consequently output error is not known. Evolutionary computation methods are ideal for this purpose [11]. A good fitness function can omit the error from the training phase. The fitness will be a measure of how well our neural network has learned its task, and network parameters can be updated until the fitness reaches a reasonable predefined value.

Having chosen an evolutionary approach, it is essential that we select one of Genetic Algorithm [13], Evolutionary Programming [14], and Evolutionary Strategy [15]. Evolutionary programming is well suited for our work, as it involves real coded entities, and we prefer not to enter number of bits-accuracy trade-off that exists in Genetic Algorithm [16].

The evolutionary algorithm, which is the basic model of the evolutionary process of nature, can be divided into the process of generating some new populations from the mutation of both parents and the natural selection of superior parameters through the competition among the parents and

generated offsprings [17]. The evolutionary algorithm is composed of structures that perform the natural adjustment of a search region, evolution, and selection from the competition between parents and offspring.

The evolutionary programming can be performed through the algorithm described below [10]:

1. Generate an initial population of individuals at random: each individual is a pair of real-valued vectors (η and w). The w is neural network weights and biases vector and η is indicating each individual variance. In fact each individual is a possible solution.
2. Create an offspring for each individual: use the mutation operator which is defined below:

$$\eta_i'(j) = \eta_i(j) \exp\left[\tau' N(0,1) + \tau N_j(0,1)\right] \qquad i = \{1, 2, \cdots, \mu\} \qquad j = \{1, 2, \cdots, n\} \tag{3}$$

   n is the length of η and w vectors, and μ is the number of individuals. The parameters τ and τ' are commonly set to $(\sqrt{2\sqrt{n}})^{-1}$ and $(\sqrt{2n})^{-1}$ [15].
3. Determine the fitness of every individual, including all parents and offspring.
4. Conduct a tournament scheme over the union of parents and offspring. For each individual, opponents are chosen uniformly at random from all the parents and offspring. For each comparison, if the individual's fitness is no smaller than the opponent's, it receives a "win." Select individuals that have most wins to form the next generation.
5. Stop if the fitness halting criterion is satisfied; otherwise, and go to step 2.

For every solution (every w vector) produced by, the evolutionary learning algorithm during training stage, a prolonged simulation is carried out. At the end of each epoch, the fitness for each possible solution is calculated based on the system performance and new solutions are generated until the desired fitness is reached.

Since the design objective is an adaptive controller, the neural network training data should contain sufficient information about any condition that the teleoperatory system may encounter. To meet this goal the environment changes during simulation. Thus, the system may experience two possibilities: free motion and time variant impedance. This impedance consists of three fundamental elements and can represent a wide range of possible physical environments from a sponge to a hard barrier. To guarantee the neural network will achieve the ability to generalize, the time variant impedance parameters, in addition to system input (human operator external force signal) changes dramatically during each simulation.

Since the training stage is an offline procedure, it is possible to substitute numerous simulations with constant parameters with one prolong simulation spanning a wide range of impedances.

3 Fitness Function

Appropriate fitness function plays a key role in how efficiently and accurately the problem is solved. An improper fitness function may cause the system to completely malfunction, for example, if we only use $\frac{1}{error}$ as fitness function the evolutionary algorithm tends to make neural network produce large PID gains at the master side and small PID gains at the slave side. This gain setting made the system operate in a bounded trajectory, and thus produces smaller error. (Here, the error is the difference between master and slave positions, not the difference between desired neural network output and its real output.) As a matter of fact, with this fitness function the neural network does not learn the rules governing the system to decrease error; it only learns to produce smaller outputs. This problem can be overcome by normalizing the error. This example depicts the fitness function impact on system performance. However, we used a different fitness function as described in (4–6).

$$\begin{gathered}\mathbf{error} = \mathbf{X}_s - \mathbf{X}_m \\ a = \left[\frac{\mathbf{error}^T \times \mathbf{error}}{\mathbf{time}}\right]^{\frac{1}{2}}\end{gathered} \tag{4}$$

$$\begin{gathered}\mathbf{diff} = \mathbf{F}_e - \mathbf{F}_h \\ b = \left[\frac{\mathbf{diff}^T \times \mathbf{diff}}{\mathbf{time}}\right]^{\frac{1}{2}}\end{gathered} \tag{5}$$

$$\text{fitness} = \frac{1}{ab} \tag{6}$$

Our fitness function combines two factors: first, error minimization is a general rule (4). Second, there must be a correlation between the force applied by the human (F_h) and the environmental force (F_e) (5). In times of ideal telepresence, F_h is equal to F_e.

$$\left.\begin{matrix} Z_e = Z_t \\ X_s = X_m \end{matrix}\right\} \Rightarrow F_e = F_h \tag{7}$$

As a matter of fact, this factor represents the rule neural network should learn.

In order to calculate fitness, it is essential to have the environment force (F_e) in advance. This force is actually a criterion of the system environment – it describes what kind of environment the slave robot is manipulating. (A bigger environment force means a harder barrier.) Razi et al. [12] proposed an environmental force estimation method based on the simplified inverse dynamic model. By using this model, approximating F_e in training phase is feasible. However, after training the neural network, there will be no need to F_e in control scheme.

4 Simulation Results

To test the designed controller, we ran a set of simulations. The simulated teleoperatory system-based on a one degree-of-freedom master salve robot, designed and implemented by Razi et al. [12]. We integrated all important factors in our simulated model such as nonlinearities, saturations, etc. We modeled each DC motors with one mechanical time constant and one electrical time constant. For simplicity, we used a PD controller instead of PID, which is quite common in robotics. In addition, we assumed our nonlinear impedance has a constant mass and parameters k and b (spring and damper parameters) are time-varying. The neural network has 15 hidden neurons, and, receives both master and slave position in past 20 sampling times as inputs. Figure 3 shows a block diagram of our system. ($F_h{}^*$ and $F_e{}^*$ are the human and the environment external forces, respectively. While the former is considered as system input, the latter is set zero for the sake of simplicity).

The simulation was divided into two stages: the training stage and the test stage, in every learning epoch, for each possible solution (each w vector) a prolonged simulation (more than 150 s) was carried out. The human external force ($F_h{}^*$), k, and b were changing dramatically during each simulation. Changes in these parameters were chosen deliberately in order to cover all possible conditions that the system may encounter. Hard contact (environment impedance) position was also time varying. At the end of each epoch, fitness was evaluated based on system performance. Having known fitness values for every possible solution, the evolutionary algorithm was able to find desired weights for the neural network. After significant amount of training (more than 500 epochs), the system was in the desired state and the learning stage ended.

Figure 4 shows system response (master and salve positions), and force signals (F_h and F_e), in addition to an arbitrary human external force in the test stage. The environment impedance settings are also chosen haphazardly.

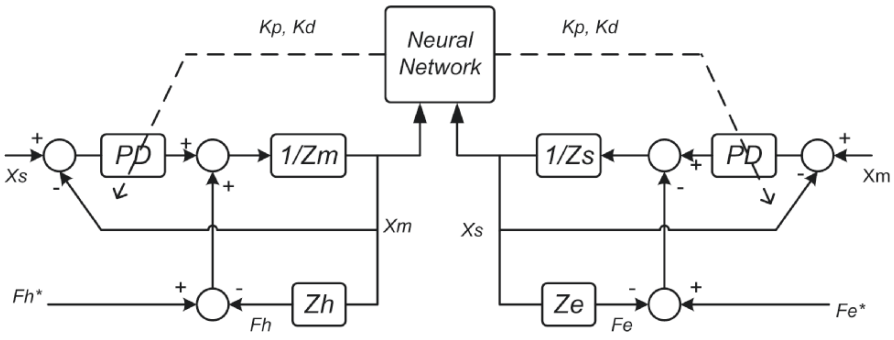

Fig. 3. System block diagram: Z_m, Z_s, Z_h and Z_e are modeling the master, slave, human and environment dynamics

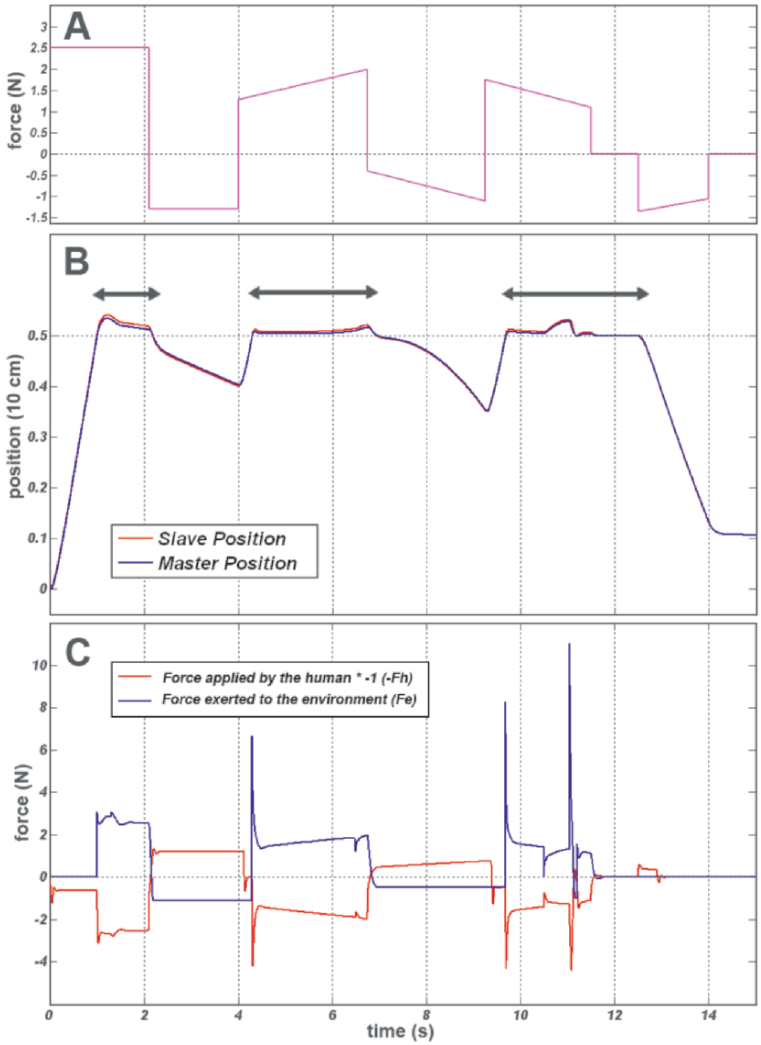

Fig. 4. Illustrates system responses to an arbitrary human external force and haphazard environment impedance setting. (**a**) Shows the human external force ($F_h{}^*$). This force makes the system move back and forth to the barrier (nonlinear impedance). (**b**) Shows both master and slave positions. *Double headed arrows* shows time spans that slave robot is in contact with nonlinear impedance. During these time spans we used a time-variant nonlinear environment impedance. The fluctuations in positions are reflecting this fact – in different loading conditions, different displacements are expected with a specific human external force. The nonlinear impedance is positioned at 5 cm. (**c**) Shows the force applied by the human with a negative sign, $-F_h$, and the force exerted to the environment, F_e (the negative sign is used for better illustration). Environment force is a criterion of what slave robot is interacting with. The fluctuations in force signals is a mirror of nonlinear impedance changes. Impulse functions observed in force signals are a result of step changes in nonlinear impedance parameters and will not exist in reality

As one may notice, master is following slave precisely while force applied by the human is approximately equal to the force exerted to the environment. Experiment with different kind of impedances shows that our system performance is acceptable for both linear and nonlinear time variant impedances. While previous methods can only deal with linear impedances [2,7], our system, due to nonlinear nature of neural networks, is able to handle impedance nonlinearities. Furthermore, this task cannot be accomplished by a traditionally tuned PID, because this scheme is sensitive to environment changes and loses its control ability dramatically when the environment impedance changes. This is what expected of a teleoperatory system with ideal telepresence. Additionally, the system tolerance against conspicuous changes in nonlinear impedance parameters is confirming its robustness.

5 Conclusion

A key element in teleoperatory systems is to transfer various impedance conditions accurately to the operator side. In this paper, we presented a new self-tuning controller based on an evolutionary trained neural network. As a matter of fact, what has been done is to allow the simple but efficient classical control framework to adapt for environmental changes. Since a traditional PID controller is very good for position and force feedback signal following if we know the impedance of slave system and the environmental conditions, our goal was to tune the PID controller using an adaptive method – a neural network – based on environmental conditions. We approved high performance of proposed controller by running a set of simulations which indicated that under any possible impedance condition the controller was able to maintain telepresence satisfactorily.

In future work we plan to examine our controller in a real world application and also to apply our method to more difficult teleoperatory problems, such as dynamically changing manipulators and delayed cases.

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A Neural-Based Method for Choosing Embedding Dimension in Chaotic Time Series Analysis

Sepideh J. Rastin and Mohammad Bagher Menhaj

Summary. This paper introduces applying a neural-based method for determining minimum embedding dimension for chaotic time series analysis. Many methods have been proposed on selecting optimal values for delay embedding parameters. Some frequently used methods are investigated and practically implemented, and then by using artificial neural networks (ANN) as one of components of the computational intelligence (CI) an approach was proposed to determine the minimum embedding dimension. This approach benefits from the multilayer feedforward neural networks ability in function approximation. The advantage of this method is that it gives a global nonlinear model for the system that can be used for many purposes such as prediction, noise reduction and control. Based on the achieved neural model an indirect algorithm for maximal Lyapunov estimation was suggested.

Key words: Neural networks, Chaos, Embedding, Time series.

1 Introduction

Analysis of time series derived from successive measurements of the underlying system is the most straightforward way to understand the nature of the underlying system. *Chaotic systems* that show extremely complex behavior and amazing structures are of the great interest for researchers because the time series data arise from such systems seem to be originated from the intrinsically random phenomena, but they come from deterministic nonlinear dynamical systems [3]. When we are encountered with a nonlinear system which behaves chaotically in some parts of its parameter space, linear data analysis fails and despite of the determinism leads to the false conclusion that the system is stochastic [1, 3]. This was a strong reason for developing some nonlinear techniques to uncover the deterministic structures. Chaotic behavior has appeared in economics, astrophysics, meteorology, biology, chemical processes and so many other real life events [1, 3].

The nonlinear time series methods studied here are based on the theory of dynamical systems which are defined by an m-dimensional map or an

m-dimensional flow in the forms presented in (1) and (2):

$$x_{n+1} = F(x_n) \tag{1}$$

$$\dot{x}(t) = f(x(t)) \tag{2}$$

Moreover, we are interested in dissipative systems for which the volume is contracted by the time evolution if the phase space is finite dimensional [2]. For such systems, a set of initial conditions of positive measure will be attracted to some invariant subspace of phase space called the *attractor* after some transient time. The set of initial conditions leading to the same nontransient behavior is referred to the *basin* of attraction [1].

Since the dynamics of such systems are defined in some phase space, it is natural to reconstruct the phase space of the investigated system from the observations taken from system's output. There are two fundamental methods for phase space reconstruction, Delay coordinates and Derivative coordinates [3]. The last is not suitable for experimental data, because Derivatives are susceptible to noise [1, 3]. Thus the Delay reconstruction is considered for practical aspects. This method has two parameters, the delay embedding and the delay time. The appropriate adjustment of these parameters is important in practice [1, 3–5, 12]. This paper discusses various conventional methods to select these parameter in an optimal manner, and then introduces a method for determining minimum embedding dimension estimation. This method utilizes the artificial neural networks (ANN) as one of the elements of computational intelligence (CI) and has called the *predictive* method. Traditional methods for embedding dimension estimation are usually exploiting from the fact that the determinism should not be violated and the invariants should not be changed due to the reconstruction process. Besides satisfying these conditions, the proposed method processes more flexibly resulting in a global nonlinear model for the underlying system.

The rest of the paper organized as follows. In Section 2 we first review the eminent features of chaotic systems, then investigate some nonlinear tools in order to distinguish chaotic time series from the others via quantifying these characteristics. To do so, we present the Lyapunov exponents in Section 3. The natural instability of a chaotic systems manifest itself in positive maximal Lyapunov exponent. A robust direct algorithm was described to measure this nonlinear statistics. Section 4 is devoted to the phase space reconstruction from the given time series and its related theorems. Sections 5 and 6 present various routines for time lag selection and embedding dimension determination for delay reconstruction. Section 7 was dedicated to neural based predictive approach and its procedure. Based on this proposed approach, an algorithm was suggested to estimate the maximal Lyapunov exponent in Section 8. This algorithm reduces the computation complexity with respect to the algorithm described in Section 3. The given methods are practically implemented and applied to the measured data of Colpitts chaotic oscillator. Simulation results are presented in Section 9. Finally Section 10 concludes the paper.

2 Characteristics of Chaotic Systems

The first key feature of chaotic systems is their determinism. Chaos theory says that the random variables are not the only possible sources of irregularity [1]. Irregularity can be seen in deterministic nonlinear dynamical systems' outputs in some part of their parameter space. Interesting attractors can occur in such deterministic systems.

The hallmark of chaos is the exponential divergence of nearby trajectories due to the instability of solutions. This property has been referred to *sensitive dependence on initial conditions*, and makes the system unpredictable in spite of the deterministic evolution. For dissipative systems exponential separation happens in the stretching directions. Other directions are so much contracted such that the dissipation condition satisfies [2].

This dynamical aspect of chaos has its corresponding side in the geometry of the attractor [1]. The nonlinearity, the dissipation and the invariance of the attractor together with the exponential divergence cause the attractor folded in the phase space and mapped to itself. This process leads to some kind of self-similarity known as *statistical* type [5]. If a piece of a strange attractor is enlarged, it will resemble itself. Due to this reason the attractors of chaotic systems have been called strange. Strange attractors show globally bounded but locally instable behavior.

The last key property is related to the power spectrum of these systems. Although their power spectra still may contain peaks, a noisy background of broadband spectrum is present [1,3]. We cannot use this feature to distinguish a noisy quasiperiodic signal from a chaotic one.

In order to verify the chaos, we can define some criteria for investigating these properties.

3 Maximal Lyapunov Exponent

According to the sensitive dependence on initial conditions in chaotic systems, an initial infinitesimal perturbation will typically grow exponentially; the averaged exponent of this growth rate is called the Lyapunov exponent that quantifies the strength of chaos [1, 15].

The number of definable Lyapunov exponents is equal to the phase space dimensions [1,15]. Such Lyapunov spectrum is denoted by $(\lambda_1, \ldots, \lambda_m)$, where subscript m denotes the phase space dimension. The maximal Lyapunov exponent, λ, is the most important element of the spectrum because it has a dominant behavior, and its positiveness is a signature of exponential divergence of nearby trajectories. The quantity λ is defined by the following equation [3]:

$$\lambda = \lim_{\Delta n \longrightarrow \infty} \lim_{\delta 0 \longrightarrow \infty} \frac{1}{\Delta n} \ln[\frac{\delta_{\Delta n}}{\delta_0}] \tag{3}$$

where δ_0 is the initial distance between two points in phase space and $\delta_{\Delta n}$ is the distance between two trajectories deriving from these points at time n. If δ_0 is finite rather than infinitesimal, $\delta_{\Delta n}$ cannot get larger than the diameter of the attractor [2].

It should be noted that the Lyapunov spectrum and thus the maximal exponent are characteristic exponents for the system because they are invariant under smooth transformations [1, 2, 8]. Oseledec (1967) studied on invariant probability measures, and proved that the Lyapunov spectrum has the property of ergodicity by means of his "Multiplicative Ergodic Theorem" [2, 16].

In accorded with the importance of maximal Lyapunov exponent, it is necessary to estimate it from a given data series. Various methods are classified in direct and indirect approaches. Here, a direct approach is presented [7]:

1. Choose a point y_{n_0} in the m-dimensional phase space.
2. Find all of its neighbors with distance smaller than ϵ.
3. Compute the average over the distances of all neighbors to the reference part of the trajectory as a function of relative time.
4. Repeat the above steps for many values of n_0.

Finally (4), has to be computed

$$S(\Delta n) = \frac{1}{N}\sum_{n_0=1}^{N} \ln\left(\frac{1}{|N_\epsilon(y_{n_0})|}\sum_{y_n \in N_\epsilon} |s_{n_0+\Delta n} - s_{n+\Delta n}|\right) \tag{4}$$

where reference points y_{n_0} are phase space vectors and N_ϵ is the neighborhood of y_{n_0} with ϵ radius. If for some ranges of Δn and for some choices of m and ϵ, the function $S(\Delta n)$ exhibits a linear increase, its slope would be an estimation of the maximal Lyapunov exponent. This method was implemented and applied to the time series of 2,000 data points obtained from observable variable x of Henon chaotic map given with (5),

$$\begin{aligned} x_{n+1} &= 1 - a{x_n}^2 + y_n \\ y_{n+1} &= bx_n \end{aligned} \tag{5}$$

for values $a = 1.4, b = 0.3$ the system is chaotic [2]. We computed (4) for different values of ϵ and m. The results for $\epsilon = 0.2$ and $m = 1, \ldots, 5$ are plotted in Fig. 1. The parameter ϵ was choose so that the 500 reference points have enough neighbors and the distances do not saturate for small Δn. Using linear regression for the linear parts of the curves, we can determine λ with a tolerence of .01 as $\lambda = 0.41 \pm 0.01$.

4 Phase Space Reconstruction, Embedding Theorems

Since we do not confront with a phase space object but a time series, we have to convert it into state vectors such that the invariant characteristics of the

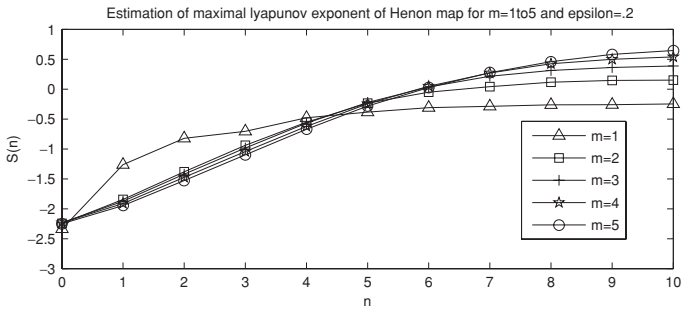

Fig. 1. Maximal Lyapunov exponent estimation for the time series of 2,000 data points obtained from observable variable x of Henon map given with (5) and choosing $\epsilon = 0.2$ and $m = 1, \ldots, 5$

original unknown attractors are preserved. Thus, an embedding of a compact smooth manifold A into R^m should be a map G that is a one to one and immersion on A [1].

As we mentioned before one of the reconstruction method is the delay coordinates established upon the Takens' *delay embedding theorem* [1, 3, 8, 9]. Let us denote the measurement function with s. A sequence of scalar measurements taken at multiples of a fixed sampling time can be shown as:

$$s_n = s(x(n\Delta t)) \tag{6}$$

then the delay reconstruction is formed by the vectors y_n,

$$y_n = (s_{n-(m-1)k}, ..., s_n)) \tag{7}$$

In the above, $k\Delta t$ is referred to the lag or delay time denoted by τ and m is the dimension of delay reconstruction. Takens proved that for an infinite noise free data series, a delay map of dimension $m \geq 2D + 1$ is an embedding of a D-dimensional compact manifold, i.e., it is a deffeomorphism [1, 9]. This theorem was generalized by Saur et al. called the *fractal delay embedding prevalence theorem*. They replaced the condition $m \geq 2D + 1$ with $m \geq 2D_f + 1$, where D_f denotes the Capacity (Box Counting Dimension) of the attractor [11]. Moreover, it has been shown that an embedding dimension $m > D_f$ suffices [1, 8, 10].

The delay reconstruction is consisted of two parameters adjustment: the embedding dimension m and the lag τ. It is easy to show that in reality with a finite number of noisy data, the estimates of the invariants depend on both m and τ [3]. Therefore, their optimal selection is of practical importance.

5 Choosing the Delay Time

Except for the fact that in *fractal delay embedding prevalence theorem* certain values for Δt and τ are not allowed to be chosen, these values are not the topic

of embedding theorem under its ideal conditions [1, 3]. Different τs result in diffeomorphically equivalent attractors. If τ is selected small compared to the time scales of the system, components of the delay vectors are strongly correlated. In such cases, all reconstructed vectors are collected around the bisectrix of R^m, unless m is very large [1, 12]. This situation gets better when τ is increased. In these cases, the attractor unfolds and its structure becomes visible on larger scales. If τ is increased to very large amounts, the successive elements get independent which may lead to self-intersection in reconstructed trajectories [8]. The most conventional method is the minimization of the redundancy of the coordinates of the reconstructed space [8]. To do so, we can choose the time at which the autocorrelation function reaches $1/e = 1/2.7183$ as the lag time. Since the autocorrelation is a linear statistical quantity, it is more sophisticated to choose the time corresponding to the first minimum of the mutual information function as the delay time. The mutual information for time delay τ is:

$$I(\tau) = \sum_{i,j} p_{ij} \ln p_{ij}(\tau) - 2\sum_{i} p_i \ln p_i \tag{8}$$

where p_i is the probability to find a time series in the ith bin of the histogram created for the probability distributions of the data p_{ij} is the joint probability. Note that there is no assurance that $I(\tau)$ has an apparent minimum [1, 3].

6 Embedding Dimension Estimation

Embedding theorem says that the choice of m needs a priori knowledge of D_f of the original attractor which is unrealistic for experimental data [1, 3, 10]. Thus several methods have been proposed on embedding dimension estimation [3, 4, 13]. The main classical method can be classified into three types [4].

The first method is the computation of some invariant quantity like the maximal Lyapunov exponent while increasing the parameter m from low values to high values. When the estimated value for the invariant stops changing, the adequate m is achieved. This method is very data intensive and time consuming [4].

The second method is the singular value decomposition based approach. This method is very subjective and also the resultant reconstruction is not always optimal [3].

The last conventional method is a geometrical approach based on finding false nearest neighbors [4, 6]. As m increases in the reconstruction of a data series, the attractor unfolds and when it gets completely unfolded, a trajectory will never cross itself. The method of false nearest neighbors (FNN) recognizes that where the trajectory has some self-intersections, two neighboring points actually will be far away in the true embedding space. Based on this approach, Kennel (1992) was proposed an algorithm to determine the minimum m [6]. This algorithm was subjective in determining whether a neighbor is false.

To avoid this problem, Cao (1997) introduced a modified version of the Kennel algorithm that has been presented below [4]:
Let

$$E1(m) = \frac{E(m+1)}{E(m)} \tag{9}$$

with

$$E(m) = \frac{1}{N - m\tau} \sum_{t=0}^{N-m\tau-1} \frac{\|y_{m+1}(t) - y_{m+1}^{NN}(t)\|}{\|y_m(t) - y_m^{NN}(t)\|} \tag{10}$$

and

$$\|y_m(t) - y_m^{NN}(t)\| = \max_{0 \le j \le m-1} |s(t + j\tau) - s^{NN}(t + j\tau)| \tag{11}$$

where N is the length of the data series and m, τ denote the embedding dimension and the lag, respectively. The superscript NN means the nearest neighbor to the other vector as defined by the metric of (11). The optimal embedding dimension is given by the value of m where $E1(m)$ stops changing. Cao also proposed a related method to distinguish deterministic signals from the stochastic ones for practical conditions. He defined

$$E2(m) = \frac{E^*(m+1)}{E^*(m)} \tag{12}$$

where

$$E^*(m) = \frac{1}{N - m\tau} \sum_{t=0}^{N-m\tau-1} |s(t + m\tau) - s^{NN}(t + m\tau)| \tag{13}$$

for random data, E2(m) will be equal to one for any m. However, for deterministic data the values of E2 (m) will not equal to 1 for any m.

7 Predictive Method for Minimum Embedding Dimension Estimation

In this section, we propose a neuro based method for minimum embedding dimension estimation. This method benefits from the multilayer feedforward neural networks ability in function approximation. It has been shown that a three layered feedforward net with sigmoid functions in hidden layer and a linear function in output layer is able to approximate all of the squared integrable function with any approximation order, provided that there are enough number of neurons in the hidden layer. This fact has been called the *universal function approximation theorem* [14]. However, this theorem has some practical limitations.

For using this method, assume that the given system can be observed through the measurement function

$$y = h(\underline{x}), \underline{x} \in R^k \tag{14}$$

where R^k denotes the original phase space. Let

$$F_m : R \longrightarrow R^M \tag{15}$$

where F_m is the delay map given below:

$$\underline{Y}_m(n) = F_m(\underline{x}) = [y(n), y(n-\tau), ..., y(n-(m-1)\tau)]^T \tag{16}$$

Furthermore, the time evolution of the dynamics of the underlying system can be described by a deterministic map like F

$$\underline{x}(n) = F(\underline{x}(n-\tau)) \tag{17}$$

We want to find m_E such that

$$\underline{Y}_{m_E}(n) \approx \underline{x}(n) \tag{18}$$

This implies that the reconstruction attractor approximates the original one such that the time evolution from $\underline{Y}_{m_E}(n)$ to $\underline{Y}_{m_E}(n+1)$ follows the time evolution from $\underline{x}(n)$ to $\underline{x}(n+1)$ in original attraction. From (16) we have

$$\underline{x}(n) = F_m^{-1}(\underline{Y}_m(n)) \tag{19}$$

Besides, we have

$$\begin{aligned} y(n) &= h(\underline{x}(n)) \\ &= h \circ F(\underline{x}(n-\tau) \\ &= g(\underline{x}(n-\tau)) \end{aligned} \tag{20}$$

Combination of (19) and (20) yields

$$\begin{aligned} y(n) &= g \circ F_m^{-1}(\underline{Y}_m(n-\tau)) \\ &= q(\underline{Y}_m(n-\tau)) \\ &= q([y(n-\tau), \ldots, y(n-m)\tau)]) \end{aligned} \tag{21}$$

therefore $y(n)$ can be approximated in the form of:

$$y(n) = \hat{q}(\underline{Y}_m(n-\tau)) \tag{22}$$

Function $\hat{q}$ as an approximation of q can be obtained using a feedforward net with *error backpropagation* (BP) training algorithm. To do so, the net architecture is made of the input layer, the hidden layer and the output layer. The input layer consists of m units, and the elements of delay vectors are distributed to the neurons. In order to determine $m = m_E$:

1. Start from $m = 1$.
2. Train the net and apply the test set to the trained net to obtain the $\hat{y}(n)$. Compare $\hat{y}(n)$ with $y(n)$. Compute the prediction error, $e = \hat{y} - y$.
3. Put $m = m + 1$ and compute e again.
4. The routine will be finished when root mean squared prediction error, $e(rms)$, has no remarkable changes as m.

The value of m for which the $e(rms)$ begins to be constant is equal to the minimum embedding dimension m_E. This approach is illustrated in Fig. 2.

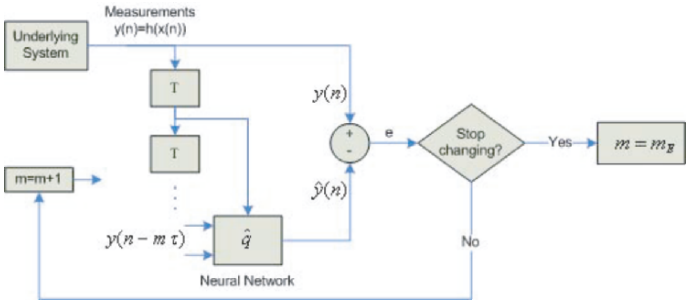

Fig. 2. The predictive approach for minimum embedding dimension estimation

8 Indirect Method for Maximal Lyapunov Estimation

The predictive method results in determination of m_E and also gives a neural model. Based on this model, an indirect algorithm for maximal Lyapunov exponent estimation is suggested. This approach is summerized below:

1. Select an arbitrary vector $\underline{Y}_m(n)$ in the delay reconstructed space. Then, use the neural model to obtain $y(n+\tau)$.
2. Compute $\tilde{\underline{Y}}_m(n) = \underline{Y}_m(n) + \underline{\epsilon}_0$, where ϵ_0 is a perturbation vector in the form of $(\epsilon_0, 0, \cdots, 0)$ with a small ϵ_0 tending to zero. Then, apply the neural model to compute the $\tilde{y}(n+\tau)$.
3. Compute $S(\Delta n\tau)$ given by (23). Plot the graph $S(\Delta n\tau)$ versus $\Delta n\tau$ and compute its slope which gives an estimation of the quantity λ.

$$S(\Delta n\tau) = \frac{1}{N}\sum_{n_0=1}^{N-m\tau}\{ln(|\tilde{y}_m(n+\Delta n\tau) - y_m(n+\Delta n\tau)|) \tag{23}$$

This approach requires less computations than that of the direct one, because there is no need of neighbor searching.

9 Simulation Results

The methods presented for choosing optimal values for m and τ were practically implemented and applied to the experimental data derived from the Colpitts chaotic oscillator [17]. A schematic of Colpitts circuit is given in Fig. 3. The given time series was made of 6,000 measured points. As mentioned before, mutual information is a powerful technique for selecting τ, the mutual information function was computed for lag from 0 to 32 units. As Fig. 4 shows, this function has an obvious minimum. Thus, we choose the corresponding time of this minimum for $\tau = 4$.

The next step is to determine m. We first computed the maximal Lyapunov exponent by the presented direct method for m = 1 to m = 6. The results are

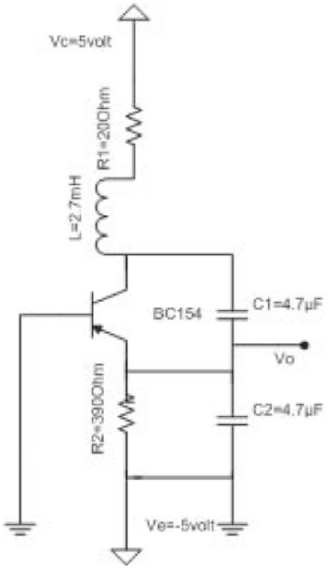

Fig. 3. The Colpitts oscillator

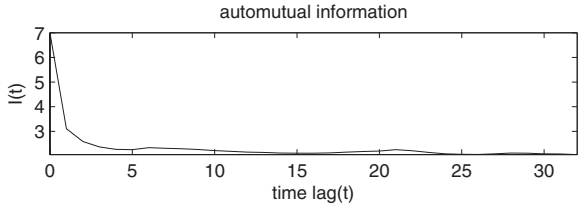

Fig. 4. Computed mutual information function for Colpitts time series

exhibited in Fig. 5. The estimated λ is independent of m for $m \geq 3$. The violations of linear growth in small scales may be due to the measurement noise or due to the lack of neighboring points. Then, we applied the Cao geometrical algorithm to the data series. The maximum value for m was set to 8. Figure 6 shows the resultant E1(m) and E2(m). Form the figure, it can be inferred that $m = 3$ will give an appropriate selection for embedding dimension. Moreover, one can figure out that the given time series is not stochastic.

As the last approach, the predictive method was implemented. A feedforward net was composed of three layers. Having tested different numbers for neurons, we put 8 units in the hidden layer. The given time series was divided into the training and test sets. We applied 5,500 data for training and 500 data for testing. The BP learning algorithm with the mean of squared errors as the index function was used for the training procedure. The value of index

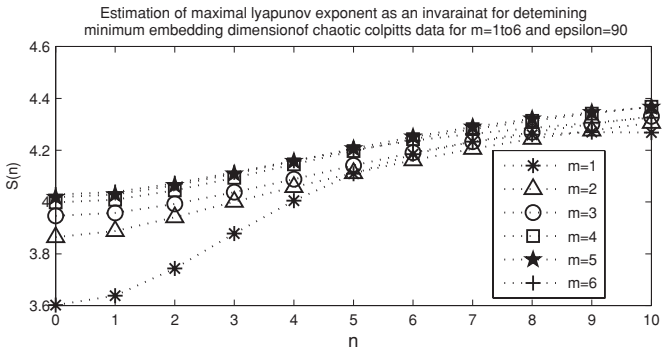

Fig. 5. Computing the maximal Lyapunov exponent as an invariant for estimating m

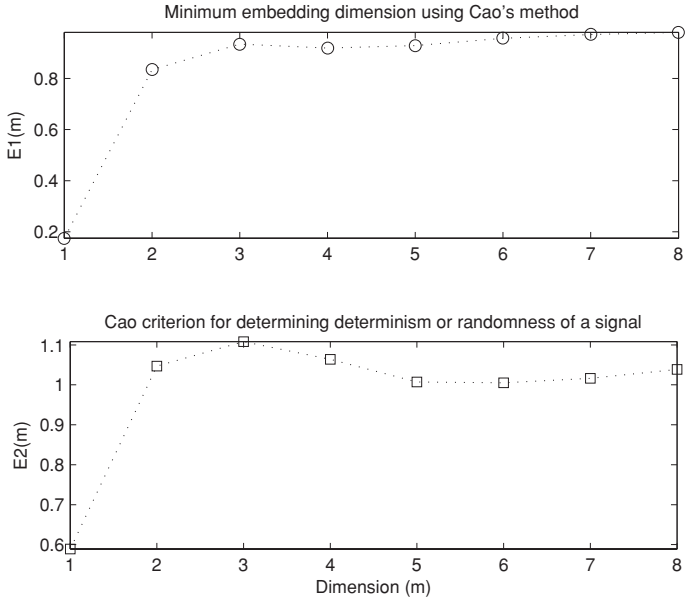

Fig. 6. Applying Cao geometrical approach for choosing m with $\tau = 4$. (**a**) E1(m) (**b**) E2(m)

function to stop the iteration was set to 10^{-8}. The learning curves of the utilized nets with m-inputs is presented in Fig. 7.

As we can see, in cases $m = 1, 2$ the learning curves did not reach to the desired value, but by increasing m to $m \geq 3$. The curves have come to the desired predetermined value. Besides, from Fig. 8 it is obvious that for $m \geq 3$ the prediction error has not remarkable changes. Thus we choose $m = 3$ again. The reconstructed attractor with $m = 3$ and $\tau = 4$ is shown in Fig. 9.

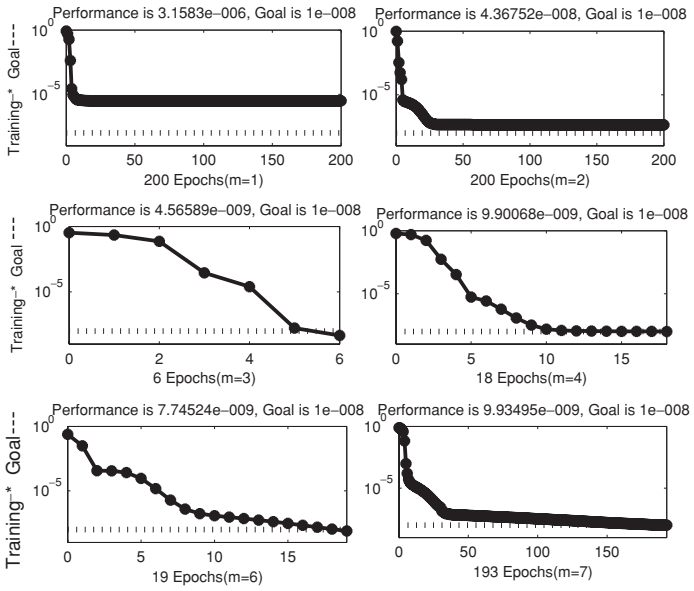

Fig. 7. Learning curves corresponding to m-inputs neural nets

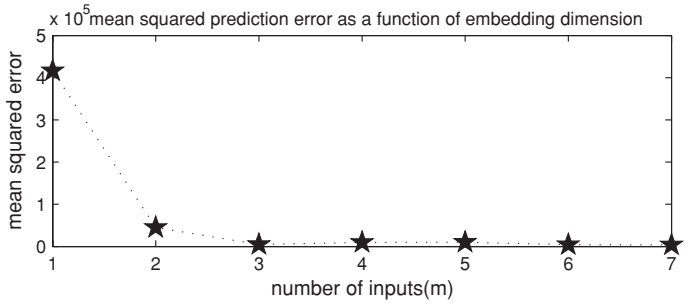

Fig. 8. Mean of squared prediction error of each used neural network as a function of m

10 Conclusion

Characteristics of chaotic time series were first studied. In order to investigate these properties, some nonlinear analysis tools were reviewed. Since the reconstruction of phase space is the basis of all nonlinear time series analysis, delay reconstruction and its parameter adjustments were studied. In addition to classical method for choosing these parameters a neuro based method known as predictive was presented. All methods were practically implemented and applied to the experimental data of Colpitts chaotic oscillator. Among methods described in the paper, the Cao geometrical approach and our suggested method were very promising. Cao algorithm is better than the other

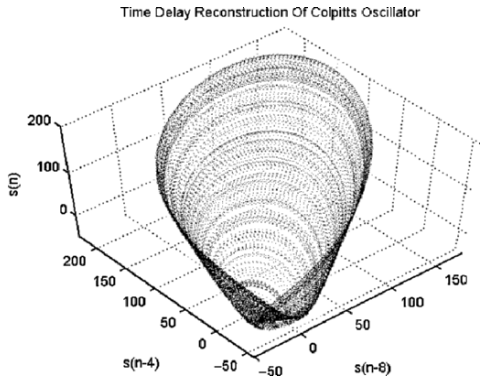

Fig. 9. The reconstructed attractor with $m = 3$ and $\tau = 4$

traditional methods, because it is not subjective and data intensive. The presented method gives a global nonlinear model for underlying system. This model can be used for many purposes such as prediction, noise reduction and control. Besides, based on the achieved neural model an indirect algorithm for maximal Lyapunov exponent estimation was suggested. This algorithm reduces remarkably the computational complexity.

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On Classification of Some Hopfield-Type Learning Rules via Stability Measures

Mohammad Reza Rajati, Mohammad Bagher Menhaj

Summary. This paper first reviews several learning methods for training Hopfield-type associative memories as well as a novel architecture with neurons of nonmonotonic stimulus functions. These learning rules are classified into three groups according to a measure of stability closely related to the storage capacity. This measure helps us better study the ability of a network to store patterns as stable states of its dynamics in case it is highly loaded. We then analyze the experimental data related to the stability measure and classify the previously studied learning methods according to the measure. We also show that the behavior of those learning rules converges to either the behavior of Hebbian learning or that of the pseudo-inverse method.

Key words: Hopfield-type neural networks, Learning rules, Stability measures, Storage capacity, Equilibrium points.

1 Introduction

Hopfield-type neural networks are shown to be amenable to thorough analysis. They have simple synthesis procedures and interesting aspects for scientific investigations including those related to the content addressability [3], storage capacity [8], robustness against noise and adaptability to the neurons' malfunction [10]. Fundamentally, networks of Hopfield-type suffer from poor capacity and performance. Many architectures are innovated to remedy the restrictions of the Hopfield associative memory, including those trying to modify the connections and updating schemes (for example [12]) and those employing new learning rules [2].

This paper tends to classify some architectures and learning rules according to a stability condition related to the storage capacity of the network. Of course, there are many other capacity and performance indices, which are open to examine for Hopfield-like architectures, and we leave them to the forthcoming articles. We first review the conventional Hopfield network model. Then we discuss the capacity issues in recurrent associative memories. Some learning methods for Hopfield networks are discussed in Sect. 3 and finally we classify them according to a stability measure by experimental analysis.

2 The Hopfield Model and Its Storage Capacity

The Hopfield network is a recurrent neural network governed by the difference equation:

$$\begin{aligned} &a_i(0) = p_i,\ n_i(t+1) = \sum_j^N w_{ij} a_j(t) + b_i,\ a_i(t+1) = f(n_i(t+1)) \\ &f(u) = \operatorname{sgn}(u) \end{aligned} \tag{1}$$

where p_i is the unknown pattern to be recognized, a_i is the output of the network, w_{ij} is the connection weight between neurons i, j, b_i is the threshold term and the stimulus function is $f(u)$. This is a discrete version of the Hopfield model. The connection weight matrix should be calculated to store the required prototype patterns as fixed states of the network dynamics so that patterns can be recalled from noisy or incomplete initial inputs.

2.1 Storage Capacity

The storage capacity of the Hopfield-like associative memories is of great consideration in the neurocomputing literature. It is formulated by either big O notation in terms of the number of neurons (M_c), or the relative capacity α_c defined as $\alpha_c = L/N$, where L is the number of patterns stored and N is the number of neurons.

Although one can place any load upon a neural system, there is obviously a value for α above which some of the vectors in the training set will not be stored as stable states. We refer to this as the maximum permissible loading (or just loading) and denote it by $\alpha_{\max}$. Both of the aforementioned formulations are deeply discussed and it has been shown that for randomly realized unbiased binary patterns, $\alpha_{\max} \approx 0.14$ and $M_c = N/\log N$ [8].

3 Some Learning Methods and Architectures

There are several methods to obtain a weight matrix with higher performance for a recurrent associative memory. We consider here, Hebbian, pseudo-inverse, Menhaj–Seifipour, and Li–Michel learning rules. Besides, we examine the architecture proposed by Yanai and Amari [12] which uses nonmonotonous stimulus functions for the neurons.

3.1 Hebbian Learning

This is the conventional learning rule for recurrent associative memories with:

$$w_{ij} = \sum_{l=1}^{L} p_i^l p_j^l \tag{2}$$

3.2 The Pseudo-Inverse Method

The Pseudo-inverse rule is introduced by Personnaz et al. [11] and studied deeply by Yen and Michel [13] to generate the weight matrix according to the rule:

$$W = PP^{+} \tag{3}$$

where P is the matrix whose columns are the $p^{l}$and P^{+} is its pseudo-inverse, the matrix with the property that: $P^{+}P = I$. It is notable that many modified methods such as Perceptron-style methods are approximate versions of the pseudo-inverse method [3–5].

3.3 Menhaj–Seifipour Algorithm

Menhaj and Seifipour propose a new algorithm, and state that it has a better storage capacity and a higher speed of convergence [9]. They build the memory matrix by:

$$r_{ij} = \frac{1}{2^{L}} \prod_{l=1}^{L} (p_i^l + p_j^l) \tag{4}$$

$$W = R^{T}R \tag{5}$$

This network is proved to minimize the energy function:

$$E(t) = -\sum_{i}\sum_{j} w_{ij}[a_i(t) + a_j(t)]^2 - 4\sum_{i} b_i(t)a_i(t) \tag{6}$$

Additionally, the matrix built by the above rule is sparse and results in a higher decrease of the energy function in each time-step than the classical Hopfield network with Hebbian learning. The architecture of sparsely connected networks is studied by Liu and Michel and the sparse nature of the networks is proved to be beneficial [7].

3.4 Li–Michel Learning Rule

Li–Michel synthesis method relies on a vigorous mathematical foundation, i.e., analysis of linear systems operating on a hypercube [6]. For the sake of clarity, we just present a brief algorithm, without thorough theoretical considerations.

To store L prototype patterns in a Hopfield-type memory as asymptotically stable equilibrium points, let:

$$X = [x_1, x_2, \ldots, x_{L-1}],\ x_i = p^i - p^L, i = 1, 2, 3, \ldots, L-1 \tag{7}$$

Then obtain a Singular Value Decomposition of X:

$$X = USV^{T}, U = [u_1, u_2, \ldots, u_N] \tag{8}$$

Suppose that k is the rank of S, then k is the dimension of the space spanned by x_i's. Then the weight matrix and the bias vector are obtained by:

$$W^+ = \sum_{j=1}^{k} u_j u_j^T \quad W^- = \sum_{i=1}^{N} u_i u_i^T \tag{9}$$

$$W^+ = \alpha W^+ - \beta W^- b = \alpha p^L - W p^L \tag{10}$$

in which α, β are properly selected constants which satisfy: $\alpha > 1, \beta < 1$.

This algorithm guarantees the system to have at most 3^Nequilibrium points. Additionally, at most $2^N$of the equilibrium points are asymptotically stable.

3.5 Yanai–Amari Architecture

Yanai and Amari [12] propose an associative memory with two stage nonlinear dynamics:

$$a(k+1) = \text{sgn}[W(a - f(Wa))] = \text{sgn}[Wa - Wf(Wa)] \tag{11}$$

They use a nonmonotonic function for their network:

$$f(u) = \begin{cases} a(u+h) - c & u < -h \\ 0 & -h \leq u \leq h \\ a(u-h) + c & u > h \end{cases} \tag{12}$$

where h and c are non-negative. We used $a = 0.4,\ h = 0.1,\ c = 0$ in this paper.

4 Empirical Analysis of Storage Capacity

To obtain the relative capacity of the models examined, we trained Hopfield-type networks of 100 neurons, with a set of random unbiased prototype patterns. Loading was increased and the response of the network to an erratic version of one of the stored patterns (with a Hamming distance of 10) is evaluated, and the normalized overlap $(1 - H_d/N)$ of the response and the stored pattern is illustrated in Fig. 1. The absolute capacity is usually defined as the maximum loading in which the network can recall a pattern more than 90% (and sometimes exactly 100%) perfectly.

It is obvious from Fig. 1 that $\alpha_{\max}$ is 0.15, 0.35, 0.25, and 0.05 for Hebbian, pseudo-inverse, Li–Michel and Menhaj–Seifipour learning rules, respectively. It is also notable that in high loadings, Hebbian learning is able to recall the patterns with a 70% overlap, but other rules recall the patterns with more than 94% overlap, although loaded highly. Yanai and Amari reported $\alpha = 0.3$ for their architecture [12].

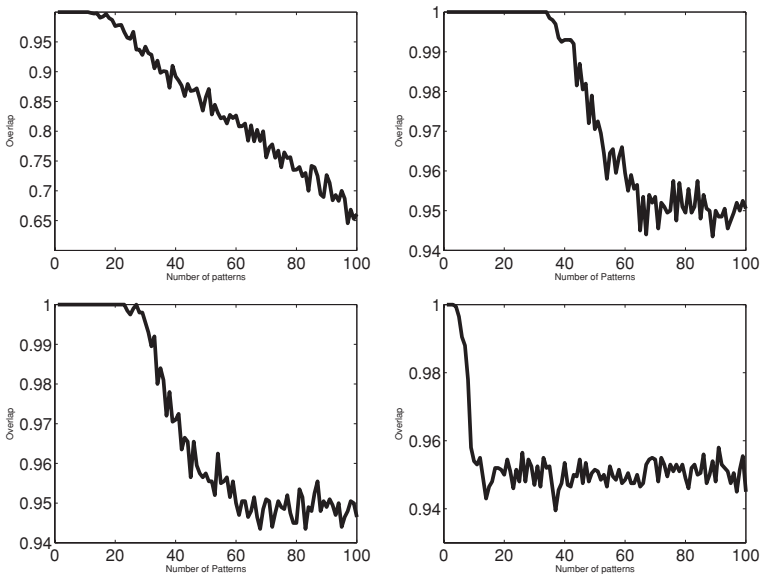

Fig. 1. Pattern recall by (**a**) Hebbian learning (**b**) pseudo–inverse rule (**c**) Li–Michel rule (**d**) Menhaj–Seifipour training method. Each network has 100 neurons and is trained by random patterns. Every point on the plots is generated by averaging over 20 runs. N = 100

5 Classification of Hopfield Memories via a Stability Measure

Abbott classified all Hopfield models into three groups. Any member of each group may have a different behavior when the loading upon it is not near $\alpha_{\max}$, but all members loaded near $\alpha_{\max}$ have the same behavior [1].

From the dynamic equations of the network, it can be seen that a state a will be stable if n_i has the same sign as a_i for all i. So, the parameter $n_i a_i$ should be non-negative for all i in order for the network to have a pattern p as its stable equilibrium point. Furthermore, assume a network with a set of stable states. The weight matrix could be scaled by any positive number, and thereby the synaptic signals will increase (and obviously the $n_i a_i$'s) but the domains of attraction of the stable states will not get wider. Thus, the following stability measure is defined to characterize the nature of stable states:

$$\gamma_i^l = \frac{n_i^l p_i^l}{\|W_i\|} \quad \|W_i\| = \sqrt{\sum_{j=1}^{N} (w_{ij})^2} \tag{13}$$

Considering the worst case analysis, the minimum value of γ_i^ls is a parameter for identification of the network's basins of attraction [5].

The global groups of recurrent networks are different in the distribution of their γ values. The first group, known as Hopfield group, has a normal distribution with a mean of $1/\sqrt{\alpha}, \alpha < 0.15$. In this group of models, negative values of γ could be present in the network, and this is a sign of the existence of unstable patterns (Hopfield network with Hebbian learning is within this group of models).

The second group has matrices of pseudo-inverse type. The γ values theoretically converge to the same value $\gamma_0 = \sqrt{(1-\alpha)/\alpha}$. So we suppose a notch distribution of γ values in our numerical results. The third group has a clipped normal distribution, with positive γ values [4].

6 Classification of the Models via Experimental Analysis

In this section, we analyze the γ distributions of different algorithms in the paper. Figure 2 depicts different γ distributions for the learning rules. The distributions are plotted by training networks consisting 1,000 neurons with a set of 500 unbiased random bipolar patterns.

It is easily observed that Hebbian learning causes a normal distribution of γ's. It is notable that many values of γ are negative, so in high loadings, some of the patterns will not be stored as stable states.

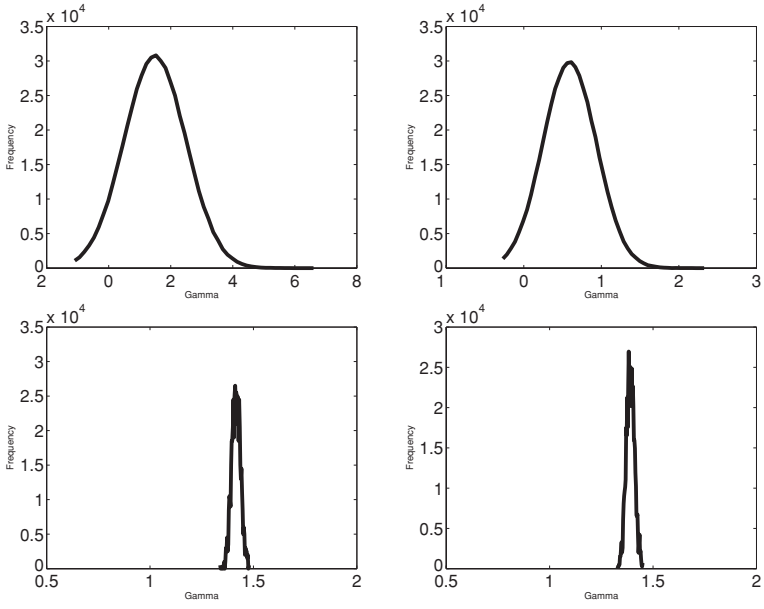

Fig. 2. Gamma distributions of (**a**) Hebbian learning (**b**) network with non-monotonic neurons (**c**) pseudo-inverse rule (**d**) Li–Michel learning rule. Each network is trained by 500 patterns and has 1,000 neurons

The pseudo-inverse rule, as the canonical model of the second group, is tested by γ values as well and a notch distribution is resulted. Li–Michel learning rule results in a very notch distribution with no negative values. Thereby, it could be concluded that this rule falls into the pseudo-inverse class of models. It may be a cause of its high performance (this procedure was performed many times and γ never became negative).

Yanai and Amari state that their proposed model is an approximation of the pseudo-inverse rule. The γ distribution of this rule has negative values, so it could not be classified into the pseudo-inverse group, although the γ values are not distributed very widely, and it is less likely that the γ's be negative (in some cases, the authors encountered distributions without negative values). The architecture is therefore classified into the first group.

About Menhaj–Seifipour learning rule, we inspected the convergence properties of the network's γ distribution (Fig. 3). We plotted the γ distributions in different loadings. In high loadings the γ values converge to the same amount of $\gamma_0 = 1$ and this fact helps us figure out that this rule could be classified in the pseudo-inverse group of models. When the number of stored patterns becomes large even though the loading is low, the matrix constructed by the

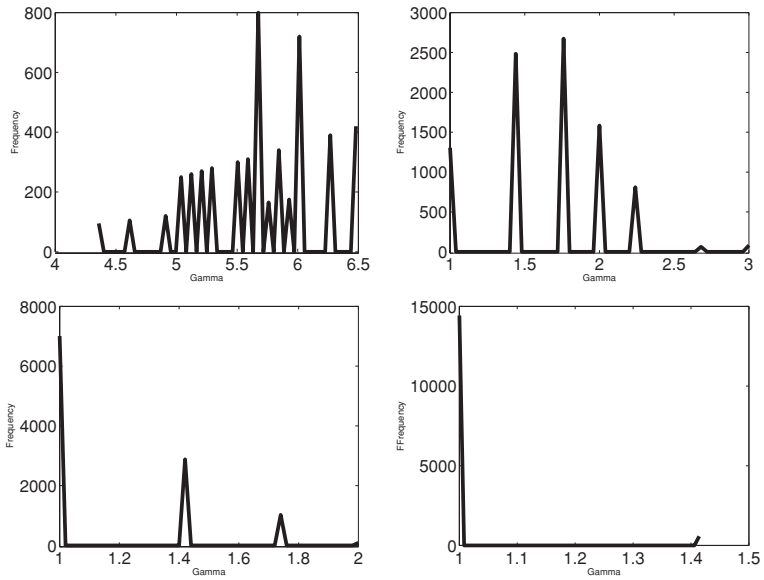

Fig. 3. Gamma distributions of Menhaj rule with (**a**) L = 5 (**b**) L = 9 the network is not loaded very highly and the distributions behavior is not similar to that in the saturation. With (**c**) L = 11 (**d**) L = 15 the network's distributions converge to a notch distribution. The notch distribution is reached when the number of patterns grows and it does not related to the number of neurons, because the resulting weight matrix converges to the identity matrix. For biased data this convergence will be slower. Here the number of neurons is N = 1,000

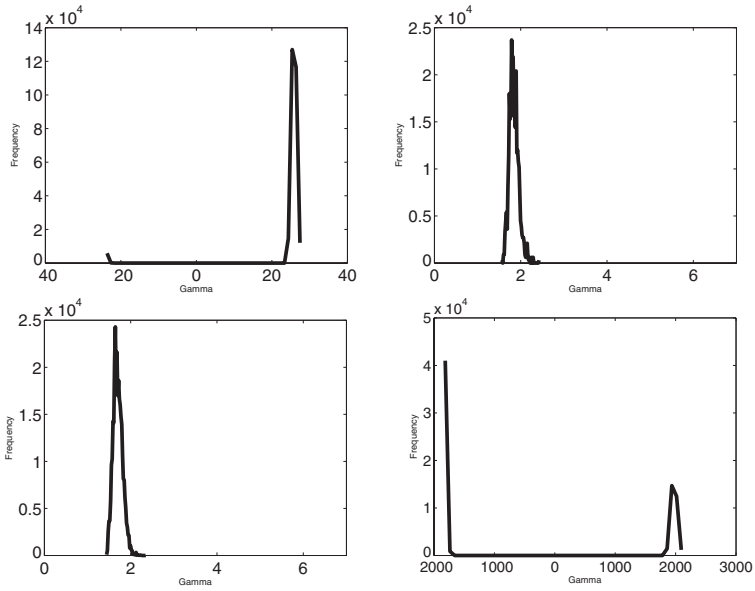

Fig. 4. Gamma distributions for biased patterns. (**a**) Hebbian (**b**) pseudo-inverse (**c**) Michel (**d**) Yanai–Amari N = 1000, L = 300

rule will converge to the identity matrix and all γ values become the same, which is similar to the theoretically derived distribution for the pseudo-inverse group of models.

We also analyzed the effect of bias on patterns in γ distributions. We trained the networks with a set of random patterns in which the probability of presence of 1 is 90%, in contrast to the case of unbiased patterns, in which this probability is 50% (Fig. 4). It is obvious that Hebbian learning and Yanai model are not tolerant to biased data, in contrast to Michel and pseudo-inverse rules which maintain their notch distribution.

7 Conclusion

In this article, we classified different algorithms of learning via stability measures proposed by Abbott. We showed that Li–Michel and Seifipour–Menhaj rules' behaviors converge to that of the pseudo-inverse method. The behavior of the Yanai–Amari network model converges to that of the Hopfield model, although because of its nature, it has a similarity with the pseudo-inverse group of models. We leave deeper discussions about the information theoretical aspects of Menhaj–Seifipour rule, finding the shape and size of the attraction basins of the discussed rules and the convergence properties of the networks trained by them to future contributions.

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A New Genetic Based Algorithm for Channel Assignment Problems

Seyed Alireza Ghasempour Shirazi and Mohammad Bagher Menhaj

Summary. In the channel assignment problem, frequencies are assigned to the requested calls in a cellular mobile network subject to cochannels, adjacent channels and cosite constraints such that a required bandwidth is minimized. In this paper, a new method based on genetic algorithm is proposed to solve these problems. The performance of the proposed method is evaluated by solving three channel assignment problems. Results show that this method can find solutions with a minimum required bandwidth in comparison with the other algorithms investigated in the paper.

Key words: Genetic algorithm, Channel assignment problem.

1 Introduction

By the emerging of cellular mobile systems and their rapid growth due to the portability and the availability of these systems provided an important alternative in the field of wireless mobile communications. The increasing demand of new services in this field, however, is in contrast to the capacity constraints inherent in the current communication systems. Hence, the use of techniques, which are capable of ensuring that the frequency spectrum assigned for use in mobile communications will be better utilized, is gaining an ever-increasing importance.

The channel assignment problem (CAP) in this paper is based on a common model. The service area of the system is divided into a number of hexagonal cells. Every user is located in one cell. When a user requests a call in this system, a channel is assigned to that user to provide the communication service. This channel must satisfy the electromagnetic compatibility (EMC) constraints to avoid the radio interference between channels. Three types of EMC constraints that are considered in this paper are cochannel, adjacent and cosite constraints.

In the simplest form of the CAP, the cochannel constraint only is considered, and the problem is known to be equivalent to a graph-coloring problem [1]. Since the graph-coloring problem is known to be nondeterministic polynomial-complete (NP-complete) [2], therefore, the CAP is also

NP-complete. Therefore, the calculation time and the computation complexity of searching for the optimum solution in the CAP grow exponentially with the problem size.

The rest of the paper is organized as follows. In Sect. 2, we formulate the CAP. In Sect. 3, we briefly discuss conventional genetic algorithm. In Sect. 4, we introduce a new method based on genetic algorithm. In Sect. 5, we assess the quality of proposed method by using it to solve three CAPs and then compare the results with two of the existing channel assignment algorithms. Finally, we conclude our work in Sect. 6.

2 Channel Assignment Problem Formulation

CAP in this paper follows the problem formulation by Gamst and Rave [3]. In 1982, Gamst and Rave [3] defined the general form of the CAP in an arbitrary inhomogeneous cellular radio network. In their definition, the EMC constraints in an n-cell network are described by a $n \times n$ symmetric matrix which is called constraint matrix C [3]. Each nondiagonal element c_{ij} in C represents the minimum separation distance between a frequency assigned to a call in cell $\#i$ and a frequency assigned to call in cell $\#j$. The cochannel constraint is represented by $c_{ij} = 1$, and the adjacent channel constraint is represented by $c_{ij} = 2$. $c_{ij} = 0$ indicates that calls in cell $\#i$ and cell $\#j$ are allowed to use the same frequency. Each diagonal element c_{ii} in C represents the minimum separation distance between any two frequencies assigned to calls in cell $\#i$, which is called cosite constraint, where $c_{ii} \geq 1$ is always satisfied.

The number of required frequencies for each cell in an n-cell network are described by an n-element vector, which is called demand vector D. Each element d_i in D represents the number of frequencies that must be assigned to calls in cell $\#i$. If f_{ik} denotes kth frequency in cell $\#i$ that is assigned to a_{ik}, the kth call in cell $\#i$, then the EMC constraints are represented by:

$$\begin{gathered} |f_{ik} - f_{j\ell}| \geq c_{ij}, \quad i,j = 1,\cdots,n, \quad k = 1,\cdots,d_i, \quad \ell = 1,\cdots,d_j, \\ if \quad i = j \Rightarrow k \neq \ell, \quad if \quad k = \ell \Rightarrow i \neq j. \end{gathered} \tag{1}$$

Each f_{ik} is represented by a positive integer. The CAP is to assign a set of frequencies f_{ik} to the set of calls a_{ik} such that the bandwidth required by the system, i.e., max f_{ik}, is minimized, subject to EMC constraints.

In addition to the constraint matrix C and the demand vector D, we consider another important parameter called lower bound (lb) in the formulation of CAP. Parameter lb determines minimum value of the maximum f_{ik} for all i and k, so that no interference is caused (i.e. $lb = \min\{\max_{i,k}\{f_{ik}\}\}$). This means if f_{ik}s can take values between 1 to lb, the values of f_{ik}s will not violate any constraints and a conflict-free channel assignment will be obtained. In fact, lb indicates the minimum required bandwidth for the CAP and if any smaller bandwidth is used, interference will be unavoidable and some constraints will be violated.

3 Conventional Genetic Algorithm

Genetic algorithm has been proposed by Holland [4,5]. He introduced a novel optimization algorithm that is in-depth different from the two major classes of classical calculus-based and enumerative techniques [6]. In solving a given optimization task, the GA starts with a collection of solutions (i.e. parameter estimates) called by chromosomes. Each individual (chromosome) is evaluated for its fitness. In each iteration of the GA, the fittest chromosomes (parents) are allowed to mate and bear offspring (produce new individuals). These individuals (children) or new parameter estimates provide the basis for the next generation. The conventional GA may be completely described by the following steps [7]:

1. Initialization
2. Generate a random population
3. Apply the selected crossover operator to the individuals
4. Apply the selected mutation operator to the individuals
5. Replace the old population with the resulting individuals
6. Repeat steps 3–5 until the termination criterion is satisfied

4 Our New Method Based on Genetic Algorithm

Our new method that is based on genetic algorithm is inspired by recombination accomplished by some insects such as bees [8]. To make it more clearly, we proceed as follows. Consider a conventional GA in which the recombination is done in a way that one chromosome is recombined with the best chromosome that exists in the present population. In other words, all selected chromosomes are recombined with the best chromosome of the current population called as the queen chromosome. The recombination procedure can be better observed from Fig. 1.

With a deep look at Fig. 1, it becomes clear that the produced chromosomes are indeed the queen with some changes. In other words, the new chromosomes may represent new queens who inherit some parent queen characteristics. Therefore, instead of employing the procedure given in Fig. 1, we use recombination procedure as shown in Fig. 2. This represents the core of our method.

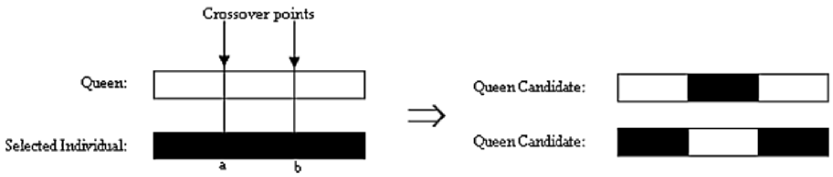

Fig. 1. Recombination procedure by applying crossover operator at crossover points a and b

Fig. 2. Recombination procedure by applying mutation operator

As observed in Fig. 2, some portions of the queen are randomly mutated to generate a new chromosome. This results in new queen candidates. Each queen candidate becomes a queen and replaces the original queen if it has a better performance with respect to the previous queen. If the performance of the queen candidate gets worse than the current queen's performance, the new chromosome (queen candidate) is disregarded and a part of the original queen is again selected and mutated randomly. Our proposed Algorithm is summarized as follows:

1. Select a coding scheme to represent adjustable parameters.
2. Generate randomly a chromosome as a queen and evaluate its performance.
3. Make an exact copy of the queen chromosome and name it queen candidate chromosome.
4. Apply the mutation operator to the queen candidate chromosome and evaluate its performance. If the performance of the queen candidate is better than the performance of the queen then replace the queen chromosome by the queen candidate chromosome, otherwise, go to step 3.
5. Repeat steps 3 and 4 until termination criterion is satisfied.

Coding scheme used in this paper is a sequence of binary numbers. The value of each gene on the queen chromosome is zero or one. A certain number of adjacent genes show the binary value of variables a_{ik}, e.g., if the range of values for a_{ik}s is from 1 to 15, to show the value of each a_{ik}, we consider four genes ($(15)_{10} = (1111)_2$) and if we have five calls then the queen chromosome should have 20 genes. The necessary number of genes for showing the value of a_{ik} in base 2 is equal to the required number of bits to show the value of lower bound (l). Actually, we use binary representation for chromosome. For example, consider a four-cell network whose constraint matrix (C), demand vector (D), and lower bound (lb) are:

$$C = \begin{bmatrix} 5\,4\,0\,0 \\ 4\,5\,0\,1 \\ 0\,0\,5\,2 \\ 0\,1\,2\,5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad lb = 11.$$

According to Sect. 2, the term $\sum_{i=1}^{n} d_i$ shows the total number of the required calls in this network. Hence, we have six calls, which are shown by

a_{11}, a_{21}, a_{31}, a_{41}, a_{42}, and a_{43}. Thus, queen is considered as (a_{11}, a_{21}, a_{31}, a_{41}, a_{42}, a_{43}).

The performance of queen or queen candidate is evaluated by (2).

$$performance = \left(lb \cdot \left(\sum_{i=1}^{n} d_i \right) \cdot b \right) + \sum_{i=1}^{n} \sum_{k=1}^{d_i} f_{ik} \tag{2}$$

where b shows the number of blocked calls (block calls are calls that their frequencies interfere with frequencies of other calls), n is the number of cells in the network, d_i is the ith element of the demand vector (D), and f_{ik} is the frequency assigned to a_{ik} that its value is obtained by converting binary value of a_{ik} to an integer value.

Mutation is done by choosing randomly some genes of the queen and changing them from 0 to 1 or 1 to 0.

The termination criterion is that the values of a_{ik}s (f_{ik}s) satisfy (1) and the maximum value of a_{ik}s is equal to the value of lower bound (lb).

5 Simulation Results

To test the proposed algorithm and compare its performance with the two existing approaches (Heuristic method proposed by Sivarajan et al. [9] and Adaptive Local Search algorithm proposed by Wang and Rushforth [10]), we used three CAPs that their constraint matrix (C), demand vector (D), and lower bound (lb) are as follows:

$$C_1 = \begin{bmatrix} 4\,3\,4\,3\,4 \\ 3\,4\,3\,2\,2 \\ 4\,3\,4\,2\,4 \\ 3\,2\,2\,4\,2 \\ 4\,2\,4\,2\,4 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 1 \\ 4 \end{bmatrix}, \quad lb_1 = 38.$$

$$C_2 = \begin{bmatrix} 4\,3\,3\,5\,4 \\ 3\,4\,4\,4\,4 \\ 3\,4\,4\,2\,2 \\ 5\,4\,2\,4\,2 \\ 4\,4\,2\,2\,4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \quad lb_2 = 59.$$

$$C_3 = \begin{bmatrix} 7\,3\,6\,1\,6\,7\,7 \\ 3\,7\,6\,4\,4\,2\,5 \\ 6\,6\,7\,3\,7\,5\,3 \\ 1\,4\,3\,7\,5\,3\,4 \\ 6\,4\,7\,5\,7\,7\,5 \\ 7\,2\,5\,3\,7\,7\,2 \\ 7\,5\,3\,4\,5\,2\,7 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 4 \\ 4 \\ 6 \\ 1 \\ 5 \\ 1 \\ 3 \end{bmatrix}, \quad lb_3 = 101.$$

We have implemented our algorithm in C. This program was run on a PC with a Pentium 4 (3.2 GHz) CPU. Table 1 shows results produced by our method and the two existing approaches for the three stated problems. First column of this table indicates the problem number. Constraint matrix (C), demand vector (D) and lower bound (lb) for each problem are given in the second, third and fourth column, respectively. The fifth, sixth, seventh and eighth column show no. of calls, the maximum frequency obtained by our algorithm, heuristic method [9], and adaptive local search algorithm [10], respectively.

As shown in Table 1, the maximum frequency (required bandwidth) obtained by our algorithm in each problem is equal or less than the maximum frequency obtained by the other two methods. Consequently, we can conclude that our new method has a good performance in solving CAPs in cellular radio networks.

Table 2 contains derived channel assignment for problems 2.

6 Conclusion

In this paper, we proposed a new method based on genetic algorithm to solve CAPs in cellular radio networks. We compared the performance of the proposed algorithm with the other two methods used in CAPs and showed through simulations that our method can find solutions with the least necessary bandwidth that does not violate any EMC constraints.

Table 1. Results obtained by two channel assignment methods and our algorithm for 3 CAPs

Problem number	Constraint matrix (C)	Demand vector (D)	Lower bound (lb)	No. of calls	Genetic based algorithm	Sivarajan et al. [9]	Wang et al. [10]
1	C_1	D_1	38	16	38	38	38
2	C_2	D_2	59	25	59	59	59
3	C_3	D_3	101	24	101	101	102

Table 2. Derived channel assignment for problem 2

Cell 1	Cell 2	Cell 3	Cell 4	Cell 5
56	59	29	25	27
50	53	23	21	19
44	47	13	17	15
38	41	7	11	9
32	35	3	5	1

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Max-Product Fuzzy Relational Equations as Inference Engine for Prediction of Textile Yarn Properties

Yordan Kyosev, Ketty Peeva, Ingo Reinbach, and Thomas Gries

Summary. This work presents first practical implementation of a new algorithm for solving max-product fuzzy relational equations as inference engine. The original, analytical provided procedure computes the greatest solution and the set of all minimal solutions, in case of consistency. In case of inconsistency, which presents not adequate knowledge base or not adequate case for solution, the equations, that correspond to the unsatisfied rules, are obtained. The algorithm is implemented for solving max-product fuzzy linear system for predicting properties of textile yarns, but these systems as inference engine are applicable in wide range of areas. Several methodology problems of the practical implementation like the type of membership functions, relation coefficients, dealing with multiple interactions are presented.

Key words: Inference engine, Inverse problem resolution, Max-prod composition, Textile yarn.

1 Introduction

There are several approaches for building the inference engine. Those of them, which follow directly programmed "if-then" rules are suitable for the diagnosis problems, but not for engineering applications for prediction of certain product properties as function of the technological process. Most popular are the neuronal networks, often combined with fuzzy input and/or output. They are very powerful because of their multilayer nonlinear approximation nature, but they do not present clearly and user-friendly the knowledge base. The most important disadvantage for the current case is the impossibility to work in inverse direction, for backward reasoning. The fuzzy linear equations present clear definitions of the relations between output and input for a given system. Due to further development of the theory, they can be successfully used in both the directions: for forward reasoning – for calculating the outputs when the inputs and the relation matrix are given, and for backward reasoning – to calculate which input has to be provided in order to receive certain output.

There are several applications of fuzzy relational equations in the textile engineering. Some of them use fuzzy max–min linear systems for diagnostics [1–4]. The max–min composition is suitable for a specific kind of reasoning,

if the main task is to establish whether some events are present or not, taking into account the fuzziness of the used data. For predicting the properties of the materials or for similar engineering applications, often proportionality between the variables has to be present. In this case max-product law of composition provides a suitable mathematical description of the relations among the physical parameters. The direct problem (calculation of the max-product composition between the input vector X and the weight matrix A) is trivial, but for solving inverse problem there are still open fields for researching. Here we present first practical implementations of the algorithm presented in [5]. It uses algebraic-logical approach by using objects for representing the way of thinking of the man by solving fuzzy equations and is based on universal algorithm [6], developed as an extension of the theory and software in [7]. In Sect. 2 is explained basically how to use the fuzzy linear systems as inference engine. After that, in Sect. 3 are described some methodology problems, which we had to solve during the practical implementation of the fuzzy linear system as inference engine. At the end is presented a short example.

2 Max-Prod Fuzzy Linear Equations as Inference Engine

2.1 Mathematical Model

The properties of the textile yarns depend on a large number of parameters. Detailed experimental investigations over influence of some preparation processes over the parameters of the fibers sliver [8], some machine construction and adjustments [9,10], as well as the complex numerical and experimental investigations of the drafting process [11] prove, that predicting the yarn properties requires a complex mathematical model.

Formalized description of the process of prediction of yarn properties is presented in Fig. 1, where are mentioned only the most important input and output parameters. Let us present the process of prediction as a general system with N inputs x_i and M outputs b_j, where $i = 1 \cdots N$ and $j = 1 \cdots M$. In the yarn production is important to have estimation of the maximum possible strength of the yarn for a certain material. At the same time, the yarn irregularity CV, as well as the yarn hairiness have to remain in certain limits. All these requirements are output parameters. They depend on a set of input parameters, like machine adjustment, working speed, material preparation, temperature and humidity in the rooms etc. The relations between all inputs and outputs are usually nonlinear and they include complex interactions among several single inputs. One full experimental investigation of these relations by using design of experiments requires a large number of tests, which is not usually possible in industrial conditions and is time and resources consuming. On the other side, the complexity and multiscaling of the real problems particularly for spinning and for other textile processes, complicates building of phenomenological models, which represent the physics of all interactions. Our

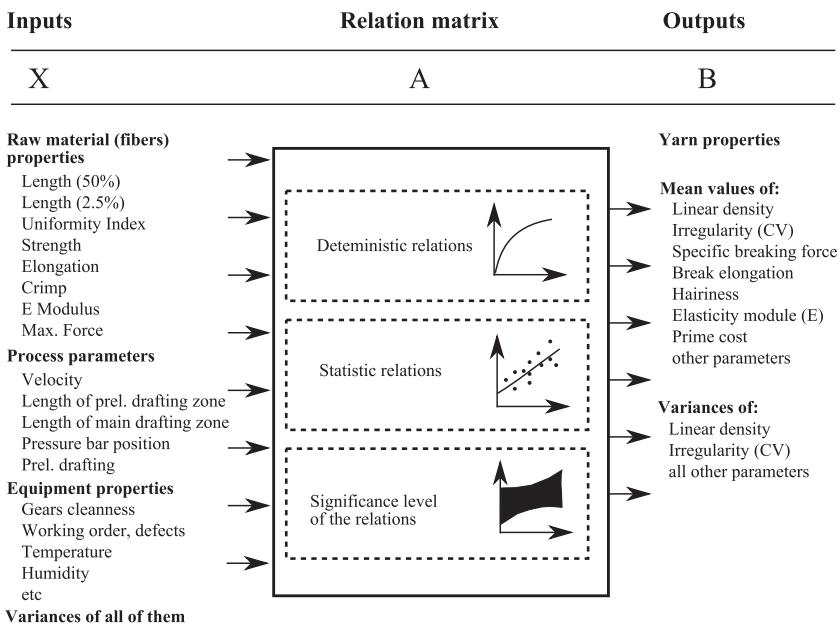

Fig. 1. Inputs, relation matrix and outputs when building a system for prediction of the properties of textile yarns

goal is to create a simple, fast and user friendly model, which consists of the main important relations between single inputs and outputs, and allows both forward and backward reasoning. The fuzzy linear system of equations fulfill these requirements, as for the case most appropriate is the use of *max-product* composition.

Let the relation between all inputs $X = x_i$ and the output b_j is presented with the equation

$$(a_{j1}.\, x_1) \vee \cdots \vee (a_{j\,n-1}.\, x_{n-1}) \vee (a_{j\,n}.\, x_n) = b_j, \tag{1}$$

where $\vee$ denotes max operator and . – multiplication. The complete system for all outputs is

$$\left| \begin{array}{cccc} (a_{11}.\, x_1) & \vee \cdots \vee & (a_{1\,n}.\, x_n) & = \; b_1 \\ \cdots & \cdots & \cdots & \cdots \cdots \\ (a_{m\,1}.\, x_1) & \vee \cdots \vee & (a_{m\,n}.\, x_n) & = \; b_m \end{array} \right. , \tag{2}$$

written in the following equivalent matrix form

$$A \odot X = B,$$

where $A = (a_{ij})_{m \times n}$ stands for the matrix of coefficients, $X = (x_j)_{n \times 1}$ stands for the matrix of unknowns, $B = (b_i)_{m \times 1}$ is the right side of the system. For

each $i, 1 \leq i \leq m$ and for each $j, 1 \leq j \leq n$, we have a_{ij}, b_i, $x_j \in [\,0,\,1\,]$ and the max-prod composition is written as $\odot$.

The coefficients a_{ij} matematically represent the influence (weight) of the input x_i over the output b_j. In industry, the experts are looking for the best performance, quality or seek the reasons for the worst cases. The best and worst cases of one output b_j correspond to its maximal and minimal value. Looking for minimum can be inverted to looking for maximum [7], thus we will work furtherly only with the maximal value.

2.2 Forward and Backward Schemes of Reasoning

We suppose that the relations' matrix A and one vector X with input parameters, for example raw material, equipment and process data are given. The estimation of the output parameters requires only computing of the composition of the left side of (1). This corresponds to the forward scheme of reasoning, or to the so called direct problem. This way of calculation is fast, because is connected to one matrix composition. It is useful for predicting of the properties of the yarn, when the different materials, process parameters or machine equipment are used. One can define goal function for some elements of the vector B and to start optimization problem, looking for the most suitable inputs X. Such optimization is often not effective, because the system (1) can have a large number of solutions and the standard optimization algorithms will find only a local solution. More effective is the optimization, when is used backward scheme of reasoning.

In this case, used also for diagnosis problems, we have to solve the *inverse* problem, finding all solutions of (1) for given outputs B and relations A. If the system (1) has solutions, it has one greatest and one or lots of lower solutions. The lower solutions can be interpretated as the cheapest and the worst material, which can be used for production of the yarn with the required in B quality and properties. The greatest solution gives the best (and expensive) material, which still will lead to producing the yarn with the same properties. The interval solutions of the system build the range of variations of the input parameters, where the output will remain unchanged. The interval solutions are of great importance for the application engineers, as they show which input parameters can be changed without loss of quality.

The application of the backward scheme of reasoning for optimization is a little bit different from the optimization with forward scheme. Here the required (maximum) values in B have to be given initially. Then, solving the system (1), all interval solutions can be calculated. For the experts remains the task to select this or these from the solutions X, which are more effective.

2.3 Solution Notes

If $A \odot X = B$ is consistent, it has unique greatest solution $X_{\mathrm{gr}} = A^t \diamond B$ [5, 13–15].

The $\diamond$-*product* of matrices A and B is in general defined as matrix $C = (c_{ij})_{m\times n} = A \diamond B$, if

$$c_{ij} = \min_{k=1}^{p}(a_{ik} \diamond b_{kj}), \text{when } 1 \leq i \leq m, 1 \leq j \leq n.$$

In our case B is a vector and C becomes vector too, and the A^t denotes transpose matrix of the A, where $A^t = (a_{ij}^t) = a_{ji}$.

A program implementation of exact method and an algorithm for solving the system $A \odot X = B$ for the unknown X is explained in [5, 7]. As much as possible improvements over the straightforward exhaustive depth search of this NP-hard problem are obtained. Rather than work with the system $A \odot X = B$, is used a matrix, whose elements capture all the properties of the equations. In depth first search, it is proposed how to drop branches that do not lead to minimal solutions. A sequence of simplification rules is defined, which brings the matrix into a new form. Once in this form, dominance is applied to remove redundancy. In this manner the time complexity of an exhaustive search is reduced merely by making a more clever choice of the objects over which the search is performed. This provides an easy finding of the complete solution of the original system.

3 Implementation Methodology

3.1 Membership Functions

The selection of the membership functions depends on the specificity of the problem.

For instance the yarn strength depends on the yarn twist nearly quadratically, as the type of the real function is presented in Fig. 2. In this case, we split the single input variable *twist* into new four input variables, which can be named like "very low","low","normal", and "high" twist, and which have also a unique range. Furtherly, we input four output variables for yarn strength, no matter that, for the forward reasoning this is not obligatory. These four output variables in this case are required, because some of the physical value of the yarn strength can be obtained for **two** different values of the variable "twist" – one before the maximum and one after the maximum. With the additional new output variables can be exactly specified, if the value is "high strength, but under the critical twist" or "high strength, but above the critical twist."

The next not typically used is the type of certain membership functions. For the above mentioned case (Fig. 2) the physical input variable increases monotonously and there is no overlapping between the local areas. For such type of parameters we used "saw" – like membership functions for normalization of the input variables (Fig. 3, left). Of course a lot of parameters work

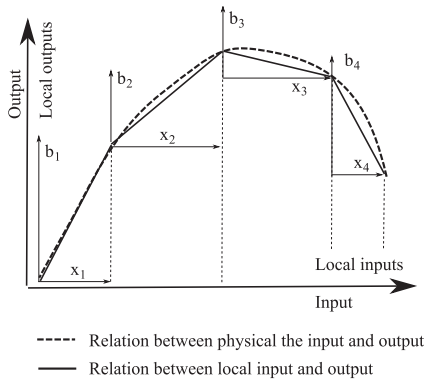

Fig. 2. Local approximation and setting up of new variables for the nonlinear relationships between input and output. The presented curve is typical for the relationship between yarn twist and yarn strength

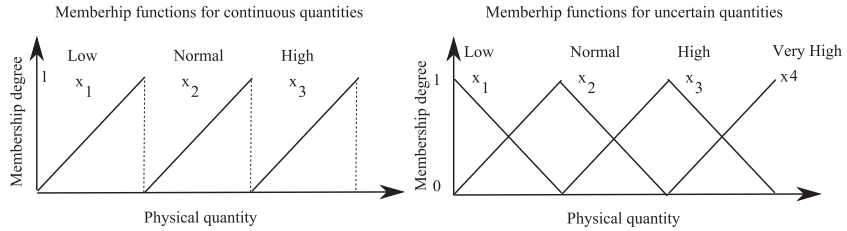

Fig. 3. Membership functions for continuous input variable without overlapping (*left*) and triangular membership functions for uncertain variables, like "state of the machine," connected with not exact definition of the state of working parts, gears, dirtiness level etc. (*right*)

well with the standard type functions (Fig. 3, right), for which more explanation can be found in almost all introductional literature about fuzzy logic, as for instance [12].

3.2 Coefficients of the Relation Matrix

The most important key point for expert system building is the selection of the proper structure of the relation matrix A. Its coefficients a_{ij} are obtained from experts, using mechanical models of the system for some of the relations or experimental results for the more complicated ones. They are divided into three groups, depending on the type of the relation between the input and output variables in the system (2):

– Physical (deterministic) relation between the inputs and outputs. Example: during the drafting process the fiber sliver becomes longer and finer. The drafting ratio I connects the input T_{inp} and output T_{out} fineness, as

$T_{out} = T_{in} \cdot I$. For this case, the corresponding coefficient in the system (1) has value $a_{ij} = I$, if the finenesses T_{out} and T_{in} are normalized.

- Stochastic correlation between the input and output parameters. Example: at higher working speed of the machine V the irregularity of the sliver CV becomes higher, too. This relation is not well (jet) deterministically described, but exists enough statistical data as a proof of its significance (Fig. 4, left). At the same time the speed of the machine does not influence significantly the mean fiber length $L_{50\%}$, and in this case we will set correspondent $a_{ij} = 0$.
- Relation between the variances of the input and output parameters. This can be the case not only for the stochastic relation between parameters, but also for the deterministic ones. The models usually do not describe all the influences, like the humidity and temperature of the room, some defects in the gears, sticking of the dust, which is usual for the textile production. We use the variances of all input and output variables, and input their relation in the corresponding coefficient a_{ij}. Here, in general can be assumed that $a_{ij} = 1 - r_{ij}^2$, where r_{ij} is the correlation coefficient of the regression equation for the connection between input x_i and output b_j. In this system all the input and output parameters are analyzed as pairs "parameter–variance": $x_i - x_{K+i}$, where K is the number of the independent input variables, $2K \leq N$. The use of the additional variables for the variances per input and output parameters makes the fuzzy linear system two times larger (actually four, but the half of the coefficients are zeros), but the variances are required for proper description of the processes. Example of confidential area of the influence between input "machine speed" and output "yarn mass irregularity" is presented on the Fig. 4 (right).

3.3 Significant Multiple Interactions

In some cases the interactions between two or more input parameters are very important. They can not be properly modeled by the system (2) and for

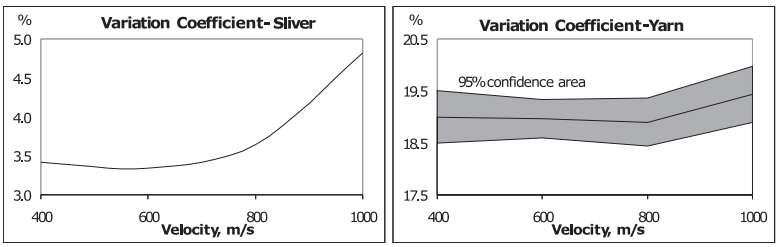

Fig. 4. Variation coefficient of the sliver (intermediate half-finished product), *left*, depending nonlinear on the machine velocity with very high degree of correlation. The same coefficient for the final product – yarn on the *right figure*, has the same trend, but with quite large confidence interval

this reason we used the idea for the multilayers from the neural networks. If the interaction between variables x_i and x_k is significant, we build new one, composite variable, $x_l = x_i \cdot x_k$. We use such composite variables during the solution of (1) formally as independent variables, but after that, during the decoding of the results, these variables require some additional operations and checks about possible logical contradictions.

4 Numerical Example

The relation matrix when working with industrial problems is usually bigger than 15×10, which is not convenient for printing. Because of this, the realization of forward and backward reasoning in the MATLAB environment are presented here as an example with highly reduced size. Let the relation matrix A is given as

$$A = \left(\begin{array}{ccc|ccc} 0.0 & 0.9 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.6 & 0.3 & 0.0 & 0.0 & 0.0 \\ 0.8 & 0.4 & 0.2 & 0.0 & 0.0 & 0.0 \\ \hline 0.0 & 0.0 & 0.0 & 0.4 & 0.8 & 0.1 \\ 0.0 & 0.0 & 0.0 & 0.2 & 0.1 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.1 & 0.8 \end{array}\right)$$

Here is demonstrated the block-architecture of this matrix, where the upper left block represents the coefficients of the relation between the input and output variables, the bottom right block – the coefficient for the variances between these variables. For forward scheme of reasoning, we need the outputs B, if the inputs X are given. For instance $X = \begin{pmatrix} 0.3 & 0.9 & 1.0 & 0.2 & 0.0 & 0.6 \end{pmatrix}^t$. Here we have to compute the *max-prod* composition $B = A \odot X$, which in the MATLAB environment using the library, described in [5] is simple:

```
>> B=fuzzy_maxprod(A,X')

B =

    1.0000
    0.5400
    0.3600
    0.0800
    0.6000
    0.4800
```

Let us solve the inverse example, typical for the backward reasoning, asking – which inputs X have to be used, in order to receive the presented output B? The solver calculates the greatest and finds two lower solutions

```
>> s=solvedot(A,B)
 greatest solution - transposed
    0.4500    0.9000    1.0000    0.2000    0.1000    0.6000

 lower solutions - transposed
         0    0.9000    1.0000    0.2000         0    0.6000
         0    0.9000    1.0000         0    0.1000    0.6000
```

which builds two interval solutions of the problem

$$X_1 = \begin{pmatrix} [0,\ 0.45] \\ 0.9 \\ 1 \\ 0.2 \\ [0,\ 0.1] \\ 0.6 \end{pmatrix}, X_2 = \begin{pmatrix} [0,\ 0.45] \\ 0.9 \\ 1 \\ [0,\ 0.2] \\ 0.1 \\ 0.6 \end{pmatrix}.$$

This example demonstrates, that for certain relation matrix A there are three inputs variables, which value can be changed and despite of this to receive the same outputs B. The variable x_1 has to be between 0 and 0.45, which means, that it has no significant influence over the outputs in this case. The variables x_4 and x_5 are connected – one of them can vary in some limits if the other one is fixed. On the language of the application engineers this solution set means, that we can obtain the same yarn properties, by relative large variation of the input parameters x_1 and careful choice between the variation of one of the inputs x_4 or x_5.

5 Discussion

In order to take into account the spread of the parameters, that is usually for the textile products, we use the variances for almost all input and output quantities as variables in the system, too. This increases the size of the system and worsens the clarity of the knowledge presentation. Better approach can be realization of the system by the means of logic [2, 7], where probably by using membership and nonmembership degrees, can be modeled the spread of the investigated variables as well. The use of intuitionistic approach formally would lead to saving the number of variables, but the amount of the data and calculations will be again almost identical with the presented here. The use of the intuitionistic approach still requires some additional development in the theory and software, which did not allow us to implement it.

6 Conclusions

The use of the *max-prod* fuzzy linear systems of equation as inference engine is explained. The practical implementation of these systems requires additional knowledge about the selection of the membership functions, presenting the highly nonlinear relations, spread of the variables, as well as the ways for filling the relation matrix. Interpretations of the mathematical model, from the point of view of the prediction of textile yarn properties are given, but the model is applicable in a wide range of areas.

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Automatic Defects Classification and Feature Extraction Optimization

Bernd Kuhlenkötter, Carsten Krewet, and Xiang Zhang

Summary. This paper introduces an automatic classification system that can identify defects on product surfaces in manufacturing, especially in processes like grinding and polishing. The identification process is based on grayscale images taken by a vision system. Some technologies that extract features from digital images are discussed. The support vector machine (SVM) is used in this paper as a multiclass classifier. It is shown that the overall classification rate can be close to the level that a skilled operator can obtain. The issues concerning the optimization of feature extraction are also covered in this paper.

Key words: Defect classification, Feature extraction, Support vector machine, Optimization.

1 Motivation

Nowadays the manufacturing process is tending to high automation level, as much as possible relieving workers from the laborious tasks and unpleasant working environment [1, 2]. Nevertheless, many inspection tasks are still done manually due to the difficulty of automatic execution. One example is that the flaw inspection and identification on the surface of fittings, e.g., water tap heads, have long been done by human operators in sanitary industries. It is very beneficial to automate this process. First of all, the efficiency of this process will be dramatically increased. Second, the job is monotonous and tedious, leading to less concentration of operators over the time, which causes classification errors. Third, operators have their own standards of inspecting and classifying the defects. It is possible that one defect, which is identified by one operator to class A, is classified by another operator into class B. It is also possible that one operator might make different judgments at different times.

The work in this paper is aimed, but not limited, to automatically classify defects on water tap heads after grinding and polishing processes. From manufacturing practice, possible defects are defined into 15 categories in advance. Figure 1 shows samples of seven kinds of defects. From practical experience, an

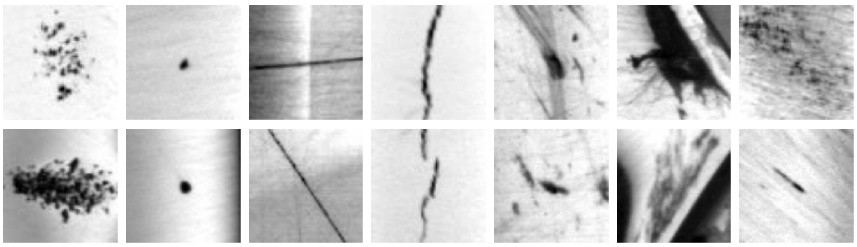

Fig. 1. Defect samples (from left to right: casting peel, pore, lined mark, crack, burned residues, grease residues, polishing shade)

operator reaches a classification rate in the range from 60% to 90% depending on their experience and on their concentration level. The wrong inspections come from the lack of concentration and subjective errors. In comparison, an automatic inspection and classification system can evaluate the defects using a constant criterion and overcome the varied standards among different operators.

2 Automatic Classification System

The vision system consists of a carrier, a camera system, a lighting system, other accessories and the software. The system hardware is responsible to provide a constant lighting environment and obtain the digital images of surfaces under this constant circumstance. The software provides the solution to examine the images from the camera system, locating and classifying the defects on workpiece surfaces.

Two steps are included in the software implementation, the feature extraction and the classifier design. The feature extraction is the most important part in the system. It defines the rules to describe and express the defects inside an image in a form that the classifier can understand and utilize to distinguish one class from others. Generally, feature extraction digitizes the defect images in a way that enlarges the distinctions among categories and discards the similarities at the same time. After that, the features are applied as the training data to the classifier. Support vector machine (SVM) [3] is an effective artificial method to solve both regression and classification problem, especially when the input dimension is very high. It has been successfully applied in many research and industrial classification tasks [4, 5]. Therefore it is also used as the classifier in this project described in this paper. The one-against-one scheme is used to combine a group of two-class classifier into a multiclass classifier. In most cases, the one-against-one scheme yields a better result than the one-against-all scheme [6].

3 Feature Extraction Technologies

The feature extraction is the most crucial step to the final accuracy of the classification. However, no single feature extraction method is consistently

superior to other methods [7] because the result of a method highly depends on the task to be solved. Therefore, several feature extraction technologies are implemented and tested, including shape features, statistical features, the local energy of some filtering channels and grayscale information.

3.1 Shape Features

Shape features indicate some values that represent the size related to the object contour. Some of the shape features are illustrated in Fig. 2, in which C is the contour of the detected defect, C_{con} is the convex hull of C and f_{max} is the maximal Feret diameter. The Feret diameter is defined as the projection length of the convex envelope of an object in a given direction. Besides f_{max}, seven shape features are used in this paper, area S, length l_a, breadth l_b, elongation e, compactness c, roughness r and area ratio s_r. The features are either shown in Fig. 2 or can be computed by following formulas

Length	$l_a + l_b = \frac{P}{2}$	Breadth	$l_a * l_b = S$
Elongation	$e = \frac{l_a}{l_b}$	Compactness	$c = \frac{P^2}{4\pi S}$
Roughness	$r = \frac{P}{P_{con}}$	Area ratio	$s_r = \frac{S}{S_{con}}$

where P is the perimeter of the contour C, P_{con} and S_{con} are the perimeter and area of its convex hull C_{con}, respectively.

The length l_a and breadth l_b are the logical length and breadth that can be calculated by the area and the perimeter. Elongation is the quotient of the length divided by the breadth, thus always greater than 1. Compactness is the square of the ratio of the perimeter of the original contour and the perimeter of a circle that has an equal area as the original contour. Ideally the compactness is 1 when the contour is a circle, otherwise it is greater than 1. Roughness and area ratio are two measures to indicate the convexity of the contour. A convex contour has the value 1 for the both measures. These eight shape features are not all independent.

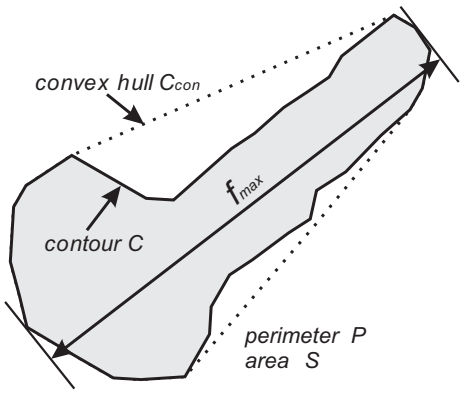

Fig. 2. Shape features

3.2 Filter Bank

Another technology to extract features from the texture image is the filter bank. The filter bank is also called multichannel spatial filtering method. The idea is to apply a sequence of filters on the image and take the local energy of the filtered images as features. The inspiration for this method comes from neurological studies. These research works suggest that the preprocessing stages in the human vision system involve a set of parallel and quasi-independent mechanisms or channels which resemble band-pass filters. Each filter in the filter bank contains intensity variations over a narrow range of frequency and orientation, specifying the regularity, coarseness and directionality of the original image [8]. One filtering transaction is computed by applying a convolution kernel to the original image and the local energy is calculated from the filtered image in a specified window. The general processing flow is shown in Fig. 3. The kernel or unit impulse response of the filter ℓ is given by a square matrix f_ℓ. The filtered image $y_\ell(i,j)$ is obtained by centrally convoluting the original image $x(i,j)$ with the filter f_ℓ, which can be written as

$$y_\ell(i,j) = x(i,j) * f_\ell(i,j) \tag{1}$$

Then the ℓ^{th} feature is specified by the local variance of the filtered image $y_\ell(i,j)$ in a $W \times W$ window and can be expressed as

$$FEA_\ell = \frac{1}{W^2} \sum_{m,n=0}^{W} \left\{ y_\ell(\frac{W}{2} - m, \frac{W}{2} - n) - u_\ell(i,j) \right\}^2 \tag{2}$$

where $u_\ell(i,j)$ is the mean value of the filtered image $y_\ell(i,j)$ in the $W \times W$ window and W is the window size which is specified by users. The different filter banks differ from each other mainly in the formulating of the filters f_ℓ.

Two kinds of filter banks are used: Laws filters [9, 10] and Gabor filters [11, 12]. Refer to our previous paper [13] for formulation and parameter configuration of these filters.

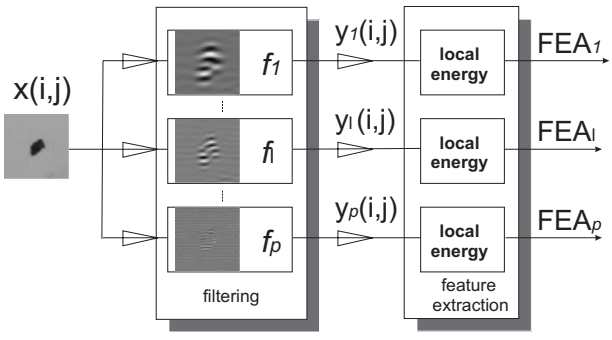

Fig. 3. Processing flow of filter bank

3.3 Statistical Features

The gray level co-occurrence matrix [14,15] is a well-known statistical tool for extracting second-order texture information from images. The co-occurrence matrix P_d is a $N_g \times N_g$ square matrix defined on a given displacement vector $\overrightarrow{d} = \{dx, dy\}$ where N_g is the grayscale level of the image. The entry (i, j) of the matrix P_d is the number of occurrences of the pair of gray level i and j which is a distance $\overrightarrow{d}$ apart. An example is given in Fig. 4 to demonstrate how to compute the co-occurrence matrix of a grayscale image. The left side of Fig. 4 shows an image of three grayscale levels, in which numbers denotes the pixel grayscales. The right side is the corresponding co-occurrence matrix P_d, which is a 3×3 square matrix, with respect to the displacement vector $\overrightarrow{d} = (1, 1)$. After that the co-occurrence matrix is calculated and a large range of features can be computed from this co-occurrence matrix. Five of them are used in this paper.

$$\begin{aligned}
\text{Energy } f_1 &\quad \textstyle\sum_i \sum_j P^2(i,j) \\
\text{Entropy } f_2 &\quad \textstyle\sum_i \sum_j P(i,j) \log_2[P(i,j)] \\
\text{Contrast } f_3 &\quad \textstyle\sum_i \sum_j (i-j)^2 P(i,j) \\
\text{Homogeneity } f_4 &\quad \textstyle\sum_i \sum_j P(i,j)/(1+|i-j|) \\
\text{Correlation } f_5 &\quad \textstyle\sum_i \sum_j (i-\mu_x)(j-\mu_y)P(i,j)/\sigma_x\sigma_y
\end{aligned}$$

where μ is the mean value of the co-occurrence matrix P, μ_x, μ_y, σ_x and σ_y are the means and the standard deviations corresponding to the vectors p_x, p_y that are expressed by

$$p_x = \sum_j P(i,j) \qquad \text{and} \qquad p_y = \sum_i P(i,j)$$

3.4 Grayscale Information

Besides the features introduced above, we use additionally the average and standard deviation of grayscale values of the defect image as grayscale features. The grayscale features should be localized considering different size of the various defects. The grayscale information are obtained in four areas in the defect image, respectively, see Fig. 5. In this case, the number of grayscale features is eight, two of each area.

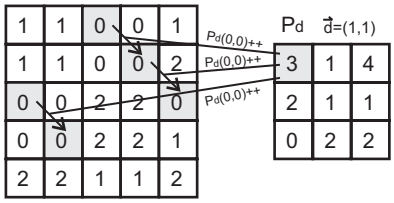

Fig. 4. Get the co-occurrence matrix of the grayscale image

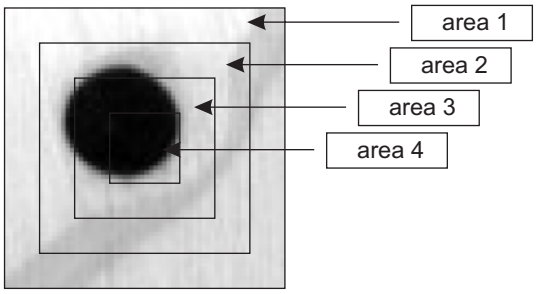

Fig. 5. Grayscale information

Table 1. Training and testing classification rate of varied features

Fea.	Fea. Num.	Tr. cl. rate(%)	Te. cl. rate(%)
Shape	8	87.5	59.5
Laws	25	99.5	69.4
Gabor	16	98.5	70.1
Statistical	15	98.0	75.2
Grayscale	8	98.5	72.3

4 Classification Results

Table 1 shows the classification results using only one kind of features. It can be concluded from the table that the shape feature is not suitable for this application. There are two reasons for that. First, there are no clear differences in the shape between some defects. The second reason is that the geometric information of some kinds of defects cannot be exactly defined. For example, it is not easy to describe the shape of a burned residues and a polishing shade. The pattern information is more effective than the simple geometric information in this sense.

The best result is obtained by using statistical features based on co-occurrence matrix, a 75.2% overall classification rate. The classification efficiency of grayscale information, Gabor features and Laws features are slightly lower than that of statistical features.

The performance of the classification system is improved when features from different technologies are combined. Table 2 shows the classification results of the combined features. The overall classification rate reaches a rate of 81.1% when the statistical features are combined with Gabor features and grayscale information.

5 Optimization of Feature Extraction

Many approaches are available to extract pattern features and many parameters can be adjusted in each approach. Thus, it is usually a troublesome task

Table 2. Training and testing classification rate of combined features

Gabor	Statistical	Grayscale	Num.	Tr. cl. rate(%)	Te. cl. rate(%)
X	X		21	100	77.2
X		X	24	100	78.4
	X	X	13	98.0	77.4
X	X	X	29	98.2	81.1

to select the most appropriate methods and parameters to obtain features that can best separate the samples. Sometimes it can be done by a lot of experiments and then by evaluation of the results of classification. However, there are often demands to have a standard for evaluating features, which does not depend on the classifier that is in use. In fact, features extraction and classification are two separate procedures though they are closely related to each other. Feature extraction is a way to represent the characteristics of a subject, while classification determines how to separate the samples based on the subject representation. The feature extraction should fulfill two principles. One is an indispensable condition of the classification task that a feature should exhibit enough differences among diverse categories to be classified. Otherwise, samples would be impossible to be separated effectively no matter which kind of classifier is applied. The other is a supplementary condition requiring that those features, which do not meet the first principle, are not used. The second principle is to optimize the input to the classifier and ensure the generalization of the model. However, the principles are quite descriptive. They make sense to select suitable features only if we can find an effective way to evaluate the quality and goodness of the features.

5.1 Bhattacharyya Distance

Suppose that we have two classes of samples that need to be separated. The feature values for m samples of the first class A are as follows

$$\underbrace{f_{A1}, f_{A2}, f_{A3}, ..., f_{Am}}_{m} \tag{3}$$

and the feature values for n samples of the class B are

$$\underbrace{f_{B1}, f_{B2}, f_{B3}, ..., f_{Bn}}_{n} \tag{4}$$

We also suppose that the feature values are normally distributed. Figure 6 illustrates two different situations of relative distributions of features f_{Ai} and features f_{Bi}. In this first case (left side), the class A can be easily separated from the class B because they are clearly different from each other that there is no overlap between features. In the second example, class A is theoretically hard to be discriminated from class B because the average value of features

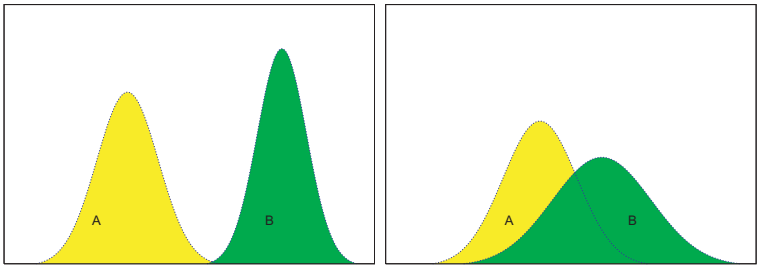

Fig. 6. Separability of two classes

are too close to each other. A good feature drags one class apart from another and the variance of this good feature should be small at the same time. An ideal situation is that the mean error $|\mu_a - \mu_b|$ is very large and two variances σ_a, σ_b are very small. Thus, the separability of a feature relates not only to the difference of the means but also to the deviation of features of each class.

The Bhattacharyya distance (BH distance) [16] is a method to statistically quantify the separability of two classes using a feature which can be written as

$$B_{dis}(A,B) = \frac{1}{4}\left\{\frac{(\mu_A - \mu_B)^2}{\sigma_A^2 + \sigma_B^2}\right\} + \frac{1}{2}\ln\left\{\frac{1}{2}\left(\frac{\sigma_B}{\sigma_A} + \frac{\sigma_A}{\sigma_B}\right)\right\} \tag{5}$$

where μ_A, μ_B, σ_A, σ_B are the features' means and standard deviations of the class A and the class B, which can be written as

$$\mu_A = \frac{1}{m}\sum_{k=1}^{m} f_{Ak} \tag{6}$$

$$\mu_B = \frac{1}{n}\sum_{k=1}^{n} f_{Bk} \tag{7}$$

$$\sigma_A = \sqrt{\frac{\sum_{k=1}^{m}(f_{Ak} - \mu_A)^2}{m}} \tag{8}$$

$$\sigma_B = \sqrt{\frac{\sum_{k=1}^{n}(f_{Bk} - \mu_B)^2}{n}} \tag{9}$$

For simplicity, the first part (Fisher ratio) in (5) can be used instead of BH distance as the measure of separability of two classes with respect to one feature. In the ideal situation, namely a large mean difference and small variances of each class, the BH distance and the Fisher ratio are both large scalars. The smaller the distance, the less separable are the two classes. Therefore, the BH distance or the Fisher ratio can be a criterion for evaluating the goodness of a feature.

5.2 Optimize Features Based on Co-Occurrence Matrix

The statistical features based on the co-occurrence matrix have given good results in the experiments above. In addition, they are flexible to be configured. The statistical features can be thought of as an weighted sum of the co-occurrence matrix elements. The features in (3) and (4) are calculated by

$$f_{A(B)k} = \sum_{i=1}^{N_g} \sum_{j=1}^{N_g} W(i,j) \cdot P_{A(B)k}(i,j) = \mathbf{w}.p_{A(B)k} \tag{10}$$

where $W(i,j)$ is the weight matrix, P_{Ak} and P_{Bk} are the co-occurrence matrixes, $\mathbf{w}$, p_{Ak}, p_{Bk} are vectors that are formulated from the matrixes W, P_{Ak} and P_{Bk}. The P_{Ak} and P_{Bk} are known. Thus, once the weight matrix $W(i,j)$ is determined, the feature extraction process is sequentially determined. A weight matrix corresponds with a feature extraction strategy.

In the experiments above, we used only some standard features, e.g., energy, contrast, homogeneity that are general to all applications. The weight matrix of each standard feature is decided beforehand and does not depend on the problem that is being worked on. The idea of the feature extraction optimization is to find the best feature for a specific application, or at least one that is superior to the standard features. As mentioned above, the form of weight matrix defines the final feature. Therefore, obtaining the optimal feature for two classes A and B is equivalent to find a specific weight matrix $W(i,j)$ that can maximize the BH distance or the Fischer ratio between f_{Ak} and f_{Bk}.

This is a nonlinear optimization problem with N_g^2 unknowns. Most of the in-use iteration algorithms, like conjugate gradient method, need not only the function values but also function gradients for a fast convergence rate. The gradients of the objective function (5) with respect to $\mathbf{w}$ can be indirectly calculated by gradients of μ_A, μ_B, σ_A, σ_B with respect to the same $\mathbf{w}$, which can be written as

$$\nabla\mu_A = \frac{\partial\mu_A}{\partial\mathbf{w}} = \frac{1}{m}\sum_{k=1}^{m} p_{Ak} \tag{11}$$

$$\nabla\mu_B = \frac{\partial\mu_B}{\partial\mathbf{w}} = \frac{1}{n}\sum_{k=1}^{n} p_{Bk} \tag{12}$$

$$\nabla\sigma_A = \frac{\partial\sigma_A}{\partial\mathbf{w}} = \frac{\sum_{k=1}^{m}(\mathbf{w}\cdot p_{Ak} - \mu_A)(p_{Ak} - \nabla\mu_A)}{m\sigma_A} \tag{13}$$

$$\nabla\sigma_B = \frac{\partial\sigma_B}{\partial\mathbf{w}} = \frac{\sum_{k=1}^{n}(\mathbf{w}\cdot p_{Bk} - \mu_B)(p_{Bk} - \nabla\mu_B)}{n\sigma_B} \tag{14}$$

with (5–9), (11–14), the optimization problem can be solved.

However, the problem is not as simple as what has been introduced so far. Suppose 256 grayscale levels are used to generate the co-occurrence matrix, i.e., $N_g = 256$. In this case 65,536 unknowns exist in the optimization problem. Even taking the symmetry into consideration, there are still 32,896 unknowns, which means an extremely large optimization problem. The weight matrix W is too flexible to ensure the generalization of the final solution. Even though we obtain a weight matrix, with which the BH distance is a large number for training samples, it cannot be proven that this matrix will also bring a large distance for testing samples. Figure 7 shows a weight matrix that is calculated by maximizing BH distance between pores and polishing shades in the training set with a grayscale level of 32. The BH distance of training set is about 2,347 with this weight matrix, but only about 0.22 for the testing set. It goes back to the generalization problem in the learning theory. The solution of this kind of problems is normally to apply constraints on the over-flexible weight matrix, for example, requiring that the weight matrix surface is smooth and not so chaotic as that in Fig. 7.

Walker et al. [17] presented a strategy to construct a weight matrix. They started with a standard feature, e.g., energy or contrast, and considered every weighted elements in the co-occurrence matrix as a feature. Then the BH distances for each elements were calculated consequently. Therefore another matrix, which was called by them as a discrimination matrix, can be obtained. The discrimination matrix is also disturbed and fragmentary. After that, they used a second order polynomial surface to approximate the discrimination matrix. The polynomial surface was then used as the weight matrix finally. It was reported that the optimized features obtained in this way performs normally a bit better than the original standard features, but not always.

This method depends on standard features because the standard weight matrix is needed to calculate the discrimination matrix. In contrast, the constrained weight matrix strategy we introduced above is more general and more configurable. The problem now is what kinds of constraints should be imposed on the weight matrix in advance. We suggest two options. One is adopting polynomials as the form of the weight matrix. In this case, Walker's method can be considered a special implementation of the strategy we put

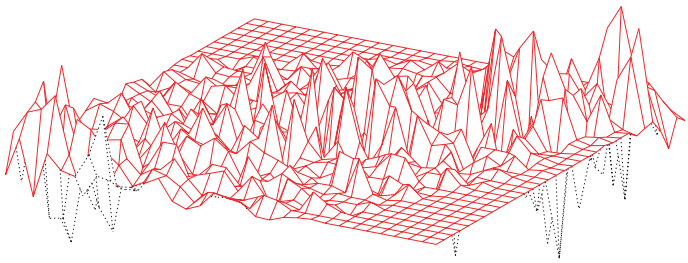

Fig. 7. Nonconstrained weight matrix

forward here. Another is using B-Spline surface representation. The optimization unknowns are the coefficients of the polynomials for the first case, while the coordinates of control points become the optimization objective when the B-Spline representation is adopted. Apparently, the B-Spline is a more adaptable representation because both the continuity of the surface and the number of control points are configurable. However, the optimization problem is much more complicated than polynomial representation because it is not easy to calculate the gradients of the control points coordinates with respect to the unknowns $\mathbf{w}$.

6 Summary

In this paper, an industrial vision system is introduced to identify and classify defects on free-form surfaces during grinding and polishing processes. The classification is based on grayscale images taken by a vision system. Some features, shape features, filter banks, statistical features and grayscale information are adopted for the classification task. SVM is served as a multiclass classifier, receiving the features as input and determine the category of the defect. In this application, the statistical features, grayscale features, and Gabor filter bank have shown better results than other kinds of features. The result is even better when these three kinds of features are combined together. With the combined features, an overall classification rate 81.1% can be reached, which is comparable to a trained operator. In addition, the optimization of the statistical features based on the co-occurrence matrix is also discussed in this paper. The statistical features based on the co-occurrence matrix can be considered as a weighted sum of the elements of the co-occurrence matrix. A general weight matrix can be adopted instead of the standard matrixes to construct a new feature. An optimized weight matrix should generate a feature, with respect to which the BH distance among defects is as large as possible. Constraints must be imposed on the weight matrix to guarantee the generalization of the weight matrix which is generated through the optimization process.

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Short-Term Load Forecasting in Power System Using Least Squares Support Vector Machine

Ganyun LV, Xiaodong Wang and Yuanyuan Jin

Summary. Short-term load forecasting is an important subject for power systems and has been studied from different points of view. Least squares support vector machine (LS-SVM) has a good generalization ability and capability of tolerating noise in nonlinear modeling. An approach based on LS-SVM is proposed for daily peak load forecasting in power distribution systems. The LS-SVM is used to learn the relationships among past, current and future temperatures and loads. The LS-SVM was trained to recognize the peak load of the day. The suitability of the proposed approach is illustrated through an application to real load shapes from Hefei Electricity Distribution Corporation in Anhui. Peak load forecasts with satisfying accuracy are reported from the testing data.

Key words: Short-term load forecasting, LS-SVM, Daily peak load forecasting, Power distribution systems, Accuracy.

1 Introduction

Short-term electric load forecasting is an important requirement for electric system operation. In general, load forecasts should be performed over a broad spectrum of time intervals, which could be classified into short term, medium term and long term forecast. Short-term load forecasting (STLF) aims at predicting electric loads for a period of minutes, hours, days, or weeks. STLF plays an important role in the real-time control and the security functions of an energy management system. Daily peak load is one important task in STLF. Moreover, an accurate peak load forecast can be helpful in developing a power supply strategy, financing planning, electricity management and market search.

Many techniques have been proposed during the last few decades regarding STLF. Traditional techniques applied to STLF include Kalman filtering, regression models, the autoregressive (AR) model [1, 2]. Time-series models employ extrapolation of historical data for the estimation of future hourly loads. A disadvantage of this type of models is that weather information or any other factors that contribute to the load behavior can not be fully utilized.

Regression models analyzed the relationship among loads and other influential factors, such as weather conditions and consumers behavior. The main disadvantage of this kind of models is that complex modeling techniques and heavy computational efforts are required to produce reasonably accurate results [3].

An enormous upwelling of interest has grown in recent years in application of artificial intelligence (AI) techniques to industrial processes. Their advantage is that no complex mathematical formulation or quantitative correlation between inputs and outputs is required. Expert systems (ES) [4,5], fuzzy logic (FL) [6] and neural networks [7–9] have been proposed for electric load forecasting. Expert system based methods capture the expert knowledge into a comprehensive database, which is then used for predicting the future load. These models exploit knowledge of human experts for the development of rules for forecasting. However, transformation of an expert knowledge to a set of mathematical rules is often a very difficult task. Fuzzy theory has similar problem. Traditional ANN method can directly acquire experience from the training data, and overcome some of the shortcomings of the expert system. However, it suffers from a number of weaknesses, including the need for a large number of controlling parameters, difficulty in obtaining a stable solution and the danger of over-fitting. As ANN, ES and FL approaches have their advantages and disadvantages, hybrid artificial intelligence approaches are also under consideration. Their disadvantages could be overcome by connecting ES, FL, and ANN as a whole [10, 11].

With an emerging technique of support vector machines (SVM) [12–14], combining the advantages of neural networks (handling large amount of highly nonlinear data) and nonlinear regression (high generalization), the issues of high dimensionality as well as the previous drawbacks from neural networks are overcome. Because of the above reason, SVM is employed to short-term load forecasting. For the first time, this paper presents a method for daily peak load forecasting using least squares support vector machine (LS-SVM).

2 Review of LS-SVM

The basic idea of the SVM regression is to map the input data into a feature space via a nonlinear map. In the feature space, a linear decision function is constructed. The SRM principle is employed in constructing optimum decision function. Then SVM nonlinearly maps the inner product of the feature space to the original space via kernels. The SVM nonlinear regression algorithms are reviewed in this section.

Given a set of training data

$$D = \{(x_1, y_1), \cdots, (x_i, y_i), \cdots (x_l, y_l)\} \in R^n \times R. \tag{1}$$

The nonlinear function $\phi(.)$ was employed to map original input space R^n to higher dimensional feature space R^k: $\phi(x) = (\varphi(x_1), \cdots, \varphi(x_l))$, where

$k(k \gg n)$ represents the dimension of feature space. Then an optimum decision function $f(x_i) = w\varphi(x_i) + b$ is constructed in this higher dimensional feature space, where $w = (w_1, \cdots, w_k)$ is a vector of weights in this feature space. Nonlinear function estimation in the original space becomes a linear function estimation in feature space. Thus, according to principle of structural risk minimization (SRM), when the quadratic ε-insensitive loss function is selected in the LS-SVM. The optimal problem can be formulated as minimization of the following objective function J:

$$minJ(w, \xi) = \frac{1}{2}\|w^2\| + \frac{1}{2}c\sum_{i=1}^{l}\xi^2. \tag{2}$$

Subject to the equality constraints

$$y_i = w\varphi(x_i) + b + \xi_i, i = 1, 2, \cdots, l. \tag{3}$$

We define the Lagrangian as

$$L(w, b, \xi, a) = \frac{1}{2}ww + \frac{1}{2}c\sum_{i=1}^{l}\xi^2 - \sum_{i=1}^{l}a_i(w\varphi(x_i) + b + \xi) \tag{4}$$

where $a_i(i = 1, \cdots, l)$ are Lagrange multipliers.

By the optimality conditions

$$\frac{\partial L}{\partial w} = 0, \frac{\partial L}{\partial b} = 0, \frac{\partial L}{\partial \xi} = 0, \frac{\partial L}{\partial a} = 0. \tag{5}$$

We have

$$\sum_{i=1}^{l}a_i = 0, w = \sum_{i=1}^{l}a_i\varphi(x_i), a_i = c\xi_i, w\varphi(x_i) + b + \varphi(x_i) = y_i. \tag{6}$$

By (2) and (6), the optimization problem can be rewritten as following,

$$\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & K(x_1, x_1) + \frac{1}{c} & \cdots & K(x_1, x_l) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & K(x_l, x_1) + \frac{1}{c} & \cdots & K(x_l, x_l) \end{pmatrix} \begin{pmatrix} b \\ a_1 \\ \vdots \\ a_l \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \\ \vdots \\ y_l \end{pmatrix} \tag{7}$$

where $K(x_i, x_j)$ is a symmetric positive definite function in original input space, called kernel function:

$$K(x_i, x_j) = (\varphi(x_i), \varphi(x_j)). \tag{8}$$

Finally, the nonlinear function takes the form:

$$f(x) = \sum_{i=1}^{l}a_iK(x, x_i) + b. \tag{9}$$

This nonlinear function is the so-called SVM.

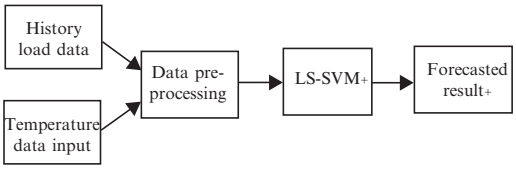

Fig. 1. Historical daily peak temperature of corresponding month

3 Short-Term Forecasting Using LS-SVM

A new method based on LS-SVM is proposed for daily peak load forecasting in the paper. Figure 1 depicted a simplified block diagram for daily peak forecasting with LS-SVM. The procedure includes three steps: data preprocessing, training of LS-SVM, and forecasting with the trained LS-SVM.

3.1 Data Preprocessing

The structure for the daily peak load forecasting with LS-SVM is given in Fig. 1. The number of input neurons of the NN structures is selected to be nine. The inputs of SVM are selected includes as follows:

(1) The peak loads of the previous seven days, as P_{i-1}, P_{i-2}, P_{i-3}, P_{i-4}, P_{i-5}, P_{i-6} and P_{i-7}.
(2) The peak temperatures of the previous seven days, as T_{i-1}, T_{i-2}, T_{i-3}, T_{i-4}, T_{i-5}, T_{i-6} and T_{i-7}.
(3) The average peak loads of the previous seven days,

$$\bar{P} = \frac{1}{7}\sum_{k=1}^{7} P_{i-k}$$

(4) The average temperature of the previous three days,

$$\bar{T} = \frac{1}{3}\sum_{k=1}^{3} T_{i-k}$$

(5) The forecasted peak temperature of the forecasted day, T_i.

Thus 17 input data were obtained by data preprocessing program from historical load database. Training and testing samples were obtained with the preprocessing program. The output of LS-SVM is daily peak load of the forecasted day.

3.2 Training Networks of SVM and Forecasting

Short-term load forecasting is a predicting problem with multivariables, which could be taken as a regression problem too. Forecasted peak load is output of the regression function, while history load data and weather information etc. is input X of the function. LS-SVM extracts the implicit nonlinear relationship among input variables and output forecasted peak load by learning from training data. With the trained LS-SVM network, LS-SVM output forecasted peak load by inputting the testing data. Thus the LS-SVM realized the load forecasting.

3.3 Choosing of Kernel Function

In this paper Gaussian RBF function is chosen to be the kernel function of SVM, as follows:

$$K(x_i, x) = \exp(-\frac{\|x - x_i\|^2}{2\sigma^2})(i = 1, 2, \cdots, l), \tag{10}$$

where σ is bandwidth of the kernel function.

3.4 Evaluating Indexes

Several indexes are adopted in this paper for evaluating the forecasting performance of proposed method. There are shown as following. Relative error of forecasting is:

$$RE_k = \frac{P'_{i+k} - P_{i+k}}{Pi + k} \times 100\%, k = 0, 1, 2, \cdots . \tag{11}$$

Maxim relative error of forecasting is:

$$RE_{max} = \max \sum_k RE_k. \tag{12}$$

Mean error of forecasting is:

$$MRE = \frac{1}{n} \sum_{k=0}^{n-1} RE_{i+k}. \tag{13}$$

Root mean square error (RMSE) is:

$$RMSE = \sqrt{\frac{1}{n} \sum_{k=1}^{n} \|P'_{i+k-1} - P_{i+k-1}\|^2}. \tag{14}$$

4 Examples

This paper presents daily peak load forecasting method applied to the electricity consumption in the Hefei region. The training data for the proposed approach is obtained from Hefei Electricity Distribution Corporation in Anhui Province. Daily peak load data from 1 March 2002 to 31 June 2002 is used as training data. Daily peak load data from 1 July 2002 to 7 July 2002 were used as testing data. With these data, training and testing samples are obtained by the above data preprocessing program. For evaluating the effect of proposed method, daily peak load forecasting with LS-SVM are compared with forecasting results by a BP neural network. Historical daily peak load of a typical month, and temperatures information in that month are shown as Figs. 2 and 3. From Fig. 2, it is shown that the peak load is much less in weekend than in working day, and the peak load grows with growth of temperature.

In this paper, Gaussian RBF function is selected as the kernel function, and bandwidth of the function is set to be 10. The regularization parameter of SVM c is set to be 75. The number of hidden layer neural of ANN is 30. The same training and testing samples are used for LS-SVM and BP Neural Network. The actual daily peak load at July 1 2002 to 7 July 2002, forecasted peak load by RBF neural networks and LS-SVM were shown as Table 1.

From Table 1, the forecasting error of LS-SVM is from 1.32% to 3.79%, while the forecasting error of BP neural network is from 0.74% to 6.74%. The

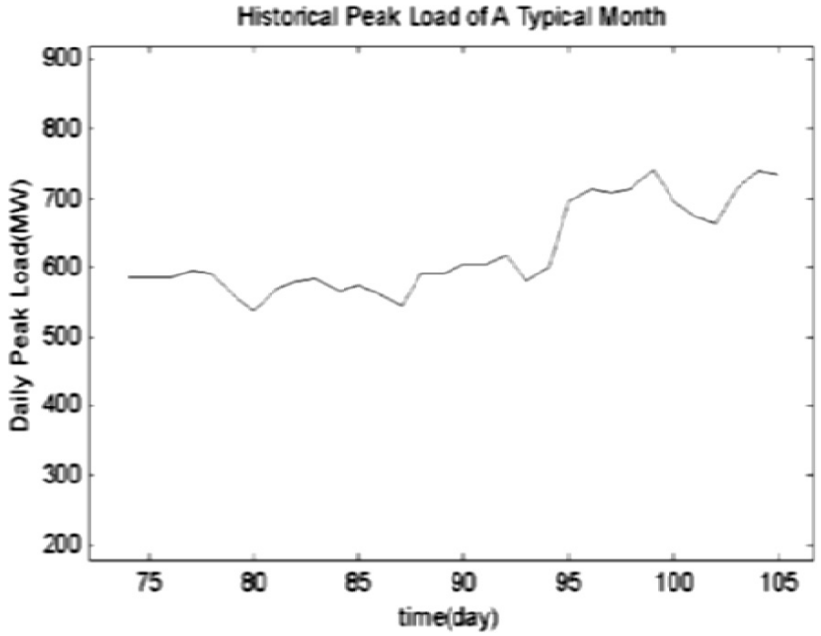

Fig. 2. Historical daily peak load of a typical month

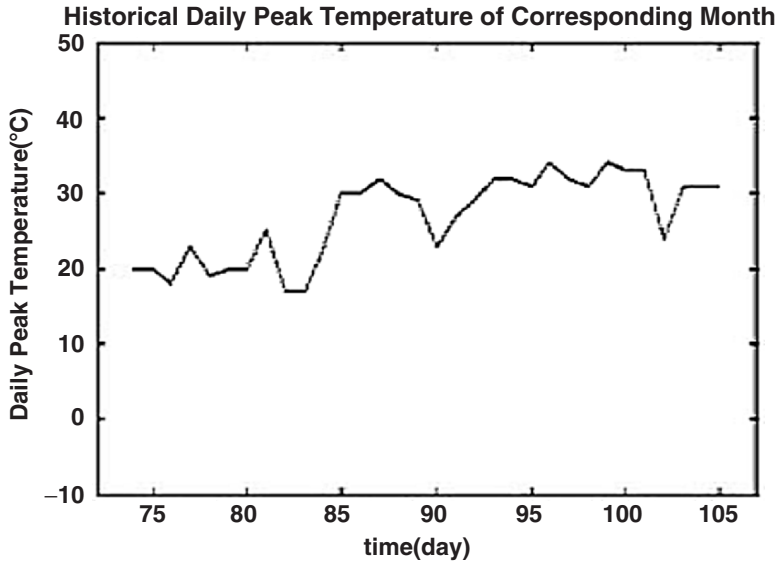

Fig. 3. Historical daily peak temperature of corresponding month

Table 1. Actual daily peak load, forecasted peak load by BP neural network and LS-SVM

July	Actual daily peak load(MW)	Forecasted by BP network(MW)	$RE_k(\%)$	Forecasted by LS-SVM(MW)	$RE_k(\%)$
1(Mon)	687	665.0	3.20	670.9	2.43
2(Tue)	721	705.8	2.11	730.5	1.32
3(Wed)	731	725.6	0.74	741.4	1.42
4(Thu)	746	759.3	1.79	762.4	2.15
5(Fri)	702	749.3	6.74	728.6	3.79
6(Sat)	654	679.2	3.86	670.7	2.56
7(Sun)	649	672.1	3.54	661.0	1.85

SVM produces less error than BP neural network method as a whole. Max and mean relative error, RMSE of forecasting by two methods are list in Table 2.

From Table 2, max relative error of forecasting (REmax) by LS-SVM method is much less than by BP neural network method. REmax by LS-SVM method reached 3.79%, while the data is 6.74% by BP neural network method. The mean relative error and RMSE of forecasting by LS-SVM is much less than by BP neural network method. It was shown that LS-SVM produces better generalization, due to the regularization factor. From above testing results, it could be conclude that the proposed method is very effective in daily peak load forecasting. From Table 2, max relative error of forecasting (REmax) by LS-SVM method is much less than by BP neural network method. REmax by LS-SVM method reached 3.79%, while the data

Table 2. Comparison of forecasting error with two methods

Forecasting Error of peak load	Forecasted by BP network	Forecasted by LS-SVM
MRE	3.14%	2.22%
REmax	6.74%	3.79%
MRSE	24.86	16.29

is 6.74% by BP neural network method. The mean relative error and RMSE of forecasting by LS-SVM is much less than by BP neural network method. It was shown that LS-SVM produces better generalization, due to the regularization factor. From above testing results, it could be conclude that the proposed method is very effective in daily peak load forecasting.

5 Conclusions

The electricity distribution utilities need accurate load data and load forecasting results for distribution network planning and operation, power production planning, load management, and customer service. LS-SVM has a good generalization ability and capability of tolerating noise in nonlinear modeling. In this paper, an approach based on LS-SVM is proposed for daily peak load forecasting in power distribution systems. For evaluating the effect of proposed method, daily peak load forecasting with LS-SVM are compared with forecasting results by BP Neural Network. The training and testing data for the ANN and LS-SVM has been obtained from Hefei Electricity Distribution Corporation in Anhui Province. To demonstrate the effectiveness of the method, four evaluating indexes, including relative error of forecasting, max and mean relative error, root mean square error of forecasting, are investigated in the paper. The proposed method has a very good performance in these evaluating indexes. Comparing with BP neural network, the LS-SVM has a better performance as whole because of its good generalization ability. It is conclude from the testing results that the proposed method is very effective in daily peak load forecasting. The proposed forecasting method could be generally applicable to most distribution utilities.

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If supplementary material is available, please provide the volume editors with:

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Fifteen Years of Fuzzy Logic in Dortmund

R. Mesiar and Vilém Novák

Summary. Fuzzy logic research in Dortmund since 1991 is described and evaluated. The main stress is paid to the basic operations, fuzzy logic in the narrow sense, inference methods (mainly IF-THEN rules), fuzzy relations, fuzzy sets with related concepts and applications. Information about some related project activities and cooperation with industrial subject is also included.

Key words: Fuzzy logic, Fuzzy relation, Fuzzy set, IF-THEN rules, Inference rules, Quantifer.

1 Introduction

The history of fuzzy sets and fuzzy logic started in 1965 by the seminal Zadeh paper [59]. Some 20 years after, fuzzy logic became popular also in Dortmund. Especially, Reusch and Moraga studied and taught fuzzy logic and in 1991 they invited Thiele to join their research group in Dortmund. Observe that though Thiele started a deep research in algebraic foundations of fuzzy logic in the former GDR, his scientific activities in this field exploded after coming to Dortmund. An important step forward not only in spreading the own results in fuzzy logic and providing a forum for the exchange of latest ideas in the field, but also in establishing contacts with several German and foreign researchers in fuzzy logic and its application was the launching of Fuzzy Days conference. Fuzzy Days in Dortmund were held for the first time in 1991 – and though the first contacts of Dortmund researchers with the fuzzy set and fuzzy logic area can be traced also to earlier periods, this moment 15 years ago can be taken as a real beginning of fuzzy logic research in Dortmund. Initially, the conference was intended for scientists and practitioners as a platform for discussions on the theory and applications of fuzzy logic. Early on, synergic links with neural networks and evolutionary algorithms were included and the conference evolved gradually to embrace the full spectrum of Computational Intelligence. Therefore, starting with the fourth Fuzzy Days in 1994, this conference was launched as a conference for Computational Intelligence – one of the world's

first conferences featuring fuzzy logic, neural networks and evolutionary algorithms in one event. Following this highly successful tradition, Fuzzy Days provide also now an international forum for reporting significant results on the theory and applications of Computational Intelligence theory, including the hot topics like the Internet or the robotics.

Note that Fuzzy Days were in nineties competing with EUFIT conference held in Aachen since 1993. It is a great success of the Dortmund group that Fuzzy Days are alive also in 2006 and after EUFIT was abolished, it remained the only important international event in the field of fuzzy logic held in West Germany. Moreover, since the sixth Fuzzy Days in 1999 the proceedings edited by Reusch were published by Springer Verlag [24–26] and thus available to a wider audience.

Another important international activity of the Dortmund group was the seminar organized to the 25th anniversary of the Department of Computer Science, chair I, in the fall of 1997. About 20 distinguished scientists from the area of fuzzy logic and other branches of Computational Intelligence have not only presented the state-of-art of discussed areas, but they have prepared their presentations also in the written form. Collection of these works was summarized and edited by Reusch and Temme in an edited volume "Computational Intelligence in Theory and Practice" [27].

The aim of this contribution is to recall some of the most important results of the group of researchers at the Department of Computer Science, chair I, University of Dortmund, in the fuzzy logic and related areas. The next section is devoted to the basic operations. Section 3 presents some important results in the fuzzy logic. Section 4 deals with IF-THEN rules and other inference methods while in Section 5 we discuss fuzzy relations. Section 6 contains several results from fuzzy sets and related concepts, including some applications. Finally, some conclusions are given.

2 Basic Operations

Among several other works dealing with basic operations with fuzzy sets, we mention [15, 34, 42, 53] due to Thiele and Vetterlein.

In [34], Mostowski's concept of generalized quantifiers was translated into fuzzy logic and it was compared with the concept of fuzzy quantifiers originally introduced by Zadeh [60] and further developed and discussed by him, Yager, Mareš and others. Based on t-norms and t-conorms, a new class of fuzzy universal and existential quantifiers was presented. Using the quantifier "almost-all" as a basic quantifier, new approaches for defining fuzzy quantifiers like "most" or "many" in arbitrary universes were proposed. Note that fuzzy linguistic quantifiers and their semantics were discussed in details in the next Thiele's work [42].

Continuation of these works in [15] is focused on the discussion of generalized "ALL" and "EXISTS" quantifiers. If the universe of discourse is finite then these quantifiers coincide with the corresponding finite extensions of the

relevant t-norm T and t-conorm S, respectively. However, this situation is different when the universe is infinite, and in such case different types of generalized "ALL" and "EXISTS" quantifiers are introduced and discussed. Observe that we meet a similar situation by infinite series (and even by uncountable series) when generalizing the standard sum of reals.

It is well known that T-fuzzy equivalences are linked to pseudo-metrics [2–4], where T is a given (continuous Archimedean) triangular norm. In [53], Vetterlein brings an interesting reverse approach. Namely, he shows how to relate a t-norm to a (pseudo-) metric by means of fuzzy equivalences. This fresh result is promissing for the further research and applications in decision-making.

3 Fuzzy Logic

Fuzzy logic in the narrow sense was developed in Dortmund mostly by Lehmke and later by Vetterlein.

Lehmke in [12] opened the problem of a gap between the automated theorem proving and logic programming in many-valued and fuzzy logics. In that paper, he closed that gap with respect to resolution theory for fuzzy logic presenting a resolution calculus for a generalization of Łukasiewicz many-valued propositional logic. Moreover, he sketched the development of a structure-preserving clausal form for this logic, he defined resolution-based rules of inference and gave some hints on how to prove the soundness and completeness of these rules with respect to the semantic consequence operator. As an important practical output of his theoretical investigations, Lehmke has designed an automatic prover for the fuzzy logic systems. As an example of its efficiency recall that this prover has shown the redundance of axiom BL3 (commutativity) in the axiomatic system for BL-logics proposed by Hájek [6] (this result was independently shown by Cintula using theoretical arguments only).

A deep research of BL-logics and BL-algebras from axiomatic point of view is due to Vetterlein. In [51] he studied partial algebras for Lukasiewicz logics and their extensions. Adding further connectives to Lukasiewicz logics, several new types of fuzzy logics are investigated, paying attention especially to the corresponding algebraic counterparts. In [50, 52], Vetterlein clarifies mutual relation of BL-algebras and effect algebras and compares BL-algebras with some algebras known from the theories of fuzzy logic and of quantum logic (e.g., MV-algebras, PL-algebras, G-algebras, BCK-algebras). Moreover, properties well known from the quantum structure theory, such as the Riesz decomposition property, compability or relative cancellation property, are studied on BL-algebras or on dual BL-algebras. Among the latest fuzzy logic results of Vetterlein, we mention only the discussion of the fuzzy logic L based on rules, see [54], and the study of MTL-algebras arising from partially ordered groups in [55].

4 Inference Methods

One of the first topics of Dortmund's research in fuzzy logic area was the study and development of various inference methods. One of its roots was the chapter [32] of Thiele dealing with the question how a cumulative inference operator can be generated by default deduction rules. The proposed codiagonal generation gave the possibility to generalize the abstract theory of monotonic inference operators developed by Tarski, Birkhoff, Hall and Schmidt. In particular, it has allowed formulating simple conditions on the given system of default deduction rules so that the generated inference operator was cumulative.

Fix-points and fix-mundis when interpreting fuzzy IF-THEN rules and rule bases by the standard Compositional Rule of Inference are discussed in [30]. In continuation of this paper [31], Temme and Fathi extended this research to the t-norm based Compositional Rule of Inference. Aggregating by Max operator, they showed that the solution for a fix-point of a single rule could be canonically extended to fix-mundis of rule base.

Moraga and Temme discussed functional equivalence between S-neural networks and fuzzy models in [19]. They introduced and characterized a family of S-functions in neural networks and allowed the interpretation of the activity of the artificial neurons as fuzzy IF-THEN rules.

When considering a fuzzy IF-THEN rule base as a system of equations for a functional, the crucial task is to ensure the solvability. This problem, including the uniqueness discussion, was performed by Thiele in [45]. Several versions of compactness inspired by the compactness of consequence operators used in the theory of formal systems from mathematical logic played a fundamental role in formulating solvability and uniqueness conditions.

Summarization of 10 years research of inference methods resulted in 2003 into a nice chapter [13] describing the mathematical foundations of fuzzy inference. Lehmke et al. brought here a comprehensive overview of several types of compositional rules of inference and processing of IF-THEN rule bases.

From recent results in this area we recall the proposal of methods for data-driven reshaping or designing the uncertainty transition of piecewise linear fuzzy sets representing the linguistic terms of the fuzzy rules. This optimization of fuzzy IF-THEN rules is presented by Moraga and Sales in [18].

New approach to extract fuzzy IF-THEN rules from data including noise by using the information matrix technique is due to Moraga and Huang in [7]. Vetterlein with Štěpnička in [58] discuss completion of fuzzy IF-THEN rule base using smoothing splines.

Interesting recent contribution to the relational model of fuzzy IF-THEN rules that relates also to results described in Section 5, is the paper [23]. Two new notions are introduced in it, namely a model of fuzzy IF-THEN rules in a structure and a continuous model of fuzzy IF-THEN rules with respect to given data. The second problem is connected with the problem of solvability of the respective system of fuzzy relation equations. It is shown the solvability

degree stands as a coefficient in the characteristic continuity inequality. If the system is solvable then the continuity of the respective model is guaranteed.

A complex view on the theory of fuzzy IF-THEN rules that encompasses not only traditional relational view on them but also their possible interpretation as special sentences of natural language is presented in [21]. This work has been inspired by works of Thiele who proposed to use tools of formal fuzzy logic for characterization of fuzzy IF-THEN rules. The theory has been formulated using the recently developed fuzzy type theory [22].

5 Fuzzy Relations

Though we have already mentioned fuzzy relations also in Section 2 in work of Vetterlein, major contributions in this field in Dortmund were done by Reusch, Moraga, Schmechel, and especially by Thiele.

A deep discussion of relationships of fuzzy equivalence relations and fuzzy partitions started in contributions [29, 33, 39, 48]. It was closed in [14] by a paper finished by Reusch and Mesiar after Thiele passed away. Recall that the transitivity of a fuzzy relation R on a given universe X is defined by

$$C(R(x,y),R(y,z)) \leq R(x,z)$$

for all $x, y, z \in X$, where $C : [0,1]^2 \to [0,1]$ is some extension of boolean conjunction.

Duality of fuzzy equivalence relations and fuzzy partitions leads to introduction of duality fitting conjunctors, i.e., commutative nondecreasing mappings $C : [0,1]^2 \to [0,1]$ with neutral element 1 (observe that duality fitting conjunctors are also called symmetric semicopulas in [1]).

Among other Thiele's results in the area of fuzzy relations we recall tolerance relations and fuzzy clustering discussed in [35–37, 41] and different characterizations of Ruspini partitions [28] presented in [38, 40].

Inconditionality of fuzzy relations was studied by Moraga et al. in [5]. To measure such inconditionality related to a given continuous t-norm T, two different methods were proposed. Moreover, the conditions when both methods result to be equivalent are given.

6 Fuzzy Sets, Related Concepts, and Applications

Since the beginning of the work of Dortmund group in fuzzy logic till now, a great attention was paid to several specific problems of the fuzzy set theory, but also to the philosophy of fuzzy sets and related concepts, as well as to the applications.

As an example for the first ring recall the recent results of Vetterlein on spline interpretation between hyperspaces of convex or fuzzy sets in [56] and a proposal of an interesting defuzzification method using Steiner points in [57] (with Navara).

The important issue of fuzzy control is treated by Moraga et al. in [20]. Motivated by an experimental scenario, basic cases of fuzzy control are presented and formally analyzed, their capabilities are discussed and their constraints are explained. Moreover, it is shown that parameterization of, either fuzzy sets or connectives used to express the rules governing a fuzzy controller allows the use of new optimization methods to improve the overall performance.

Concepts related to fuzzy sets are discussed in numerous contributions of the Dortmund group. For example, algebraic foundations of information granulation are deeply discussed by Thiele in [47]. This paper considers generation of granulations by means of equivalence relations, tolerance relations, partial order and linear order crisp and fuzzy relations.

In [43], Thiele investigated conditions under which a closure operator on the power set of a given universe may be represented by an upper approximation operator within the rough set framework, and by a modal diamond operator. Similar questions were considered in the cases of a lower approximation operator and a modal box operator, rough fuzzy sets, fuzzy rough sets, and fuzzy diamond and box operators. In a continuation of this paper [44], Thiele developed an axiomatic characterization of approximation operators that are defined using the concepts of fuzzy rough sets.

Rough sets are discussed also in another work of Thiele [46], where the explicit concept of rough set on the basis of modal logic is elaborated. The main attention is paid to the generation of lower and upper approximations and their processing exploiting readability relations modeled by equivalence relations.

Concerning the applications, we give only few examples. Moraga and Heider have reviewed in [17] the contributions to multiple-valued logic at the light of development in the area of artificial neural networks. It is shown there that it is possible to adapt methods of design of feedforward neural networks to generate networks of multiple-valued neurons to realize any multiple-valued function (compare also [16]).

In [9], Moraga et al. brought a tutorial review of spectral methods in switching and multiple-valued logic theory and the design of digital systems developed from 1991 until 2000.

Another application of Moraga et al. in [8] concerns calculation versus subjective assessment with respect to fuzzy probability. Recall that for an infinite population it is impossible to get precisely its probability distribution from the sample. Particularly, if the size of the sample is small then estimated values of the probabilities need not be so precise and so, they can be represented by some fuzzy numbers. In that case, it is possible to use the interior–outer-set model to calculate a fuzzy probability distribution, or invite some experts to review the sample and to subjective by assess. In this paper, authors, with simulation experiments and inquiring experts, have proved that the results from the calculation and the subjective assessment are very near in terms of the fuzzy expected value and the standard deviation. Thus they showed that the interior–outer-set model can replace experts to give fuzzy probabilities.

Identification of some optimal fuzzy model by means of search technique based on evolationary algorithms was considered by Moraga and Vergara in [49]. The authors considered fuzzy multiple input single output (MISO) models to identification problems and parameter estimation methods, and they have expanded the theoretical results to MIMO (multiple output) fuzzy models.

Medical applications were presented by Kiseliova, Moraga and Wagner in [10,11]. In [10], various ways of incorporation of time in an inference mechanism within the formalism of fuzzy logic was introduced. In [11], a rather expressive fuzzy temporal logic for linear time is introduced. This logic is a multivalued generalization (in Łukasiewicz style) of a two-valued linear-time temporal logic. For example, the "until" quantifier is exploited. Furthermore, this logic is obtained by introducing a generalized time quantifier applied to fuzzy time sets. In the introduced fuzzy temporal logic, generalized computational rules of inference, suitable for approximate reasoning in a temporal setting, are presented as valid formulas. Moreover, the presented approach is illustrated by some medical examples.

7 Concluding Remarks

Though our description of results of the Dortmund researchers in the fuzzy logic area is not completely exhaustive, it sufficiently illustrates the richness, depth and importance of 15 years of fuzzy logic research in Dortmund. Not only these results have formed Dortmund to become a real center of fuzzy logic based foundations of Intelligent Computing. Up to Fuzzy Days Conferences and the 25th Anniversary seminar, "Dortmunders" have succeeded to create a rich international research visiting program, inviting several distinguished scientists to collaborate with them and their students. One of its outputs is also a recent special issue of Journal of MultiValued Logic & Soft Computing dedicated to Thiele with Gottwald and Moraga as the guest editors.

Scientific activities of the Dortmund researchers in the fuzzy logic area were realized in the framework of several projects and applied in cooperation with some industry units. We mention only some of these application activities. They can be roughly splitted into the following areas:

1. Expert systems using fuzzy logic rules (in cooperation with the mechanical engineering departments of the Universities of Dortmund and Bochum, as well as with the chemical engineering department of the University of Dortmund)
   - Applications in the design of composite materials
   - Special composites made of metal and ceramics
2. Optimization of fuzzy expert systems
   - Modeling of 1D and 2D functions using fuzzy controllers
   - Improvement of the rule set

   - Evolutionary concepts for the improvement of the performance
3. Fuzzy Logic in industrial image processing (in cooperation with industry partners Mannesmann, Demag)
   - Development of operators for image processing tasks
   - Evolutionary optimization of digital filter kernels
   - Development of a new way of describing colors: Fuzzy color processing
   - Estimation of 3D features using stereo camera systems
4. Evolution strategies for the optimization of fuzzy systems (in cooperation with Mannesmann, Degussa)
   - Optimization of the fuzzy rules
   - Optimization of membership functions
   - Applications in industry
5. Fuzzy Logic and robot soccer
   - Embedded in the FIRA robot systems
   - Development of robots
   - Fuzzy logic for the control of the robots and the estimation of the current situation on the playfield
6. Fuzzy Logic and medicine (in cooperation with the University of Essen and University of Witten/Herdecke and University of Bochum)
   - Fuzzy logic based descriptions of human tissues
   - Fuzzy image segmentation
   - Fuzzy based diagnosis

Concerning the research projects, recall (using the original titles in German whenever was the case):

1. Special research projects
   - SFB 531 “Design und Management komplexer technischer Prozesse und Systeme mit Methoden der Computational Intelligence” founded by B. Reusch
   - SFB 531 “Design und Management komplexer technischer Prozesse und Systeme mit Methoden der Computational Intelligence”, Subproject A1: “Mathematische Grundlagenuntersuchungen zur Theorie der Fuzzy-IF-THEN-Regelbasen”
   - SFB 531 “Design und Management komplexer technischer Prozesse und Systeme mit Methoden der Computational Intelligence”, Subproject C1: “Unscharfe Modellierung von grosstechnischen Anlagen der Chemietechnik zur Verbesserung der Zuverlässigkeit”
   - SFB 559 “Modellierung groer Netze in der Logistik”, Subproject M12 “Multikriterielle Entscheidungsfindung”
2. Projects funded by the EC
   - “GDOES Expert System for at-the-line-control of Coated Steel Products”, Cooperation between Lehrstuhl I, Fachbereich Informatik, Universität, Dortmund, Thyssen Krupp Stahl AG, Duisburg, Voest-Alpine

Stahl Linz GmbH, Linz, Centre de Recherches Metallurgiques (CMR) Liège, Institutet för Metallforskning, Stockholm
– "RapCoat-Rapid Prototyping for Coatings" Cooperation between Lehrstuhl I, Fachbereich Informatik, Universität Dortmund, Institut für Spektrochemie und angewandte Spektroskopie, Dortmund, Voest-Alpine Stahl Linz GmbH, Linz, Centre de Recherches Metallurgiques (CMR), Liège, Institutet för Metallforskning, Stockholm

Concerning the cooperation with industry partners, it was realized with the next partners:

1. Mannesmann Dematic Engineering GmbH, "Analysis of welding points using fuzzy logic and fuzzy color processing"
2. ThyssenKrupp Stahl AG, "Quality analysis of coated steel sheets using fuzzy image processing"
3. BMW AG, "Forecasting of car faults for improved reliability, application of fuzzy based time series predicition, modeling of expert knowledge using fuzzy logic"
4. Degussa AG, Oxeno GmbH, "Prediction of pump faults using signal processing and fuzzy classification"

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Intuitionistic Fuzzy Graphs

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Summary. A new definition for intuitionistic fuzzy graph is given. Some properties of intuitionistic fuzzy graphs are considered and the authors introduced the notions of various concepts. These concepts are analyzed through suitable illustrations.

Key words: Intuitionistic fuzzy graph, Semi-μ strong path, Semi-γ strong path, Bridge, Composition.

1 Introduction

Fuzzy set [4] has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest. The concept of a fuzzy relation was defined by Zadeh [9] and it has found applications in the analysis of cluster patterns [3]. Rosenfeld [6] considered fuzzy relations on fuzzy sets and developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Then Bhattacharya [2] introduced some remarks on fuzzy graphs. Later, complement of fuzzy graphs and some operations on fuzzy graphs are introduced by Mordeson and Peng [5]. Further, Sunitha and Vijayakumar [7] defined the complement of a fuzzy graph in a different way and studied some operations on it. Yeh and Banh [8] have also introduced various connectedness concepts in fuzzy graphs. After the pioneering work of Rosenfeld [6], Yeh and Banh [8] in 1975, when some basic fuzzy graph theoretic concepts and applications have been indicated.

Atanassov [1] introduced the concept of intuitionistic fuzzy (IF) relations and intuitionistic fuzzy graphs (IFGs). Research on the theory of intuitionistic fuzzy sets (IFSs) has been witnessing an exponential growth in Mathematics and its applications. This ranges from traditional Mathematics to Information Sciences.

This leads to consider IFGs and their applications. In this paper, we introduced IFG and analyzed its components. It is further proposed by the authors that these concepts can be extended to other types of IFSs and analyzing various components.

2 Preliminaries

Definition 1. *An IFG is of the form* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *where*

(i) $\mathrm{V} = \{v_1, v_2, ..., v_n\}$ *such that* $\mu_1 : \mathrm{V} \longrightarrow [0,1]$ *and* $\gamma_1 : \mathrm{V} \longrightarrow [0,1]$ *denote the degree of membership and nonmembership of the element* $v_i \in \mathrm{V}$*, respectively, and*

$$0 \leq \mu_1(v_i) + \gamma_1(v_i) \leq 1, \quad \ldots\ldots (1)$$

for every $v_i \in \mathrm{V}$*,* $(i = 1, 2, \ldots \mathrm{n})$*,*

(ii) $\mathrm{E} \subseteq \mathrm{V} \times \mathrm{V}$ *where* $\mu_2 : \mathrm{V} \times \mathrm{V} \longrightarrow [0,1]$ *and* $\gamma_2 : \mathrm{V} \times \mathrm{V} \longrightarrow [0,1]$ *are such that*

$$\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)], \quad \ldots\ldots (2)$$
$$\gamma_2(v_i, v_j) \leq \max[\gamma_1(v_i), \gamma_1(v_j)] \quad \ldots\ldots (3)$$
$$\textit{and} \quad 0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1 \quad \ldots\ldots (4)$$

for every $(v_i, v_j) \in \mathrm{E}$*,* $(i, j = 1, 2, \ldots \mathrm{n})$*.*

Notations

The triple $\langle v_i, \mu_{1i}, \gamma_{1i} \rangle$ denotes the degree of membership and nonmembership of the vertex v_i. The triple $\langle \mathrm{e}_{ij}, \mu_{2ij}, \gamma_{2ij} \rangle$ denotes the degree of membership and nonmembership of the edge relation $\mathrm{e}_{ij} = (v_i, v_j)$ on V.

Note 1.

(i) When $\mu_{2ij} = \gamma_{2ij} = 0$, for some i and j, then there is no edge between v_i and v_j.
(ii) When either one of the following is true, then there is an edge relation between v_i and v_j.
   - $\mu_{2ij} > 0$ or $\gamma_{2ij} > 0$.
   - $\mu_{2ij} = 0$ or $\gamma_{2ij} > 0$.
   - $\mu_{2ij} > 0$ or $\gamma_{2ij} = 0$.
(iii) If one of the inequalities (1) or (2) or (3) or (4) is not satisfied, then G is not an IFG.

Example 1. Consider $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ where $\mathrm{V} = \{v_1, v_2, v_3, v_4, v_5\}$. (refer Fig. 1)

Example 2. Consider $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ where $\mathrm{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. (refer Fig. 2)

Definition 2. *An IFG* $\mathrm{H} = \langle \mathrm{V}', \mathrm{E}' \rangle$ *is said to be an IF subgraph (IFSG) of the IFG,* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *if* $\mathrm{V}' \subseteq \mathrm{V}$ *and* $\mathrm{E}' \subseteq \mathrm{E}$*.*

In other words, if $\mu'_{1i} \leq \mu_{1i}$ *;* $\gamma'_{1i} \geq \gamma_{1i}$ *and* $\mu_{2ij}{}' \leq \mu_{2ij}$ *;* $\gamma_{2ij}{}' \geq \gamma_{2ij}$ *for every* $i, j = 1, 2, \ldots, \mathrm{n}$*.*

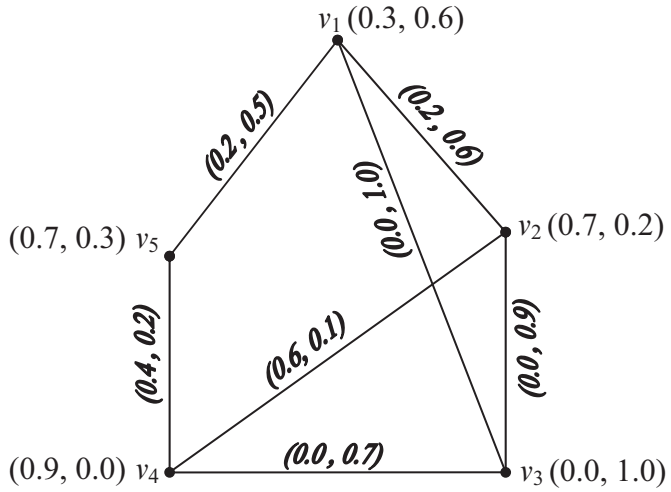

Fig. 1. Intuitionistic fuzzy graph

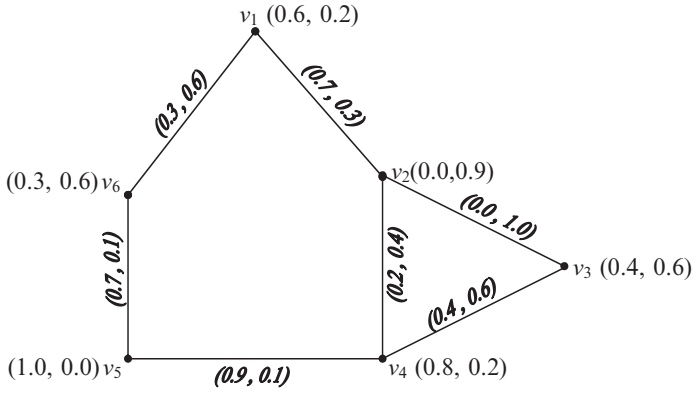

Fig. 2. G is not an intuitionistic fuzzy graph

Definition 3. *An IFG ,* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *is said to be a semi-*μ *strong IFG if*

$$\mu_{2ij} = \min\,(\mu_{1i}, \mu_{1j}),\ \ for\ every\ (v_i, v_j) \in \mathrm{E}.$$

Definition 4. *An IFG ,* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *is said to be a semi-*γ *strong IFG if*

$$\gamma_{2ij} = \max\,(\gamma_{1i}, \gamma_{1j}),\ \ for\ every\ (v_i, v_j) \in \mathrm{E}.$$

Definition 5. *An IFG,* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *is said to be a strong IFG if*

$$\mu_{2ij} = \min\,(\mu_{1i}, \mu_{1j})\ \ and\ \gamma_{2ij} = \max\,(\gamma_{1i}, \gamma_{1j})\ \ for\ all\ (v_i, v_j) \in \mathrm{E}.$$

Example 3. Let $\mathrm{V} = \{v_1, v_2, v_3, v_4, v_5\}$. (refer Fig. 3)

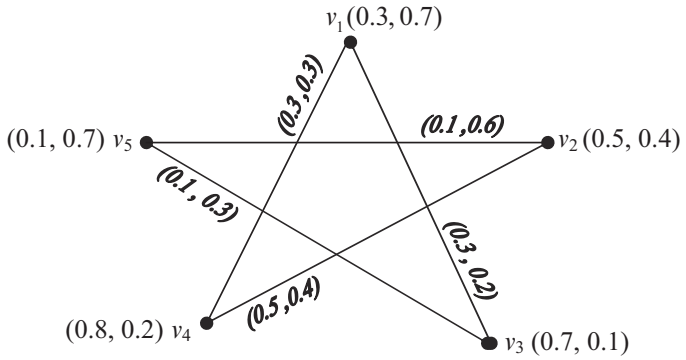

Fig. 3. Semi-μ strong IFG

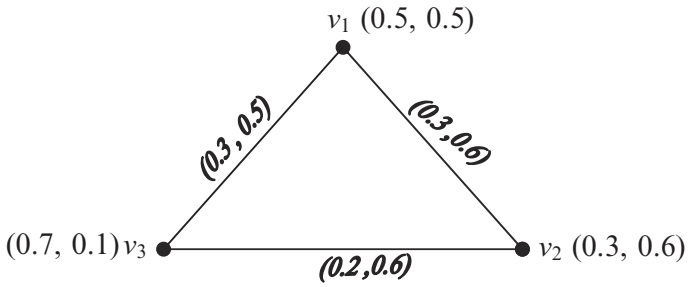

Fig. 4. Semi-γ strong IFG

Example 4. Let V = $\{v_1, v_2, v_3\}$. (refer Fig. 4)

Definition 6. *A path P in an IFG is a sequence of distinct vertices* v_1, $v_2 \ldots v_n$ *such that either one of the following conditions is satisfied:*

(a) $\mu_{2ij} > 0$ *and* $\gamma_{2ij} = 0$ *for some* i *and* j,
(b) $\mu_{2ij} = 0$ *and* $\gamma_{2ij} > 0$ *for some* i *and* j,
(c) $\mu_{2ij} > 0$ *and* $\gamma_{2ij} > 0$ *for some* i *and* j $(i, j = 1, 2, \ldots \mathrm{n})$.

Example 5. Let V = $\{v_1, v_2, v_3, v_4, v_5\}$. (refer Fig. 5)
Here $v_1v_4v_3v_2$ is a path.

Definition 7. *The length of a path* P = $v_1v_2 \ldots v_{n+1}$ $(n > 0)$ *is* n.

Definition 8. *A path* P = $v_1v_2 \ldots v_{n+1}$ *is called a cycle if* $v_1 = v_{n+1}$, *and* $n \geq 3$.

Definition 9. *Two vertices that are joined by a path are said to be connected.*

Definition 10. *The* μ*-strength of a path* P = $v_1v_2 \ldots v_n$ *is defined as*

$$\min_{i,\mathrm{j}} \{\mu_{2ij}\} \qquad \ldots\ldots \quad (5)$$

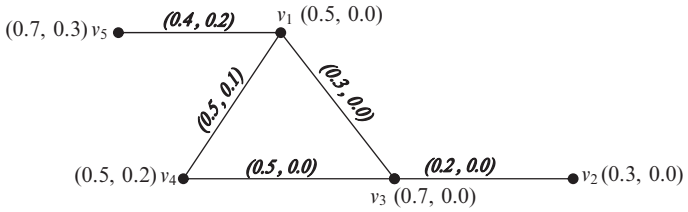

Fig. 5. A path in an IFG

and is denoted by S_μ.

The γ*-strength of a path* $P = v_1 v_2 \dots v_n$ *is defined as*

$$\max_{i,j} \{\gamma_{2ij}\} \qquad \dots\dots \quad (6)$$

and is denoted as S_γ.

Note 2.

If an edge possesses both the values (5) and (6), then it is the *strength* of the path P and is denoted by S_P.

Definition 11. *For any t,* $0 \leq t \leq 1$*, the set of triples* $\langle V_t, \mu_{1t}, \gamma_{1t} \rangle$*, where*

$$\mu_{1t} = \{v_i \in V : \mu_{1i} \geq t\} \qquad \dots\dots (7)$$

$$\textit{or} \qquad \gamma_{1t} = \{v_i \in V : \gamma_{1i} \leq t\} \qquad \dots\dots (8)$$

for some $i = 1, 2, \dots n$*, is a subset of* V
and the set of triples $\langle E_t, \mu_{2t}, \gamma_{2t} \rangle$*, where*

$$\mu_{2t} = \{(v_i, v_j) \in V \times V : \mu_{2ij} \geq t\} \qquad \dots\dots (9)$$

$$\textit{or} \qquad \gamma_{2t} = \{(v_i, v_j) \in V \times V : \gamma_{2ij} \leq t\} \qquad \dots\dots (10)$$

for some $i, j = 1, 2, \dots n$*, is a subset of* E.

Example 6. Let $V = \{v_1, v_2, v_3, v_4, v_5\}$. (refer Fig. 6)
Here, $V_{0.6} = \{v_1, v_2, v_4, v_5\}$,
$E_{0.6} = \{v_1v_2, v_2v_5, v_4v_5, v_5v_1\}$.

3 Properties

Theorem 1. *If* $0 \leq x \leq y \leq 1$*, then* (V_x, E_x) *is a subgraph of* (V_y, E_y).

Proof. Let $G = \langle V_y, E_y \rangle$ and $H = \langle V_x, E_x \rangle$.

To prove H is a subgraph of G, it is enough to prove that $V_x \subseteq V_y$ and $E_x \subseteq E_y$.

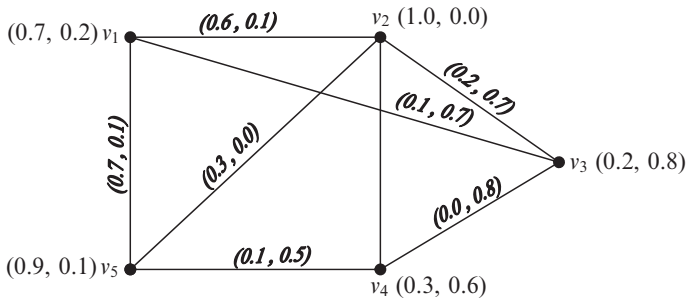

Fig. 6. $V_{0.6}$ and $E_{0.6}$

Let $v_i \in V_x$. Therefore, $\gamma_{1i} \leq x$

$$\leq y, \text{ since } x \leq y.$$
$$\Rightarrow v_i \in V_y. \text{ Hence } V_x \subseteq V_y.$$

Let $(v_i, v_j) \in E_x$. Therefore, $\gamma_{2ij} \leq x$

$$\leq y, \text{ since } x \leq y.$$

Thus, we have $(v_i, v_j) \in E_y$. Hence, $E_x \subseteq E_y$.
Hence, (V_x, E_x) is a subgraph of (V_y, E_y). □

Theorem 2. *If* $H = \langle V', E' \rangle$ *is an IF subgraph of* $G = \langle V, E \rangle$*, then for any* $0 \leq x \leq 1$, $\langle V'_x, E'_x \rangle$ *is an IF subgraph of* $\langle V_x, E_x \rangle$.

Proof. Given $V' \subseteq V$ and $E' \subseteq E$.
To prove $V'_x \subseteq V_x$; $E'_x \subseteq E_x$, it is enough to prove (7)–(10) for μ or γ.

Let $v_i \in V'_x$
$\Rightarrow \mu'_{1i} \geq x$
$\Rightarrow \mu_{1i} \geq x$, since $\mu'_1 \leq \mu_1$
$\Rightarrow v_i \in V_x$
$\Rightarrow V'_x \subseteq V_x$

Let $(v_i, v_j) \in E'_x$
Therefore, $\mu'_{2ij} \geq x$

$\Rightarrow \mu_{2ij} \geq x$, since $\mu'_2 \leq \mu_2$
$\Rightarrow (v_i, v_j) \in E_x$

Hence, $E'_x \subseteq E_x$.
Therefore, $\langle V'_x, E'_x \rangle$ is an IF subgraph of $\langle V_x, E_x \rangle$. □

Definition 12. *Let* $\langle e_{ij}, \mu_{2ij}, \gamma_{2ij}\rangle$ *and* $\langle e_{jk}, \mu_{2jk}, \gamma_{2jk}\rangle$ *be two edge relations on* V. *The composition of these two edge relations is an IFS, denoted by* $e_{ij} \bullet e_{jk}$, *is of the form* $\langle e_{ik}, \mu_{2ik}, \gamma_{2ik}\rangle$ *where*

$$\mu_{2ik} = \max\{\min_j[\mu_{2ij}, \mu_{2jk}]\} \text{ and}$$
$$\gamma_{2ik} = \min\{\max_j[\gamma_{2ij}, \gamma_{2jk}]\}, \text{ for all } v_i, v_k \in V.$$

Definition 13. *Let* $\langle e_{ij}, \mu_{2ij}, \gamma_{2ij}\rangle$ *be an edge relation on* V. *Then it is said to be*

(i) reflexive if $\langle e_{ii}, \mu_{2ii}, \gamma_{2ii}\rangle = \langle v_i, \mu_{1i}, \gamma_{1i}\rangle$ *for all* $v_i \in V$.
(ii) symmetric if $\langle e_{ij}, \mu_{2ij}, \gamma_{2ij}\rangle = \langle e_{ji}, \mu_{2ji}, \gamma_{2ji}\rangle$, *for all* $v_i, v_j \in V$.
(iii) transitive if the edge relations (v_i, v_j) *and* (v_j, v_k) *imply the edge relation* (v_i, v_k).

Definition 14. *The powers of edge relation* e_{ij} *are defined as*

$$e_{ij}^1 = e_{ij} = \langle e_{ij}, \mu_{2ij}, \gamma_{2ij}\rangle$$
$$e_{ij}^2 = e_{ij} \bullet e_{ij} = \langle e_{ij}, \mu_{2ij}^2, \gamma_{2ij}^2\rangle$$
$$e_{ij}^3 = e_{ij} \bullet e_{ij} \bullet e_{ij} = \langle e_{ij}, \mu_{2ij}^3, \gamma_{2ij}^3\rangle \text{ and so on.}$$

Also,

$$e_{ij}^\infty = \langle e_{ij}, \mu_{2ij}^\infty, \gamma_{2ij}^\infty\rangle$$

where $\mu_{2ij}^\infty = \max_{k=1,2,\ldots n}\{\mu_{2ij}^k\}$ *and* $\gamma_{2ij}^\infty = \min_{k=1,2,\ldots n}\{\gamma_{2ij}^k\}$ *are the* μ*-strength and* γ*-strength of connectedness between any two vertices* v_i *and* v_j.

Also,

$$e_{ij}^0 = \begin{cases} 0, & \text{if } v_i \neq v_j, \\ \langle v_i, \mu_{1i}, \gamma_{1i}\rangle, & \text{if } v_i = v_j. \end{cases}$$

Theorem 3. *If* $H = \langle V', E'\rangle$ *is an IF subgraph of* $G = \langle V, E\rangle$, *then for some* $(v_i, v_j) \in E$, $\mu_{2ij}^{'\infty} \leq \mu_{2ij}^\infty$ *and* $\gamma_{2ij}^{'\infty} \geq \gamma_{2ij}^\infty$.

Proof. By given, $V' \subseteq V$ and $E' \subseteq E$.

$$\Rightarrow \mu'_{1i} \leq \mu_{1i}; \gamma'_{1i} \geq \gamma_{1i}, \text{for every } v_i \in V \quad \ldots\ldots(11)$$

and

$$\mu'_{2ij} \leq \mu_{2ij}; \quad \ldots\ldots (12)$$
$$\gamma'_{2ij} \geq \gamma_{2ij} \quad \ldots\ldots (13)$$

for every $v_i, v_j \in V$.

Consider a path $v_1 v_2 \dots v_n$ of H.
Here,

$$\mu'^{\infty}_{2ij} = \min_{k=1,2,\dots n} \left\{ (\mu'_{2ij})^k \right\} \quad \dots\dots \quad (14)$$

$$\gamma'^{\infty}_{2ij} = \max_{k=1,2,\dots n} \left\{ (\gamma'_{2ij})^k \right\} \quad \dots\dots \quad (15)$$

and

$$\mu^{\infty}_{2ij} = \min_{k=1,2,\dots n} \left\{ (\mu_{2ij})^k \right\} \quad \dots\dots \quad (16)$$

$$\gamma^{\infty}_{2ij} = \max_{k=1,2,\dots n} \left\{ (\gamma_{2ij})^k \right\} \quad \dots\dots \quad (17)$$

Therefore, we have

$$\begin{aligned} \mu'^{\infty}_{2ij} &= \min_{k=1,2,\dots n} \left\{ (\mu'_{2ij})^k \right\} \\ &\leq \min_{k=1,2,\dots n} \left\{ (\mu_{2ij})^k \right\}, \text{ by (12)} \\ &= \mu^{\infty}_{2ij}. \end{aligned}$$

Also,

$$\begin{aligned} \gamma'^{\infty}_{2ij} &= \max_{k=1,2,\dots n} \left\{ (\gamma'_{2ij})^k \right\} \\ &\geq \max_{k=1,2,\dots n} \left\{ (\gamma_{2ij})^k \right\}, \text{by (13)} \\ &= \gamma^{\infty}_{2ij}. \end{aligned}$$

Hence proved. □

Definition 15. *Let* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *be an IFG. Let* v_i, v_j *be any two distinct vertices and* $\mathrm{H} = \langle \mathrm{V}', \mathrm{E}' \rangle$ *be an IF subgraph of* G *obtained by deleting the edge* (v_i, v_j).

That is, $\mathrm{H} = \langle \mathrm{V}', \mathrm{E}' \rangle$, *where*

$$\begin{aligned} & \mu'_{2ij} = 0 \text{ and } \gamma'_{2ij} = 0 \\ \text{and} \quad & \mu'_2 = \mu_2 \\ & \gamma'_2 = \gamma_2 \text{ for all other edges.} \end{aligned}$$

Now, (v_i, v_j) *is said to be a bridge in* G, *if either* $\mu'^{\infty}_{2\mathrm{xy}} < \mu^{\infty}_{2xy}$ *and* $\gamma'^{\infty}_{2xy} \geq \gamma^{\infty}_{2xy}$ *or* $\mu'^{\infty}_{2\mathrm{xy}} \leq \mu^{\infty}_{2xy}$ *and* $\gamma'^{\infty}_{2xy} > \gamma^{\infty}_{2xy}$, *for some* v_x, $v_y \in \mathrm{V}$.

In other words, deleting an edge (v_i, v_j) reduces the strength of connectedness between some pair of vertices (or) (v_i, v_j) is a bridge if, there exists v_x, v_y such that, (v_i, v_j) is an edge of every strongest path from v_x to v_y.

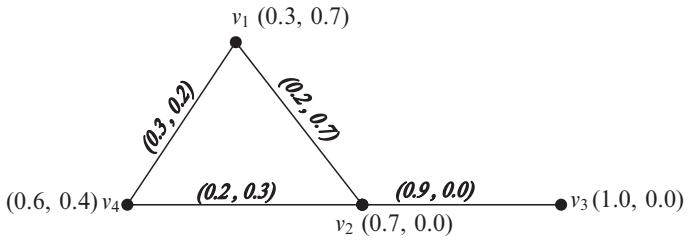

Fig. 7. (v_1, v_4) is a bridge

Example 7. Let $\mathrm{V} = \{v_1, v_2, v_3, v_4\}$.

In Fig. 7, the strength of $\mathrm{P} = v_1v_4$ in G is (0.3, 0.2). Also, the strength of $\mathrm{P}' = v_1v_2v_4$ is (0.2, 0.3). Here, (v_1, v_4) is a bridge, because if we delete (v_1,v_4) from G, the strength of the connectedness between v_1 and v_4 in $\mathrm{G} - (v_1, v_4)$ is decreased.

Theorem 4. *Let* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *be an IFG. For any two vertices* v_i, v_j *in* G*, the following conditions are equivalent:*

(i) (v_i,v_j) *is a bridge.*
(ii) $\mu^{'\infty}_{2ij} < \mu_{2ij}$ *and* $\gamma^{'\infty}_{2ij} > \gamma_{2ij}$.
(iii) (v_i ,v_j) *is not an edge of any cycle.*

Proof. (ii) $\Rightarrow$ (i).

$$\text{Assume } \mu^{'\infty}_{2ij} < \mu_{2ij} \text{ and } \gamma^{'\infty}_{2ij} > \gamma_{2ij}.$$

To prove (v_i ,v_j) is a bridge. If (v_i , v_j) is not a bridge, then

$$\mu^{'\infty}_{2ij} = \mu^{\infty}_{2ij} \geq \mu_{2ij}, \text{ and } \gamma^{'\infty}_{2ij} = \gamma^{\infty}_{2ij} \leq \gamma_{2ij}$$

which implies $\mu^{'\infty}_{2ij} \geq \mu_{2ij}$ and $\gamma^{'\infty}_{2ij} \leq \gamma_{2ij}$, a contradiction.

Hence, (v_i, v_j) is a bridge.

(i) $\Rightarrow$ (iii)

Assume (v_i, v_j) is a bridge. To prove (v_i, v_j) is not an edge of any cycle.

If (v_i, v_j) is an edge of a cycle, then any path involving the edge (v_i, v_j) can be converted into a path not involving (v_i, v_j) by using the rest of the cycle as a path from v_i to v_j. This implies (v_i, v_j) cannot be a bridge which is a contradiction to our assumption. Therefore, (v_i, v_j) is not an edge of any cycle.

(iii) $\Rightarrow$ (ii)

Assume (v_i, v_j) is not an edge of any cycle.

To prove $\mu^{'\infty}_{2ij} < \mu_{2ij}$ and $\gamma^{'\infty}_{2ij} > \gamma_{2ij}$.

Assume that $\mu'^{\infty}_{2ij} \geq \mu_{2ij}$ and $\gamma'^{\infty}_{2ij} \leq \gamma_{2ij}$. Then, there is a path from v_i to v_j not involving (v_i, v_j) that has strength greater than or equal to μ_{2ij} and less than or equal to γ_{2ij} and this path together with (v_i, v_j) forms a cycle which is a contradiction. Hence, $\mu'^{\infty}_{2ij} < \mu_{2ij}$ and $\gamma'^{\infty}_{2ij} > \gamma_{2ij}$. Therefore, the statements (i), (ii) and (iii) are equivalent. □

Theorem 5. *Let* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *be an IFG with the set of vertices* V. *Then*

(i) If μ_{2ij} *and* γ_{2ij} *are constants for all* v_i , $v_j \in \mathrm{V}$, *then* G *has no bridge.*
(ii) If μ_{2ij} *and* γ_{2ij} *are not constants for all* $(v_i, v_j) \in \mathrm{E}$, *then* G *has at least one bridge.*

Proof. (i) Let μ_{2ij} and γ_{2ij} are constants for all v_i , $v_j \in \mathrm{V}$.

Let $\mu_{2ij} = \mathrm{c}_1$ and $\gamma_{2ij} = \mathrm{c}_2$ for all v_i , $v_j \in \mathrm{V}$, where $0 \leq \mathrm{c}_1 \leq 1$ and $0 \leq \mathrm{c}_2 \leq 1$.

In this IFG, since each edge has the same weight (the degree of membership and nonmembership values of an edge), deleting any edge does not reduce the strength of connectedness between any pair of vertices. Hence, G has no bridge.

(ii) Assume that μ_{2ij} and γ_{2ij} are not constants for all $(v_i, v_j) \in \mathrm{E}$.

Choose an edge $(v_x, v_y) \in \mathrm{E}$ such that

$$\mu_{2\mathrm{xy}} = \max\{\mu_{2ij}\}$$
$$\gamma_{2\mathrm{xy}} = \min\{\gamma_{2ij}\}, \text{ for all } v_i, v_j \in \mathrm{V}.$$
$$\text{Therefore, } \mu_{2\mathrm{xy}} > 0 \text{ and } \gamma_{2\mathrm{xy}} < 1.$$

There exists at least one edge $(v_\mathrm{s}, v_\mathrm{t})$ distinct from $(v_\mathrm{x}, v_\mathrm{y})$ such that

$$\mu_{2\mathrm{st}} < \mu_{2\mathrm{xy}} \text{ and } \gamma_{2\mathrm{st}} > \gamma_{2\mathrm{xy}}.$$

We claim that $(v_\mathrm{x}, v_\mathrm{y})$ is a bridge of G. For, if we delete the edge $(v_\mathrm{x}, v_\mathrm{y})$, then the strength of connectedness between v_x and v_y in the IF subgraph thus obtained is decreased. In other words, $\mu'^{\infty}_{2\mathrm{xy}} < \mu_{2\mathrm{xy}}$ and $\gamma'^{\infty}_{2\mathrm{xy}} > \gamma_{2\mathrm{xy}}$.

Therefore, by Theorem 4, $(v_\mathrm{x}, v_\mathrm{y})$ is a bridge of G. □

Corollary 1. *In an IFG,* $\mathrm{G} = \langle \mathrm{V}, \mathrm{E} \rangle$ *for which* $\mu_2 : \mathrm{V} \times \mathrm{V} \longrightarrow [0, 1]$ *and* $\gamma_2 : \mathrm{V} \times \mathrm{V} \longrightarrow [0, 1]$ *are not constant mapping, an edge* (v_i, v_j) *for which* μ_{2ij} *is maximum and* γ_{2ij} *is minimum . Therefore it is a bridge of* G.

Definition 16. *A vertex* v_i *is said to be a cut-vertex in* G *if deleting a vertex* v_i *reduces the strength of connectedness between some pair of vertices or* v_i *is a cut vertex if and only if there exists* v_x, v_y *such that* v_i *is a vertex of every strongest path from* v_x *to* v_y.

In other words, $\mu'^{\infty}_{2\mathrm{xy}} \leq \mu_{2\mathrm{xy}}$ *and* $\gamma'^{\infty}_{2\mathrm{xy}} < \gamma_{2\mathrm{xy}}$ *(or)* $\mu'^{\infty}_{2\mathrm{xy}} < \mu_{2\mathrm{xy}}$ *and* $\gamma'^{\infty}_{2\mathrm{xy}} \leq \gamma_{2\mathrm{xy}}$ *for some* v_x, $v_y \in \mathrm{V}$.

Example 8. Let $\mathrm{V} = \{v_1, v_2, v_3, v_4, v_5\}$.

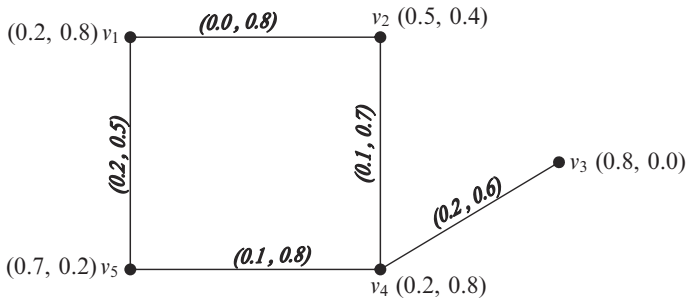

Fig. 8. v_1 is a cut-vertex

4 Conclusion

In this paper, the intuitionistic fuzzy extension of some known concepts of fuzzy graphs has been investigated. Much more work could be done to investigate the structure of IFG. It would be useful, since IFGs have applications in pattern clustering and network analysis which in turn would have applications in telecommunications. In this work, we have restricted our discussion to the first type IFS. It is also proposed to extend these concepts on the other extensions of IFSs.

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On Some Intuitionistic Properties of Intuitionistic Fuzzy Implications and Negations

Trifon A. Trifonov and Krassimir T. Atanassov

Summary. In a series of papers 23 different intuitionistic fuzzy implications were constructed. They generate five intuitionistic fuzzy negations. In our current work we summarize the properties of these operations by checking a list of axioms, including those of intuitionistic logic.

1 Introduction: on Some Previous Results

A series of 23 different intuitionistic fuzzy implications were constructed in [3–7, 11] and some of their properties were noted. Here we shall study the properties of these implications systematically and extensively.

In intuitionistic fuzzy logic if x is a variable then its truth value is represented by the ordered couple

$$V(x) = \langle a, b\rangle, \tag{1}$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of nonvalidity of x. For simplicity of presentation we shall use the following three variables x, y and z with their corresponding truth values: $V(x) = \langle a, b\rangle, V(y) = \langle c, d\rangle, V(z) = \langle e, f\rangle$ $(a, b, c, d, e, f, a+b, c+d, e+f \in [0, 1])$. We shall also consider connectives of propositional logic over variables instead of connectives over well-formed propositional formulas. It is clear that the former approach can be easily extended to the latter.

For the needs of the discussion below we shall define the notion of *Intuitionistic Fuzzy Tautology* (IFT, see [1, 2]) by:

$$x \text{ is an IFT if and only if } a \geq b, \tag{2}$$

while x will be a (classical) *tautology* iff $a = 1$ and $b = 0$.

Obviously, the notion of IFT is weaker than the notion of tautology. Subsequently, any formula, which is a tautology is also an IFT.

Let us also consider the standard partial ordering of IF truth values:

$$V(x) \leq V(y) \text{ if } a \leq c \text{ and } b \geq d. \tag{3}$$

The operations "conjunction" (&) and "disjunction" ($\vee$) are defined (see [1,2] by:

$$V(x\,\&\,y) = \langle \min(a,c), \max(b,d)\rangle, \quad V(x \vee y) = \langle \max(a,c), \min(b,d)\rangle. \tag{4}$$

In some definitions we shall use functions sg and $\overline{sg}$:

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad \overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases} \tag{5}$$

We will consider 23 possible definitions of the "implication" operation, listed in Table 1.

Each implication operation $I(x,y)$ can define a negation operation $N(x)$ by the following equality:

$$N(x) = I(x,F), \text{ where } V(F) = \langle 0,1\rangle. \tag{6}$$

The negations, generated by the implication operations are given in Table 1.

2 Main Results

2.1 Klir and Yuan's Axioms

In a book by Georg Klir and Bo Yuan [8] nine axioms for fuzzy implications are introduced. They are the following:

Axiom 1 $(\forall x,y)(x \leq y \rightarrow (\forall z)(I(x,z) \geq I(y,z))$.
Axiom 2 $(\forall x,y)(x \leq y \rightarrow (\forall z)(I(z,x) \leq I(z,y))$.
Axiom 3 $(\forall y)(I(0,y) = 1)$.
Axiom 4 $(\forall y)(I(1,y) = y)$.
Axiom 5 $(\forall x)(I(x,x) = 1)$.
Axiom 6 $(\forall x,y,z)(I(x,I(y,z)) = I(y,I(x,z)))$.
Axiom 7 $(\forall x,y)(I(x,y) = 1$ iff $x \leq y)$.
Axiom 8 $(\forall x,y)(I(x,y) = I(N(y),N(x)))$, where N is a negation.
Axiom 9 I is a continuous function.

Table 3 summarizes which of Klir and Yuan's axioms are satisfied by the 23 implications. The condition "= 1" should be interpreted as "is a tautology." If the axiom is valid using an interpretation "is an IFT" instead, the number of the axiom is marked by an asterisk (*). We should note that the validity of Axiom 7 does not imply the validity of Axiom 7*. We should also note that Axiom 8 is checked using the classical intuitionistic fuzzy negation ($\neg_1$); if it is valid using the respective implication-generated negation as $N(x)$, then the axiom is listed as 8^N.

The validity of each of these assertions can be checked directly. In some cases this is a long and tedious procedure, which can be automatically performed by a proof-checking program as the one suggested in [10].

Table 1. List of intuitionistic fuzzy implications

Notation	Name	Form of implication
$\rightarrow_1$	Zadeh	$\langle \max(b, \min(a, c)), \min(a, d)) \rangle$
$\rightarrow_2$	Gaines-Rescher	$\langle 1 - \mathrm{sg}(a - c), d.\mathrm{sg}(a - c) \rangle$
$\rightarrow_3$	Gödel	$\langle 1 - (1 - c).\mathrm{sg}(a - c), d.\mathrm{sg}(a - c) \rangle$
$\rightarrow_4$	Kleene-Dienes	$\langle \max(b, c), \min(a, d) \rangle$
$\rightarrow_5$	Lukasiewicz	$\langle \min(1, b + c), \max(0, a + d - 1) \rangle$
$\rightarrow_6$	Reichenbach	$\langle b + ac, ad \rangle$
$\rightarrow_7$	Willmott	$\langle \min(\max(b, c), \max(a, b), \max(c, d)),$ $\max(\min(a, d), \min(a, b), \min(c, d)) \rangle$
$\rightarrow_8$	Wu	$\langle 1 - (1 - \min(b, c)).\mathrm{sg}(a - c),$ $\max(a, d).\mathrm{sg}(a - c).\mathrm{sg}(d - b) \rangle$
$\rightarrow_9$	Klir and Yuan 1	$\langle b + a^2c, ab + a^2d \rangle$
$\rightarrow_{10}$	Klir and Yuan 2	$\langle c.\overline{sg}(1 - a) + \mathrm{sg}(1 - a),$ $(\overline{sg}(1 - c) + b.\mathrm{sg}(1 - c))$ $d.\overline{sg}(1 - a) + a.\mathrm{sg}(1 - a).\mathrm{sg}(1 - c) \rangle$
$\rightarrow_{11}$	Atanassov 1	$\langle 1 - (1 - c).\mathrm{sg}(a - c), d.\mathrm{sg}(a - c).\mathrm{sg}(d - b) \rangle$
$\rightarrow_{12}$	Atanassov 2	$\langle \max(b, c), 1 - \max(b, c) \rangle$
$\rightarrow_{13}$	Atanassov and Kolev	$\langle b + c - b.c, a.d \rangle$
$\rightarrow_{14}$	Atanassov and Trifonov	$\langle 1 - (1 - c).\mathrm{sg}(a - c) - d.\overline{sg}(a - c).\mathrm{sg}(d - b),$ $d.\mathrm{sg}(d - b) \rangle$
$\rightarrow_{15}$	Atanassov 3	$\langle 1 - (1 - \min(b, c)).\mathrm{sg}(\mathrm{sg}(a - c) + \mathrm{sg}(d - b))$ $- \min(b, c).\mathrm{sg}(a - c).\mathrm{sg}(d - b),$ $1 - (1 - \max(a, d)).\mathrm{sg}(\overline{sg}(a - c) + \overline{sg}(d - b))$ $- \max(a, d).\overline{sg}(a - c).\overline{sg}(d - b) \rangle$
$\rightarrow_{16}$		$\langle \max(1 - \mathrm{sg}(a), c), \min(\mathrm{sg}(a), d) \rangle$
$\rightarrow_{17}$		$\langle \max(b, c), \min(a.b + a^2, d) \rangle$
$\rightarrow_{18}$		$\langle \max(b, c), \min(1 - b, d) \rangle$
$\rightarrow_{19}$		$\langle \max(1 - \mathrm{sg}(\mathrm{sg}(a) + \mathrm{sg}(1 - b)), c),$ $\min(\mathrm{sg}(1 - b), d) \rangle$
$\rightarrow_{20}$		$\langle \max(1 - \mathrm{sg}(a), 1 - \mathrm{sg}(1 - \mathrm{sg}(c))), \min(\mathrm{sg}(a),$ $\mathrm{sg}(1 - sg(c))) \rangle$
$\rightarrow_{21}$		$\langle \max(b, c(c + d)), \min(a(a + b), d(c^2 + d + cd)) \rangle$
$\rightarrow_{22}$		$\langle \max(b, 1 - d), \min(1 - b, d) \rangle$
$\rightarrow_{23}$		$\langle \max(1 - \mathrm{sg}(\mathrm{sg}(a) + \mathrm{sg}(1 - b)),$ $1 - \mathrm{sg}(\mathrm{sg}(1 - \mathrm{sg}(\mathrm{sg}(c) + \mathrm{sg}(1 - d)))$ $+\mathrm{sg}(1 - \mathrm{sg}(1 - d)))),$ $\min(\mathrm{sg}(1 - b), \mathrm{sg}(1 - \mathrm{sg}(1 - d))) \rangle$

2.2 Intuitionistic Logic Axioms

The next and more important question is which of the introduced implications satisfy all the axioms of Propositional Intuitionistic Logic (IL) (see for example [9]).

The validity of the IL axioms was already checked for some implications in [6]. Here we shall give an full list of valid axioms for each one of the

Table 2. List of intuitionistic fuzzy negations

		Name	Form of negation
$\rightarrow_1$	$\neg_1$	Zadeh	$\langle b, a\rangle$
$\rightarrow_2$	$\neg_2$	Gaines-Rescher	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$
$\rightarrow_3$	$\neg_2$	Gödel	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$
$\rightarrow_4$	$\neg_1$	Kleene-Dienes	$\langle b, a\rangle$
$\rightarrow_5$	$\neg_1$	Lukasiewicz	$\langle b, a\rangle$
$\rightarrow_6$	$\neg_1$	Reichenbach	$\langle b, a\rangle$
$\rightarrow_7$	$\neg_1$	Willmott	$\langle b, a\rangle$
$\rightarrow_8$	$\neg_2$	Wu	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a).\mathrm{sg}(1-b)\rangle$
$\rightarrow_9$	$\neg_3$	Klir and Yuan 1	$\langle b, a.b+a^2\rangle$
$\rightarrow_{10}$	$\neg_1$	Klir and Yuan 2	$\langle \mathrm{sg}(1-a).b, \overline{sg}(1-a)+a.\mathrm{sg}(1-a)\rangle$
$\rightarrow_{11}$	$\neg_2$	Atanassov 1	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a).\mathrm{sg}(1-b)\rangle$
$\rightarrow_{12}$	$\neg_4$	Atanassov 2	$\langle b, 1-b\rangle$
$\rightarrow_{13}$	$\neg_1$	Atanassov and Kolev	$\langle b, a\rangle$
$\rightarrow_{14}$	$\neg_5$	Atanassov and Trifonov	$\langle 1-\mathrm{sg}(a)-\overline{sg}(a).\mathrm{sg}(1-b), \mathrm{sg}(1-b)\rangle$
$\rightarrow_{15}$	$\neg_5$	Atanassov 3	$\langle 1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)), 1-\overline{sg}(a).\overline{sg}(1-b)\rangle$
$\rightarrow_{16}$	$\neg_2$		$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$
$\rightarrow_{17}$	$\neg_3$		$\langle b, a.b+a^2\rangle$
$\rightarrow_{18}$	$\neg_4$		$\langle b, 1-b\rangle$
$\rightarrow_{19}$	$\neg_5$		$\langle 1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)), \mathrm{sg}(1-b)\rangle$
$\rightarrow_{20}$	$\neg_2$		$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$
$\rightarrow_{21}$	$\neg_3$		$\langle b, a.b+a^2\rangle$
$\rightarrow_{22}$	$\neg_4$		$\langle b, 1-b\rangle$
$\rightarrow_{23}$	$\neg_5$		$\langle 1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)), \mathrm{sg}(1-b)\rangle$

23 implications. We will again verify the validity axioms in two variants – tautological validity (Table 4) and IFT validity (Table 5).

We use the following list of axioms for propositional intuitionistic logic:

(a) $A \rightarrow A$,
(b) $A \rightarrow (B \rightarrow A)$,
(c) $A \rightarrow (B \rightarrow (A\&B))$,
(d) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$,
(e) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,
(f) $A \rightarrow \neg\neg A$,
(g) $\neg(A\&\neg A)$,
(h) $(\neg A \vee B) \rightarrow (A \rightarrow B)$,
(i) $\neg(A \vee B) \rightarrow (\neg A\&\neg B)$,
(j) $(\neg A\&\neg B) \rightarrow \neg(A \vee B)$,
(k) $(\neg A \vee \neg B) \rightarrow \neg(A\&B)$,
(l) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$,
(m) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$,
(n) $\neg\neg\neg A \rightarrow \neg A$,
(o) $\neg A \rightarrow \neg\neg\neg A$,
(p) $\neg\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg\neg B)$,
(q) $(C \rightarrow A) \rightarrow ((C \rightarrow (A \rightarrow B)) \rightarrow (C \rightarrow B))$.

Table 3. List of axioms of Klir and Yuan that are satisfied by intuitionistic fuzzy implications

Notation	
$\to_1$	2,3,4,5*,9
$\to_2$	1,2,3,5
$\to_3$	1,2,3,4,5,6
$\to_4$	1,2,3,4,5*,6,8,9
$\to_5$	1,2,3,4,5*,6,8,9
$\to_6$	2,3,4,5*,9
$\to_7$	3*,4,5*,8,9
$\to_8$	1,2,3,5
$\to_9$	2,3,4,5*
$\to_{10}$	2,3,4
$\to_{11}$	1,2,3,4,5,6
$\to_{12}$	1,2,3,6,8,9
$\to_{13}$	1,2,3,4,5*,6,8,9
$\to_{14}$	1,2,3,4,5,6,7
$\to_{15}$	1,2,3,5,7,7*,8
$\to_{16}$	1,2,3,4,6
$\to_{17}$	2,3,4,5*,6,9
$\to_{18}$	1,2,3,4,5*,6,9
$\to_{19}$	1,2,3,4,6
$\to_{20}$	$1,2,3,5,6,8^N$
$\to_{21}$	?
$\to_{22}$	$1,2,3,5^*,6,8^N,9$
$\to_{23}$	$1,2,3,5,6,8^N$

The most important of the results collected in Table 4 can be formulated as the following:

Theorem 1. *Implications* $\to_3$, $\to_{11}$, $\to_{14}$, $\to_{20}$, $\to_{23}$ *satisfy all intuitionistic logic axioms as tautologies.*

The validity of axioms for cells marked by a question mark (?) in Table 5 and in the tables below is not yet clear and is an open problem.

The most important of the results collected in Table 5 can be formulated as following

Theorem 2. *Implications* $\to_1$, $\to_3$, $\to_4$, $\to_5$, $\to_{11}$, $\to_{14}$, $\to_{18}$, $\to_{20}$, $\to_{22}$, $\to_{23}$ *satisfy all intuitionistic logic as IFTs.*

Finally, let us consider the Modus Ponens rule in the following two forms

$$I(x \,\&\, I(x,y), y) \tag{7}$$

$$\frac{x, \quad I(x,y)}{y}, \tag{8}$$

Table 4. List of axioms of the intuitionistic logic that are satisfied by intuitionistic fuzzy implications as tautologies

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	M1	M2
$\to_1$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_2$	+	−	−	−	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+
$\to_3$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
$\to_4$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_5$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_6$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_7$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_8$	+	−	−	−	−	+	+	−	+	+	+	+	+	+	+	+	−	+	+
$\to_9$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_{10}$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_{11}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
$\to_{12}$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_{13}$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_{14}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
$\to_{15}$	+	−	−	−	−	+	+	−	+	+	+	+	+	+	+	+	−	+	+
$\to_{16}$	−	−	−	−	−	+	+	−	+	+	+	+	+	+	+	+	−	+	−
$\to_{17}$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_{18}$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−
$\to_{19}$	−	−	−	−	−	+	+	−	+	+	+	+	+	+	+	+	−	+	−
$\to_{20}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_{22}$	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−
$\to_{23}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+

marked in Tables 4 and 5 by M1 and M2, respectively. In Table 4, M1 and M2 are checked for tautological validity, while in Table 5 they are checked for IFT validity.

The following assertions can be proved:

Theorem 3. *(a) Implications* $\to_1, \to_2, \to_3, \to_4, \to_5, \to_6, \to_7, \to_8, \to_9, \to_{10}, \to_{11}, \to_{12}, \to_{13}, \to_{14}, \to_{15}, \to_{16}, \to_{17}, \to_{18}, \to_{19}$ *satisfy M1 (7) as a tautology.*

(b) Implications $\to_2, \to_3, \to_8, \to_{11}, \to_{14}, \to_{15}, \to_{20}, \to_{23}$ *satisfy M2 (8) as a rule with tautologies.*

Theorem 4. *(a) Implications* $\to_{15}, \to_{19}$ *satisfy M1 (7) as an IFT.*

(b) Implications $\to_1, \to_2, \to_3, \to_4, \to_5, \to_8, \to_{11}, \to_{12}, \to_{13}, \to_{14}, \to_{15}, \to_{18}, \to_{20}, \to_{22}, \to_{23}$ *satisfy M2 (8) as a rule with IFTs.*

3 Conclusion

Some months ago the concept of intuitionistic fuzzy set was criticized to be not related to intuitionism. Truly, for a long time all researchers in this area used

Table 5. List of axioms of the intuitionistic logic that are satisfied by intuitionistic fuzzy implications as IFTs as tautologies

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	M1	M2
$\to_1$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_2$	+	−	−	−	+	+	+	−	+	+	+	+	+	+	+	+	+	−	+
$\to_3$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_4$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_5$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_6$	+	+	+	?	−	+	+	+	+	+	+	+	+	+	+	+	?	−	−
$\to_7$	+	−	−	−	−	+	+	−	+	+	+	?	?	+	+	?	−	−	?
$\to_8$	+	−	−	?	?	+	+	−	+	+	+	+	+	+	+	+	+	−	+
$\to_9$	+	+	+	?	?	+	+	?	?	?	?	?	?	+	+	?	?	−	?
$\to_{10}$	−	−	−	−	−	−	+	−	−	−	−	−	−	−	−	−	−	−	−
$\to_{11}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_{12}$	−	−	−	+	+	+	+	−	+	+	+	+	+	+	+	+	+	−	+
$\to_{13}$	+	+	+	?	?	+	+	?	+	+	+	+	+	+	+	+	?	−	+
$\to_{14}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_{15}$	+	−	−	−	−	+	+	−	+	+	+	+	+	+	+	+	−	+	+
$\to_{16}$	−	−	−	−	−	+	+	−	+	+	+	+	+	+	+	+	−	−	−
$\to_{17}$	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
$\to_{18}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_{19}$	−	−	−	−	−	+	+	−	+	+	+	+	+	+	+	+	−	+	−
$\to_{20}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_{21}$	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
$\to_{22}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+
$\to_{23}$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+

only the classical negation ($\neg_1$). Currently we have a more comprehensive list of intuitionistic fuzzy negations and implications. Some of them, as we saw above, do satisfy the axioms of intuitionistic logic.

The new operations open a very large field for future research. We hope that soon many other properties will be clarified and new results will be obtained. There are some possible directions:

1. Study the relationships between the different *implications* and order them as vertices of an oriented graph with respect to the ordering "$\leq$".
2. Study the relationships between the different *negations* and order them as vertices of an oriented graph with respect to the ordering "$\leq$".
3. Study all couples $(\to_i, \neg_j)$ and determine which of them have nice properties, e.g., satisfy all axioms of intuitionistic logic.
4. Construct two new sets of variations of the above 23 implications using the schemes:

$$P \to_{23+i} Q = \Box P \to_i \Diamond Q, \qquad P \to_{46+i} Q = \Diamond P \to_i \Box Q \tag{9}$$

and study their properties.

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On Intuitionistic Fuzzy Negations

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Summary. Up to now four intuitionistic fuzzy negations were constructed. In the present paper one new negation is described and its specific properties are discussed. The relations between it and the other negations are studied. The standard and modified Laws for Excluded Middle, the standard and modified De Morgan's Laws are checked for the new negation.

1 Introduction: on Some Previous Results

Variants of intuitionistic fuzzy implications are discussed in [3, 7–9, 11–13]. The implications from [7] are intuitionistic fuzzy versions of the fuzzy implications defined in [1]. In [8,9] the introduced implications are used as basis for obtaining of intuitionistic fuzzy negations. Here we will introduce for a new negation. Below we will study some properties of all negations and will show that they satisfy the properties of the intuitionistic negation.

Let x be a variable. Then its intuitionistic fuzzy truth-value is represented by the ordered couple

$$V(x) = \langle a, b\rangle, \tag{1}$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of nonvalidity of x. Any other formula is estimated by analogy. Obviously, when V is ordinary fuzzy truth-value estimation, for it $b = 1 - a$.

Everywhere below we shall assume that for the three variables x, y and z equalities: $V(x) = \langle a, b\rangle, V(y) = \langle c, d\rangle, V(z) = \langle e, f\rangle$ $(a, b, c, d, e, f, a + b, c + d, e + f \in [0, 1])$ hold.

For the needs of the discussion below we shall define the notion of intuitionistic fuzzy tautology (IFT, see, [1,3]) by:

$$x \text{ is an IFT if and only if } a \geq b, \tag{2}$$

while x will be a (classical) *tautology* if $a = 1$ and $b = 0$.

In some definitions we shall use functions sg and $\overline{sg}$:

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad \overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases} \tag{3}$$

In ordinary intuitionistic fuzzy logic (see [1, 3]) the negation of variable x is $N(x)$ such that

$$V(N(x)) = \langle b, a \rangle.$$

The operations "conjunction" (&) and "disjunction" ($\vee$) are defined (see [1, 3] by:

$$V(x \,\&\, y) = \langle \min(a, c), \max(b, d) \rangle, \quad V(x \vee y) = \langle \max(a, c), \min(b, d) \rangle. \tag{4}$$

In [11] (see also [15]) the explicit forms of all 23 implication are given and their 23 corresponding negations are obtained, using as a basis equality

$$N(x) = I(x, F), \text{ where } V(F) = \langle 0, 1 \rangle. \tag{5}$$

The negations, generated by the implication operations are given in Table 1.

For these negations and for their corresponding implications the following three properties are checked in [8, 9, 9]:

Property P1: $A \to \neg\neg A$,
Property P2: $\neg\neg A \to A$,
Property P3: $\neg\neg\neg A = \neg A$.

Obviously, negation $\neg_1$ is a classical negation (it satisfies simultaneously properties P1 and P2), while for the four other ones it is shown that they have intuitionistic behavior (they satisfy property P1 and do not satisfy property P2). All negations satisfy property P3.

In [10] the validity of the Law for Excluded Middle (LEM) in the following forms is studied:

$$\langle a, b \rangle \vee \neg\langle a, b \rangle = \langle 1, 0 \rangle \qquad \text{(tautology form)} \tag{6}$$

and

$$\langle a, b \rangle \vee \neg\langle a, b \rangle = \langle p, q \rangle, \qquad \text{(IFT form)} \tag{7}$$

and a Modified LEM in the forms:

$$\neg\neg\langle a, b \rangle \vee \neg\langle a, b \rangle = \langle 1, 0 \rangle \qquad \text{(tautology form)} \tag{8}$$

and

$$\neg\neg\langle a, b \rangle \vee \neg\langle a, b \rangle = \langle p, q \rangle, \qquad \text{(IFT form)} \tag{9}$$

where $1 \geq p \geq q \geq 0$.

Usually, De Morgan's Laws have the forms:

$$\neg x \wedge \neg y = \neg(x \vee y), \qquad \neg x \vee \neg y = \neg(x \wedge y). \tag{10}$$

Table 1. List of intuitionistic fuzzy negations

		Name	Form of negation
$\rightarrow_1$	$\neg_1$	Zadeh	$\langle b, a\rangle$
$\rightarrow_2$	$\neg_2$	Gaines-Rescher	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$
$\rightarrow_3$	$\neg_2$	Gödel	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$
$\rightarrow_4$	$\neg_1$	Kleene-Dienes	$\langle b, a\rangle$
$\rightarrow_5$	$\neg_1$	Lukasiewicz	$\langle b, a\rangle$
$\rightarrow_6$	$\neg_1$	Reichenbach	$\langle b, a\rangle$
$\rightarrow_7$	$\neg_1$	Willmott	$\langle b, a\rangle$
$\rightarrow_8$	$\neg_2$	Wu	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a).\mathrm{sg}(1-b)\rangle$
$\rightarrow_9$	$\neg_3$	Klir and Yuan 1	$\langle b, a.b+a^2\rangle$
$\rightarrow_{10}$	$\neg_1$	Klir and Yuan 2	$\langle \mathrm{sg}(1-a).b, \overline{sg}(1-a)+a.\mathrm{sg}(1-a)\rangle$
$\rightarrow_{11}$	$\neg_2$	Atanassov 1	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a).\mathrm{sg}(1-b)\rangle$
$\rightarrow_{12}$	$\neg_4$	Atanassov 2	$\langle b, 1-b\rangle$
$\rightarrow_{13}$	$\neg_1$	Atanassov and Kolev	$\langle b, a\rangle$
$\rightarrow_{14}$	$\neg_5$	Atanassov and Trifonov	$\langle 1-\mathrm{sg}(a)-\overline{sg}(a).\mathrm{sg}(1-b), \mathrm{sg}(1-b)\rangle$
$\rightarrow_{15}$	$\neg_5$	Atanassov 3	$\langle 1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)), 1-\overline{sg}(a).\overline{sg}(1-b)\rangle$
$\rightarrow_{16}$	$\neg_2$		$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$
$\rightarrow_{17}$	$\neg_3$		$\langle b, a.b+a^2\rangle$
$\rightarrow_{18}$	$\neg_4$		$\langle b, 1-b\rangle$
$\rightarrow_{19}$	$\neg_5$		$\langle 1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)), \mathrm{sg}(1-b)\rangle$
$\rightarrow_{20}$	$\neg_2$		$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$
$\rightarrow_{21}$	$\neg_3$		$\langle b, a.b+a^2\rangle$
$\rightarrow_{22}$	$\neg_4$		$\langle b, 1-b\rangle$
$\rightarrow_{23}$	$\neg_5$		$\langle 1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)), \mathrm{sg}(1-b)\rangle$

The above mentioned change of the LEM inspired the idea from [11] to study the validity of De Morgan's Laws that the classical negation $\neg$ (here it is negation $\neg_1$) satisfies. Really, easy it can be proved that the expressions

$$\neg_1(\neg_1 x \vee \neg_1 y) = x \wedge y, \qquad \neg_1(\neg_1 x \wedge \neg_1 y) = x \vee y \tag{11}$$

are IFTs, but the other negations do not satisfy these equalities. For them the following assertions are valid for every two propositional forms x and y:

$$\neg_i(\neg_i x \vee \neg_i y) = \neg_i\neg_i x \wedge \neg_i\neg_i y, \qquad \neg_i(\neg_i x \wedge \neg_i y) = \neg_i\neg_i x \vee \neg_i\neg_i y \tag{12}$$

for $i = 2, 4, 5$, while negation $\neg_3$ does not satisfy these equalities.

2 Main Results

Here we shall introduce a set of new negations. They are not connected with the previous ones. They will generalize the classical negation, but on the other hand, they will have some nonclassical properties. The set will have the form

$$\mathcal{N} = \{\neg^{\varepsilon} \mid 0 \leq \varepsilon < 1\}. \tag{13}$$

Below we shall study some basic properties of an arbitrary element of $\mathcal{N}$.

Let everywhere below $0 \leq \varepsilon < 1$ be fixed. We define:

$$\neg^{\varepsilon}\langle a, b\rangle = \langle \min(1, b+\varepsilon), \max(0, a-\varepsilon)\rangle. \tag{14}$$

Obviously, $\neg_1 = \neg^0 \in \mathcal{N}$, i.e., $\mathcal{N}$ contains at least one element.

Figure 1 shows x and $\neg_1 x$, while Figs. 2 and 3 show y and $\neg^{\varepsilon} y$ and z and $\neg^{\varepsilon} z$, respectively.

We show that the couple $\langle \min(1, b+\varepsilon), \max(0, a-\varepsilon)\rangle$ is an intuitionistic fuzzy one. Indeed, if $a - \varepsilon \leq 0$, then

$$\min(1, b+\varepsilon) + \max(0, a-\varepsilon) = \min(1, b+\varepsilon) \leq 1. \tag{15}$$

If $a - \varepsilon > 0$, i.e., $a > \varepsilon$, then $b + \varepsilon < a + b \leq 1$ and

$$\min(1, b+\varepsilon) + \max(0, a-\varepsilon) = b + \varepsilon + a - \varepsilon = a + b \leq 1. \tag{16}$$

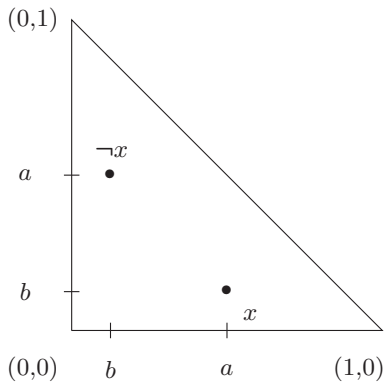

Fig. 1. Classical IFS Negation ($\neg_1$)

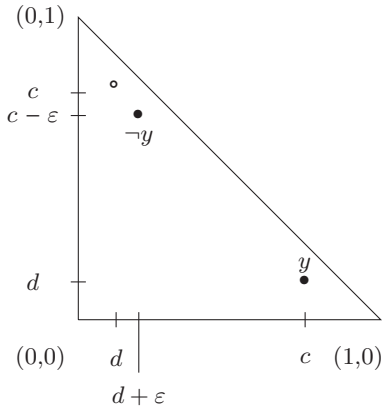

Fig. 2. ε-negation

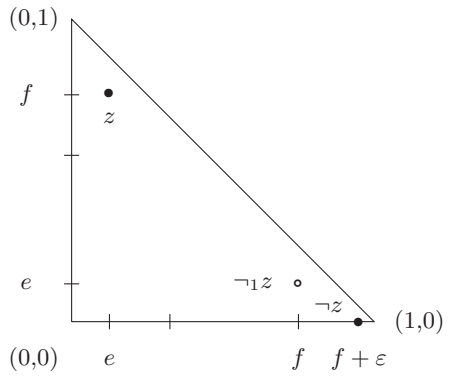

Fig. 3. ε-Negation, $e - \varepsilon < 1$

By analogy with above, we can construct two new implications, generated by the new negation. The first of them is based on $x \to y = \neg x \vee y$ or

$$V(x \to y) = \neg\langle a, b\rangle \vee \langle c, d\rangle \tag{17}$$

and has the form:

$$\begin{aligned}\langle a, b\rangle \to^{\varepsilon} \langle c, d\rangle &= \langle \max(c, \min(1, b + \varepsilon)), \min(d, \max(0, a - \varepsilon))\rangle \\ &= \langle \min(1, \max(c, b + \varepsilon)), \max(0, \min(d, a - \varepsilon))\rangle.\end{aligned} \tag{18}$$

Now, we see that

$$\langle a, b\rangle \to^{\varepsilon} \langle 0, 1\rangle = \langle \min(1, b + \varepsilon), \max(0, a - \varepsilon)\rangle, \tag{19}$$

i.e., the negation generated by implication $\to^{\varepsilon}$ coincides with negation $\neg^{\varepsilon}$.

The second implication that we can construct with negation $\neg^{\varepsilon}$ is based on $x \to y = \neg x \vee \neg\neg y$ or

$$V(x \to y) = \neg\langle a, b\rangle \vee \neg\neg\langle c, d\rangle \tag{20}$$

and has the form:

$$\begin{aligned}&\langle a, b\rangle \to^{\varepsilon} \langle c, d\rangle = \neg^{\varepsilon}\langle a, b\rangle \vee \neg^{\varepsilon}\neg^{\varepsilon}\langle c, d\rangle \\ &= \langle \min(1, b + \varepsilon), \max(0, a - \varepsilon)\rangle \vee \neg^{\varepsilon}\langle \min(1, d + \varepsilon), \max(0, c - \varepsilon)\rangle \\ &= \langle \min(1, b + \varepsilon), \max(0, a - \varepsilon)\rangle \vee \\ &\quad \langle \min(1, \max(0, c - \varepsilon) + \varepsilon), \max(0, \min(1, d + \varepsilon) - \varepsilon)\rangle \\ &= \langle \min(1, b + \varepsilon), \max(0, a - \varepsilon)\rangle \vee \langle \max(\varepsilon, c), \min(1 - \varepsilon, d))\rangle \\ &= \langle \max(\min(1, b + \varepsilon), \varepsilon, c)), \min(1 - \varepsilon, d, \max(0, a - \varepsilon)\rangle \\ &= \langle \min(\max(1, \varepsilon, c), \max(b + \varepsilon, \varepsilon, c)), \\ &\quad \max(\min(0, 1 - \varepsilon, d), \min(1 - \varepsilon, d, a - \varepsilon)\rangle \\ &= \langle \min(1, \max(b + \varepsilon, c)), \max(0, \min(a - \varepsilon, d)\rangle.\end{aligned} \tag{21}$$

Therefore, the two implications generated by negation $\neg^{\varepsilon}$ coincide.

By direct checking we see that the new negation is different than the other five and by this reason already we have six intuitionistic fuzzy negations. Also, we see that the new implication is different with the other 23 ones.

Now, we shall formulate similar assertions as in [8], but for the new negation.

Theorem 1. *Negation $\neg^{\varepsilon}$ satisfies Property 1 for its generated implication in as an IFT, but not as a tautology.*

Proof. Let x be a given propositional form.

$$\begin{aligned}\langle a,b\rangle \rightarrow^{\varepsilon} \neg^{\varepsilon}\neg^{\varepsilon}\langle a,b\rangle &= \langle a,b\rangle \rightarrow^{\varepsilon} \langle \max(\varepsilon,a), \min(1-\varepsilon,b)\rangle \\ &= \langle \min(1,\max(\varepsilon,a,b+\varepsilon)), \max(0,\min(1-\varepsilon,b,a-\varepsilon))\rangle \\ &= \langle \min(1,\max(a,b+\varepsilon)), \max(0,\min(b,a-\varepsilon))\rangle. \\ &= \langle \max(a,\min(1,b+\varepsilon)), \min(b,\max(1,a-\varepsilon))\rangle.\end{aligned} \tag{22}$$

Obviously, the latter expression cannot be a tautology. On the other hand

$$\max(a,\min(1,b+\varepsilon)) - \min(b,\max(1,a-\varepsilon)) \geq \min(1,b+\varepsilon) - b \geq 0, \tag{23}$$

i.e., Property 1 is an IFT. □

Theorem 2. *Negation $\neg^{\varepsilon}$ satisfies Property 2 for its generated implication as an IFT, but not as a tautology.*

Proof. Let x be a given propositional form.

$$\begin{aligned}&\neg^{\varepsilon}\neg^{\varepsilon}\langle a,b\rangle \rightarrow^{\varepsilon} \langle a,b\rangle = \langle \max(\varepsilon,a), \min(1-\varepsilon,b)\rangle \rightarrow^{\varepsilon} \langle a,b\rangle \\ &= \langle \min(1,\max(a,\min(1-\varepsilon,b)+\varepsilon), \max(0,\min(b,\max(\varepsilon,a)-\varepsilon))\rangle \\ &= \langle \min(1,\max(a,\min(1,b+\varepsilon))), \max(0,\min(b,\max(0,a-\varepsilon))\rangle \\ &= \langle \max(a,\min(1,b+\varepsilon)), \min(b,\max(0,a-\varepsilon))\rangle.\end{aligned} \tag{24}$$

Obviously, the latter expression cannot be a tautology. On the other hand

$$\max(a,\min(1,b+\varepsilon)) - \min(b,\max(0,a-\varepsilon)) \geq a - \max(0,a-\varepsilon) \geq 0, \tag{25}$$

i.e., Property 2 is an IFT. □

Therefore, we have constructed an example of a couple of a negation and an implication for which both properties P1 and P2 are IFTs for arbitrary propositional form x, but from this fact does not follow that x coincide with $\neg^{\varepsilon}\neg^{\varepsilon}x$. This is the third example for such a couple along with couples $(\neg_2, \rightarrow_{20})$ and $(\neg_5, \rightarrow_{23})$, described in [11].

Theorem 3. *Negation $\neg^{\varepsilon}$ satisfies Property 3.*

Proof. We shall use the above results:

$$\begin{aligned}\neg^{\varepsilon}\neg^{\varepsilon}\neg^{\varepsilon}\langle a,b\rangle &= \neg^{\varepsilon}\langle \max(\varepsilon,a), \min(1-\varepsilon,b)\rangle \\ &= \langle \min(1, \min(1-\varepsilon,b)+\varepsilon), \max(0, \max(\varepsilon,a)-\varepsilon)\rangle \\ &= \langle \min(1, \min(1,b+\varepsilon), \max(0, \max(0,a-\varepsilon)\rangle \\ &= \langle \min(1,b+\varepsilon), \max(0,a-\varepsilon)\rangle = \neg\langle a,b\rangle.\end{aligned} \tag{26}$$

Therefore Property 3 is valid. □

Now, we can classify each couple $(\neg, \rightarrow)$ as:

- Classical – it satisfies properties P1, P2, P3 and for each x: $V(x) = V(\neg\neg x)$;
- Intuitionistic – it satisfies properties P1, P3 and does not satisfy property P2;
- Nonstandard – it satisfies properties P1, P2, P3 and there is x: $V(x) \neq V(\neg\neg x)$.

Open problem 1 *Classify all different couples $(\neg, \rightarrow)$ to the three groups.*

Now we shall study the validity of the LEM and the De Morgan's Laws in the different forms, described above.

Theorem 4. *Negation $\neg^{\varepsilon}$ satisfies the LEM in its IFT form (7), but not in its tautological form (6).*

Theorem 5. *Negation $\neg^{\varepsilon}$ satisfies the Modified LEM in its IFT form (9), but not in its tautological form (8).*

Theorem 6. *Negation $\neg^{\varepsilon}$:*

(a) Does not satisfy the De Morgan's Laws in the form (10);
(b) Satisfies the De Morgan's Laws in the form (11);
(c) Satisfies the De Morgan's Laws in the form (12).

Finally, we shall study the relations between the different negations. By direct checks we can see the validity of the following Table 2.

The lack of relation between two implications is noted in Table 2 by "$*$".

The values from Table 3 are also interesting.

3 Conclusion: a New Argument that the Intuitionistic Fuzzy Sets Have Intuitionistic Nature

The above assertions show that all negations but the first one satisfy properties conditions of the intuitionistic logic not of the classical logic. A part

Table 2. List of the relations between the different intuitionistic fuzzy negations

	$\neg_1$	$\neg_2$	$\neg_3$	$\neg_4$	$\neg_5$	$\neg^{\varepsilon}$
$\neg_1$	$=$	$*$	$\leq$	$\geq$	$\geq$	$\leq$
$\neg_2$	$*$	$=$	$*$	$*$	$\geq$	$*$
$\neg_3$	$\geq$	$*$	$=$	$\geq$	$\geq$	$*$
$\neg_4$	$\leq$	$*$	$\leq$	$=$	$\geq$	$\leq$
$\neg_5$	$\leq$	$\leq$	$\leq$	$\leq$	$=$	$\leq$
$\neg^{\varepsilon}$	$\geq$	$*$	$*$	$\geq$	$\geq$	$=$

Table 3. List of the values of some special constants for the different intuitionistic fuzzy negations

$V(x)$	$\neg_1 V(x)$	$\neg_2 V(x)$	$\neg_3 V(x)$	$\neg_4 V(x)$	$\neg_5 V(x)$	$\neg^{\varepsilon} V(x)$
$\langle 1, 0\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle \varepsilon, 1-\varepsilon\rangle$
$\langle 0, 1\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$
$\langle 0, 0\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0\rangle$	$\langle 0, 0\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle \varepsilon, 0\rangle$

Table 4. List of the fuzzy negations, generated by intuitionistic fuzzy negations

Notation	Form of the intuitionistic fuzzy negation	Form of the fuzzy negation
$\neg_1$	$\langle b, a\rangle$	$1-a$
$\neg_2$	$\langle 1-\mathrm{sg}(a), \mathrm{sg}(a)\rangle$	$1-\mathrm{sg}(a)$
$\neg_3$	$\langle b, a.b+a^2\rangle$	$1-a$
$\neg_4$	$\langle b, 1-b\rangle$	$1-a$
$\neg_5$	$\langle 1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)), \mathrm{sg}(1-b)\rangle$	$1-\mathrm{sg}(a)$
$\neg^{\varepsilon}$	$\langle \min(1, b+\varepsilon), \max(0, a-\varepsilon)\rangle$	$1-\max(0, a-\varepsilon)$

of these negations were generated by implications, that were generated by fuzzy implications. Now, let us return from the intuitionistic fuzzy negations to ordinary fuzzy negations. The result is shown on Table 4, where $b = 1 - a$.

Therefore, from the intuitionistic fuzzy negations we can generate fuzzy negations, so that two of them ($\neg_3$ and $\neg_4$) coincide with the standard fuzzy negation ($\neg_1$). Therefore, there are intuitionistic fuzzy negations that lose their properties when they are restricted to the ordinary fuzzy case. In other words, the construction of the intuitionistic fuzzy estimation

$$\begin{aligned}&\langle \text{degree of membership/validity},\\ &\ \text{degree of nonmembership/nonvalidity}\rangle\end{aligned} \tag{27}$$

that is specific for the intuitionistic fuzzy sets, is the reason for the intuitionistic behavior of these sets. Over them we can define intuitionistic as well as classical negations.

In the fuzzy case the negations $\neg_2$ and $\neg_5$ coincide, generating a fuzzy negation that satisfies Properties 1 and 3 and does not satisfy Property 2, i.e., it has intuitionistic character. As we see above, the new negation $\neg^{\varepsilon}$ has more strange behavior.

In [4] two other classes of negations are introduced. We will formulate the following interesting

Open problem 2 *What are the relation betweens these three sets of negations?*

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A Simulation Model for Trust and Reputation System Evaluation in a P2P Network

Roberto Aringhieri and Daniele Bonomi

Summary. A peer-to-peer (P2P) network is an exchange community of anonymous peers or individuals. The evolution of P2P network has determined the need of trust and reputation systems (TRS) in order to improve the reliability of local interactions: collecting in some way the local experiences, the TRS assesses the possibility that an individual has a malicious behavior, i.e., he cheats other individuals.

In this paper we present an agent-based simulation model for the evaluation of a generic TRS within a decentralized P2P network. We describe some minimal requirements that, in our opinion, every P2P simulators should have. Moreover, we propose a complete model of peers in which the behavior of both good and malicious peers is accurately defined.

Key words: Agent-based simulation, Trust and reputation system, P2P network.

A peer-to-peer (P2P) network is an exchange community of anonymous peers or individuals. Each individual can join or leave the community freely. When joined, it plays the role of client and server at the same time. Such a community is characterized by the absence of a central authority (decentralized environment) and by the fact that each individual has a local view of the whole community. A global behavior can emerge from local interactions, i.e., exchanges between pairs of individuals.

The evolution of P2P network has determined the need of trust and reputation systems (TRS) in order to improve the reliability of local interactions: collecting *in some way* the local experiences, the TRS assesses the possibility that an individual has a *malicious* behavior, i.e., he cheats other individuals. The way of collecting or aggregating the local experiences defines different TRS (see e.g. [5, 9, 11, 14, 18]).

In this paper we present a simulation model for the evaluation of a generic TRS within a decentralized P2P network. In Sect. 1 we review four different TRSs, for which a simulation model or a numerical evaluation is reported,

in order to define some minimal requirements for a generic simulation model. From this review, three main components of a simulation model are highlighted, that is the network model, the content distribution model and the peer behavior model. The latter is discussed in Sect. 2. The simulation model is described in Sect. 3. Conclusions are discussed in Sect. 4.

1 Model Requirements

Reputation has been extensively studied in economics especially using game theory to model the behavior of economic agents acting in a market to maximize their utilities. Usually, the agents work under the incomplete information and the "looking forward" assumptions. The latter consists in maximizing the agent utility considering a long time period using past events to predict the future needs of the agent thus modifying the current utility.

The need of improving the reliability of local interactions among peers determines the increasing relevance of trust and reputation topic in the field of P2P research. A typical successful example is the "Feedback Forum", the eBay reputation mechanism [1], which is deeply analyzed in literature [16,18].

The problem of managing trust in a decentralized environment is formally described in [2]. Here, we simply describe the basic TRS environment. Each peer stores its *local reputations*, i.e., the result of the interactions with other peers. These reputations are usually represented by values in $[0, 1]$, where 0 and 1 represent, respectively, the worst and the best reputation. To decide if an offerer peer j is trustworthy or not, the peer i uses its local reputation about j or, when i has not previous experience with j or the local reputation is still considered not reliable enough, it tries to obtain the *global reputation* of j from the community. The global reputation is usually a value obtained by aggregating the local reputation of other peers about j.

In [10], the authors review and describe several tools to manage TRS. They identify three broad classes of tools: social network formation, probabilistic estimation techniques and game-theoretic reputation models. For each tools they analyze, when possible, the "trust related model semantics" and the "incurred implementation costs." In particular, the latter concerns the performance analysis and the implementation overhead.

In our review, we consider the "EigenTrust" **a**lgorithm [14], the "Maximum Likelihood Estimation" method [9, 11], the "P2PRep" protocol with fuzzy aggregation [4, 5] and the method based on fuzzy logic inference proposed in [18]. All these papers report numerical results to evaluate the goodness of the proposed method. These results are usually obtained by running a simulation model. For each paper, we briefly describe the method and then we analyze the main characteristic of the simulation model. Our concern is to define some minimal requirements of a generic simulation model for the evaluation of a TRS.

1.1 EigenTrust Algorithm

In EigenTrust, each peer i rates another peer j from which it tries to download a file by keeping track of the numbers of successful *sat*(*i*,*j*) and unsuccessful *unsat*(*i*,*j*) downloads. A *local trust value* s_{ij} is then defined as the difference between *sat*(*i*,*j*) and *unsat*(*i*,*j*). To aggregate these local trust values around the P2P network, they are normalized so that malicious peers will not be able to assign arbitrarily high trust values to other malicious peers and subvert the EigenTrust algorithm. The normalized local trust value c_{ij} is defined as:

$$c_{ij} = \frac{\max(s_{ij}, 0)}{\sum_j \max(s_{ij}, 0)}$$

The normalized trust values are aggregated using the concept of *transitive trust*: peer i can know about the trust of peer k by asking all peers j with which peer i has interacted. However, since not all peers j are trustworthy, their opinions is weighed with the trust peer i places in them:

$$t_{ik} = \sum_j c_{ij} c_{jk}$$

To compute the trust value, the authors proposed the distributed EigenTrust algorithm which works as follows.

Foreach peer i do:

1. Query all peers j who have downloaded files from i for their opinions about him ($t_j^{(0)} = p_j$)
2. *Repeat*
   2a. Compute i's current global trust value $t_i^{(k+1)}$
   2b. Send opinion $c_{ij} t_i^{(k+1)}$ to j from which i has downloaded files
   2c. Wait for all j to send their updated trust values $c_{ji} t_j^{(k+1)}$

   until $\mid t_i^{(k+1)} - t_i^{(k)} \mid < \epsilon$.

In order to evaluate the EigenTrust method, the authors proposed the following simulation model. They consider a typical P2P network in which a query is propagated by broadcast with hop-count horizon throughout the network as done in a Gnutella network [17].

For each malicious node, the authors proposed several *threat model* in which several aspects are taken into account, i.e., individual malicious or collectives malicious, camouflage behavior, malicious spies and so on. Note that malicious peers connect to the most highly connected peers and they are supposed to have a large bandwidth allowing them to answer to the top 20% of queries received. In order to guarantee the convergence of the method, the authors introduce the concept of pretrusted peers, i.e., peers that are known to be trustworthy.

The author also provide a content distribution model in which each peer shares a subset of content types (categories) in which the popularity of single content is governed by a Zipf distribution. An experiment consists of a certain number of repeated query cycles; each cycle terminates when the peer retrieves the required content, otherwise it continues to query the network. A simulation is composed of a number of repeated experiments depending on the scenario evaluated.

1.2 Maximum Likelihood Estimation

The "Maximum Likelihood Estimation" is a probabilistic technique whose main concern is to reduce the implementation overhead. The authors assumes to consider a P2P network composed of peers having high probabilities of performing honestly during their transactions.

Let θ_j be the probability of peer j to act honestly. A peer j interacts with peers $p_1, p_2, \ldots, p_n$ and the variables $x_1, x_2, \ldots, x_n \in \{0, 1\}$ denote the honest performance ($x_j = 1$) and the dishonest one ($x_j = 0$). Assuming that peers $p_1, p_2, \ldots, p_n$ can lie with specific probability ℓ_k for peer p_k, the probability of observing report y_k from peer p_k can be calculated as:

$$\mathbb{P}[Y_k = y_k] = \begin{cases} \ell_k(1-\theta_j) + (1-\ell_k)\theta_j & \text{if } y_k = 1 \\ \ell_k\theta_j + (1-\ell_k)(1-\theta_j) & \text{if } y_k = 0 \end{cases}.$$

Given a random sample of independent reports $y_1, y_2, \ldots, y_n$, the likelihood function of this sample is

$$L(\theta_j) = \mathbb{P}[Y_1 = y_1]\ \mathbb{P}[Y_2 = y_2] \ldots \mathbb{P}[Y_n = y_n].$$

The maximum likelihood estimation procedure requires to find the value of θ_j maximizing $L(\theta_j)$.

In [9], the authors reports the results obtained by some simulation experiments. First of all, note that the function $L(\theta_j)$ implies the independence of reports $Y_1, \ldots, Y_n$. Thus in their simulation they assumed a noncollusive behavior of peers. They do not consider any particular structure of the network in which the interactions among peers were generated at random. Other settings are the constant number of peers (128), the number of interaction per peer varying in $\{20, 40, 60, 80, 100\}$ and the fraction of liars varying in $\{0.1, 0.2, 0.3, 0.4, 0.5\}$. All the results is the average value of 20 repeated experiments.

1.3 P2PRep and Fuzzy Aggregation

P2PRep is a reputation-based protocol which formalizes the way of each peer stores and shares with the community the reputation of other peers [6, 8, 12]. It runs in a fully anonymous and decentralized P2P environment. A more

detailed description of the protocol is given in [5]. Here, we report only the reputation model.

Let $r_{i,j}$ be the local reputation resulting from direct interactions between peer i and peer j. A fuzzy value expresses local reputations to take into consideration the fact that transactions can be heterogeneous for importance, resource value, and so on. At any time $n > 1$, based on the outcome of the nth transaction ($t_{i,j}^{(n)} = 1$ if the outcome was satisfactory, $t_{i,j}^{(n)} = 0$ otherwise), the local reputation is updated as follows

$$r_{i,j}^{(n)} = \alpha^{(n)} r_{i,j}^{(n-1)} + (1 - \alpha^{(n)}) t_{i,j}^{(n)} \quad \text{and} \quad r_{i,j}^{(1)} = t_{i,j}^{(1)}.$$

The value of $\alpha^{(n)} \in [0, 1]$ is a feedback measure varying during the time following a well-known technique for feedback control, that quickly stabilizes to a fair and efficient setting [13].

The global reputation of peer j can be computed as follows. The peer i runs a poll by using P2PRep and inquires other peers for collecting their local reputation $r_{k,j}$ of j. Under the assumption of *unanimity* [3], the global reputation can be computed aggregating all $r_{k,j}$ values using the ordered weighted average (OWA) operator [20] which allows the decision maker to give different importance to the values of a criteria. Technically, an OWA operator is a weighted average that acts on an ordered list of arguments and applies a set of weights to tune their impact on the final result. Namely, in their setting, the authors get

$$\lambda_{OWA} = \frac{\sum_{k=1}^{n} w_k r_{t_k,j}}{\sum_{k=1}^{n} w_k}$$

where n is the number of reputations to be aggregated considered in decreasing order, that is, assuming $r_{t_1,j} \geq r_{t_2,j} \geq ... \geq r_{t_n,j}$ and $[w_1 \; w_2 \ldots \; w_n]$ is a weighting vector.

The authors set the OWA weights *asymmetrically*, since the aggregation operator needs to be biased toward the lower end of the interval, increasing the impact of low local reputations on the overall result. The reason is that the authors assume that peers are usually trustworthiness and a malicious behavior is the exception.

Especially in [4], the authors proposed an extensive numerical evaluation of P2PRep. The underlying P2P network is such that each peer is reachable from all others and in which delays due to message routing are not take into account. Over this broadcast network, a set of queries are simulated, each asking for a randomly chosen resource. For each query, the peer querying the network is randomly chosen (with a uniform probability distribution) over all available peers. Then, a preferred offerer o is selected in two different ways: the first one, the *random policy*, selects o randomly choosing some peers among those having the resource required whilst the second one selects o using P2PRep.

The authors model the behavior of both well-behaved and malicious peers. They assume that (1) malicious peers provide only malicious resources; (2) malicious peers respond to the polling on a peer o by always providing a (malicious) 1 reputation if $o \in M$, and by providing a genuine opinion, otherwise. On the other side, all well-behaved peers i participate in a poll on offerer o by returning their local reputation $r_{i,o}$ if such a value is recorded; no response is returned otherwise.

The main settings of this simulation model are the following: the number of peers P in the network is uniformly distributed in $[300, 400]$; the number of malicious peers M, $M \subset P$ is the 40% of $|P|$; the number of different kinds of resources is 20; the max poll cardinality is uniformly distributed in $[5, 15]$. A simulation consists of 50 repeated experiments, each one evaluating a different and randomly generated scenario in which the number of queries for each experiment ranges from $1,000$ to $10,000$ with an increment of $1,000$.

1.4 Method Based on Fuzzy Logic Inference

The authors proposed a method based on fuzzy logic inferences, which can handle uncertainty, fuzziness and incomplete information in peer trust reports. Moreover, this method aggregates peer reputations with affordable message overhead. The authors start their development from an accurate analysis of eBay transaction. Thus, their method is well-suited for a centralized community, which is not the target of our simulation. On the other side, the relevance of this method depends on two facts. The first one is the definition of a set of rules to determine the weights of the fuzzy aggregator; from this point of view, this method can be viewed as a variant of P2PRep. The second one is to populate the simulation using data concerning real transaction obtained by the analysis of eBay transactions.

1.5 Remarks from the Literature

Looking at this review, we can define some minimal requirements to be considered when devising a simulation model for a TRS, i.e., a trust and reputation systems. Three major components can be identified: the network model, the content distribution model, the peer behavior model.

A good model representing the network connecting the P2P community become necessary when the dissemination of contents and local reputations is a crucial point to evaluate the performance of a TRS. To correctly model the dissemination is needed to identify the neighbors of a given peer. From our review, the most sophisticated network model is the one proposed for EigenTrust evaluation. The minimal requirement is that of the network model should allow the concept of neighborhood based on a given measure of distance between peers, e.g., hop-count distance.

The TRS evolves analyzing the positive or negative result of the transactions among peers. Each transaction is composed of a peer requiring a content to its neighborhood. It can be assumed that peers are interested in sharing and asking a subset of the total available contents in the network. From this point of view, the description given in [14] should be considered as the minimal requirement.

The peer behavior model is the crucial component of any TRS simulator. From this point of view, our review highlights several attempts for modeling the behavior of peers. The model should take into account several aspects of malicious peer behavior. Most of them (collusive or not, camouflage, malicious spies and so on) are highlighted in our review. We observe that there is no effort to model the behavior of well-behaving peers. The definition of a complete model of peer behavior is a challenging topic and it is approached in Sect. 2.

2 The Peer Behavior Model

In this section, we discuss the main assumptions modeling the behavior of peers. We assume that peers belong to two distinct classes: the class of *malicious peers* M and the class of well-behaved or *good peers* G.

2.1 Malicious Peers

A malicious peer $p \in M$ tries to distribute dangerous contents such as virus, worms, and so on. Its main objective is to distribute the maximum number of malicious contents. To reach it, the peer can follow the following phases:

distribution: p continues to distribute malicious contents until it reach a given number F_p of distributed content or its reputation is greater than a given threshold θ_p;

camouflage or disconnection: if the reputation of p is considered acceptable, the peer can hide temporarily its maliciousness acting for a certain time as a good peer in order to improve its reputation; otherwise p abandons the community.

Finally, after exiting, the peer p can connect again using the same identifier or to use a new one.

We observe that malicious peers can operate in a group. So we assume that small cliques $C_M \subset M$ of malicious peers can adopt a common strategy in order to cheat the reputation system. For instance, in a system using P2PRep protocol, they can give highest vote to peers belonging to C_M and neutral vote to the others. More formally, the clique contains both malicious peers and spies: a spy provides not malicious contents when selected as offerer but returns highest local reputations to all peers belonging to C_M.

2.2 Good Peers

A good peer $g \in G$ joins the community in order to find and to retrieve a list of required contents. For each contents, the peer g queries the community obtaining a list of offering peers O from which the offerer $o \in O$ having the best reputation is selected. Then, the peer g starts the file download from the offerer o.

After each download, the peer g checks the file to verify its integrity in terms of correctness – is it the required content? – and security – is it a malicious file? A fast and accurate check depends on the *fanatic* level associated to each peer: the probability of a peer $g \in G$ to identify malicious contents is proportional to its fanatic level. In other words, the fanatic level models the ability of a peer to recognize malicious contents after each download.

We observe that peer g becomes distributor of malicious contents every time the check fails. This situation can be interrupted when a periodic and more accurate check will be performed.

Finally, after obtaining all required contents, the good peer g temporarily exit from the community.

2.3 Peer Dynamics

The P2P community is not fixed: new peers arrive, existing peers can temporary disconnect or definitively abandon the community. Therefore, we need to model the community dynamics.

We introduce some parameters to model the rate of arrivals and the rate of abandons in such a way to allow different dynamics for peers belonging to M and G, respectively. For instance, we can have malicious peers more dynamic than the good ones, or vice versa.

3 The Simulation Model

Our model has been developed using the AnyLogic platform which allows to create models using several methodologies such as discrete event, agent-based and many others [15]. Moreover, it provides the optimization engine OptQuest [12] which can be used to optimize the model parameters.

The agent-based simulation seems well-suited to implement a decentralized community composed of autonomous individuals such as that populating a P2P network. Moreover, the *statechart* (see Fig. 1), which is the basic tool to define an agent, is the proper instruments to describe the behavior of a peer during the simulation.

The minimal requirements, described in Sect. 1.5, are implemented in our simulation model. The network model allows to define the neighborhoods of a given peer by using an hop-count distance over a grid network. The content distribution model assigns to each peer a small amount of content categories

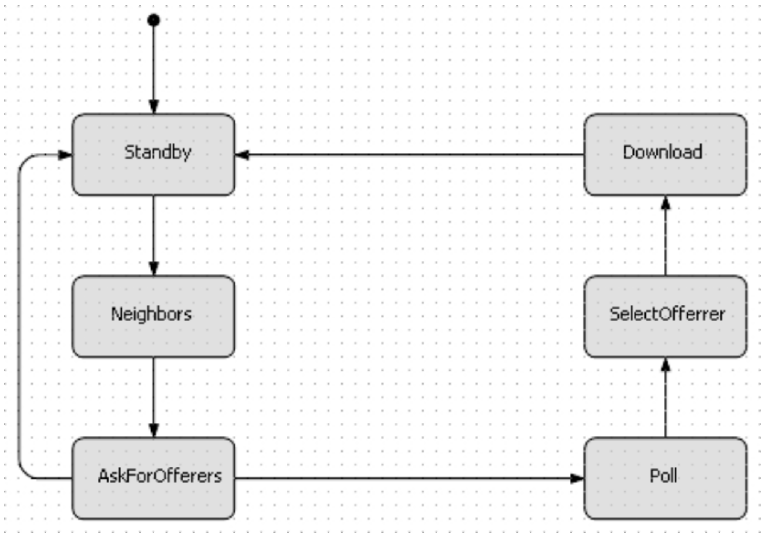

Fig. 1. The statechart modeling the good peer behavior

and the contents belonging to each category is distributed by using a Zipf function. For each content, the corresponding dimension is also randomly generated.

The main component of our agent-simulation model is the description of the agent implementing a generic peer: a peer is an autonomous agent described by a Java class which contains attributes and statecharts. The attributes model the parameters determining the peer characteristics whilst the statecharts model the peer behavior such as described in Sect. 2.

For instance, the statechart reported in Fig. 1 describes the behavior of the agent modeling a good peer: the agent leaves the `Standby` state when it decides to retrieve a content; then, it selects a list of offerers belonging to its neighborhood (`Neighbors` and `AskForOfferer` states); if this list is empty, the agent returns in the `Standby` state, otherwise it proceeds with the download; in the following two states, i.e., `Poll` and `SelectOfferer`, the agent tries to form an opinion about all the offerer peers taking into account local and global reputations; finally, the agent starts the download from the offerer having the best reputation; the download is an independent task thus the agent comes back to the `Standby` state immediately after the download starts.

The simulation experiment starts creating and arranging N_p peers over the network as depicted in Fig. 2: circles and squares represent, respectively, good and malicious peers while thin and large arrows represent, respectively, a good and a malicious content exchange among peers. The N_p peers are replicated with different attributes determining its basic characteristics. For instance, the fact of a peer is malicious or not is defined by the boolean attribute `isMalicious`.

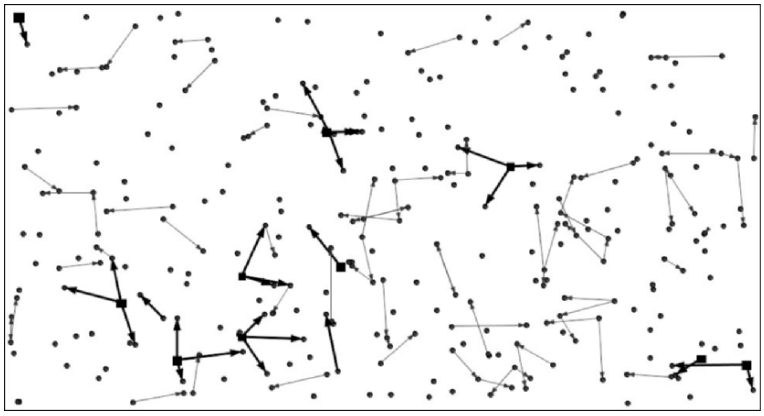

Fig. 2. The P2P network and some downloads among peers

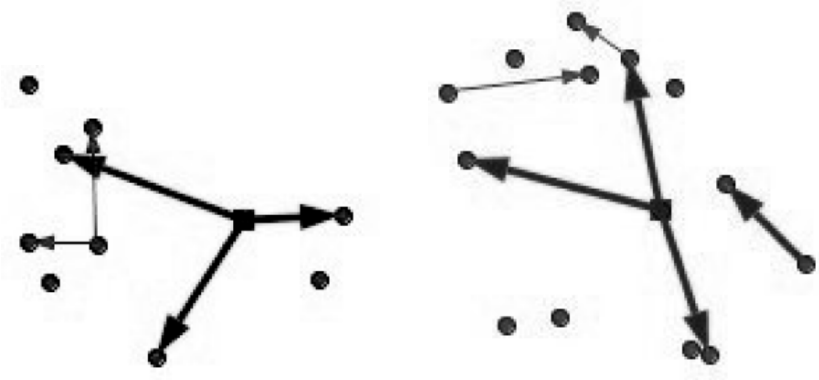

Fig. 3. Download details and distribution of malicious contents

To model the fact that each peer can have a different bandwidth, a maximum number of parallel downloads is allowed. Each download has a duration proportional to the content dimension. When a download finishes, as described in Sect. 2.2, the peer checks the content retrieved. If the content is malicious and the check fails, the peer can distribute this malicious content until the periodic and more accurate check discovers it.

Figure 3 details two possible download situation. On the left, a good peer distributes two contents to its neighbors whilst a malicious peer tries to distribute three malicious contents. If one of the receiving peer fails its check, it becomes a distributor of malicious content as depicted on the right part of the figure.

The TRS is implemented as external Java library in order to allow an independent development both to improve the existing TRSs or to add new ones. The interface between model and library is implemented in such a way

to allow changes in the library without modifying anything in the model. In particular, the interface is used within `Poll` and `SelectOfferer` states (see Fig. 1). Currently, the library implements the P2PRep protocol using two different aggregator for computing the global reputation: the OWA operator described in Sect. 1.3 and its weighted version [19].

The main quality index is defined as the number of malicious transactions executed during the simulation by the whole community. The model also collects several statistics both a local level (statistic about a single peer) and a global level. Moreover, the model compares the TRS results with those obtained by a basic random policy in which the peer chooses randomly a peer belonging to the offerer list O.

A preliminary validation of our model has been performed comparing the simulation outcomes with those reported in [4] obtaining a positive results.

4 Conclusions

In this paper we have discussed a simulation model for the evaluation of a generic TRS within a decentralized P2P network. From the literature review, restricted to the papers in which a simulator is proposed, we have devised some minimal requirements that, in our opinion, every P2P simulators should have. Moreover, we have observed that no efforts have been made in literature to model the behavior of well-behaved peers while several considerations concerning the malicious peers behavior have been discussed. We have proposed a complete model of peers in which the behavior of both good and malicious peers is accurately defined. We have presented our agent-based simulation model having the minimal requirements discussed above and implementing the complete model of peer behavior. A Java library provides the TRS in a transparent way for the model user.

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A Fuzzy Trust Model Proposal to Ensure the Identity of a User in Time

Antonia Azzini and Stefania Marrara

Summary. Access controls ensure that all direct accesses to objects are authorized by means of user identification. However, in some scenarios it is also necessary to continuously check the identity of the user in order to avoid malicious behaviors such as person exchanges immediately after the initial authentication phase.

Aim of this work is to propose a methodology based on a balanced mix of strong and weak authentication techniques studied to guarantee a high and prolonged in time level of security combining the advantages of each authenticator.

1 Introduction

Access controls ensure that all direct accesses to objects are authorized. By regulating the reading, changing, and deletion of data and programs, access controls protect against accidental and malicious threats to secrecy, authenticity, and system availability. The effectiveness of access controls rests on one important premise, the proper user identification [3]: no one should be able to acquire the access rights of another. Traditionally, access control relies on profile information associated to users and resources in a given domain. However, in some scenarios it is also necessary to continuously check the identity of the user in order to avoid malicious behaviors such as person exchanges immediately after the authentication phase used for accessing the system. An example can be a system for university course examinations from remotely connected pc stations: in this situation we can be interested in being sure that the authenticated student is not substituted by another person just after the initial identification process, but she is the one that compiles the entire course test.

Aim of this work is to propose a methodology based on a balanced mix of strong and weak authentication techniques studied to guarantee a high and prolonged in time level of security avoiding the excessive cost of using only biometric devices.

For this reason, remote access is initially provided by means of biometric devices but then it is granted in time by means of other authentication

methods. In such a scenario, the system must distinguish between the initial authentication phase, in which it recognizes the user profile and allows the access, and the following authentication steps in which the system decides if its trust in user's identity is enough high to allow the user to continue to perform the activity she is doing. Focus of this paper is not the semantics used to describe the users profile, but the description of the fuzzy logic based methodology used by a system to continuously check and confirm its trust in the identity of a user.

The structure of the paper is as follows. Section 2 presents a brief overview of the authentication devices used to ensure the identity of a user and compares advantages and drawbacks of the different techniques, Sect. 3 describes the general architecture of the fuzzy methodology used to ensure the user identity during time, Sect. 4 presents the fuzzy rules used by the methodology engines to compute the user identity trust level during time and, finally, Sect. 5 reviews the conclusions of this work and propose some future work and open issues.

2 User Authentication Systems and Their Trustfulness

User authentication is the process of positively verifying the identity of an user, often as a prerequisite to allowing access to resources in a system. User authentication is then essential for reliable access control and rights management systems determine a user authorization to access the content [3].

2.1 Traditional Systems

Traditional cryptosystems do not identify the user as such. The authentication is *knowledge-based*, answering the question: 'What you know' such as a password, or *token-based*, answering the question: 'What you have' such as a key, magnetic or chip card.

A password includes single words, phrases, and personal identification numbers (PINs) that are closely kept secrets used for authentication. The basic problem with this technique is that a memorable password can often be guessed or searched by an attacker and a long, random, changing password is difficult to remember. As result they are stored and released on some alternative authentication mechanism and they can be shared with other users.

An identity or security token is a physical device that can contain passwords, such as a bankcard, or smartcard, that includes tamper-resistant packaging and special hardware that disables the token if it is tampered with or if the number of failed authentication attempts exceeds a chosen threshold. The main problem is that these devices can be lost, stolen, forgotten or disclosed.

Strong authentication methods are usually developed to solve the drawbacks the traditional techniques. Biometric systems implement human authentication and identification in rights management systems. They are defined as

ID-based authenticators, answering the question: 'Who you are'. They are characterized by the uniqueness to one person. The main security defense is that they are difficult to copy or forge.

2.2 Biometric Systems

Biometrics are automated methods of authentication based on measurable human physiological or behavioral characteristics. Common physical biometrics include fingerprints, hand or palm geometry and retina, iris or facial characteristics. Behavioral features include signature, voice (which has also a physical component), keystroke pattern and gait.

Biometric technologies most commonly implemented are based on:

- *Fingerprint*, based on matching numeric information of finger minutia. It is easy, fast of use and low cost and it has considered the higher authentication form from the people.
- *Hand Geometry*, which involves analyzing and measuring the shape of the hand. It offers a good balance of performance characteristics and is relatively easy of use, the accuracy can be very high.
- *Iris*, which analyzes features found in the iris, uses a fairly conventional camera element and requires no close contact between the user and the reader. It has the potential for higher than average template-matching performance, even though easy of use and system integration have not traditionally been strong points with iris scanning devices.
- *Face*, which analyzes facial characteristics. It requires a digital camera to develop a facial image of the user for authentication.
- *Voice*, which is not based on voice recognition, but on voice-to-print authentication, where complex technology transforms voice into text.
- *Signature*, which analyzes the way a user signs her name. Signing features such as speed and pressure are as important as the finished signature's static shape.

These methods are inherently more reliable than password-based authentication, as biometric features cannot be borrowed, stolen, or forgotten; furthermore they are extremely difficult to copy, share and distribute. The main issue in biometric authentication system is performance, defined considering different factors, depending on critical issues in the data acquisition phase.

A comparison between different techniques is in [1] and briefly reported in Table 1.

Once enrolled in a biometric system, a user can be successfully authenticated. The overall process, presented in detail in [3, 4], is the same for each different biometric approach, and it is represented with a first enrollment phase and a second matching phase. The result is typically explained in terms of a *matching score*; the higher the matching score, the better comparison result is obtained.

In a such identification system, acceptance is determined considering two types of biometric errors:

Table 1. Biometric comparison

Characteristics	Fingerprint	Hand geometry	Iris	Face	Voice	Signature
Ease of use	High	High	Medium	Medium	High	High
Accuracy	High	High	Very high	Very high	High	High
Use acceptance	Medium	Medium	Medium-low	Medium	Very high	High
Required security level	High	Medium	Very high	Medium	Medium	Medium
Long term stability	High	Medium	High	Medium	Medium	Medium

- FAR – False Acceptance Rate – that defines the percentage of impostors incorrectly matched to a valid user's biometric.
- FRR – False Rejection Rate – that defines the percentage of incorrectly rejected valid users.

There is a trade off between FAR and FRR in every biometric system, since they are functions of the system threshold t: if t is decreased to make the system more tolerant to input variations and noise, FAR increases. For each biometric technology these rates are calculated by experimental tests. Phenotypic features do not set limits on the FAR, but clearly, over time the phenotypic variation imposes a lower limit on the FRR.

2.3 Critical Issue

Some systems incorrectly assume that biometric measurements are secret and grant access to any user presenting matching measurements. On the other hand, as sensitive data, biometrics should be properly protected, but they cannot be considered secret. The only way to secure a biometrics system is to ensure that the characteristics presented come from a real person and they are obtained and authenticated during verification from the person. For this reason it should be defined a *liveness test*, in which, before granting a user access, a system must make sure that the authentication device is verifying a living person; this tests are usually performed by the core biometric technology.

Another critical aspect is that a biometric system must believe that the biometric measurements presented come from a trusted input device and they have been captured at a certain time. If authentication is performed on-device, the device should be trustworthy; otherwise, if it is performed off-device, the software operating environment and the communication link between the software and the device must be secure.

2.4 Advantages and Shortcomings

Biometric characteristics are essentially permanent and unchangeable and users cannot pass them to other users as easily as they do with cards or passwords. Furthermore these techniques are based on features that cannot be lost or forgotten. A biometric authentication systems is also fast. The authentication of an user in a fingerprint reader system can take under two

seconds, whereas finding a key ring, locating the right key and using it can take as long as 10s.

Some issues remain jet unresolved. In some cases, if the input sample quality is not sufficient for further processing, the system must reacquire data, and the resulting system might be more complicated or more expensive. Furthermore some biometric sensors, particularly those having contact with users, have a limited lifetime. The most important drawback is that biometric systems could violate user privacy. Biometric characteristics are sensitive data containing personal information: for example a DNA sample contains the user's susceptibility to disease. A biometric system can imply loss of anonymity, and users may consider it intrusive or personally invasive.

2.5 Traditional Versus Strong Authentication Techniques

Different authentication categories may be appropriate for different applications, depending on perceived user profiles, the need to interface with other systems or database, environmental conditions, and a host of other application specific parameters. The attributes of the three categories of user authentication, described in the previous sections, are compared in Table 2.

The different authentication technologies are compared in detail in [2], giving a number of some potential attacks against user authentication and relative defenses by each technique; however, important issue for each of them can be summarized as follows:

– Knowledge-based: its secrecy and high keyspace defend well against search and host attacks. Its ability to participate in challenge–response protocols protects against replay and transmission attacks, with nonexpensive costs. The main problem is the difficult to remember passwords for the user. This technique does not provide a compromise detection and does not offer much defense against repudiation.

Table 2. Basic user authentication attributes

Attributes	User authentication		
	Knowledge based	Token based	ID based
Identification	Password, secret	Token	Biometric
Supports	Secrecy or obscurity	Possession	Uniqueness and personalization
Security defence	Closely kept	Closely held	Forge resistant
Security drawback	Less secret	Lost, stolen	Difficult to replace
Examples	Combinational lock, password	Metal key, smart card	Fingerprint, face

- Token-based: it can store or generate multiple passcodes (also if combined with a password). It provides compromise detection and added protection against denial-of-service attacks. The two main shortcomings are inconvenience and cost, and vulnerability to theft. Equipment cost is higher than a password and comparable to a much secure biometric that requires a reader. A token with biometric combination has similar security characteristics to a token plus password, however, the inconvenience of FRR for a biometric, defined in Sect. 2.2, with respect to the inconvenience of remembering a password is matter of user preference.
- Biometrics: one advantage of biometric is that it is less easily lent or stolen than the other authenticators, so it provides a stronger defense against reputation. The relative simplicity also improves a better security and trustworthy authentication process. The stability of such system refers to the fact that a good biometric maintains its distinctive features over time, without compromising information. A problem is the limited lifetime for particular biometrics, but the main drawback is the possible violation of the user privacy.

An appropriate authentication solution depends upon the particular application, each system has its strength and weakness and no a single technique is expected to effectively meet all requirements of all the applications like accuracy, security, trustworthy, and cost. Although, few combinations of authenticators are recommended, in order to provide secure and trustworthy authentication systems.

3 Architecture of the Model

This section introduces an access control model based on a balanced mix of strong and weak authentication techniques studied to guarantee a high level of security combining the advantages of each authenticator. The proposed model describes a trust evaluation process implemented by a system which needs to be continuously confirmed about the identity of the user who is performing a certain activity. As an example, we can imagine an on-line degree system which needs to be sure of the identity of the student who is making an examination, not only before the test takes place, but also during the test itself, in order to avoid people replacements after the initial identification process. Figure 1 shows the basic steps of our trust process: after an initial authentication, the server can require a second or third (or even more) step of authentication based on two parameters, the level of trust previously computed and the time passed from the last authentication. We suppose the first authentication acquired by strong techniques while the following steps can be acquired by strong or weak techniques on the basis of the trust level we have in a certain time.

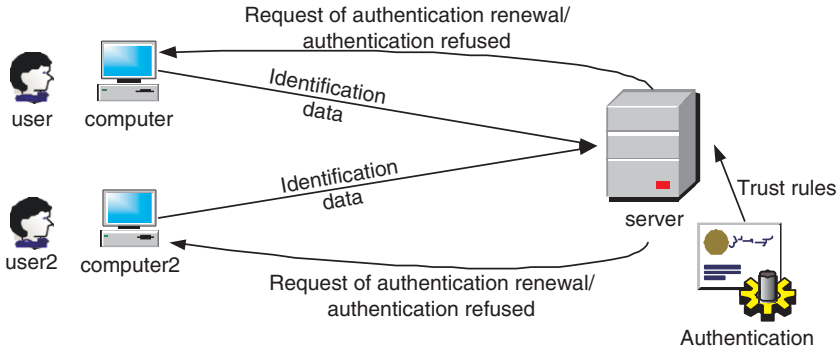

Fig. 1. Context for trust evaluation model

3.1 Trustworthiness Evaluation Parameters

After receiving an initial strong authentication, the server accepts or refuses the user on the basis of the biometric value (*BIO*) which has to be higher than a certain threshold (*th*) fixed for the application. Indeed, we suppose that our strong acquisition techniques use an internal fuzzy matching function between the actual enrollment and the template stored. In case the user is authenticated, the system receives a fuzzy value (e.g., 0.85) which represents how the biometric enrollment matches the user's template. The timeliness function, which shows how the system's trust in the identity of the user decays in time, is shown in (1) where the value BIO_{max} represents the initial value obtained at the initial authentication at time t_0 and D is the rate of decay.

$$BIO(t) = BIO_{max} * e^{-(t-t_0)/D} \tag{1}$$

Additionally, the system takes into account another parameter *TOK* that represents the boolean output (high/low or authenticated/denied) of the weak authentication system which supports the evaluation of the trust in the user's identity during the activity. At the initial authentication step, the weak techniques are not directly involved, and the parameter *TOK* is automatically set to *high*. Prior to the processing of the inputs, it is necessary to create fuzzy membership functions which define the degree of membership of each input parameter in the context of the proposed model. Furthermore, sets of fuzzy rules, based on linguistic variables, which combine the fuzzy sets, are defined in order to characterize the output of the model.

After the preprocessing step, the information obtained by the biometric engine and the parameter *TOK* are fed into a first fuzzy inference engine *Start* in order to calculate a trustworthiness value *trust* that provides the level of trust of the system in the user's identity after the initial authentication at

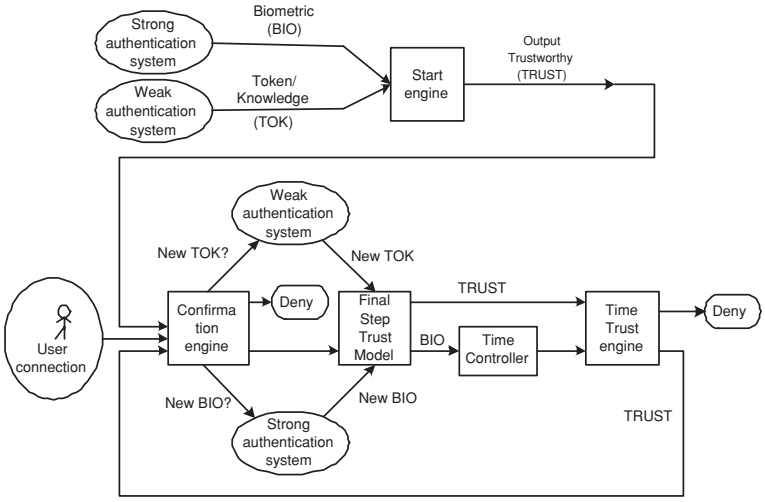

Fig. 2. Trust Model combining Strong and Weak Authentication Methods and Fuzzy Systems

time t_0. The output $TRUST$ is then fed to another engine, *Confirmation*, that checks if the user is active ($us(t_0) = OK$) and in case decides if it is necessary a new biometric or weak enrollment to enforce the trust of the system before the user can continue his activity. The enrollments provide new parameters BIO_{t_0} or TOK_{t_0} that are used by another engine, *FinalStepTrust*, to compute the definitive level of trust at time t_0. If the level of trust is higher then the threshold value defined for the application the user is authenticated and can start to work, otherwise she is refused by the system.

After a certain time interval Δt, the system checks if the trust acquired at time t_0 has been affected by the decay rate of the initial biometric authentication and then needs to be confirmed. The trust level achieved by the user at time t_0 and the new value of the parameter BIO at time $t_1 = t_0 + \Delta t$ ($BIO(t_1)$) are now fed to the last fuzzy inference engine *TimeTrust*, which decides the trust level at time t_1 which can cause the system to refuse the user or to ask for trust enforcement by going back to the *Confirmation* engine.

The process, shown in Fig. 2, stops when the user is not more active or the trust level decays dramatically to the value of *very low*.

4 Trust Model Rules

Each model previously described in Sect. 3, has been implemented with different fuzzy rules, in order to control the trustworthy value at each time step t with respect to different evolved parameters. An example of fuzzy rules, defined for each implemented model, is reported in Table 3.

Table 3. Sample Fuzzy Rules defined for each Trust Model

Model	Fuzzy rules
Start Model	**IF** BIO is high **AND** TOK is high **THEN** TRUST is high **IF** BIO is medium **AND** TOK is high **THEN** TRUST is medium ... **IF** BIO is low **AND** TOK is low **THEN** TRUST is very low
Confirmation Model	**IF** USER is ok **AND** TRUST is high **THEN** TRUST is high **IF** USER is ok **AND** TRUST is low **THEN** TRUST is medium **AND** New BIO ... **IF** USER is ok **AND** TRUST is medium **THEN** TRUST is medium **AND** New TOK
Final-Step Model	**IF** New BIO is high **THEN** TRUST is high **IF** New TOK is high **THEN** TRUST is medium ... **IF** New BIO is low **THEN** TRUST is very low
Time-Trust Model	**IF** BIO is high **AND** TRUST(t_0) is high **THEN** TRUST(t_1) is high **IF** BIO is medium **AND** TRUST(t_0) is high **THEN** TRUST(t_1) is medium ... **IF** BIO is low **AND** TRUST(t_0) is medium **THEN** TRUST(t_1) is low

The Start Model is carried out at first time, giving a trustworthy value depending on biometric and token/knowledge based acceptance rates, that have been acquired at the initial user login step.

The trust output is then carried out at each step in the other models, and it will be checked: if its value is lower than a fixed threshold value, than the system rejects further user authentication and stops the entire fuzzy model; otherwise the trust value will become one of the inputs for the further models, in order to obtain a new trustworthy value at the new step.

The trustworthy value will go into a loop in which timed checks will be implemented in order to obtain, respectively, user rights and user status connection.

5 Conclusions

In this work we propose a fuzzy logic based methodology based on a balanced mix of strong and weak authentication techniques studied to guarantee a high and prolonged in time level of security combining the advantages of each authenticator.

In such a scenario, the system, after an initial authentication phase in which it recognizes the user profile and allows the access, performs some other

authentication steps in which it decides if its trust in user's identity is enough high to allow the user to continue to perform the activity she is doing. Focus of this paper is the description of the fuzzy logic based methodology used to continuously check and confirm the trust level in the identity of a user.

Future work will include research studies in order to avoid biometric attacks and weak malicious authentication at first access and during the overall examination time.

Acknowledgments

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Quantification of the Effectiveness of the Markov Model for Trustworthiness Prediction

Farookh Khadeer Hussain, Elizabeth Chang, and Tharam S. Dillon

Summary. In this paper we propose a method for determining the effectiveness of the Markov Model for predicting the future trustworthiness value of a given agent by utilizing simulation methods. This paper presents in detail the simulation method that we employed in order to determine the effectiveness of the Markov Model. Additionally the paper presents the results that we obtained form the simulation experiments.

1 Introduction

In order determine the effectiveness of the Markov Model that we had proposed in an earlier publication [2] for predicting the future trustworthiness value of a given agent, we created a prototype application. The application itself was programmed as a GUI, for clarity and ease of use, using Net Beans IDE with Java SE 1.5.0_03 as programming technology.

The prototype application is designed to simulate the process of a trusting agent, making a trusted based decision of whether or not to make an interaction or transaction with the trusted agent. A number of steps are involved in order to reach the stage of the trusting agent making a trust based decision of whether or not to make a trust based decision about the trusted agent. In this section, we will enumerate and discuss the steps involved in a given simulation cycle:

1. The trusting agent issues a resource query, containing the context in which the trusting agent wishes to carry out the transaction.
2. The other agents present in the network, who feel that they can satisfy the request from the trusting agent reply to the resource query.

At the beginning of the simulation, each agent present in the network is assigned a set of contexts by the application, in which it is the expert. Similarly when the application starts, each agent in the network is assigned a trustworthiness value in the range of [1,6]. We intend to make use of the trustworthiness scale proposed in our earlier publications [1].

When ever an agent, which has been assigned a trustworthiness value of either "5" or "6" gets a resource query from another agent, requesting for interaction in a given context, it checks whether it is an expert in that context. If it is an expert in that context then it replies to the resource querying agent, registering/expressing its interest in interacting with the resource requesting agent in the specified context. However, if is not an expert in the specified context then the agent does not reply.

When ever an agent, which has been assigned a trustworthiness value of either "3" or "4" gets a resource query from another agent, requesting for interaction in a given context, it checks whether it is an expert in that context. If it is an expert in that context then it replies to the resource querying agent, registering/expressing its interest in interacting with the resource requesting agent in the specified context. However, if is not an expert in the specified context then the agent replies 50% of the times, i.e., it replies to every second resource query stating and expressing its interest in interacting with a trusting agent in the context specified by the trusting agent in the resource query, even though the context does not fall in its expertise.

When ever an agent, which has been assigned a trustworthiness value of either "1" or "2" gets a resource query from another agent, requesting for interaction in a given context, it checks whether it is an expert in that context. If it is an expert in that context then it replies to the resource querying agent, registering/expressing its interest in interacting with the resource requesting agent in the specified context. However, unlike agents assigned with trustworthiness value of "5" and "6" or "3" and "4", the agent which has been assigned a trustworthiness value of its interest in interacting with a trusting agent in the context specified by the trusting agent in the resource query, even though the context does not fall in its expertise.

3. The trusting agent then chooses an agent from the set of agents who have replied to the resource query. The trusting agent makes use of the Markov Model to choose an agent from among the set of agents who may have possible replied to his resource query. The rest of this paper, explain the working and the way in which the Markov Model has been implemented in detail and quantifies using the metrics defined in this paper the effectives of the Markov Model.
4. The trusting agent then notifies the prototype simulation set up about the agent that it has chosen for interaction. Based on the information communicated by the trusting agent to the prototype simulation set up, it can then determine the correctness of a decision made by the trusting agent.

This process (Step 1–Step 4) could be repeated as many times as desired by the user. The effectiveness of the Markov Model in making a trust based decision by predicting the future trustworthiness value of an agent is determined based on the effectiveness or accuracy of the decisions made in Step 4.

This paper is organised as follows, In Sect. 2, we explain in detail the exogenous parameters used in the simulation set up. Sections 3 and 4 summarize and explain each of the phases in the simulation set briefly. Finally the results are presented in Sec. 4. Section 5 concludes the paper along with future work.

2 Exogenous Parameters Used in the Simulation

Table 1 presents to the user the exogenous parameters that have been used in the application. Exogenous parameters are parameters controlled by the user of the system. They allow the user to set up a custom test environment for which to perform the required simulations.

3 Behaviors of the Agents in the Prototype Simulation

The simulation is set up in such a way that agents with different trustworthiness values would exhibit different behaviors. As mentioned before, the total number of agents that the user wants or desires in the simulation set is specified by the user. The prototype simulation set up then assigns a trustworthiness value to each agent in the simulation set up. Agents are classified according to their behavior. We define the behavior of an agent as *"the way in which it conducts it self in a given situation or circumstance."* The behavior of an agent in turn depends directly on the trustworthiness value assigned to it by the simulation set up. We have three different classes of behavior and each class of behavior varies form the others in terms of how an agents responds or reacts to a given situation. Agents who have been assigned a trustworthiness value of either "5" or "6" would take the behavior of *Trustworthy Agents of Good Agents.* On the other hand agents who have been assigned a trustworthiness value of either "3"or "4" would take up the behavior of *Neutrally Trustworthy Agents.* On the other hand agents who have been assigned a trustworthiness value of either "1"or "2" would take up the behavior of *Malicious Agents or Bad Agents or Untrustworthy Agents.* The behavior of each agent is a collection of activities that that agents performs in a given situation.

An important point to be noted here that, the objective of this simulation is to measure or quantify the effectiveness of the Markov Model in making a correct trust based decision and not to determine the effectiveness in which the behavior of an agent can be modeled. Hence it is not necessary that we have six different kinds of behaviors corresponding to six different trustworthiness values, in order to determine the effectiveness of the Markov Model. In order to determine the effectiveness of the proposed Markov Model, all we need is a means by which the prototype simulation system can assign a particular class of behavior to a given agent (trusted agent) (as mentioned before each

Table 1. Exogenous parameters

Parameter	Description	Possible values
# Network agents	Sets the total number of agents to be used in the system	0<x<100000
% Malicious agents	Sets the percentage of total agents to be regarded as untrustworthy	$-1 < x < 100$
# Neutral	Sets the number of total peers to be regarded as neutrally trustworthy	$-1 < x <$ Total Peers
# Number of simulations	Sets the number of iterations that the simulation will do before stopping	$0 < x < 100000$
# Simulation increment	Sets an interval for the number of times the results will be compiled while the simulation is running	$0 < x <$ #Number of simulations

 # Network agents: The total number of agents with which the user wishes to run the simulation. This value must be greater than 0 and less than 100,000.

% Malicious: This field corresponds to the percentage of *Network Agents* that would be assigned a trustworthiness value of either "1" or "2" in the simulation. These agents would hence exhibit and display a behavior that corresponds to the trustworthiness of "1" and "2". The simulation set up can have zero or more malicious peers but there can never be more than 99% of the total agents in the simulation set up.

 # Neutral: This field corresponds to the number of agents that would be assigned a trustworthiness value of either "3" or "4" in the simulation. These agents would hence exhibit and display a behavior that corresponds to the trustworthiness of "3" and "4". A simulation set up can have zero or more neutral peers and no more than the total peers in the system.

 # Number of simulations: This is the total number of simulations cycles in the whole simulation. A simulation consists of a candidate agent being selected, soliciting a resource request and gathering replies, and then processing reputation replies. A Markov model is then used to determine the agent which the trusting agent (or selected agent) should choose to interact with Results are gathered at the end of each simulation cycle. The application only allows for 99,999 as the maximum number of simulations.

 # Simulation increment: This is the interval of simulation cycles that are run, during the simulation, which the simulation time will increment and results will be processed for that set of simulations, in the interval. Since the Markov model is dynamic and has a time aspect, this is used to increment the fictional time element of the Markov model. At the end of each cycle, misrepresentation and transaction results are processed and calculated for number of simulations in the given interval.

class of behavior is associated with a unique trustworthiness value/s) and then as the simulation progress we need to find out whether or using the Markov Model the trusting agent can determine the trustworthiness value or the class of behavior of the trusted agent. If using the Markov Model the trusting agent has been able to successfully determine the trustworthiness value of the trusted agent then the prediction process is said to be accurate. Additionally, if using the Markov Model the trusting agent has not been able to successfully determine the trustworthiness value of the trusted agent then the prediction process is said to be inaccurate.

We have defined three classes of behaviors based on how an agent responds in the following three situations:

1. When it is presented with a resource query.
2. When it is presented with a reputation query.
3. When it has been chosen to carry out a transaction.

We classify and derive the three kinds of behavior based the different responses to the above three situations.

As can be seen from above, we have defined ONLY those behaviors for the agents to which they have to respond to during the simulation of the Markov Model. As has been explained in Sec. 1, once the simulation set up has chosen a trusting agent, it then issues a resource query. So the behavior an agent exhibits on receiving this resource would vary and various sorts of behaviors can be seen in Table 2. Once the trusting agents compiles a list of all the agents who are willing to provide it with the requested resource, then it issues a reputation query for each of them. The way in which an agent replies to a given reputation query varies and depends on its behaviors. Finally once an agent has been chosen by the trusting agent for interacting the way in which it interacts again depends on its behavior. As can be seen there are only three situations or circumstances in which the behavior of a given agent could vary during the running of the Markov Model. We have defined three different for each circumstance or situation corresponding to the three different types of agents.

During simulation we have to consider the behavior at a certain level of abstraction. Our level of abstraction is composed of the three situations mentioned above. In reality however, the behavior of an agent in the real world would be an exhaustive list of activities which correspond to the ways in which an agent responds to different situations. Since we are building a simulation set up for the Markov Model we have considered only those situations which an agent may face during the Markov Model.

4 Effectiveness of Markov Model on Trust Based Decision Making

This section presents the effectiveness of the Markov Model in predicting future trustworthiness value of a given agent and walks through an example scenario by taking a set of inputs and outputs. As listed in Sect. 2 the user needs to specify the exogenous parameters in order to start the simulation.

Table 2. Behaviors of the different agents in the system

Circumstance	Trustworthy agents	Neutral agents	Untrustworthy Agents
Presented with a resource query	Check their expertise set and reply ONLY if they have expertise in the context of the resource query	Reply to the resource expressing their willingness to provide the expected resource irrespective of whether or not they have expertise in the resource 50% of the times On the rest 50% of the times check their expertise set and reply ONLY if they have expertise in the context of the resource query	*Always* reply to the resource expressing their willingness to provide the expected resource irrespective of whether or not they have expertise in the resource
Presented with a reputation query	Check their local database to determine whether or not they have interacted with the reputation queried agent in the context and time slot provided in the reputation query and then give a correct referral	50% of the times they reply to the reputation query, giving a random trustworthiness value in between [1,6] irrespective of whether or not they have interacted with the reputation queried agent in the context and time slot provided in the reputation query On the rest 50% of the times, Check their local database to determine whether or not they have interacted with the reputation queried agent in the context and time slot provided in the reputation query and then give a correct referral	*Always* reply to the reputation query, giving a random trustworthiness value in between [1,6] irrespective of whether or not they have interacted with the reputation queried agent in the context and time slot provided in the reputation query
Chosen to carry out a transaction	Once they have been chosen to interact in a context (to which they have replied their willingness to interact with the resource querying agent) and since each of them are assumed to be experts in their own contexts they always provide a good service to the trusting agent *always* provide *bad service* to the trusting agent once the trusting agent has made a decision to interact with them	*Always* provide *bad service* to the trusting agent once the trusting agent has made a decision to interact with them	(*Always* provide *good service* once a trusting agent has made a decision to interact with them)

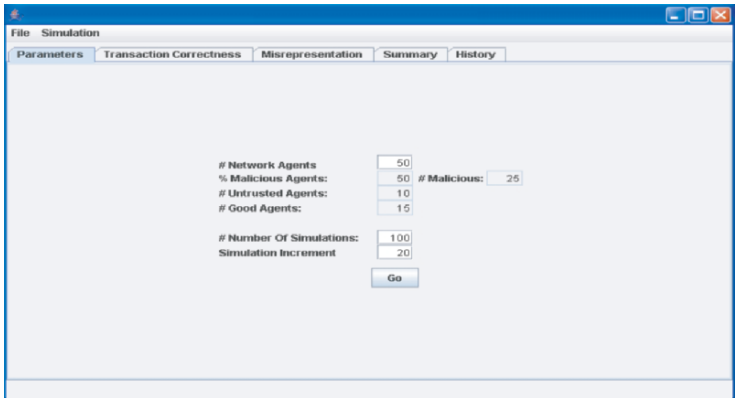

Fig. 1. Figure showing the parameters that the user needs to specify

From this screen (see Fig. 1), it can be seen that there are 50 network agents, which is the total number of agents in the prototype simulation set up. The user wishes 50% of these total agents to take up the behavior of malicious agents (which is specified in the *%Malicious* textbox). Once the percentage of the malicious agents and the number of neutral agents have been specified by the user, the prototype system automatically updates the number of agents in the network that should take up the behavior of the good agents. The prototype simulation system is set up so the user cannot specify the number of agents with malicious behavior or the number of agents with neutral behavior more than the number of network agents. Additionally there has to be at least one good agent in the system.

The last two parameters that the user needs to specify are the total number of simulation transactions (specified in the #number of simulations textbox) and the number of simulation transactions after which the results should be updated (specified in the #simulation increment textbox). In this case the user has specified the total of simulation transactions to be 100. This means there will be 100 total simulations transactions and after every 20 simulation transactions, a new time interval occurs and results for those 20 simulations are compiled into a single reference point of data.

Using these parameters the simulation takes about 15 s to run through and then all results are available. Note: more agents (in particular malicious agents) and more simulations would mean a longer simulation time.

In order to determine the whether or not the Markov Model aids in the process of making a correct decision, we have classified the transactions taking place in the system into two classes, namely

– Good Transactions: those transactions in which a correct decision (the resource querying agent ended up making a decision of interacting with an agent of trustworthiness value of either “5” or “6”) was made by the resource querying agent by making use of the proposed Markov Model.

– Bad Transactions: those transactions in which a incorrect decision (the resource querying agent ended up making a decision of interacting with an agent whose trustworthiness value was neither "5" nor "6") was made by the resource querying agent by making use of the proposed Markov Model.

The percentage of good transactions in a given interval is computed using the following formulae

%Good Transactions =

(Transactions in which correct decision was made/Total Transactions) *100

The first section is the results section of the prototype simulation setup, has the results of the good and bad transactions which took place. When the user clicks on the transaction correctness tab, he/she can see two subtabs appear in the panel and you can choose between the percentage of good transactions and the percentage of bad transactions.

The Good Transactions (Fig. 2) graph's x-axis is represented as the number of transactions and goes from 1 to 5. As you recall from the parameters we entered 100 simulations with an increment of 20. This means every 20 simulations a new data point is formed to represent the previous collection of 20 simulations. Therefore five data points form to represent the results of the simulation. In this case, for all sets of 20 simulations it showed the percentage of good transactions in each lot was 100% which means overall a good transaction was made 100% of the time, which means the Markov model is working accordingly.

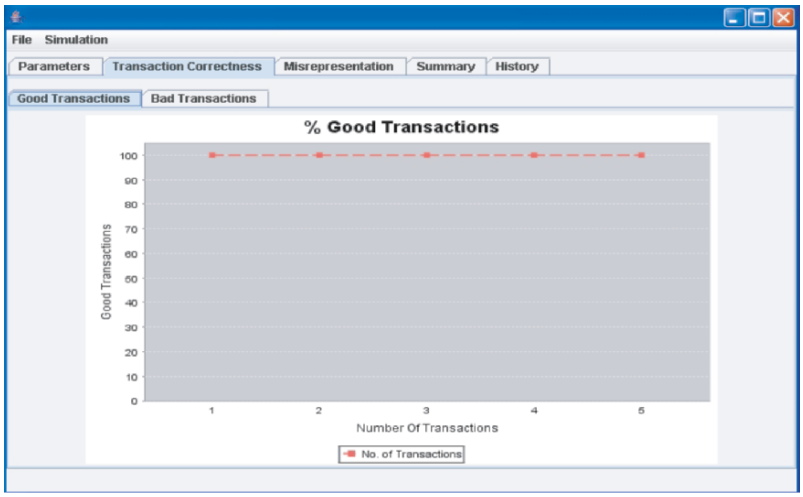

Fig. 2. Graph showing the percentage of correct (good) transactions

Fig. 3. Graph showing the percentage of bad transactions

On the flipside is the percentage of bad transactions which is shown in Fig. 3.

%Bad Transactions =

(Transactions in which incorrect decision was made/Total Transactions) *100

As you can see from the picture, at all of the five sets of data points no bad transactions (i.e., 0%) were made, which complies with the 100% success rate of the agent.

5 Conclusions and Future Work

In this paper we presented a simulation based method in order to test the effectiveness of the Markov Model for determining the effectiveness of the Markov Model. We presented and explained in detail the various steps involved as well. We presented the three different classes of agents based on their trustworthiness values and their behaviors as well.

Finally, we presented the results that we got form the simulation set up. We found that the Markov Model is very effective to the extent of 100% determining the future trustworthiness of an agent assuming that the system is stationary. We intend to do future work along several directions. We intend to explore how the effectiveness of the system in case it is nonstationary. Additionally we intend to find out the effectiveness of the Markov Model in determining the witness trustworthiness value of the agents accurately. Additionally we intend to explore as future work the accuracy of the trust based decisions made by an agent by employing the Markov Model, when the trustworthiness value of the reputation queried agent or the trusted agent tends to be dynamic over different time slots.

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Fuzzy-Genetic Methodology for Web-based Computed-Aided Diagnosis in Medical Applications

F. de Toro, J. Aroba, J.M. Lopez

Summary. This paper presents an integrated fuzzy-genetic methodology to address web-based computed-aided diagnosis by using bio-signal processing in medical applications. A deterministic crowding genetic algorithm is used for obtaining different subsets of features that provide high performance classification in a K-Nearest Neighbor classifier. These subsets of features are then used as training data in a rule generator based on fuzzy clustering to obtain a performance qualitative model that can give information about the more suitable features to use in the diagnosis. This model can also be used to assess the (performance) accuracy that will be reached by using a given set of features – possible those ones available at a specific medical centre. The overall methodology is applied to Paroxysmal Atrial Fibrillation (PAF) – the heart arrhythmia that causes more frequently cerebrovascular incidents – Diagnosis based on analysis of nonfibrillation ECGs

Key words: Computer-aided diagnosis, Evolutionary algorithms, Machine learning, Fuzzy clustering, Decision making.

1 Introduction

We can define computed-aided diagnosis (CAD) as the diagnosis a physician makes using output from a computerized analysis of medical data. Multiple features are used to classify an observation as normal or abnormal. A radiologist may, for example, note the size, shape, and margin sharpness of a potential breast in a mammogram and somehow use this information to determine whether a cancer is present. The goal in training a diagnostic classifier is to employ a limited dataset including normal (without disease) and abnormal (with disease) cases to determine the classifier parameter values so that it correctly classifies other datasets of unknown pathology. The training of a classifier can be viewed as an optimization problem where the quantity to be maximized is the performance of the classification on an independent dataset. Binary classifiers [1] separate two classes of observations and assigns new observations to one of the two classes: the normal (no disease evident) and abnormal (indicative of disease) class. As mentioned, certain characteristics

of the observations, called features, are used in making the classification decision. The process of choosing the characteristics used in a diagnostic classifier takes place in the so-called feature selection stage [2]. Evolutionary algorithms (EAs) [3] and, above all, one of its most-applied paradigms – genetic algorithms [3, 4] – have been applied successfully for feature selection [5] and also for determining the classifier parameters values in medical diagnostic applications: e.g., weights of an artificial neural network [6] or thresholds in rule-based detection schemes [7]. The ability to find multiple solutions in a single iteration gives EAs a privileged position to address multimodal optimization problems [8]. There are two good, practical reasons which may prompt the location of multiple optima in such types of problems. First, by encouraging the location of multiple optima, the chances of situating the global optimum are increased. Secondly, in a design context, identifying a diverse set of high-quality solutions (global and local optima) will provide an insight into the nature of the design space and suggest alternative solutions. This second possibility has been poorly exploited so far [8]. Nevertheless, in [9] we addressed the utilization of a genetic (evolutionary) algorithm in the feature selection stage of a classifier for Paroxysmal Atrial Fibrillation (PAF) diagnosis, the heart arrhythmia that causes more frequently cerebrovascular incidents [10]. A deterministic crowding [11, 12] technique was embodied in the design of the genetic algorithm in order to improve the chances of obtaining diversified solutions (different subsets of solutions) from this natural multimodal optimisation problem. Here, a solution to the problem is a subset of features that provides good performance classification (so high-quality local optima can also be considered for this purpose). This enables specialists to make a diagnosis based on different subsets of features. This is of great interest, since some characteristics may be unreliable due to the interference of other medical disorders that the patient may have. It also allows specialists from different centers to adopt a concrete subset of characteristic criteria depending on the equipment available at a particular center. Sometimes, the specialist may not have access to the instrument required to measure a certain characteristic so some of the solutions cannot be used. After the multimodal optimization process, the diagnostic performance based on different subsets of characteristics is known. In this paper, we step forward from our previous work presented in [9] by using a fuzzy-clustering tool that inferences qualitative knowledge from the population of solutions (different subsets of features) obtained from the optimization process carried out during the training of the classifier. In a more specific way, the fuzzy rule generator integrated in the methodology diagnosis, called PREFURGE [13], provides some qualitative rules regarding the performance accuracy that can be reached in the diagnostic classification. Each rule obtains information about the performance accuracy as a function of the degree of participation of every feature in the classification process. The information contained in these rules can be useful in different ways: (1) to extract some conclusions about which are the more important features in order to get a good performance classification; (2) to predict which performance classification can be obtained by using a given subset of features available at

a specific medical center. In the following section, the overall PAF computer-aided diagnosis methodology presented in this paper is explained. First of all, Sect. 2.1 summarizes some previous work concerning the automatic diagnosis of the PAF by using nonfibrillating ECGs. Then Sect. 2.2 revisits the fuzzy rule generator PREFURGE and shows how it is integrated in the aforementioned methodology. Section 2.3 discusses a prospective web-based implementation of the methodology described in Sect. 2.2. In the last section, experimental work addressing the extraction of fuzzy rules in a PAF diagnosis application is given. We finalize this work with a summary of the conclusions drawn from this investigation.

2 Materials and Methods

In many situations, the medical diagnosis of health disorders (e.g., heart disease) can be addressed through the extraction of characteristic parameters from bio-signals such as ECGs. In many cases, this kind of noninvasive diagnosis is more appropriate than other solutions involving surgery. The extracted characteristics (also called features) are used in an algorithm for classification to produce a computed-aided diagnosis. Computed-aided diagnosis can be used with other diagnostic strategies, e.g., as a second reading or second opinion in making diagnostic decisions [14]. Next section reviews our previous work addressing PAF Diagnosis.

2.1 Previous Work in the Field of Automatic PAF Diagnosis

In [15,16], we addressed the diagnosis of the PAF by using a K-Nearest Neighbor (KNN) algorithm to make the classification decision (healthy or ill patient). The features used for the classifier were selected from an initial group of features extracted from ECGs noncontaining explicit fibrillation episodes so a preventive diagnosis is addressed. A detailed explanation of the extracted features is given in [16]. In [15], the feature selection process is performed using a forward stepwise search (FSS) algorithm . One method of FSS begins by selecting the single best performing feature as a seed. It then steps through each subsequent feature, adding it to the subset if it improves the classification accuracy, and discarding it otherwise. Each feature has only one chance to survive, which limits the possible combinations. In addition to this, from the optimization process performed by the FSS algorithm only one solution (subset of features) is obtained, so the medical specialist is restricted to use only one subset of features to make the classification decision, furthermore the FSS performs a one track process that easily discards a feature entirely after a single consideration of its usefulness. In [9], the aforementioned approach is improved by using an evolutionary algorithm in the feature selection stage previous to the KNN classification algorithm (see Fig. 1). A Deterministic Crowding [11, 12] technique was embodied in the design of the evolutionary

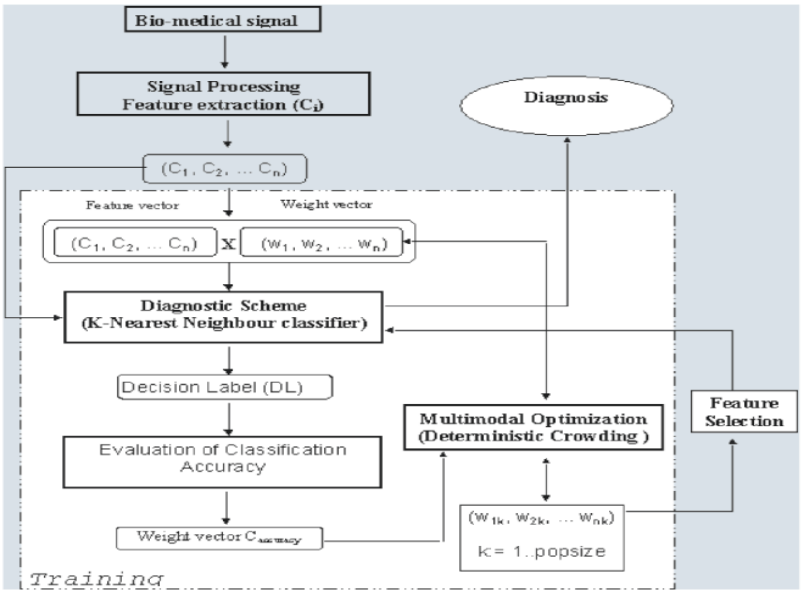

Fig. 1. Methodology for PAF diagnosis using a multimodal evolutionary algorithm for feature selection

algorithm in order to improve the chances of obtaining diversified solutions (different subsets of solutions) from this natural multimodal optimization problem. There are two main advantages of this approach regarding the FSS algorithm used in [15]. (1) Different search paths are explored concurrently and a feature is not discarded after a single consideration of its usefulness. That way, the chances of being trapped in local optima are lowered. (2) After the optimization process, different subsets of features that provided a good performance classification are obtained, so the specialist can use those features available at a specific medical center or those one more robust to the interfering action of other heart diseases. This methodology (Fig. 1) involves a first stage of the general parameter definition [16], followed by the extraction of a set of characteristics (feature vector), which are used by the diagnostic scheme (KNN classifier) to obtain the decision label (normal/abnormal case). The characteristics are related to the physical properties of the biomedical signal. Thus, the diagnostic decision is a label obtained as a function of certain input items or characteristics (C_i). The diagnostic scheme is applied to the input items, after which the diagnosis (DL) is obtained for each subject.

This diagnostic methodology is modular so that different researchers are able to easily add (or remove) new features to the vector. After this parameter analysis, the basic feature vector containing all defined parameters is multiplied by a weight vector. A value ranging between 0 and 1 is assigned to each component of the weight vector, according to the degree of influence

that each component of the parameter vector has on the diagnostic scheme. To optimize the performance accuracy, the values of the weights are iteratively adjusted by an evolutionary algorithm. Moreover, we are able to decide whether the components of the weight vector will take real values (ranging from 0 to 1) or binary values (0 or 1). In the first case, each solution (weight vector) shows the importance of each component of the characteristic vector. In the second case, each solution indicates which components of the characteristic vector should be considered in the diagnosis. Due to the fact that EAs work with a population of candidate solutions, different choices of weight vectors can be explored in a single iteration of the algorithm. If the necessary diversity mechanism [17] is incorporated into the EAs, at the end of the convergence, the set of solutions obtained (each solution is a different subset of weights) is diverse and provides the specialist with the necessary flexibility. Deterministic crowding is chosen as diversity maintaining technique for two main reasons: (1) it shows a good performance in several comparative studies regarding other methods [12,17]; and (2) in contrast with other techniques like clearing [18] or fitness sharing [4], there is no need to determine any user parameter. Distance metric is defined in the parameter space (genotype distance) to encourage dissimilarity between features contained in the solutions.

2.2 Fuzzy-Genetic Methodology for PAF Diagnosis

In this work, we step forward from this previous work by adding a final fuzzy rule generation stage that enables to extract qualitative information from the set of solutions obtained from the training of the classifier. In a more specific way, the fuzzy rules obtain the performance classification as a function of the degree of importance given to each feature in the diagnostic scheme-measured with the weight vector. The fuzzy rule generator explores all possible information contained in the set of solutions obtained from the training of the classifier. This information can be used to determine which are the features more suitable in order to get a good performance classification. These rules could also be used to predict the performance classification that can be obtained by using the features available at a specific medical center. Lastly, the information found by the PREFURGE tool can be used to speedup the training of the classifier by providing a good guess for the initial population of the evolutionary algorithm. The overall methodology is depicted in Fig. 2. In what follows we review PREFURGE [13] used as a fuzzy rule generator in this methodology.

PREFURGE (Predictive Fuzzy Rules Generator)

Classical clustering algorithms generate a partition of the population in a way that each case is assigned to a cluster c. These algorithms use the so-called "rigid partition" derived from the classical sets theory: the elements of the partition matrix obtained from the data matrix (with n elements) can only

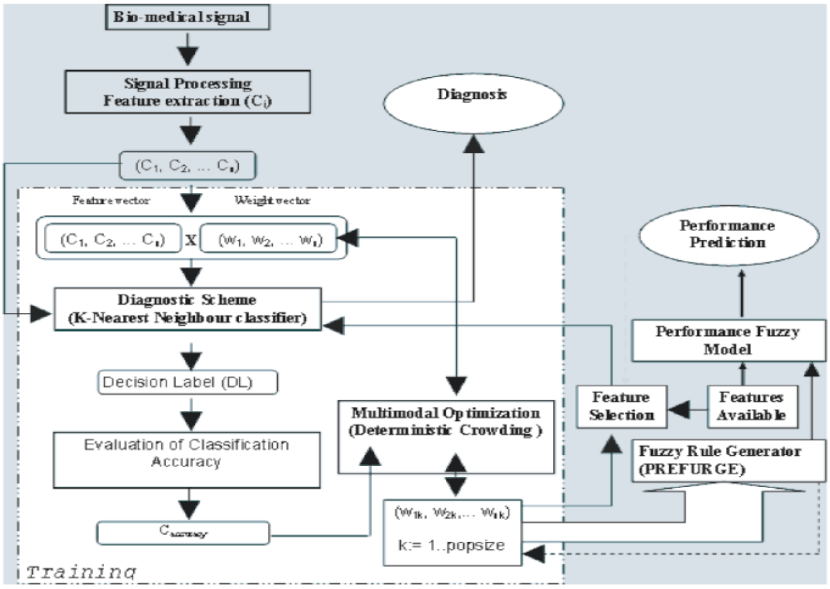

Fig. 2. Methodology for the Paroxysmal Atrial Fibrillation diagnosis by using an integrated Fuzzy-evolutionary approach

contain values 0 or 1; with zero indicating null membership and one indicating whole membership to each one of the c partitions. That is, the elements must fulfill:

$$\begin{aligned} &\text{(a)} \quad \mu_{ik} \in \{0,1\}, \quad 1 \le i \le c,\ 1 \le k \le n, \\ &\text{(b)} \quad \sum_{i=1}^{c} \mu_{ik} = 1, \quad 1 \le k \le n, \\ &\text{(c)} \quad 0 \le \sum_{k=1}^{n} \mu_{ik} \le n, \quad 1 \le i \le c. \end{aligned} \tag{1}$$

Fuzzy partition is a generalization of the previous one, so that it holds the same conditions and restraints for its elements, except that in this case real values between 0 and 1 are allowed (partial membership grade). Therefore, samples may belong to more than one group, so that the selecting and clustering capacity of the samples increases. From this we can deduce that the elements of a fuzzy partition fulfill the conditions given in (1), except that now condition (a) will be written as:

$$\mu_{jk} \in [0,1], \quad 1 \le i \le c,\ 1 \le k \le n. \tag{2}$$

The best known general-purpose fuzzy clustering algorithm is the so-called Fuzzy C-Means (FCM) [19]. It is based on the minimization of distances

between two points (data) and the prototypes of cluster centers (c-means). For this purpose, the following cost function is used

$$J(x,u,v) = \sum_{i=1}^{c}\sum_{k=1}^{n}(\mu_{ik})^m \mid x_k - v_j \mid_A^2, \tag{3}$$

where U is a fuzzy partition matrix of X, $V = (v_1, v_2, ..., v_n)$ is a vector of cluster center prototypes which must be determined and $m \in [1, \infty]$ is a weighting exponent which determines the degree of fuzziness of the resulting clusters. Finally,

$$D_{ikA}^2 = \mid x_k - v_i \mid_A^2 = (x_k - v_j)^T A(x_k - v_i) \tag{4}$$

is the norm used for measuring distances (matrix A induces the rule to be used – provided that it is the unit matrix, which is very frequent – i.e., the Euclidean norm). The described algorithm was used [20] to build a fuzzy model based on rules of the form

$$R' : \text{ IF } x \in A' \text{ THEN } y \in B', \tag{5}$$

where $x = (x_1, x_2, ..., x_n) \in$ R are input variables, $A = (A_1, A_2, ..., A_n)$ are n fuzzy sets, y $\in$ R is the output variable and B is the fuzzy set for this variable. The fuzzy clustering tool used in this work, PREFURGE [13], uses the algorithm described in [20] improved in the following aspects:

- It allows working with quantitative databases, with n input and m output parameters.
- The different variables object of study can be weighted by assigning them weights for the calculation of distances between points of the space being partitioned.
- The achieved fuzzy clusters are processed by another algorithm to obtain graphic rules trapeziums (Fig. 3).
- An algorithm processes and solves cases of multiple projections in the input space (mounds).
- The output provided in the original method has been improved with a graphic interface showing the graphic of the achieved rules.
- An algorithm provides automatically the interpretation of the fuzzy graphic rules in natural language.

Furthermore, it is important to note that the graphic output provided by PREFURGE enable an easy interpretation of the fuzzy rules in a natural language. As an example, Fig. 4 shows two rules generated by PREFURGE. In the rule of Fig. 4a, the fuzzy set assigned to each parameter is represented by a polyhedron. The parameter values are represented on the x-axis of each fuzzy set, and the value of membership to a cluster on the y-axis. This fuzzy rule would be interpreted as follows:

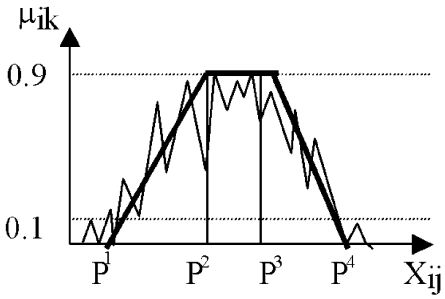

Fig. 3. Approach by trapezes carried out by PREFURGE

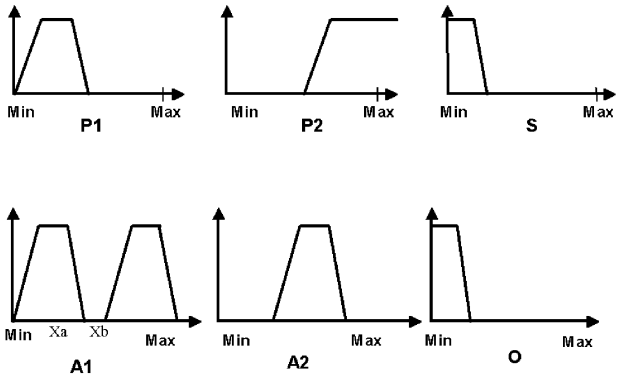

Fig. 4. Two examples of fuzzy rules generated by PREFURGE

"IF P1 is small and P2 is bigger or equal to average THEN S is very small."

When applying the fuzzy clustering algorithm [13] to the generated databases, it is possible to obtain multiple projections in the input parameters. In the fuzzy rule of Fig. 4b, a multiple projection is represented in the input parameter A1. In this case we observe how the parameter A1 can take different types of values for a certain kind of output. This fuzzy rule can be interpreted as follows:

"IF A1 is small or big and A2 is average THEN O is very small."

2.3 Web-Based Implementation

The immense possibilities that the Internet provides for information remote access and global connectivity has led to an increasing effort in applying the so-called new technologies in the field of health care as well as others. In this context, telemedicine is defined in [23] as "the combined use of telecommunications and computer technologies to improve the efficiency and effectiveness of healthcare services by liberating caregivers from traditional

constraints of place and time and by empowering consumers to make informed choices in a competitive marketplace." Finally, new technologies enable the easy sharing of resources and databases for medical investigation purposes. All this reasons justify the increasing effort in exploring the benefits that the Internet can bring to both the medical specialist and health care researchers.

In this sense, we can see that a web-based implementation of the methodology presented in Sect. 2.2 remotely accessible through the Internet is very beneficial. Final users, such as medical specialists can extract the features from the patient bio-signal data and send the extracted features via on-line towards the web diagnostic applications which would deliver the computed diagnosis within a relatively small response time. Furthermore, this centralized approach for the implementation of the discussed methodology of this paper enables an easy upgrade of the features used in the diagnostic. Researchers can find new features improving the current classification accuracy on test data, these new features could be also send on-line to the remote web-application for an immediate upgrade.

3 Case Study

As an example of the application of the diagnostic methodology described in this work, a public database provided by Physiobank [21] and used in our previous works related to PAF diagnosis [9] has been used. It comprises the ECG records of 25 healthy individuals (n files) and 25 patients diagnosed with PAF (p files). The records are labeled (healthy or ill). We have used 14 definable characteristics from the ECG records. Real coded weight vector (each component is a real number between 0 and 1) has been considered instead of the binary representation used in [9]. This way, the weight vector gives information of how important is each feature in order to obtain a good performance classification. For biomedical diagnostic applications, the final diagnosis is that either a patient is ill (suffering a certain pathology) or healthy (free from this particular pathology). This means that the classification result can be one of the following cases: (1) the algorithm classifies the subject as ill and the subject is in fact ill (true positive, TP); (2) the algorithm classifies the subject as healthy and the subject is in fact healthy (true negative, TN); (3) the algorithm classifies the subject as ill but the subject is healthy (false positive, FP); and (4) the algorithm classifies the subject as healthy but the subject is ill (false negative, FN). Within these cases, different functions have been considered as performance criteria in the fuzzy rule generator.

Classification accuracy:

$$C = \frac{TP + TN}{TP + TN + FP + FN}. \tag{6}$$

Sensitivity: it represents the ratio between the detected ill patients and the total ill patients.

$$SENSI = \frac{TP}{TP + FN}. \tag{7}$$

Specificity: it represents the ratio between the detected healthy subjects and the total healthy subjects

$$SPECI = \frac{TN}{TN + FP}. \tag{8}$$

Due to the small size of the test database (25 PAF patients and 25 nonPAF subjects), the evaluation of the classification accuracy (and sensitivity) is calculated in 50 cycles by the leaving one out method [21], i.e., in each cycle, one vector is selected from the database as the test element. This vector is classified according to the scheme described above, with the other 49 labeled vectors serving as classification references. In each cycle the classification results are updated in four counters: true_positive (TP), true_negative (TN), false_positive (FP), and false_negative (FN). Finally the Classification accuracy, sensitivity, and specificity are calculated following the equations (6)–(8). The crossover operator used in the deterministic crowding procedure is a single-point real-coded operator [22]. Two different types of mutation have been considered: uniform mutation (9) and Gaussian mutation (10). Then, the mutated weigh components are obtained from a uniform distribution function and a normal distribution function, respectively. Each type of mutation is used with a probability of 0.5. The mutation rate probability has been set to 0.6.

$$w'_j = U(0,1), \tag{9}$$

$$\begin{aligned} \mu &= w_j, \\ \sigma &= \sqrt{\min((1 - w_j), w_j)}, \\ w'_j &= N(\mu, \sigma). \end{aligned} \tag{10}$$

For the training of the classifier, the deterministic crowding algorithm has been run during 10^{14} weight vector fitness evaluations. After this optimization period more than 200 different subsets of features providing a performance classification above 80% were obtained. These solutions were used as training data in PREFURGE, and six different rules were discovered (Figs. 5, 6). For a better visualization of the results, the rules appear splitted into two figures involving parameters P1–P7 and P8–P14, respectively. Each rule contains fuzzy input weight vector information of each of the 14 definable weight components and three fuzzy output performance indicators: classification accuracy (label S1), sensitivity (label S2), and specificity (label S3). From a medical point of view, information concerning to parameters that lead to a "high classification accuracy" and "high sensitivity" – first rule in Figs. 5 and 6 – is of interest. The sensitivity gives information about how accurate the classifier diagnoses ill patients. From the observation of the first rule in Figs. 5 and 6 in our experiment, we can conclude that the presence of P8 and P12 features and the absence of P1 and P2 features lead to a high classification accuracy and a high sensitivity.

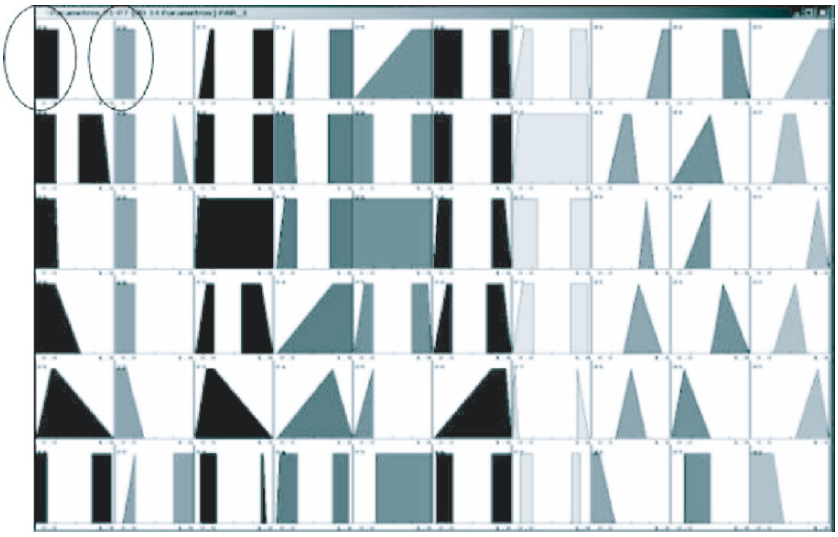

Fig. 5. The discovered six-rules for PAF application with the seven definable parameters (P1–P7) and 3 output variables: classification accuracy (S1), sensitivity (S2), and specificity (S3)

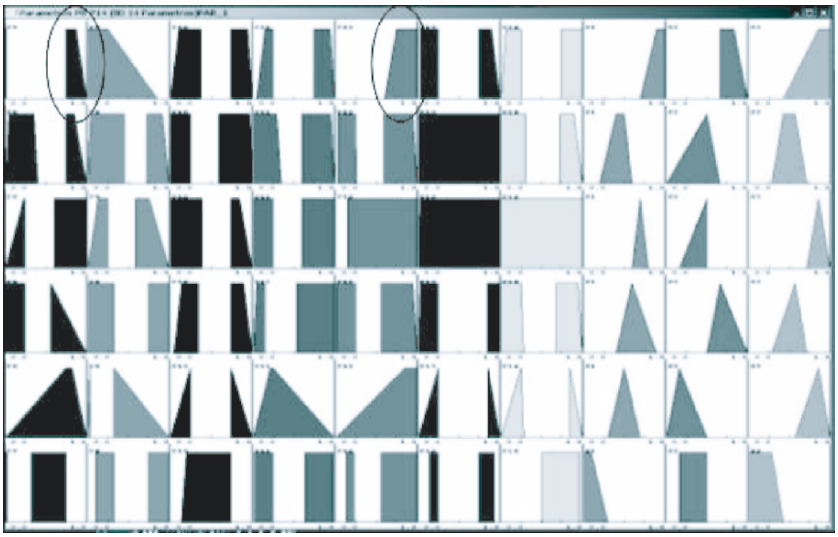

Fig. 6. The discovered six-rules for PAF application with the seven definable parameters (P8–P14) and three output variables: classification accuracy (S1), sensitivity (S2), and specificity (S3)

4 Concluding Remarks

This work presents a methodology to address computed-aided diagnosis. This methodology consist of an initial training of a KNN classifier by using a deterministic crowding genetic algorithm in order to obtain a set of solutions to adjust the classifier. A solution is compound by a set of features to base the diagnostic classification on. Then, the set of solutions is used as an input of a fuzzy clustering algorithm in order to obtain a model based on IF–THEN rules. These rules can be used to predict the performance classification than will be obtained by using a specific set of features and also to extract information about possible patterns in the set of found solutions such as relevant/indifferent features for the diagnosis. The methodology has been applied to a PAF diagnosis experiment.

5 Acknowledgments

This work has been supported by the Spanish Ministry of Science and Technology (TIN2004-06689-C03-03), and the EU DRIVSCO.

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Weight Optimization for Loan Risk Estimation with Genetic Algorithm

Irina Lovtsova

Summary. This paper considers an application of genetic algorithm for finding some weight values for loan risk estimation, used for objective principles analyzing the creditworthiness and payment ability of private individuals. To provide genetic algorithm working in this kind of task, some modifications were made in genetic operators, fitness function calculation, and data representation. During optimal weight values finding process, statistical data are divided in two parts – training and testing data sets. The generated population consists of possible weight values. Crossover and mutation operators work only with the generated data. Fitness calculation is provided for generated data of the individuals, respectively, statistical data. The increase of fitness function value depends on successful choice of weight values combination, respectively, the statistical data set. For the search progress it was decided to provide the insert and further keep the individuals with worse fitness in the special table during the algorithm run.

Key words: Genetic algorithm, Fitness function, Crossover, Mutation.

1 Introduction

This paper presents genetic algorithm (GA) application for weights finding to estimate the possible risk to give a loan – is it acceptable or not [1, 4]. The decision has the relative importance for the statistical data that are represented with nine criteria. If all criteria were to receive the same weight, then all criteria are equally important. So, this is not the fact and some different weighting scheme would be chosen as a common method for the loan risk suggestion.

The use of genetic algorithm to solve this kind of task requires some modification in genetic operators, fitness function calculation and data representation, particularly, if statistical data have continuous and numerical representation [3], and it is stored in database. To solve data representation problem it was decided to describe the statistical data with points that estimate each criteria value.

Crossover and mutation operators work only with the generated data. To provide the ability to get 100% after generated weights sum calculation, some

modifications in crossover operator were used. Modified mutation can provide only good solutions – in case if a better solution is not generated, mutation will never be accepted. The knowledge about weakness individuals is kept in the table of weak individuals and any generation of the same individual is restricted – it will be replaced by the fittest one. The crossover operator uses the aforementioned table too – in case if the obtained offspring is the same as the individual's from the *table of weak individuals*, then crossover results must be rejected and operator will run one more time. It will increase the search efficiency because the individuals with worse fit will not be reviewed in the further generations.

To make the calculation of fitness easy all possible computations concerning the statistical data were used before. The database represents the statistical data that are represented by estimation points and decision regarding the loan risk. The decision concerning the loan risk acceptability is based on knowledge that is obtained from the statistics.

2 GA in Weight Optimization Task

This task provides common principles as objective as possible for analyzing the loan worthiness and payment ability of private individuals that are clients of the bank that gives loans. The statistical data have continuous and numerical representation [3]. So, to make the data representation easy it was decided to describe the statistical data with points that estimate each criteria value. Some of the represented criteria values, such as net incoming margin, are calculated using some formulas and only then estimated by point system (0–5) (See Table 1).

Statistical data are represented with estimation points (minimum point value is 0 and maximum is 5) and loan risk estimation (acceptable or not acceptable risk)

$$p_1, p_2, \ldots, p_9, y.$$

In the weight optimization task, the independent variables are represented as a set of point values and the dependent variable y is the decision regarding the loan risk (to give the loan or not).

Initial population is generated by random and represented as a set of nine weight values [1] with precision 2 signs after comma that in the sum will give 100%

$$w_1, w_2, \ldots, w_9.$$

After the obtained weights multiplication by respective point values

$$w_1^* p_1 + w_2^* p_2, \ldots, w_9^* p_9,$$

the decision about loan risk acceptability would be provided using threshold criteria (that is determined by loan keeper).

The algorithm activity is divided in two steps – the training and the testing processes. During the training process the fittest weight values combination

Table 1. The name of criteria, criteria values, points values, and weights

Criteria	Criteria values	Points	Weights
Age of the borrower	<18, 18–24, 25–35, 36–65, >65	0–5	w_1
Education	University (degree), university (student), special, secondary school, lower	0–5	w_2
Marital status	Married, living together, single, divorced, widow (-er)	0/5	w_3
Saving/investment products	Yes/no	0/5	w_4
Net income margin	<0%, 0–5%, 5–10%, 10–15%, 15–20%, >20%	0–5	w_5
Employment	High, acceptable, medium, low	0–5	w_6
Estate	Significant, insignificant, not holder	0–5	w_7
Internal loan history	Past loan overdue	0–5	w_8
External loan history	No any overdue, paid-up overdue, no information, existing overdue	0–5	w_9

will be found to estimate the loan risk with high accuracy. This process provides the detection of fitted weight values. The testing process is used to prove the fitness of training results. In case if the found weight values cannot provide the respective results during the testing process the training process will be repeated.

The next modification that was applied solving this task is a *table of weak individuals*, in which the information about weak individuals, obtained during mutation and also crossover, is kept during the algorithm run. This table contains the following individuals:

- An individual, that was selected for mutation, but the mutated individual has better fitness
- Offspring value, that is generated during crossover and whose values are the same as the individual's from the *table of weak individuals*

3 An Individual Fitness

Fitness function is a criterion that determines the fitness of each individual of the population regarding the database of statistics. The fitness of each individual was calculated by running through all the data of statistical database that is divided into training and testing parts.

The fitness function calculation for training process has simple structure due to the database data representation (in the database all the data are represented using the points that estimate each statistical date for loan):

$$Fitness = \frac{MNP}{CNP}. \tag{1}$$

The condensation MNP means the maximal number of points that may be collected after comparison of the individual with each loan risk data from the database before and after applying weights in case of they all are true. Second condensation CNP describes the number of collected points after comparison of the individual with each loan risk data from the database before and after applying weights.

The number of collected points is evaluated, respectively, percentage to the number of maximum number of points that are possible to collect.

The fitness during the testing process will be calculated in the same way reposing on the testing statistical database. The main criteria, on which the result of weights accuracy will be based, is the possible error value. So, if the difference between training and testing fitness of the obtained weight values would be wide (>10%), then the obtained result cannot be accepted. It means that the training process will be run one more time.

4 Crossover

The basic operator for producing new individuals in the GA is crossover [2]. Weight optimization task has a particular point – the sum of weight will be equal to 100%. So, after individuals crossing this sum may be changed. To keep the sum of weight values the crossover operator has the following modifications:

- Take a pair of two individuals chosen by selection
- Randomly choose the crossover point
- Calculate a sum of weight variables from the individuals that are located before and after crossover point
- Shift vice verse the respective parts of the pair of individuals in the rough guide of the crossover point
- Compare a sum of weight values that are located in the shifted individual part before and after crossover. In case if these sums are different, the part which belongs to the individual with the worst fitness will have some correction (after crossover the difference value will be subtracted or added to the part of weakness individual that is located before/after crossover point)
- In case one or both of the obtained offspring is the same as the individual's from the *table of weak individuals*, then crossover results must be rejected and operator will run one more time

5 Mutation

Mutation operator is used for finding new points in the search space under evaluation. However, the mutation operator may also cause a loss of a very good solution. To solve this problem, the mutation operator was modified to help to generate only a good solution [5].

The new individual creation process applying the modified mutation has the following steps:

- Randomly choose an individual
- Randomly choose some weight values from the individual. The number of chosen weight values is generated by random
- Calculate a sum of all chosen weight values from the individual and then randomly generate the same number to the chosen weight values. The sum of generated weighs will not be changed
- Calculate the fitness value for the obtained individual
- Compare the chosen and the obtained individuals fitness function values
- Check the individual with worse fitness in the *table of weak individuals*
- In case if the obtained individual is fitter than the chosen one, mutation will be continued. The next mutation happens in the obtained individual. This operator will mutate until the obtained individual fitness does not cause a loss of fitness.

6 Experimental Results

The accomplished experiments have been carried out in order to observe the behavior of GA in the aforementioned weight optimization task. The results of all experiments produced by the GA were evaluated together.

The weight value combinations that were obtained during the experiments are very similar each other to, respectively, of its content (w1 of one experiment is similar to the w1 from the other one). One of the experimental results is represented graphically in the Fig. 1.

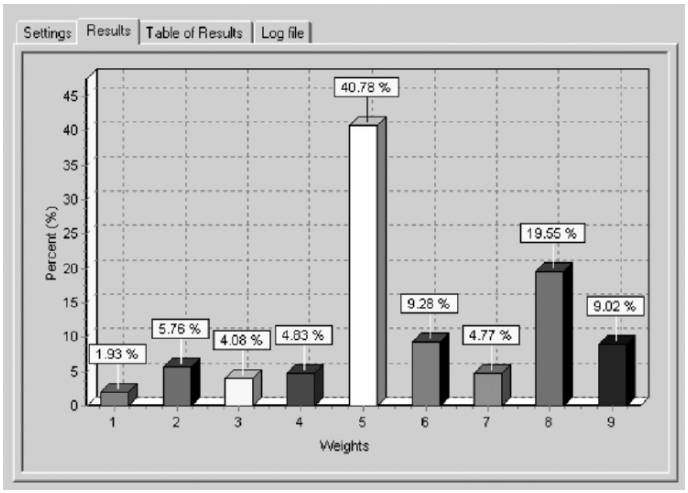

Fig. 1. Obtained weight value deviation

All obtained experimental results are located around weight values that are represented in the figure (see Fig. 1).

7 Conclusions

This paper reviews the weight optimization task solving with GA. The first step in development of the statistical data was to decide the number assignment for each of nine criteria. The methodology chosen for estimation point values assignment is based on statistic data analysis. So, the values of the criteria were ranged according to possible estimation point values from 0 to 5.

To make the population individuals much easier and more convenient for fitness calculation the individuals consist only from weight values. Hence, the crossover and mutation operators only work with the generated data, respectively.

During this work some modification was applied to the generated individuals – it is a *table of weak individuals*, in which the information about weak individuals, obtained during mutation and also crossover, is kept during the algorithm run.

To provide the accuracy of the results carried out from the experiments the testing process was applied on the special scheduled database data. The increase of fitness function value depends on successful choice of weight values combination.

The results that are carried out from the experiments have similar combination of weight values. During the experiments statistical data were used only about the borrowers that are clients of the bank that loans can potentially be given to. To estimate the credit risk of new borrowers it is necessary to find another kind of weights when the importance for the statistical data that are represented with nine criteria is different.

Acknowledgments

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A Fuzzy Feature Extractor Neural Network and its Application in License Plate Recognition

Modjtaba Rouhani

Summary. This paper presents a fuzzy neural network model to extract and classify selected features in subregions of a two-dimensional signal (e.g., image signal). As a representative example, we applied this model to the problem of Iranian automobiles license plate recognition (LPR).

Key words: Fuzzy neural networks, Feature extractor, License plate recognition.

1 Introduction

Fuzzy neural networks are hybrid systems that possess the advantages of both neural networks and fuzzy systems. The integration of fuzzy systems and neural networks combines the human inference style and natural language description of fuzzy systems with the learning and parallel processing of neural networks. There are numerous approaches to integrate fuzzy systems and neural networks. Extensive bibliography on fuzzy neural network and an introduction to it can be found in [1].

We propose a new fuzzy neural network model that performs excellent in noisy environment and is more robust for distorted or shifted patterns. The model aims to preserve the advantages and capabilities of Neocognitron, proposed by Fukushima et al. [2,3], in a mush simpler structure, by introducing fuzzy logic concepts. The main concept of fuzzy neurons (FNs) is introduced in [4]. This model has been successfully used for recognition of handwritten Persian characters [5]. The proposed model extracts prespecified features in subregions of the two-dimensional input signal and compares them to those of stored patterns in a fuzzy manner.

The paper is organized as follow: Sect. 2 describes the proposed fuzzy neural network in detail. The third section is dedicated to license plate recognition and more especially Iranian license plate type recognition. Sect. 4 illustrates the results.

2 Feature Extractor Fuzzy Neural Network

The structure of the fuzzy neural network (FNN) is depicted in Fig. 1. The proposed FNN model is a feed forward neural network with four processing layers. The first layer containing fuzzy neurons arranged and indexed in two dimensions. Each FN in the first layer represents a subregion of the input pattern and extracts prespecified features in it. The second layer has the same dimensions as the first layer and computes the membership of every feature in every first layer FN to stored classes in a fuzzy manner. In other words, this layer determines how close the input pattern is to the prototype patterns. In the third layer, the overall similarity of input pattern to each stored pattern is computed by fuzzy max–min composition. As visualized in Fig. 1, third layer is arranged in one dimension and has as many FNs as stored patterns. The output of third layer neurons is the fuzzy membership of input pattern to each stored classes. The forth layer, i.e., output layer, determines the input class as ones with the maximum membership in the third layer. To prevent misclassifying of input patterns which do not belong to any stored classes, the outputs of forth layer are all zero, if the maximum output of the third layer is below a threshold level.

Assume that the input layer has $n_{10} \times n_{20}$ pixels and the first layer contains $n_{11} \times n_{21}$fuzzy neurons (FNs). Each FN in layer 1 represent a region of size $L \times L$ in the input space and has K fuzzy outputs. The kth output of neuron (i,j) in layer 1,

$$a^1_{i,j}(k), \quad i = 1, \ldots, n_{11}, \quad j = 1, \ldots, n_{21}, \quad l = 1, \ldots, K$$

measures the kth feature in the $L \times L$ region in input layer. These features have to be selected, based on the problem must be solved by FNN, such that FNN be able to distinguish among output classes. Selected features for the problem of Iranian license plate type recognition will be discussed in the next section. Layer 1 has no adjustable parameter.

Layer 2 determines the amount of similarities between the outputs of layer 1 and the basic shapes corresponding to that region of the prototype patterns.

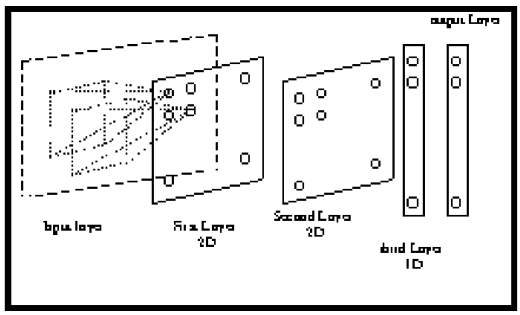

Fig. 1. The schematic diagram of FNN

Each neuron in this layer corresponds to a region in the input layer. The outputs of layer 2 calculate the similarities between the basic shapes located in the given region and those of the M prototype patterns. These outputs are governed by the following equations:

$$a^2_{i,j}(m) = \min_k f(a^1_{i,j}(k), \beta^m_{i,j}(k), \alpha^m_{i,j}(k)) \quad k = 1, \ldots, K, \quad \mathrm{m} = 1, \ldots, \mathrm{M}, \quad (1)$$

$$f(a, \beta, \alpha) = \exp\left(-\left(\frac{a-\beta}{\alpha}\right)^2\right). \quad (2)$$

Four-dimensional matrices α and β are the only adjustable parameters of FNN and are determined by the algorithm given below.

The outputs of the third layer are described by following equation and give the overall similarity of input pattern to mth prototype pattern:

$$a^3_m = \min_{i,j}(a^2_{i,j}(m)). \quad (3)$$

The single output of the forth layer determines the most likely prototype in the input layer as:

$$a^4 = \arg\max_m(a^3_m). \quad (4)$$

The proposed FNN model has a straightforward supervised learning rule. The only adjustable parameters are the center and the variance of Gaussian like membership functions (2) in the second layer. Once all learning patterns $p = 1,\ldots,\mathrm{P}$ are applied to FNN and output activities of first layer FNs, $\mathrm{a}^1_{\mathrm{i,j}}(k;p)$, computed for patterns belong to each class $m = 1,\ldots,\mathrm{M}$, these parameters are adjusted in a single iteration:

$$\beta^m_{i,j}(k) = mean_p(a^1_{i,j}(k;p)), \text{ pattern } p \text{ belongs to class } m, \quad (5)$$

$$\alpha^m_{i,j}(k) = \max_p(-\frac{\log(Tr)}{a^1_{i,j}(k;p)}), \text{ pattern } p \text{ belongs to class } m. \quad (6)$$

Tr in last equation is a threshold parameter, determines minimum output of layer 2 for patterns belong to each class.

3 License Plate Type Recognition

To demonstrate the capabilities of the proposed FNN, it has been used to classify Iranian license plates. License plate recognition (LPR) systems are usually composed of plate locator, plate segmentation, and character recognition (OCR) [6]. However, Iranian license plates vary widely in their size, color, and number of rows and characters as shown in Fig. 2 and summarized

Fig. 2. Iranian license plate types

Table 1. Iranian license plate types summary (see Fig. 2)

Type	Size (cm)	Number of rows	Number of columns	Background color(s)	Text color
1	55 × 15	1	2	White	Black
2	55 × 15	1	2	Red	White
3	55 × 15	1	2	Yellow	Black
4	40 × 20	2	1	White	Black
5	40 × 20	2	1	Yello/white	Black
6	40 × 20	2	2	Yellow/white	Black

in Table 1. These variations made the process of plate recognition much more complicated. To overcome this problem, a plate type recognition subsystem is implemented using the proposed FNN. The plate type recognition subsystem receives a license plate candidate (which has a fixed size) from plate locator subsystem and determines the license plate type, if it really contains a license plate. To recognize different Iranian plate types, a number of features have to be considered, namely: contrast between hi-level and low-level pixels, horizontal gradients, vertical gradients, colors, and the number of passes from hi-level pixels to low-level pixels in prespecified subregions of license plate candidate.

The input pattern to layer 0 (input layer) of FNN is a RGB image and has three values at each pixel, $a^0_{I,j}(r, g, b)$. Every neuron in layer 1 calculates K=7 different features in subregions of input image. These seven features are carefully selected to represent different aspects of M=6 license plate types.

4 Results

To demonstrate the ability of propose FNN, over hundred of automobile images are presented to Iranian license plate recognition software developed. A plate locator subsystem locates candidate regions may contain license plate. This is done basically, by bi-levelizing input image and searching for rectangle

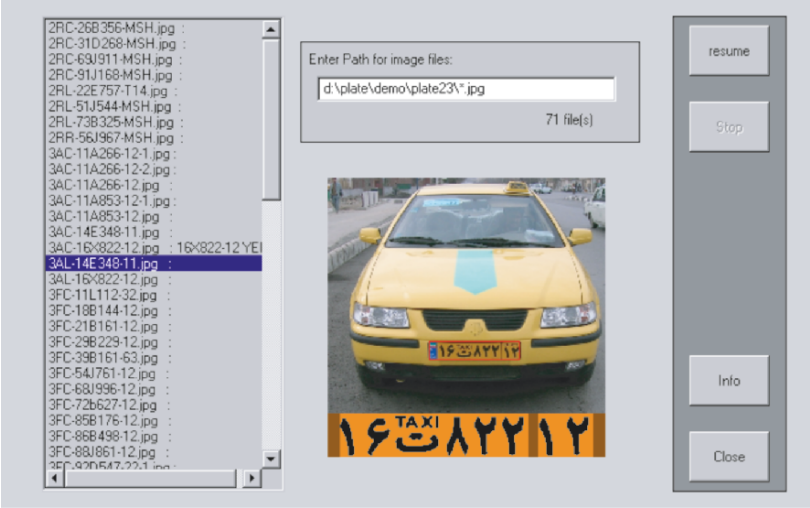

Fig. 3. Iranian license plate recognition software

like regions using Hough transform techniques. Then, these candidate regions are presented to FNN to determine if it really contains a license plate and if so, of what type. Based on recognized plate type, an appropriate segmentation algorithm is called and each character is passed to OCR subsytem (see Fig. 3).

Input images have approximately 600×400 pixels so that license plate candidates are 180×60 pixel regions. Layer 1 has 8×3 fuzzy neurons, each corresponding to a 100×40 region of input layer.

To learn FNN, license plate candidates from plate locator together with shifted candidates are applied to learning algorithm as mentioned by (5) and (6).

The accuracy of proposed FNN license plate type recognizer was 100% for main patterns and over 98% for shifted patterns. We conclude that the performance of FNN has been evaluated to be excellent in recognition of Iranian license plate types.

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Nearest Interval Approximation of an Intuitionistic Fuzzy Number

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Summary. The problem of approximation of a fuzzy number by a real number or real interval was studied mainly to introduce defuzzifying methods and to define ranking procedures between fuzzy numbers. The study of the problem in the intuitionistic fuzzy case is justified in the same way. In this paper we suggest a natural method of approximation of the intuitionistic numbers by real intervals with respect to Euclidean and Tran–Duckstein distances. Sometimes it is useful to obtain crisp solutions even if the initial data and the used methods are intuitionistic fuzzy.

1 Intuitionistic Fuzzy Numbers

We consider the following description of a fuzzy number u

$$u(x) = \begin{cases} 0, & \text{if } x \leq a_1, \\ l_u(x), & \text{if } a_1 \leq x \leq a_2, \\ 1, & \text{if } a_2 \leq x \leq a_3 \\ r_u(x), & \text{if } a_3 \leq x \leq a_4, \\ 0, & \text{if } a_4 \leq x, \end{cases}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}, l_u : \mathbb{R} \to [0,1]$ is a nondecreasing continuous function, $l_u(a_1) = 0, l_u(a_2) = 1$, called the left side of the fuzzy number and $r_u : \mathbb{R} \to [0,1]$ is a nonincreasing continuous function, $r_u(a_3) = 1, r_u(a_4) = 0$, called the right side of the fuzzy number. The α-cut, $\alpha \in]0,1]$, of a fuzzy number u (in fact it can be introduced for any fuzzy set) is a crisp set defined as

$$u_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}.$$

Every α-cut of a fuzzy number is a closed interval $u_\alpha = [u_L(\alpha), u_U(\alpha)]$, where

$$u_L(\alpha) = \inf\{x \in \mathbb{R} : u(x) \geq \alpha\},$$
$$u_U(\alpha) = \sup\{x \in \mathbb{R} : u(x) \geq \alpha\}.$$

If the sides of the fuzzy number u are strictly monotone then one can see easily that u_L and u_U are inverse functions of l_u and r_u, respectively. Throughout in this paper we consider fuzzy numbers with strictly monotone sides.

Definition 1. *An intuitionistic fuzzy set* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle ; x \in X\}$ *such that* μ_A *and* $1 - \nu_A$ *are fuzzy numbers is called an intuitionistic fuzzy number.*

We denote by $A = \langle \mu_A, \nu_A\rangle$ an intuitionistic fuzzy number and by $IF(\mathbb{R})$ the space of all intuitionistic fuzzy numbers.

Remark 1. With respect to the α-cuts of the fuzzy number $1 - \nu_A$ are immediate the equalities

$$(1 - \nu_A)_L(\alpha) = \nu_{A_L}(1 - \alpha)$$

and

$$(1 - \nu_A)_U(\alpha) = \nu_{A_U}(1 - \alpha),$$

for every $\alpha \in]0, 1]$.

If $A = \langle \mu_A, \nu_A\rangle, B = \langle \mu_B, \nu_B\rangle$ are intuitionistic fuzzy numbers it is natural to define the addition by

$$A + B = \langle \mu_{A+B}, \nu_{A+B}\rangle,$$

where $\mu_{A+B} = \mu_A + \mu_B$ and ν_{A+B} such that

$$(1 - \nu_{A+B})_L(\alpha) = (1 - \nu_A)_L(\alpha) + (1 - \nu_B)_L(\alpha),$$
$$(1 - \nu_{A+B})_U(\alpha) = (1 - \nu_A)_U(\alpha) + (1 - \nu_B)_U(\alpha),$$

for every $\alpha \in]0, 1]$. In fact,

$$\begin{aligned}(\nu_{A+B})_L(\alpha) &= (1 - \nu_{A+B})_L(1 - \alpha)\\ &= (1 - \nu_A)_L(1 - \alpha) + (1 - \nu_B)_L(1 - \alpha)\\ &= \nu_{A_L}(\alpha) + \nu_{B_L}(\alpha)\end{aligned}$$

and, analogously,

$$(\nu_{A+B})_U(\alpha) = \nu_{A_U}(\alpha) + \nu_{B_U}(\alpha),$$

for every $\alpha \in [0, 1[$.

We obtain the scalar multiplication of intuitionistic fuzzy numbers in a similar way.

2 Distances between Intuitionistic Fuzzy Numbers

For two arbitrary fuzzy numbers u and v with α-cuts $[u_L(\alpha), u_U(\alpha)]$ and $[v_L(\alpha), v_U(\alpha)]$ we define the quantities

$$d_E^2(u,v) = \int_0^1 (u_L(\alpha) - v_L(\alpha))^2 d\alpha + \int_0^1 (u_U(\alpha) - v_U(\alpha))^2 d\alpha$$

and

$$d_{TD}^2(u,v) = \int_0^1 \left(\frac{u_L(\alpha) + u_U(\alpha)}{2} - \frac{v_L(\alpha) + v_U(\alpha)}{2} \right)^2 d\alpha + \frac{1}{3} \int_0^1 \left[\left(\frac{u_U(\alpha) - u_L(\alpha)}{2} \right)^2 + \left(\frac{v_U(\alpha) - v_L(\alpha)}{2} \right)^2 \right] d\alpha.$$

Then $d_E(u,v)$ and $d_{TD}(u,v)$ are distances between the fuzzy numbers u and v and d_E, d_{TD} define metrics on the set of fuzzy numbers. The metric d_E is well known in fuzzy set theory, it is an extension of the Euclidean distance. The metric d_{TD} was introduced in [3] by Tran and Duckstein for ranking fuzzy numbers.

The following result helps us to obtain metrics on the set of intuitionistic fuzzy numbers.

Theorem 1. *If d is a metric on the set of fuzzy numbers then $\widetilde{d}$ defined by*

$$\widetilde{d}(A,B) = \sqrt{\frac{1}{2} d^2(\mu_A, \mu_B) + \frac{1}{2} d^2(1-\nu_A, 1-\nu_B)},$$

where $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ is a metric on the set of intuitionistic fuzzy numbers.

Proof. The triangle inequality is equivalent to

$$\sqrt{d^2(\mu_A,\mu_B) + d^2(1-\nu_A, 1-\nu_B)} \le \sqrt{d^2(\mu_A,\mu_C) + d^2(1-\nu_A, 1-\nu_C)} + \sqrt{d^2(\mu_C,\mu_B) + d^2(1-\nu_C, 1-\nu_B)},$$

for every intuitionistic fuzzy numbers $A = \langle \mu_A, \nu_A \rangle, B = \langle \mu_B, \nu_B \rangle$, and $C = \langle \mu_C, \nu_C \rangle$. The inequality is, in fact,

$$\sqrt{a_1^2 + b_1^2} \le \sqrt{a_2^2 + b_2^2} + \sqrt{a_3^2 + b_3^2},$$

for any positive real numbers $a_1, b_1, a_2, b_2, a_3, b_3$ under the restrictions $a_1 \le a_2 + a_3$ and $b_1 \le b_2 + b_3$. Because

$$\sqrt{(a_2 + a_3)^2 + (b_2 + b_3)^2} \le \sqrt{a_2^2 + b_2^2} + \sqrt{a_3^2 + b_3^2}$$

for every real numbers a_2, b_2, a_3, b_3, the theorem is proved. □

Remark 2. To obtain objects consistent with the fuzzy case the constant $\frac{1}{2}$ is added.

According to Theorem 1 and Remark 1 in the intuitionistic fuzzy case the metrics d_E and d_{TD} become

$$\widetilde{d}_E^2(A,B) = \frac{1}{2}\int_0^1 \left(\mu_{A_L}(\alpha) - \mu_{B_L}(\alpha)\right)^2 d\alpha + \frac{1}{2}\int_0^1 \left(\mu_{A_U}(\alpha) - \mu_{B_U}(\alpha)\right)^2 d\alpha + \frac{1}{2}\int_0^1 \left(\nu_{A_L}(\alpha) - \nu_{B_L}(\alpha)\right)^2 d\alpha + \frac{1}{2}\int_0^1 \left(\nu_{A_U}(\alpha) - \nu_{B_U}(\alpha)\right)^2 d\alpha$$

and

$$\begin{aligned}\widetilde{d}_{TD}^2(A,B) =& \frac{1}{2}\int_0^1 \left(\frac{\mu_{A_L}(\alpha)+\mu_{A_U}(\alpha)}{2} - \frac{\mu_{B_L}(\alpha)+\mu_{B_U}(\alpha)}{2}\right)^2 d\alpha \\ &+ \frac{1}{6}\int_0^1 \left[\left(\frac{\mu_{A_U}(\alpha)-\mu_{A_L}(\alpha)}{2}\right)^2 + \left(\frac{\mu_{B_U}(\alpha)-\mu_{B_L}(\alpha)}{2}\right)^2\right] d\alpha \\ &+ \frac{1}{2}\int_0^1 \left(\frac{\nu_{A_L}(\alpha)+\nu_{A_U}(\alpha)}{2} - \frac{\nu_{B_L}(\alpha)+\nu_{B_U}(\alpha)}{2}\right)^2 d\alpha \\ &+ \frac{1}{6}\int_0^1 \left[\left(\frac{\nu_{A_U}(\alpha)-\nu_{A_L}(\alpha)}{2}\right)^2 + \left(\frac{\nu_{B_U}(\alpha)-\nu_{B_L}(\alpha)}{2}\right)^2\right] d\alpha.\end{aligned}$$

3 Nearest Interval Approximation of Intuitionistic Fuzzy Numbers

In the paper [2] the nearest interval approximation of a fuzzy number with respect to metric d_E was introduced. In the following we use the same idea to determine the nearest interval approximation of an intuitionistic fuzzy number with respect to the metrics $\widetilde{d}_E$ and $\widetilde{d}_{TD}$. As a consequence, the nearest interval approximation of a fuzzy number with respect to the metric d_{TD} is obtained.

Let us suppose $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy number and

$$\left[\mu_{A_L}(\alpha), \mu_{A_U}(\alpha)\right], \alpha \in]0,1],$$
$$\left[\nu_{A_L}(\alpha), \nu_{A_U}(\alpha)\right], \alpha \in [0,1[,$$

where

$$\nu_{A_L}(\alpha) = (1-\nu_A)_L(1-\alpha)$$

and

$$\nu_{A_U}(\alpha) = (1-\nu_A)_U(1-\alpha),$$

are its α-cuts. We try to find a closed interval $C_{\widetilde{d}_E}(A) = [C_L, C_U]$ which is the nearest to A with respect to the metric $\widetilde{d}_E$. It is obvious that each real interval can also be considered as an intuitionistic fuzzy number with constant α-cuts $[C_L, C_U]$, for all $\alpha \in]0,1[$.

Now we have to minimize

$$\widetilde{d}_E\left(A, C_{\widetilde{d}_E}(A)\right)$$

with respect to C_L and C_U, that is to minimize

$$F_1(C_L, C_U) = \int_0^1 \left(\mu_{A_L}(\alpha) - C_L\right)^2 d\alpha + \int_0^1 \left(\mu_{A_U}(\alpha) - C_U\right)^2 d\alpha + \int_0^1 \left(\nu_{A_L}(\alpha) - C_L\right)^2 d\alpha + \int_0^1 \left(\nu_{A_U}(\alpha) - C_U\right)^2 d\alpha$$

with respect to C_L and C_U. We find the partial derivatives

$$\frac{\partial F_1(C_L, C_U)}{\partial C_L} = -2\int_0^1 \left(\mu_{A_L}(\alpha) - C_L\right) d\alpha - 2\int_0^1 \left(\nu_{A_L}(\alpha) - C_L\right) d\alpha$$

$$= -2\int_0^1 \left(\mu_{A_L}(\alpha) + \nu_{A_L}(\alpha)\right) d\alpha + 4C_L,$$

$$\frac{\partial F_1(C_L, C_U)}{\partial C_U} = -2\int_0^1 \left(\mu_{A_U}(\alpha) - C_U\right) d\alpha - 2\int_0^1 \left(\nu_{A_U}(\alpha) - C_U\right) d\alpha$$

$$= -2\int_0^1 \left(\mu_{A_U}(\alpha) + \nu_{A_U}(\alpha)\right) d\alpha + 4C_U,$$

and then we solve the system

$$\begin{cases} \dfrac{\partial F_1(C_L, C_U)}{\partial C_L} = 0, \\ \dfrac{\partial F_1(C_L, C_U)}{\partial C_U} = 0. \end{cases}$$

The solution is

$$C_L = \int_0^1 \frac{\mu_{A_L}(\alpha) + \nu_{A_L}(\alpha)}{2} d\alpha,$$

$$C_U = \int_0^1 \frac{\mu_{A_U}(\alpha) + \nu_{A_U}(\alpha)}{2} d\alpha.$$

Since

$$\det \begin{pmatrix} \dfrac{\partial^2 F_1(C_L, C_U)}{\partial C_L^2} & \dfrac{\partial^2 F_1(C_L, C_U)}{\partial C_L \partial C_U} \\ \dfrac{\partial^2 F_1(C_L, C_U)}{\partial C_U \partial C_L} & \dfrac{\partial^2 F_1(C_L, C_U)}{\partial C_U^2} \end{pmatrix} = \det \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 16 > 0$$

and

$$\frac{\partial^2 F_1(C_L, C_U)}{\partial C_L^2} = 4 > 0,$$

then C_L and C_U given above minimize $F_1(C_L, C_U)$.

We have

Theorem 2. *The nearest interval of the intuitionistic fuzzy number* $A = \langle \mu_A, \nu_A \rangle$ *with respect to the metric* $\widetilde{d}_E$ *is*

$$C_{\widetilde{d}_E}(A) = \left[\int_0^1 \frac{\mu_{A_L}(\alpha) + \nu_{A_L}(\alpha)}{2} d\alpha, \int_0^1 \frac{\mu_{A_U}(\alpha) + \nu_{A_U}(\alpha)}{2} d\alpha \right]. \quad (1)$$

Remark 3. Because $\mu_A(x) + \nu_A(x) = 1$ for every $x \in X$ implies

$$\nu_{A_L}(\alpha) = \mu_{A_L}(1 - \alpha)$$

and

$$\nu_{A_U}(\alpha) = \mu_{A_U}(1 - \alpha),$$

for any $\alpha \in [0, 1[$, from the above theorem we obtain the nearest interval of a fuzzy number u with respect to metric d_E (see [2])

$$\left[\int_0^1 u_L(\alpha)\, d\alpha, \int_0^1 u_U(\alpha)\, d\alpha \right].$$

Passing to the second metric $\widetilde{d}_{TD}$ we have to minimize the mapping

$$\begin{aligned} F_2(C_L, C_U) = & \int_0^1 \left(\frac{\mu_{A_L}(\alpha) + \mu_{A_U}(\alpha)}{2} - \frac{C_L + C_U}{2} \right)^2 d\alpha \\ & + \frac{1}{3} \int_0^1 \left[\left(\frac{\mu_{A_U}(\alpha) - \mu_{A_L}(\alpha)}{2} \right)^2 + \left(\frac{C_U - C_L}{2} \right)^2 \right] d\alpha \\ & + \int_0^1 \left(\frac{\nu_{A_L}(\alpha) + \nu_{A_U}(\alpha)}{2} - \frac{C_L + C_U}{2} \right)^2 d\alpha \\ & + \frac{1}{3} \int_0^1 \left[\left(\frac{\nu_{A_U}(\alpha) - \nu_{A_L}(\alpha)}{2} \right)^2 + \left(\frac{C_U - C_L}{2} \right)^2 \right] d\alpha, \end{aligned}$$

with respect to C_L and C_U. We find the partial derivatives

$$\begin{aligned} \frac{\partial F_2(C_L, C_U)}{\partial C_L} = & - \int_0^1 \frac{\mu_{A_L}(\alpha) + \mu_{A_U}(\alpha) + \nu_{A_L}(\alpha) + \nu_{A_U}(\alpha)}{2} d\alpha \\ & + \frac{4C_L + 2C_U}{3}, \end{aligned}$$

$$\frac{\partial F_2(C_L, C_U)}{\partial C_U} = -\int_0^1 \frac{\mu_{A_L}(\alpha) + \mu_{A_U}(\alpha) + \nu_{A_L}(\alpha) + \nu_{A_U}(\alpha)}{2} d\alpha + \frac{4C_U + 2C_L}{3}$$

and then we solve the system

$$\begin{cases} \dfrac{\partial F_2(C_L, C_U)}{\partial C_L} = 0, \\ \dfrac{\partial F_2(C_L, C_U)}{\partial C_U} = 0. \end{cases}$$

The solution is

$$C_L = C_U = \int_0^1 \frac{\mu_{A_L}(\alpha) + \mu_{A_U}(\alpha) + \nu_{A_L}(\alpha) + \nu_{A_U}(\alpha)}{4} d\alpha.$$

Since

$$\det \begin{pmatrix} \dfrac{\partial^2 F_2(C_L, C_U)}{\partial C_L^2} & \dfrac{\partial^2 F_2(C_L, C_U)}{\partial C_L \partial C_U} \\ \dfrac{\partial^2 F_2(C_L, C_U)}{\partial C_U \partial C_L} & \dfrac{\partial^2 F_2(C_L, C_U)}{\partial C_U^2} \end{pmatrix} = \det \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} = \frac{4}{3} > 0$$

and

$$\frac{\partial^2 F_2(C_L, C_U)}{\partial C_L^2} = \frac{4}{3} > 0$$

then C_L and C_U given above minimize $F_2(C_L, C_U)$.

We obtain

Theorem 3. *The nearest interval of the intuitionistic fuzzy number* $A = \langle \mu_A, \nu_A \rangle$ *with respect to the metric* $\widetilde{d}_{TD}$ *reduces to a point, namely*

$$C_{\widetilde{d}_{TD}}(A) = \int_0^1 \frac{\mu_{A_L}(\alpha) + \mu_{A_U}(\alpha) + \nu_{A_L}(\alpha) + \nu_{A_U}(\alpha)}{4} d\alpha. \tag{2}$$

Remark 3 and Theorem 3 imply the following new result of interval approximation of the fuzzy numbers.

Corollary 1. *The nearest interval of the fuzzy number* $u, [u_L(\alpha), u_U(\alpha)], \alpha \in]0, 1]$ *with respect to the metric* d_{TD} *reduces to a point, namely*

$$\int_0^1 \frac{u_L(\alpha) + u_U(\alpha)}{2} d\alpha.$$

Sometimes it is important to know the value of the distance between an intuitionistic fuzzy number A and its nearest interval approximation with respect to a metric $\widetilde{d}$. We denote $I_{\widetilde{d}}(A)$ this value which represents the lost of information passing from the intuitionistic fuzzy number A to real intervals. We have

Theorem 4. 1.

$$I^2_{\widetilde{d}_E}(A) = \frac{1}{2}\int_0^1 \left(\mu^2_{A_L}(\alpha) + \mu^2_{A_U}(\alpha) + \nu^2_{A_L}(\alpha) + \nu^2_{A_U}(\alpha)\right) d\alpha - \frac{1}{4}\left(\int_0^1 \left(\mu_{A_L}(\alpha) + \nu_{A_L}(\alpha)\right) d\alpha\right)^2 - \frac{1}{4}\left(\int_0^1 \left(\mu_{A_U}(\alpha) + \nu_{A_U}(\alpha)\right) d\alpha\right)^2;$$

2.

$$I^2_{\widetilde{d}_{TD}}(A) = \frac{1}{6}\int_0^1 \left(\mu^2_{A_L}(\alpha) + \mu^2_{A_U}(\alpha) + \nu^2_{A_L}(\alpha) + \nu^2_{A_U}(\alpha)\right) d\alpha + \frac{1}{6}\int_0^1 \left(\mu_{A_L}(\alpha)\mu_{A_U}(\alpha) + \nu_{A_L}(\alpha)\nu_{A_U}(\alpha)\right) d\alpha - \frac{1}{16}\left(\int_0^1 \left(\mu_{A_L}(\alpha) + \mu_{A_U}(\alpha) + \nu_{A_L}(\alpha) + \nu_{A_U}(\alpha)\right) d\alpha\right)^2.$$

Proof. It is immediate because with the above notations

$$I^2_{\widetilde{d}_E}(A) = \frac{1}{2}F_1\left(\int_0^1 \frac{\mu_{A_L}(\alpha) + \nu_{A_L}(\alpha)}{2} d\alpha, \int_0^1 \frac{\mu_{A_U}(\alpha) + \nu_{A_U}(\alpha)}{2} d\alpha\right)$$

and

$$I^2_{\widetilde{d}_{TD}}(A) = \frac{1}{2}F_2\left(\int_0^1 \frac{\mu_{A_L}(\alpha) + \mu_{A_U}(\alpha) + \nu_{A_L}(\alpha) + \nu_{A_U}(\alpha)}{4} d\alpha, \int_0^1 \frac{\mu_{A_L}(\alpha) + \mu_{A_U}(\alpha) + \nu_{A_L}(\alpha) + \nu_{A_U}(\alpha)}{4} d\alpha\right).$$

□

In the particular case of fuzzy numbers we obtain

Corollary 2. 1.

$$I^2_{d_E}(u) = \int_0^1 \left(u^2_L(\alpha) + u^2_U(\alpha)\right) d\alpha - \left(\int_0^1 u_L(\alpha)\, d\alpha\right)^2 - \left(\int_0^1 u_U(\alpha)\, d\alpha\right)^2;$$

2.

$$I^2_{d_{TD}}(u) = \frac{1}{3}\int_0^1 \left(u^2_L(\alpha) + u_L(\alpha)\, u_U(\alpha) + u^2_U(\alpha)\right) d\alpha - \frac{1}{4}\left(\int_0^1 \left(u_L(\alpha) + u_U(\alpha)\right) d\alpha\right)^2.$$

Proof. It is immediate because

$$\nu_{A_L}(\alpha) = \mu_{A_L}(1-\alpha) = u_L(1-\alpha)$$

and

$$\nu_{A_U}(\alpha) = \mu_{A_U}(1-\alpha) = u_U(1-\alpha),$$

for every $\alpha \in [0, 1[$. □

A fuzzy number u with the functions l_u and r_u (see Sect. 1) defined by

$$l_u(x) = \left(\frac{x - a_1}{a_2 - a_1}\right)^r$$

and

$$r_u(x) = \left(\frac{a_4 - x}{a_4 - a_3}\right)^r,$$

respectively, where $r > 0$, will be denoted $u = (a_1, a_2, a_3, a_4)_r$ (see [1]). If $u = (a_1, a_2, a_3, a_4)_r$ then

$$u_\alpha = \left[a_1 + \alpha^{1/r}(a_2 - a_1), a_4 - \alpha^{1/r}(a_4 - a_3)\right], \quad \alpha \in]0, 1].$$

If $r = 1$ then u is a trapezoidal fuzzy number. An intuitionistic fuzzy number $A = \langle \mu_A, \nu_A \rangle$ such that μ_A and $1 - \nu_A$ are trapezoidal fuzzy numbers is called a trapezoidal intuitionistic fuzzy number.

We obtain

Corollary 3. *Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number such that $\mu_A = (a_1, b_1, c_1, d_1)_{r_1}$ and $1 - \nu_A = (a_2, b_2, c_2, d_2)_{r_2}$. Then*

1.
$$C_{\widetilde{d}_E}(A) = \left[\frac{a_1 + b_1 r_1}{2(r_1 + 1)} + \frac{a_2 + b_2 r_2}{2(r_2 + 1)}, \frac{d_1 + c_1 r_1}{2(r_1 + 1)} + \frac{d_2 + c_2 r_2}{2(r_2 + 1)}\right];$$

2.
$$C_{\widetilde{d}_{TD}}(A) = \frac{a_1 + d_1 + (b_1 + c_1) r_1}{4(r_1 + 1)} + \frac{a_2 + d_2 + (b_2 + c_2) r_2}{4(r_2 + 1)}.$$

Proof. In (1) and (2) we put

$$\mu_{A_L}(\alpha) = a_1 + \alpha^{1/r_1}(b_1 - a_1),$$
$$\mu_{A_U}(\alpha) = d_1 - \alpha^{1/r_1}(d_1 - c_1),$$
$$\nu_{A_L}(\alpha) = (1 - \nu_A)_L(1 - \alpha) = a_2 + (1 - \alpha)^{1/r_2}(b_2 - a_2),$$

and

$$\nu_{A_U}(\alpha) = (1 - \nu_A)_U(1 - \alpha) = d_2 - (1 - \alpha)^{1/r_2}(d_2 - c_2),$$

for every $\alpha \in]0, 1[$. □

Corollary 4. *Let $A = \langle \mu_A, \nu_A \rangle$ be a trapezoidal intuitionistic fuzzy number, $\mu_A = (a_1, b_1, c_1, d_1)$ and $1 - \nu_A = (a_2, b_2, c_2, d_2)$. Then*

1.
$$C_{\widetilde{d}_E}(A) = \left[\frac{a_1 + b_1 + a_2 + b_2}{4}, \frac{c_1 + d_1 + c_2 + d_2}{4}\right];$$

2.
$$C_{\widetilde{d}_{TD}}(A) = \frac{a_1 + b_1 + c_1 + d_1 + a_2 + b_2 + c_2 + d_2}{8}.$$

Proof. $r_1 = r_2 = 1$ in the above formulas. □

4 Properties of the Nearest Intervals Approximation of Intuitionistic Fuzzy Numbers

In this section we study the continuity and linearity of the nearest intervals approximation introduced in the previous section.

Let us denote $[\mathbb{R}]$ the family of all closed intervals on the real line.

Definition 2. *An interval approximation operator* $C : IF(\mathbb{R}) \to [\mathbb{R}]$ *satisfying*

$$\forall \varepsilon > 0, \exists \delta > 0 : \widetilde{d}(A, B) < \delta \Rightarrow \widetilde{d}(C(A), C(B)) < \varepsilon,$$

where $\widetilde{d}$ *is a metric in the family of all intuitionistic fuzzy numbers, is called continuous with respect to* $\widetilde{d}$.

The above definition expresses, in fact, the natural requirement: if two intuitionistic fuzzy numbers are close then their interval approximations are also close.

Theorem 5. 1. *The interval approximation operator* $C_1 : IF(\mathbb{R}) \to [\mathbb{R}]$ *defined by*

$$C_1(A) = C_{\widetilde{d}_E}(A)$$

is continuous with respect to the metric $\widetilde{d}_E$;

2. *The interval approximation operator* $C_2 : IF(\mathbb{R}) \to [\mathbb{R}]$ *defined by*

$$C_2(A) = C_{\widetilde{d}_{TD}}(A)$$

is continuous with respect to the metric $\widetilde{d}_{TD}$.

Proof. 1. For any given two intuitionistic fuzzy numbers $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ we have

$$\begin{aligned}
&\widetilde{d}_E^2\left(C_{\widetilde{d}_E}(A), C_{\widetilde{d}_E}(B)\right) \\
&= \left(\int_0^1 \frac{\mu_{A_L}(\alpha) + \nu_{A_L}(\alpha)}{2} d\alpha - \int_0^1 \frac{\mu_{B_L}(\alpha) + \nu_{B_L}(\alpha)}{2} d\alpha\right)^2 \\
&+ \left(\int_0^1 \frac{\mu_{A_U}(\alpha) + \nu_{A_U}(\alpha)}{2} d\alpha - \int_0^1 \frac{\mu_{B_U}(\alpha) + \nu_{B_U}(\alpha)}{2} d\alpha\right)^2 \\
&= \frac{1}{4}\left(\int_0^1 (\mu_{A_L}(\alpha) - \mu_{B_L}(\alpha) + \nu_{A_L}(\alpha) - \nu_{B_L}(\alpha)) d\alpha\right)^2 \\
&+ \frac{1}{4}\left(\int_0^1 (\mu_{A_U}(\alpha) - \mu_{B_U}(\alpha) + \nu_{A_U}(\alpha) - \nu_{B_U}(\alpha)) d\alpha\right)^2 \\
&\leq \frac{1}{4}\int_0^1 (\mu_{A_L}(\alpha) - \mu_{B_L}(\alpha) + \nu_{A_L}(\alpha) - \nu_{B_L}(\alpha))^2 d\alpha \\
&+ \frac{1}{4}\int_0^1 (\mu_{A_U}(\alpha) - \mu_{B_U}(\alpha) + \nu_{A_U}(\alpha) - \nu_{B_U}(\alpha))^2 d\alpha
\end{aligned}$$

$$\leq \frac{1}{2}\int_0^1 \left(\mu_{A_L}(\alpha)-\mu_{B_L}(\alpha)\right)^2 d\alpha + \frac{1}{2}\int_0^1 \left(\nu_{A_L}(\alpha)-\nu_{B_L}(\alpha)\right)^2 d\alpha$$
$$+ \frac{1}{2}\int_0^1 \left(\mu_{A_U}(\alpha)-\mu_{B_U}(\alpha)\right)^2 d\alpha + \frac{1}{2}\int_0^1 \left(\nu_{A_U}(\alpha)-\nu_{B_U}(\alpha)\right)^2 d\alpha$$
$$= \widetilde{d}_E^2(A,B)$$

and the continuity is proved.

2. Let $A=\langle \mu_A, \nu_A\rangle$ and $B=\langle \mu_B, \nu_B\rangle$ be two intuitionistic fuzzy numbers. We get

$$\widetilde{d}_{TD}^2\left(C_{\widetilde{d}_{TD}}(A), C_{\widetilde{d}_{TD}}(B)\right)$$
$$= \left(\int_0^1 \frac{\mu_{A_L}(\alpha)+\mu_{A_U}(\alpha)+\nu_{A_L}(\alpha)+\nu_{A_U}(\alpha)}{4} d\alpha\right.$$
$$\left. - \int_0^1 \frac{\mu_{B_L}(\alpha)+\mu_{B_U}(\alpha)+\nu_{B_L}(\alpha)+\nu_{B_U}(\alpha)}{4} d\alpha\right)^2$$
$$\leq \frac{1}{4}\int_0^1 \left(\frac{\mu_{A_L}(\alpha)+\mu_{A_U}(\alpha)}{2} - \frac{\mu_{B_L}(\alpha)+\mu_{B_U}(\alpha)}{2}\right.$$
$$\left. + \frac{\nu_{A_L}(\alpha)+\nu_{A_U}(\alpha)}{2} - \frac{\nu_{B_L}(\alpha)+\nu_{B_U}(\alpha)}{2}\right)^2 d\alpha$$
$$\leq \frac{1}{2}\int_0^1 \left(\frac{\mu_{A_L}(\alpha)+\mu_{A_U}(\alpha)}{2} - \frac{\mu_{B_L}(\alpha)+\mu_{B_U}(\alpha)}{2}\right)^2 d\alpha$$
$$+ \frac{1}{2}\int_0^1 \left(\frac{\nu_{A_L}(\alpha)+\nu_{A_U}(\alpha)}{2} - \frac{\nu_{B_L}(\alpha)+\nu_{B_U}(\alpha)}{2}\right)^2 d\alpha$$
$$\leq \widetilde{d}_{TD}^2(A,B)$$

and the continuity is proved. □

Theorem 6. *The interval approximation operators* $C_i : IF(\mathbb{R}) \to [\mathbb{R}], i \in \{1,2\}$, *defined above, are linear.*

Proof. It is immediate. For example, the proof of the additivity of C_1 is the following

$$C_{\widetilde{d}_E}(A+B)$$
$$= \left[\int_0^1 \frac{(\mu_{A+B})_L(\alpha)+(\nu_{A+B})_L(\alpha)}{2} d\alpha, \int_0^1 \frac{(\mu_{A+B})_U(\alpha)+(\nu_{A+B})_U(\alpha)}{2} d\alpha\right]$$
$$= \left[\int_0^1 \frac{\mu_{A_L}(\alpha)+\nu_{A_L}(\alpha)}{2} d\alpha + \int_0^1 \frac{\mu_{B_L}(\alpha)+\nu_{B_L}(\alpha)}{2} d\alpha,\right.$$
$$\left. \int_0^1 \frac{\mu_{A_U}(\alpha)+\nu_{A_U}(\alpha)}{2} d\alpha + \int_0^1 \frac{\mu_{B_U}(\alpha)+\nu_{B_U}(\alpha)}{2} d\alpha\right]$$

$$= \left[\int_0^1 \frac{\mu_{A_L}(\alpha) + \nu_{A_L}(\alpha)}{2} d\alpha, \int_0^1 \frac{\mu_{A_U}(\alpha) + \nu_{A_U}(\alpha)}{2} d\alpha\right]$$
$$+ \left[\int_0^1 \frac{\mu_{B_L}(\alpha) + \nu_{B_L}(\alpha)}{2} d\alpha, \int_0^1 \frac{\mu_{B_U}(\alpha) + \nu_{B_U}(\alpha)}{2} d\alpha\right]$$
$$= C_{\widetilde{d}_E}(A) + C_{\widetilde{d}_E}(B). \qquad \square$$

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On Intuitionistic Fuzzy Expert Systems With Temporal Parameters

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Summary. An extension of the concept of an intuitionistic fuzzy expert system with temporal components is constructed and some of its modifications are discussed.

Key words: Expert systems, Intuitionistic fuzzy set, Temporal logic.

1 Introduction

The ideas for putting together the concepts of fuzzy set and expert system (ES) has been discussed in details in the research literature. In [1, 4] the concepts of Intuitionistic Fuzzy Logic (IFL; see, e.g., [6,7,9,10]) and ES united, defining a new object called *Intuitionistic Fuzzy Expert System* (IFES).

The main components of a production system are:

- A Data Base (DB) containing facts about the problem to be solved
- A Knowledge Base (KB) containing the rules that are to be used in the reasoning process
- An inference engine which operates through the KB using the DB for proving or rejecting hypotheses

Here we shall describe an ES with facts having simultaneously intuitionistic fuzzy estimations and temporal parameters. So, these ES can answer temporal questions concerning the existence-validity of the recorded facts in the DB(s) and can estimate the facts validity or nonvalidity using intuitionistic fuzzy estimations.

2 Short Remarks on IFL

Following [6, 7, 9] we define the basic elements of IFL.

Two real numbers, $\mu(p)$ and $\nu(p)$, are assigned to the proposition p with the following constraint to hold:

$$\mu(p) + \nu(p) \leq 1. \tag{1}$$

They correspond to the "truth degree" and to the "falsity degree" of p.

Let this assignment be provided by an evaluation function V, defined over a set of propositions S in such a way that:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle. \tag{2}$$

When values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be also extended for the operations "negation" (now there are six different types of negation, the simplest of which is the classical one is $\neg$; another is $\neg^*$), "conjunction" ($\&$), "disjunction" ($\vee$), "implication" (already there are 24 different implications, two of which are the classical one $\supset$ and $\rightarrow$) and others, e.g., through the definitions:

$$\begin{aligned}
V(\neg p) &= \neg V(p) = \langle \nu(p), \mu(p) \rangle, && (3)\\
V(\neg^* p) &= \neg^* V(p) = \langle 1 - sg(\mu(p)), sg(\mu(p)) \rangle, && (4)\\
V(p) \& V(q) &= V(p \& q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle, && (5)\\
V(p) \vee V(q) &= V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle. && (6)\\
V(p) \supset V(q) &= V(p \supset q) = \langle \max(\nu(p), \mu(q)), \min(\mu(p), \nu(q)) \rangle, && (7)\\
V(p \supset q) &= \langle 1 - (1 - \mu(q)).sg(\mu(p) - \mu(q)),\\
&\qquad \nu(q).sg(\mu(p) - \mu(q)).sg(\nu(q) - \nu(p)) \rangle. && (8)
\end{aligned}$$

We can define also:

$$\langle \mu(p),\ \nu(p) \rangle \geq \langle \mu(q),\ \nu(q) \rangle \text{ if and only if } \mu(p) \geq \mu(q) \text{ and } \nu(p) \leq \nu(q). \tag{9}$$

Let for every proposition p, if it is a *(standard) tautology* then: $V(p) = \langle \mu(p), \nu(p) \rangle$ if and only if $\mu(p) = 1$ and $\nu(p) = 0$. It is an Intuitionistic Fuzzy Tautology (IFT) if and only if $\mu(p) \geq \nu(p)$.

The evaluation function V can be extended also for the modal operators "$\Box$" and "$\Diamond$" as follows

$$\begin{aligned}
V(\Box p) &= \Box V(p) = \langle \mu(p), 1 - \mu(p) \rangle, && (10)\\
V(\Diamond p) &= \Diamond V(p) = \langle 1 - \nu(p), \nu(p) \rangle. && (11)
\end{aligned}$$

It can be seen easily that for each proposition p such that $V(p) = \langle \mu(p), 1 - \mu(p) \rangle$, i.e., the estimation is fuzzy, but not intuitionistic fuzzy, then $V(\Diamond p) = V(p) = \Box V(p)$.

Let p be a fixed proposition and let $\alpha, \beta \in [0, 1]$. Following [6, 7, 9], we define operators D_α, $F_{\alpha,\beta}$ (for $\alpha + \beta \leq 1$), $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$, and $J^*_{\alpha,\beta}$ by:

$$\begin{aligned}
V(D_\alpha(p)) &= \langle \mu(p) + \alpha.(1 - \mu(p) - \nu(p)), && (12)\\
&\qquad \nu(p) + (1 - \alpha).(1 - \mu(p) - \nu(p)) \rangle,
\end{aligned}$$

$$V(F_{\alpha,\beta}(p)) = \langle \mu(p) + \alpha.(1 - \mu(p) - \nu(p)), b + \beta.(1 - \mu(p) - \nu(p)) \rangle, \quad (13)$$
$$\text{for } \alpha + \beta \leq 1,$$
$$V(G_{\alpha,\beta}(p)) = \langle \alpha.\mu(p), \beta.\nu(p) \rangle, \quad (14)$$
$$V(H_{\alpha,\beta}(p)) = \langle \alpha.\mu(p), b + \beta.(1 - \mu(p) - \nu(p)) \rangle, \quad (15)$$
$$V(H^*_{\alpha,\beta}(p)) = \langle \alpha.\mu(p), b + \beta.(1 - \alpha.\mu(p) - \nu(p)) \rangle, \quad (16)$$
$$V(J_{\alpha,\beta}(p)) = \langle \mu(p) + \alpha.(1 - \mu(p) - \nu(p)), \beta.\nu(p) \rangle, \quad (17)$$
$$V(J^*_{\alpha,\beta}(p)) = \langle \mu(p) + \alpha.(1 - \mu(p) - \beta.\nu(p)), \beta.\nu(p) \rangle. \quad (18)$$

Following [2], the main elements of temporal IFL will be introduced.

Let T be a fixed set of real numbers which we shall call "time scale" and it is strictly oriented by the relation "$<$."

Let p be a proposition and V be a truth-value function, which maps the ordered pair:

$$V(p,t) = \langle \mu(p,t), \nu(p,t) \rangle \quad (19)$$

to the proposition p and to the time moment $t \in T$.

Let $x \in E$ be a fixed proposition and $A \subset E$, where here and below E is a set of propositions. Firstly, following [8] we shall introduce one new (for the IFS theory) operator as follows:

$$\tau(A(T), x) = \{t \mid \mu_A(x,t) > \nu_A(x,t) \ \& \ t \in T\}. \quad (20)$$

Obviously, for all $x \in E$:

$$\emptyset \subset \tau(A(T), x) \subset T. \quad (21)$$

For x we can assert that it is "Intuitionistic Fuzzy Valid" (IFV) in time moment t, if and only if

$$\mu_A(x,t) \geq \nu_A(x,t). \quad (22)$$

Numbers $\mu_A(x,t)$ and $\nu_A(x,t)$ can be, respectively, interpreted as a "degree of validity" and a "degree of nonvalidity."

Let us assume that in E for each element x there exists an element $\neg x$ and let for it be valid:

$$\tau(A(T), \neg x) = \{t | \nu_A(x,t) > \mu_A(x,t) \ \& \ t \in T\}. \quad (23)$$

Therefore, the predicate

$$\varphi(x) = \text{"}x \text{ has always been true"} \quad (24)$$

will be IFV, if (22) holds for all $t \in T$. Obviously, φ coincide with the above-mentioned operator $\mathcal{A}$.

By similarity, we can define the following predicates, too:

$$\psi(x) = \text{"}x \textit{ has sometimes been true, but not always,}\text{"} \quad (25)$$
$$\chi(x) = \text{"}\textit{once } x \textit{ was true,}\text{"} \quad (26)$$
$$\omega(x) = \text{"}x \textit{ has never been true.}\text{"} \quad (27)$$

Obviously, χ coincide with the above-mentioned operator $\mathcal{O}$. It can be easily seen that

$$\varphi(x) = 1, \quad \text{if and only if } \tau(A(T), x) = T, \tag{28}$$

$$\psi(x) = 1, \quad \text{if and only if } \emptyset \neq \tau(A(T), x) \neq T, \tag{29}$$
$$\text{and } (\exists t_1, t_2 \in \tau(A(T), x))(\exists t_3 \in T - \tau(A(T), x))(t_1 < t_3 < t_2),$$

$$\chi(x) = 1, \quad \text{if and only if } \emptyset \neq \tau(A(T), x) \neq T, \tag{30}$$
$$\text{and } (\forall t_1, t_2 \in \tau(A(T), x))(\neg\exists t_3 \in T - \tau(A(T), x))(t_1 < t_3 < t_2),$$

$$\omega(x) = 1, \quad \text{if and only if } \tau(A(T), x) = \emptyset. \tag{31}$$

All the above predicates $\varphi, \psi, \chi, \omega$ have values in set $\{0, 1\}$. Now, we can construct their IFVs.

Let below $card(X)$ be the cardinality of set X. Therefore, for the fixed elements $x \in X$ we can define the couple

$$\rho(x) = \langle \frac{card(\tau(A(X), x))}{card(T)}, \frac{card(\tau(A(X), \neg x))}{card(T)} \rangle. \tag{32}$$

It is an intuitionistic fuzzy couple, because

$$0 \leq \frac{card(\tau(A(X), x))}{card(T)} + \frac{card(\tau(A(X), \neg x))}{card(T)} \leq 1. \tag{33}$$

The second inequality will become an equality, if there was no time moment when for $x : \mu_A(x, t) = \nu_A(x, t)$. The set of all time moments for which the latter equality is not valid (let us note it by Δ_x) determines the "degree of uncertainty" for x, and of course,

$$\frac{card(\tau(A(X), x))}{card(T)} + \frac{card(\tau(A(X), \neg x))}{card(T)} + \frac{\Delta_x}{card(T)} = 1. \tag{34}$$

We can define the following two new predicates:

$$\xi(x) = 1, \quad \text{if and only if } \rho(x) \text{ is an IFT}, \tag{35}$$

$$\sigma(x) = 1, \quad \text{if and only if } \rho(\neg x) \text{ is an IFT}. \tag{36}$$

These predicates can be interpreted as follows:

$$\xi(x) = \text{"}x \textit{ is often true,}\text{"} \tag{37}$$

$$\sigma(x) = \text{"}x \textit{ is rarely true.}\text{"} \tag{38}$$

These two predicates can be generalized. For example, we can use the two real numbers $\lambda, \mu \in [0, 1]$ and we can define that $\langle \lambda, \mu \rangle$ is (λ, μ)-IFT if and only if $a \geq \lambda$ and $b \leq \mu$. Then

$$\xi^*(x) = \text{"}x \textit{ is } (\lambda, \mu)\textit{-often true,}\text{"} \tag{39}$$

$$\sigma^*(x) = \text{"}x \textit{ is } (\lambda, \mu)\textit{-rarely true.}\text{"} \tag{40}$$

For them there will hold

$$\xi^*(x) \text{ is } (\lambda,\mu)\text{-often if and only if } \mu(\rho(x)) \geq \lambda \;\&\; \nu(\rho(x)) \leq \mu, \quad (41)$$

$$\sigma^*(x) \text{ is } (\lambda,\mu)\text{-rarely if and only if } \mu(\rho(\neg x)) \geq \lambda \;\&\; \nu(\rho(\neg x)) \leq \mu. \quad (42)$$

In the present research we shall use the following important restriction: in the expressions that we can construct and use, the temporal operators will be the nearest to the variables. For example, the expression

$$\Box\,(\xi(x))\&F_{\alpha,\beta}(\rho(y) \vee \chi(z)) \quad (43)$$

is correct, while the expression

$$\xi(\,\Box\,(x))\&\rho(F_{\alpha,\beta}(y) \vee G_{\gamma,\delta}(z)) \quad (44)$$

is not.

3 Main Results

Following the idea for an IFES from [5] we will introduce IFES with Temporal Parameters (IFESTP).

To define the new concept, here we will base on one of the most general types of ESs definitions.

Here, as [5], we will add new ESs' components: priorities and the degrees μ and ν of truth and falsity (correctness and incorrectness) of the facts. Therefore, every fact A of the DB will have the form: $[A, p_A, \mu_A, \nu_A]$ where $\mu_A, \nu_A \in [0,1]$ and $\mu_A + \nu_A \leq 1$ are the above-mentioned degrees, $p_A \in [0,1]$ is the priority of A and A is a standard ES fact.

Let a fact A with the three above components be included in the DB. Let a new fact B be generated at a certain time moment of the ES functioning with priority p_B. If the two facts are not related, then the new fact enters the DB. In an ordinary ES, the new fact B replaces the old fact A when B coincides with, or contradicts A. Now the ES will function in another way, based on the new component. When the facts A and B coincide, their representative (in particular, A or B) remains in the DB, but with a new priority that can be determined by different ways, e.g.,

1. It is equal to $\max(p_A, p_B)$
2. It is equal to $p_A + p_B - p_A.p_B$
3. It is equal to $f(p_A, p_B)$, where function $f : [0,1]^2 \to [0,1]$ is defined so that $\max(x,y) \leq f(x,y) \leq 1$

On the other hand, the fact having the maximum priority among p_A and p_B remains in the DB when the facts A and B are in a contradiction. The change can be realized, e.g., by formulae:

1. It is equal to $\min(p_A, p_B)$,
2. It is equal to $p_A.(1 - p_B)$,
3. It is equal to $g(p_A, p_B)$, where function $g : [0,1]^2 \to [0,1]$ is defined so that $\min(x, y) \geq f(x, y) \geq 0$.

The components μ_A, ν_A, and p_A of the fact allow for an interpretation in which they are independent. For example, when we describe different facts and their estimations generated by a group of n experts, every one of which estimates some information, the ith one can estimate the fact A by the values $\mu_{A,i}$ and $\nu_{A,i}$ $(1 \leq i \leq n)$, but every fact will have its priority p_A.

The last ES parameter can depend not only on the priority of the corresponding expert, but also on other factors. In the last case, the three parameters will be independent. If every one of the experts estimates some fact, facts $[A, \mu_{A,1}, \nu_{A,1}, p_A]$, $[A, \mu_{A,2}, \nu_{A,2}, p_A]$, ..., $[A, \mu_{A,n}, \nu_{A,n}, p_A]$ will enter the DB. After this, the DB can store the fact A with these (μ, ν)-parameters μ_{i^*} and ν_{i^*}, for which

$$p_{i^*} = \max_{1 \leq i \leq k} p_i, \tag{45}$$

or these (μ, ν)-parameters for which

$$\mu_{i^*}.p_{i^*} = \max_{1 \leq i \leq k} \mu_i.p_i \tag{46}$$

(if the maximum is reached at some values of i, then the value of i at which ν_i is minimum among the other ν-values is determined).

Now, we shall extend the concept of an IFES, adding temporal components.

Let T be a time scale. Let each fact keeps the first time moment in which it starts being valid as its first temporal component. It will obtain the form

$$[A, \mu_{A,1}, \nu_{A,1}, p_A, t_1]. \tag{47}$$

When (if) it loses its validity, it will obtain as a second parameter the respective time moment and will obtain the form

$$[A, \mu_{A,1}, \nu_{A,1}, p_A, t_1, t_2], \tag{48}$$

etc. When the fact obtains for a sth time truth-value "true" (or, it obtains a tautological or an IFT-value), it will have the form

$$[A, \mu_{A,1}, \nu_{A,1}, p_A, t_1, t_2, ..., t_{2s-1}], \tag{49}$$

and when it again loses this value, it will obtain the form

$$[A, \mu_{A,1}, \nu_{A,1}, p_A, t_1, t_2, ..., t_{2s-1}, t_{2s}]. \tag{50}$$

Therefore, we can already apply over this fact each one of the temporal operators from Sect. 2, having in mind the final remark from this section.

Now we return to the IFES. As we mentioned above, we call them IFESTC, because over their variables we can apply some temporal operators.

The KB-rules in the IFESTC have either of the following forms.

1.

$$[\langle M_H, N_H\rangle\ H : -e(\tau_1 B_1, \tau_2 B_2, \ldots, \tau_n B_n)\ \langle M_B, N_B\rangle], \tag{51}$$

where $M_H, N_H, M_B, N_B \subset [0,1]$ and $\sup M_H + \sup N_H \leq 1$ and $\sup M_B + \sup N_B \leq 1$, $e(B_1, B_2, \ldots, B_n)$ is a logical expression for the variables (some of which can be ES facts) $B_1, B_2, \ldots, B_n$ and $\tau_1, \tau_2, ..., \tau_n$ are either some temporal operator, or an empty symbol, i.e., over the respective variable no temporal operator is applied.
The expression $e(B_1, B_2, \ldots, B_n)$ may contain operations "&," "$\vee$," "$\subset$," "$\rightarrow$," "$\neg$," "$\neg^*$," standard modal ("$\Box$," "$\Diamond$"), extended modal (D_α, $F_{\alpha,\beta}$, $H_{\alpha,\beta}$, $J_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J^*_{\alpha,\beta}$), and level ($P_{\alpha,\beta}$, $Q_{\alpha,\beta}$) operators. Therefore, $e(\tau_1 B_1, \tau_2 B_2, \ldots, \tau_n B_n)$ can have very complex form.
The intervals have the forms

$$M_H = [\mu_i^H, \mu_s^H], \tag{52}$$
$$N_H = [\nu_i^H, \nu_s^H], \tag{53}$$
$$M_B = [\mu_i^B, \mu_s^B], \tag{54}$$
$$N_B = [\nu_i^B, \nu_s^B]. \tag{55}$$

They can be given the following interpretation. For each assignment of each variable occurring in the rule, if B_i are all true with degrees within the intervals $[\mu_i^B, \mu_s^B]$ (for the degree of truth) and $[\nu_i^B, \nu_s^B]$ (for the degree of falsity) the consequent H has values μ_H and ν_H within the intervals $[\mu_i^H, \mu_s^H]$ and $[\nu_i^H, \nu_s^H]$, respectively. Naturally, the calculated degrees μ_H and ν_H satisfy the constraint $0 \leq \mu_H + \nu_H \leq 1$.
Let μ_B and ν_B be the already calculated truth and falsity degrees of the rule. The degrees' calculation of the consequent H, μ_H, and ν_H in terms of the interval rule is the following

$$\mu_H = \mu_i^H + \alpha_\mu.(\mu_s^H - \mu_i^H), \tag{56}$$
$$\nu_H = \nu_i^H + \alpha_\nu.(\nu_s^H - \nu_i^H), \tag{57}$$

where

$$\alpha_\mu = \begin{cases} \dfrac{\mu_B - \mu_i^B}{\mu_s^B - \mu_i^B}, & \text{if } \mu_s^B > \mu_i^B, \\ \dfrac{1}{2}, & \text{otherwise}, \end{cases} \tag{58}$$

$$\alpha_\nu = \begin{cases} \dfrac{\nu_B - \nu_i^B}{\nu_s^B - \nu_i^B}, & \text{if } \nu_s^B < \nu_i^B, \\ \dfrac{1}{2}, & \text{otherwise}. \end{cases} \tag{59}$$

2.
$$[\langle M_H, N_H\rangle H : -e(\tau_1 B_1, \tau_2 B_2, \ldots, \tau_n B_n)\langle \mu_B, \nu_B\rangle], \tag{60}$$

where M_H, N_H, μ_B, and ν_B are as above. The interpretation is again as above, except that the truth and falsity degrees of B_1, B_2, ..., B_n must be greater than or equal to μ_B and less than or equal to ν_B, respectively. The calculation is as follows

$$\alpha_\mu = \begin{cases} \dfrac{\mu_B - b_\mu}{1 - b_\mu}, & \text{if } b_\mu < 1, \\ \dfrac{1}{2}, & \text{otherwise,} \end{cases} \tag{61}$$

$$\alpha_\nu = \begin{cases} \dfrac{\nu_B}{b_\nu}, & \text{if } b_\nu > 0, \\ \dfrac{1}{2}, & \text{otherwise,} \end{cases} \tag{62}$$

where degrees of $e(B_1, B_2, \ldots, B_n)$ are b_μ and b_ν.
μ_H and ν_H are calculated in the same way.
The next two cases are modifications of the first ones:

3.
$$[Y_{\alpha,\beta,\ldots}]H : -e(\tau_1 B_1, \tau_2 B_2, \ldots, \tau_n B_n)\langle M_B, N_B\rangle], \tag{63}$$

4.
$$[Y_{\alpha,\beta,\ldots}]H : -e(\tau_1 B_1, \tau_2 B_2, \ldots, \tau_n B_n)\langle \mu_B, \nu_B\rangle], \tag{64}$$

where all but the last components are equal; the last components are the same as above.

The meaning of the first components in both types of rules (the next components are the same as their counterparts above) is as follows: Y is an operator identifier, i.e., $Y \in \{\Box, \Diamond, D_\alpha, F_{\alpha,\beta}, \ldots\}$ and $\alpha, \beta, \ldots$ are its necessary components (their number is 0, 1, 2, or 6, depending on the identifier).

For example, the calculation of the degrees of the clause head is based on operator $F_{\alpha,\beta}$, for $0 \leq \alpha + \beta \leq 1$ and proceeds in the following way:

$$\begin{aligned} \langle \mu_H, \nu_H\rangle &= F_{\alpha,\beta}(\mu_B, \nu_B) \\ &= \langle \mu_B + \alpha.\pi_B, \nu_B + \beta.\pi_B\rangle, \end{aligned} \tag{65}$$

where $\pi_B = (1 - \mu_B - \nu_B)$ can be interpreted as the uncertainty factor.

Thus we defined an ES capable of a more fine-grained process description than the classical ESs.

4 Conclusion

In [5] a series of nine Generalized Net (GN; extension of Petri net; see [3, 11]) models of different types of ESs (already existing or possible in principle) are described. The ordinary IFES is represented by the eight GN-model. In the near future the defined ES will be simulated with the aid of GN topologies.

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Generalized Fuzzy Cardinalities of IF Sets

Pavol Kráĺ

Summary. The paper by Casasnovas and Torrens (Fuzzy sets and systems 133:193–209, 2003) presents the axiomatic theory of fuzzy cardinalities of finite fuzzy sets. The aim of our contribution is to extend this axiomatic theory to the finite intuitionistic fuzzy sets using t-norms on a lattice $\mathcal{L}^I$, the underlying lattice of interval-valued fuzzy sets.

Key words: Fuzzy cardinality, Interval-valued fuzzy sets.

1 Introduction

The axiomatic theory of fuzzy cardinalities of finite intuitionistic fuzzy sets, as a straightforward generalization of the axiomatic cardinality theory of finite fuzzy sets (cf. [2]), can be found in [10]. There the cardinality is defined as a mapping $\mathcal{C}_I : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$, where $\mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$ denotes the set of all general convex interval-valued fuzzy number (icgnn), i.e., the finite convex interval-valued fuzzy sets on $\mathbb{N}$.

The aim of our contribution is to extend this axiomatic theory using the t-sums based on generalized t-norms (t-norms on a lattice $\mathcal{L}^I$, the underlying lattice of interval-valued fuzzy sets) and to study the properties of such cardinalities.

2 Preliminaries

In this section we will summarize basic notions which will be needed in the next discussion. Throughout, let X denotes the universal set.

Definition 1. (Goguen [7] *An $\mathcal{L}$-fuzzy (LF) set A on a universe X is a function $A : X \longrightarrow L$.*

Remark 1. The lattice L from the previous definition is usually a complete distributive lattice equipped with the standard operations $\vee, \wedge$, bottom element $0_{\mathcal{L}}$, top element $1_{\mathcal{L}}$, and a unary, involutive, order-reversing operator $\mathcal{N}_{\mathcal{L}}$.

Definition 2. [1]. *An IF set on the universe X is a set of the form*

$$A = \{(x, \mu_A(x), \nu_A(x))| \ \ x \in X\},$$

where μ_A, $\nu_A : X \longrightarrow [0,1]$ satisfy the following condition

$$(\forall x \in X)(\mu_A(x) + \nu_A(x) \leq 1).$$

$\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$ are called the membership degree and the nonmembership degree, respectively, of $x \in A$.

Definition 3. [11]. *An interval-valued fuzzy (IVF) set in X is a set A given by*

$$A = \{\langle x, M_A(x)\rangle| \ x \in X\}, \tag{1}$$

where the function $M_A : X \longrightarrow D[0,1]$ ($D[0,1]$ is the set of all closed subintervals on the interval $[0,1]$) defines the degree of membership of an element x to A.

It is obvious that

$$M_A(x) = [M_A(x)_L, M_A(x)_U],$$

where $M_A(x)_L$ and $M_A(x)_U$ are the lower and the upper bound of interval $M_A(x)$.

Remark 2. An intuitionistic fuzzy set can be seen as an $\mathcal{L}$-fuzzy set (see [5]) for the complete lattice $\mathcal{L}^* = (L^*, \leq_{L^*})$ defined by

$$\begin{aligned} &L^* = \{(x_1, x_2) \in [0,1]^2| \ \ x_1 \leq 1 - x_2\},\\ &(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow (x_1 \leq y_1 \text{ and } x_2 \geq y_2),\\ &\text{for all } (x_1, x_2), (y_1, y_2) \in L^*. \end{aligned}$$

The top and bottom element of $\mathcal{L}^*$ we denote $1_{\mathcal{L}^*}, 0_{\mathcal{L}^*}$, respectively. We will write $(x_1, x_2) \in L^*$, x when no confusion can arise.
The mapping $\mathcal{N}^s_{\mathcal{L}^*} : L^* \longrightarrow L^*$ defined by, for all $x \in L^*$,

$$\mathcal{N}^s_{\mathcal{L}^*}(x) = (x_2, x_1)$$

is a negation on $\mathcal{L}^*$ which is called the standard negation on $\mathcal{L}^*$. Interval-valued fuzzy sets can be seen as an $\mathcal{L}$-fuzzy set [5] for the complete lattice $\mathcal{L}^I = (L^I, \leq_{L^I})$ defined by

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \quad \text{iff} \quad (x_1 \leq y_1 \text{ and } x_2 \leq y_2),$$
$$\text{for all } [x_1, x_2], [y_1, y_2] \in L^I.$$

In the sequel, if $x \in L^I$, then we denote its bounds by x_1 and x_2, i.e., $x = [x_1, x_2]$. The top and bottom element of $\mathcal{L}^I$ we denote $1_{\mathcal{L}^I}, 0_{\mathcal{L}^I}$, respectively.

From now on, we will denote the class of IF sets $\mathcal{F}_{\mathcal{L}^*}(X)$ and the class of IVF sets $\mathcal{F}_{\mathcal{L}^I}(X)$. If no confusion can arise, we also use $[0,1]$ as an abbreviation of the lattice $([0,1], \leq)$ and denote the class of ordinary fuzzy sets by $\mathcal{F}_{[0,1]}(X)$ or shortly by $\mathcal{F}(X)$.

Let $A \in \mathcal{F}_{\mathcal{L}}(X)$, then we define for further usage the following sets.

Let $0_{\mathcal{L}} \leq_L \alpha \leq_L 1_{\mathcal{L}}$, $\alpha \neq 0_{\mathcal{L}}$. Then the α-cut of A is the set

$$A_\alpha = \{x \mid x \in X \text{ and } A(x) \geq_L \alpha\}.$$

Let $0_{\mathcal{L}} \leq_L \alpha \leq_L 1_{\mathcal{L}}$, $\alpha \neq 0_{\mathcal{L}}, 1_{\mathcal{L}}$. Then the strong α-cut of A is the set

$$A^\alpha = \{x \mid x \in X \text{ and } A(x) >_L \alpha\}.$$

The core of $\mathcal{L}$-fuzzy set A is the set $\text{core}(A) = A_1$. The support of $\mathcal{L}$-fuzzy set A is the set $\text{supp}(A) = A^0$.

We will also use the following notation:

$$[A]_i = \sup\{\alpha \in L \setminus 0_{\mathcal{L}} \mid \ |A_\alpha| \geq i\}, \quad i \in \mathbb{N}.$$

An $\mathcal{L}$-fuzzy set A on X with finite support will be called a finite $\mathcal{L}$-fuzzy set. The class of all finite $\mathcal{L}$-fuzzy sets (where L is L^*, L^I or [0,1]) on X will be denoted by $\mathcal{F}^F_{\mathcal{L}}(X)$.

We will use only $\mathcal{L}$-fuzzy sets with finite supports in the rest of paper.

In the sequel we will sometimes denote, for an arbitrary $x \in X$ and $a \in L$, the $\mathcal{L}$-fuzzy set $A : X \longrightarrow L$ given by

$$A(y) = \begin{cases} a, & \text{if } y = x, \\ 0_{\mathcal{L}}, & \text{if } y \in X \setminus \{x\}, \end{cases}$$

shortly by a/x. From the context it will be clear whether this is a fuzzy set or an IF set.

Deschrijver, et al. [5] have extended triangular norm, triangular conorm a negation to the lattice $\mathcal{L}^*$.

Definition 4.

- *A t-norm on $\mathcal{L}^*$ is a commutative, associative mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ which is increasing in both arguments and which satisfies $\mathcal{T}(1_{\mathcal{L}^*}, x) = x$, for all $x \in L^*$.*

- *A t-conorm on $\mathcal{L}^*$ is a commutative, associative mapping $\mathcal{S} : (L^*)^2 \longrightarrow L^*$ which is increasing in both arguments and which satisfies $\mathcal{S}(0_{\mathcal{L}^*}, x) = x$, for all $x \in L^*$.*
- *A negation on $\mathcal{L}^*$ is a decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ which satisfies $\mathcal{N}(0_{\mathcal{L}^*}) = 1_{\mathcal{L}^*}$ and $\mathcal{N}(1_{\mathcal{L}^*}) = 0_{\mathcal{L}^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called involutive.*

Some triangular norms and conorms on $\mathcal{L}^*(\mathcal{L}^I)$ can be characterized using t-norms T and t-conorms S on [0,1] which satisfy the condition $T(x, y) \leq 1 - S(x, y)$, for all $x, y \in [0, 1]$.

Definition 5 [5]. *A t-norm $\mathcal{T}$ on $\mathcal{L}^*$ is called t-representable iff there exist a t-norm T and a t-conorm S on* [0,1] *such that, for all $x, y \in L^*$,*

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2)).$$

Definition 6. *A t-conorm $\mathcal{S}$ on $\mathcal{L}^*$ is called t-representable iff there exist a t-norm T and a t-conorm S on* [0,1] *such that, for all $x, y \in L^*$,*

$$\mathcal{S}(x, y) = (S(x_1, y_1), T(x_2, y_2)).$$

The following example contains non-t-representable t-norms and t-conorms on L^*.

Example 1 [5].

1. $\mathcal{T}_W(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$
   $\mathcal{S}_W(x, y) = (\min(1, x_2 + 1 - y_1, y_2 + 1 - x_1), \max(0, x_2 + y_2 - 1))$
2. $\mathcal{T}_1(x, y) = (\max(0, x_1 + y_1 - x_2 y_2 - 1), \min(1, x_2 + y_2))$
   $\mathcal{S}_1(x, y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - x_1 y_1 - 1))$
3. $\mathcal{T}_2(x, y) = (\max(0, \min(x_1 - y_2, y_1 - x_2)), \min(1, x_2 + y_2))$
   $\mathcal{S}_2(x, y) = (\min(1, x_1 + y_1), \max(0, \min(x_2 - y_1, y_2 - x_1)))$
4. $\mathcal{T}_3(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, y_2 + 2(1 - x_1), x_2 + 2(1 - y_1), 1 - x_1 + 1 - y_1))$
   $\mathcal{S}_3(x, y) = (\min(1, y_1 + 2(1 - x_2), x_1 + 2(1 - y_2), 1 - x_2 + 1 - y_2), \max(0, x_2 + y_2 - 1))$
5. $\mathcal{T}_4(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2 + \frac{1}{2}, 1 - x_1 + y_2, 1 - y_1 + x_2))$
   $\mathcal{S}_4(x, y) = (\min(1, x_1 + y_1 + \frac{1}{2}, 1 - x_2 + y_1, 1 - y_2 + x_1), \max(0, x_2 + y_2 - 1))$

The previous definitions can be simple rewritten also for the lattice $\mathcal{L}^I$.

The intersection, union, and complement of two IF(IVF) sets A and B can be modeled using these general t-norms, t-conorms, and negations in the following way.

Definition 7 [4]. *The generalized intersection $\cap_{\mathcal{T}}$, union $\cup_{\mathcal{S}}$ and complement $\mathrm{co}_{\mathcal{N}}$ of IF sets is defined as follows: for all $A, B \in \mathcal{F}_{\mathcal{L}^*}(X)$ ($\mathcal{F}_{\mathcal{L}^I}(X)$) and for all $x \in X$,*

$$A \cap_{\mathcal{T}} B(x) = \mathcal{T}(A(x), B(x)),$$
$$A \cup_{\mathcal{S}} B(x) = \mathcal{S}(A(x), B(x)),$$
$$\mathrm{co}_{\mathcal{N}} A(x) = \mathcal{N}(A(x)).$$

If no confusion can arise, the intersection and union of IF sets modeled using $\mathcal{T} = (T_M, S_M), \mathcal{S} = (S_M, T_M)$ we denote simply $\cup$ and $\cap$.

A generalized natural number (gnn) $\overline{n}$ is a fuzzy set on $\mathbb{N}$, $\overline{n} : \mathbb{N} \longrightarrow [0,1]$. The gnn is convex if $\overline{n}(k) \geq \min(\overline{n}(i), \overline{n}(j))$ whenever $i \leq k \leq j$. The set of all gnn we denote $\mathcal{F}(\mathbb{N})$. The set of all convex finite gnn we denote by $\mathcal{F}^{CF}(\mathbb{N})$. Let $\overline{n}$, $\overline{m}$ be gnn. The following operation is called the extended addition (see [17]):

$$(\overline{n} \oplus \overline{m})(k) = \sup\{\min(\overline{n}(i), \overline{m}(j));\ i + j = k\}.$$

A generalized interval-valued fuzzy number (ignn) $\overline{n}_I$ is an interval-valued fuzzy set on $\mathbb{N}$, $\overline{n}_I : \mathbb{N} \longrightarrow \mathcal{L}^I$. The ignn will be called convex if $\overline{n}_{I1}$ and $\overline{n}_{I2}$ are convex gnn. The set of all ignn we denote $\mathcal{F}_{\mathcal{L}^I}(\mathbb{N})$. The set of all convex finite ignn we denote by $\mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$. Let $\overline{n}_I$, $\overline{m}_I$ be ignn. The extended addition of ignn is defined componentwise (see [10]):

$$(\overline{n}_I \oplus^I \overline{m}_I)(k) = [(\overline{n}_{I1} \oplus \overline{m}_{I1})(k), (\overline{n}_{I2} \oplus \overline{m}_{I2})(k)].$$

We can also define the t-norm-based extended addition of ignn in the following way:

$$(\overline{n}_I \oplus^I_{\mathcal{T}} \overline{m}_I)(k) = \sup\{\mathcal{T}(\overline{n}_I(i), \overline{m}_I(j));\ i + j = k\}.$$

The fuzzy cardinality of a fuzzy set A will be denoted by $\mathcal{C}(A)$. The fuzzy cardinality of an IF set A will be denoted by $\mathcal{C}_I(A)$.

3 Fuzzy Cardinality of Fuzzy Sets

The well-known (convex) fuzzy cardinalities of fuzzy sets are FGCount, FLCount, and FECount which can be expressed as follows [18]:

$$\mathrm{FGCount}(A)(k) = [A]_k,$$
$$\mathrm{FLCount}(A)(k) = 1 - [A]_{k+1},$$
$$\mathrm{FECount}(A)(k) = \min([A]_k, 1 - [A]_{k+1}),$$

where $k \in \mathbb{N}$.

$\mathrm{FGCount}(A)(k)$ can be interpreted as the possibility that the cardinality of A is at least k, $\mathrm{FLCount}(A)(k)$ can be interpreted as the possibility that the cardinality of A is at most k and $\mathrm{FECount}(A)(k)$ can be interpreted as the possibility that the cardinality of A is exactly k.

In works [16,18] we can find the generalization of FGCount, FLCount, and FECount for finite fuzzy sets defined using t-norms and negations.

The axiomatic theory of fuzzy cardinality of finite fuzzy sets was proposed by Casasnovas and Torrens [2] in the following way:

Definition 8 [2]. *A mapping $\mathcal{C} : \mathcal{F}^F(X) \longrightarrow \mathcal{F}^{CF}(\mathbb{N})$ is a fuzzy cardinality iff it satisfies the following conditions:*

1. *Additivity: if A and B are finite fuzzy sets defined on the universe X and $\mathrm{supp}(A) \cap \mathrm{supp}(B) = \emptyset$, then $\mathcal{C}(A \cup B) = \mathcal{C}(A) \oplus \mathcal{C}(B)$.*
2. *Variability: if A and B are finite fuzzy sets defined on the universe X and $i > |\mathrm{supp}\, A|$, $j > |\mathrm{supp}\, B|$, then $\mathcal{C}(A)(i) = \mathcal{C}(B)(j)$.*
3. *Consistency: if A is a crisp subset of X, then $\mathcal{C}(A)(i) \in \{0, 1\}$, for all $i \in \mathbb{N}$ and if $n = |\mathrm{supp}\, A|, \mathcal{C}(A)(n) = 1$.*
4. *Monotonicity: if $x \in X$, $y \in X$, $a \in [0, 1]$, $b \in [0, 1]$ and $a \leq b$, then:*

$$\mathcal{C}(a/x)(0) \geq \mathcal{C}(b/y)(0),$$

$$\mathcal{C}(a/x)(1) \leq \mathcal{C}(b/y)(1).$$

It is easy to see that the FGCount satisfies the conditions of previous definition.

The fuzzy cardinality has some interesting properties. The proofs of the following propositions can be found in [2].

Proposition 1 [2]. (Valuation property) *Let $\mathcal{C} : \mathcal{F}^F(X) \longrightarrow \mathcal{F}^{CF}(\mathbb{N})$ be a fuzzy cardinality and let A, $B \in \mathcal{F}^F(X)$. Then*

$$\mathcal{C}(A \cup B) \oplus \mathcal{C}(A \cap B) = \mathcal{C}(A) \oplus \mathcal{C}(B).$$

Proposition 2 [2]. *Let $f, g : [0, 1] \longrightarrow [0, 1]$ be functions such that $f_{|\{0,1\}}$ and $g_{|\{0,1\}}$ take values in $\{0, 1\}$, $f(1) = 1$, $g(0) = 1$, f is increasing and g is decreasing.*

Then the function $\mathcal{C}_{f,g} : \mathcal{F}^F(X) \longrightarrow \mathcal{F}(\mathbb{N})$ defined on singletons by

$$\mathcal{C}_{f,g}(a/x)(0) = g(a); \quad \mathcal{C}_{f,g}(a/x)(1) = f(a);$$

$$\mathcal{C}_{f,g}(a/x)(i) = f(0) \quad \text{for all } i > 1$$

for all $a \in [0, 1]$ and extended to any finite fuzzy set A with $\mathrm{supp}(A) = \{x_1, ..., x_n\}$ by

$$\mathcal{C}_{f,g}(A) = \underset{i=1,..,n}{\oplus} \{\mathcal{C}_{f,g}(A(x_i)/x_i)\}$$

is a fuzzy cardinality.

Fuzzy cardinality from the previous proposition will be called the fuzzy cardinality generated by f, g. Each fuzzy cardinality from Definition 8 is the fuzzy cardinality generated by f, g.

Proposition 3 [2]. *A function $\mathcal{C} : \mathcal{F}^F(X) \longrightarrow \mathcal{F}(\mathbb{N})$ is a fuzzy cardinality iff there exist functions $f, g : [0, 1] \longrightarrow [0, 1]$ such that $f_{|\{0,1\}}$ and $g_{|\{0,1\}}$ take values in $\{0, 1\}$, $f(1) = 1$, $g(0) = 1$, f is increasing, g is decreasing and such that $\mathcal{C} = \mathcal{C}_{f,g}$.*

4 Fuzzy Cardinality of IF Sets

We generalize the previous definition to the case of IF sets.

Definition 9. *A mapping* $\mathcal{C}_I : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$ *is a fuzzy cardinality of IF sets iff it satisfies the following conditions:*

1. *Additivity: if* A *and* B *are finite IF sets defined on the universe* X *and* $\operatorname{supp}(A) \cap \operatorname{supp}(B) = \emptyset$, *then* $\mathcal{C}_I(A \cup B) = \mathcal{C}_I(A) \oplus^I_{\mathcal{T}} \mathcal{C}_I(B)$.
2. *Variability: if* A *and* B *are finite IF sets defined on the universe* X *and* $i > |\operatorname{supp} A|$, $j > |\operatorname{supp} B|$, *then* $\mathcal{C}_I(A)(i) = \mathcal{C}_I(B)(j)$.
3. *Consistency: if* A *is a crisp subset of* X, *then* $\mathcal{C}_I(A)(i) \in \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$, *for all* $i \in \mathbb{N}$ *and if* $n = |\operatorname{supp} A|$, $\mathcal{C}_I(A)(n) = 1_{\mathcal{L}^I}$.
4. *Monotonicity: if* $x \in X$, $y \in X$, $a \in L^*$, $b \in L^*$, *and* $a \leq_{L^*} b$, *then:*

$$\mathcal{C}_I(a/x)(0) \geq_{L^I} \mathcal{C}_I(b/y)(0),$$

$$\mathcal{C}_I(a/x)(1) \leq_{L^I} \mathcal{C}_I(b/y)(1).$$

Example 2. The mapping $\mathcal{C}_{I1} : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$ given by

$$\mathcal{C}(A)(k) = [A]_k$$

is a fuzzy cardinality of IF sets.

Proposition 4. *Let* $\mathcal{C}_I : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$ *be a fuzzy cardinality of IF sets and, let* $\mathcal{T}$ *be t-norm on* $\mathcal{L}^I$, *let* A *be a finite IF sets with* $\operatorname{supp}(A) = \{x_1, \ldots, x_n\}$. *Then*

1. *we have for all* $k \in \mathbb{N}$,

$$\mathcal{C}_I(A)(k) = \sup\{\mathcal{T}(\mathcal{C}_I(A(x_1)/x_1)(i_1), \ldots, \mathcal{C}_I(A(x_n)/x_n)(i_n)) |\ i_1 + \cdots + \ i_n = k\},$$

   where $\mathcal{T}(x_1, \ldots, x_n) = \mathcal{T}(x_1, \mathcal{T}(x_2, \ldots, x_n))$, *for all* $x_i \in L^I$.
2. *if* A *is a crisp set, then we have, for all* $k < |\operatorname{supp}(A)|$,

$$\mathcal{C}_I(A)(k) = \mathcal{C}_I(1_{\mathcal{L}^*}/x_i)(0).$$

The fuzzy cardinality of IF sets defined above has similar properties to those of fuzzy sets.

Proposition 5. (Valuation property) *Let* $\mathcal{C}_I : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$ *be a fuzzy cardinality of IF sets, let* $\mathcal{T}$ *be t-norm on* $\mathcal{L}^I$ *represented by* T_M *and let* A, $B \in \mathcal{F}^F_{\mathcal{L}^*}(X)$. *Then*

$$\mathcal{C}_I(A \cup B) \oplus^I_{\mathcal{T}} \mathcal{C}_I(A \cap B) = \mathcal{C}(A)_I \oplus^I_{\mathcal{T}} \mathcal{C}_I(B).$$

Proposition 6. *Let $f_I, g_I : L^* \longrightarrow L^I$ be mappings such that $f_{I|\{0_{\mathcal{L}^*},1_{\mathcal{L}^*}\}}$ and $g_{I|\{0_{\mathcal{L}^*},1_{\mathcal{L}^*}\}}$ take values in $\{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$, $f(1_{\mathcal{L}^*}) = 1_{\mathcal{L}^I}$, $g(0_{\mathcal{L}^*}) = 1_{\mathcal{L}^I}$, f is increasing and g is decreasing. Then the function $\mathcal{C}_{I\,f_I,g_I} : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}_{\mathcal{L}^I}(\mathbb{N})$ defined on singletons by*

$$\mathcal{C}_{I_{f_I,g_I}}(a/x)(0) = g_I(a); \quad \mathcal{C}_{I_{f_I,g_I}}(a/x)(1) = f_I(a);$$

$$\mathcal{C}_{I_{f_I,g_I}}(a/x)(i) = f_I(0_{\mathcal{L}^*}) \quad \text{for all } i > 1$$

for all $a \in L^*$ and extended to any finite IF set A with $\mathrm{supp}(A) = \{x_1, ..., x_n\}$ by

$$\mathcal{C}_{I_{f_I,g_I}}(A) = \underset{i=1,..,n}{\oplus^I_{\mathcal{T}}} \left\{\mathcal{C}_{I_{f_I,g_I}}(A(x_i)/x_i)\right\}$$

is a fuzzy cardinality of IF sets.

Fuzzy cardinality of IF sets from the previous proposition will be called the fuzzy cardinality of IF sets generated by f_I, g_I. Each fuzzy cardinality of IF sets from Definition 9 is the fuzzy cardinality generated by f_I, g_I.

Example 3. The mapping $\mathcal{C}_{I\,1} : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$ is generated by $g_I(a) = [1, 1]$, $f_I(a) = [a_1, a_1]$, for all $a \in L^*$.

Proposition 7. *A function $\mathcal{C}_I : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}_{\mathcal{L}^I}(\mathbb{N})$ is a fuzzy cardinality of IF sets iff there exist functions $f_I, g_I : L^* \longrightarrow L^I$ such that $f_{I|\{0_{\mathcal{L}^*},1_{\mathcal{L}^*}\}}$ and $g_{I|\{0_{\mathcal{L}^*},1_{\mathcal{L}^*}\}}$ take values in $\{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$, $f_I(1_{\mathcal{L}^*}) = 1_{\mathcal{L}^I}$, $g_I(0_{\mathcal{L}^*}) = 1_{\mathcal{L}^I}$, f_I is increasing, g_I is decreasing and such that $\mathcal{C}_I = \mathcal{C}_{I\,f_I,g_I}$.*

Definition 10. *The fuzzy cardinality of IF sets $\mathcal{C}_I$ will be called representable iff there exist fuzzy cardinalities of fuzzy sets $\mathcal{C}_1, \mathcal{C}_2$ such that, for all $A \in \mathcal{F}^F_{\mathcal{L}^*}(X)$, $\mathcal{C}_I(A)(k) = [\mathcal{C}_1(A_1)(k), \mathcal{C}_2(A_2)(k)]$, for each $k \in \mathbb{N}$, where $A_1, A_2 \in \mathcal{F}^F_{[0,1]}(X)$ are given by $A_1(x) = \mu_A(x)$, $A_2(x) = 1 - \nu_A(x)$, for all $x \in X$. We denote a representable cardinality by $\mathcal{C}^r_I$.*

The functions f_I, g_I are called representable iff there exist functions f_1, f_2 and g_1, g_2 such that, for all $x \in L^*$,

$$f_I(x) = [f_1(x_1), f_2(1 - x_2)],$$

$$g_I(x) = [g_1(x_1), g_2(1 - x_2)].$$

We denote representable functions f_I, g_I by f^r_I, g^r_I.

The following proposition shows that a representable cardinality $\mathcal{C}_I$ with representants $\mathcal{C}_1$ and $\mathcal{C}_2$ is well defined iff $\mathcal{C}_1 \leq \mathcal{C}_2$.

Proposition 8. *Let $\mathcal{C}_1, \mathcal{C}_2 : \mathcal{F}^F_{[0,1]}(X) \longrightarrow \mathcal{F}^{CF}(\mathbb{N})$ be fuzzy cardinalities of fuzzy sets. The mapping $\mathcal{C}^r_I : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$ defined by, for all $A \in \mathcal{F}^F_{\mathcal{L}^*}(X)$*

$$\mathcal{C}^r_I(A) = [\mathcal{C}_1(A_1), \mathcal{C}_2(A_2)], \tag{2}$$

where $A_1, A_2 \in \mathcal{F}^F_{[0,1]}(X)$ are given by $A_1(x) = (A(x))_1$ and $A_2(x) = 1 - (A(x))_2$, for all $x \in X$, is a fuzzy cardinality of IF sets iff $\mathcal{C}_1 \leq \mathcal{C}_2$.

If $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{T}$ represented by T_M, then $\mathcal{C}_I^r(A)(k)$ is an element of D, for any IF set A for which $A_1 = A_2$. Hence, we obtain straightforward and natural generalization of fuzzy cardinality of fuzzy set in the sense of Definition 8, i.e., the fuzzy cardinality of IF sets applied to an arbitrary fuzzy set is a generalized natural number.

Proposition 9. *Let $\mathcal{C}_I$ be a fuzzy cardinality of IF sets and let f_I, g_I be its associated functions. Then $\mathcal{C}_I$ is representable iff f_I, g_I are representable. Furthermore, if $\mathcal{C}_I = [\mathcal{C}_1, \mathcal{C}_2]$ and $f_I = [f_1, f_2], g_I = [g_1, g_2]$, then f_1, g_1 and f_2, g_2 are the functions associated to $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.*

Proposition 10. *Function $f_I(g_I)$ is representable iff*

$$(f_I(g_I)((a_1, a_2)))_1 = (f_I(g_I)((a_1, a_2')))_1$$

and

$$(f_I(g_I)((a_1, a_2))) = (f_I(g_I)((a_1', a_2)))_2,$$

for all (a_1, a_2), (a_1, a_2'), $(a_1', a_2) \in L^*$.

The interesting question is whether the fuzzy cardinality of IF sets satisfies the valuation property and complementarity rule which are defined using t-norms, t-conorms a negations on $\mathcal{L}^*$. We formulate the generalized valuation property and complementarity rule as follows.

$\mathcal{T}, \mathcal{S}$-valuation property: for each $A, B \in \mathcal{F}^F_{\mathcal{L}^*}(X)$,

$$\mathcal{C}_I(A \cap_{\mathcal{T}} B) \oplus^I_{\mathcal{T}'} \mathcal{C}_I(A \cup_{\mathcal{S}} B) = \mathcal{C}_I(A) \oplus^I_{\mathcal{T}'} \mathcal{C}_I(B).$$

$\mathcal{N}$-complementarity rule: for each $A \in \mathcal{F}^F_{\mathcal{L}^*}(X)$ and for a negation $\mathcal{N}$ on $\mathcal{L}^*$,

$$\mathcal{C}_I(A) \oplus^I_{\mathcal{T}'} \mathcal{C}_I(\mathrm{co}_{\mathcal{N}} A) = \mathcal{C}_I(X).$$

For representable fuzzy cardinalities and the generalized sum defined using minimum, we can give the partial characterization of $\mathcal{T}, \mathcal{S}, \mathcal{N}, f_I^r, g_I^r$ satisfying the properties defined above.

Proposition 11. *Let $C_I^r : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}_I}(\mathbb{N})$ be a representable fuzzy cardinality of IF sets and let A, $B \in \mathcal{F}^F_{\mathcal{L}^*}(X)$ be IF sets. Let $\mathcal{T}$, $\mathcal{S}$ be t-representable t-norm and t-conorm. Then the $\mathcal{T}, \mathcal{S}$-valuation property is satisfied iff at least one from the following conditions holds:*

1. *$\mathcal{T} = (T_M, S_M), \mathcal{S} = (S_M, T_M)$,*
2. *$f_I^r = [f_1, f_2]$,$g_I^r = [g_1, g_2]$, where $g_i(a) = 1$ or $g_i(a) = 0$ and $f_i(b) = 1$ or $f_i(b) = 0$, for all $i \in \{1, 2\}$, $a \in]0, 1]$, $b \in [0, 1[$.*

Proposition 12. *The $\mathcal{N}$-complementarity rule holds for a representable cardinality induced by functions f_I^r, g_I^r, IF singleton a/x and a negation $\mathcal{N}(x) = (1 - N(x_2), N(1 - x_1))$ iff one from the following conditions is satisfied*

1. *For* $i \in \{1,2\}$, $g_i(1) = 0$ *and for all* $a \in [0,1[$, $f_i(a) = 0$, $g_i(a) = 1$, *and* $N(a) = 0$.
2. *For* $i \in \{1,2\}$, $f_i(0) = 0$, *for all* $a \in [0,1]$, $g_i(a) = 1$ *and for all* $a \in]0,1]$, $f_i(a) = 1$ *and* $N(a) = 1$.
3. *For* $i \in \{1,2\}$, *for all* $a \in [0,1]$, $g_i(a) = 1$ *and for all* $a \in [0,1[$, $f_i(a) = 0$ *and* $N(a) = 0$.
4. *For* $i \in \{1,2\}$, *for all* $a \in [0,1]$, $f_i(a) = 1$, $g_i(a) = 1$.
5. *For* $i \in \{1,2\}$, $f_i(0) = 1$, $g_i(1) = 0$ *and for all* $a \in [0,1[$, $g_i(a) = 1$, *and* $N(a) = 0$.

For more general cardinalities, it is really difficult to find some similar conditions. But using singletons, it is possible to find also some necessary conditions for the $\mathcal{T},\mathcal{S}$-valuation property and the $\mathcal{N}$-complementarity rule to be satisfied.

Proposition 13. *Let* $C_I : \mathcal{F}^F_{\mathcal{L}^*}(X) \longrightarrow \mathcal{F}^{CF}_{\mathcal{L}^I}(\mathbb{N})$ *be a fuzzy cardinality of IF sets. Let* $\mathcal{T}$, $\mathcal{S}$ *be a t-norm and a t-conorm on* $\mathcal{L}^*$, *let* $\mathcal{T}'$ *be a t-norm on* $\mathcal{L}^I$. *If cardinality satisfies the* $\mathcal{T},\mathcal{S}$*-valuation property then the following conditions hold for all* $a, b \in L^*$, $a \leq_{L^*} b$:

1. $\mathcal{T}'(g_I(\mathcal{T}(a,b)), g_I(\mathcal{S}(a,b))) = \mathcal{T}'(g_I(a), g_I(b))$,
2. $\mathcal{T}'(g_I(\mathcal{T}(a,b)), f_I(\mathcal{S}(a,b))) = \mathcal{T}'(g_I(a), f_I(b))$,
3. $$\max(\mathcal{T}'(f_I(\mathcal{T}(a,b))f_I(\mathcal{S}(a,b))), \mathcal{T}'(f_I(0), g_I(\mathcal{T}(a,b))))$$ $$= \max(\mathcal{T}'(f_I(a), f_I(b)), \mathcal{T}'(f_I(0), g_I(a))),$$
4. $\mathcal{T}'(f_I(0), g_I(\mathcal{T}(a,b))) = \mathcal{T}'(f_I(0), g_I(a))$.

Proposition 14. *Let* $\mathcal{T}'$ *be a t-norm on* $\mathcal{L}^I$, *let* $\mathcal{N}$ *be a negation on* $\mathcal{L}^*$. *If* $\mathcal{N}$*-complementarity rule holds for a cardinality induced by functions* f_I, g_I, *then the following conditions are satisfied, for all* $a \in L^*$:

1. $\mathcal{T}'(g_I(a), g_I(\mathcal{N}(a))) = g_I(1)$,
2. $\max(\mathcal{T}'(g_I(a), f_I(\mathcal{N}(a))), \mathcal{T}'(g_I(\mathcal{N}(a)), f_I(a))) = f_I(1)$.

Conclusion

In this paper we have extended the axiomatic theory of fuzzy cardinality to the IF sets using t-norms on $\mathcal{L}^I$. We have studied the basic properties of such cardinality (representability, valuation property, complementarity rule). Further description of generalized fuzzy cardinalities of IF sets (especially the construction of a corresponding equipotency relation) will be an object of future research.

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Towards Usage Policies for Fuzzy Inference Methodologies for Trust and QoS Assessment

Stefan Schmidt, Robert Steele, Tharam Dillon

Summary. In this paper, we discuss the benefits of several fuzzy inference system design methodologies and evaluate their characteristics in regard to our trustworthiness and QoS measurement models. Our analysis shows that Mamdani–Assilian or Larsen type and Takagi–Sugeno–Kang type fuzzy inference methods have their merits in different situations. We propose to equip an autonomous agent which acts on behalf of a human being with a policy table enabling the agent to dynamically decide which fuzzy inference system it will select during the trustworthiness evaluation process. We argue that in most situations the Mamdani–Assilian or Larsen type fuzzy inference system represents the preferred choice. However, in situations where the fuzzy rulebase is large, the Takagi–Sugeno–Kang type fuzzy inference system should be chosen due to its superior performance characteristics. This way the agent can perform its tasks more efficiently by choosing the appropriate calculation method depending on the given circumstances.

Key words: Fuzzy inference, Autonomous trustworthiness evaluation, Quality of service.

The assessment of trust and credibility is part of our daily life – it happens subconsciously and is based on recommendations, past experiences, and vague feelings. Reliable and precise measurement of trust and credibility is especially important if we want to achieve autonomous interactions of intelligent agents in unsupervised distributed environments. However, the replication of such social behavior in information systems represents a major challenge.

In an ideal scenario, a person, who wants to purchase goods or consume a service, would instruct his intelligent agent to execute this time-consuming task on his behalf. The agent's duties and responsibilities would involve service discovery, service selection, contract negotiations, service execution or consumption, payments, and reviewing of the delivered service quality.

In previous work, we have proposed models for both, trustworthiness evaluation in distributed environments to support selection of potential services [1], as well as a quality of service (QoS) review model [2]. Both models are based on fuzzy logic [3] which offers a mathematical concept to deal with uncertainty for the calculation of outputs. This ability to offer reasoning capabilities based on uncertain or incomplete information makes it suitable to simulate

human reasoning which is based on similar principles. In both models, we have chosen the Mamdani–Assilian (MA) [4] approach for the fuzzy inference process of our models. In separate research, our group has also proposed a fuzzy model based on the Takagi–Sugeno–Kang (TSK) [5] inference method to determine the trustworthiness and credibility of peer agents in distributed environments [6].

In this paper we discuss the benefits of these and other fuzzy inference system (FIS) design methodologies and evaluate their characteristics in regard to our trustworthiness and QoS measurement models. First, we briefly introduce our fuzzy trust and QoS evaluation models to establish the context for our suitability analysis. Second, we provide details on the fuzzy inference methods before comparing their benefits in the different situations that the agent might encounter. Based on this analysis we will finally introduce a policy-based model which assists the agent to select the appropriate FIS for the specific situation the agent encounters.

1 Related Work

A number of researchers have proposed models based on fuzzy logic concepts to offer solutions for the computation of trust, credibility, reputation, or QoS. For example, Falcone et al. [7] use Fuzzy Cognitive Maps (FCM) [8] to model the dynamic influence of measured attributes before and during the trust calculation. A different approach is the Regret system [9] which integrates fuzzy concepts into the analysis of social networks in electronic marketplaces. Other fuzzy logic-based approaches for the calculation of trust in distributed systems has been developed by [10, 11]. In previous research, we have also proposed a fuzzy logic-based model to offer flexible and efficient approaches for the computation of variables like trustworthiness, credibility, and QoS [2].

Manara et al. [12] have implemented different models in MATLAB to measure the performance between three different fuzzy controllers. They compared their own approach named Conditionally Firing Rules, and the standard approaches Mamdani–Assilian, and Takagi–Sugeno–Kang. Their tests show that the Takagi–Sugeno–Kang controller performs faster in most tests compared to the MA controller. However, they limit their comparisons to response time measurements and precision measurements. Youssef [13] compared MA, TSK, Larsen [14], Singleton, and Tsukamoto type fuzzy controllers in a 1-input, 1-output power system relaying system and found the TSK type inference engine preferable due to its computational efficiency resulting in faster response times required for real-time systems.

Most of these fuzzy logic-based models are designed to suit a specific environment or measure a set of model-specific variables. However, none of these papers have so far investigated the suitability of different fuzzy inference methodologies for the tasks of trust and credibility evaluation and QoS measurement. In this paper, we compare the major approaches (MA or Larsen

type and TSK type) used for the defuzzification during the inference process and determine their suitability in different situations.

2 Fuzzy Trust and QoS Assessment

In this section, we will briefly discuss our model for trust evaluation and QoS measurement to establish the context of our research.

2.1 Fuzzy Trust Evaluation

In an unsupervised multi-agent environment, the measurement of trust in other agents plays a crucial role during the service selection process where an agent needs to choose between a number of potential business partners which it previously discovered. The selection of a future business partner or service no longer only depends on matching the tangible criteria a service offers, but also on the willingness and capability of a potential business partner to deliver quality of service in a given context at a given timeslot. We use the notion *trustworthiness*, as a measure, to quantify the trust level an agent has in a potential business partner in a given context at a given timeslot. Our model describes a trust evaluation process implemented by an agent to measure trust in a future negotiation partner before the negotiation process takes place. In our model, we define the service consumer as *Trusting Agent*, potential business partners as *Recommendation Queried Agents*, peer agents who share their opinions about Recommendation Queried Agents as *Recommending Agents*, and the actually selected business partner as *Trusted Agent*.

Given that the Trusting Agent has no sufficient information about Recommendation Queried Agents, he asks Recommending Agents to deliver their opinions about them within a given context and a given timeslot. These opinions are composed of one or more datasets which contain a trustworthiness value, context, timestamp, and a trustworthiness value range for each business interaction which the Recommending Agent had with the Recommendation Queried Agent in the past. The Trusting Agent creates a weighted average according to the age of the records within a dataset before feeding this value as one of three input variables into its fuzzy inference engine. The second input for the fuzzy inference engine is the agent's credibility value, and the third input is the weight of this opinion resulting from the number of records in the dataset which the Recommending Agent delivered.

In a next step, these inputs are fuzzified and mapped to the rulebase in order to calculate a crisp output using either the MA or the TSK approach during the fuzzy inference process. After having computed a crisp trustworthiness output value from each opinion delivered, we calculate the average overall trustworthiness values for a particular Recommendation Queried Agent. Furthermore, if the agent holds trustworthiness records about the Recommendation Queried Agent in his individual database, he combines this value

with the aggregated trustworthiness value from the delivered opinions using weight factors for his own data and the externally delivered data. Through this aggregation the Trusting Agent finally gains a trustworthiness value for each Recommendation Queried Agent.

These final composite trustworthiness values for each Recommendation Queried Agent are then used as support during the service selection process. The agent representing the selected service is then called Trusted Agent.

2.2 Service Quality Review after a Business Interaction

The measurement of the QoS after a business interaction serves several purposes. First and foremost, the Trusting Agent requires a QoS value to update his trustworthiness value for the Trusting Agent. This trustworthiness value can then be used for future trust evaluation processes. It is also useful for providing opinions about the Trusted Agent to peer agents which might ask for this value in future. Second, the QoS value will be used to adjust the credibility value of Recommending Agents which have delivered their opinions on the Trusting Agent within the context of this business interaction. If their opinions were close to the calculated QoS value (within a predefined threshold) then their credibility value will be increased and otherwise decreased. It is noteworthy that our model reduces credibility values to a significantly higher extend if the trustworthiness review calculations have a negative outcome than vice versa.

The calculation of the QoS value is based on the extended fuzzy logic-based model [2] of CCCI metrics introduced by Chang, et al. [6]. During the contract negotiations between the Trusting Agent and the Trusted Agent a number of quality assessment criteria are defined, communicated and agreed upon by both parties in the contract. After the completion of the business interaction, these criteria are used for the assessment of the contract fulfillment (QoS). The CCCI metrics define three measurement variables for each quality assessment criterion.

Commitment – The commitment variable measures the actual degree of fulfillment of every specified criterion. That is, the commitment to each criterion (service condition), to which the Trusted Agent and the Trusting Agent mutually agreed upon, before the business interaction.

Clarity – The clarity variable provides a measure to establish if each criterion was clearly specified, commonly understood, and mutually agreed to between the Trusting and the Trusted Agent.

Influence – The influence variable measures the impact of each criterion on the overall investigated quality as perceived by the agent owner and published in the service contract.

Similar to our fuzzy trust evaluation model, we use a fuzzy inference engine to compute the desired output service quality value for each predefined criterion. We fuzzify the predefined value for influence, the observed value for

commitment, as well as the clarity value which can only be determined after the business interaction took place and possible ambiguities in the service contract become clear. The fuzzified inputs for each quality criterion will then be mapped to the predefined rulebase before being inferred in order to gain the service quality value. The two approaches for fuzzy inference are introduced in Sect. 3 in greater detail. After having computed service quality values for each quality assessment criterion, we calculate the average overall service quality values for the assessed business interaction.

The overall service quality value represents the *correlation* between the expected service quality (as defined in the contract) and the actually delivered quality of the product or service. This value can then be used to adjust both, the trustworthiness value for the Trusted Agent as well as the credibility values for the Recommending Agents which supplied their opinions during the trust evaluation process.

3 Fuzzy Rules and Inference Methodologies

In this section, we will discuss and compare the Mamdani–Assilian (MA) or Larsen type and Takagi–Sugeno–Kang (TSK) type fuzzy inference methodologies. First, we must understand the nature of all rules expressed in FIS. Fuzzy rules allow us to characterize imprecise dependencies between the input variables in our trust and QoS assessment models using linguistic variables rather then crisp sets.

The usage of linguistic variables allows the desired freedom for the agent owner to apply his personal understanding and experience to model the system behavior. Furthermore, the design of rules based on linguistic variables is more suitable for agent owners which possess the required knowledge in the domain or context in which the assessment takes place but they may not possess sufficient mathematical expertise to encode their knowledge in a complex program or data structure. The imprecision involved in the usage of linguistic variables which are represented by overlapping fuzzy sets allow a high level of readability and comprehensibility for human beings.

3.1 MA and Larsen Type Fuzzy Inference Systems

An exemplary MA or Larsen type fuzzy rule which we have used in previous work has the following form [15]

$$\text{IF } x_1 \text{ IS } A1^j \text{ AND } x_2 \text{ IS } A2^j \ldots \text{AND } x_n \text{ IS } An^j \text{ THEN } y \text{ IS } Bj, \quad j = 1, 2, \ldots, M,$$

where x_i for $i = 1, 2, ..., n$ are linguistic input variables, such as credibility or clarity in the trust evaluation model and the QoS assessment model, respectively; Ai^j for $i = 1, 2, ..., n$ are input fuzzy sets such as "high," "medium," or "low;" y is the linguistic output variable such as trustworthiness or service

quality; Bj is the output fuzzy set, and M is the number of fuzzy rules. Noteworthy for MA and Larsen type inference engines is the fact that the consequent of each rule is also composed of fuzzy sets which are represented as linguistic variables. For each firing fuzzy rule the output of the rule inference (implication) will be mapped to its corresponding output fuzzy set, i.e., the result is described in terms of membership in fuzzy sets.

The main difference between Mamdani–Assilian and Larsen type methods lies in the implication of fuzzified variables within a antecedent part of the fuzzy rule and the composition of the resulting membership functions defined by fuzzy variables in the rule consequents. MA proposed a max–min approach which uses the minimum operator for rule implication and the maximum operator for rule composition. Larsen proposed the max-product method which uses the product operator for rule implication and the max operator for rule composition. Apart from those different rule implication and membership function composition methods both approaches have identical characteristics and, therefore, they exhibit a similar performance.

In order to receive the desired crisp output value, a process called defuzzification must be applied. Popular defuzzification approaches include the "centre of area" method, the "centre of maxima" method, or the "mean of maxima" approach [16].

3.2 Takagi–Sugeno Type Fuzzy Inference

The newer TSK fuzzy inference approach takes a somewhat different path. While the antecedent block of each fuzzy rule remains the same, the consequent block employs a simple equation which takes the input fuzzy variables into account. This equation can be of linear or quadratic type and is referred to as type-1 or type-2 TSK models [17], respectively. Fuzzy inference systems can also be modeled using fuzzy rules with singleton consequents [18]. However, limited modeling capabilities [17] of singleton type FIS result in more coarse grained results and thus affect the quality of the model.

An example fuzzy rule which represents a type-2 TSK model has the following form:

$$\text{IF } x_1 \text{ IS } A1^j \text{ AND } x_2 \text{ IS } A2^j \ldots \text{AND } x_n \text{ IS } An^j \text{ THEN } y = f_j(x_1, x_2, ..., x_n),$$
$$for\ j = 1, 2, \ldots, M.$$

The function f_j accomplishes a direct mapping between the fuzzified input variables to the output space y. Normally, each of the r rules of the fuzzy system is assigned a weight factor w during the aggregation of all output singletons resulting in the following weighted sum $y_{overall}$:

$$y_{overall} = \frac{\sum_r^{i=1} w_i \cdot y_i}{\sum_r^{i=1} w_i} \tag{1}$$

The obvious difference to the MA or Larsen Fuzzy inference methods is that the consequent of each rule is not a fuzzy set but instead a singleton.

3.3 Comparison Between MA or Larsen and TSK Fuzzy Inference

Now that we briefly discussed the foundation of fuzzy rules and the MA or Larsen type and TSK type fuzzy inference methodologies, we need to compare their respective advantages.

Clearly, the MA or Larsen approaches represent a more intuitive and interpretable approach since the knowledge of the agent owner is applied using linguistic variables rather than linear or quadratic equations which are harder to extract from analyzing human experiences and feelings. However, when using linguistic variables within the consequent block of fuzzy rules it becomes necessary to apply additional calculations in order to generate a crisp output. These additional calculations (defuzzification) require more computational resources and can thus result in slower system performance compared to TSK inference systems. This is especially the case if the fuzzy inference engine is used for a large number of calculations or contains a large set of rules. Furthermore, the MA approach offers the flexibility to choose the defuzzification method which the agent owner finds most suitable in a given context. However, one should always remember that this option might lead to inconsistent calculation results. On the other hand, the TSK approach has a continuous output surface and offers better performance which makes it more suitable for real time (possibly self-adjusting) control applications.

In conclusion, we observe that MA or Larsen type fuzzy inference engines are more suitable for analytic applications where agent owners can express their existing knowledge without in depth mathematical knowledge whereas the TSK type fuzzy inference engine is capable of processing a larger amount of data and is thus more suitable for real time control applications.

4 Usage Policies for Fuzzy Inference Methodologies

In this section, we will discuss the suitability of the above-discussed methodologies for the different tasks within our models. Also we will highlight a number of possible scenarios and analyze their impact on the performance of the models. Our previously introduced fuzzy models for the evaluation of trustworthiness and the measurement of service quality both have very different usage scenarios which need to be considered for the decision of the appropriate fuzzy inference method.

4.1 Fuzzy Trust Evaluation Considerations

The Fuzzy Trust Evaluation model needs to consider a potentially large number of opinions during the trust calculation process. Each of the delivered

opinions will be fed into the fuzzy inference engine in order to receive a trustworthiness value for the Recommendation Queried Agent as perceived by the Recommending Agent. Furthermore, the Trusting Agent will evaluate the trustworthiness of several potential business partners during the service selection process. Since the Trusting Agent performs the trustworthiness evaluations before the business interaction with the final business partner, the performance of the fuzzy inference engine is a significant factor for the overall performance of the agent. Long delays caused by a slow performing trust evaluation model may lead to problems during the business interaction, especially in real time environments. The fuzzy rulebase is unlikely to change frequently since it is defined once before the agent is deployed. Apart from minor adjustments the rulebase will not be altered during future activities of the agent.

4.2 Service Quality Review Considerations

The service quality review process has a different purpose. This calculation process takes place after the business interaction took place and, thus, system performance is not of significant importance. Furthermore, the fuzzy-based QoS review inference engine will only be used once and not several hundred times as is the case during the trustworthiness evaluation process. However, the fuzzy rulebase may be altered before each business interaction in order to reflect and comply with the individual service agreement between the service consumer (agent owner represented by the Trusting Agent) and the service provider (selected service represented by the Trusted Agent).

4.3 Experimental Settings for Performance Tests

In order to evaluate the performance of the MA type, the Larsen type, the TSK type, and the singleton type fuzzy inference engines, we have set up a test environment using the Java-based jFuzzyLogic API [19]. We have designed the following fuzzy inference systems:

1. A Mamdani–Assilian type FIS
2. A Larsen type f FIS.
3. A Takagi–Sugeno–Kang type FIS
4. A Singleton type FIS

All other settings are identical to ensure comparability. The underlying model for our FIS performance tests is based on our fuzzy trustworthiness model which has three input variables where two of the variables contain three fuzzy sets and one variable contains six fuzzy sets. The output fuzzy variable is composed of six fuzzy sets. All fuzzy sets are represented by overlapping Gaussian functions, except for the TSK output variable which is represented by six first-order TSK functions and the singleton output variable which is represented by six (singleton) values.

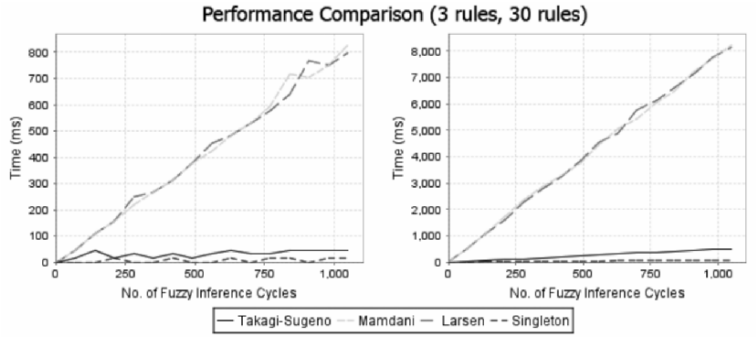

Fig. 1. Performance comparison of four FIS with a small rulebase and a large rulebase

The output variable for the FIS is defined separately for the two methodologies. For the MA or Larsen type FIS we have defined one output fuzzy variable which contains three fuzzy sets. For the TSK type fuzzy inference engine we have represented the three linguistic terms (representing fuzzy sets) for the output variable as a first-order TSK function. For all modeled FIS we have chosen the *center of gravity* defuzzification method as depicted in (Fig. 1).

Our main goal was to measure the performance of the different types of fuzzy inference systems for a large number of cycles in the trust evaluation model. We have chosen to simulate the calculation of trustworthiness values for 15 potential business partners (Recommendation Queried Agents). We furthermore assume that neighboring agents (Recommending Agents) deliver 70 opinions in average for each Recommendation Queried Agent. Therefore, the FIS must process 1,050 opinions overall.

4.4 Policy Table

Our simulation results show that the number of rules play a significant role for the performance of an FIS. The investigated FIS types exhibit a fast response time of time of less than one second when only three fuzzy rules are defined. For a larger rulebase of 30 rules we observe expected longer response times for both systems. However, the TSK type FIS performs ∼16 times faster compared to MA or Larsen type fuzzy inference engines. In all tests the singleton type FIS exhibits the best performance, however this type of FIS was not considered during the design of the policy table. The limited modeling capabilities of singleton type FIS result in more coarse grained trustworthiness and QoS values and thus affect the quality of our fuzzy models.

In situations where the tested fuzzy inference methodologies perform similar or where the response times are very high, we favor the MA or Larsen type FIS because its set up requires less domain knowledge and allows consistent

Table 1. Policy table for FIS methodologies

Fuzzy rulebase	No. of inference cycles	Trustworthiness evaluation model	Quality of service review model
Large	Many	Takagi–Sugeno–Kang	Mamdani–Assilian or Larsen
Large	Few	Mamdani–Assilian or Larsen	Mamdani–Assilian or Larsen
Small	Many	Mamdani-Assilian or Larsen	Mamdani–Assilian or Larsen
Small	Few	Mamdani–Assilian or Larsen	Mamdani–Assilian or Larsen

usage of linguistic terms rather than mathematical functions. The set up of a TSK type FIS requires the existence of experimental data or extensive domain knowledge.

This analysis results in a policy table (Table 1) with which the agent will be equipped. This policy table supports the selection of an appropriate FIS for the different tasks the agent fulfills and the different situations the agent finds itself in.

5 Conclusion

Our analysis shows that Mamdani–Assilian (MA) or Larsen type and Takagi–Sugeno–Kang (TSK) type fuzzy inference methods have their merits in different situations. The Trusting Agent is equipped with a policy table to support dynamic decisions about which type of fuzzy inference system (FIS) it will select during the trustworthiness evaluation and quality of service (QoS) measurement process. Our tests show that in most situations the MA or Larsen type FIS is represents the preferred choice.

To measure the QoS after a business interaction, an MA or Larsen type FIS allows higher flexibility and a more intuitive approach for the agent owner during his initial system set up. The increased level of flexibility is given through a number of defuzzification methods from which the agent owner can choose, according to his personal preferences. Furthermore, an MA or Larsen type fuzzy inference system allows a linguistic modeling approach which offers a more intuitive approach in situations where extensive domain knowledge is absent. Also, since only a small set of data is processed during the QoS measurement within our fuzzy model, computational efficiency is not a crucial requirement.

In situations where the fuzzy rulebase is large and many inference cycles are required, the TSK type FIS is used due to its superior performance characteristics. The reduced computing efforts as a result of the more compact TSK type fuzzy inference approach are especially suitable for the increased number of calculations in our fuzzy trust evaluation model. This high volume of calculations occurs if the agent operates in an information-rich environment

where a large number of witness agents deliver their opinions about a potential business partner. The TSK type fuzzy inference approach significantly reduces the amount of computational power required to calculate overall trustworthiness values for all potential business partners in a timely manner and, thus, results in faster selection of business partners. We assume that in cases where the agent owner wants to increase the precision of the FIS by deploying a large rulebase, sufficient domain knowledge, or previously recorded data already exist in order to design appropriate TSK type functions.

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Simulating a Trust-Based Peer-to-Peer Metadata Publication Center

Paolo Ceravolo, Alessio Curcio, Ernesto Damiani and Micol Pinelli

Summary. This paper introduces a distributed Trust Layer that can be superimposed to metadata generators. By means of a simulator of the Trust Layer we developed an experimentation aimed at validating the role of a Trust Layer as a technique for automatically screening high-quality metadata in a set of assertions coming from sources with different level of trustworthiness.

1 Introduction

Nowadays, communication technologies have increased the opportunities to cooperate and share information among parties. Communication technologies cancel geographical distance, support self-organizing systems, and extend interaction processes towards a distributed dynamics. According to [5], groups of people sharing a common intent and spending time in achieving this intent are called communities of practices. CoPs exist within businesses and across business units and company boundaries. In order to manage interactions in such kind of community it is crucial to define an organizational knowledge. In ordinary communities this knowledge is spontaneously managed by means of informal learning and mutual engagement. In a domain where interactions are supported by communication technologies, organizational knowledge must be formalized and memorized in a predefined format. Moreover, in current business contexts and in distributed environment that require multidisciplinary approaches and competencies, this stress the relevance of user's role, reputation, and trust. For these reasons, generic knowledge management techniques in CoPs have to be evolved towards a source oriented evaluation of the acquired knowledge. The knowledge extracted during the analysis of the information flow produced by the community must be filtered by the relevance of the node producing it. Also the composition of nodes can evolve and the knowledge is continuously under the evolution pressure.

Typically, knowledge management techniques use metadata in order to specifying content, quality, type, creation, and spatial information of a data

item. A number of specialized formats exist for the creation of metadata. A typical example is the resource description framework (RDF). But metadata can be stored in any format such as free text, extensible markup language (XML), or database entries. All of these format must relay on a vocabulary that can have different degree of formality. If this vocabulary is compliant to a set of logical axioms it is called an ontology.

There are a number of well-known advantages in using information extracted from data instead of data themselves. On one hand, because of their small size compared to the data they describe, metadata are more easily shareable than data. Thanks to metadata sharing, information about data becomes readily available to anyone seeking it. Thus, metadata make data discovery easier and reduce data duplication. On the other hand, metadata can be created by a number of sources (the data owner, other users, automatic tools) and may or may not be digitally signed by their author.

The present paper briefly outlines our current research work (for a more detailed description, see [2]) on how to validate such assertions by means of a Trust Layer, including a Trust Manager able to collect votes from the different nodes and to compute variations to trust values on metadata. In order to test the validity of our algorithms tasked to the computation of trust values and to the aggregation of different values from different sources, we developed a Trust Layer simulator able to return the progression of system according to different configuration of the community and according to different trust aggregator functions. This paper is organized as follows: in Sect. 2 we outline the architecture of our Trust Layer, while Sect. 3 we focus our attention on the parameters allowed in order to set a simulations; finally in Sect. 4 we expose the result of some simulations.

2 The Trust Layer Architecture

Before describing our proposed Trust Layer, let us make some short remarks on related works. Current approaches distinguish between two main types of trust management systems [1], namely Centralized Reputation Systems and Distributed Reputation Systems. In centralized reputation systems, trust information is collected from members in the community in the form of ratings on resources. The central authority collects all the ratings and derives a score for each resource. In a distributed reputation system there is no central location for submitting ratings and obtaining resources reputation scores; instead, there are distributed stores where ratings can be submitted. In our approach trust is attached to metadata in the form of assertions rather than to generic resources. While trust values are expressed by clients, our Trust Layer includes a centralized Metadata Publication Center that acts as an index, collecting and displaying metadata assertions, possibly in different formats and coming from different sources. It is possible to assign different trust values to assertions, depending on their origin: assertions manually provided by a

domain expert are more reliable than automatically generated ones. Metadata in the Publication Center are indexed and clients interact with them by navigating documents indexed by means of metadata. Users provide implicitly (with their behavior) or explicitly (by means of an explicit vote) an evaluation about metadata trustworthiness. This trust-related information is provided by the Publication Center to the Trust Manager in the form of new assertions expressing the trust of an assertions, which we call Trust Metadata. Trust Metadata are built using the well-known technique of reification. This choice allows our system to interact with heterogeneous sources of metadata: our Trust Metadata are not dependent on the format of the original assertions. Also, all software modules in our architecture can evolve separately; taken together, they compose a complete Trust Layer, whose components communicate by means of web services interfaces. This makes it possible to test the whole system despite the fact that single models can evolve with different speeds. Summarizing our architecture, the Trust Manager is composed of two functional modules:

- Trust evaluator: examines metadata and evaluates their reliability;
- Trust aggregator: aggregates all the inputs coming from the trust evaluators by means of a suitable aggregation function.

This system allows to integrate large amount of assertions produced from different sources. Trust aggregation algorithms provides a self-running mechanism allowing high-quality assertion to emerge in the whole set of produced assertions. Figure 1 describes the architecture of our Trust Layer. More details on Trust Manger can be found in [3].

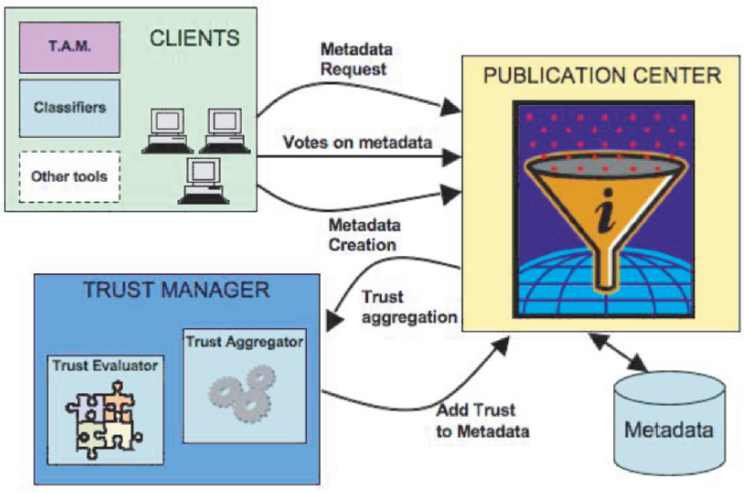

Fig. 1. The Trust Layer architecture

3 Setting a Trust Layer Simulator

The scope of the simulation tests we implemented is testing various types of trust aggregators used to compute the evolution of Trust Metadata. The evolution of a metadata base is strictly dependent to the characteristics of the community accessing the resources available in the Publication Center. Users of the community can have different roles, with different levels of expertise about the content expressed by the resources. Roles are also characterized by different attitudes, for instance users can spend more time in producing metadata or in navigating the metadata base. Also in the community we can have different user groups grouping users with the same role. The logical schema of the software simulating Trust Layer is the following (Fig. 2).

In order to execute a test we need to define all the elements of this logical schema. The configuration of the whole system starts by instantiating a `Project`. A `Project` must contain all the informations about `ConceptList`, `Usergroups`, `Role`, `StateMachine`, `Schema`, and `Resources` entities. A `ConceptList` represents a set of concepts composing the vocabulary used to create the metadata on the resources of the system. `UserGroups` cluster users with the same features, such as having a `Role`, a `StateMachine` of the attitudes and other parameters. The `UserGroup` defines also the population that interact in the system, setting a number of simulated users for each `Role`. The `Role` maps an expertise value (in a range from 0 to 1) to each concept of the `ConceptList`. The states in which users of a simulation can move are represented by one or more `StateMachine` that differ one each other by

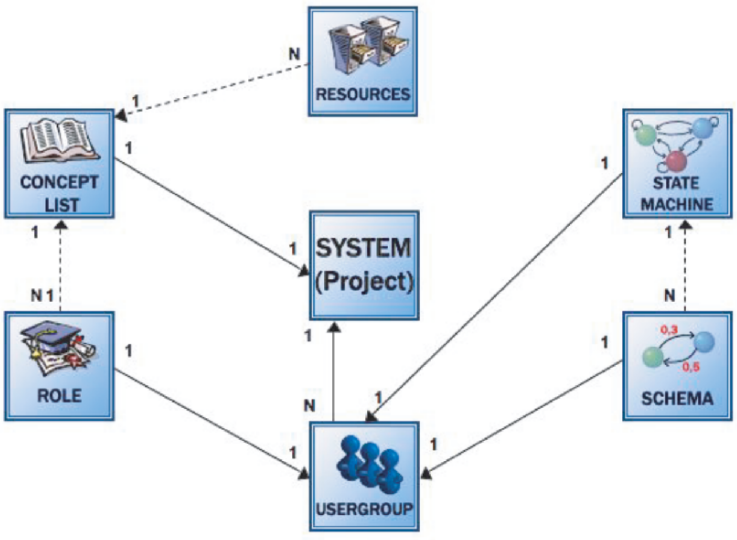

Fig. 2. The logical schema of the simulator

the number and type of states. The probability to change state is defined in the `Schema`. `Resources` are entities (documents or other kind of materials) on which the population of users creates assertions. The last information needed to configure a `Project` is to set the aggregator type and relative parameters.

4 Some Examples of Simulations

When we execute a simulation, each simulated user, moves from a state to another, performing a defined behavior; for example the behavior can cause the production of assertions (Metadata), or assessments on the assertion trust (Trust Metadata). This way simulated users can provide inputs to both Trust Evaluator and Trust Aggregator modules. Every simulated user has a fault probability, depending on his role, to make wrong assessments on metadata assertions. Trust values contained in the Trust Metadata produced by the system depend to the expertise value associated to users' role. These metadata are produced by the Trust Evaluator that is a client module corresponding to each single user. Every fixed time, Trust Evaluators send metadata to the Trust Aggregator that is a service available on a central server. Trust Aggregator collects Trust Metadata to obtain a final aggregated trust value for each assertion.

As result of the simulation we can observe some trust trends for each assertion associated to a single resource. We represent these trends by means of three functions that show the overall system trend:

- Trustworthiness represents the overall truth level of the system obtained as the ratio between the sum of correct metadata assertions and the sum of all Trust Metadata with height trust values.
- Precision calculated as the ratio between number of Trust Metadata that should be produced and the number of Trust Metadata that have been produced.
- Recall is the ratio between the number of correct Trust Metadata that should be produced and the number of correct Trust Metadata that have been produced.

By means of these functions we have an objective quantification for evaluating the effectiveness of our aggregation algorithms.

Acknowledgments

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The Complex Facets of Reputation and Trust

Karl Aberer, Zoran Despotovic, Wojciech Galuba and Wolfgang Kellerer

Summary. Trust and reputation systems have proven to be essential to enforcing cooperative behavior in peer-to-peer networks. We briefly describe the current approaches to building reputation systems: social networks formation, probabilistic estimation, and game theoretic models. We then observe that all of the current models make a number of simplifying assumptions that may not necessarily hold in real networks, such as either irrational (probabilistic) or completely rational behavior, instant propagation of reputation information and homogeneity of interactions. We argue that dropping those assumptions and allowing more degrees of freedom is necessary in order to construct more realistic and richer reputation models. We support our argument by citing reputation research done in economics, evolutionary psychology, biology, and sociology, and consider models that take into account adaptive behavior changes, co-evolution of behaviors, bounded rationality, and variable interaction patterns. We then outline how those complexities can be dealt with and point out main directions for the future study of more realistic and less constrained reputation models that can potentially lead to construction of more secure, responsive, and cooperative peer-to-peer systems.

1 Introduction

Reputation systems have proven to be essential to enforcing cooperative behavior in peer-to-peer networks. Many solutions have been proposed [3, 7, 13, 18, 22, 31], each employing a different model of computing trust, disseminating and storing reputation data, and responding to lack of cooperation in the network [25]. In this paper we focus on the reputation and trust models themselves rather than practical considerations of implementing and deploying a reputation system. We begin with the description of the basic concepts, then survey the current approaches, examine the different assumptions commonly made by the different reputation and trust models and propose ways in which they can be relaxed or extended.

2 Fundamentals

Assume a set of *nodes* continuously engaging in bilateral *interactions*. For simplicity we assume that a single interaction always involves a pair of nodes and that interactions involving a larger group of nodes can always be decomposed into a set of binary interactions.

Each interaction has an associated *benefit* and *cost*. These two values are normally such that nodes face a Prisoner's Dilemma (PD) [12]. It is beneficial for the node to cooperate only if the other node cooperates as well, otherwise it is better to defect.

In this setting, when Alice interacts with Bob it can gain more if it is able to predict that Bob will cooperate. The extent to which a node believes the other will cooperate is the extent to which a node *trusts* the other node. There are a number of ways this belief can be inferred and they are captured by the different *trust models*. One of the inferences that can be made is: if Alice cooperated with Bob then it implies Alice will also cooperate with Carol. If this inference is applied universally, the collective actions of Alice form a commonly shared belief among the other nodes of how likely Alice is to cooperate. This belief is what is termed *reputation*. In the paper we will focus on reputation-based models of trust, whose computation solely depends on the actions of the peers instead of relying on other elements such as third party guarantors of trust (e.g., PKI) or virtual currency for which trust can be purchased, etc.

3 State-of-the-Art

In reputation-based models trust towards a given node A is determined based on the past actions of A. Every node V_i only has information about the actions of A that V_i itself experienced. To compute the reputation of A, nodes need to exchange the information about the actions of A that they have observed. This exchange and the subsequent computation of reputation can proceed in many ways.

There are four classes of approaches [8]: social networks, probabilistic estimation, game-theoretic models, and evolutionary approaches.

3.1 Social Networks

The social network approach assumes an existence of a digraph of social links between nodes. The interactions between the nodes proceed along the links and each link has a trust value associated with it. That value is updated based on the interactions between the nodes at the two ends of the link. A node V can compute the trust value for another non-neighbor node W by aggregating trust values from other nodes in the following way:

1. Enumerate (all) paths from W to V
2. Aggregate trust values along the paths
3. Merge the results of aggregation at V as the final trust value

The social network approaches vary in the details of the three above steps: what domain is used to represent trust, what the selected paths are, what are the aggregation and merging functions. Trust values are either computed on demand between specific W and V or simultaneously for all nodes using some form of iterative methods that converge on the eigenvector of trust values.

3.2 Probabilistic Estimation

The computations in social networks produce trust values that are hard to interpret. In particular, given a trust value for the node A it is hard to translate that value into the probability that A will cooperate. But this can be rectified if the assumption about probabilistic behavior of the nodes is made explicit and then well-known probabilistic estimation techniques such as Bayesian estimation and maximum likelihood estimation are used to compute the trust of a given peer. This is what probabilistic estimation methods do. As an example, consider a network consisting of peers having associated innate probabilities of cooperating. Denote by θ_j the probability of peer j. Assume that peer j interacted with n other peers $p_1, \ldots, p_n$ and its performances in these interactions were $x_1, \ldots, x_n$, where $x_i \in \{0, 1\}$ (1 denoting the honest performance and 0 the dishonest one). When asked to report on peer j's performances witnesses $p_1, p_2, \ldots, p_n$ may lie and misreport. Assuming that they lie with specific probabilities, say l_k for peer p_k, the probability of observing report y_k from peer p_k can be calculated as:

$$P[Y_k = y_k] = \begin{cases} l_k(1-\theta_j) + (1-l_k)\theta_j \text{ if } y_k = 1, \\ l_k\theta_j + (1-l_k)(1-\theta_j) \text{ if } y_k = 0. \end{cases} \quad (1)$$

By definition, the likelihood function associated with a random sample of reports $y_1, y_2, \ldots, y_n$ is:

$$L(\theta_j) = P[Y_1 = y_1]P[Y_2 = y_2] \cdots P[Y_n = y_n]. \quad (2)$$

After collecting the reports on the peer it is about to interact with, the trust computing peer just has to make this product and find θ_j that maximizes it. This number is the maximum likelihood estimate of the unknown probability. To do this, the computing peer must have good estimates of the parameters $l_1, \ldots, l_n$. They can be made by comparing own performances with reports on them. Note also that the own experiences are seamlessly integrated into this model – the trust computing source peer i just has to put $p_i = 1$ for his own experiences x_i. As another advantage of the probabilistic methods, we emphasize that, when compared to social networks, they bring a substantial reduction of the communication overhead. The reason is that they deal only with feedback on the target peer, while social networks essentially aggregate all available feedback, i.e., opinions of all peers about all other peers.

3.3 Game-Theoretic Approach

In game-theoretic approaches to reputation systems it is often assumed that the players are perfectly rational in the sense that they are only interested in maximizing their own payoffs. These assumptions allow the computation of Nash equilibria as strategy profiles where peers have no incentive to deviate. Normally, game-theoretic modeling of reputation effects requires repeated interaction and uncertainties among the players with respect to their opponents' payoffs [21]. More recently, there have been attempts to extend these models in order to more closely model real world settings. Most notably, the two important models are: private and public monitoring games. In these games players do not observe each other's actions but only their signals. In private monitoring games [20], the signals are different for different players, while in public monitoring games [23], all peers observe the same signals about the actions of other peers.

However, we see a number of problems with respect to the application of game-theoretic reputation models. One is related to the behavior. There are plenty of settings where the full rationality of the players cannot be expected. Any setting with human players would be an example. The second is the difficulty of introducing the rationality assumption into the reputation mechanism implementation itself.

3.4 Evolutionary Approach

Game theorists have also approached the problem of cooperation in a population of PD-players from a more experimental angle. Most notably, Axelrod [4] has demonstrated the success of the tit-for-tat strategy in an Evolutionary Prisoner's Dilemma setting. In this setting pairs of players are involved in repeated PD games. Each player maintains a score, which is updated after every game round according to the PD payoff matrix. The players with the highest score are considered most fit and their strategies are replicated replacing other unfit strategies. The winning tit-for-tat strategy follows three simple rules:

1. *Initially cooperate* – when interacting with an opponent for the first time, always cooperate,
2. *Punish* – if the opponent defected in the previous round, punish him by defecting, and
3. *Forgive* – if the opponent cooperated in the previous round, cooperate even if there is a history of opponent's defection.

The tit-for-tat strategy has been shown to be evolutionary stable, being able to drive into extinction small populations of invading defectors, that try to exploit cooperators. At the same time groups of tit-for-taters are always cooperating within the group, which allows them to accumulate score surplus which in turn can be used to fight against transient groups of defectors.

To be successful, the tit-for-tat strategy needs a setting in which the PD interactions are repeated many times for the same pair of players, which allows

punishment to occur. In a large population of infrequently interacting individuals this may not be possible (e.g., eBay and its transactions). This observation led to the definition of a new setting in which every pair of players can only play one round of PD and never meet again. Building cooperation in this setting relies on the rule: "If A cooperates with B then B can reciprocate and cooperate with some other player C." This rule is termed *indirect reciprocity*, as opposed to the *direct reciprocity* rule followed by tit-for-tat. In this case, to build cooperation players can no longer rely on private observation of the actions of the opponent. Once an observation is made, remembering that observation is pointless since all interactions are one-shot and such an observation can never be used to make cooperation decisions. Hence there arises the need to exchange observations with other players. This can be implemented by associating a public label with each player. All players can read the label, and all players except the owner of the label are allowed to change it. It has been shown that to enable sustainable cooperation only two states of the label are sufficient [19]. The two states correspond to good and bad reputation. When a pair of players interacts, their labels are modified according to their actions. The behavior of the player can be succinctly described as two functions: the *action function* and the *assessment function*. The action function takes the label of self and the opponent and produces the decision to either cooperate or defect. The assessment function is executed after the actions of both agents have taken place. The assessment function takes the label of self, the label of the opponent, and the action of the opponent, and produces the new value for the opponent's label. Since the outputs of the functions are binary, there is a relatively small number of all possible functions. There are exactly 16 possible action functions and 256 possible assessment functions, which together results in 4,096 possible behaviors. Ohtsuki et al. [15] have performed a systematic experimental study of all those 4,096 behaviors. Out of these they have found eight evolutionary stable cooperative strategies, termed "the leading eight" (Table 1).

A population of agents using one of these strategies is able to sustain cooperation and drive out of existence any small population of defectors and/or reputation liars (i.e., players that set the labels to "bad" value even though their opponent cooperated).

There is a remarkable similarity between tit-for-tat and the leading eight strategies. The leading eight strategies exhibit all the properties of tit-for-tat: initial cooperation, forgiveness, and punishment for defection. Tit-for-tat can be implemented with one bit of local state in the player, leading eight strategies make this state public by storing it in the player's label.

4 Propagation of Reputation Information

If we compare the two cases – direct reciprocity and indirect reciprocity – they are two extremes in reputation information propagation. In the case of direct

Table 1. The "leading eight" behaviors in the evolutionary indirect reciprocity game

assessment function:

	GG	BG	GB	BB
C	G	*	G	*
D	B	G	B	*

action function:

GG	GB	BG	BB
C	D	C	*

G and B stand for good and bad reputation labels, respectively. C and D stand for cooperation and defection. The GG, GB, BG, BB encode four possible states of the labels. The first letter is the label of self and the second letter is the label of the opponent. The three asterisks in the fields of the assessment function can take any value, hence eight possible assessment functions are possible. The value at the asterisk in the action function is uniquely determined based on the choice of one of the eight assessment functions (for details refer to [14]). As can be seen from the tables, the leading eight behaviors are similar to tit-for-tat, bad behavior is forgiven after it is punished. In addition to that, punishment of bad behavior is justified, a good player defecting with a bad player is assessed as good

reciprocity it is sufficient to rely on privately gathered history of interactions with players, no propagation of reputation is necessary. On the other hand, in the case of indirect reciprocity, once two players interact, their reputation labels are updated and immediately available to all other players, the reputation information propagates instantaneously. When a reputation system is implemented in a peer-to-peer setting the assumptions about the propagation of reputation no longer hold. The character of reputation propagation is determined by the implementation. The question that arises is whether the delayed reputation propagation influences the performance of the reputation system. There is at least one piece of evidence [5] which suggests that delaying the communication of reputation in games with imperfect private monitoring leads to more efficient equilibria. Taking the propagation of reputation information into account might lead to the discovery of entirely new phenomena.

Question 1 *How do the reputation propagation dynamics influence the performance of the reputation system?*

The propagation of reputation cannot only be delayed, it may also be possible to propagate it partially while still maintaining the reputation system performance at an acceptable level. Participating in a reputation system incurs a cost to the peers, the smaller the fraction of nodes that need to participate in each reputation update the smaller the load on the system. We have performed simulations to test the impact of limited reputation information propagation on the performance of the reputation system described in Sect. 3.4 (Fig. 1). Experiments indicate that it is sufficient to make the reputation information available to 30% of the agents to obtain performance that is close to the performance of the system with full propagation. This suggests that there are substantial communication savings to be gained by simply limiting the propagation of the reputation information.

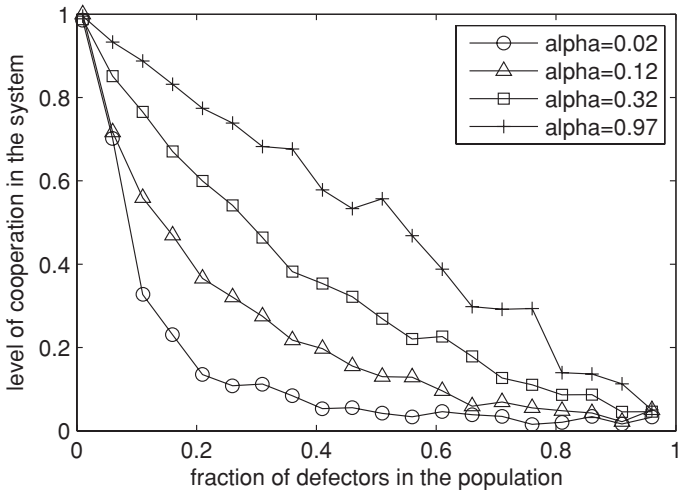

Fig. 1. A population of honest agents using one of the leading eight strategies (see Sect. 3.4) is pitted against a population of defectors, who always defect and propagate negative reputation information about others. We vary the number of defectors in the population and observe the level of cooperation in the system measured by the fraction of interactions amongst the honest agents in which both agents cooperate. Every reputation update is propagated to a fraction α of the whole population chosen uniformly at random. We repeat the experiment for different values of α. We can observe that if the reputation updates are propagated to a few agents only (2%) even a small number of badmouthing defectors can subvert cooperation. On the other hand, the reputation propagation rate set to 30% is sufficient to allow practically linear graceful decrease in cooperation level as the number of defectors increases

Question 2 *Is it necessary to propagate the reputation information to all the nodes to have a robust reputation system?*

Question 3 *How does the fraction of nodes to which reputation information is propagated influence the performance of the system?*

Question 4 *How to choose the fraction of nodes to which the reputation information is propagated?*

5 Bounded Rationality

Game theorists have considered imperfect monitoring games in which noise is allowed to occur in the system: imperfect observation of other players' actions, imperfect action execution, error-prone reputation information exchange, etc. While considering limitations of perception of the players, game theory still usually assumes that the players are absolutely rational. However, they may

have limited resources available to compute their behavior. Nash equilibria have been shown to be NP-hard to compute [11]. In the extreme, being unable to compute their behavior peers can behave entirely irrationally (randomly). This set of limitations is commonly termed *bounded rationality*. Conlisk [6] provides a plethora of empirical evidence from economics and experimental psychology in support of bounded rationality. The key observations are that:

- Bounded rationality can explain a number of empirical anomalies in economics for which unbounded rationality models fail
- Rationality is scarce, good decisions, are costly, they require both reliable information, which is difficult to obtain and computational power
- Bounded rationally leads people to imitate behaviors of others, which is cheaper than computing the behavior on their own

Given the predictive success of bounded rationality models, questions arise:

Question 5 *How can we incorporate bounded rationality into reputation models?*

Question 6 *What are the bounds on rationality in peer-to-peer systems and how can they influence the dynamics of cooperation and reputation?*

6 Behavioral Evolution

In the previous section we have already mentioned how imitation plays a role in selection of behaviors by agents. When a behavior is replicated its utility is locally evaluated by the agent. If the utility of the behavior is low it is promptly replaced by another behavior. This creates an evolutionary setting in which behaviors are replicated by imitation and selected by the agents for utility. An agent might use a set of behaviors (rules of behavior) and each of them can be individually imitated, creating a setting in which groups of mutually dependent behaviors co-evolve. The two main mechanisms of behavioral imitation in human societies are: payoff-biased transmission – imitating the behavior of the most successful individuals and conformist transmission – imitating the most frequent behavior [15].

How can we relate the above facts about behavioral evolution to interacting populations of selfish peers in peer-to-peer systems? First, we must clarify that it is not the peers that are selfish, but the human users of the peer-to-peer software. It is the users themselves who decide how their peers should behave. Hence, we could conjecture that a lot of the social mechanisms described above are driving the evolution of peer behaviors. This conjecture is confirmed by the following empirical evidence. Peer-to-peer file sharing software called eMule [2] is open source, which allows anyone to make modifications to it and distribute them. This has given rise to a number of mutated versions of the base eMule client, the so-called "mods" [1]. There are mods that protect the user privacy by encrypting downloaded data, there are mods which

implement various bandwidth saving heuristics, there are extremely uncooperative mods that cut off uploads to other peers to conserve bandwidth, there are mods that detect uncooperative mods and disconnect from them, there are even mods that detect those policing mods and use stealth techniques to hide their defection, etc. These mods are constantly created and propagated via numerous websites and evaluated by users on various electronic forums. The social network of peer-to-peer system users selecting behaviors for their peers is tightly interrelated with the overlay network providing an arena for the execution of those behaviors selected by the users. Up to our knowledge there have been no attempts to study these two networks as one entity with all their dependencies.

Question 7 *How can we model behavioral evolution in peer-to-peer systems?*

Question 8 *How can we model the peer-to-peer software choices and modifications made by humans and how do they affect the performance of the system?*

7 Second-Order Defection Problem

In an indirect reciprocity setting with cooperation being sustained by the means of reputation, there exists the following problem: in order for the reputation system to work, agents need to cooperate on exchanging reputation information and the information about the actions of other agents they have observed. Moreover, for the reputation system to be effective agents need to punish defectors which incurs additional costs. This creates a second-order cooperation problem, which could be solved by adding yet another reputation system on top of the existing one, but this in turn would lead to a third-order cooperation problem.

In peer-to-peer reputation systems research the problem is rarely explicitly addressed. The usual practice is to test the robustness of the system by introducing subpopulations of second-order defectors, i.e., peers that withhold or provide false reputation information. These evaluations only show that first-order cooperation can be sustained under a second-order defector invasion but it does not show that second-order cooperation is sustainable.

One of the game-theoretic solutions to this problem is the introduction of incentives [17] to motivate agents to share their reputation information truthfully. However the solution relies on a third party to handle the payments. This and many other similar approaches simply reformulate the problem of second-order defection and delegate it to another, normally centralized system component. Up to now there has been no self-contained distributed reputation system proposed that is free from the second-order defection problem.

There is, however, a natural system that appears to have solved that problem – human society. Biologists and psychologists studying indirect reciprocity among humans have been trying to find the exact reasons for the remarkable

stability of reputation and how it evolved [26,30]. Many hypotheses have been proposed, most notably:

- Group selection – Boyd et al. [27] suggest that cooperation can evolve by natural selection at the level of groups. Those groups that use reputation are more cooperative and hence more fit.
- Conformist transmission – Heinrich et al. [15] show how weak conformity in populations can lead to the stabilization of reputation exchange and cooperation.
- Costly signaling – Gintis et al. [29] show how using costly signals agents can advertise their quality as cooperators and in this way increase their reproductive success.

These mechanisms could be implemented and studied in artificial reputation systems potentially leading to increased performance and stability of second-order cooperation.

Question 9 *How can we apply the known reputation stability mechanisms from natural systems to engineering peer-to-peer systems free from the second-order defection problem?*

8 Inhomogeneous Interactions

In models of reputation systems it is frequently assumed that the structure of interactions between agents is homogeneous, i.e., each agent is equally likely to interact with any other agent. This assumption allows the construction of tractable analytical models. However, in practice the pattern of interactions in the system may not be homogeneous, which may produce large deviations from the predictions of the models. For example nodes that interact with a large number of other nodes may need to rely more on reputation information exchange and nodes that frequently interact with a small subset of nodes may rely more on bilateral tit-for-tat strategies and may have no incentive to share the reputation. These two types might need to coexist in the same network. More complex behaviors are possible. A group of nodes that are highly interacting with each other may choose to collude by artificially increasing each other's reputations but defecting with other nodes that are not part of the group. Once nonhomogeneous interactions are allowed there is no single winning behavior, such as Ohtsuka's leading eight or Axelrod's tit-for-tat. A complex set of mutually dependent behaviors can successfully coexist.

In overlay routing substrates the structure of interactions is normally determined by the underlying overlay maintenance algorithm – the interactions are packets forwarded by the nodes to their neighbors. In the case of DHTs the interactions are the key access and insertion requests, which are determined by the particular data placement strategy, normally a hash function. The inhomogeneities in the structure of interactions in any of those cases may warrant the existence of different equilibrium behaviors for different nodes.

Question 10 *What is the character of interaction inhomogeneities in peer-to-peer networks?*

Question 11 *How can those inhomogeneities influence the behavior of the selfish peers exchanging reputation information?*

So far, we have assumed that inhomogeneities arise from some external mechanism outside the peer's control. In general, however, a peer might decide what peers it interacts with based on its selfish choice. For example, a peer may choose to interact less frequently with low reputation peers. Selection of who to interact with becomes part of the peer behavior, which leads to a recursive problem: the structure of interactions determines the optimal behaviors at every node and the behaviors of nodes determine the structure of interactions.

A number of studies have looked at network formation by selfish peers [10,16,28]. However, all of the studies assume behavioral homogeneity of peers, i.e., all peers having the same utility function. Also, none of the studies consider both network formation and cooperation building via reputation as a single problem.

Question 12 *How can peers use the reputation information to choose what peers they want to interact with? What is the structure and dynamics of the resulting interaction network?*

9 Identity Stability

Most reputation systems rely on the assumption that identities of the agents are stable and can be reliably used. However, in contrast to human societies, identities in a peer-to-peer system are low cost and easy to change. A malicious peer whose reputation is low can leave the system and rejoin under a different identity thus clearing the whole history of its defections. A malicious peer can also assume a number of identities to have significant presence in the network [9]. Identity can also be stolen to take advantage of the reputation of the previous owner.

A well-known solution to the problem of identities is public key infrastructure. However, maintaining a hierarchy of trusted third parties creates scalability problems as well as introduces a single point of failure. Another widely employed solution is increasing the cost of identities by initializing the reputation of newly coming peers to a low value and making the peers gradually build their reputation. This, however, creates a disadvantage for short-lived peers who loose their identity every time they depart from the system and during their short lifetime are not able to accumulate enough reputation to gain any benefit from participation in the system.

When considering identity, researchers commonly assume one of the two extremes: either cheap, easy to change identities, or expensive, reliable ones. However, there exist cases which lie in between. For example, when two peers

open a TCP connection to communicate through it, the stability of the identities at both ends of the TCP link is guaranteed. This concept of pairwise identity stability can be extended to arbitrary groups of communicating peers within which peer identities are stable. Identity stability needs to be associated with a particular scope.

We may also add assumptions about partial perception of identity, i.e., a peer might only be able to determine that a node belongs to some larger group but not pinpoint exactly which node it is. For example, a node might be identified as belonging to a university campus, but the individual identity of the peer might be unknown. This creates new challenges and adds more complexity to the already wide range of possible behaviors in a reputation system. Up to our knowledge, partial perception of identities has not been considered in the context of reputation systems.

Question 13 *What are the minimal assumptions on the stability and perception of identity needed to construct a robust reputation system?*

Identity is inextricably linked with anonymity and privacy in peer-to-peer networks. Having accurate identity models might enable the designers to make more precise statements about the anonymity guarantees in their peer-to-peer systems [24].

Question 14 *Can cooperation be sustained while maintaining anonymity in a peer-to-peer system? What are the tradeoffs?*

10 Conclusions

Each of state-of-the-art approaches to reputation systems for peer-to-peer networks is based on a set of assumptions about the target deployment environment. We have demonstrated how breaking some of these fundamental assumptions leads to unexpected phenomena and complex peer behavior. Clearly, there exists no single universal solution that can work well in all distributed environments. Instead the properties of the environment should be precisely determined before designing a reputation system. We have identified the main dimensions along which these environment properties can be categorized:

- *Communication model* – how information propagates in the environment, how costly the propagation is. This influences the speed at which reputation information can be disseminated and how many peers it may reach.
- *Computational constraints* – how costly computation and local storage are. These assumptions determine the degree to which peers' rationality is bounded.
- *Peer software dynamics* – how selfish users deploy new software, how software is modified. These processes drive the behavioral evolution of the

system and put constraints on how fast new behaviors can be deployed or enforced in an existing system.
- *Interaction model* – how peers interact, to what degree they can choose their interaction partners. These properties of the system can strongly influence the reputation dynamics and the choice of the optimal behavior.
- *Identity model* – whether identity might change, how identity is created and represented and to what degree it can be accessed by other peers. This determines the level of privacy and anonymity and the precision at which statements about the reputation of individual peers can be made.
- *Peer goal dynamics* – what the goals of the peers are, how they change over time. This describes the behavioral heterogeneity of the population and at the same time groups of peers with malicious goals can be used to model many forms of attacks on the system.

All of these environment properties strongly influence the design choices that need to be made when constructing a reputation system. How do the environment properties constrain the performance of the reputation system? What are the combinations of environment properties that fundamentally prevent from building any cooperation in the system? What is the best formal model of the distributed target environment which allows to make precise statements about all of its properties? These and many other problems constitute a new and exciting agenda for trust and reputation research in peer-to-peer systems.

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Fuzzy Covering Relation and Ordering: An Abstract Approach

Branimir Šešelja

Summary. The aim of the paper is to introduce an abstract fuzzy relation generalizing "covering." These types of relations frequently appear in human activities and can be used for the construction of a fuzzy order. We start with the known connection between a crisp ordering relation and a corresponding covering relation on a same set. We also establish some new, less known connections among these.

Our aim is to define a fuzzy covering relation independently, as it appears in applications, and then to define the corresponding fuzzy ordering.

We consider fuzzy sets in a general way, as mappings from a set to a complete lattice. Then we define a fuzzy covering relation on a set deduced from the given partial order on the same set. We prove some properties of these. Next we start other way around: we take an abstract fuzzy (binary) relation θ, satisfying some of the mentioned properties. We prove that a fuzzy ordering relation can be defined, so that θ is precisely its fuzzy covering relation, provided that the underlying set is finite and the lattice is distributive.

Key words: Fuzzy order, Fuzzy covering.

1 Covering and Order in Crisp Case

As it is known, a crisp ordering relation $\leqslant$ on a set X is a subset of X^2 satisfying

reflexivity: for all $x \in X$, $x \leqslant x$;

antisymmetry: for all $x, y \in X$, $x \leqslant y$ and $y \leqslant x$ imply $x = y$ (or equivalently $x \neq y$ and $x \leqslant y$ imply $y \not\leqslant x$;

transitivity: for all $x, y, z \in X$, $x \leqslant y$ i $y \leqslant z$ imply $x \leqslant z$.

The relation $<$ is deduced from the ordering as usual:

$x < y$ if and only if $x \neq y$ and there is no z, such that $z \notin \{x, y\}$ and $x \leqslant z \leqslant y$.

If $\leqslant$ is a crisp ordering relation on a set X, then the covering relation $\prec$ on the same set is defined as follows:

$x \prec y$ if and only if $x < y$ and there is no z, such that $x < z < y$.

Connection of a covering relation and the order from which it is deduced is usually given by the following lemma.

Lemma 1. *If a and b are distinct elements of a finite ordered set $(P, \leqslant)$, then $a \leqslant b$ if and only if either $a \prec b$, or $a \prec c_1 \prec \cdots \prec c_n \prec b$, for some $c_1, \ldots, c_n \in P$.*

The following property of the covering relation deduced from the given order is needed in the sequel. It is not common in the literature, therefore we prove it here.

Lemma 2. *Let R be a crisp ordering relation on a set X, and $\prec$ the corresponding covering relation. Then, the following holds.*

For all $x, y_1, y_2, \ldots, y_n \in P$

if $x \prec y_1 \prec y_2 \prec \cdots \prec y_n$ then $(*)$

$x \neq y_i$ for $i = 1, 2, \ldots, n$ and $x \not\prec y_i$, for $i = 2, 3, \ldots, n$.

Proof. The first part is proved by induction on i. We prove that $x < y_i$ for all $i = 1, \ldots, n$, whence, by the definition of the relation $<$ it follows that $x \neq y_i$. Indeed, by $x \prec y_1$ it follows that $x < y_1$. Assume that $x < y_i$. Then $y_i \prec y_{i+1}$ implies $y_i < y_{i+1}$, whence, by transitivity of the relation $<$, $x < y_{i+1}$. Therefore $x < y_i$, then also $x \neq y_i$ for all $i = 1, \ldots, n$.

Since $x < y_1$ and $y_1 < y_i$ (for $i > 1$) (by the previously proven part), it is not true that $x \prec y_i$, proving the proposition. □

For the converse, we define a particular (crisp) binary relation on a set X, and we prove that it is possible to use it for the definition of an order on X, in the same way as it is done in Lemma 1 for the covering relation deduced from the given order.

Theorem 1. *Let $\prec$ be a binary relation on a set X, satisfying the property $(*)$ (preceding lemma). Then it is possible to define an ordering relation on X which for finite X coincides with the order whose covering relation is $\prec$.*

Proof. Let $\prec$ be a relation on a nonempty set X satisfying the property $(*)$. Define the relation $\leqslant$ on X as it is done in Lemma 1: for $a, b \in X$

$a \leqslant b$ if and only if either $a \prec b$, or $a \prec c_1 \prec \cdots \prec c_n \prec b$, for some $c_1, \ldots, c_n \in X$.

Now $\leqslant$ is obviously reflexive. It is antisymmetric since by the definition $(*)$, $x \leqslant y$ and $y \leqslant x$ holds if and only if $x = y$. Transitivity holds since two chains of elements appearing in $x \leqslant y$ and $y \leqslant z$ can be connected in a single chain from x to z.

If $\leqslant$ is the order constructed above by means of the relation $\prec$, then it is straightforward to check that the covering relation deduced from $\leqslant$ coincides with $\prec$. □

2 Lattice-Valued Fuzzy Order and Covering

If $(P, \leqslant)$ is a partially ordered set, poset, then infimum and supremum of $a, b \in P$ (if they exist) are denoted, respectively, by $a \wedge b$ and $a \vee b$. For infimum or supremum of a subset or a family of elements of P, we use the notation $\bigwedge Q$, $\bigvee x_i$, and so on. A poset in which every two-element subset has infimum and supremum is a *lattice*. A lattice L is *complete* if infimum and supremum exist for every subset of L. A complete lattice has the top and the bottom element, denoted, respectively, by 1 and 0. In particular, in a complete lattice $\bigvee \emptyset = 0$.

In this paper *fuzzy sets* are considered to be mappings from a finite or denumerable nonempty set X (domain) into a lattice L (co-domain). Consequently, fuzzy (binary) relations on a set X are mappings from X^2 to L. For more details about lattice-valued fuzzy sets as used in the present paper, see [3, 4].

A *fuzzy ordering* relation on a set X (there are several definitions, see, e.g., [1, 2], we adopt this one) is a mapping ρ from X^2 to L satisfying

reflexivity: for all $x \in L$, $\rho(x, x) = 1$;
antisymmetry: for all $x, y \in X$, if $x \neq y$, then $\rho(x, y) \wedge \rho(y, x) = 0$;
transitivity: for all $x, y, z \in X$, $\rho(x, y) \geqslant \rho(x, z) \wedge \rho(z, y)$.

In the following we define and investigate fuzzy covering relation deduced from a given fuzzy order.

Definition 1. *Let $\rho : X^2 \to L$ be a fuzzy ordering relation on the set X.*

Define a fuzzy relation $\theta_\rho : X^2 \to L$, as a subrelation of ρ, as follows:

$$\theta_\rho(x, y) := \begin{cases} 0, & \text{if } x = y \text{ or } \rho(x, z) \wedge \rho(z, y) > 0 \text{ for some } z \notin \{x, y\}; \\ \rho(x, y), & \text{if } \rho(x, z) \wedge \rho(z, y) = 0 \text{ for all } z \notin \{x, y\}. \end{cases}$$

We call the relation θ_ρ the fuzzy covering relation induced by the fuzzy ordering ρ.

Theorem 2. *Fuzzy covering relation θ_ρ induced by a fuzzy relation ρ as defined above, fulfils the following.*

For every $n \in \mathbb{N}$,

$$\begin{aligned} &\text{if } \theta_\rho(x_1, x_2) \wedge \theta_\rho(x_2, x_3) \wedge \cdots \wedge \theta_\rho(x_{n-1}, x_n) > 0, \text{ then} \\ &\quad x_1 \neq x_i, \text{ for all } i = 2, \ldots, n \quad \text{and} \\ &\quad \theta_\rho(x_1, x_i) = 0, \text{ for all } i = 3, \ldots, n. \end{aligned} \tag{$**$}$$

Proof. If

$$\theta_\rho(x_1, x_2) \wedge \theta_\rho(x_2, x_3) \wedge \cdots \wedge \theta_\rho(x_{n-1}, x_n) > 0,$$

then obviously we have that $\theta_\rho(x_i, x_{i+1}) > 0$ for every $i = 1, \ldots, n-1$, and hence $\theta_\rho(x_i, x_{i+1}) = \rho(x_i, x_{i+1}) > 0$ for each i. Therefore, since ρ is transitive, if $x = x_i$ for some $i \in \{2, \ldots, n\}$, we would have $\rho(x_1, x_{i-1}) \wedge \rho(x_{i-1}, x) > 0$, contradicting the antisymmetry of ρ. Thus, $x_1 \neq x_i$, for all $i = 2, \ldots, n$.

Suppose now that $\theta_\rho(x_1, x_i) > 0$, for some $i \in \{3, \dots, n\}$. Then, as before, $\theta_\rho(x_1, x_i) = \rho(x_1, x_i) > 0$. Again by the transitivity of ρ, we have that

$$\begin{aligned}\rho(x_1, x_i) &\geqslant \rho(x_1, x_{i-1}) \wedge \rho(x_{i-1}, x_i) \\ &\geqslant \cdots \geqslant \rho(x_1, x_2) \wedge \rho(x_2, x_3) \wedge \cdots \wedge \rho(x_{i-1}, x_i) > 0.\end{aligned}$$

But then, $\theta_\rho(x_1, x_i) = 0$, by the definition of θ_ρ. Therefore, the assumption $\theta_\rho(x_1, x_i) > 0$ leads to a contradiction, which proves the theorem. □

An abstract approach to covering, without referring to any order, is the following. Observe that we were able to formulate it under particular conditions on L.

Theorem 3. *Let X be a nonempty set, L a complete infinitely distributive lattice and θ a fuzzy relation on X satisfying the property $(**)$ above. Then there exists a fuzzy ordering relation ρ_θ defined by relation θ so that its fuzzy covering relation is θ.*

Proof. Let X be a finite set, L a complete distributive lattice, and $\theta : X^2 \to L$ an L-fuzzy relation on X, satisfying the condition $(**)$. Define the fuzzy relation $\rho_\theta : X^2 \to L$ in the following way: for all $x, y \in X$

$$\begin{aligned}&\rho_\theta(x, x) := 1, \quad \text{and if } x \neq y, \text{ then} \\ &\rho_\theta(x, y) := \bigvee_{n \in \mathbb{N}, x_i \in X} (\theta(x, x_1) \wedge \theta(x_1, x_2) \wedge \cdots \wedge \theta(x_{n-1}, x_n) \wedge \theta(x_n, y)),\end{aligned}$$

where $\bigvee$ runs over all finite sequences $x_1, \dots, x_n \in X$, for each $n \in \mathbb{N}$; we allow also the empty sequence in which case there is only $\theta(x, y)$ on the right side.

By the definition ρ_θ is reflexive.

It is antisymmetric: if $x \neq y$, then, since L is distributive,

$$\begin{aligned}\rho_\theta(x, y) \wedge \rho_\theta(y, x) &= \\ &= \bigvee(\theta(x, x_1) \wedge \cdots \wedge \theta(x_n, y)) \wedge \bigvee(\theta(y, y_1) \wedge \cdots \wedge \theta(y_m, x)) \\ &= \bigvee(\theta(x, z_1) \wedge \cdots \wedge \theta(z_p, y) \wedge \theta(y, z_{p+1}) \wedge \cdots \wedge \theta(z_q, x)) = 0,\end{aligned}$$

by the condition $(**)$ (supremum is taken over all corresponding sequences).

Transitivity: Observe first that

$$\rho_\theta(x, y) = \bigvee(\theta(x, x_1) \wedge \cdots \wedge \theta(x_n, y)).$$

On the other hand, applying distributivity of L, similarly as in the proof of antisymmetry, we get

$$\rho_\theta(x, z) \wedge \rho_\theta(z, y) = \bigvee(\theta(x, z_1) \wedge \cdots \wedge \theta(z_p, z) \wedge \theta(z, z_{p+1}) \wedge \cdots \wedge \theta(z_q, y)). \quad (1)$$

Obviously,

$$\rho_\theta(x,y) \geqslant \rho_\theta(x,z) \wedge \rho_\theta(z,y).$$

Next we prove that the fuzzy covering relation deduced from ρ_θ, denoted by θ_{ρ_θ}, is θ. Indeed, by Definition 1, $\theta_{\rho_\theta}(x,x) = 0$, but also $\theta(x,x) = 0$ (this is implicit in $(**)$). If $x \neq y$ and $\rho_\theta(x,z) \wedge \rho_\theta(z,y) > 0$ for some $z \notin \{x,y\}$, then also $\theta_{\rho_\theta}(x,y) = 0$. On the other hand, since $\rho_\theta(x,z) \wedge \rho_\theta(z,y) > 0$, then by (1) and by $(**)$, $\theta(x,y) = 0$, equally as θ_{ρ_θ}. Finally, if $\rho_\theta(x,z) \wedge \rho_\theta(z,y) = 0$ for all $z \notin \{x,y\}$, then $\theta_{\rho_\theta}(x,y) = \rho_\theta(x,y)$. But $\rho_\theta(x,z) \wedge \rho_\theta(z,y) = 0$ means that the join on the right-hand side of (1) equals 0 whenever there is a corresponding chain from x to y. Therefore, $\rho_\theta(x,y) = \theta(x,y)$, proving that in this final case also $\theta_{\rho_\theta}(x,y) = \theta(x,y)$. Hence, we have $\theta_{\rho_\theta} = \theta$. □

3 Examples

Example 1. Let $X = \{x,y,z,u,v\}$, and let the lattice L be the unit interval $[0,1]$. A fuzzy ordering relation ρ and the corresponding fuzzy covering are given by the tables.

ρ	x	y	z	u	v
x	1	0.2	0.2	0.2	0
y	0	1	0	0.5	0
z	0	0	1	0.5	0
u	0	0	0	1	0
v	0	0	0.1	0.4	1

θ_ρ	x	y	z	u	v
x	0	0.2	0.2	0	0
y	0	0	0	0.5	0
z	0	0	0	0.5	0
u	0	0	0	0	0
v	0	0	0.1	0	0

Example 2. Let $X = \{x,y,z,u,v\}$ as above, and let L be the distributive lattice presented by its diagram in Fig. 1.

An L-fuzzy relation $\theta : X^2 \to L$, satisfying condition $(**)$ is given in the first table; the second is the table of the corresponding ordering relation ρ_θ.

It is easy to check that the covering relation θ_{ρ_θ}, induced by the fuzzy order ρ_θ, equals θ.

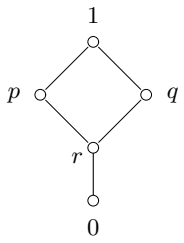

Fig. 1. Distributive lattice L

θ	x	y	z	u	v
x	0	p	q	0	0
y	0	0	0	0	q
z	0	0	0	p	q
u	0	0	0	0	0
v	0	0	0	0	0

ρ_θ	x	y	z	u	v
x	1	p	q	r	q
y	0	1	0	0	q
z	0	0	1	p	q
u	0	0	0	1	0
v	0	0	0	0	1

4 Conclusion

It is known that the covering relation induced by an order gives some local properties of the order (e.g., being a parent in the ordered tree of someone's ancestors). Fuzzy orders and hence also the corresponding coverings are even more connected with applications. Therefore, the approach presented here might give a new insight into this important part of fuzzy relational calculus: construction of a fuzzy order by means of an abstract relation which turns out to be the corresponding cover.

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Measures of Differentiability

Martin Kalina and Alexander Šostak

Summary. In this paper measures of differentiability, based on nearness relations, are defined. The behavior of these measures with respect to derivatives of a sum, product and quotient of functions, as well as their behavior with respect to the chain rule is studied.

1 Introduction and Preliminaries

Burgin and Šostak in [2] introduced measures of continuity, which in some sense fuzzified the notion of continuity. This was further developed in [1]. This concept was also studied by Janiš in [4–6]. In this paper we intend to introduce measures of differentiability, based on the notion of a nearness relation. Nearness-based derivatives were defined in [8] and further developed in [9]. A slightly different way to nearness-based differentiability was followed by Janiš in [7]. This paper describes a new approach to the nearness-based differentiability of crisp functions introducing the measures of differentiability.

First, let us recall some notions, which are essential for our considerations.

Definition 1 *[See e.g. [5]] Let $f : [0,1] \longrightarrow [0,\infty]$ be a non-increasing function. Then $f^{(-1)} : [0,\infty] \longrightarrow [0,1]$ is said to be the pseudoinverse of f iff*

$$f^{(-1)}(z) = \sup\{x \in [0,1]; f(x) > z\}.$$

Lemma 1 *Let $f : [0,1] \longrightarrow [0,\infty]$ be a continuous non-increasing function. Then*

$$f\left(f^{(-1)}(y)\right) = y. \tag{1}$$

Proof. Assume f is strictly decreasing at $f^{(-1)}(y)$. Then formula (1) obviously holds. Let $f^{(-1)}(y) \in [a,b]$ such that f is constant in $[a,b]$. Then, by Definition 1, $f^{(-1)}(y) = a$ (since f is continuous) and hence $f(a) = y$, what was to be proved.

For the purpose of this paper we slightly modify the definition of a nearness relation. We denote $R^* = R \cup \{\pm\infty\}$

Definition 2 $\mathcal{N} : R^* \times R^* \longrightarrow [0;1]$ *is a nearness relation iff the following hold*

1. $\mathcal{N}(x,x) = 1$ *for all* $x \in R^*$
2. $\mathcal{N}(x_1,x_2) = \mathcal{N}(x_2,x_1)$ *for all* $x_1, x_2 \in R^*$
3. *Let* $x_1 \le x_2 \le x_3 \le x_4$. *Then there holds*
$$\mathcal{N}(x_1,x_4) \le \mathcal{N}(x_2,x_3)$$
4. *All* $x \in R$ *yield*
$$\lim_{t \longrightarrow \pm\infty} \mathcal{N}(x,t) = 0$$

Property 4 of nearness relations implies the following

Lemma 2 *Let* $\mathcal{N} : R^* \times R^* \longrightarrow [0;1]$ *be a nearness relation. Then for all* $x \in R$ *the following hold*

$$\mathcal{N}(x,\infty) = 0, \quad \mathcal{N}(x,-\infty) = 0$$

There are two important classes of nearness relations:

Definition 3 *A nearness relation* $\mathcal{N}$ *is called strict iff the following is satisfied:*

$$\mathcal{N}(x_1,x_2) = 1 \qquad \Leftrightarrow \qquad x_1 = x_2.$$

The nearness relation $\mathcal{N}$ *is called shift-invariant iff the following is satisfied for all* $z \in X$*:*

$$\mathcal{N}(x_1,x_2) = \mathcal{N}(x_1+z, x_2+z).$$

Let us give some examples of nearness relations:

Example 1 a. Let $\rho : R^* \times R^* \longrightarrow [0,\infty]$ be a shift-invariant metric with the property

$$\rho(\infty,x) = \infty \quad \text{if} \quad x \neq \infty$$

and the same holding also for $-\infty$. Then

$$\mathcal{N}_1(x_1,x_2) = \max\{0, 1-\rho(x_1,x_2)\}$$
$$\mathcal{N}_2(x_1,x_2) = e^{-\rho(x_1,x_2)}$$
$$\mathcal{N}_3(x_1,x_2) = \begin{cases} 1, & \text{if } x_1 = x_2 \\ \max\{0, \frac{1}{2} - \rho(x_1,x_2)\} & \text{if } x_1 \neq x_2 \end{cases}$$
$$\mathcal{N}_4(x_1,x_2) = \begin{cases} 1, & \text{if } \rho(x_1,x_2) \le 1 \\ \max\{0, 2-\rho(x_1,x_2)\} & \text{if } \rho(x_1,x_2) > 1 \end{cases}$$

are nearness relations based on the metric. They are shift-invariant.

b.

$$\mathcal{N}_5(x_1,x_2) = \begin{cases} 1, & \text{if } x_1 = x_2 = 0 \\ \min\{\frac{x_1}{x_2}; \frac{x_2}{x_1}\} & \text{if } x_1 x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

is a nearness relation. Here, the degree, to which two elements are near to each, is not given by their distance. This nearness is not shift-invariant.
All presented nearness relations, except $\mathcal{N}_4$, are strict.

As a direct corollary to Definitions 1, 2, and 3 we get the following:

Lemma 3 *Let $f : [0,1] \longrightarrow [0,\infty]$ be a non-increasing function and $g : [0,\infty] \longrightarrow [0,1]$ be its pseudoinverse. If (R^*, ρ) is a metric space, then $\mathcal{N} : R^* \times R^* \longrightarrow [0,1]$ defined by*

$$\mathcal{N}(x_1, x_2) = g\left(\rho(x_1, x_2)\right)$$

is a shift-invariant nearness relation.

2 Measures of Differentiability

Definition 4 *Let $g : R \longrightarrow R$ be a continuous function. Let us fix a point $x \in R$. Let $S = (s_n)_n$ be a sequence of real numbers such that*

$$(\forall n)(s_n \neq x) \qquad \lim_{n \longrightarrow \infty} s_n = x \tag{2}$$

Take some nearness relation $\mathcal{N}$. Denote

$$g'_m(S) = \liminf_{n \longrightarrow \infty} \frac{g(s_n) - g(x)}{s_n - x}, \qquad g'_M(S) = \limsup_{n \longrightarrow \infty} \frac{g(s_n) - g(x)}{s_n - x}. \tag{3}$$

Then we say that $\mathcal{N}_S(g'(x)) = \mathcal{N}\left(g'_m(S), g'_M(S)\right)$ is the sequential measure of differentiability of g at x.

Example 2 Consider the nearness relations $\mathcal{N}_1$ and $\mathcal{N}_2$ from Example 1. Let

$$S = \left((-1)^n \frac{1}{n}\right)_n.$$

Let $f_1(x) = |x|$. Then the sequential measures of differentiability of f_1 at 0 with respect to the sequence S and the nearness relations $\mathcal{N}_1$ and $\mathcal{N}_2$ are, respectively,

$$\mathcal{N}_{1S}(f'_1(0)) = \mathcal{N}_1(-1,1) = 0$$
$$\mathcal{N}_{2S}(f'_1(0)) = \mathcal{N}_2(-1,1) = \exp(-2)$$

Let $f_2(x) = \frac{1}{3}|x|$. Then the corresponding sequential measures at 0 are

$$\mathcal{N}_{1S}(f'_2(0)) = \mathcal{N}_1\left(-\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$$
$$\mathcal{N}_{2S}(f'_2(0)) = \mathcal{N}_2\left(-\frac{1}{3}, \frac{1}{3}\right) = \exp\left(-\frac{2}{3}\right)$$

Let

$$f_3(x) = \begin{cases} \sqrt{2x - x^2} \text{ if } x \in [0,2] \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

Then both corresponding sequential measures at 0 are equal to zero.
Let

$$f_4(x) = \begin{cases} \sqrt{2x - x^2} & \text{if } x \in [0, 2] \\ -\sqrt{-2x - x^2} & \text{if } x \in [-2, 0[\\ 0 & \text{otherwise} \end{cases}$$

Then both corresponding sequential measures at 0 are equal to 1 since

$$\mathcal{N}_1(\infty, \infty) = \mathcal{N}_2(\infty, \infty) = 1.$$

Theorem 1 *Let $g : R \longrightarrow R$ and $h : R \longrightarrow R$ be continuous functions, x some fixed point, S a sequence fulfilling condition (2) and let f be a continuous additive generator of a t-norm $*_f$. Put $\mathcal{N}(x_1, x_2) = f^{(-1)}(\rho(x_1, x_2))$, where ρ is a given metric. Then the following hold:*

$$\mathcal{N}_S(g'(x) + h'(x)) \geq \mathcal{N}_S(g'(x)) *_f \mathcal{N}_S(h'(x)) \tag{4}$$

$$\mathcal{N}_S((g(x).h(x))') \geq \mathcal{N}_S(g'(x).h(x)) *_f \mathcal{N}_S(g(x).h'(x)) \tag{5}$$

$$\mathcal{N}_S\left(\left(\frac{g(x)}{h(x)}\right)'\right) \geq \mathcal{N}_S\left(\frac{g'(x)}{h(x)}\right) *_f \mathcal{N}_S\left(\frac{g(x).h'(x)}{h^2(x)}\right) \tag{6}$$

In the last item we assume $h(x) \neq 0$.

Proof. The left-hand-side of formula (4) is the following:

$$\mathcal{N}((g+h)'_m(S), (g+h)'_M(S)) = f^{(-1)}(\rho((g+h)'_m(S), (g+h)'_M(S))) \geq$$
$$f^{(-1)}(\rho(g'_m(S) + h'_m(S), g'_M(S) + h'_M(S))) \geq$$
$$f^{(-1)}(\rho(g'_m(S), g'_M(S)) + \rho(h'_m(S), h'_M(S)))$$

since f, and hence also $f^{(-1)}$, is a non-increasing function. Applying Lemma 1 we get

$$f^{(-1)}(\rho(g'_m(S), g'_M(S)) + \rho(h'_m(S), h'_M(S))) =$$
$$f^{(-1)}\left(f\left(f^{(-1)}(\rho(g'_m(S), g'_M(S)))\right) + f\left(f^{(-1)}(\rho(h'_m(S), h'_M(S)))\right)\right) =$$
$$\mathcal{N}_S(g'(x)) *_f \mathcal{N}_S(h'(x))$$

since f is the additive generator of $*_f$.

The left-hand-side of formula (5) is the following:

$$\mathcal{N}((g \cdot h)'_m(S), (g \cdot h)'_M(S)) = f^{(-1)}(\rho((g \cdot h)'_m(S), (g \cdot h)'_M(S))) \geq$$
$$f^{(-1)}(\rho(g'_m(S) \cdot h(x) + h'_m(S) \cdot g(x), g'_M(S) \cdot h(x) + h'_M(S) \cdot g(x))) \geq$$
$$f^{(-1)}(\rho(g'_m(S) \cdot h(x), g'_M(S) \cdot h(x)) + \rho(h'_m(S) \cdot g(x), h'_M(S) \cdot g(x)))$$

since g and h are continuous functions at x and $f^{(-1)}$ is a non-increasing function. Now, we can proceed exactly as in the previous case and we get formula (5).

Formula (6) is just a modification of formula (5), just applied to functions g and $\frac{1}{h}$. □

Theorem 2 *Let $g : R \longrightarrow R$ and $h : R \longrightarrow R$ be continuous functions, x some fixed point. Put, for some $k > 0$, the following nearness relation:*

$$\mathcal{N}^k(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 = x_2 = 0 \\ \min\left\{ \left(\frac{x_1}{x_2}\right)^k ; \left(\frac{x_2}{x_1}\right)^k \right\} & \text{if } x_1 x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then for a sequence $S = (s_n)_n$, fulfilling condition (2), we get

$$\mathcal{N}_S^k\left((g(h(x)))'\right) \geq \mathcal{N}_S^k(g'(z)) \cdot \mathcal{N}_S^k(h'(x)) \tag{7}$$

where $z = h(x)$.

Proof. Let us denote $V = (h(s_n))_n$. We will discuss several cases.

- First, assume that $h'_m(S) > 0$ and $g'_m(V) > 0$. Then we get

$$\mathcal{N}_S^k\left((g(h(x)))'\right) = \mathcal{N}^k(g(h)'_m(S), g(h)'_M(S)) = \frac{g(h)'_m(S)}{g(h)'_M(S)} \geq$$
$$\frac{g'_m(V)}{g'_M(V)} \cdot \frac{h'_m(S)}{h'_M(S)} = \mathcal{N}^k(g'_m(V), g_M(V)) \cdot \mathcal{N}^k(h'_m(S), h'_M(S)) =$$
$$\mathcal{N}_S^k(g'(z)) \cdot \mathcal{N}_S^k(h'(x))$$

- Assume that $h'_M(S) < 0$ and $g'_M(V) < 0$. This case can be treated similarly as the first one, just we must keep in mind that now

$$|h'_M(S)| < |h'_m(S)| \quad \text{and} \quad |g'_M(V)| < |g'_m(V)|.$$

- Assume that $h'_M(S) < 0$ and $g'_m(V) > 0$. Then $g(h)'_M(S) < 0$ and we get

$$\mathcal{N}_S^k\left((g(h(x)))'\right) = \mathcal{N}^k(g(h)'_m(S), g(h)'_M(S)) = \frac{g(h)'_M(S)}{g(h)'_m(S)} \geq$$
$$\frac{g'_m(V)}{g'_M(V)} \cdot \frac{h'_M(S)}{h'_m(S)} = \mathcal{N}^k(g'_m(V), g_M(V)) \cdot \mathcal{N}^k(h'_m(S), h'_M(S)) =$$
$$\mathcal{N}_S^k(g'(z)) \cdot \mathcal{N}_S^k(h'(x))$$

- Assume that $h'_m(S) > 0$ and $g'_M(V) < 0$. Then $g(h)'_M(S) < 0$ and we get

$$\mathcal{N}_S^k\left((g(h(x)))'\right) = \mathcal{N}^k(g(h)'_m(S), g(h)'_M(S)) = \frac{g(h)'_M(S)}{g(h)'_m(S)} \geq$$
$$\frac{g'_M(V)}{g'_m(V)} \cdot \frac{h'_m(S)}{h'_M(S)} = \mathcal{N}^k(g'_m(V), g_M(V)) \cdot \mathcal{N}^k(h'_m(S), h'_M(S)) =$$
$$\mathcal{N}_S^k(g'(z)) \cdot \mathcal{N}_S^k(h'(x))$$

In all other cases we get either $\mathcal{N}^k(h'_m(S), h'_M(S)) = 0$ or $\mathcal{N}^k(g'_m(V), g'_M(V)) = 0$ and hence formula (7) is fulfilled. □

Theorem 3 *Let $\mathcal{N}$ be a strict and shift-invariant nearness relation such that there exist some points $a \neq b$ with $\mathcal{N}(a,b) > 0$. Further, let $x \in R$ be some fixed point. Then for each t-norm $*$ there exist a couple of continuous functions $g : R \longrightarrow R$ and $h : R \longrightarrow R$ and sequence $S = (s_n)_n$, fulfilling condition (2), such that the following formula holds:*

$$\mathcal{N}_S\left((g(h(x)))'\right) < \mathcal{N}_S(g'(z)) \ * \ \mathcal{N}_S(h'(x)), \tag{8}$$

where $z = h(x)$.

Proof. Without loss of generality we can assume that $\mathcal{N}(-1,1) > 0$. Put $h(x) = |x|$ and $g_k(x) = kx$ for $k > 0$. Then, for a sequence S converging to 0 and possessing infinitely many elements bigger then 0 and infinitely many elements less then 0, the following holds

$$\begin{aligned} \mathcal{N}_S\left((g(h(0)))'\right) &= \mathcal{N}(-k,k) \\ \mathcal{N}_S\left(g'(0)\right) &= 1 \\ \mathcal{N}_S\left(h'(0)\right) &= \mathcal{N}(-1,1) > 0, \end{aligned}$$

hence

$$\mathcal{N}_S(g'(0)) \ * \ \mathcal{N}_S(h'(0)) = \mathcal{N}(-1,1) > 0. \tag{9}$$

It is enough to choose k such that

$$\mathcal{N}(-k,k) < \mathcal{N}(-1,1) \tag{10}$$

and the assertion in question is proved. The existence of such k is implied by property 4 of nearness relations. □

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Lipschitz Continuity of Triangular Norms

Andrea Mesiarová

Summary. In this contribution we will study t-norms which are stable with respect to l_p norms (where $p \in [1, \infty]$) and the properties of classes of l_p-stable t-norms. We will also study transformations which preserve the l_1-stability of t-norms.

Key words: Triangular norm, Lipschitz property, l_p norm.

1 Introduction

Estimation of errors in outputs of processing by means of t-norms heavily depends on the input errors. However, to measure the deviation of observed input (x_1, y_1) and the real input (x_0, y_0), several types of norms on $\mathbb{R}^2$ can be chosen. For a given norm $D : \mathbb{R}^2 \to [0, \infty[$, a t-norm $T : [0, 1]^2 \to [0, 1]$ is called D-Lipschitz if and only if there is a real constant $k \in]0, \infty[$ such that

$$|T(x_1, y_1) - T(x_0, y_0)| \leq kD(x_1 - x_0, y_1 - y_0), \tag{1}$$

for all $(x_1, y_1), (x_0, y_0) \in [0, 1]^2$. Vice versa a t-norm satisfying (1) is called k-D-Lipschitz. Obviously, each D-Lipschitz t-norm is continuous.

The aim of this contribution is to discuss D-Lipschitz t-norms for $D = l_p$, where $p \in [1, \infty[$ and l_p is a norm given by

$$l_p(x, y) = \sqrt[p]{|x|^p + |y|^p},$$

and for the Chebyschev norm $D = l_\infty$, $l_\infty(x, y) = \max(|x|, |y|)$. Note that for $p = 1$, we obtain in (1) the standard Lipschitz property (in this case we will say that the corresponding t-norm is Lipschitz). We will give the characterization (in the case of $p = 1$ full and partial otherwise) of the class of k-l_p-Lipschitz t-norms for $k \in]0, \infty[, p \in [1, \infty]$ and study the boundary properties of these classes. Note that for $p = 1, k = 1$ we obtain the class of associative copulas, which was already completely characterized

in [6, 7] as ordinal sums of t-norms with convex additive generators. Note also that the first steps in discussion of 1-l_p-Lipschitz t-norms were introduced in [2]. The characterization of the class of k-Lipschitz t-norms for $k > 1$ which can be found in [3, 5] is an answer to an open problem posed by Alsina, Frank and Schweizer in [1]. Note that a partial answer to this problem for k-Lipschitz t-norms generated by differentiable additive generators was given by Shyu [8]. Finally, for k-Lipschitz t-norms we will study transformations which preserve Lipschitz property (possibly with different constant k).

Because of neutral element it is obvious that for all $p \in [1, \infty]$ a t-norm can be k-l_p-Lipschitz only for $k \in [1, \infty[$. Let us denote by $\mathcal{T}_{k,l_p}$ ($\mathcal{T}_k$) the class of all k-l_p-Lipschitz (k-Lipschitz) t-norms. Since for $p, q \in [1, \infty]$, $p \leq q$, l_p-norm is stronger then l_q-norm, i.e., $l_p(x, y) \geq l_q(x, y)$ for all $(x, y) \in [0, 1]^2$ we know that $\mathcal{T}_{k,l_q} \subset \mathcal{T}_{k,l_p}$ for all $k \in [1, \infty[$. Moreover, also $\mathcal{T}_{k,l_p} \subset \mathcal{T}_{m,l_p}$ for all $k, m \in [1, \infty[$, $k \leq m$, $p \in [1, \infty]$. All classes $\mathcal{T}_{k,l_p}$ are compact ($\mathcal{T}_{k,l_q} \subset \mathcal{T}_k$ which is compact).

Because of continuity, each k-l_p-Lipschitz t-norm can be represented as an ordinal sum of Archimedean t-norms. It is obvious that if an ordinal sum t-norm is k-l_p-Lipschitz every summand in ordinal sum is k-l_p-Lipschitz, too.

Proposition 1. *Let $(T_\alpha)_{\alpha \in \mathcal{A}}$ be a family of k-l_p-Lipschitz t-norms and let T be an ordinal sum of these t-norms, i.e., $T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}$ for some family of pairwise disjoint open subintervals $(]a_\alpha, e_\alpha[)_{\alpha \in A}$. Then t-norm T is a k-l_p-Lipschitz t-norm.*

Proof. From the properties of ordinal sums we know that

$$|T(x_1, y_1) - T(x_0, y_0)| \leq k \cdot l_p(x_1 - x_0, y_1 - y_0)$$

for all $(x_0, y_0), (x_1, y_1) \in [a_\alpha, e_\alpha]^2$ for some $\alpha \in \mathcal{A}$ (this follows from the fact that, for each $\alpha \in \mathcal{A}$, T_α and $T|_{[a_\alpha, e_\alpha]^2}$ have the same Lipschitz properties with respect to any l_p norm) and for all $(x_0, y_0), (x_1, y_1) \notin]a_\alpha, e_\alpha[^2$ for all $\alpha \in \mathcal{A}$ (this follows form the fact that min is 1-l_p-Lipschitz for any $p \in [1, \infty]$). Assume that $(x_0, y_0) \in [a_{\alpha_0}, e_{\alpha_0}]^2$ and $(x_1, y_1) \in [a_{\alpha_1}, e_{\alpha_1}]^2$ for some $\alpha_0, \alpha_1 \in \mathcal{A}, \alpha_0 \neq \alpha_1$. Then the line connecting points (x_0, y_0) and (x_1, y_1) can be divided to several parts which are subsets of squares $[a_\alpha, e_\alpha]^2$ and the rest of the line which is not a subset of any of such squares. Assume that this line consist of three parts: the first part is a subset of $[a_{\alpha_0}, e_{\alpha_0}]^2$, the second is a subset of $[a_{\alpha_1}, e_{\alpha_1}]^2$ and the rest is not a subset of any square $[a_\alpha, e_\alpha]^2$ (all other cases can be proved analogically). Denote by $(x_2, y_2), (x_3, y_3)$ the points which separate the first and the third part, and the third and the second part of the line, respectively. Then $(x_2, y_2) = \beta(x_0, y_0) + (1 - \beta)(x_1, y_1)$ and $(x_3, y_3) = \gamma(x_0, y_0) + (1 - \gamma)(x_1, y_1)$ for some $\beta, \gamma \in [0, 1], \beta \geq \gamma$. Moreover,

$$\begin{aligned}|T(x_1,y_1)-T(x_0,y_0)| &= |T(x_1,y_1)-T(x_3,y_3)|+|T(x_3,y_3)\\ &\quad -T(x_2,y_2)|+|T(x_2,y_2)-T(x_0,y_0)|\\ &\le k\cdot(l_p(x_1-x_3,y_1-y_3)+l_p(x_3-x_2,y_3-y_2)\\ &\quad +l_p(x_2-x_0,y_2-y_0))=k\cdot(\gamma l_p(x_1-x_0,y_1-y_0)\\ &\quad +(\beta-\gamma)l_p(x_1-x_0,y_1-y_0)\\ &\quad +(1-\beta)l_p((x_1-x_0,y_1-y_0))\\ &= k\cdot l_p(x_1-x_0,y_1-y_0).\end{aligned}$$

Proposition 1 shows that the problem of characterization of k-l_p-Lipschitz t-norms can be reduced to the characterization of continuous strictly decreasing functions $t:[0,1]\to[0,\infty]$, $t(1)=0$, such that the two-place function $T:[0,1]^2\to[0,1]$ given by

$$T(x,y)=t^{-1}(\min(t(x)+t(y),t(0))) \tag{2}$$

is a k-l_p-Lipschitz t-norm.

2 k-Lipschitz t-Norms

In the case of k-Lipschitz t-norms it is important that Lipschitz property can be reduced to one coordinate, i.e., that a t-norm T is k-Lipschitz if and only if $|T(x+\varepsilon,y)-T(x,y)|\le k\varepsilon$ for all $x,y\in[0,1]$, $\varepsilon\in[0,1-x]$. The following definition of a k-convex function as well as subsequent results about k-Lipschitz t-norms can be found in [3].

Definition 1. *Let $t:[0,1]\to[0,\infty]$ be a strictly monotone function and let $k\in]0,\infty[$ be a real constant. Then t will be called k-convex if*

$$t(x+k\varepsilon)-t(x)\le t(y+\varepsilon)-t(y) \tag{3}$$

holds for all $x\in[0,1[\,,y\in]0,1[\,,$ with $x\le y$ and $\varepsilon\in\left]0,\min(1-y,\frac{1-x}{k})\right]$.

It is evident that if a strictly monotone function $t:[0,1]\to[0,\infty]$ is k-convex then it is continuous on $]0,1[$ (more precisely, each decreasing k-convex function is continuous on $]0,1]$ and each increasing k-convex function is continuous on $[0,1[$). Note that a decreasing function can be k-convex only for $k\ge 1$. Moreover, when a decreasing function t is k-convex it is l-convex for all $l\ge k$.

Theorem 1. *Let $T:[0,1]^2\to[0,1]$ be an Archimedean t-norm and let $t:[0,1]\to[0,\infty]$ be an additive generator of T. Then T is k-Lipschitz if and only if t is k-convex.*

Corollary 1 ([8]). *Let $t : [0,1] \to [0,\infty]$ be an additive generator of a t-norm T, differentiable on $]0,1[$ and let $t'(x) < 0$ for all $0 < x < 1$. Then T is k-Lipschitz if and only if $t'(y) \geq kt'(x)$ whenever $0 < x < y < 1$.*

Theorem 1 imply that generated k-Lipschitz t-norms satisfy $T(x+k\varepsilon, y) \geq T(x, y+\varepsilon)$ for all $x, y \in [0,1], x \leq y$, $\varepsilon \in \left]0, \min(1-y, \frac{1-x}{k})\right]$, what can be (because of ordinal sum structure) shown also for all k-Lipschitz t-norms. For $k = 1$ this property is exactly the Schur-concavity of a t-norm.

Unfortunately in the case of k-l_p-Lipschitz t-norms the situation cannot be reduced to one coordinate (for one coordinate we obtain the standard Lipschitz property) and thus a similar proof is out of the use. In [2] the following conjecture can be found:

Conjecture 1 *A t-norm $T : [0,1]^2 \to [0,1]$ generated by an additive generator t is 1-l_p-Lipschitz if and only if there exist a convex function $g : [0,1] \to [0,\infty]$ such that $g^p(x) = t(x)$.*

When we assume the same increment in both coordinates we obtain the following necessary condition:

Proposition 2. *An additive generator $t : [0,1] \to [0,\infty]$ of a k-l_p-Lipschitz t-norm T satisfy for all $(x,y) \in [0,1]^2$ such that $t(x) + t(y) \leq t(0)$, $\varepsilon \in \left]0, \min(1-x, 1-y, \frac{1-T(x,y)}{k \cdot 2^{\frac{1}{p}}})\right]$*

$$t(x+\varepsilon) - t(x) + t(y+\varepsilon) - t(y) \geq t(T(x,y) + k \cdot 2^{\frac{1}{p}}\varepsilon) - t(T(x,y)). \qquad (4)$$

Proof. Assume $(x,y) \in]0,1[^2$ such that $t(x) + t(y) \leq t(0)$, and $0 < \varepsilon \leq \min(1-x, 1-y, \frac{1-T(x,y)}{k \cdot 2^{\frac{1}{p}}})$. Then $t(T(x,y)) = t(x) + t(y)$. Since T is k-l_p-Lipschitz we have

$$T(x+\varepsilon, y+\varepsilon) - T(x,y) \leq k2^{\frac{1}{p}}\varepsilon,$$

i.e.,

$$T(x+\varepsilon, y+\varepsilon) \leq T(x,y) + k2^{\frac{1}{p}}\varepsilon.$$

Subsequently

$$t(x+\varepsilon) + t(y+\varepsilon) = t(T(x+\varepsilon, y+\varepsilon)) \geq t(T(x,y) + k2^{\frac{1}{p}}\varepsilon).$$

If we subtract from the left side of the above inequality the term $t(x) + t(y)$ and from the right side the term $t(T(x,y))$ we obtain the desired inequality.

Corollary 2. *A differentiable additive generator $t : [0,1] \to [0,\infty]$ of a k-l_p-Lipschitz t-norm T satisfy for all $(x,y) \in]0,1[^2$*

$$t'(x) + t'(y) \geq k \cdot 2^{\frac{1}{p}} t'(T(x,y)).$$

Proof. Since T is k-l_p-Lipschitz, the additive generator t fulfills inequality (4) and we have:

$$t'(x) + t'(y) = \lim_{\varepsilon \to 0} \frac{t(x+\varepsilon) - t(x)}{\varepsilon} + \lim_{\varepsilon \to 0} \frac{t(y+\varepsilon) - t(y)}{\varepsilon} \geq$$
$$k2^{\frac{1}{p}} \lim_{\varepsilon \to 0} \frac{t(T(x,y) + k2^{\frac{1}{p}}\varepsilon) - t(T(x,y))}{k2^{\frac{1}{p}}\varepsilon} = k2^{\frac{1}{p}} t'(T(x,y)).$$

Note that each differentiable additive generator t such that $g = t^{\frac{1}{p}}$ is a convex function satisfies Corollary 2 for all $k \in [1,\infty[$.

3 Boundaries of the Class of k-l_p-Lipschitz t-Norms

The minimum t-norm $T_{\mathbf{M}}$ is k-l_p-Lipschitz for all $k \in [1,\infty[$, $p \in [1,\infty]$, i.e., for all $k \in [1,\infty[$, $p \in [1,\infty]$ the minimum t-norm $T_{\mathbf{M}}$ is the maximum of the class of all k-l_p-Lipschitz t-norms. On the other hand, in the class of k-Lipschitz t-norms though there are several minimal elements there is no weakest k-Lipschitz t-norm for $k > 1$. The following proposition can be found in [4].

Proposition 3. *Let $A_{*k} : [0,1]^2 \to [0,1]$ be given by*

$$A_{*k}(x,y) = \inf\{T(x,y) \mid T \text{ is a } k\text{-Lipschitz t-norm}\},$$

i.e., A_{*k} is the pointwise infimum of all k-Lipschitz t-norms. Then A_{*k} is the weakest k-Lipschitz aggregation operator with neutral element 1, i.e.,

$$A_{*k}(x,y) = \max(x + ky - k, y + kx - k, 0).$$

The aggregation operator from the above proposition is a t-norm only for $k = 1$. Note that since $\mathcal{T}_k$ is compact we know that for $k > 1$ A_{*k} cannot be obtained as a limit of k-Lipschitz t-norms.

In the case of 1-l_p-Lipschitz t-norm the situation is the following.

Proposition 4. *Let $p \in [1,\infty]$. Then the weakest 1-l_p-Lipschitz t-norm is the Yager t-norm with parameter p, i.e., t-norm given by $T_p^Y(x,y) = \max(1 - \sqrt[p]{(1-x)^p + (1-y)^p}, 0)$.*

Proof. Let $T : [0,1]^2 \to [0,1]$ be a 1-l_p-Lipschitz t-norm then

$$T(1,1) - T(x,y) \leq \sqrt[p]{(1-x)^p + (1-y)^p},$$

i.e., $1 - \sqrt[p]{(1-x)^p + (1-y)^p} \leq T(x,y)$. Now to complete the proof we need to show that the Yager t-norm with parameter p is 1-l_p-Lipschitz. Let us assume points $(x_0,y_0),(x_1,y_1) \in [0,1]$, and let $T_p^Y(x_0,y_0) \leq T_p^Y(x_1,y_1)$. If $T_p^Y(x_0,y_0) > 0$ then the inequality

$$\begin{aligned} T_p^Y(x_1,y_1) - T_p^Y(x_0,y_0) &= 1 - l_p(1-x_1,1-y_1) - (1 - l_p(1-x_0,1-y_0)) \\ &\leq \sqrt[p]{(x_1-x_0)^p + (y_1-y_0)^p} = l_p((x_1-x_0,y_1-y_0)) \end{aligned}$$

follows from the triangle inequality of l_p norm. If $T_p^Y(x_0,y_0) = 0$ then $1 \leq \sqrt[p]{(1-x_0)^p + (1-y_0)^p}$ and the inequality

$$\begin{aligned} T_p^Y(x_1,y_1) - T_p^Y(x_0,y_0) &= 1 - l_p(1-x_1,1-y_1) \\ &\leq \sqrt[p]{(x_1-x_0)^p + (y_1-y_0)^p} = l_p(x_1-x_0,y_1-y_0), \end{aligned}$$

i.e., $1 \leq l_p(x_1-x_0,y_1-y_0) + l_p(1-x_1,1-y_1)$ follows from the inequality $l_p(1-x_0,1-y_0) \leq l_p(x_1-x_0,y_1-y_0) + l_p(1-x_1,1-y_1)$ what is again the triangle inequality of l_p norm.

For example, the only basic t-norm which is 1-l_2-Lipschitz is the minimum. Drastic product is not even continuous and both the Łukasiewicz and the product t-norm are 1-Lipschitz but not 1-l_2-Lipschitz.

For $p > 1$ and $k \geq 1$, k-l_p-Lipschitz t-norms are bounded from below by the following conjunctors.

Proposition 5. (i) *Let $p \in]1,\infty[$ and $k \in \left[\sqrt[p]{2^{p-1}},\infty\right[$ then the weakest k-l_p-Lipschitz conjunctor is a commutative conjunctor $A_{k,p}$, given for $x \leq y$ by*

$$A_{k,p}(x,y) = \max(x + (y-1)(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}, 0).$$

(ii) *Let $p \in]1,\infty[$ and $k \in \left[1, \sqrt[p]{2^{p-1}}\right[$ then the weakest k-l_p-Lipschitz conjunctor is a commutative conjunctor $B_{k,p}$, given for $x \leq y$ by*

$$B_{k,p}(x,y) = \begin{cases} \max(x + (y-1)(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}, 0) \\ \qquad\qquad\qquad\qquad \text{if } y \geq 1 + (x-1)(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}} \\ \max(1 - k\sqrt[p]{(1-x)^p + (1-y)^p}, 0) \quad \text{otherwise.} \end{cases}$$

Proof. Let $C : [0,1]^2 \to [0,1]$ be a k-l_p-Lipschitz conjunctor. Assume $x,y \in [0,1], x \leq y$. Then

$$z - T(x,y) = T(z,1) - T(x,y) \leq k\sqrt[p]{|z-x|^p + (1-y)^p}$$

holds for all $z \in [0,1]$ and consequently $\sup_{z\in[0,1]} z - k\sqrt[p]{|z-x|^p + (1-y)^p} \leq T(x,y)$. Since for $z < x$ the inequality $x - k(1-y) \leq T(x,y)$ imply $z - k\sqrt[p]{|z-x|^p + (1-y)^p} \leq T(x,y)$ and since the function

$$f(z) = z - k\sqrt[p]{|z-x|^p + (1-y)^p}$$

is continuous on $[x,1]$ it is enough to assume $\max_{z\in[x,1]} z - k\sqrt[p]{|z-x|^p + (1-y)^p}$. We will compare values of the function f in points $x, 1$ and in all stationary points of f. We have $f(x) = x - k(1-y)$, $f(1) = 1 - k\sqrt[p]{(1-x)^p + (1-y)^p}$ and $f'(z) = 1 - k(|z-x|^p + (1-y)^p)^{\frac{1-p}{p}}|z-x|^{p-1}$. Then $f'(z^*) = 0$ whenever $k = (|z^*-x|^p + (1-y)^p)^{\frac{p-1}{p}}|z^*-x|^{1-p}$, i.e., $z^* = x + (1-y)(k^{\frac{p}{p-1}} - 1)^{-\frac{1}{p}}$. We know that $z^* \in [0,1]$ if and only if $y \geq 1 + (x-1)(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}$. Then $f(z^*) = x + (y-1)(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}$. In the following we will prove that $f(z^*) \geq \max(f(1), f(x))$ (note that z^* can be greater than 1). Inequality $f(z^*) \geq f(x)$ is evident since $k \geq (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}$. Now assume $f(z^*) \geq f(1)$. If $y = 1$ then $1 - k\sqrt[p]{(1-x)^p + (1-y)^p} \leq x + (y-1)(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}$ holds since $k \geq 1$. If $y < 1$ then denote $a = \frac{1-x}{1-y}$, $a \in [0,\infty]$. We need to prove that

$$a + (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}} \leq k\sqrt[p]{a^p + 1}.$$

Define a function $h(x) = k\sqrt[p]{a^p+1} - a - (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}$. Then $h(0) = k - (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}} > 0$ and $\lim_{a\to\infty} f(a) \geq 0$ since $k \geq 1$. Further $h'(a) = k(a^p+1)^{\frac{1-p}{p}}a^{p-1} - 1$ and $h(a^*) = 0$ whenever $a^* = (k^{\frac{p}{p-1}} - 1)^{-\frac{1}{p}}$. Then $h(a^*) = \frac{k^{\frac{p}{p-1}}}{(k^{\frac{p}{p-1}}-1)^{\frac{1}{p}}} - (k^{\frac{p}{p-1}} - 1)^{-\frac{1}{p}} - (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}$, i.e., $h(a^*) = 0$. This means that $h(a) \geq 0$ for all $a \in [0,\infty]$, i.e., $f(z^*) \geq f(1)$. In the case when $y \leq 1 + (x-1)(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}$ (note that in the case when $x \leq y$ this is possible only for $k \leq 2^{\frac{p-1}{p}}$) we need to prove that $f(1) \geq f(x)$, i.e., that

$$x - k(1-y) \leq 1 - k\sqrt[p]{|z-x|^p + (1-y)^p},$$

i.e., that $k\sqrt[p]{(1-x)^p + (1-y)^p} \leq (1-x) + k(1-y)$. If $x = y = 1$ the inequality trivially holds. Otherwise $x < 1$ and we can denote $b = \frac{1-y}{1-x}$. Then since $y \leq 1 + (x-1)(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}$ we have $b \in [(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}, 1]$ We have to prove $k\sqrt[p]{b^p + 1} \leq 1 + kb$. Define $g(b) = 1 + kb - k\sqrt[p]{b^p + 1}$. Then $g(1) = 1 + k - k2^{\frac{1}{p}} \geq 0$ for $k \leq \frac{1}{2^{\frac{1}{p}} - 1}$, i.e., for all $k \leq 2^{\frac{p-1}{p}}$ it is satisfied. Further $g((k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}) = 1 + k(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}} - k^{\frac{p}{p-1}} \geq 0$. Since $g'(b) = k - k(1+b^p)^{\frac{1-p}{p}}b^{p-1} > 0$ for all $b \in [(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}, 1]$ function g has no stationary point, i.e., $g(b) \geq 0$ for all $b \in [(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}, 1]$, i.e., $f(1) \geq f(x)$. Since conjunctors cannot attain negative value we have proved that every k-l_p-Lipschitz conjunctor is greater either

than $A_{k,p}$ or $B_{k,p}$. Evidently, $A_{k,p}$ and $B_{k,p}$ are conjunctors. To complete the proof we need to show that conjunctors $A_{k,p}$ and $B_{k,p}$ are k-l_p-Lipschitz. Similarly as in the case of ordinal sums it can be shown that conjunctor $A_{k,p}$ is k-l_p-Lipschitz whenever it is k-l_p-Lipschitz on the set $\{(x,y) \in [0,1]^2 \mid x \leq y\}$ and conjunctor $B_{k,p}$ is k-l_p-Lipschitz whenever it is k-l_p-Lipschitz on the sets $\{(x,y) \in [0,1]^2 \mid x \leq y, y \geq 1 + (x-1)(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}\}$ and $\{(x,y) \in [0,1]^2 \mid x \leq y, y \leq 1 + (x-1)(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}\}$ (parts where $x \geq y$ are then k-l_p-Lipschitz because of commutativity).

Let us assume $(x_0, y_0), (x_1, y_1) \in [0,1]^2$. Then if in these points $T(x_0, y_0) = 1 - k\sqrt[p]{(1-x_0)^p + (1-y_0)^p}$ and $T(x_1, y_1) = 1 - k\sqrt[p]{(1-x_1)^p + (1-y_1)^p}$, $T(x_1, y_1) \geq T(x_0, y_0)$ the inequality

$$1 - k\sqrt[p]{(1-x_1)^p + (1-y_1)^p} - (1 - k\sqrt[p]{(1-x_0)^p + (1-y_0)^p}) \leq k\sqrt[p]{|x_1 - x_0|^p + |y_1 - y_0|^p}$$

follows from the triangle inequality of l_p norm. In the case when $T(x_0, y_0) = x_0 + (y_0 - 1)(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}$ and $T(x_1, y_1) = x_1 + (y_1 - 1)(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}$ the inequality $x_1 + (y_1 - 1)(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}} - (x_0 + (y_0 - 1)(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}) \leq k\sqrt[p]{|x_1 - x_0|^p + |y_1 - y_0|}$ means that $|x_1 - x_0| + |y_1 - y_0|(k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}} \leq k\sqrt[p]{|x_1 - x_0|^p + |y_1 - y_0|}$. In the case when $y_1 = y_0$ the inequality holds trivially. Otherwise denote $c = \left|\frac{x_1 - x_0}{y_1 - y_0}\right|$, $c \in [0, \infty]$ We need to prove that $c + (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}} \leq k(1 + c^p)^{\frac{1}{p}}$. Define $u(c) = k(1 + c^p)^{\frac{1}{p}} - c - (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}}$. Then $u(0) = k - (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}} > 0$ and $\lim_{c \to \infty} \geq 0$ since $k \geq 1$. Further $u'(c) = k(1 + c^p)^{\frac{1-p}{p}} c^{p-1} - 1$ and $u'(c^*) = 0$ whenever $c^* = (k^{\frac{p}{p-1}} - 1)^{-\frac{1}{p}}$. Then $u(c^*) = \frac{k^{\frac{p}{p-1}}}{(k^{\frac{p}{p-1}} - 1)^{\frac{1}{p}}} - (k^{\frac{p}{p-1}} - 1)^{-\frac{1}{p}} - (k^{\frac{p}{p-1}} - 1)^{\frac{p-1}{p}} = 0$. This means that $u(c) \geq 0$ for all $c \in [0, \infty]$ and thus the proof is finished.

Evidently, the only 1-l_∞-Lipschitz t-norm is the minimum $T_{\mathbf{M}}$. Note that since for k-l_p-Lipschitz t-norm T we have

$$|T(x_1, y_1) - T(x_2, y_2)| \leq k\sqrt[p]{|x_1 - x_2|^p + |y_1 - y_2|^p} \leq k2^{\frac{1}{p}}\delta,$$

where $\delta = \max(|x_1 - x_2|, |y_1 - y_2|)$, we have $\mathcal{T}_{k \cdot 2^{\frac{1}{p}}, l_\infty} \supset \mathcal{T}_{k,l_p}$. As a special case then $\mathcal{T}_{2,l_\infty} \supset \mathcal{T}_1$, i.e., the class of 2-$l_\infty$-Lipschitz t-norms is a superset of the class of associative copulas. Though minimal element of both of these classes is the Łukasiewicz t-norm $T_{\mathbf{L}}$, we will show that these two classes are not equal.

Example 1. Let $t : [0,1] \to [0,\infty]$ be given by

$$t(x) = \begin{cases} 2.225 - 3x & \text{if } x \in [0, 0.55], \\ 1.4 - \frac{3}{2}x & \text{if } x \in]0.55, 0.6], \\ 1.7 - 2x & \text{if } x \in]0.6, 0.7], \\ 1 - x & \text{otherwise.} \end{cases}$$

Then t is a (non-convex) additive generator of a t-norm T which is for $x \leq y$ given by

$$T(x,y) = \begin{cases} x + y - 1 & \text{if } (x,y) \in [0.7,1]^2, x + y \geq 1.7, \\ \frac{1}{2}x + \frac{1}{2}y - 0.15 & \text{if } (x,y) \in [0.7,1]^2, x + y \geq 1.5, \\ \frac{2}{3}x + \frac{2}{3}y - 0.4 & \text{if } (x,y) \in [0.7,1]^2, x + y \geq 1.425, \\ \frac{1}{3}x + \frac{1}{3}y + 0.075 & \text{if } (x,y) \in [0.7,1]^2, x + y \leq 1.425, \\ \frac{2}{3}x + \frac{2}{3}y - \frac{1.175}{3} & \text{if } (x,y) \in [0.6, 0.7[^2, \\ x + \frac{1}{2}y - 0.5 & \text{if } (x,y) \in [0.6, 0.7[\times [0.7,1], 2x + y \geq 2.2, \\ \frac{4}{3}x + \frac{2}{3}y - \frac{2.6}{3} & \text{if } (x,y) \in [0.6, 0.7[\times [0.7,1], 2x + y \geq \frac{17}{8}, \\ \frac{2}{3}x + \frac{1}{3}y - \frac{0.475}{3} & \text{if } (x,y) \in [0.6, 0.7[\times [0.7,1], 2x + y \leq \frac{17}{8}, \\ \frac{1}{2}x + \frac{1}{2}y - \frac{0.575}{3} & \text{if } (x,y) \in [0.55, 0.6[^2, \\ \frac{1}{2}x + \frac{2}{3}y - \frac{0.875}{3} & \text{if } (x,y) \in [0.55, 0.6[\times [0.6, 0.7[, \\ x + \frac{2}{3}y - \frac{2}{3} & \text{if } (x,y) \in [0.55, 0.6[\times [0.7,1], \frac{3}{2}x + y \geq \frac{73}{40}, \\ \frac{1}{2}x + \frac{y}{3} - \frac{0.175}{3} & \text{if } (x,y) \in [0.55, 0.6[\times [0.7,1], \frac{3}{2}x + y \leq \frac{73}{40}, \\ \max(0, x + y - \frac{2.225}{3}) & \text{if } (x,y) \in [0, 0.55[^2, \\ \max(0, x + \frac{1}{2}y - \frac{1.4}{3}) & \text{if } (x,y) \in [0, 0.55[\times [0.55, 0.6[, \\ \max(0, x + \frac{2}{3}y - \frac{1.7}{3}) & \text{if } (x,y) \in [0, 0.55[\times [0.6, 0.7[, \\ \max(0, x + \frac{1}{3}y - \frac{1}{3}) & \text{if } (x,y) \in [0, 0.55[\times [0.7,1]. \end{cases}$$

The t-norm T is 2-l_∞-Lipschitz (evidently, $\frac{\partial T}{\partial x}(x,y) + \frac{\partial T}{\partial y}(x,y) \leq 2$ in all points where both partial derivatives exist), but not 1-Lipschitz.

Proposition 6. *Let $T : [0,1]^2 \to [0,1]$ be a continuous t-norm continuously differentiable on $]0,1[^2$. Then T is k-l_p-Lipschitz for some $p > 1$ whenever*

$$\left(\frac{\partial T}{\partial x}(x,y)\right)^{\frac{p}{p-1}} + \left(\frac{\partial T}{\partial y}(x,y)\right)^{\frac{p}{p-1}} \leq k^{\frac{p}{p-1}} \tag{5}$$

holds for all $(x,y) \in]0,1[^2$, and T is k-Lipschitz whenever $\frac{\partial T}{\partial x}(x,y) \leq k$ and $\frac{\partial T}{\partial y}(x,y) \leq k$ holds for all $(x,y) \in]0,1[^2$.

Proof. A continuous t-norm T is k-l_p-Lipschitz whenever for all $(x,y) \in]0,1[^2$, $\varepsilon \in [0, 1-x]$, $\delta \in [0, 1-y]$

$$T(x+\varepsilon, y+\delta) - T(x,y) \leq k \sqrt[p]{\varepsilon^p + \delta^p},$$

i.e.,

$$T(x+\varepsilon, y+\delta) - T(x+\varepsilon, y) + T(x+\varepsilon, y) - T(x,y) \leq k \sqrt[p]{\varepsilon^p + \delta^p}.$$

Denote $m = \frac{\delta}{\epsilon}$ (in the case when $\epsilon = 0$ we will denote $m = \frac{\epsilon}{\delta}$ and continue analogically). Then $m \in [0, \infty[$ and we obtain

$$T(x+\varepsilon, y+m\varepsilon) - T(x+\varepsilon, y) + T(x+\varepsilon, y) - T(x,y) \leq k\varepsilon \sqrt[p]{1+m^p},$$

i.e., for $m \neq 0$

$$m \cdot \frac{T(x+\varepsilon, y+m\varepsilon) - T(x+\varepsilon, y)}{m\varepsilon} + \frac{T(x+\varepsilon, y) - T(x,y)}{\varepsilon} \leq k \sqrt[p]{1+m^p}$$

and subsequently

$$m \cdot \lim_{\varepsilon \to 0} \frac{T(x+\varepsilon, y+m\varepsilon) - T(x+\varepsilon, y)}{m\varepsilon} + \lim_{\varepsilon \to 0} \frac{T(x+\varepsilon, y) - T(x,y)}{\varepsilon} \leq k \sqrt[p]{1+m^p}.$$

This means that T is k-l_p-Lipschitz whenever for all $(x,y) \in]0,1[^2$ and all $m \in]0, \infty[$

$$m \cdot \frac{\partial T}{\partial y}(x,y) + \frac{\partial T}{\partial x}(x,y) \leq k \sqrt[p]{1+m^p} \tag{6}$$

and $\frac{\partial T}{\partial y}(x,y) \leq k$, $\frac{\partial T}{\partial x}(x,y) \leq k$ holds (this follows from the case when $\varepsilon = 0$ or $\delta = 0$). In the case when $p = 1$ inequalities $\frac{\partial T}{\partial y}(x,y) \leq k$, $\frac{\partial T}{\partial x}(x,y) \leq k$ are equivalent with inequality $m \cdot \frac{\partial T}{\partial y}(x,y) + \frac{\partial T}{\partial x}(x,y) \leq k \sqrt[p]{1+m^p}$, $m \in [0, \infty]$. For the case when $p > 1$ is finite we define a function $f : [0, \infty] \to \mathbb{R}$ by

$$f(m) = k \sqrt[p]{1+m^p} - m \cdot \frac{\partial T}{\partial y}(x,y) - \frac{\partial T}{\partial x}(x,y).$$

The inequality (6) is fulfilled if and only if $\min\limits_{m \in [0,\infty]} f(m) \geq 0$. Since $f(0) \geq 0$ and $f(\infty) \geq 0$ whenever $\frac{\partial T}{\partial y}(x,y) \leq k$ and $\frac{\partial T}{\partial x}(x,y) \leq k$ are fulfilled and f is differentiable on $]0, \infty[$, to see the non-negativity of f we need to investigate only values of f in stationary points. We have $f'(m) = k\,(1+m^p)^{\frac{1-p}{p}}\, m^{p-1} - \frac{\partial T}{\partial y}(x,y)$ and thus $f'(m^*) = 0$ whenever $k \cdot (1+(m^*)^p)^{\frac{1-p}{p}} \cdot (m^*)^{p-1} = \frac{\partial T}{\partial y}(x,y)$, i.e., when

$$m^* = \frac{\left(\frac{\partial T}{\partial y}(x,y)\right)^{\frac{1}{p-1}}}{\left(k^{\frac{p}{p-1}} - \left(\frac{\partial T}{\partial y}(x,y)\right)^{\frac{p}{p-1}}\right)^{\frac{1}{p}}}.$$

Note that whenever $\frac{\partial T}{\partial y}(x,y) \leq k$ we have $m^* \in [0,\infty]$. We obtain

$$f(m^*) = \left(k^{\frac{p}{p-1}} - \left(\frac{\partial T}{\partial y}(x,y)\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} - \frac{\partial T}{\partial x}(x,y)$$

and thus $f(m^*) \geq 0$ whenever

$$k^{\frac{p}{p-1}} \geq \left(\frac{\partial T}{\partial y}(x,y)\right)^{\frac{p}{p-1}} + \left(\frac{\partial T}{\partial x}(x,y)\right)^{\frac{p}{p-1}}.$$

In the case when $p = \infty$, T is k-l_p-Lipschitz whenever for $m \geq 1$

$$m \cdot \frac{\partial T}{\partial y}(x,y) + \frac{\partial T}{\partial x}(x,y) \leq k \cdot m$$

and for $m \leq 1$

$$m \cdot \frac{\partial T}{\partial y}(x,y) + \frac{\partial T}{\partial x}(x,y) \leq k$$

hold. Extreme points in this case are $m = 0, 1, \infty$ (in the case when $k = \frac{\partial T}{\partial y}(x,y)$ all points m are extreme points, but in such a case $\frac{\partial T}{\partial x}(x,y) = 0$ and we get $\frac{\partial T}{\partial y}(x,y) + \frac{\partial T}{\partial x}(x,y) = k$). For $m = 0$ we get $\frac{\partial T}{\partial x}(x,y) \leq k$ for $m = \infty$ we get $\frac{\partial T}{\partial y}(x,y) \leq k$ and for $m = 1$ we get

$$\frac{\partial T}{\partial y}(x,y) + \frac{\partial T}{\partial x}(x,y) \leq k$$

what is exactly the inequality (5) for $p = \infty$. Finally note that since t-norm T is non-decreasing inequality (5) implies inequalities $\frac{\partial T}{\partial y}(x,y) \leq k$ and $\frac{\partial T}{\partial x}(x,y) \leq k$ for all $p > 1$.

Note that this proposition can be generalized also for continuous t-norms which are continuously differentiable on $]0,1[^2 \setminus R$, where the set R consists of isolated points and isolated segments of the type $x \times [0,1]$ (or $[0,1] \times x$), $x \in [0,1]$.

4 Transformations of k-Lipschitz t-Norms

Recall that in the case if 1-Lipschitz t-norms, i.e., in the class of associative copulas, it holds that a transformation C_f of a copula C, $C_f(x,y) = f^{-1}(C(f(x), f(y)))$, where $f : [0,1] \to [0,1]$ is an increasing bijection is again a copula (for any C) if and only if f is concave.

Proposition 7. (i) *Let $T : [0,1]^2 \to [0,1]$ be an associative copula (1-Lipschitz t-norm) and let $f : [0,1] \to [0,1]$ be an increasing bijection. Then T_f is a k-Lipschitz t-norm if and only if f is k-concave, i.e., $f(x + k\varepsilon) - f(x) \geq f(y + \varepsilon) - f(y)$ holds for all $x \in [0,1[\,, y \in]0,1[\,$, with $x \leq y$ and $\varepsilon \in \left]0, \min(1 - y, \frac{1-x}{k})\right]$.*

(ii) *Let $T : [0,1]^2 \to [0,1]$ be a k-Lipschitz t-norm and let $f : [0,1] \to [0,1]$ be an increasing bijection. Then T_f is a k-Lipschitz t-norm if and only if f is concave.*

Proof. Assume $x, y \in [0,1], \varepsilon \in \left]0, \min(1-x, \frac{1-T_f(x,y)}{k})\right]$. Denote $z = T_f(x,y)$. Then $T_f(x+\varepsilon, y) - T_f(x,y) \leq k\varepsilon$ whenever $T(f(x+\varepsilon), f(y)) \leq f(z+k\varepsilon)$. Since $f(z) = T(f(x), f(y))$ this inequality is equivalent to $T(f(x+\varepsilon), f(y)) - T(f(x), f(y)) \leq f(z+k\varepsilon) - f(z)$. Since $f(z) \leq \min(f(x), f(z))$ we have $z \leq \min(x, y)$.

(i) The 1-Lipschitz property of T, i.e., $T(f(x+\varepsilon), f(y)) - T(f(x), f(y)) \leq f(x+\varepsilon) - f(x)$ and k-concavity of f, i.e., $f(z+k\varepsilon) - f(z) \geq f(x+\varepsilon) - f(x)$ imply that T_f is a k-Lipschitz t-norm.
Vice versa assume Łukasiewicz t-norm $T_{\mathbf{L}}$ (which is 1-Lipschitz) with additive generator $t : [0,1] \to [0,\infty]$, $t(x) = 1 - x$. $(T_{\mathbf{L}})_f$ is k-Lipschitz whenever its additive generator $t \circ f$ is k-convex, i.e., when $t(f(x+k\varepsilon)) - t(f(x)) \leq t(f(y+\varepsilon)) - t(f(y))$ for all $x \in [0,1[\,, y \in]0,1[$, with $x \leq y$ and $\varepsilon \in \left]0, \min(1-y, \frac{1-x}{k})\right]$. We get $1 - f(x+k\varepsilon) - (1 - f(x)) \leq 1 - f(y+\varepsilon) - (1 - f(y))$, i.e., $f(y+\varepsilon) - f(y) \leq f(x+k\varepsilon) - f(x)$ what means that f is k-concave.

(ii) Let f be concave. If $T(f(x), f(y)) = f(x)$ then $T(f(x+\varepsilon), f(y)) - T(f(x), f(y)) \leq f(x+\varepsilon) - f(x) \leq f(z+\varepsilon) - f(z) \leq f(z+k\varepsilon) - f(z)$. If $T(f(x), f(y)) = f(y)$ then also $T(f(x+\varepsilon), f(y)) = f(y)$, i.e., $T(f(x+\varepsilon), f(y)) - T(f(x), f(y)) = 0$ and the result is trivial. Suppose that $f(z) < \min(f(x), f(y))$. Then $z < x$ and for each $0 < \varepsilon \leq \min(\frac{x-z}{k}, 1-x)$ it holds

$$f(z+k\varepsilon) - f(z) \geq k(f(x+\varepsilon) - f(x)).$$

This inequality together with the k-Lipschitz property of T, i.e., $T(f(x+\varepsilon), f(y)) - T(f(x), f(y)) \leq k(f(x+\varepsilon) - f(x))$ ensures the result, i.e., T_f is k-Lipschitz in some neighborhood of (x,y). However, due to the compactness of $[0,1]^2$, k-Lipschitz property valid for each $(x,y) \in [0,1]^2$ in some neighborhood of this point is equivalent to the k-Lipschitz property of T_f on entire $[0,1]^2$.
Vice versa let $a, b \in [0,1], a < b$. Denote $a = x, b = y + \varepsilon, y = x + k\varepsilon = \frac{a}{k+1} + \frac{kb}{k+1}$. Assume an additive generator $t : [0,1] \to [0,\infty]$ of a k-Lipschitz t-norm given by

$$t(x) = \begin{cases} 1 - \frac{x}{k} + y\frac{1-k}{k} & \text{if } x \in [0,y] \\ 1 - x & \text{otherwise.} \end{cases}$$

Then $t(f(x+k\varepsilon)) - t(f(x)) \leq t(f(y+\varepsilon)) - t(f(y))$ means that $1 - \frac{f(x+k\varepsilon)}{k} + y\frac{1-k}{k} - (1 - \frac{f(x)}{k} + y\frac{1-k}{k}) \leq 1 - f(y+\varepsilon) - (1 - f(y))$, i.e., $k(f(y+\varepsilon) - f(y)) \leq f(x+k\varepsilon) - f(x)$. Since $x + k\varepsilon = y$ we have $k \cdot f(y+\varepsilon) + f(x) \leq (k+1)f(y)$, i.e., $\frac{kf(b)+f(a)}{k+1} \leq f(\frac{a}{k+1} + \frac{kb}{k+1})$ what means that f is concave.

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Formal Models of Knowledge Operators: Rough-Set-Style and Fuzzy-Set-Style Approaches

Ewa Orłowska

A concept of knowledge of the agents about a set of objects of an information system with incomplete information is discussed. A rough-set-style semantics of the corresponding knowledge operators is presented, following [2]. An abstract characterization of the semantic postulates is provided in terms of a class of relational systems referred to as plain K-frames and a modal-like logic, K-logic, based on the K-frames is developed. On an algebraic side, a class of K-algebras with a knowledge operator is introduced.

Next, the refined formalisms of relative K-frames and relative K-algebras are developed which capture the situation of multiple agents which are not necessarily independent. An adequate multimodal logic based on relative K-frames is presented. A methodology of reasoning with incomplete information developed in [1] is applied here.

A duality of the classes of plain K-frames and K-algebras and a duality of classes of relative K-frames and relative K-algebras are proved along the lines of [3]. Finally, fuzzy-set-style versions of knowledge operators are presented and discussed based on lattice-based fuzzy sets.

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Using a Fuzzy Model for Combining Search Results from Different Information Sources to Build a Metasearch Engine

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Summary. Summary: In this paper it is presented an algorithm for combining and reranking search results from different information sources with different techniques to retrieve and rank their results. The proposed metasearch is based in the yagers work on "aggregate operators". The developed metasearch engine in Java and Ciao Prolog is evaluated using an available set of documents collection, queries and their corresponding answers (the TIME collection). Finally some concluding remarks are given on the accuracy of the metasearch presented.

Key words: Reranking search results, Fuzzy models, Distributed information retrieval.

1 Introduction

Soft computing offers appropriate handling of vagueness, subjectivity, uncertainty, imprecission, partial truth, and approximation. The aim of information retrieval system is to obtain relevant documents with respect to the query expressing user needs, where vagueness, imprecission, and subjectivity are involved. Therefore, information retrieval is a typical application field of soft computing. As mentioned in [1], some of the main soft computing approaches in information retrieval are the following:

- Fuzzy sets and logic: information fusion, text extraction, query language models, and document clustering.
- Neural networks: document and term classification and clustering, and multimedia retrieval.
- Genetic algorithms: document classification, image retrieval, relevance feedback, and query learning.
- Probabilistic techniques: ranking, web mining.
- Rough sets and multivalued logics: document clustering.
- Bayesian networks: retrieval model, ranking, thesaurus construction, and relevance feedback.

This work[1] employs fuzzy sets and logic approach on ranked lists fusion, as suggested in [3] and [4]. This approach offer alternative solution for information retrieval when there are many information sources and each source has its own search engine. Knowing characteristics of each information source and its search engine, it is possible to obtain relevant documents without having to build a new search engine.

2 Foundations

In distributed information retrieval, there are two main problems to merge ranked result lists from each information source. First problem is to determine how many retrieved documents should be selected from each individual list, and second problem is how to combine these ranked lists into a single ranked list. In the case where information sources have overlapping documents, combining ranked lists should also consider removing duplicated documents appearing in different lists.

The approach for the multisource information retrieval can be formulated by the following steps:

1. Determine the number of documents to be retrieved in multisource information retrieval, let this be N.
2. Determine the number of retrieved documents to be selected from each information source.
   Let L_{iq} be the list containing $|L_{iq}|$ selected documents from information source S_i. The task is to determine $|L_{iq}|$ such that

$$N = \sum_{i=1}^{k} |L_{iq}| \tag{1}$$

   The intuition behind determination of $|L_{iq}|$ is that more documents should be retrieved from the information source that is more appropriate with respect to the given query. It could depend on the type of information contained in each information source, the topics expressed in the user query, and the appropriateness of the information source as collector of relevant document according to the query. $|L_{iq}|$ is defined as follows:

$$|L_{iq}| = \alpha_i N \tag{2}$$

$$\alpha_i = \frac{b_i^*}{\sum_{i=1}^{m} b_i^*} \tag{3}$$

   where fitness score b_i^* indicates how much an information source S_i is a good collector of the relevant documents for a given query. Vector B^* denotes fitness score values for all information sources with respect to a

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given query. There are several approaches to calculate B^*, some of them are based on fuzzy rules, based on fuzzy prototypal set, or based on the prototypal query. These approaches are explained briefly in [3].

3. Send the query to each information source, asking the first $|L_{iq}|$ result from each source S_i.
4. Define a set of document D where $D = \{d|d \in \cup_{i=1..k} L_i\}$ be the set of all documents appearing in the retrieved lists.
5. Calculate local performance judgement for each document.
   Document with higher performance judgement indicates more relevant document. Let C_{ji} be the performance judgement of the document d_j according to information source S_i. C_{ji} is defined as follows:

$$C_{ji} = |L_{iq}| - p_{ji} + 1 \tag{4}$$

   where p_{ji} is the position of document d_j in the list L_{iq}. If the document d_j does not exist in L_{iq}, $p_{ji} = 0$. As it can be seen, documents from longer list is considered more relevant than those from shorter list when they occupy same position in the list.
6. Determine fitness score f_h which represents user preference of the h-th search engine where $0 \leq f_h \leq maxL_q$.
   The most preferred engine has the highest fitness score and should have a greater chance to determine the final judgement. In case of complete uncertainty on fitness score of each engine, all the fitness score can be fixed equal to the average cardinality of the lists.
7. Define the weighting vector W of IOWA operator [11].

$$W = [w_1, ..., w_k], \sum_{i=1}^{k} w_i = 1 \tag{5}$$

   The *orness* of the IOWA operator is defined by

$$orness(W) = (\frac{1}{K-1}) \sum_{j=1}^{K} ((K-j) * w_j) \tag{6}$$

   $orness(W)$ range from 0 to 1 where 0 denotes AND operator or fuzzy quantifier *all*, and 1 denotes OR operator or fuzzy quantifier *at least 1*.
8. Calculate overall performance judgement C_j for each document by aggregating C_{ji} using IOWA operator as follows:

$$C_j = IOWA(\langle u_{j1}, C_{j1}\rangle, ..., \langle u_{jk}, C_{jk}\rangle) = W^T B_{jU} \tag{7}$$

   with $u_j h$ defined based on fitness score f_h of the hth search engine and performance judgement C_{jh}:

$$u_{jh} = 1 - |\frac{C_{jh} * f_h}{(maxL_q)^2} - orness(W)|, (orness(W) > 0.5) \tag{8}$$

$$u_{jh} = 1 - |\frac{C_{jh} * (maxL_q - f_h)}{(maxL_q)^2} - orness(W)|, (orness(W) \leq 0.5) \tag{9}$$

   where $maxL_q$ is the cardinality of the largest list.

9. Order the documents according to their overall performance judgment, descendingly. This is the merged ranked list.

A learning algorithm can be applied to adjust the fitness score f_h. The user can analyze the result of first query, and then identify those documents that is considered relevant to the user needs. With learning rate $s \in (0, 1)$, the fitness score can be modified as follows:

$$f_h(t_n) = min[f_h(t_{n-1}) + s * C_{rh}(t_{n-1}), maxL_q], \tag{10}$$

where $C_{rh}(t_{n-1})$ is the performance judgement of the relevant document identified by the user in the last run. User can also identify those documents that is considered irrelevant to the user needs, and the fitness score can be modified as:

$$f_h(t_n) = max[f_h(t_{n-1}) - s * C_{rh}(t_{n-1}), 0], \tag{11}$$

where $C_{rh}(t_{n-1})$ is the performance judgement of the irrelevant document identified by the user in the last run.

3 Metasearch

The idea of metasearch is to submit an user query to several search engine and merge their results to be presented as the result of the metasearch engine. Eventhough the ideal goal of a metasearch engine is to increase quantity and quality of the search result, there are arguments that the quality of the metasearch result will not be better than the quality of each search engine behind it [2]. In [10] for example, quality of a popular metasearch engine Vivisimo, is questioned since the engines behind it are not considered as the best ones. However, one of the main motivation for metasearch is the idea "More heads is better than one" [9], and as mentioned in [7] the most obvious advantage is that user can get results from multiple search engines without having to visit each in turn.

As we will use services from Google and Yahoo!, we will give overviews of techniques employed by these search engines to retrieve and rank their search result. Google claims to be a fully automated search engine. A software known as "spiders" is used to crawl the web on regular basis to find sites to be added to their index. To perform the search, Google combines the measures of overall importance and query specific relevance of a page, to ensure the most relevant and reliable results appears on the first positions. Google uses PageRank^{TM}to examine the entire link structure of the web and determine which pages are the most important. It then conducts hypertext-matching analysis to determine which pages are relevant to the specific search being conducted [6]:

– **PageRank Technology**: PageRank performs an objective measurement of the importance of web pages. PageRank interprets a link from Page A to Page B as a vote for Page B by Page A. PageRank then assesses a page's importance by the votes it receives. PageRank also considers the importance of each page that casts a vote, as votes from some pages are

considered to have greater value, thus giving the linked page greater value. In another word, links from an "important" page to other pages will help them to be "important." Important pages receive a higher PageRank and appear at the top of the search results.

– **Hypertext-Matching Analysis**: Google's search engine also analyzes page content. However, instead of simply scanning for a page-based text which can be manipulated by site publishers through meta-tags, Google's technology analyzes the full content of a page and factors in fonts, subdivisions and the precise location of each word. Google also analyzes the content of neighboring web pages to ensure the results returned are the most relevant to a user's query.

Yahoo! Search crawls the web using Yahoo! Slurp every 2–4 weeks and automatically finds new content for indexing. If the pages that are already in their index link to a site, this site will be considered for inclusion in the next update of the index. Yahoo! Search rank result according to their relevance to a particular query by analyzing the web page text, title and description accuracy as well as its source, associated links, and other unique document characteristics [12].

4 Design and Implementation

4.1 Architecture

The system architecture is shown in Fig. 1. Package `com.google.soap.search` and `com.yahoo.search` are APIs provided by Google and Yahoo! to

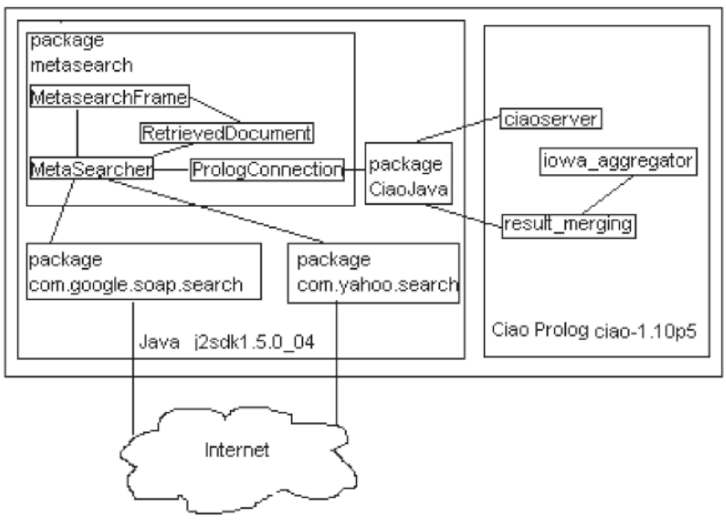

Fig. 1. System Architecture

perform web search. Package `CiaoJava` is provided by The Ciao Prolog System [5] as the interfacing package between Java and Ciao Prolog. `Ciaoserver` is a prolog query server, and `result_merging` and `iowa_aggregator` are the implementation of list fusion algorithm. Package `metasearch` is a Java implementation of the metasearch engine, based on the list fusion algorithm presented in Sect. 5.

4.2 Evaluation

Metasearcher

The Metasearcher interface is shown in Fig. 2. We take the query "information retrieval" as test sample, and ask for ten best results from each search engine. The left and right upper part of the screen in Fig. 2 shows the search results from Google and Yahoo!, respectively. Merged result is shown in the lower part of the screen. Using $W = [0.5, 0.5]$ and mean value of lists size as fitness score for each engine $f_h = \frac{10+10}{2} = 10$, we calculate local performance judgements and overall performance judgements. In the Table 1, the first column contains collection of documents retrieved by both search engines. Second and third columns contains local performance judgements from Google and Yahoo, respectively. Finally, the last column contains overall performance judgments which are calculated using IOWA operator. Thus, the metasearch result are these documents ordered by their overall performance judgements descendingly.

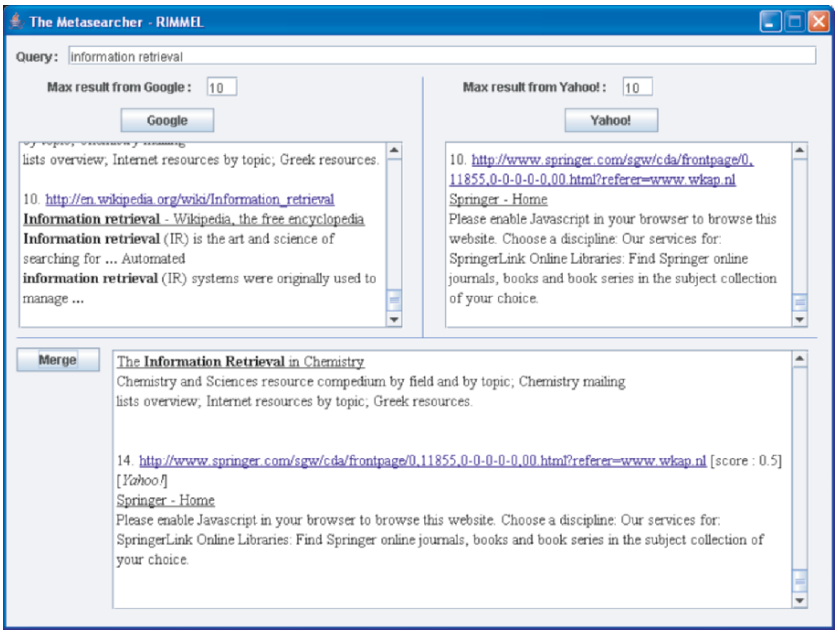

Fig. 2. Metasearch result

Table 1. Performance judgement for each webpage

Web Page	Google	Yahoo	Overall
http://www.dcs.gla.ac.uk/Keith/Preface.html	10	10	10
http://www.dcs.gla.ac.uk/∼iain/keith/	9	7	8
http://www.searchtools.com/info/info-retrieval.html	5	9	7
http://www.sims.berkeley.edu/∼hearst/irbook	8	6	7
http://www.springerlink.com/(rgjutuzz0zyeuhjmf1kyupzp) /app/home/journal.asp?referrer=parent &backto=linkingpublicationresults,1:103814,1	0	8	4
http://www.acm.org/sigir	6	2	4
http://ciir.cs.umass.edu	7	0	3.5
http://ir.dcs.gla.ac.uk	0	5	2.5
http://en.wikipedia.org/wiki/Information_retrieval	0	4	2
http://www.kluweronline.com/issn/1386-4564	4	0	2
http://dmoz.org/Computers/Software/Information_Retrieval	0	3	1.5
http://www.virage.com	3	0	1.5
http://macedonia.chem.demokritos.gr	2	0	1
http://web.syr.edu/∼diekemar/ir.html	0	1	0.5
http://www.budsir.org	1	0	0.5

Merging Module

As we have no benchmark for web search result, we evaluate the merging module using a given set of document collection, some queries and their results. Provided with collection of 423 documents from TIME [8] and 20 set of query and its corresponding ranked list of correct answers, we use information retrieval tool developed by Intelligent System Research Group (ISYS-GSI) at the Artificial Intelligence Department of Universidad Politécnica de Madrid (UPM) to do indexing and searching. Table 2 shows the set of queries and answers from TIME collection, and the answers given by our search engine.

Since we have only one search engine, it is not possible to use the merging module. Therefore, we divided the collection into two subcollection, namely part A and part B, in such a way so that the search results from part A always have greater rank than part B in the TIME query-answer samples. This partition mimics the condition where we have two collection of documents, where one is more preferred than the others, but sometimes the preferred collection has no answer for the query.

As described in sect. 5, in the ranked list fusion there are several free variables that act as parameter for the merging algorithm. These are :

- **Number of retrieved documents to be selected from each information source**
  In this case we will take all answers given by each subcollection.
- **Weighting vector of IOWA operator**
  We set the weighting vector W=[1,0] representing the fuzzy quantifier *at least 1*, considering that both subcollections are reliable information sources.

Table 2. Set of queries, its corresponding ranked list of correct answers from TIME collection, and answers given by search engine

query No.	TIME answers	search result
1.	268,288,304,308,323,326,334	308,370,378,257,326, 334,304,323, 288,268
2.	326,334	334,326,349
3.	326,350,364,385	316,349,339,421,326, 409,308,385,350
4.	370,378,385,409,421	334,349,370,385,350, 421,409,397,378
5.	359,370,385,397,421	350,385,421,397,409, 349,304,370
6.	257,268,288,304,308,323,324, 326,334	323,326,308,334,257, 349,397,268, 304,324
7.	386,408	386,408,423
8.	339,358	338,358
9.	61,155,156,242,269,315,339,358	156,61,269,242,358, 155,339
10.	61,156,242,269,339,358	358,61,339,303,155, 269,258,242,156
11.	195,198	198,135
12.	61,155,156,242,269,339,358	358,155,269,156,242, 303,61,339,258
13.	87,170,185	185,170,79,87,169
14.	269	279,269
15.	94,118,128,164,424	118,164,128,424,94
16.	169,170,239	350,185,169,239
17.	303,358	358,303
18.	356,99	356
19.	99,100,195,267,344	195,159,100,99,267, 344
20.	356	389,425

– **Fitness score f_h for each information source**
We set zero as fitness score for part B, and the maximum length between both ranked lists as the fitness score for part A. In this way, we try to represent our preference on part A.

Table 3 shows the search result on each partition, fitness score for each server, and the merged results. It is interesting to see that the ranked results after the merging procedure is nearly similar to those produced by querying the original collection. In most cases, rearranging documents having equal overall rank may produce identical ranked results between merged results and original collection results. This shows that the merging procedure can give answers almost as good as if the search is performed on a single collection.

5 Conclusion

In this work we have implemented a fuzzy approach for ranked list fusion, and apply it to combine search result from different web search engines, creating a meta web search engine. The approach is to calculate overall performance judgement for each document using IOWA operator, which aggregates local performance judgements. The search result are these documents ordered by overall performance judgement descendingly, because the greater performance

Table 3. Search results on each partition, fitness scores, and merged results

Query No.	Part A	Part B	f_1	f_2	Merging Results
1.	-	308,370,378,257, 326,334,304,323, 288,268	10	0	(308,5.0),(370,4.5),(378,4.0), (257,3.5),(326,3.0),(334,2.5), (304,2.0),(323,1.5),(288,1.0), (268,0.5)
2.	-	334,326,349	3	0	(334,1.5),(326,1.0),(349,0.5)
3.	-	316,349,339,421, 326,409,308,385, 350	9	0	(316,4.5),(349,4.0),(339,3.5), (421,3.0),(326,2.5),(409,2.0), (308,1.5),(385,1.0),(350,0.5)
4.	-	334,349,370,385, 350,421,409,397, 378	9	0	(334,4.5),(349,4.0),(370,3.5), (385,3.0),(350,2.5),(421,2.0), (409,1.5),(397,1.0),(378,0.5)
5.	-	350,385,421,397, 409,349,304,370	8	0	(350,4.0),(385,3.5),(421,3.0), (397,2.5),(409,2.0),(349,1.5), (304,1.0),(370,0.5)
6.	-	323,326,308,334, 257,349,397,268, 304,324	10	0	(323,5.0),(326,4.5),(308,4.0), (334,3.5),(257,3.0),(349,2.5), (397,2.0),(268,1.5),(304,1.0), (324,0.5)
7.	-	386,408,423	3	0	(386,1.5),(408,1.0),(423,0.5)
8.	-	338,358	2	0	(338,1.0),(358,0.5)
9.	156,61,155	269,242,358,339	4	0	(156,3.0),(269,2.0),(61,2.0), (242,1.5),(358,1.0),(155,1.0), (339,0.5)
10.	61,155,156	358,339,303,269, 258,242	6	0	(358,3.0),(61,3.0),(339,2.5), (303,2.0),(155,2.0),(269,1.5), (258,1.0),(156,1.0), (242,0.5)
11.	198,135	-	2	0	(198,2.0),(135,1.0)
12.	155,156,61	358,269,242,303, 339,258	6	0	(358,3.0),(155,3.0),(269,2.5), (242,2.0),(156,2.0),(303,1.5), (339,1.0),(61,1.0), (258,0.5)
13.	185,170,79,87, 169	-	5	0	(185,5.0),(170,4.0),(79,3.0), (87,2.0),(169,1.0)
14.	-	279,269	2	0	(279,1.0),(269,0.5)
15.	118,164,128,94	424	4	0	(118,4.0),(164,3.0),(128,2.0), (94,1.0),(424,0.5)
16.	185,169	350,239	2	0	(185,2.0),(350,1.0),(169,1.0), (239,0.5)
17.	-	358,303	2	0	(358,1.0),(303,0.5)
18.	-	356	1	0	(356,0.5)
19.	195,159,100,99	267,344	4	0	(195,4.0),(159,3.0),(100,2.0), (267,1.0),(99,1.0),(344,0.5)
20.	-	389,425	2	0	(389,1.0),(425,0.5)

judgements indicates the more relevant document. The ranked list fusion algorithm might be an alternative solution for information retrieval over a very large document collection. Our evaluation of the merging module shows that with appropriate partition and merging parameter, the result from a merging procedure can be almost as good as those obtained by performing a query on the original single collection.

Acknowledgment

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Some Fuzzy Counterparts of the Language uses of *And* and *Or*

Sergio Guadarrama, Eloy Renedo, and Enric Trillas

Summary. In this paper we want to explore some of the different uses of the conjunctions *and* and *or* in language, and how they are related with the corresponding theoretical models [2, 11]. To do this, we must study both behavior in language and the concrete meaning of its use. And after that test the models against their use in language.

Key words: Conjunction, Disjunction, Coherency, Uses of *and* and *or*, Lattices, Relations.

1 Introduction

As it was suggested in [5] fuzzy logic not only deals with issues in the technological side of computational intelligence but with what is known as the *Gordian Knot* of computational intelligence: the problem of meaning. A fuzzy set represents a concrete use of a predicate (or linguistic label) in the language and, attending to L. Wittgenstein assertion, "the meaning of a word is its use in the language", fuzzy logic also deals whit its meaning.

At the same time, fuzzy logic recognizes that there is no a single way of using the conjunction 'and', the disjunction 'or' and the negation 'not', and there are countless theories of fuzzy sets to represent them. These diverse representations of linguistic connectives are based on particular properties of their current meanings.

In language we find different uses of *and* and *or*, and therefore, different meanings of them. Any model willing to be useful to represent the linguistic *and* or linguistic *or* must capture the properties and the concrete meaning of its use. To develop these models we must study the behavior of *and*(*or*) in language in the first place.

The rest of the paper is organized as follows: first we show a list of different uses of *and* and *or* with some instructive examples extracted from dictionaries. In the following section we introduce the concept of qualified statements.

Next, it will be set out needed definitions: sets, operations, relation, and so on. Finally it will be described what a "coherent" model is, and how it must capture the conjunctive nature of the *and*'s uses and the disjunctive of the *or* one.

1.1 Uses of *and* and *or* Extracted from Dictionary

Let us transcribe the several uses of the conjunction *and* extracted from some English dictionaries (see [3, 4, 13]):

Copulative (also)

Used to join words, phrases, sentences or parts together. Examples:
I have socks and shoes.
I live in Madrid and she lives in Barcelona.
We have many flowers and plants.
John is tall and rich.

Copulative (in addition to)

Used with numbers. Examples:
One hundred and ten.
Three and two are five.
She walked one mile and half.

Copulative (very)

Used to join the same word, making their meaning stronger. Examples:
She walked miles and miles. (increase)
I tried and tried. (repetition)
He talked and talked. (continuation)

Copulative (distinction)

Used to make distinctions within the same word. Example:
There are lawyers and lawyers.

Copulative (despite)

Used to express surprise or some contradiction. Example:
You're a vegetarian and you eat fish.
You're tired and you are working.

Consecutive (then)

In this case the *and* has the meaning of *then* and links statements that are consecutive. It can express a temporal sequence, p *before than* q , can denote a consequence, p *cause that* q, or can be a necessity, p *in order to* q.

He came and went. (before than)
I was late and she got angry. (cause that)
I will go and see him. (in order to)

Now, let us transcribe the several uses of the conjunction *or* extracted from some English dictionaries (see [3, 4, 13]):

Disjunctive (possibilities)

Used to connect different possibilities; Used to indicate an alternative, usually only before the last term of a series . Examples:

Is the water hot or cold?
This, that, or the other.
You can pay now or when you come back to pick up the paint.
It doesn't matter whether you win or lose - it's taking part that's important.

Disjunctive (not either)

Used after a negative verb between a list of things to mean not any of those things or people:

Tim doesn't eat meat or fish.
She doesn't have a telephone or a fax machine.

Disjunctive and/or (logical or)

Used to mean an strictly inclusive or is required, i.e. either one of two things or both of them is possible:

Many pupils have extra classes in the evenings or [and/or] at weekends

Disjunctive either-or (logical xor)

Express an unavoidable choice or exclusive division between only two alternatives.

We can either eat now or after the show - it's up to you.

Descriptive (uncertainty)

Used to indicate uncertainty or indefiniteness:

Two or three.
There were ten or twelve people in the room.

She lives in Chicago or thereabouts.
Is it Tuesday or Wednesday today?
In some way that is not known yet:
We'll get out of this mess one way or another.

Descriptive (change)

Used to change or correct something you have said:
We told the truth, or most of it.

Descriptive (explain)

Used to show that a word or phrase means the same as, or explains or limits or corrects, another word or phrase:
Rosalind, or Roz to her friends, took the initiative.
Things have been going quite well recently, or they were up until now.
Used to indicate a synonymous or equivalent expression:
Acrophobia, or fear of great heights.
The culinary art or art of cookery.

Consecutive (if not)

In this case the *or* has the meaning of *if not* and links statements that are consecutive:
You should eat more, or you'll make yourself ill.

Consecutive (reason)

Used to give a reason for something you have said:
She must love him or she wouldn't have stayed with him all these years.

2 Standard Models of *And* (*Or*) in Fuzzy Logic

In Standard Theories of fuzzy sets *and* and *or* are typically represented by means of t-norms and t-conorms mainly due to their interesting properties, as it is explained below. But this is not mandatory as have been studied in [12] [3]. In these papers non standard theories having interesting properties were shown, and the Classical Preservation Principle was introduced, which we recall it here:

> Since crisp sets are particular instances of fuzzy sets, any newly defined fuzzy concept, when applied to crisp sets, should provide the same results as in the classical setting.

One should realize that any Standard Theory satisfy the Classical Preservation Principle, and the proposed conjunctions and disjunctions of this paper must also satisfy it.

2.1 T-norms

In fuzzy logic there are different models of *and*, which are typically represented by t-norms $T : [0,1] \times [0,1] \to [0,1]$, that are functions verifying:

- Commutativity: $T(a,b) = T(b,a)$.
- Monotonicity: $a \leq b \Rightarrow T(a,c) \leq T(b,c)$.
- Associativity: $T(a,T(b,c)) = T(T(a,b),c)$.
- Boundary conditions: $T(a,1) = a,\ T(a,0) = 0$.

2.2 T-conorms

In fuzzy logic there are different models of *or*, which are typically represented by t-conorms $S : [0,1] \times [0,1] \to [0,1]$, that are functions verifying:

- Commutativity: $S(a,b) = S(b,a)$.
- Monotonicity: $a \leq b \Rightarrow S(a,c) \leq S(b,c)$.
- Associativity: $S(a,S(b,c)) = S(S(a,b),c)$.
- Boundary conditions: $S(a,1) = 1,\ S(a,0) = a$.

The continuity of operations is important in the applications. Continuity implies that the t-norms T are only in either one of the three-families:

- $min_\varphi = \varphi^{-1} \circ min \circ (\varphi \times \varphi) = min$.
- $Prod_\varphi = \varphi^{-1} \circ Prod \circ (\varphi \times \varphi)$.
- $W_\varphi = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$.

with φ an order-automorphism of $([0,1], \leq)$. Also T can be an ordinal-sum of them, i.e. built using a set of continuous t-norms with idempotent elements different than 0 and 1.

Continuity also implies that the t-conorms S are only in either one of the three families:

- $max_\varphi = \varphi^{-1} \circ max \circ (\varphi \times \varphi) = max$.
- $Prod^*_\varphi = \varphi^{-1} \circ Prod^* \circ (\varphi \times \varphi)$.
- $W^*_\varphi = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$.

Also S can be an ordinal-sum of them, i.e. built using a set of continuous t-conorms with idempotent elements different than 0 and 1. (see [3]).

After reviewing the different uses of *and* (*or*) we realize that standard models are not enough to represent them. So, we have to search for new models of *and*(*or*) that are not necessarily commutative, idempotent or monotonic, as we do in the following sections introducing the concepts of conjunction, weak conjunction, disjunction, weak disjunction and exclusive or. In order to define conjunction and disjunction let us first introduce the concept of relation induced by an operator.

3 Relations Induced by an Operation

Given a universe U of elements, a relation R on U is a subset of $U \times U$, that is, $R \subseteq U \times U$.

Definition 1. *Given a universe U and a binary operation $* : U \times U \to U$, the $*$-relation R^l_* left induced by the operation $*$, is defined by*

$$(a, b) \in R^l_*, \text{ if it exists } c \in U, \text{ such that, } a = b * c \tag{1}$$

or equivalently

$$\forall\, a, b \in U : (a * b, a) \in R^l_* \tag{2}$$

The $*$-relation R^r_* right induced by the operation $*$ is defined by

$$(a, b) \in R^r_*, \text{ if it exists } c \in U, \text{such that, } a = c * b$$

or equivalently

$$\forall\, a, b \in U : (a * b, b) \in R^r_*.$$

Lemma 1. *If the operation $*$ is commutative, then the relations R^l_* and R^r_* induced by $*$ are identical, and we will denote it by $R_* = R^l_* = R^r_*$. (see the proof [7])*

Lemma 2. *If the operation $*$ is associative and verifies that for each $a \in U$ there exists an $e \in U$, such that $a = a * e$ (conversely, $a = e * a$), then, R^l_* (conversely R^r_*) is a preorder. (see the proof [7])*

4 Conjunction and Weak Conjunction

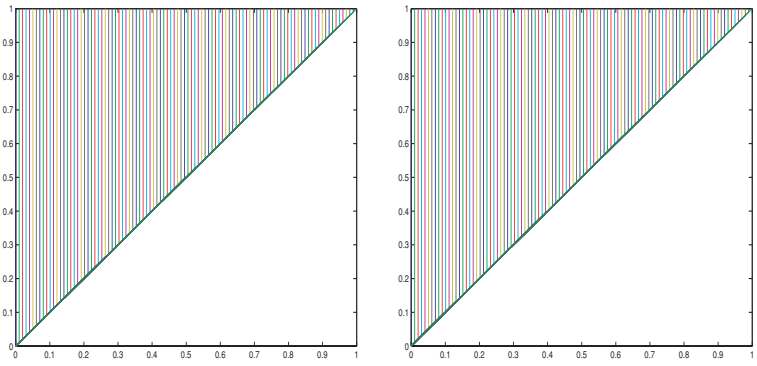
In what follows, we will use the concepts of conjunction and weak conjunction defined on the ordered set $([0, 1], \leq)$ (see [7, 11]).

Definition 2. *Given a ordered set $L = [0, 1]$ endowed with an order $\leq$ we will say that a binary operation $*$ in $([0, 1], \leq)$ is:*

A weak conjunction if $R_\leq \subset R^l_* = R^r_*$; conjunction if $R_\leq = R^l_* = R^r_*$
A weak left conjunction if $R_\leq \subset R^l_*$; left conjunction if $R_\leq = R^l_*$
A weak right conjunction if $R_\leq \subset R^r_*$; right conjunction if $R_\leq = R^r_*$

where $R_\leq$ is the order relation, i.e. $(x, y) \in R_\leq \Leftrightarrow x \leq y$.

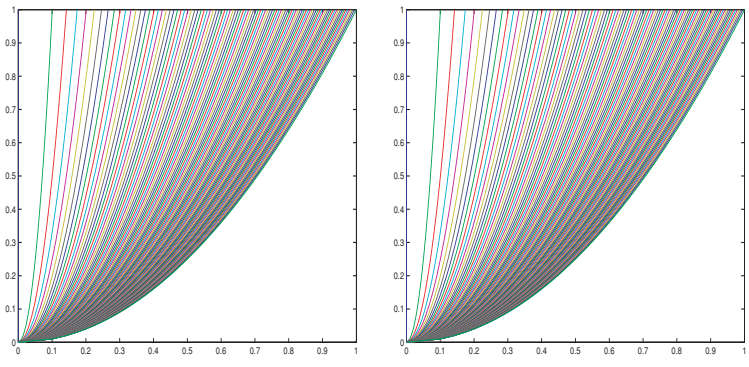
Example 1. Let it $L = [0,1]$ be endowed with the usual linear order $([0,1], \leq)$, and $*$ a continuous t-norm T ([8, 14])in $[0,1]^{[0,1]^2}$, then the diagram of the relations R_T and $R_{\leq}$ are:



R_T $R_{\leq}$

T is a conjunction in $([0,1], \leq)$, since $R_{\leq} = R_T$

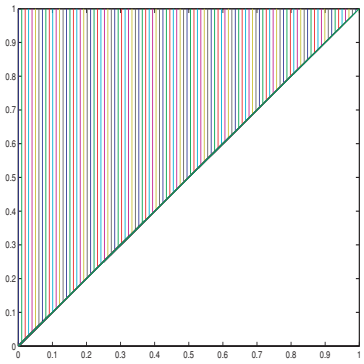
Example 2. Let it $L = [0,1]$ be endowed with the usual order $([0,1], \leq)$ and $* = gm$, (*geometric mean*) in $[0,1]^{[0,1]^2}$, then the inducted relation by gm is: $(a,b) \in R_{gm}$ if $\exists\, c = \frac{a^2}{b} \in [0,1]; a = \sqrt{b.c} \Leftrightarrow a^2 \leq b$



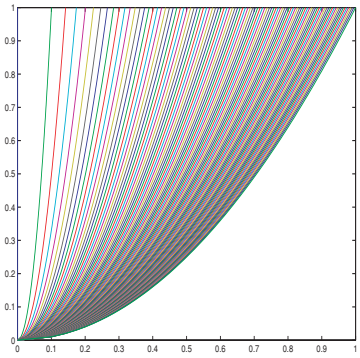
R_{gm} $R_{\leq}$

$R_{\leq} \subset R_{gm}$, therefore mg is a weak conjunction in $([0,1], \leq)$,

Example 3. Let it $L = [0,1]$ be endowed with the usual linear order $([0,1], \leq)$ and $* = am$, (*arithmetic mean*) in $[0,1]^{[0,1]^2}$, then the inducted relation by am is: $(a,b) \in R_{am}$ if $\exists\, c = 2a - b \in [0,1]; a = \frac{b+c}{2} \Leftrightarrow \frac{b}{2} \leq a \leq \frac{b}{2} + \frac{1}{2}$.



$R_{\leq}$



R_{am}

Since $R_{\leq} \nsubseteq R_{am}$ y $R_{am} \nsubseteq R_{\leq}$, *am* IS NOT a weak conjunction in $(L, \leq)$

4.1 Coherent Models

We will name truth assignment to any mapping $\mathcal{T} : E \to [0, 1]$ that assign to each element of a set of statements E a truth value of $[0, 1]$

Definition 3. *Let E be a set of statements and an operator $\& : E \times E \to E$, such that for any two statements p_1, p_2, give us the $\&$-statement $\&(p_1, p_2) =$ "p_1 and p_2". A truth assignment $\mathcal{T} : E \to [0, 1]$ is coherent with a use of* and *if the induced relation $R_{\&}$ is conjunction, and incorporates as properties the linguistic features of the use.*

Remark 1. Usually in fuzzy logic, it is assumed that the assignment of truth values to the &-statements is functionally expressible by an function $F : [0, 1] \times [0, 1] \to [0, 1]$, such that,

$$\mathcal{T}(p\&q) = F(\mathcal{T}(p), \mathcal{T}(q))$$

A particular case of F in $[0, 1]$ are t-norms $T : [0, 1] \times [0, 1] \to [0, 1]$, that all verify that for any truth assignment:

$$\mathcal{T}(p\&q) = T(\mathcal{T}(p), \mathcal{T}(q)) \leq min(\mathcal{T}(p), \mathcal{T}(q))$$

and therefore by definition (see [7]) are $\mathcal{T}$-conjunctions for any $\mathcal{T}$.

The case of the copulative *and* with the meaning of *also*

In this case the *and* can be commutative, and the statements p and q must be distinct, because otherwise it will change its meaning (see 1.1 or 1).

$$\mathcal{T}(p\&q) = \mathcal{T}(q\&p)$$

And also, it looks reasonable to assume that if "p" is true and "q" is true then "p and q" is true. And this with truth values in $([0,1],\leq)$ can be restated as:

$$\mathcal{T}(p)=1,\mathcal{T}(q)=1\Rightarrow\mathcal{T}(p\&q)=1,$$

Example 4. Let $p=$"I have socks" and $q=$"I have shoes" be, then $p\&q=$"I have socks and shoes" we can have two different truth assignments:

- One coherent: $\mathcal{T}_1(p)=1,\mathcal{T}_1(q)=1$, and being $\mathcal{T}_1(p\&q)=min(\mathcal{T}_1(p),\mathcal{T}_1(q))$ then
$$\mathcal{T}_1(p\&q)=1=min(\mathcal{T}_1(p),\mathcal{T}_1(q))$$
- One non coherent: $\mathcal{T}_2(p)=0$, $\mathcal{T}_2(q)=1$, and being $\mathcal{T}_2(p\&q)=am(\mathcal{T}_2(p),\mathcal{T}_2(q))=0.5$
$$\mathcal{T}_2(p\&q)=0.5\not\leq min(\mathcal{T}_2(p),\mathcal{T}_2(q))=0$$

Example 5. Let $p=$"John is handsome" and $q=$"John is rich" be, then $p\&q=$"John is handsome and rich" can be understood as "John is attractive".

- Given the truth assignment $\mathcal{T}(p)=0.1$, $\mathcal{T}(q)=0.7$ then this truth assignment is more coherent with $\mathcal{T}_1(p\&q)=gm(\mathcal{T}(p),\mathcal{T}(q))=\sqrt{0.1*0.7}=0.26$ than with $\mathcal{T}_2(p\&q)=am(\mathcal{T}(p),\mathcal{T}(q))=0.4$ because gm is a weak conjunction and because
$$\mathcal{T}_1(p\&q)=0.26\leq am(\mathcal{T}_1(p),\mathcal{T}_1(q))=0.4$$

The case of the copulative *and* with the meaning of *very*

In this case the statements are the same, its truth value have a restrictive behavior and usually is not idempotent.

$$\mathcal{T}(p\&p)\leq\mathcal{T}(p)$$

Example 6. (See 1.1)

- Let be $p=$"He talked" and $p\&p=$ "He talked and talked" then the truth assignment $\mathcal{T}_1(p)=0.8$ and $\mathcal{T}_1(p\&p)=\mathcal{T}_1(p)*\mathcal{T}_1(p)$ is coherent, since
$$\mathcal{T}_1(p\&p)=0.64\leq\mathcal{T}_1(p)$$
- Let be $p=$"She walked miles" and $p\&p=$ "She walked miles and miles" then the truth assignment $\mathcal{T}_2(p\&p)=W(\mathcal{T}(p),\mathcal{T}(p))=0.5$ to be coherent will force that $\mathcal{T}_2(p)\geq 0.75$ since
$$\mathcal{T}_2(p\&p)=min(0,\mathcal{T}_2(p)+\mathcal{T}_2(p)-1)=0.5\leq\mathcal{T}_2(p)\Rightarrow\mathcal{T}_2(p)\geq 0.75$$

The case of the consecutive *and* with the meaning of *then*

In this case the use of *and* links statements that are consecutive and it introduces a relation between the statements. If we have an operator $\rhd : U \times U \rightarrow U$ that represents the relation *"p then q,"* then the truth assignment to be coherent must verify:

$$\mathcal{T}(p\&q) \leq \mathcal{T}(p) \cdot \mathcal{T}(p \rhd q)$$

And if we want that this relation $\rhd$ is conditional (see [1]) must verify:

$$\mathcal{T}(p) \cdot \mathcal{T}(p \rhd q) \leq \mathcal{T}(p) \cdot \mathcal{T}(q)$$

Those can be comprised in:

$$\mathcal{T}(p\&q) \leq \mathcal{T}(p) \cdot \mathcal{T}(p \rhd q) \leq \mathcal{T}(p) \cdot \mathcal{T}(q)$$

Example 7. (See 1.1)

- Let be $p =$"He came," $q =$"He went," $p\&q =$ "He came and went" and $p \rhd q =$"He came before he went," then the truth assignment $\mathcal{T}_1(p) = 1$, $\mathcal{T}_1(q) = 1$, and $\mathcal{T}_1(p\&q) = 1$ is coherent only if $\mathcal{T}_1(p \rhd q) = 1$, since

  $$1 = \mathcal{T}_1(p\&q) \leq min(\mathcal{T}_1(p), \mathcal{T}_1(p \rhd q)) \leq min(\mathcal{T}_1(p), \mathcal{T}_1(q)) = 1$$

  In the case that "he came, he went and he went before came," the truth assignment $\mathcal{T}_1(p) = 1$, $\mathcal{T}_1(q) = 1$, and $\mathcal{T}_1(p \rhd q) = 0$ is coherent only if $\mathcal{T}_1(p\&q) = 0$, since

  $$\mathcal{T}_1(p\&q) \leq min(\mathcal{T}_1(p), \mathcal{T}_1(p \rhd q)) = 0$$

- Let be $p =$"I was late," $q =$"She got angry," $p\&q =$"I was late and she got angry" and $p \rhd q =$"She got angry because I was late," then the truth assignment $\mathcal{T}_2(p) = 0.8$, $\mathcal{T}_2(q) = 0.7$, and $\mathcal{T}_2(p \rhd q) = 0.5$ is coherent if

  $$\mathcal{T}_2(p\&q) \leq min(\mathcal{T}_2(p), \mathcal{T}_2(p \rhd q)) = 0.5$$

  In the case that she got angry because other causes then $\mathcal{T}_2$ will be coherent if

  $$\mathcal{T}_2(p\&q) \leq min(\mathcal{T}_2(p), \mathcal{T}_2(q)) = 0.7$$

5 Inclusive or: Disjunction and Weak Disjunction

The use of *or* in common language differs to its use in classical logic. Mainly due to that *or* in language usually behaves as an "exclusive or" while in classical logic behaves as an "inclusive or". In fact, to express in common

language that the *or* is inclusive we use the "and/or" particle, as we can see in the following example:

– Many pupils have extra classes in evenings and/or at weekends. (*inclusive or*).

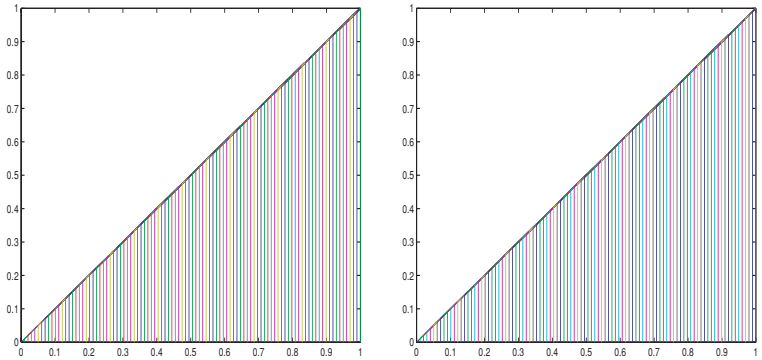
In what follows, we will use the concepts of disjunction and weak disjunction defined on the ordered set $([0,1],\leq)$ (see [7,11]) to define the "inclusive or."

Definition 4. *Given a ordered set* $L=[0,1]$ *equipped with an order* $\leq$ *we will say that a binary operation* $*$ *in* $([0,1],\leq)$ *is:*

A weak disjunction if $R_{\geq}\subset R^l_*=R^r_*$ *; disjunction if* $R_{\geq}=R^l_*=R^r_*$
A weak left disjunction if $R_{\geq}\subset R^l_*$ *; left disjunction if* $R_{\geq}=R^l_*$
A weak right disjunction if $R_{\geq}\subset R^r_*$ *; right disjunction if* $R_{\geq}=R^r_*$

where $R_{\geq}$ *is the opposite of the order relation, i.e.* $(x,y)\in R_{\geq}\Leftrightarrow x\geq y$.

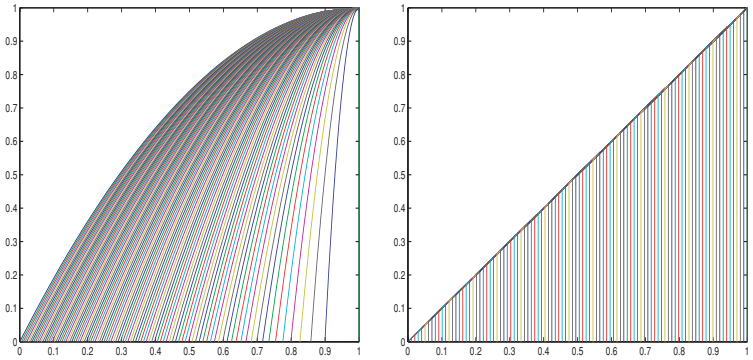
Example 8. Let $L=[0,1]$ be endowed with the usual order $([0,1],\leq)$ and $*$ a continuous t-conorm S ([8,14])in $[0,1]^{[0,1]^2}$, then the diagram of the relations R_S and $R_{\geq}$ are:



R_S $\qquad\qquad$ $R_{\geq}$

S is a disjunction in $([0,1],\leq)$, since $R_{\geq}=R_S$

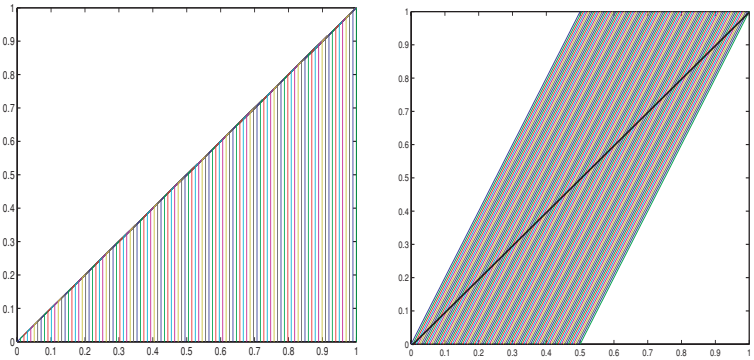
Example 9. Let $L=[0,1]$ be endowed with the usual order $([0,1],\leq)$ and $*=dgm$, (*dual of geometric mean*) in $[0,1]^{[0,1]^2}$, then the inducted relation by $dgm(x,y)=1-\sqrt{(1-x).(1-y)}$ is: $(a,b)\in R_{dgm}$ if $\exists\, c=\frac{a^2}{b}\in[0,1]; a=1-\sqrt{(1-b).(1-c)}\Leftrightarrow a^2\leq b$



R_{dgm} $R_{\geq}$

$R_{\geq} \subset R_{dgm}$, therefore dmg is a weak disjunction in $([0,1], \leq)$,

Example 10. Let $L = [0,1]$ be endowed with the usual order $([0,1], \leq)$ and $* = am$, (*arithmetic mean*) in $[0,1]^{[0,1]^2}$, then the inducted relation by am is: $(a,b) \in R_{am}$ if $\exists\, c = 2a - b \in [0,1]; a = \frac{b+c}{2} \Leftrightarrow \frac{b}{2} \leq a \leq \frac{b}{2} + \frac{1}{2}$.



$R_{\geq}$ R_{am}

$R_{\geq} \nsubseteq R_{am}$ y $R_{am} \nsubseteq R_{\geq}$, therefore am IS NOT a weak disjunction in $(L, \leq)$

Remark 2. In any lattice $(L, \leq, \cdot, +)$, the operator $\cdot$ is a conjunction and the operator $+$ is a disjunction since for all $x, y \in L, x = x \cdot y, y = x + y$ is equivalent to $x \leq y$, and therefore, the induced relations verify, $R_{\cdot} = R_{\leq}$ and $R_{+} = R_{\geq}$.

5.1 Coherent Models

Definition 5. *Let E be a set of statements and an operator $\vee : E \times E \to E$, such that for any two statements p_1, p_2, give us the $\vee$-statement $\vee(p_1, p_2) =$ "p_1 or p_2". A truth assignment $\mathcal{T} : E \to [0,1]$ is coherent with a use of* or *if the induced relation $R_{\vee}$ is a disjunction, and incorporates as properties the linguistic features of the use.*

Remark 3. Usually in fuzzy logic, it is assumed that the assignment of truth values to the $\vee$-statements is functionally expressible by an function $G : [0,1] \times [0,1] \to [0,1]$, such that,

$$\mathcal{T}(p \vee q) = G(\mathcal{T}(p), \mathcal{T}(q))$$

A particular case of G in $[0,1]$ are t-conorms $S : [0,1] \times [0,1] \to [0,1]$, that all verify that for any truth assignment:

$$\mathcal{T}(p \vee q) = S(\mathcal{T}(p), \mathcal{T}(q)) \geq max(\mathcal{T}(p), \mathcal{T}(q))$$

and therefore by definition (see [7]) are $\mathcal{T}$-disjunctions for any $\mathcal{T}$.

The case of the disjunctive *and/or* expressing *logical or*

In this case the *and/or* can be commutative and express that both possibilities can be simultaneous, it is a inclusive-or (logical or)

$$\mathcal{T}(p \vee q) = \mathcal{T}(q \vee p)$$

And also, it looks reasonable to assume that if "p or q" is true then "p" is true or "q" is true. And this with truth values in $([0,1], \leq)$ can be restated as:

$$max(\mathcal{T}(p), \mathcal{T}(q)) \leq \mathcal{T}(p \vee q)$$

Example 11. Let $p =$"Many pupils have extra classes in the evenings" and $q =$"Many pupils have extra classes at weekends" be, then $p \vee q =$"Many pupils have extra classes in the evenings and/or at weekends" we can have two different truth assignments:

- One coherent: $\mathcal{T}_1(p) = 1, \mathcal{T}_1(q) = 1$, and being $\mathcal{T}_1(p \vee q) = max(\mathcal{T}_1(p), \mathcal{T}_1(q))$ then
$$\mathcal{T}_1(p \vee q) = 1 = max(\mathcal{T}_1(p), \mathcal{T}_1(q))$$
- One non coherent: $\mathcal{T}_2(p) = 0$, $\mathcal{T}_2(q) = 1$, and being $\mathcal{T}_2(p \vee q) = am(\mathcal{T}_2(p), \mathcal{T}_2(q)) = 0.5$

$$\mathcal{T}_2(p \vee q) = 0.5 \ngeq max(\mathcal{T}_2(p), \mathcal{T}_2(q)) = 1$$

6 Exclusive or: Symmetric Difference

One can realize from the following examples that "exclusive or" in these examples have different degrees of exclusiveness. They are:

- The children always smiles or laughs. (*less exclusive or*)
- The patent was granted in 1962 or 1963. (*more exclusive or*)
- Either you leave now or I call the police. (*most exclusive or*)

The "exclusive or"= *xor* in classical logic is modeled by the well known symmetric difference operator.

A binary operator Δ defined in a Fuzzy Set Theory $(\mathcal{F}(X), \wedge, \vee, ')$ will be called a symmetric difference operator if the following condition is satisfied:

For any $\mu, \sigma \in \mathcal{F}(X), \mu\Delta\sigma \leq \mu \vee \sigma$ and $\mu\Delta\sigma \leq \mu' \vee \sigma'$
Or equivalently $\mu\Delta\sigma = (\mu \vee \sigma) \wedge (\mu' \vee \sigma')$

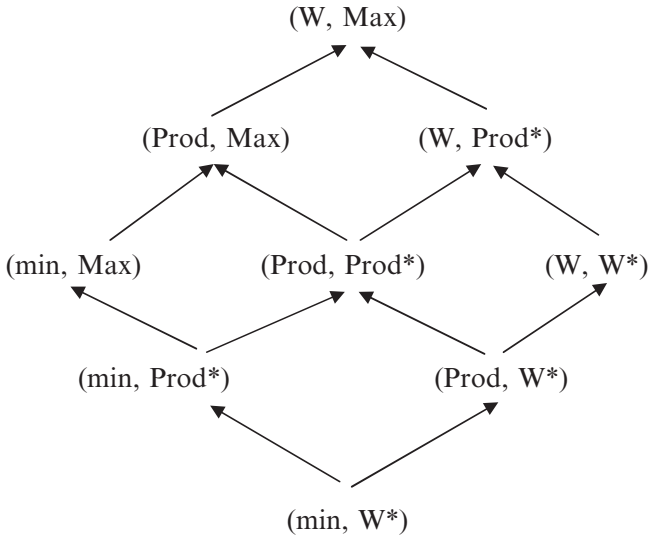


Fig. 1. Exclusiveness order of different fuzzy xor, (T, S)

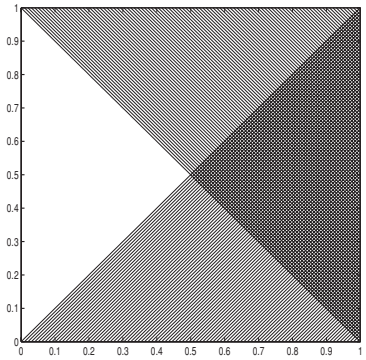
Definition 6. *Now we can functionally express xor in a Standard Fuzzy Theory* $([0,1]^X, T, S, N)$ *as:*

$$\Delta(a,b) = T(S(a,b), S(N(a), N(b)))$$

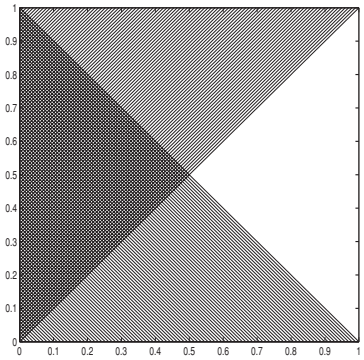
This *xor* operators can be ranged from the most exclusive to the less exclusive one (see Fig. 1):

The most exclusive or is $\Delta(a,b) = W(max(a,b), max(1-a, 1-b)$
The less exclusive or is $\Delta(a,b) = min(W^*(a,b), W^*(1-a, 1-b)$
One example in between is $\Delta(a,b) = min(max(a,b), max(1-a, 1-b)$

Example 12. Let $L = [0,1]$ be endowed with the usual order $([0,1], \leq)$ and *xor* operators $\Delta_{(min,W^*)} = min(W^*(a,b), W^*(1-a, 1-b))$ $\Delta_{(W,max)} = W(max(a,b), max(1-a, 1-b))$, then the inducted relations by $\Delta_{(min,W^*)}$ and $\Delta_{(W,max)}$ are:



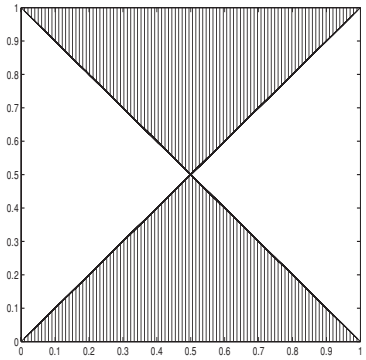
$R_{\Delta_{(min,W^*)}}$



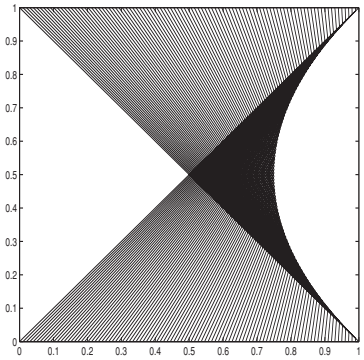
$R_{\Delta_{(W,max)}}$

One can see that $R_{\Delta_{(min,W^*)}}$ is a weak disjunction in $([0,1],\leq)$, since $R_{\geq} \subset R_{\Delta_{(min,W^*)}}$, and $R_{\Delta_{(W,max)}}$ is a weak conjunction in $([0,1],\leq)$, since $R_{\leq} \subset R_{\Delta_{(W,max)}}$.

Example 13. Let $L = [0,1]$ be endowed with the usual order $([0,1],\leq)$ and *xor* operators $\Delta_{(min,max)} = min(max(a,b), max(1-a,1-b))$ $\Delta_{(min,Prod^*)} = min((a+b-a.b),(1-a+1-b-(1-a).(1-b))$, then the inducted relations by them are:



$R_{\Delta_{(min,max)}}$



$R_{\Delta_{(min,Prod^*)}}$

One can realize that neither $R_{\Delta_{(min,max)}}$ nor $R_{\Delta_{(min,Prod^*)}}$ are a weak disjunction or conjunction in $([0,1],\leq)$, since $R_{\leq} \nsubseteq R_{\Delta_{(min,max)}} \ngeq R_{\geq}$, and $R_{\leq} \nsubseteq R_{\Delta_{(min,Prod^*)}} \ngeq R_{\geq}$.

The case of the disjunctive *or* expressing *alternatives*

In this case the *or* express different alternatives that should be in some way exclusive, it is an exclusive-or.

$$\mathcal{T}(p \Delta q) = \mathcal{T}((p \vee q)\&(p' \vee q'))$$

And also, it looks reasonable to assume that if "p or q" is true then "p" is true or "q" is true but not both. And this with truth values in $([0,1],\leq)$ can be restated as:

$$\mathcal{T}(p\Delta q) = T(S(\mathcal{T}(p),\mathcal{T}(q)), S(1-\mathcal{T}(p),1-\mathcal{T}(q)))$$

Example 14. Let $p =$"The children always smiles" and $q =$"The children always laughs" be, then $p \vee q =$"The children always smiles or laughs" we can have the coherent truth assignments with the less exclusive-or given by:

- $\mathcal{T}(p) = 0.5$, $\mathcal{T}(q) = 0.5$, and being $\mathcal{T}(p\Delta q) = \Delta_{(min,W^*)}$ then
$$\mathcal{T}(p\Delta q) = 1 \leq W^*(\mathcal{T}(p),\mathcal{T}(q))$$
- $\mathcal{T}(p) = 0.8$, $\mathcal{T}(q) = 0.7$, and being $\mathcal{T}(p\Delta q) = \Delta_{(min,W^*)}$ then
$$\mathcal{T}(p\Delta q) = 0.5 \leq W^*(\mathcal{T}(p),\mathcal{T}(q))$$

Example 15. Let $p =$"The patent was granted in 1962" and $q =$"The patent was granted in 1963" be, then $p\Delta q =$"The patent was granted in 1962 or 1963" we can have the coherent truth assignments the more exclusive-or given by:

- $\mathcal{T}(p) = 0.5$, $\mathcal{T}(q) = 0.5$, and being $\mathcal{T}(p\Delta q) = \Delta_{(min,max)}$ then
$$\mathcal{T}(p\Delta q) = 0.5 \leq max(\mathcal{T}(p),\mathcal{T}(q))$$
- $\mathcal{T}(p) = 0.8$, $\mathcal{T}(q) = 0.7$, and being $\mathcal{T}(p\Delta q) = \Delta_{(min,max)}$ then
$$\mathcal{T}(p\Delta q) = 0.3 \leq max(\mathcal{T}(p),\mathcal{T}(q))$$

Example 16. Let $p =$"Either you leave now" and $q =$"I call the police" be, then $p\Delta q =$"Either you leave now or I call the police" we can have the coherent truth assignments with the most exclusive-or given by:

- $\mathcal{T}(p) = 0.5$, $\mathcal{T}(q) = 0.5$, and being $\mathcal{T}(p\Delta q) = \Delta_{(W,max)}$ then
$$\mathcal{T}(p\Delta q) = 0 \leq max(\mathcal{T}(p),\mathcal{T}(q))$$
- $\mathcal{T}(p) = 0.8$, $\mathcal{T}(q) = 0.7$, and being $\mathcal{T}(p\Delta q) = \Delta_{(W,max)}$ then
$$\mathcal{T}(p\Delta q) = 0.1 \leq max(\mathcal{T}(p),\mathcal{T}(q))$$

7 Conclusions

Proper models of linguistic *and*(*or*) must capture its conjunctive (disjunctive) nature as well as the concrete meaning of its use. The pros and cons of each model and how to get the coherency between models and uses have been shown. This framework helps to study and to expand the representations of the use of words in language.

In this sense, the aim of this paper is to study those particular properties allowing to capture different uses of *and* ([5,6])and of *or*. That implies finding models which can incorporate the properties characteristic of each use, and this can by made by computationally testing them against linguistic examples which contain concrete uses of words and phrases.

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Fuzzy Sets Versus Language

Enric Trillas, Eloy Renedo, and Sergio Guadarrama

Summary. This paper just tries to stimulate some reflection to extend the current theories of fuzzy sets to wider areas of language, with the objective of reaching a better knowledge of the links between language and its representation by means of fuzzy sets, when possible. This for the progress of computing with words that, sooner or later, will pose the theoretically challenging, and practically important, problem of the linguistic credit, or soundness in language, of the theories of fuzzy sets. A problem that fuzzy logic cannot avoid to become a basic representation's tool for computing with words. To this end, the strategy of reconsidering the current knowledge of fuzzy sets theories does not seem far from scope.

Key words: Fuzzy sets, Theories of fuzzy sets, Imprecision, Language.

1 Introduction

Although the set of functions $[0,1]^X$ is called that of fuzzy sets in X (see [7]) each function $\mu : X \to [0,1]$ is only a purely mathematical entity until, for some predicate P on X can be recognized that

$$\text{Degree up to which "x is P"} = \mu(x),$$

for all $x \in X$. In this case it is written $\mu = \mu_P$.

To know that for all $x \in X$ the degree up to which "x is P" is $\mu(x)$ is to know how P is used on X and, hence, the function μ_P represents a use of P on X, that is, following Ludwig Wittgestein's *Philosophical Investigation*, μ_P represents the current *meaning* of P on X. Of course, it is not the case that all predicates on X do accept a degree in [0,1] (see [5]) but, in this paper, we will only consider such case.

Once it is recognized that the current use of P on X can be represented by $\mu_P \in [0,1]^X$, one can change the name of things and say that

$$x \underset{r}{\in} \underset{\sim}{P}, \text{ provided that degree up to which "x is P" is } \mu_P(x) = r$$

The new object $\underset{\sim}{P}$, to which elements in X belong with degree $\mu_P(x)$, is named as the fuzzy set with linguistic label P. Is in this way that the set $\mathcal{F}(X)$ of fuzzy sets in X is identified with $[0,1]^X$ once a definition of inclusion and equality is defined. In general, it is accepted that a fuzzy set μ_P is *included* in a fuzzy fuzzy set μ_Q, $\mu_P \leq \mu_Q$, if and only if, (see [2,3,7]):

$$\mu_P(x) \leq \mu_Q(x), \text{ for all } x \in X,$$

and that the fuzzy sets μ_P and μ_Q are *equal* if and only if $\mu_P \leq \mu_Q$ and $\mu_Q \leq \mu_Q$, that is

$$\mu_P(x) = \mu_Q(x) \Leftrightarrow \mu_P(x) = \mu_Q(x), \text{ for all } x \in X,$$

These definitions are possibly excessive, since a single small variation in a couple of values $\mu_P(x), \mu_Q(x)$, can change the relative status of two fuzzy sets $\underset{\sim}{P}$ and $\underset{\sim}{Q}$. Anyway, such definitions are the immediate generalization of the corresponding concepts with classical sets, and since

$$(\mathbb{P}(X), \cup, \cap, ^c) \text{ is isomorphic to } (\{0,1\}^X, min, max, 1 - id)$$

by means of

$$A \to \mu_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A, \end{cases}$$

it results that $\mathbb{P}(X)$ is a part of $\mathcal{F}(X)$.

Contrarily to the case of classical sets in $\mathbb{P}(X)$, where $\cap$, $\cup$ and ^{c}, are unique, it was recognized from the very beginning that, with imprecise predicates, the connectives "and", "or", and "not", have different uses. Hence, with fuzzy sets, intersection, union, and complement are not unique, and there is not a single theory of fuzzy sets. Actually, there are many theories of fuzzy sets $(\mathcal{F}(X), \cdot, +, ')$ that depart form the structure $(\mathcal{F}(X), \leq, =)$, the inclusion $\{0,1\}^X \subset [0,1]^X$, and the restriction

$$\mu_P \cdot \mu_Q = \mu_{P \cap Q}, \mu_P + \mu_Q = \mu_{P \cup Q}, \mu'_P = \mu_{P^c},$$

if P, Q are classical subsets of X, once the functions

$$\cdot : [0,1]^X \times [0,1]^X \to [0,1]^X, + : [0,1]^X \times [0,1]^X \to [0,1]^X, ' : [0,1]^X \to [0,1]^X$$

that, respectively, represent the uses of "and" (intersection), "or" (union), and "not" (complement), are defined.

Notice that because of the inclusion $\leq$ (partial order), *the only theories* $(\mathcal{F}(X), \cdot, +, ')$ *that are lattices is with* $\cdot = min$, $+ = max$. Hence, *no theory of fuzzy sets is a boolean algebra.*

2 On General Theories of Fuzzy Sets

2.1

In the classical setting each *precise* predicate P on X *specifies* a single set $\underset{\sim}{P}$ in X such that: "x is P" $\Leftrightarrow x \in \underset{\sim}{P}$, "x is not P" $\Leftrightarrow x \notin \underset{\sim}{P}$. Even more,

- "x is P" and "x is Q" $\Leftrightarrow$ "x is P *and* Q", or $[x \in \underset{\sim}{P}$ and $x \in \underset{\sim}{Q} \Leftrightarrow x \in \underset{\sim}{P} \cap \underset{\sim}{Q}]$
- "x is P" or "x is Q" $\Leftrightarrow$ "x is P *or* Q", or $[x \in \underset{\sim}{P}$ or $x \in \underset{\sim}{Q} \Leftrightarrow x \in \underset{\sim}{P} \cup \underset{\sim}{Q}]$
- "not (x is P)" $\Leftrightarrow$ "x is *not* P", or $[x \notin \underset{\sim}{P} \Leftrightarrow x \in \underset{\sim}{P}^c]$

In the fuzzy setting the situation is more complicated since *it is also needed to know how are used* "and", "or", and "not". That is, given imprecise predicates P, Q, ... on X, once given the corresponding functions μ_P, μ_Q, ..., that represent their current uses, it is needed to define the connectives $\cdot$, $+$, and $'$, in such a way that

$$\mu_{P \text{and} Q} = \mu_P \cdot \mu_Q, \mu_{P \text{or} Q} = \mu_P + \mu_Q, \mu_{\text{not} P} = \mu'_P.$$

Hence, with imprecise predicates, *each family of predicates on X does specify not only the corresponding fuzzy sets but also a theory of fuzzy sets* $(\mathcal{F}(X), \cdot, +, ')$.

2.2

A basic requisite for any theory of fuzzy sets is

- If $\mu, \sigma \in \{0,1\}^X$, then $\mu \cdot \sigma = min \circ (\mu \times \sigma), \mu + \sigma = max \circ (\mu \times \sigma), \mu' = 1 - \mu$,

to be sure that $(\mathbb{P}(X), \cap, \cup, ^c)$ is part of such theory. This is a requisite that comes from the necessity of working jointly with precise and imprecise predicates since it is not always the case that the family of predicates is also of imprecise ones.

Apart from that previous condition, that can be called as the *Principle of Preservation of Classical Case*, which laws could be taken as common for all theories of fuzzy sets? Of course, there is no a single set of basic laws but several minimal sets of these laws (see [3]).

Representing by μ_r the constant fuzzy sets $\mu_r(x) = r$, for all $x \in X$, the functions μ_0 and μ_1 do represent, respectively, the empty set ϕ and the total set X. With this, a *minimal set of common laws* for all theories of fuzzy sets $([0,1]^X, \cdot, +, ')$ is:

- $\mu_0 \cdot \mu = \mu \cdot \mu_0 = \mu_0$, $\mu_1 \cdot \mu = \mu \cdot \mu_1 = \mu$
- $\mu_0 + \mu = \mu + \mu_0 = \mu$, $\mu_1 + \mu = \mu + \mu_1 = \mu_1$
- If $\mu \leq \sigma$, then $\mu \cdot \rho \leq \sigma \cdot \rho, \rho \cdot \mu \leq \rho \cdot \sigma$ for all $\rho \in [0,1]^X$
- If $\mu \leq \sigma$, then $\mu + \rho \leq \sigma + \rho, \rho + \mu \leq \rho + \sigma$ for all $\rho \in [0,1]^X$

- $\mu'_0 = \mu_1$, $\mu'_1 = \mu_0$
- If $\mu \leq \sigma$, then $\sigma' \leq \mu'$

Given a family of predicates, some additional laws could be added to these seven laws, depending on the concrete uses of the connectives $and(\cdot)$, $or(+)$, $not(')$ between the predicates. For example, the following laws:

- *Commutative Laws:* $\mu \cdot \sigma = \sigma \cdot \mu$, $\mu + \sigma = \sigma + \mu$,
- *Associative Laws:* $\mu \cdot (\sigma \cdot \rho) = (\mu \cdot \sigma) \cdot \rho$, $\mu + (\sigma + \rho) = (\mu + \sigma) + \rho$,
- *Idempotency Laws:* $\mu \cdot \mu = \mu$, $\mu + \mu = \mu$,
- *Distributive Laws:* $\mu \cdot (\sigma + \rho) = \mu \cdot \sigma + \mu \cdot \rho$, $\mu + \sigma \cdot \rho = (\mu + \sigma) \cdot (\mu + \rho)$,
- *Involution Law:* $(\mu')' = \mu'' = \mu$,
- *Duality Laws:* $(\mu + \sigma)' = \mu' \cdot \sigma'$, $(\mu \cdot \sigma)' = \mu' + \sigma'$,

verified by classical sets, are *optional* for fuzzy sets. Concerning the two last of these laws, notice that duality with the involutive one give

$$\mu + \sigma = (\mu + \sigma)'' = (\mu' \cdot \sigma')'$$

$$\mu \cdot \sigma = (\mu \cdot \sigma)'' = (\mu' + \sigma')',$$

that is, either $+$ or $\cdot$ is defined from $\cdot$ and $'$, or from $+$ and $'$, respectively. Also, in this case, for example, from $(\mu' + \sigma')' = \mu'' \cdot \sigma'' = \mu \cdot \sigma$ follows $\mu' + \sigma' = (\mu \cdot \sigma)'$, and from $(\mu' \cdot \sigma')' = \mu + \sigma$ follows $\mu' \cdot \sigma' = (\mu + \sigma)'$, that is, just one of the two laws of duality is enough.

In general, the theory of fuzzy sets specified by a family of predicates will inherit the laws these predicates do verify with respect to the connectives and, or, not. For example, if for all P and all $x \in X$, "x is P" and "x is P" coincides with "x is P", then $\mu_P \cdot \mu_P = \mu_P$ implies that for all $\mu \in [0,1]^X$ it should be $\mu \cdot \mu = \mu$ (idempotency of $\cdot$). Notice that this laws, jointly with the law of duality $(\mu + \sigma)' = \mu' \cdot \sigma'$ implies $\mu + \mu = (\mu' \cdot \mu')' = (\mu')'$ that, if there is also the law of involution, gives $\mu + \mu = \mu$ (idempotency of $+$). Some laws can imply some other laws.

3 Decomposable Theories of Fuzzy Sets

3.1

A theory $([0,1]^X, \cdot, +, ')$ is *decomposable* (see [3]) provided there exist numerical functions

$$F : [0,1] \times [0,1] \to [0,1], G : [0,1] \times [0,1] \to [0,1], N : [0,1] \to [0,1],$$

such that, $\mu \cdot \sigma = F \circ (\mu \times \sigma), \mu + \sigma = G \circ (\mu \times \sigma), \mu' = N \circ \mu$. Of course, a theory can be *partially decomposable* when only some of the three operations $\cdot, +, '$ are decomposable.

A typical example of a decomposable theory is with $F = min, G = max$, and $N = 1-id$, a theory that verifies the totality of the laws listed in Sect. 2.2, plus the law of Kleene $\mu \cdot \mu' \leq \sigma + \sigma'$, for all $\mu, \sigma \in [0,1]^X$, since $min(a, 1-a) \leq max(b, 1-b)$ for all $a, b \in [0,1]$.

In the decomposable theories, all its laws can be reduced to laws in [0,1]. For example, the law of involution is $N^2(a) = a$; the law of duality is $N(G(a,b)) = F(N(a), N(b))$; the commutative laws are $F(a,b) = F(b,a)$ and $G(a,b) = G(b,a)$; the laws of idempotency are $F(a,a) = a$, and $G(a,a) = a$; the associative laws are $F(a, F(b,c)) = F(F(a,b),c)$, and $G(a, G(b,c)) = G(G(a,b),c)$; etc.

Notice that the principle of preservation and the minimal set of laws translate into:

- $F(0,0) = F(0,1)) = F(1,0) = 0$, $F(1,1) = 1$; $G(1,1) = G(1,0) = G(0,1) = 1$, $G(0,0) = 0$; $N(0) = 1$, $N(1) = 0$.
- $F(0,a) = F(a,0) = 0$, $F(1,a) = F(a,1) = a$; $G(1,a) = G(a,1) = 1$, $G(0,a) = G(a,0) = a$.
- If $a \leq b$, then $F(a,c) \leq F(b,c), F(c,a) \leq F(c,b)$ for all $c \in [0,1]$.
- If $a \leq b$, then $G(a,c) \leq G(b,c), G(c,a) \leq G(c,b)$ for all $c \in [0,1]$.
- If $a \leq b$, then $N(b) \leq N(a)$.

In general, the study of the laws of fuzzy sets is reduced to the study of Functional Equations and Inequations.

3.2

An important kind of decomposable theories are those that verify the laws of associativity, commutativity, and involution. In this case, as it is well known, F is a t-norm (T), G is a t-conorm (S), and N is a strong negation. Usually, such a theory $([0,1]^X, \cdot, +, ')$ is written as $([0,1]^X, T, S, N)$. When T and S are continuous in both variables, the corresponding theory is called an *Standard Theory* of fuzzy sets. Notice that because of $N^2 = id$, strong negations are always continuous function that verify $N = N^{-1}$.

The continuity of F, S and N is important in the applications. If μ or σ are continuous functions (as it is usual in the applications), then $\mu \cdot \sigma = T \circ (\mu \times \sigma)$, $\mu + \sigma = S \circ (\mu \times \sigma)$ and $\mu' = N \circ \mu$ are also continuous (no discontinuities are added).

Continuity implies that the t-norms T are only in either one of the three-families:

- $min_\varphi = \varphi^{-1} \circ min \circ (\varphi \times \varphi) = min$.
- $Prod_\varphi = \varphi^{-1} \circ Prod \circ (\varphi \times \varphi)$.
- $W_\varphi = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$.

with φ an order-automorphism of $([0,1], \leq)$, $Prod(a,b) = a \cdot b$, and $W(a,b) = max(0, a+b-1)$, or T is an ordinal-sum (set of continuous t-norms with

idempotent elements different than 0 and 1). Continuity also implies that the t-conorms S are only in either one of the three families:

- $max_\varphi = \varphi^{-1} \circ max \circ (\varphi \times \varphi) = max$.
- $Prod^*_\varphi = \varphi^{-1} \circ Prod^* \circ (\varphi \times \varphi)$.
- $W^*_\varphi = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$.

with $Prod^*(a, b) = a + b - a \cdot b$, and $W^*(a, b) = min(1, a + b)$, or S is an ordinal-sum (set of continuous t-conorms with idempotent elements different than 0 and 1). Analogously, all strong negations N are of the form $N = N_\varphi = \varphi^{-1} \circ (1 - id) \circ \varphi$ (see [3, 4]).

The only Standard theories that are lattices, namely De Morgan–Kleene Algebras, are $([0, 1]^X, min, max, N)$.

3.3

Two fuzzy sets μ, σ are *contradictory* when $\mu \leq \sigma'$. That is, when for all $x \in X$, is $\mu(x) \leq \varphi^{-1}(1 - \varphi(\sigma(x)))$, with given by N_φ, or $\varphi(\mu(x)) + \varphi(\sigma(x)) \leq 1$. A fuzzy set μ is *self-contradictory* when $\mu \leq \mu_{\varphi^{-1}(1/2)}$: the constant fuzzy set $\mu_{\varphi^{-1}(1/2)}$ is the upper-bound of self-contradictory fuzzy sets and, obviously, the fuzzy sets μ that are never self-contradictory (for any strong negation) are those with $Sup\, \mu = 1$, in particular those that are normalized (it exists $x \in X$ with $\mu(x) = 1$). Of course, the only classical set $\mu \in \{0, 1\}^X$ that is self-contradictory is the empty set ϕ (μ_0).

If $\mu \in [0, 1]^X$ has $Sup\, \mu < 1$, there are always some strong negations N_φ such that $\mu(x) < \varphi^{-1}(1/2)$ for all $x \in X$: *These are strong-negations for which μ is self-contradictory.*

3.4

All Standard theories $([0, 1]^X, T, S, N)$ verify (see [3]):

- The *Law of Kleene*, $\mu \cdot \mu' \leq \sigma + \sigma'$, for all μ, σ in $[0, 1]^X$, since it always holds the functional inequality $T(a, N(a)) \leq S(b, N(b))$ for a, b in $[0, 1]$.
- The *Law of Noncontradiction* $\mu \cdot \mu' \leq (\mu \cdot \mu')'$, in the sense that $\mu \cdot \mu'$ is always self-contradictory, since it always holds the functional inequality $T(a, N(a)) \leq N(T(a, N(a)))$, for all a, b in [0,1].
- The *Law of Excluded-Middle* $(\mu + \mu')' \leq ((\mu + \mu')')'$, in the sense that $(\mu + \mu')'$ is always self-contradictory, since it always holds the functional inequality $N(S(a, N(a))) \leq S(a, N(a)$, for all a, b in [0,1].

3.5

For any basic law of Boolean Algebras there are some Standard theories of fuzzy sets verifying it (see [3]). For example:

- The *classical law of Noncontradiction* $\mu \cdot \mu' = \mu_0$, is verified if and only if $T = W_\varphi$, $N_\varphi \leq N$, and any t-conorm S.
- The *classical law of Excluded-Middle* $\mu + \mu' = \mu_1$, is verified if and only if $S = W^*_\psi$, $N \leq N_\psi$, and any t-norm T.
- *Both* former *laws* are jointly verified, if and only if $T = W_\varphi$, $S = W^*_\psi$, $N_\varphi \leq N \leq N_\psi$.
- The *law of Von Neumann* $\mu = \mu \cdot \sigma + \mu \cdot \sigma'$ and the law $(\mu \cdot \sigma')' = \sigma + \mu' \cdot \sigma'$, are verified if and only if $T = Prod_\varphi, S = W^*_\varphi, N = N_\varphi$, that give nondual theories.
- The *distributive law* $\mu \cdot (\sigma + \rho) = \mu \cdot \sigma + \mu \cdot \rho$ is verified if and only if $S = \max$ and any t-norm T.
- The *distributive law* $\mu + (\sigma \cdot \rho) = (\mu + \sigma) \cdot (\mu + \rho)$ is verified if and only if $T = \min$ and any t-conorm S.
- *Both* former *distributive laws*, if and only if $T = \min$ and $S = \max$.
- The *law of idempotency* $\mu \cdot \mu = \mu$, is verified if and only if $T = \min$.
- The *law of idempotency* $\mu + \mu = \mu$, is verified if and only if $S = \max$,

etc.

3.6

Nevertheless there are nonbasic (derived) boolean laws or formulas that are not verified by any Standard Theory.

For example, in boolean algebras it is $(a + b) \cdot (a + b') = a + b \cdot b' = a$, and $a \cdot (a + b') = a$ (since $a \leq a + b'$), hence: $(a + b) \cdot (a + b') = a \cdot (a + b')$. Are there Standard theories where the formula $(\mu + \sigma) \cdot (\mu + \sigma') = \mu \cdot (\mu + \sigma')$ does hold? The problem lies in solving the functional equation

$$T(S(a, b), S(a, N(b))) = T(a, S(a, N(b))).$$

With $b = 1$ results $T(a, a) = a$ for all $a \in [0, 1]$, that is, $T = \min$. With $a = 0$, results $T(b, N(b)) = 0$ or $T = W_\varphi$, that is absurd. Then, *for no Standard Theory the formula holds.*

In the same vein, in boolean algebras holds $a \cdot b + a \cdot b' = a + a \cdot b'$, but the law $\mu \cdot \sigma + \mu \cdot \sigma' = \mu + \mu \cdot \sigma'$ does not hold in any Standard Theory, since the functional equation

$$S(T(a, b), T(a, N(b))) = S(a, T(a, N(b))),$$

gives the absurd $S = W^*_\varphi$ (with $a = 1$), $S = max$ (with $b = 0$).

3.7

Let us say something else on the insufficiencies of the theories $([0, 1]^X, F, G, N)$. In the language the connective *and* is not always commutative. For example,

since "He was judged and hanged" is not the same that "He was hanged and judged", no t-norm can be used for representing this sentences.

Analogously, in the language the predicates "S=silly" and "S and S=silly and silly," are sometimes used in such a way that Degree up to which "x is silly" < Degree up to which "x is silly and silly," but with t-norms $\mu_{SandS}(x) = T(\mu_S(x), \mu_S(x)) \leq \mu_S(x)$.

Apart all this, it is not written neither that all predicates do have degrees in [0,1], nor that all theories of fuzzy sets should be decomposable, nor that all function in $[0,1]^X$ are always needed. For example, in some cases it could be interesting not to have more self-contradictory fuzzy sets than μ_0 that is, to not consider the functions $\mu \neq \mu_0$ such that $\mu \leq \mu_{\varphi^{-1}(\frac{1}{2})}$ and, consequently, their negation $\mu' \geq \mu_{\varphi^{-1}(\frac{1}{2})}$ with the exception of μ_1. Unfortunately this would mean to suppress all the constant functions μ_r with $r \in (0,1)$.

4 Examples Suggesting a Generalization of the Theories (T,S,N)

4.1

It is well known that in boolean algebras it is $a \cdot (a' + b) = a \cdot b$, and $a + a' \cdot b = a + b$, but in the theory $(W, W^*, 1 - id)$, is:

$$W(a, W^*(1-a, b)) = \min(a, b)$$
$$W^*(a, W(1-a, b)) = \min(a, b)$$

That is, in such theory $\mu \cdot (\mu' + \sigma))$ and $\mu + \mu' \cdot \sigma$ are not reducible to $\mu \cdot \sigma$ like it happens in boolean algebras. What results are

$$\mu \cdot (\mu' + \sigma) = \mu \cdot_1 \sigma,\ \mu + \mu' \cdot \sigma = \mu +_1 \sigma$$

with two new connectives $\cdot_1(\min)$ and $+_1(\max)$.

4.2

The boolean formula $a \cdot b = a \cdot (a \cdot b)$, gives the Pexider functional equation

$$T_1(a, b) = T_2(a, T_3(a, b)).$$

Whit $b = 1$, follows $T_1(a, 1) = a = T_2(a, T_3(a, 1)) = T_2(a, a)$, and $T_2 = \min$.

Hence,

$$T_1(a, b) = \min(a, T_3(a, b)) = T_3(a, b),$$

and $T_1 = T_3$. Hence, the formula

$$\mu \cdot_1 \sigma = \mu \cdot_2 (\mu \cdot_3 \sigma)$$

will hold with $T_2 = \min$ and any t-norm $T_1 = T_3$.

Analogously , the boolean formula $a + b' = a + (a + b')$ suggests $\mu +_1 \sigma'^1 = \mu +_2 (\mu +_3 \sigma'^2)$, that gives the functional equation

$$S_1(a, N_1(b)) = S_2(a, S_3(a, N_2(b))),$$

giving $S_2 = \max, S_1 = S_3$, and $N_1 = N_2$.

4.3

Hence, there is the possibility of increasing the ways of representing formulas by mixing several t-norms, t-conorms and strong negations. For example, the law of classical sets

$$(\mu + \mu) \cdot (\mu \cdot \mu') = \mu_0$$

for no theory (T, S, N) is verified, but written as

$$(\mu + \mu) \cdot_1 (\mu \cdot_2 \mu') = \mu_0$$

gives the equation $T_1(S(a, a), T_2(a, N(a))) = 0$ that has infinite solutions, for example, $T_1 = \min, T_2 = W, S = \max$, and any strong negation N.

Unfortunately, by mixing connectives are not captured all boolean formulas, for example, the law

$$\mu +_1 \sigma = [(\mu +_2 \sigma) \cdot_1 (\mu +_3 \varrho)] +_4 [(\mu +_5 \sigma) \cdot_2 \mu']$$

that holds for classical sets with $+_1 = +_2 = +_3 = +_4 = +_5 = \cup$, $\cdot_1 = \cdot_2 = \cap$, and $' = ^c$, conducts to the equation

$$S_1(a, b) = S_4(T_1(S_2(a, b), S_3(a, c)), T_2(S_5(a, b), N(a)))$$

implying $N(a) = 1$ for all $a \in [0, 1)$, which is not a strong negation because is notcontinuous (see [1]).

4.4

Hence with the theories of the form $([0, 1]^X, T_1, \ldots, T_n, S_1, \ldots, S_m, N_1, \ldots, N_p)$, there is the possibility of capturing more boolean formulas, although not all of them, that by means of standard theories of fuzzy sets. That is with these theories we can capture the representation of some more ways of saying things.

4.5

In the language there are fuzzy events, those reflected by imprecise phrases to which the word “probable” can be applied, for example, “probably she will arrive late on the night”. Notwithstanding it is not clear enough which axioms the numerical probability of a fuzzy event does verify.

Let us consider a theory $([0,1]^X, \cdot, +, ')$ and made the perhaps abusive hypotheses that a "fuzzy probability" is a function $p : [0,1]^X \to [0,1]^X$ – taking all functions in $[0,1]^X$ as fuzzy events – that follows rules analogous to those of Kolmogorov. In principle, this is like the case of classical probability that assigns numbers in $[0,1]$ to some subsets in X (a boolean algebra of sets), or to the case of quantum probabilities that assign numbers in $[0,1]$ to the elements of a lattice with relative complements (orthomodular lattices).

Let us consider the axioms:

1. $p(\mu_1) = 1$
2. $p(\mu + \mu') = 1$, for all $\mu \in [0,1]^X$
3. If $\mu \leq \sigma'$, then $p(\mu + \sigma) = p(\mu) + p(\sigma)$
4. $\max(p(\mu) + p(\sigma)) \leq p(\mu + \sigma)$, for all $\mu, \sigma \in [0,1]^X$

for a fuzzy probability "p." From them immediately follows:

5. Since $\mu \leq (\mu')'$, $p(\mu + \mu') = p(\mu) + p(\mu') = 1 : \ p(\mu') = 1 - p(\mu)$
6. Since $\mu_0 = \mu_1'$, $p(\mu_0) = 1 - p(\mu_1) = 0$
7. Provided the connectives verify the law of duality: $p(\mu \cdot \mu') = p((\mu + \mu')') = 1 - p(\mu + \mu') = 0$.
8. If $\mu \leq \sigma$, since it implies that for all t-conorm S it exists $\rho \in [0,1]^X$ such that $\sigma = \mu + \rho = S \circ (\sigma \times \rho)$, it follows:

$$p(\sigma) = p(\mu + \rho) \geq \max(p(\mu), p(\rho)) \geq p(\mu),$$

that is, $p(\mu) \leq p(\sigma)$.

Obviously, axioms 1–4, when applied to fuzzy sets in $\{0,1\}^X$ – crisp or classical subsets – capture a good part of the classical definition of a probability in the sense of Kolmogorov. Nevertheless, what does not hold in general is the property

9. $p(\mu + \sigma) + p(\mu.\sigma) = p(\mu) + p(\sigma)$,

whose failure could drive us too far from an approximation *à la* Kolmogorov.

In ([8]), Zadeh noticed that:

– With $X = \mathbb{R}^n$
– Taking as fuzzy events those $\mu \in [0,1]^{\mathbb{R}^n}$ that are Borel-measurable in $\mathbb{R}^n$
– With $p(\mu) =$ the Lebesgue–Stieljes integral of μ in $\mathbb{R}^n$

it results:

– If $\cdot_1 = prod, +_1 = prod^* : \ p(\mu +_1 \sigma) + p(\mu \cdot_1 \sigma) = p(\mu) + p(\sigma)$
– If $\cdot_2 = \min, +_2 = \max : \ p(\mu +_2 \sigma) + p(\mu \cdot_2 \sigma) = p(\mu) + p(\sigma)$

formulas that, in general, do not hold with $\cdot = W, + = W^*$.

Notice that with $T = W, S = W^*$, and $N = 1-\mathrm{id}$, the eight first axioms will be reduced to six. First, since now it is $\mu \leq \sigma'$ equivalent to $\mu \cdot \sigma = \mu_0$, axiom 3 changes to the classical expression:

$3'$. If $\mu \cdot \sigma = \mu_0$, then $p(\mu + \sigma) = p(\mu) + p(\sigma)$.

Second, axiom 2 is superfluous since $\mu + \mu' = W^* \circ (\mu \times (1 - id) \circ \mu) = \mu_1$ implies $p(\mu + \mu') = p(\mu_1) = 1$. Axiom 5 reduces to $p(\mu + \mu') = p(\mu_1) = 1 = p(\mu) + p(\mu')$ since now $\mu \cdot \mu' = \mu_0$, and axiom 7 is also superfluous because of $p(\mu \cdot \mu') = p(\mu_0) = 0$

Then, working within any theory (T, S, N), with duality, we can count with properties 1–8, but to have property 9 it is needed to perhaps consider a different T or S. Hence, to preserve the basic laws of a probability *à la* Kolmogorov, it seems that a theory of the type $([0,1]^X, T, T_1, S, S_1, N)$ could be needed. For example, if we start working within the theory $([0,1]^X, W, W^*, 1 - id)$, axioms 1–8 will be preserved, but to have axiom 9 we should take either $T_1 = \min, S_1 = \max$, or $T_1 = prod, S_1 = prod^*$.

4.6

The theories that mix connectives $(T_1, \ldots, T_n, S_1, \ldots, S_m, N_1, \ldots, N_p)$, allow one the possibility to represent more derived laws than within the theories (T, S, N). Let as show few examples:

1. Provided $N_j \leq N_r$. Kleene's Law $T_i(a, N_j(a)) \leq S_k(b, N_r(b))$, does hold for all T_i and all S_k.
2. Von Neumann's Law $a = S(T_1(a, b), T_2(a, N(b)))$ admits more solutions than $T_1 = T_2 = prod_\varphi, S = W^*, N = N_\varphi$, for example, $S = W^*, T_1 = \min, T_2 = W, N = 1 - id$
3. The principle of noncontradiction: $\mu \cdot_1 \mu'^1 \leq (\mu \cdot_2 \mu'^2)'^3$, that is, $T_1(a, N_1(a)) \leq N_3(T_2(a, N_2(a)))$ holds for $N_1 \leq N_2 = N_3$, for all T_1, T_2 (that captures the case (T, S, N). But, for example with $N_1 = 1 - id$, $N_2 = N_3 = \frac{1-id}{1+id}, T_1 = prod, T_2 = \min$, does not hold.
4. The principle of Excluded-Middle: $(\mu +_1 \mu'^1)'^2) \leq ((\mu +_2 \mu'^3)'^4)'^5$, that is, $N_2(S_1(a, N_1(a)) \leq N_5(N_4(S_2(a, N_3(a))))$, holds if $N_2 = N_3 \leq N_1$, $N_5 = N_4$ for all S_1, S_2 (that captures the case (T,S,N)). But, for example, with $N_2 = N_3 = N_5 = N_4 = 1-\text{id}$, $N_1 = \frac{1-id}{1+id}, S_1 = S_2 = W^*$ does not hold.
5. Of course, in a theory $([0,1]^X, T_1, T_2, S_1, S_2, N_1, \ldots, N_p)$ such that $N_1 = N_5 \leq N_2 = N_3 = N_4$, the two laws 3 (NC) and 4 (EM) do hold for all T_1, T_2, S_1, S_2.

5 Antonyms

The negation of a linguistic term is not itself a linguistic term. In dictionaries we can find the words *rich* and *poor*, an antonym or opposite of *rich*, but not the negation *not rich*. Antonymy is important in language and, in fuzzy logic, is basic for constructing linguistic variables; many concepts are better managed once pairs of words (P, antonym of P) have being used.

Antonym is not independent of negation since is it always the case that if "x is $antP$, then x is *not* P" (but not reciprocally). Let us suppose, as it is usual, that *not-P* is the biggest antonym of P: $\mu_{antP} \leq \mu_{not\ P}$.

Oppositeness is a word requiring some kind of order and some kind of symmetry on X. Actually, it seems that oppositeness could be understood as a kind of symmetry with respect to a way of ordering X (the universe of discourse) given by the use of the predicate. In fact, discoursing is, in part, to introduce some "order" in the universe to which we refer to.

To simplify, let us consider the case where P is used on a closed interval $[a, b]$ in the real line by μ_P (see [6]). Provided:

(1) $[a, b] = [b_0, b_1] \cup [b_1, b_2] \cup \ldots \cup [b_{n-1}, b_n],\ b_0 = a, b_n = b.$
(2) μ_P is either nondecreasing or nonincreasing at each subinterval $[b_k, b_{k+1}]$,

consider the preorder $\leq_P$ defined by

$$x \leq_P \ \textbf{iff}\ x, y \in [b_k, b_{k+1}] \text{ and } \mu_P(x) \leq \mu_P(y),$$

Then, we will say that $antP$ (used by μ_{antP}) is an antonym of P (used by μ_P) whenever:

(1) There exist a strong-negation N such that $\mu_{aP} \leq \mu_{not\ P} = N \circ \mu_P$
(2) There are mappings $\alpha_k : [b_k, b_{k+1}] \to [b_k, b_{k+1}]$, $0 \leq k \leq n-1$, such that
   - $\alpha_k^2 = id$
   - If $x \leq_P y$, then $\alpha_k(y) \leq_P \alpha_k(x)$
(3) $\mu_{aP}(x) = \mu_P(\alpha_k(x))$, for all $x \in [b_k, b_{k+1}]$

For example, consider $X = [0, 10]$ and P= *close to 4* used by

$$\mu_P(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 3,\ 5 \leq x \leq 10 \\ x - 3, & \text{if } 3 \leq x \leq 4 \\ 5 - x, & \text{if } 4 \leq x \leq 5. \end{cases}$$

Since the decomposition $[0, 10] = [0, 4] \cup [4, 10]$ agrees with what was said, let us take, $\alpha_1 : [0, 4] \to [0, 4], \alpha_1(x) = 4 - x$, and $\alpha_2 : [4, 10] \to [4, 10], \alpha_2(x) = 14 - x$, that reverse the order $\leq_P$ and are idempotent. Then

$$\mu_Q(x) = \left\{ \begin{matrix} \mu_P(\alpha_1(x)), & \text{if } 0 \leq x \leq 4 \\ \mu_P(\alpha_2(x)), & \text{if } 4 \leq x \leq 10 \end{matrix} \right\} = \begin{cases} 0, & \text{if } 1 \leq x \leq 9 \\ 1 - x, & \text{if } 0 \leq x \leq 1 \\ x - 9, & \text{if } 9 \leq x \leq 10, \end{cases}$$

verifies $\mu_Q \leq 1 - \mu_P$ and allow one to recognize Q as a use of *further from 4*, that is an antonym of *a(close to 4)*, and conclude $\mu_Q = \mu_{antP}$

Let us apply the method to the case of the crisp predicate P=*bigger than 4* on $[0, 10]$, always used by

$$\mu_P(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 4 \\ 1 & \text{if } 4 < x \leq 10. \end{cases}$$

Decompose $[0,10] = [0,4] \cup [4,10]$ and take $\alpha_1(x) = 10 - \frac{3x}{2}, \alpha_2(x) = \frac{2}{3}(10-x)$. We get

$$\mu_Q(x) = \left\{ \begin{array}{l} \mu_P(\alpha_1(x)), \text{ if } 0 \leq x \leq 4 \\ \mu_P(\alpha_2(x)), \text{ if } 4 \leq x \leq 10 \end{array} \right\} = \left\{ \begin{array}{l} 1, \text{ if } 0 \leq x \leq 4 \\ 0, \text{ if } 4 \leq x \leq 10, \end{array} \right.$$

that verifies $\mu_Q(\alpha(x)) < 1 - \mu_P(x)$. Then, $Q =$ *less than* 4, is an antonym of *bigger than* 4, and *not(bigger than 4)* results to be *less or equal than 4*.

All that shows that fuzzy sets are able to model well enough phenomena of language that, like the case of antonymy, are not yet well done.

6 Last Comments

Fuzzy logic not only deals with problems of the technological side of computational intelligence. Since what is a fuzzy set does represent a concrete use of a predicate or linguistic label, that is its meaning, fuzzy logic also deals with the core of computational intelligence, a side of fuzzy logic that, in the way towards computing with words, is of a great interest. Words are context dependent and, if there are uncountable many theories of fuzzy sets, it is because fuzzy logic recognized, from the very beginning (see [7]), that there is no a single way for using the connectives (and, or, not, antonym, rules, etc.), and the representations of them are based on the properties exhibited by its current meaning, or their use in a given context and for some goal.

Perhaps the time to rethinking fuzzy says is coming. What really matters is imprecision and mathematical models can help to clarify some aspects, and to base applications in more solid grounds. Mathematics are important in that they can help us in the study of imprecision with as much precision as possible, once questions on the phenomena are well posed. Only in conjunction with good questions an fine observations of them, are mathematical models actually interesting, and useful, for a deeper understanding of the phenomena. In that sense, to go ahead with computing with words by means of fuzzy logic, a good deal of experimentation in language is to be done to find, among the multiplicity of fuzzy sets theories, the more adequate to the corresponding situation by testing them against some linguistic reality, against the concrete use of words and phrases.

What this paper tried to suggest is not only an intentional review of the existing knowledge on fuzzy sets, either to find more or less paradigmatic examples in the language, or to improve linguistic's methodologies, but the adoption of a new experimentally based view that, eventually, could lead to a renewal of fuzzy logic's theoretical way of working. Towards a typical experimental science of imprecision, where it will be important not to forget neither Occkam's Razor: never introduce more entities than those that are strictly necessary, nor Menger's addition: nor less.

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Some Properties of Fuzzy Languages

Claudio Moraga

Summary. Some properties of fuzzy formal languages associated to Chomsky, Petri and Lindenmayer formal languages will be presented. Generalized alpha cuts lead to the generation of sets of crisp languages of different type within their hierarchies of origin. A fuzzy formal language may serve as a model for several apparently different phenomena of the real world.

Key words: Formal languages, Fuzzy logic.

1 Introduction

A crisp formal language is a set of words over a given finite alphabet built according to specific rules for that language. Formal languages mainly study the structure of the words and the relationship among languages. A fuzzy formal language is a formal language where each word has a *degree of membership* to the language.

Three families of formal languages have been chosen, which are representative of different options to generate the words belonging to them. They are the Chomsky languages [1], the Petri languages [5] and the Lindenmayer languages [4]. Chomsky languages are characterized by *sequences* of elementary transformations which produce a word out of an initial symbol. Petri languages are based on the behavior of Petri nets [6] and the elementary transformations are applied *concurrently*. In the case of Lindenmayer languages, elementary transformations are applied *simultaneously* to all symbols of a word to obtain a new word. To every formal language there is associated a generating structure, which specifies the finite nonempty set of symbols or alphabet to be used, possibly a finite nonempty set of auxiliary symbols, and a finite nonempty set of elementary transformations or rules to generate the words. The generating structure of a Chomsky language is a grammar; that of a Petri language is a Petri net, and that of a Lindenmayer language is an L-system.

In order to generate a fuzzy formal language from a crisp one, there are basically two alternatives (i) replace the crisp set(s) of symbols of the generating

structure by fuzzy sets [8] or (ii) keep the crisp set(s) of symbols but create a fuzzy set of rules by assigning to each rule a "*degree of strength*" in the unit interval. In both cases the generating structures should be extended with appropriate operations for the selected fuzzy sets. These operations will belong to the family of *norms* [3]. In this paper only t-norms and t-conorms [2, 7] will be considered. It becomes apparent that to every crisp formal language an infinite number of fuzzy languages may be associated. In what follows, the symbol $\otimes$ will be used to denote t-norms and the symbol $\oplus$ to denote t-conorms.

The rest of the paper is structured as follows. First, some properties of fuzzy Chomsky languages and of the corresponding fuzzy automata will be discussed. In Sect. 3, fuzzy languages will be considered. Some aspects of fuzzy Lindenmayer languages will be the subject of Sect. 4. The paper will be concluded with some general remarks.

2 Fuzzy Chomsky Languages

Definition 1. *A fuzzy grammar G is specified by the 7-tuple (N, T, S, P,* $\omega, \otimes, \oplus$*), where (N, T, S ,P) is a Chomsky grammar,* ω*: P* $\longrightarrow$ *[0,1] associates to every production in P a weight from the unit interval,* $\otimes$ *denotes a t-norm and* $\oplus$*, a t-conorm. The fuzzy language L(G) generated by this fuzzy grammar is* $\{(w, \mu_L(w)) | w \in T^*, S \Rightarrow^* w, \mu_L(w) = \bigotimes_i p_i\}$*, where* $\mu_L(w)$ *represents the degree of membership of the word w to the language L and is obtained by applying the t-norm* $\otimes$ *to the weights of all productions involved in the generation of w. Should the grammar be ambiguous, and a word w be reachable from S by different sequences of productions, then t-conorm* $\oplus$ *will be used to calculate the final degree of membership from the degrees of membership obtained through different sequences of productions.*

Definition 2. *A fuzzy finite automaton* $\mathcal{A}$ *is specified by the 7-tuple (*Σ*, Q, F,* $q_0, \delta, \varphi, \otimes$*), where (*$\Sigma, Q, F, q_0, \delta$*) is a crisp finite automaton,* $\varphi : (\Sigma \times Q \times Q) \longrightarrow [0, 1]$ *associates a weight to every transition of the automaton and* $\otimes$ *is a t-norm. A sequence of transitions from* q_0 *to some state in F will be called a trajectory.*

It is easy to see that a deterministic fuzzy finite automaton $\mathcal{A}$ may be used to generate a fuzzy regular language $\mathrm{L}(\mathcal{A})$ by taking Σ to be the alphabet of the language, by taking the symbols of Σ associated to a sequence of transitions from q_0 to some state in F to constitute a word of the language and by applying the t-norm $\otimes$ to the weights of the corresponding transitions to determine a global weight that will be taken as the degree of membership of the generated word to the fuzzy language $\mathrm{L}(\mathcal{A})$. This process is reversible in the following sense: a deterministic fuzzy finite automaton $\mathcal{A}$ accepts a fuzzy regular language L' if and only if (Σ, Q, F, q_0, δ) accepts the crisp regular language deducible from L' (by setting all degrees of membership of the words

to be 1), and for every w in the language there exists a trajectory in $\mathcal{A}$ which supports the word and whose global weight equals $\mu_{\mathrm{L}}(\mathrm{w})$.

Definition 3. *Let G = (N, T, S, P, ω, $\otimes$) be a nonambiguous fuzzy grammar and let P_w be the subset of productions of P which are used to generate the word w. Then:*

$$(L(G), \#\alpha) = \{w \in T^* \mid \mu_L(w) \# \alpha\} \tag{1}$$

$$(L(G); \#\alpha) = \{w \in T^* \mid \forall p \in P_w,\ \omega(p) \# \alpha\} \tag{2}$$

where $\# \in \{\leq, >, =, <, \leq\}$ when $\alpha \in$ [0,1]. If α is a subset of [0,1], then $\#$ may represent $\in$ or $\notin$.

It is simple to realize that # represents a generalization of the concept of alpha cut in fuzzy sets. In the case of (L(G), #α), it represents a generalized alpha cut on the fuzzy set of words that constitute the language. For (L(G); #α), it represents a generalized alpha cut on the fuzzy set $\{(\mathrm{p}, \boldsymbol{\omega}(\mathrm{p})) \mid \mathrm{p} \in \mathrm{P}\}$ of weighted productions. Notice that both (L(G), #α) and (L(G); #α) are *crisp* formal languages deduced from the fuzzy language L(G). Furthermore, even though this is not purpose in the theory of fuzzy sets, it is also possible to use (generalized or classical) alpha cuts to obtain fuzzy subsets (which in this paper are fuzzy sublanguages), by including in (1) and (2) not only the words, but also their corresponding degrees of membership to the fuzzy language.

Example 1. Let G = (N, T, S, P, $\boldsymbol{\omega}, \otimes$), where (N, T, S, P) is a regular grammar, $\mathrm{N} = \{\mathrm{S}, \mathrm{A}, \mathrm{B}\}, \mathrm{T} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ and $\otimes$ denotes the t-norm product. Finally let the productions and their respective weights be as shown in Table 1.

It is simple to see that $\mathrm{S} \Rightarrow^* \mathrm{a}^{\mathrm{i}}\mathrm{b}^{\mathrm{j}}\mathrm{c}^{\mathrm{k}}$, with $\mathrm{i}, \mathrm{j}, \mathrm{k} \geq 1$ and accordingly, $\mathrm{L(G)} = \{\mathrm{a}^{\mathrm{i}}\mathrm{b}^{\mathrm{j}}\mathrm{c}^{\mathrm{k}} | \mathrm{i}, \mathrm{j}, \mathrm{k} \geq 1\}$. Besides $\mu_{\mathrm{L}}(\mathrm{a}^{\mathrm{i}}\mathrm{b}^{\mathrm{j}}\mathrm{c}^{\mathrm{k}}) = (0.7)^{\mathrm{i}} \cdot (0.5)^{\mathrm{j}} \cdot (0.3)^{\mathrm{k}}$. Since the numbers 3, 5 and 7 are primes and the product of $0.7 \cdot 0.5 \cdot 0.3 = 0.105$, then $(0,105)^{\mathrm{n}}$ with $\mathrm{n} \geq 1$, has a unique decomposition in terms of its prime factors smaller than 1, which is $(0.7)^{\mathrm{n}} \cdot (0.5)^{\mathrm{n}} \cdot (0.3)^{\mathrm{n}}$. It follows that

$$\mathrm{L(G, \in \{(0.105)^n | n \geq 1\}) = \{a^n b^n c^n | n \geq 1\}.} \tag{3}$$

Similarly $(0.35)^{\mathrm{n}}$ has a unique decomposition in terms of its prime factors smaller than 1, namely $(0.7)^{\mathrm{n}} \cdot (0.5)^{\mathrm{n}}$, from where

$$\mathrm{L(G, \in \{(0.35)^n \cdot (0.3)^m | m \geq 0, n \geq 1\}) = \{a^n b^n c^m | m \geq 0, n \geq 1\}.} \tag{4}$$

Notice that in the context of the Chomsky hierarchy $\{\mathrm{a}^{\mathrm{n}}\mathrm{b}^{\mathrm{n}}\mathrm{c}^{\mathrm{n}} | \mathrm{n} \geq 1\}$ is a context sensitive language and $\{\mathrm{a}^{\mathrm{n}}\mathrm{b}^{\mathrm{n}}\mathrm{c}^{\mathrm{m}} | \mathrm{m} \geq 0, \mathrm{n} \geq 1\}$ is context free, meanwhile the grammar G is only regular.

Table 1. Productions and weights for example 1

$\mathrm{p_1 : S \longrightarrow aS}$	$\boldsymbol{\omega}(\mathrm{p_1}) = 0.7$	$\mathrm{p_2 : S \longrightarrow aA}$	$\boldsymbol{\omega}(\mathrm{p_2}) = 0.7$
$\mathrm{p_3 : A \longrightarrow bA}$	$\boldsymbol{\omega}(\mathrm{p_3}) = 0.5$	$\mathrm{p_4 : A \longrightarrow bB}$	$\boldsymbol{\omega}(\mathrm{p_4}) = 0.5$
$\mathrm{p_5 : B \longrightarrow cB}$	$\boldsymbol{\omega}(\mathrm{p_5}) = 0.3$	$\mathrm{p_6 : B \longrightarrow c}$	$\boldsymbol{\omega}(\mathrm{p_6}) = 0.3$

3 Fuzzy Petri Net Languages

Definition 4. *A Petri net is a finite directed graph whose nodes are partitioned into two disjoint nonempty sets called places (P) and transitions (T), such that* $P \cap T = \varnothing$.

Let E denote the set of edges. Edges connect places to transitions and transitions to places. Formally, $\mathrm{E} \subseteq \mathrm{P} \times \mathrm{T} \cup \mathrm{T} \times \mathrm{P}$. A function $\mu : \mathrm{P} \longrightarrow \mathrm{N} \cup \{0\}$ assigns to each place a number of tokens (or leaves it empty). Let P be an ordered set. A marking of a net is a set ordered by places, which shows the prevailing number of tokens in each place. The dynamic behavior of a net is characterized by the changes in the marking. Notice that the marking of a Petri net identifies the state of the net. This suggests a certain analogy with finite automata. The modeling power of a Petri net is, however, much stronger than that of finite automata. It is enough to mention that every regular language can be generated by a Petri net, (see definition 5 below), however, not every language generated by a Petri net is accepted by a finite automaton [5]).

A transition is *enabled*, when all places connected to it have at least one token. In such a case the transition can *switch*. (Some authors prefer to say that a transition can *fire*.) When a transition is enabled and switches, it removes one token from every one of its input places and adds one token to every one of its output places. It is easy to see that every time that a transition switches, the marking of the net may change.

At a given time several transitions of a Petri net might be enabled. The formalism of the Petri nets (considered in this section) does not specify a *temporal ordering* governing the switching of the enabled transitions. Every possible temporal ordering, including simultaneous switching is acceptable. It is said that transitions switch in a *concurrent* way.

Definition 5. *A language generating Petri net is a 5-tuple* $(\Pi, \Sigma, \rho, M_0, M_F)$, *where* Π *is a Petri net after definition 4;* $\rho : T \longrightarrow \Sigma$ *is a labeling function which assigns a symbol of the finite, nonempty, alphabet* Σ *to each transition,* M_0 *denotes the initial marking and* M_F *is the set of final markings. Once a final marking is reached, the generated word finishes. The generating process is very simple: it begins with an empty symbol and starting with* M_0, *each time that a transition switches, the symbol associated to that transition is appended to the word under generation.*

Definition 6. *A fuzzy language generating Petri net is specified by the 7-tuple* $(\Pi, \Sigma, \rho, M_0, M_F, \omega, \otimes)$ *where* $(\Pi, \Sigma, \rho, M_0, M_F)$ *is a language generating Petri net, (recall definition 5),* $\omega : T \longrightarrow [0, 1]$ *assigns a weight ("degree of strength") to each transition and* $\otimes$ *denotes a t-norm to calculate the weight of the generated word based on the weights of the switched transitions. This global weight will be interpreted as the degree of membership of the generated word to the language.*

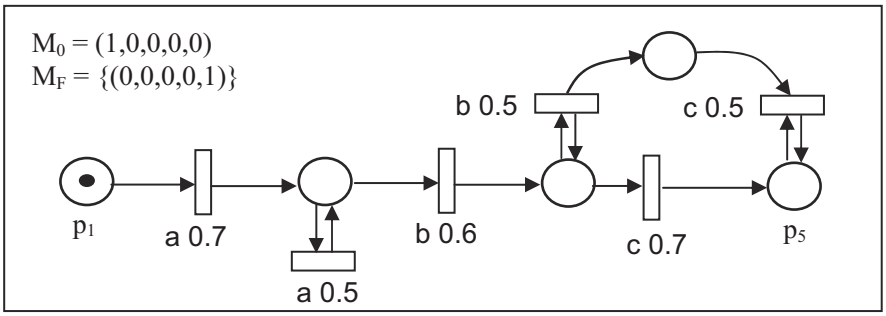

Fig. 1. Fuzzy language generating Petri net. Example 2

Example 2. Consider the fuzzy language generating Petri net of Fig. 1 and let the t-norm be the product.

From the structure of the Petri net in Fig. 1 it may be deduced that the degree of membership of the generated words $w = a^p b^q c^q$ to the language $L(\Pi)$ is given by the following equation:

$$\begin{aligned}\mu_{L(\Pi)}(w) &= 0.7 \otimes (0.5)^{p-1} \otimes 0.6 \otimes (0.5)^{q-1} \otimes 0.7 \otimes (0.5)^{q-1} \\ &= 0.294 \cdot (0.5)^{p-1} \cdot (0.5)^{2(q-1)} = 0.294 \cdot (0.5)^{p+2q-3} \end{aligned} \tag{5}$$

A short case analysis of (5) shows that:

$$\begin{aligned} p = q = 1 &\Rightarrow \mu_{L(\Pi)}(w) = 0.294 \cdot (0.5)^0 = 0.294 \\ p = q = 1 &\Rightarrow \mu_{L(\Pi)}(w) = 0.294 \cdot (0.5)^1 = 0.147 \\ p = q = 1 &\Rightarrow \mu_{L(\Pi)}(w) = 0.294 \cdot (0.5)^2 = 0.0735 \\ p = q = 1 &\Rightarrow \mu_{L(\Pi)}(w) = 0.294 \cdot (0.5)^3 = 0.03675 \end{aligned} \tag{6}$$

from where

$$\begin{aligned} (L, > 0.2) &= \{abc\} \\ (L, > 0.1) &= \{abc, a^2bc \\ (L, > 0.07) &= \{abc, a^2bc, ab^2c^2\} \\ (L, > 0.03) &= \{abc, a^2bc, ab^2c^2, a^2b^2c^2\} \end{aligned} \tag{7}$$

It is simple to observe that the value of $\mu_{L(\Pi_p)}$ (w) decreases as the value of p and q increases and then, the languages obtained with alpha cuts grow including new ("weaker") words as the level of alpha diminishes. In other words, the level of alpha controls the growth of the corresponding crisp languages derived from the fuzzy one.

4 Fuzzy Lindenmayer Languages

The generating structure of a Lindenmayer language is known as "L-system." The simplest L-system is called 0L.

Definition 7. *A 0L system consists of a finite nonempty alphabet V, an initial word $w_0 \in V^+$, and a finite nonempty set of productions $P \subset (V \times V^*)$, which represent transformation rules for symbols according to the following scheme: Let u and v be words from V^* such that $u = u_1 u_2 ... u_n$, where $u_i \in V$, and $v = v_1 v_2 ... v_n$, where $v_i \in V^*$, with i = 1, 2, ..., n. Then $u \Rightarrow v$ (v is derivable from u) if and only if there exist productions p in P such that $p(u_i) = v_i$ with i = 1, 2, ..., n.*

It is simple to see that in 0L systems $|P| \geq |V|$, (which allows a 0L to be nondeterministic), i.e., there must be at least as many elementary productions as symbols in the alphabet. Moreover the corresponding elementary productions are applied *simultaneously* to all symbols of a word. For this reason, 0L systems are sometimes called (word) rewriting or (word) replacing systems. A 0L system is called deterministic, in short D0L, if all productions in P are functions. In this case $|P| = |V|$. If the productions show at the right side words from V^+, the system is called λ-free or propagating, abbreviated P0L. A DP0L system is both deterministic and propagating. The language generated by a 0L system is given by $£(0L) = \{w \in V^* \mid w_0 \Rightarrow^* w\}$. The symbol £ is used for the language in order to distinguish it from the "L" of a 0L system. Since there is only one alphabet V, on each stage of a derivation process a new word of the language is obtained. The language £ obviously inherits the main features of the generating system. Thus, £(D0L) is a deterministic language, £(P0L) a propagating one and £(DP0L), one with both properties.

Definition 8. *A weighted system ω0L has the following structure: $(V, w_0, P, \omega, \otimes, \oplus)$, where (V, w_0, P) is a 0L system and $\omega: P \longrightarrow [0,1]$ assigns a weight to each production in P. An ω0L system generates a fuzzy Lindenmayer language £(ω0L). At each stage of a derivation the specified t-norm will be used to calculate the (possibly preliminary) weight of each word, based on the weights of the productions that were used. If a word may be generated with different sequences of derivation stages, then the t-conorm will be used to calculate the final weight of that word considering all its preliminary weights. The final weight will be considered to be the membership degree of that word to the language. The initial word has per definition membership degree 1 to the language.*

Table 2. Productions and weights for example 3

$p_1 : A \longrightarrow AA$	$p_2 : B \longrightarrow BB$	$p_3 : C \longrightarrow CC$
$\omega(p_1) = 0.95$	$\omega(p_2) = 0.93$	$\omega(p_3) = 0.89$

Example 3. Let an ωDP0L system be given as ({A, B, C}, ABC, P, ω, product, $\oplus$), with the productions and weights shown in Table 2.

The corresponding crisp language £(PD0L) is $\{A^kB^kC^k|\ k=2^n,\ n \geq 0\}$. Let it be assumed that A, B, and C are the sides of a triangle with length 1. Then £(DP0L) would represent "the set of all equilateral triangles with sides of length 2^n, where n is an integer and $n > 0$." In the fuzzy Lindenmayer language £(ωDP0L) it becomes evident that every new word will have a lower degree of membership to the language, than that of the former words; i.e., as n increases, $\mu_{£(\omega\text{PD0L})}(w)$ decreases. £(ωPD0L) could then represent "the set of all *small* triangles with side of length 2^n, where n is an integer larger than 0." It is simple to see that a 3D interpretation is also possible: £(PD0L) would represent "the set of all cubes of side 2^n," meanwhile £(ωPD0L) would represent "the set of all *small* cubes with side of length 2^n." Another interpretation outside of geometry could be the following: A, B, and C represent (the sounds of) three bells; 2^n, the number of rings of the bells and $\mu_{£(\omega\text{PD0L})}(w) = (0.95)^k(0.93)^k(0.89)^k$, the acoustic intensity of the carillon. Then £(ωPD0L) could represent "the set of all *loud* chords repeated 2^n times produced by three bells." Notice that these are three *different* interpretations associated to the *same* fuzzy language.

This example allows the claim that a fuzzy formal language may serve as an abstract model for *several* (apparently very) different phenomena of the real world.

5 Closing Remarks

It is possible to build fuzzy formal languages by assigning weights in the interval [0,1] to the elementary transformations belonging to the generating system, needed to derive words, and by using t-norms (and eventually t-conorms) to calculate the degree of membership of the words to the language. Generalized alpha cuts allow the derivation of different crisp (or fuzzy) formal languages associated to each fuzzy formal language. This allows to visualize (a posteriori) a fuzzy formal language as a weighted aggregation of different crisp formal languages over the same alphabet, in the sense that if B is a fuzzy set and $B_\alpha$is an alpha cut, then $B = \cup_{\alpha\in[0,1]}\alpha B_\alpha u$ Moreover, the resulting crisp languages may be of different type within their respective hierarchies.

Further results on this subject may be found in [9].

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General Form of Lattice Valued Intuitionistic Fuzzy Sets

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Summary. Taking into account families of cut sets of intuitionistic lattice valued fuzzy sets, it is proved that the general form of such fuzzy sets is the one with the codomain $[0,1]^I$, I being a set of an arbitrary cardinality. It turns out that in applications of intuitionistic fuzzy sets it is convenient to consider fuzzy sets in such a form.

Key words: Lattice valued intuitionistic fuzzy set, Cut sets, Cutworthy approach.

1 Introduction

1.1 Earlier Definitions of Lattice Valued Intuitionistic Fuzzy Sets

The original concept of fuzzy sets (introduced by Zadeh in 1965) has been generalized in several directions. One is obtained by replacing a codomain of the membership function (originally $[0,1]$ interval) by a richer structure (Boolean algebra, complete lattice, etc.). The best known generalization of this type is the notion of lattice valued fuzzy set introduced by Goguen in 1967. Another possibility of generalization is to consider two functions instead of one: membership and nonmembership function. This type of generalizations led to the introduction of notion of intuitionistic fuzzy sets [1, 2, 4] by Atanassov.

By the original definition, an intuitionistic fuzzy set A in a set E is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\}$, where μ_A and ν_A are both functions from E to $[0,1]$ interval, such that for every $x \in E$, $\mu_A(x) + \nu_A(x) \leq 1$. μ_A and ν_A are the functions representing belonging and nonbelonging of an element x to set E, respectively.

As a combination of two directions of generalization, Atanassov and Stoeva defined a lattice valued intuitionistic fuzzy set (intuitionistic L-fuzzy set) [3], using a complete lattice L with an involutive order reversing unary operation

$\mathcal{N} : L \to L$. An intuitionistic L-fuzzy set is an object of the form $A = \{(x, \mu(x), \nu(x)) \mid x \in X\}$, where μ and ν are functions $\mu : X \to L$, $\nu : X \to L$, such that for all $x \in X$,

$$\mu(x) \leq \mathcal{N}(\nu(x)). \tag{1}$$

In the sequel we consider intuitionistic L-fuzzy set in an equivalent form: as an ordered triple (A, μ, ν), where μ and ν are both functions from A to $[0, 1]$ interval, satisfying (1) for all $x \in A$. Besides, we use abbreviation LIFS for an lattice valued intuitionistic fuzzy set.

As it has been pointed out in paper [7], this definition has one disadvantage. Namely, requirement of existence of an involutive order reversing unary operation for a lattice is rather a strong condition. Even for some six element lattices there is no unary operations satisfying the desiring properties. Therefore, LIFS could not be defined for a large class of lattices. This was the main reason for proposal of new definitions of lattice valued intuitionistic fuzzy sets in papers [6, 7].

The first of the proposed definitions use a linearization function and the second one a lattice homomorphism (as a special case of linearization function) as tools for connection of membership and nonmembership functions.

The definition of LIFS with linearization function is not convenient for defining set operations. This disadvantage has been overcome in the definition with lattice homomorphism [7]. The notion of LIFS with lattice homomorphism is a generalization of the ordinary intuitionistic valued fuzzy set and its structure is richer. To every L-valued intuitionistic fuzzy set there correspond two families of level subsets, which are lattices under inclusion. A classical intuitionistic fuzzy set is obtained by this homomorphism, in the natural way, which is not the case with other definitions.

Still for the simple lattices and for some other similar types, there is no a lattice homomorphism, so this definition has disadvantages similar as the original one.

Finally, in this context we mention the paper [5], where relationship between intuitionistic fuzzy sets (among them also lattice valued ones) and some other extensions of fuzzy sets theory have been given.

1.2 Proposal of a New Definition

In this paper we consider a notion of the lattice valued intuitionistic fuzzy set on set X as an ordered triple (X, μ, ν), where μ and ν are mappings from X to $L = [0, 1]^I$, where I is an arbitrary indexed set, such that

$$\mu(x)(i) + \nu(x)(i) \leq 1,$$

for all $x \in X$ and all $i \in I$.

We prove that the earlier proposed definitions of lattice valued intuitionistic fuzzy sets (using the cutworthy approach and equivalence of fuzzy sets in the sense of papers [11, 12]) can be united in the framework of the definition given above.

1.3 Definitions of Basic Notions and Their Relevant Properties

The notions of cut sets for L-valued intuitionistic fuzzy sets are defined in the sequel.

Let L be a complete lattice and (X, μ, ν) a lattice valued intuitionistic fuzzy set, where μ and ν are functions from X to L.

For each $p \in L$, there are two cut sets defined by:

$$\mu_p = \{x \in X \mid \mu(x) \geq p\} \quad \text{and} \quad \nu_p = \{x \in X \mid \nu(x) \leq p\}.$$

ν_p is sometimes called a $\leq$-p-cut.

By $\mathcal{M}_L$ and $\mathcal{N}_L$ we denote two families of cut sets of LIFS:
$\mathcal{M}_L = \{\mu_p \mid p \in L\}$ and $\mathcal{N}_L = \{\nu_p \mid p \in L\}$.

These definitions are independent of the concept of LIFS and they can be used in all the mentioned frameworks, in old as well as in new definitions. The properties of cuts of LIFS that are listed in the sequel are also independent of the approach.

If 0 is the bottom element and 1 the top element of the lattice L, the following is true:

1. $\mu_0 = X$, and $\nu_1 = X$.
2. If $p \leq q$, then $\mu_q \subseteq \mu_p$, and $\nu_p \subseteq \nu_q$.
3. $$\mu(x) = \bigvee\{p \in L \mid x \in \mu_p\};$$
   $$\nu(x) = \bigwedge\{p \in L \mid x \in \nu_p\}.$$
4. If $M \subseteq L$, then
   $$\bigcap(\mu_p \mid p \in M) = \mu_{\bigvee\{p \mid p \in M\}}$$
   and
   $$\bigcap(\nu_p \mid p \in M) = \nu_{\bigwedge\{p \mid p \in M\}}.$$
5. $\mathcal{M}_L$ and $\mathcal{N}_L$ ordered by inclusion are complete lattices.

2 Results

2.1 Ordinary Lattice Valued Fuzzy Sets

The next three theorems are formulated for classical lattice valued fuzzy sets and they are used in proofs of theorems (for intuitionistic fuzzy sets) given in the sequel.

Theorem 1. *[8] Let L_1 and L_2 be complete lattices, such that $L_1 \subseteq L_2$, all infima in L_1 and L_2 coincide and the top elements are the same. Let $\mu : X \to L_1$ and $\nu : X \to L_2$ be fuzzy sets, such that $\mu(x) = \nu(x)$ for all $x \in X$. Then, fuzzy sets μ and ν have the same collections of cut sets.*

Theorem 2. *[8] Let L and L_1 be complete lattices and let $\varphi : L \to L_1$ be the injection from L to L_1 which maps the top element of L to the top element of L_1, such that for all $x, y \in L$, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$. Let $\mu : X \to L$ be a fuzzy set on X. Let fuzzy set $\nu : X \to L_1$ be defined by $\nu(x) = \varphi(\mu(x))$. Then fuzzy sets μ and ν have the same families of cuts and $\mu_p = \nu_{\varphi(p)}$ for all $p \in L$.*

In the sequel, we will use the theorem also in the dual form:

Theorem 3. *Let L and L_1 be complete lattices and let $\varphi : L \to L_1$ be the injection from L to L_1 which maps the bottom element of L to the bottom element of L_1, such that for all $x, y \in L$, $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$. Let $\mu : X \to L$ be a fuzzy set on X. Let fuzzy set $\nu : X \to L_1$ be defined by $\nu(x) = \varphi(\mu(x))$. Then fuzzy sets μ and ν have the same families of $\leq$-cuts and $\mu^{\leq}_p = \nu^{\leq}_{\varphi(p)}$ for all $p \in L$.*

Theorem 4. *[8] Necessary and sufficient conditions under which $\mathcal{F} \subseteq \mathcal{P}(X)$ is a collection of cut sets of a fuzzy set $\mu : X \to L$, for a fixed complete lattice L is that $\mathcal{F}$ is closed under intersections, contains X and its dual poset under inclusion can be embedded into L, such that all infima and the top element are preserved under the embedding.*

Theorem 5. *[8] Let $\mu : X \to L$ be a lattice valued fuzzy set. Then there exists a cardinal number c and a fuzzy set $\nu : X \to [0,1]^c$ such that fuzzy sets μ and ν have identical collections of cut sets.*

The following theorem is well known Theorem of synthesis for lattice valued fuzzy sets (see e.g. [9] or [10]).

Theorem 6. *Let F be a family of subsets of a nonempty set X, which is closed under intersection and contains X. Let $\mu : X \to F$ be defined by*

$$\mu(x) = \bigcap (p \in F \mid x \in p).$$

Then, μ is a fuzzy set on X, where $(F, \leq)$ is a complete lattice anti-isomorphic with $(F, \subseteq)$, its family of p-cuts is F and for every $p \in F$, $p = \mu_p$.

Here it is the statement that is analogous to the previous one and which deals with $\leq$ - cuts.

Theorem 7. *Let F be a family of subsets of a nonempty set X, which is closed under intersection and contains X. Let $\nu : X \to F$ be defined by*

$$\nu(x) = \bigcap (p \in F \mid x \in p).$$

Then, ν is a fuzzy set on A, its family of $\leq$-p-cuts is F and for every $p \in F$, $p = \nu_p$.

2.2 General form of LIFS

In the following theorem we prove that the intuitionistic lattice valued fuzzy sets with the codomain $[0,1]^I$ is the most general concept of LIFS (considering cutworthy approach). This is the main theorem of this paper.

Theorem 8. *Let (X, μ, ν) be a LIFS with an involutive order reversing unary operation $\mathcal{N} : L \to L$, where μ and ν are functions from X to a complete lattice L, satisfying (1). Then there exists an index set I and a fuzzy set (X, μ', ν'), where $\mu' : X \to [0,1]^I$ and $\nu' : X \to [0,1]^I$ such that fuzzy sets (X, μ, ν) and (X, μ', ν') have identical collections of cut sets.*

Proof. Let (X, μ, ν) be a lattice valued intuitionistic fuzzy set and for $p \in L$, μ_p and ν_p its cut sets. Let $\mathcal{N} : L \to L$ be the corresponding involutive order reversing unary operation . Then for all $x \in X$,

$$\mu(x) \leq \mathcal{N}(\nu(x)). \tag{2}$$

Since $\mathcal{N}$ is order reversing and involutive, we have that $\nu(x) \leq \mathcal{N}(\mu(x))$ is valid as well.

Let 1 be the top and 0 the bottom element of the lattice L. By $\mathcal{N}(1) = 0$, and $\nu(x) \leq \mathcal{N}(\mu(x))$, it follows that:

$$\text{if } \mu(x) = 1 \text{ then } \nu(x) = 0.$$

This fact is used in the sequel.

Let $\overline{L}$ be a lattice dually isomorphic to L under an isomorphism δ, such that $L \cap \overline{L} = \emptyset$. We consider set $\mathcal{P}(L \cup \overline{L})$ under the set inclusion, and this lattice will serve as the lattice L_1 in applications of Theorem 2 and Theorem 3 in this proof.

Next we define an injection φ from L into $L_1 = \mathcal{P}(L \cup \overline{L})$, as follows:

$$\varphi(1) = L \cup \overline{L} \text{ and } \varphi(p) = \downarrow p := \{z \in L \mid z \leq p\}, \text{for all } p \in L, p \neq 1.$$

Now, it is straightforward to prove that for all $x, y \in L$, $\varphi(x \wedge y) = \varphi(x) \cap \varphi(y)$ and conditions of Theorem 2 are satisfied.

Further, we define another injection ψ from L into L_1: $\psi(p) = \xi(\downarrow \delta(p))$, where ξ is a mapping from $\mathcal{P}(\overline{L})$ to $\mathcal{P}(L \cup \overline{L})$ defined by $\xi(X) = \overline{L} \setminus X$ and δ is the dual isomorphism from L to $\overline{L}$ defined above. Here (as above) $\downarrow \delta(p) = \{z \in \overline{L} \mid z \leq \delta(p)\}$, where $\leq$ is an ordering relation in lattice $\overline{L}$.

We have that $\psi(0) = \xi(\downarrow \delta(0)) = \xi(\overline{L}) = \emptyset$ and $\psi(p \vee q) = \xi(\downarrow (\delta(p \vee q))) = \xi(\downarrow (\delta(p) \wedge \delta(q))) = \xi((\downarrow \delta(p)) \cap (\downarrow \delta(q))) = \overline{L} \setminus ((\downarrow \delta(p)) \cap (\downarrow \delta(q))) = (\overline{L} \setminus \downarrow \delta(p)) \cup (\overline{L} \setminus \downarrow \delta(q)) = \psi(p) \cup \psi(q)$. We have proved that the conditions of Theorem 3 are satisfied.

The well known fact for lattice $\mathcal{P}(L \cup \overline{L})$ (which is a Boolean lattice) is that it can be naturally embedded into $[0,1]^I$ for a set I of suitable cardinality c (using an auxiliary isomorphism from $\mathcal{P}(L \cup \overline{L})$ to $\{0,1\}^I$). To construct this

isomorphism, we start from a set I of the same cardinality as $L \cup \overline{L}$ and denote by γ a bijection $\gamma : L \cup \overline{L} \to I$. Let σ be the induced (by γ) mapping that maps every subset of $L \cup \overline{L}$ to a subset of I. Further, we take mapping κ from $\mathcal{P}(I)$ to $\{0,1\}^I$, which maps each subset to its characteristic function. By all the mentioned mappings, the top and bottom elements are mapped into corresponding top and bottom elements, and infima and suprema are preserved.

Let α be the mentioned embedding from $\mathcal{P}(L \cup \overline{L})$ to $[0,1]^I$:

$$\alpha(M) = \kappa(\sigma(M)).$$

We finally construct mappings $\mu' : X \to [0,1]^I$ and $\nu' : X \to [0,1]^I$ as a composition of mappings, as follows:

$$\mu'(x) = \alpha(\varphi(\mu(x)))$$

and

$$\nu'(x) = \alpha(\psi(\nu(x))),$$

for all $x \in X$.

Since α is an embedding and φ and ψ satisfy conditions of Theorems 2 and 3, the corresponding families of cuts are identical.

Now, we only have to prove that the condition

$$\mu'(x)(i) + \nu'(x)(i) \leq 1,$$

is satisfied for all $x \in X$ and all $i \in I$, i.e., that (X, μ', ν') is the intuitionistic fuzzy set.

Let $x \in X$. We distinguish two cases: $\mu(x) \neq 1$ and $\mu(x) = 1$.

1. Let $\mu(x) \neq 1$ and let $i \in I$. Then for all $i \in \gamma(\overline{L})$, $\mu'(x)(i) = \alpha(\varphi(\mu(x)))(i) = 0$, where $\gamma(\overline{L}) = \{\gamma(z) \mid z \in \overline{L}\}$. On the other hand, for all $i \notin \gamma(\overline{L})$, $\nu'(x)(i) = \alpha(\varphi(\mu(x)))(i) = 0$. Hence,

$$\mu'(x)(i) + \nu'(x)(i) \leq 1.$$

2. Let $\mu(x) = 1$ and let $i \in I$. Then, by assumption, $\nu(x) = 0$. We have that:

$$\mu'(x)(i) = \alpha(\varphi(\mu(x)))(i) = \alpha(L \cup \overline{L})(i) = 1$$

and

$$\nu'(x)(i) = \alpha(\psi(\nu(x)))(i) = \alpha(\emptyset)(i) = 0.$$

Therefore, the statement of the theorem is proved.

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A Note on Generated Pseudo-Operations with Two Parameters as a base for the Generalized Pseudo-Laplace Type Transform

Ivana Štajner-Papuga

1 Introduction

Approach proposed in this paper has been done in the pseudo-analysis' framework, where by the pseudo-analysis mathematical theory that is a generalization of the classical analysis has been considered. Over the years, this theory has proved itself to be a powerful tool for solving problems in different aspects of mathematics, as well as in various practical problems. Using this apparatus, some important notions that are analogous to their classical counterparts, i.e., notions such as $\oplus$-measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc., have been introduced [7, 10, 14, 18–20].

As already mentioned, pseudo-Laplace transform is one of the important notion from the pseudo-analysis' framework, and it is often used in dealing with differential or integral equation [5, 10, 14–16, 18, 19]. The generalized $(\oplus, \odot)$-Laplace transform presented in this paper is a generalization of the pseudo-Laplace transform, based on a special class of generalized pseudo-operations which need not be commutative nor associative. It should be stressed that this class of operations has been introduced in [24,25] and it has been used to extend the pseudo-linear superposition principle ([4,8,10,14–18]) on generalized burger's type nonlinear partial differential equations [25].

Another aspect of the generalization proposed in this paper has been focused on the domain of functions that pseudo-Laplace type transform has been applied to. This type of generalization has already been applied on pseudo-convolution resulting with generalized pseudo-convolution that has taken an important role in theory of fuzzy numbers (operations with fuzzy numbers), as well as in optimization, information theory, system theory, etc. [20]. Also, this approach has been applied on Laplace type transforms based on a semiring from the first or second class (pseudo-operations are commutative and associative, see [7, 10, 14, 17–20]), which, combined with the generalized pseudo-convolution, led to the extension of classical limit theorems for triangle functions [23].

Another important problem addressed by pseudo-analysis is construction of aggregation operators by means of different types of pseudo-integrals. Some of integrals that have been used for this constructions are Lebesgue integral, Choquet and Sugeno integral, monotone set functions-based integrals, Choquet-like integrals, (S,U)-integral, etc. (see [1–3, 6, 11]). Therefore, this paper proposes a further step in this direction by investigating possible role of the generalized $(\oplus,\odot)$-Laplace transform as a aggregation type operator.

Preliminary notions, such as generalized pseudo-operations, $\oplus$-integral and corresponding pseudo-convolutions, are given in Sect. 2. The third section contains definition of the generalized $(\oplus,\odot)$-Laplace transform, where $\oplus$ and $\odot$ are generated pseudo-operations with two parameters. Also, some basic properties and limit properties of the generalized $(\oplus,\odot)$-Laplace transform are investigated in Sect. 3. Aggregation type operator constructed by means of the generalized $(\oplus,\odot)$-Laplace transform is presented in Sect. 4.

2 Preliminary Notions

The first part of generalization proposed in this paper is based, as already mentioned, on a special class of generalized pseudo-operations (see [24, 25]). This class is given by the following definition.

Definition 1. *Let ε and γ be arbitrary but fixed positive real numbers and let g be a positive strictly monotone continuous function defined on $\mathbb{R}$ or $[0,\infty)$. Generated pseudo-addition and pseudo-multiplication with two parameters, denoted with $\oplus$ and $\odot$, respectively, are*

$$x \oplus y = g^{-1}(\varepsilon g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x)^{\gamma} g(y)). \tag{1}$$

Since operations $\oplus$ and $\odot$ need not be commutative nor associative operations, it is necessary to define pseudo-sum of n elements $\alpha_i \in [a,b], i \in \{1,2,\dots n\}$:

$$\bigoplus_{i=1}^{n} \alpha_i = (\dots((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \dots) \oplus \alpha_n.$$

Neutral elements from the left for $\oplus$ and $\odot$ are $\mathbf{0} = g^{-1}(0)$ and $\mathbf{1} = g^{-1}(1)$, respectively, i.e., $\mathbf{0} \oplus x = x$ and $\mathbf{1} \odot x = x$.

Remark 1. For $\varepsilon = \gamma = 1$, commutative and associative pseudo-operations from g-semiring are obtained ([9, 12, 14]).

Remark 2. Operations of this type have been successfully used in dealing with nonlinear PDE (see [24,25]), e.g., in dealing with the Burger's type of nonlinear partial differential equation $u_t - \alpha u_{xx} = \alpha \Phi(u) u_x^2$, where Φ is a given continuous function and $\alpha \in \mathbb{R}$. In this case, pseudo-linear combination of solutions, based on the generated pseudo-operations with two parameters with a generating function $g(x) = \pm \int_0^x \exp(\int_0^t \Phi(s)\, ds)\, dt$, is, again, a solution (see [25]).

Now, let $(a,b]$ be a subinterval of the real line and let, for some $n \in \mathbb{N}$, $P_n = \{(x_i, x_{i+1}]\}_{i=0}^{n-1}$ be its n-partition where $a = x_0 < x_1 < \ldots < x_n = b$. The $\oplus$-measure $\mu_{P_n} : P_n \to [0,\infty)$ is given by

$$\mu_{P_n}((x_i, x_{i+1}]) = g^{-1}\left(\frac{x_{i+1} - x_i}{\varepsilon^{n-i-1}}\right).$$

Some properties of this family of measures has been proved in [22]. Among them is the following type of pseudo-linearity:

$$\mu_{P_{n-r+j}}\left(\bigcup_{i=j}^{r} A_i\right) = \bigoplus_{i=j}^{r} \mu_{P_n}(A_i),$$

where $1 \le j \le r \le n$, $P_n = \{A_i\}_{i=1}^{n} = \{(x_{i-1}, x_i]\}_{i=1}^{n}$ is a n-partition of interval $(a,b]$ and $P_{n-r+j} = \{B_s\}_{s=1}^{n-r+j}$ is new $(n-r+j)$-partition, such that $B_s = A_s$ while $s = 1, 2, \cdots, j-1$, $B_j = \cup_{i=j}^{r} A_i$ and $B_s = A_{s+r-j}$ for $s = j+1, \cdots, n-r+j$.

Further on, with P_n' is denoted an $(n+1)$-partition of interval $(a,b]$ obtained from n-partition P_n in the following manner: we keep all the points from previous partition and add one more point and renumerate the points of the new partition in the increasing order. After s-repetition of this procedure an $(n+s)$-partition $P_n^{(s)}$ is obtained (see [22]). Now, if $f : [a,b] \to [0,\infty)$ is a continuous function, the $\oplus$-integral of function f is

$$\int_{[a,b]}^{(\oplus,\odot)} f d\mu_{P_n} = \lim_{\substack{\mu_{P_n^{(s)}} \to \mathbf{0} \\ (s \to +\infty)}} \left(\bigoplus_{i=0}^{n+s-1} \left(f(x_{i+1}) \odot \mu_{P_n^{(s)}}((x_i, x_{i+1}]) \right) \right),$$

if the limit exists.

Since it has been proved in [22] that the $\oplus$-integral does not depend on the partition of the interval $[a,b]$ and that it can be represented in the following manner

$$\int_{[a,b]}^{(\oplus,\odot)} f d\mu_{P_n} = g^{-1}\left(\int_a^b g^{\gamma} \circ f(x) dx \right),$$

the $\oplus$-integral will be denoted by $\int_{[a,b]}^{(\oplus,\odot)} f$.

Corresponding pseudo-convolution of continuous functions $f, h : [0,\infty) \longrightarrow [0,\infty)$ is

$$f \star h(x) = \int_{[0,x]}^{(\oplus,\odot)} \left([f]_g(x-t) \odot h(t) \right), \tag{2}$$

where $[\cdot]_g$ is a transform of the following form $[f]_g(x) = g^{-1}\left(g^{1/\gamma}\left(f(x)\right) \right)$.

The second aspect of generalization presented here concerns the domain of functions that generalized pseudo-Laplace type transform has been applied to. In order to do this second part of generalization, classical addition

has been substituted by L, where L is a binary operation on $[0,+\infty)$ which is nondecreasing in both coordinate, continuous on $[0,+\infty)^2$, commutative, associative, has 0 as identity and fulfills cancellation law. This operation is given by multiplicative generator $l:[0,\infty)\to(0,1]$ as $L(x,y)=l^{-1}\left(l(x)l(y)\right)$, where l is continuous, decreasing function such that $l(0)=1$. Additionally, let $\diamondsuit:[0,\infty)^2\longrightarrow[0,\infty)$ be another binary operation that is distributive with respect to L. It is easy to see that if L is given by multiplicative generator, operation $\diamondsuit$ is $x\diamondsuit y=l^{-1}\left(\exp\left(-\alpha\ln l(x)\ln l(y)\right)\right)$, where $\alpha\in(0,\infty)$.

Remark 3. Operation L has been introduced in the style of Schweizer and Sklar [26]. Some further generalizations in this direction are possible.

3 The Generalized $(\oplus,\odot)$-Laplace Transform Based on Generated Pseudo-Operations with Two Parameters

Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by generating function g, and let L and $\diamondsuit$ be binary operations given by generating function l.

Definition 2. T*he generalized* $(\oplus,\odot)$*-Laplace transform based on generated pseudo-operations with two parameters of a continuous function* $f:[0,\infty)\longrightarrow[0,\infty)$ *is*

$$L_{\odot}^{\oplus}(f)(z)=\lim_{b\longrightarrow\infty}\int_{[0,b]}^{(\oplus,\odot)}\left(\left[g^{-1}\circ l\right]_g(x\diamondsuit z)\odot f(x)\right), \tag{3}$$

if the limit exists.

It should be emphasized that through this paper, by generalized $(\oplus,\odot)$-Laplace transform, the pseudo-Laplace type transform given by previous definition will be considered.

Using connection between the $\oplus$-integral and the Riemann integral, following form of the generalized $(\oplus,\odot)$-Laplace transform is obtained:

$$Ł_{\odot}^{\oplus}(f)(z)=g^{-1}\left(\int_0^{\infty}\left(l\left(x\diamondsuit z\right)g\left(f(x)\right)\right)^{\gamma}\,dx\right).$$

Example 1. Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by generating function $g(x)=x^p,\ x\in[0,\infty)$, for some $p>0$. Under this assumption, corresponding generalized $Ł_{\odot}^{\oplus}$-transform of function $f:[0,\infty)\longrightarrow[0,\infty)$ is

$$Ł_{\odot}^{\oplus}(f)(z)=\left(\int_0^{\infty}\left(l\left(x\diamondsuit z\right)f(x)\right)^{p\gamma}\,dx\right)^{1/p},$$

and for $l(x)=\left(\ln(x+e)\right)^{-1}$

$$Ł_{\odot}^{\oplus}(f)(z) = \left(\int_0^{\infty} e^{-p\gamma\alpha \ln(\ln(x+e)) \ln(\ln(z+e))} f^{p\gamma}(x)\, dx \right)^{1/p}.$$

Remark 4. (a) For $\varepsilon = \gamma = 1$, pseudo-Laplace type transform based on a semiring from the second class (pseudo-operations are commutative and associative, see [7, 10, 14, 17–20]), presented in [23] can be obtained.
(b) For $\varepsilon = \gamma = 1$ and $L(x, y) = x + y$, pseudo-Laplace transform from [19] can be obtained. In this case, the pseudo-exchange formula in cooperation with the inverse pseudo-Laplace transform has been used for determination of utility functions' extreme values [5, 19].

Remark 5. Generalization of Laplace type transform of a measurable function $f : [0, \infty) \longrightarrow [0, 1]$ known as the (S, T)-Laplace transform, where $([0, 1], S, T)$ is the conditionally distributive semiring, can be found in [5].

3.1 Basic Properties of the Generalized $(\oplus, \odot)$-Laplace Transform

Some basic properties of the generalized $(\oplus, \odot)$-Laplace transform are given by the following theorem.

Theorem 1. *Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by generating function g, l generating function for operations L and $\diamondsuit$ and $Ł_{\odot}^{\oplus}$ corresponding transform given by (3).*

(i) Following type of pseudo-linearity holds:

$$Ł_{\odot}^{\oplus}\left([a \odot f \oplus b \odot h]_g \right)(z) = a \odot Ł_{\odot}^{\oplus}\left([f]_g(t) \right)(z) \oplus b \odot Ł_{\odot}^{\oplus}\left([h]_g(t) \right)(z).$$

(ii) For some $u, v \in [0, \infty)$ such that $L(u, v) = z$ following holds:

$$Ł_{\odot}^{\oplus}(f)(u) = Ł_{\odot}^{\oplus}(\varphi_v \odot f)(z),$$

where $\varphi_v(x) = \left[g^{-1}\left(1/l(x \diamondsuit v)\right) \right]_g$.

Proof. (i) This property follows from (3) and properties of used class of generalized pseudo-operations:

$$\begin{aligned}
&Ł_{\odot}^{\oplus}\left([a \odot f \oplus b \odot h]_g \right)(z) \\
&= g^{-1}\left(\int_0^{\infty} l^{\gamma}(x \diamondsuit z) g\left(a \odot f(x) \oplus b \odot h(x)\right)\, dx \right) \\
&= g^{-1}\left(\int_0^{\infty} l^{\gamma}(x \diamondsuit z) \left(\varepsilon g(a \odot f(x)) + g(b \odot h(x))\right)\, dx \right) \\
&= g^{-1}\left(\varepsilon g^{\gamma}(a) \int_0^{\infty} l^{\gamma}(x \diamondsuit z) g(f(x))\, dx + g^{\gamma}(b) \int_0^{\infty} l^{\gamma}(x \diamondsuit z) g(h(x))\, dx \right) \\
&= a \odot Ł_{\odot}^{\oplus}[f]_g(z) \oplus b \odot Ł_{\odot}^{\oplus}[h]_g(z).
\end{aligned}$$

(ii) Properties of operations L and $\diamondsuit$ and assumption that $L(u,v) = z$, for some $u, v \in [0,\infty)$, insure that $l(u) = l(z)\,(l(v))^{-1}$ and $l^{\gamma}(x \diamondsuit u) = l^{\gamma}(x \diamondsuit z) l^{-\gamma}(x \diamondsuit v)$. Now, following holds:

$$\begin{aligned} Ł_{\odot}^{\oplus}(f)(u) &= g^{-1}\left(\int_0^{\infty} l^{\gamma}(x \diamondsuit u) g^{\gamma}\left(f(x)\right)\, dx\right) \\ &= g^{-1}\left(\int_0^{\infty} l^{\gamma}(x \diamondsuit z) l^{-\gamma}(x \diamondsuit v) g^{\gamma}\left(f(x)\right)\, dx\right) \\ &= Ł_{\odot}^{\oplus}\left(\varphi_v \odot f\right)(z), \end{aligned}$$

where $\varphi_v(x) = \left[g^{-1}\left(1/l(x \diamondsuit v)\right)\right]_g$. □

The generalized $(\oplus, \odot)$-Laplace transform also has some nice properties when applied to integrals of functions. Following theorem deals with this problem for $l(x) = e^{-x}$, i.e., for $L(x,y) = x + y$.

Theorem 2. *Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by generating function g, $l(x) = e^{-x}$ and $L_{\odot}^{\oplus}$ corresponding transform given by (3).*

(i) For $f : [0,\infty) \longrightarrow [0,\infty)$ being a continuous function, following holds:

$$L_{\odot}^{\oplus}\left[F\right]_g(z) = g^{-1}\left((1/z\gamma)^{1/\gamma}\right) \odot L_{\odot}^{\oplus} f(z),$$

where $F(x) = \int_{[0,x]}^{(\oplus,\odot)} f$.

(ii) For $\star$ being a pseudo-convolution given by (2) and $f_1, f_2 : [0,\infty) \longrightarrow [0,\infty)$ continuous functions, following holds:

$$L_{\odot}^{\oplus}\left[f_1 \star f_2\right]_g(z) = \left[L_{\odot}^{\oplus} f_1\right]_g(z) \odot L_{\odot}^{\oplus} f_2(z).$$

Proof. Proof of this theorem is based on properties of the classical Laplace transform, pseudo-operations given by (1), $\oplus$-integral and (3):

$$\begin{aligned} Ł_{\odot}^{\oplus}\left[F\right]_g(z) &= g^{-1}\left(\int_0^{\infty} e^{-\gamma x z}\left(\int_0^{x} \left(g \circ f(t)\right)^{\gamma} dt\right) dx\right) \\ &= g^{-1}\left(\frac{1}{z\gamma}\int_0^{\infty} e^{-\gamma x z}\left(g \circ f(x)\right)^{\gamma}\, dx\right) \\ &= g^{-1}\left((1/z\gamma)^{1/\gamma}\right) \odot Ł_{\odot}^{\oplus} f(z). \end{aligned}$$

Proof for (ii) can be found in [27]. □

3.2 Limits of Generalized $(\oplus, \odot)$-Laplace Transforms

Let g be a generating function from Definition 1 and let λ be some positive real parameter. Now, with $\oplus_\lambda$ is denoted a generated pseudo-addition given by

$$x \oplus_\lambda y = (g^\lambda)^{-1}(\varepsilon g^\lambda(x) + g^\lambda(y)).$$

Of special interest for this paper is a fact that a family of such operations is converging to an idempotent operation (see [25]), i.e.,

$$\lim_{\lambda \longrightarrow \infty} x \oplus_\lambda y = \max\{x, y\}$$

while g is strictly increasing. Strictly decreasing generating function g will lead to min. For corresponding generated pseudo-multiplication holds $x \odot_\lambda y = x \odot y$, which is easily checked.

Remark 6. This property of generated pseudo-operatios given by (1) has been investigated in [25], where it has been applied on nonlinear PDE.

Remark 7. Limit property of g-operations ($\varepsilon = \gamma = 1$) has been investigated in [13]. Also, this limit property has been expanded to the g-integral (see [13]) and the g-convolution [21].

Now, the generalized $(\oplus_\lambda, \odot_\lambda)$-Laplace transform of some continuous function f is defined as

$$Ł_{\odot_\lambda}^{\oplus_\lambda}(f)(z) = Ł_{\odot}^{\oplus_\lambda}(f)(z) = \lim_{b \longrightarrow \infty} \int_{[0,b]}^{(\oplus_\lambda, \odot)} \left(\left[(g^\lambda)^{-1} \circ l^\lambda \right]_{g^\lambda} (x \diamondsuit z) \odot f(x) \right). \quad (4)$$

Remark 8. Since function l^λ also generates operation L, operations L and $\diamondsuit$ are remaining as operations on the domain of function f in (4).

Theorem 3. *Let $(\oplus_\lambda)_{\lambda \in [0,+\infty)}$ be a family of generated pseudo-additions given by continuous strictly increasing generator $g : [0, +\infty) \to [0, +\infty)$, and let $L_{\odot}^{\oplus_\lambda}$ be corresponding generalized pseudo-Laplace type transforms. Than, following holds*

$$\lim_{\lambda \to \infty} Ł_{\odot}^{\oplus_\lambda}(f)(z) = \sup_{x \geq 0} \left(f(x) \odot g^{-1}\left(l^\gamma (x \diamondsuit z) \right) \right). \quad (5)$$

Proof. For g strictly increasing generating function, $\lambda \in (0, +\infty)$ and pseudo-operations $x \oplus_\lambda y = (g^\lambda)^{-1}(\varepsilon g^\lambda(x) + g^\lambda(y))$ and $x \odot_\lambda y = x \odot y$, corresponding generalized pseudo-Laplace type transform is

$$Ł_{\odot}^{\oplus_\lambda}(f)(z) = \lim_{b \longrightarrow \infty} \int_{[0,b]}^{(\oplus_\lambda, \odot)} \left(\left[(g^\lambda)^{-1} \circ l^\lambda \right]_{g^\lambda} (x \diamondsuit z) \odot f(x) \right)$$

$$= g^{-1} \left(\left(\int_0^\infty \left(l\left(x \diamondsuit z\right) g\left(f(x)\right) \right)^{\lambda\gamma} dx \right)^{1/\lambda} \right).$$

The last integral will be denoted with $I_{\lambda,[0,\infty]}$. For all $b > 0$ and integral $I_{\lambda,[0,b]} = g^{-1}\left(\left(\int_0^b \left(l\left(x \diamondsuit z\right) g\left(f(x)\right)\right)^{\lambda\gamma}\, dx\right)^{1/\lambda}\right)$ following holds

$$I_{\lambda,[0,b]} = g^{-1}\left(\left(\lim_{n\to+\infty}\left(\sum_{i=1}^{n}\left(\left(l\left(x_i \diamondsuit z\right) g\left(f(x_i)\right)\right)^{\lambda\gamma}\frac{b}{n}\right)\right)\right)^{1/\lambda}\right),$$

where $0 = x_0 < x_1 < \cdots < x_n = b$ and length of each subinterval $[x_i, x_{i+1}]$ is b/n. Now, for a fixed n, we have

$$\lim_{\lambda\longrightarrow\infty}\left(\sum_{i=1}^{n}\left(l\left(x_i \diamondsuit z\right) g\left(f(x_i)\right)\right)^{\lambda\gamma}\frac{b}{n}\right)^{1/\lambda} = \sup_{i\in\{1,\ldots,n\}}\left(l\left(x_i \diamondsuit z\right) g\left(f(x_i)\right)\right)^{\gamma}$$

and due to continuity of generating function,

$$\lim_{\lambda\longrightarrow\infty} I_{\lambda,[0,b]} = \lim_{n\longrightarrow\infty} g^{-1}\left(\sup_{i\in\{1,\ldots,n\}}\left(l\left(x_i \diamondsuit z\right) g\left(f(x_i)\right)\right)^{\gamma}\right)$$

$$= g^{-1}\left(\sup_{x\in[0,b]}\left(l\left(x \diamondsuit z\right) g\left(f(x)\right)\right)^{\gamma}\right).$$

Since previous equation holds for all $b > 0$, we have $\lim_{\lambda\longrightarrow\infty} I_{\lambda,[0,\infty]} = g^{-1}\left(\sup_{x\geq 0}\left(l\left(x \diamondsuit z\right) g\left(f(x)\right)\right)^{\gamma}\right) = \sup_{x\geq 0}\left(f(x) \odot g^{-1}\left(l^{\gamma}(x \diamondsuit z)\right)\right)$. □

Generalized pseudo-Laplace type transform obtained in the previous theorem as a limit case will be denoted with $Ł_{\odot}^{\max}$. Similar result holds for strictly decreasing generating function, i.e., generalized pseudo-Laplace type transform of the form $Ł_{\odot}^{\min}(f)(z) = \inf_{x\geq 0}\left(f(x) \odot g^{-1}\left(l^{\gamma}(x \diamondsuit z)\right)\right)$ can be easily obtained.

Limit case obtained in Theorem 3 is a generalization of Laplace type transform proposed in [23]. Since the following result is analogous to the exchange formula proved in [23], the proof will be omitted.

Proposition 1. *Let $\oplus$ and $\odot$ be generated pseudo-operations given by (1) and let $L_{\odot}^{\max}$ be a limit case from Theorem 3. Then, for continuous functions $f_1, f_2 : [0,\infty) \longrightarrow [0,\infty)$, following holds*

$$L_{\odot}^{\max}\left[f_1 \star_{L,\gamma} f_2\right]_g(z) = \left[Ł_{\odot}^{\max} f_1\right]_g(z) \odot Ł_{\odot}^{\max} f_2(z),$$

where $f_1 \star_{L,\gamma} f_2(x) = g^{-1}\left(\sup_{L(u,v)=x} g^{\gamma}(f_1(u)) g^{\gamma}(f_2(v))\right)$.

Remark 9. Operation $\star_{L,\gamma}$ can be obtained as a limit of generalized pseudo-convolutions given by (2) where addition on domains of functions f_1 and f_2 has been substituted with L.

4 Pseudo-Aggregation Operators Based on the Generalized $(\oplus, \odot)$-Laplace Transforms

By an aggregation operator [2] is usually considered a function A such that $A : \cup_{n\in\mathbb{N}}[0,1]^n \longrightarrow [0,1]$ and

(i) $A(u_1,\ldots,u_n) \leq A(v_1,\ldots,v_2)$ when $u_i \leq v_i$ for all $i \in \{1,\ldots,n\}$,
(ii) $A(u) = u$ for all $u \in [0,1]$,
(iii) $A(1,\ldots,1) = 1$ and $A(0,\ldots,0) = 0$.

A large class of aggregation operators have been constructed by different types of integrals [1, 2, 6]. With a method similar to the construction of (S,U)-integral-based aggregation operators [6], it is possible to construct following pseudo-aggregation operator $\widetilde{A} : \cup_{n\in\mathbb{N}}[0,\infty)^n \longrightarrow [0,\infty)$ based on $\oplus$-integral:

$$\widetilde{A}(u_1,\ldots,u_n) = \int_{[0,1]}^{(\oplus,\odot)} \varphi,$$

where $\varphi : (0,1] \longrightarrow [0,\infty)$ is a function given by $\varphi(x) = u_i,\ \ x_{i-1} < x \leq x_i,$ $i \in \{1,\ldots,n\},$ for some n-partition $0 = x_0 < x_1 \ldots < x_n = 1$ (see [2, 6]). For each input value corresponding associated interval can be considered as an area of influence of the input value in question.

Problem addressed in this paper is whether operators of aggregation type can be induced by the means of the generalized $(\oplus, \odot)$-Laplace transforms.

Let $u_1, u_2, \ldots, u_n$ be n input values from $[0,\infty)$. For each n input values and each n-partition where $0 = x_0 < x_1 < \ldots < x_n = 1$ of interval $(0,1]$ is possible to form a step function $\varphi : (0,\infty) \longrightarrow [0,\infty)$ as

$$\varphi(x) = \begin{cases} u_i, & \text{for } x \in (x_{i-1}, x_i], \\ g^{-1}(0), & \text{for } x > 1, \end{cases} \tag{6}$$

where g is a generating function for pseudo-operations $\oplus$ and $\odot$ given by (1).

Definition 3. *Pseudo-aggregation operator* $\widetilde{A_{GL}} : \cup_{n\in\mathbb{N}}[0,\infty)^n \longrightarrow [0,\infty)$ *based on the generalized* $(\oplus, \odot)$*-Laplace transform is*

$$\widetilde{A_{GL}}(u_1,\ldots,u_n) = \mathcal{L}_{\odot}^{\oplus}(\varphi)(z), \tag{7}$$

where φ *is a step function for input values* $u_1, u_2, \ldots, u_n$ *given by (6) and* z *is some real positive parameter.*

Since generalized $(\oplus, \odot)$-Laplace transform is based on nonassociative and noncommutative pseudo-operations, the impact of some input value to the result can be determined by its index and by length of associated subinterval of the unite interval.

It can be easily shown that pseudo-aggregation operator $\widetilde{A_{GL}}$ with parameter z has the following form

$$\widetilde{A_{GL}}(u_1,\ldots,u_n)=\bigoplus_{i=1}^{n} u_i \odot \omega_{(x_{i-1},x_i],z}, \tag{8}$$

where $(x_{i-1},x_i]$ is subinterval of the unite interval associated to input value u_i and

$$\omega_{(x_{i-1},x_i],z}=g^{-1}\left(\varepsilon^{i-n}\int_{x_{i-1}}^{x_i} l^{\gamma}(t\diamondsuit z)\,dt\right).$$

Example 2. Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by generating function $g(x)=x^p,\ x\in[0,\infty)$ for some $p>0$, and let $L(x,y)=x+y$. Now, $\omega_{(x_{i-1},x_i],z}=g^{-1}\left((e^{-z\gamma x_{i-1}}-e^{-z\gamma x_i})/\varepsilon^{n-i}\gamma z\right)$ and corresponding pseudo-aggregation operator $\widetilde{A_L}$ with parameter z for input values $u_1,\ldots,u_n$ is

$$\widetilde{A_L}\,(u_1,\ldots,u_n)=\left(\frac{1}{z\gamma}\sum_{i=1}^{n} u_i^{p\gamma}\left(e^{-z\gamma x_{i-1}}-e^{-z\gamma x_i}\right)\right)^{\frac{1}{p}}.$$

Basic properties of pseudo-aggregation operator $\widetilde{A_{GL}}$ with parameter z are given by next proposition.

Proposition 2. *Let $\widetilde{A_{GL}}$ be a pseudo-aggregation operator given by (7). Then*

(i) $\widetilde{A_{GL}}(u_1,\ldots,u_n)\le\widetilde{A_{GL}}(v_1,\ldots,v_2)$ *when* $u_i\le v_i$ *and* u_i *and* v_i *are associated to the same subinterval* $(x_{i-1},x_i]$, $i\in\{1,\ldots,n\}$,

(ii) $\widetilde{A_{GL}}(u)=u\odot\omega_{(0,1],z}$ *for all input values* u,

(iii) $\widetilde{A_{GL}}(\mathbf{1},\ldots,\mathbf{1})=\mathbf{1}\odot\omega_{(0,1],z}$ *and* $\widetilde{A_{GL}}(\mathbf{0},\ldots,\mathbf{0})=\mathbf{0}\odot\omega_{(0,1],z}$.

Some other properties of pseudo-aggregation operators $\widetilde{A_{GL}}$ are given by the next proposition. This proposition is a generalization of result from [27] with similar proof based on (8), therefore the proof will be omitted.

Proposition 3. *Let $\widetilde{A_{GL}}$ be pseudo-aggregation operator given by (7). For input values $u,u_1,\ldots,u_n$ and $v_1,\ldots,v_n$ and real parameters $\alpha,b\in[0,\infty)$, following hold:*

(i) $\widetilde{A_{GL}}(u,\ldots,u)=u\odot\omega_{(0,1],z}$,

(ii) $\widetilde{A_{GL}}\left([u_1\oplus b]_g,\ldots,[u_n\oplus b]_g\right)=\widetilde{A_{GL}}\left([u_1]_g,\ldots,[u_n]_g\right)\oplus\left([b]_g\odot\omega_{(0,1],z}\right)$,

(iii) $\widetilde{A_{GL}}\left([\alpha]_g\odot u_1,\ldots,[\alpha]_g\odot u_n\right)=\alpha\odot\widetilde{A_{GL}}\left(u_1,\ldots,u_n\right)$,

(iv) $\widetilde{A_{GL}}\left([u_1\oplus v_1]_g,\ldots,[u_n\oplus v_n]_g\right)$
$=\widetilde{A_{GL}}\left([u_1]_g,\ldots,[u_n]_g\right)\oplus\widetilde{A_{GL}}\left([v_1]_g,\ldots,[v_n]_g\right)$.

5 Conclusion

The main aim of this paper has been to present some further possible steps in generalizations, based on the pseudo-analysis' apparatus, of well known notions as Laplace transform and aggregation operators, that could broaden the area of applications. Some further research of this problem should concern properties of generalize $(\oplus, \odot)$-Laplace transform and corresponding pseudo-aggregation operators and possible applications.

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Fuzzy All-Pairs Shortest Paths Problem

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Summary. In this paper, we deal with the All-Pairs Shortest Paths Problem (APSPP) on a graph in which a fuzzy number, instead of a real number, is assigned to each edge. Since the fuzzy min operator based on the extension principle leads to nondominated solutions, we propose another approach to solving the APSPP using a suitable fuzzy ranking method. We also show that the efficiency of computations may be improved by the proposed APSPP modification of the Dijkstra algorithm based on a binary heap data structure.

Key words: Shortest path problem, Fuzzy ranking, Binary heap, Priority queue.

1 Introduction

The shortest path problems are among the most important tasks of graph theory with many practical applications, e.g., in transportation, routing, and communication. They include such problems as finding the shortest path between two given vertices of a graph, finding the shortest paths from a given vertex to all other vertices, and finding the shortest paths between all pairs of vertices. While geographical distances can be stated deterministically, costs or times can fluctuate with traffic conditions, payload, and so on. In the last two cases, deterministic values for representing the edge weights cannot be used. A typical way of expressing these uncertainties in the edge weights is to utilize fuzzy numbers based on fuzzy set theory.

In the literature, several different approaches can be found for solving fuzzy graph problems. Zadeh [1] shows that fuzzy graphs may be viewed as a generalisation of the calculi of crisp graphs. Blue et al. [2] give a taxonomy of graph fuzziness that distinguishes five basic types combining fuzzy or crisp vertex sets with fuzzy or crisp edge sets and fuzzy weights and fuzzy connectivity. The paper also introduces an approach to finding the shortest path based on level graphs. Boulmakoul [3] proposed a new algebraic structure to solve the problem of the K-best fuzzy shortest paths and showed that the generalized Gauss–Seidel algorithm solving this problem always converges. However, all

these approaches are rather theoretical and do not address the implementation point of view that will be in the center of our considerations.

2 Crisp Problem and Its Time Complexity

The All-Pairs Shortest Paths Problem (APSPP) for graphs with crisp edge weights is usually solved by the Floyd–Warshall algorithm [4,5], which is based on the idea of gradual improvement so that the set of intermediate vertices of considered paths grows with every iteration until it encompasses all the vertices.

2.1 Floyd–Warshall Algorithm

Let $G = (V, E)$ be a connected weighted graph with non-negative edge lengths and $d_{ij}^{(k)}$ be the length of the shortest path from i to j such that any intermediate vertices on the path are chosen from the set $\{1, 2, \ldots, k\}$. A path consisting of a single edge has no intermediate vertices.

In the initial step we define $d_{ij}^{(0)} = w_{ij}$ where W is the adjacency matrix.

To pass from $d_{ij}^{(k-1)}$ to $d_{ij}^{(k)}$, we use the following reasoning:

1. If the shortest path from i to j with intermediate vertices from the set $\{1, 2, \ldots, k\}$ does not pass through k then $d_{ij}^{(k)} = d_{ij}^{(k-1)}$
2. If the shortest path from i to j with intermediate vertices from the set $\{1, 2, \ldots, k\}$ passes through k then $d_{ij}^{(k)} = d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

Summarizing, we get a recursive rule (the dynamic programming formulation) for computing $d_{ij}^{(k)}$:

$$d_{ij}^{(0)} = w_{ij} \tag{1}$$

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) \quad \text{for } k \geq 1 \tag{2}$$

The final answer is $d_{ij}^{(n)}$ because this allows all possible vertices as intermediate vertices. A straightforward transcription of this recursive rule leads to the following algorithm in which $mid(i, j)$ are mid-vertex pointers for extracting the final shortest paths.

```
FLOYD-WARSHALL(G,d,mid)
  input    : connected weighted graph G = (V, E) with nonnegative edge lengths
             w : E(G) → ℝ⁺;
  output   : d[i, j], i, j ∈ V;
             mid[i, j], i, j ∈ V;
  for i := 1 to |V| do
    for j := 1 to |V| do
      begin  d[i, j] := w[i, j];
             mid[i, j] :=null
      end;
```

```
for k := 1 to |V| do
  for i := 1 to |V| do
    for j := 1 to |V| do
      if d[i,k] + d[k,j] < d[i,j]
        then begin d[i,j] := d[i,k] + d[k,j];
                   mid[i,j] := k
             end;
```

Theorem 1. *The Floyd–Warshall algorithm runs in* $O(|V|^3)$ *time and needs* $O(|V|^2)$ *space.*

Proof. Time complexity of the algorithm is determined by the three nested loops in its second part and space complexity is given by the size of the adjacency matrix and a matrix of mid-vertex pointers for extracting the final shortest paths. □

2.2 Single-Source Shortest Paths

In this section, we describe Dijkstra's algorithm for finding the shortest paths from a given vertex (called *source*) to all other vertices. Since this algorithm is the key part of our all-pairs shortest paths algorithm proposed in the next section of this paper, we only focus on its efficient implementation using a priority queue data structure and skip its traditional descriptions.

A *priority queue* [6] supports these operations:

1. *Insert*(Q,u,*key*): insert u with the key value *key* in Q.
2. $ExtractMin(Q)$: extract the item with the minimum key value in Q.
3. *DecreaseKey*(Q,u,*new_key*): decrease the value of u's key value to *new_key*.

A priority queue can be easily implemented by a *binary heap*. It is a binary tree with vertices numbered by integers and satisfying the following conditions:

1. Each vertex of a binary heap that is not included in the last two levels has two successors.
2. In the last level, all vertices are placed from the left. This means that, passing vertices in the last but one level from left to right, only some of them (or none) may have two successors. In the latter case, at most one vertex may exist with one successor and all other vertices of this level are leaves.
3. The number of each vertex is not higher than the numbers of its successors.

The root of the binary heap is numbered by 1, other vertices at lower levels from left to right are assigned consecutive integers starting from 2.

It can be proved that, for the defined numbering of the binary heap vertices, the jth element in the ith level of a binary heap corresponds to position $2^{i-1} + j - 1$ of the array; left and right successors of vertex i have positions $2i$

and $2i+1$, respectively, and its predecessor has position $\lfloor i/2 \rfloor$ (that is position i div 2 in the Pascal notation).

Operations *Insert*, *ExtractMin* and *DecreaseKey* are implemented by a binary heap as follows:

$\boxed{Insert(Q,u,key)}$

1. [*Inserting a vertex.*] A new vertex is inserted into a binary heap by one of the following steps:
   (a) If the heap is empty, a tree with one vertex is created.
   (b) If all vertices from the first to last but one level have two successors, then, to the leftmost vertex in the last level, left successor is assigned (so that a new level is created).
   (c) If there is a vertex in the last but one level that has only the left successor, then its right successor is added.
   (d) If the previous condition is not satisfied and there is a vertex in the last but one level that contains no successors, then the left successor is added.
2. [*Determination of key value.*] Key value of the element u is assigned to the key of the inserted vertex.
3. [*Updating key values.*] If the inserted vertex u differs from the root and its key value is lower than the key value of its predecessor p, then we swap u and p, move to the predecessor position and repeat this step; otherwise the algorithm ends.

$\boxed{ExtractMin(Q)}$

1. [*Extracting the root.*] We remove the root from the binary heap because it has the lowest key value.
2. [*Creating a new root.*] We extract the vertex that is placed in the lowest level in the rightmost position and insert it in the position of the old root.
3. [*Updating key values.*] If the new root r has a successor x with a lower key value, then we swap r and x, move to the position of the successor and repeat this step; otherwise stop.

$\boxed{DecreaseKey(Q,u,new\_key)}$

After modifying the key value of a vertex, we have to check its neighbors to see whether their key values need to be modified, too. This is accomplished in the same way as in the last step of *Insert* operation.

Theorem 2. *Operations Insert, ExtractMin and DecreaseKey using a binary heap with n vertices run in $O(\log n)$ time.*

Proof. Let a binary heap containing n vertices have h levels. According to its properties we have $n = 2^0 + 2^1 + 2^2 + \ldots + 2^{h-1} + x$ where x is an integer from interval $[1, 2^h]$. Hence we get $n = 2^h - 1 + x \geq 2^h$ and therefore $h \leq \log_2 n$. Operations *Insert* and *ExtractMin* pass along a path from the root to a leaf. The *DecreaseKey* operation may pass from the root up to the vertex with a

modified key value. The proposition is stemming from the fact that all these operations in each level perform only a constant number of statements. □

Let $d[v]$ be the length of the shortest path from the source s to the vertex v. Evidently $d[s] = 0$. The lengths of unknown paths from the source to the other vertices are initialized as ∞. The algorithm is based on gradually improving the estimates of these lengths. Let (u, v) be an edge with length $w(u, v)$ and current estimates of the shortest path lengths to vertices u and v be $d[u]$ and $d[v]$. If $d[u] + w(u, v) < d[v]$, then $d[u] + w(u, v)$ becomes the new estimate of $d[v]$.

The process by which an estimate is updated is called *relaxation*. The vertices of the shortest path are determined by means of the currently saved pointers to predecessors. If we apply the relaxation repeatedly to all edges of the given graph, then the values $d[v]$ converge to the lengths of the shortest paths from v to the source s.

Let $S \subseteq V$ be a set of vertices for which we know the shortest distance to the source. Initially S is empty. The question is how we decide which vertex among the vertices of $V - S$ should be added to S. The algorithm uses the *greedy* strategy. In each step, it selects from $V - S$ the vertex u for which $d[u]$ is minimum.

In order to perform this selection efficiently, the vertices of $V - S$ are stored in a priority queue and the key value of each vertex u is $d(u)$.

Now we can formulate the algorithm. The proof of its correctness can be found, e.g., in [6]. The information about vertices contained in S, i.e., the vertices with the final value of the shortest distance from the source is stored in the Boolean variables *determined* and pointers in the array *pred* define the inverted tree of the shortest paths pointing back to s.

```
DIJKSTRA(G,s,d,pred)
  input               : connected weighted graph G = (V, E)
                        with nonnegative edge lengths w : E(G) → IR+;
                        s - source;
  output              : d[u]; u ∈ V
                        pred[u]; u ∈ V
  auxiliary variables : Adj[u] - set of neighbours of vertex u.
  for ∀u ∈ V do
    begin d[u] := +∞;
          determined[u] :=False;
          pred[u] := nil
    end;
  d[s] := 0;
  Q :=priority_queue(V);   { push all vertices into Q }
  while NonEmpty(Q) do
    begin u := ExtractMin(Q);
          for ∀v ∈ Adj[u] do
            if d[u] + w(u, v) < d[v]
              then begin d[v] := d[u] + w(u, v);
```

```
                                DecreaseKey(Q,v,d[v]);
                                pred[v] := u
                        end;
            determined[u] :=True
    end; { Dijkstra s → V }
```

Theorem 3. *Dijkstra's algorithm runs in $O(|E| \log |V|)$ time.*

Proof. By Theorem 2 the operation ExtractMin that extracts the vertex with the minimal key value from the priority queue takes $O(\log |V|)$ time. To decrease the key of the neighboring vertex for each incident edge, we may need up to $O(\log |V|)$ time. Thus the time spent is $O(\log |V| + deg(u) \log |V|)$. The other steps of the update are performed in constant time. So the overall running time is

$$\begin{aligned} T(V,E) &= \sum_{u \in V} (\log |V| + deg(u) \log |V|) = \sum_{u \in V} (1 + deg(u)) \log |V| \\ &= \log |V| \sum_{u \in V} (1 + deg(u)) = \log |V| (|V| + 2|E|) \\ &= O((|V| + |E|) \log |V|) \end{aligned}$$

Since G is connected, $|V|$ is not asymptotically greater than $|E|$, so the time complexity of Dijkstra's algorithm is $O(|E| \log |V|)$. □

2.3 "Repeated" Dijkstra's Algorithm

In Sect. 2.1, we introduced the Floyd–Warshall algorithm for solving the APSPP. However, this function can also be provided using Dijkstra's algorithm if we use as a source all vertices from the set V. This approach needs only slight modifications in data structures and, as we will prove later, is more efficient that the previous one. Here is the algorithm.

```
REPEATED_DIJKSTRA(G,d,pred)
  input    : connected weighted graph G = (V, E)
             with nonnegative edge lengths w : E(G) → IR+;
  output   : d[i, j];
             pred[i, j];
  for i := 1 to |V| do
    begin  DIJKSTRA(G,i,d0,pred0);
           for j := 1 to |V| do
             begin  d[i, j] := d0[j]
                    pred[i, j] := pred0[j]
             end
    end;
```

Theorem 4. *The previous algorithm for All-Pairs Shortest Paths Problem runs in $O(|V||E| \log |V|)$ time and needs $O(|V|^2)$ space.*

Proof. By Theorem 3 Dijkstra's algorithm runs in $O(|E|\log|V|)$ *time and this time dominates the time* $O(|V|)$ *of the nested loop* **for** j*. So the overall time complexity is given by the loop* **for** i *and nested Dijkstra's algorithm. The space complexity is given by the size of the squared matrices used.* □

Since we find the shortest paths between all the pairs most frequently in road networks where the relation $|E| < O(|V|^2)$ is mostly satisfied, our approach here is more efficient than the application of the Floyd–Warshall algorithm.

3 Fuzzy Version of the APSPP

Let us assume that the weights of the edges be given by *linear triangular fuzzy numbers*. Mathematically, a linear triangular fuzzy number $\tilde{A}$ can be represented by a triple (a_1, a_2, a_3) and its membership function $\mu_{\tilde{A}}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{, if } 0 \le x \le a_1 \\ \frac{x-a_1}{a_2-a_1} & \text{, if } a_1 \le x \le a_2 \\ 1 & \text{, if } x = a_2 \\ \frac{x-a_3}{a_2-a_3} & \text{, if } a_2 \le x \le a_3 \\ 0 & \text{, if } x \ge a_3 \end{cases} \tag{3}$$

The *addition* of these fuzzy numbers can be derived using Zadeh's extension principle and is determined as follows:

$$\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \tag{4}$$

This operation always results in a triangular fuzzy number. In the APSPP, we must also evaluate minimum operators. This means that it is necessary to have a method for ranking or comparing fuzzy numbers. An *ordering relation* $\preceq$ of fuzzy numbers can be defined, e.g., as follows:

$$\tilde{A} \preceq \tilde{B} \iff (a_1 \le b_1) \wedge (a_2 \le b_2) \wedge (a_3 \le b_3) \tag{5}$$

However, this relation is not a complete ordering, as fuzzy numbers $\tilde{A}$, $\tilde{B}$ satisfying

$$(\exists i, j \in \{1, 2, 3\}) : \ (a_i < b_i) \wedge (a_j > b_j) \tag{6}$$

are not comparable by $\preceq$.

Let us consider fuzzy min operation defined like the fuzzy addition in the following way:

$$\widetilde{\min}(\tilde{A}, \tilde{B}) = (\min(a_1, b_1), \min(a_2, b_2), \min(a_3, b_3)) \tag{7}$$

It is evident that, for noncomparable fuzzy numbers $\tilde{A}$, $\tilde{B}$, this fuzzy min operation results in a fuzzy number different from both of them. For example, for $\tilde{A} = (5, 10, 13)$ and $\tilde{B} = (6, 9, 14)$, we get from (7) a fuzzy min $(5, 9, 13)$ which differs from $\tilde{A}$ and $\tilde{B}$.

The previous remarks demonstrate the difficulties with comparisons of fuzzy numbers. For this reason, the ranking or ordering methods of fuzzy quantities have been proposed by many authors. Most of them were summarized in [7, 8]. Unfortunately, none of these methods is commonly accepted. In this paper, we use, for simplicity, the fuzzy ranking method described in [9], modified for the case of triangular fuzzy numbers. This method uses the inverse functions $g_{\tilde{A}}^L : [0,1] \to [a_1, a_2]$ and $g_{\tilde{A}}^R : [0,1] \to [a_2, a_3]$ derived from the functions $f_{\tilde{A}}^L : [a_1, a_2] \to [0,1]$ and $f_{\tilde{A}}^R : [a_2, a_3] \to [0,1]$, respectively. From $y = \frac{x-a_1}{a_2-a_1}$ we can easily derive that

$$g_{\tilde{A}}^L = a_1 + (a_2 - a_1)y \tag{8}$$

Similarly we get

$$g_{\tilde{A}}^R = a_3 + (a_2 - a_3)y. \tag{9}$$

The ranking function is defined as the distance between the *centroid point* $(\tilde{x}_0, \tilde{y}_0)$ and the origin, i.e.

$$R(\tilde{A}) = \sqrt{(\tilde{x}_0)^2 + (\tilde{y}_0)^2} \tag{10}$$

where

$$\tilde{x}_0 = \frac{\int\limits_{\mathrm{Supp}\,\tilde{A}} x\mu_{\tilde{A}}(x)dx}{\int\limits_{\mathrm{Supp}\,\tilde{A}} \mu_{\tilde{A}}(x)dx}, \quad \tilde{y}_0 = \frac{\int\limits_0^1 (y\, g_{\tilde{A}}^L)dy + \int\limits_0^1 (y\, g_{\tilde{A}}^R)dy}{\int\limits_0^1 (g_{\tilde{A}}^L)dy + \int\limits_0^1 (g_{\tilde{A}}^R)dy} \tag{11}$$

and $\mathrm{Supp}\,\tilde{A}$ is the support of $\tilde{A}$.

Fuzzy numbers $\tilde{A}$, $\tilde{B}$ are then ranked by their ranking function values $R(\tilde{A})$ and $R(\tilde{B})$.

FUZZY-DIJKSTRA(G,s,$\tilde{d}$,*pred*)
  Input : connected weighted graph $G = (V, E)$ with fuzzy edge lengths $\tilde{w}(e)$, $e \in E$
    s - source (root);
  output : $\tilde{d}[u]$;
    pred$[u]$; $u \in V$
  auxiliary variables : *Adj*$[u]$ - set of neighbors of vertex u.
  for $\forall u \in V$ **do**
    begin $\tilde{d}[u] := (0, \infty, \infty)$;
      determined$[u]$:=`False`;
      pred$[u]$:= `nil`
    end;
  $\tilde{d}[s] := (0, 0, 0)$;

```
Q :=priority_queue(V);   { push all vertices into Q ordered by d̃[u] }
while NonEmpty(Q) do
  begin  u := ExtractMin(Q);
         for ∀v ∈ Adj[u] do
           if R(d̃[u] ⊕ w̃(u,v)) < R(d̃[v])
             then begin  d̃[v] := d̃[u] ⊕ w̃(u,v);
                         DecreaseKey(Q,v,d̃[v]);
                         pred[v] := u
                  end;
         determined[u] :=True
  end; { Fuzzy Dijkstra s → V }
```

Theorem 5. *Fuzzy Dijkstra's algorithm runs in $O(|E|\log|V|)$ time.*

Proof. Let $O(T_R)$ be the time of the centroid point evaluation. Then, as in Theorem 3, we can determine the overall running time as follows:

$$\begin{aligned} T(V,E) &= \sum_{u\in V}(\log|V| + T_R deg(u)\log|V|) = \sum_{u\in V}(1 + T_R deg(u))\log|V| \\ &= \log|V|\sum_{u\in V}(1 + T_R deg(u)) = \log|V|(|V| + T_R\cdot 2|E|) \\ &= O((|V| + |E|)\log|V|) \end{aligned}$$

Since G is connected, $|V|$ is not asymptotically greater than $|E|$, so this is $O(|E|\log|V|)$. □

Similarly as in the deterministic case of the APSPP, we can solve its fuzzy modification using fuzzy Dijkstra's algorithm if we use, as a source, all vertices from the set V, and the weights of the edges are given by fuzzy numbers.

Hence we get:

Theorem 6. *The algorithm for fuzzy All-Pairs Shortest Paths Problem runs in $O(|V||E|\log|V|)$ time.*

4 Conclusions

In this paper, the problem of finding the all-pairs shortest paths was studied. We assumed that the weights of edges were given by linear triangular fuzzy numbers. First, we proposed an APSPP modification stemming from the Dijkstra algorithm and showed where it was more efficient that the classical Floyd–Warshall algorithm. Then we generalized it for the fuzzy APSPP using a unique fuzzy ranking method to avoid generating the set of nondominated paths (or Pareto Optimal paths) because the number of nondominated paths derived from a large network can be too numerous, and it could be difficult for a decision maker to choose a preferable path.

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Optimal Toll Charges in a Fuzzy Flow Problem

Stephan Dempe and Tatiana Starostina

Summary. The problem of computing best toll charges is modelled as a bilevel programming problem where, depending on the toll charge selected by the leader in the upper level, a minimum cost network flow is computed in the lower level. Due to fuzzyness of the costs for passing the edges in the network, this lower level problem is a parametric fuzzy linear programming problem. In the paper we suggest an approach for constructing a surrogate crisp optimization problem and an algorithm for solving the latter one.

Key words: Fuzzy bilevel programming, Optimal toll charges, Fuzzy network flow problem, Fuzzy costs.

1 Introduction

Toll charges are collected in many countries with different aims. Besides maximizing the collected charges and using them for covering expenses of the government especially the use of toll charges for control of the traffic is of interest. In both cases, a function $f : X \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ is used to measure the quality of the toll charges $c^t \in \mathbb{R}^n_+$ and the related traffic $x \in X$. To compute the traffic flow different graph theoretic models can be used [8]. We will consider a simplified problem of this type, where we assume that only the traffic from one origin to one destination is of interest (which then reduces to a minimum cost network flow problem). The generalization of this to the general problem is straightforward.

To compute the traffic flow (or an optimal solution of the minimum cost network flow problem) the costs for passing one edge of the underlying graph need to be known. In general these costs depend on the flow on this edge itself, they are not constant. Explicit formulas to compute these costs are not known up to now [5]. To circumvent the resulting vagueness in the model formulation, we model the costs as fuzzy numbers.

Let $x(c^t)$ denote an optimal solution of the minimum cost network flow problem depending on the toll charges c^t. Then, $f(x(c^t), c^t)$ denotes the quality of this flow together with the toll charges. The aim of the toll setting

authority is it now to maximize the function $g(c^t) := f(x(c^t), c^t)$ subject to $c^t \in C$. The topic of this paper is to suggest an algorithm solving this problem.

Related crisp optimization problems have been investigated, e.g., in [1, 3, 13].

2 The Fuzzy Flow Problem

To model the flow problem first, consider a graph $G = (V, A)$ describing a transportation network, where the set of nodes V of the graph G ($|V| = n$) stands for the junctions and the set of arcs A is used for the streets. Assume that w vehicles have to drive from node $s \in V$ (the source) to node $d \in V$ (the destination). Each arc $(v_i, v_j) \in A$ has a capacity U_{ij}. This capacity defines the maximum number of vehicles, which can pass an appropriate part of the road between two nodes in given time period.

Numbers $x_{ij},\ i, j = 1, 2, ..., n$, which are defined on the arcs $(v_i, v_j) \in A$ of graph G are called flows if they satisfy the following conditions:

$$\sum_{v_j \in \Gamma^+(v_i)} x_{ij} - \sum_{v_k \in \Gamma^-(v_i)} x_{ki} = b_i := \begin{cases} w, & \text{if } v_i = s, \\ -w, & \text{if } v_i = d, \\ 0, & \text{if } v_i \notin \{s, d\} \end{cases} \tag{1}$$

for all $v_i \in V$ and

$$0 \le x_{ij} \le U_{ij},\ \forall (v_i, v_j) \in A. \tag{2}$$

Here, $\Gamma^+(v_i)$ and $\Gamma^-(v_i)$ denote the set of successors resp. predecessors of node v_i. Equations (1) are the flow conservation conditions and inequalities (2) describe the need to satisfy the capacity conditions [11].

Some weight c_{ij} is assigned to the arc $(v_i, v_j) \in A$ representing the cost needed for traveling through the arc. This cost c_{ij} depends, e.g., on the distance between two nodes v_i and v_j, on the time needed to cover the distance between these nodes, etc.

Having in mind the applied problem of computing optimal tolls for using the network, let us assume that the costs c_{ij} for $(v_i, v_j) \in A$ are sums $c_{ij} = c^0_{ij} + c^t_{ij}$ of toll independent (i.e., e.g., only user, time or distance dependent) costs and the toll charges. Now, we can formulate the minimal cost flow problem in which a flow is searched for in the graph having minimal costs:

$$\left.\begin{array}{l} \displaystyle\sum_{(v_i, v_j) \in A} (c^0_{ij} + c^t_{ij}) x_{ij} \to \min \\ \displaystyle\sum_{v_j \in \Gamma^+(v_i)} x_{ij} - \sum_{v_k \in \Gamma^-(v_i)} x_{ki} = b_i,\ \forall v_i \in V \\ 0 \le x_{ij} \le U_{ij},\ \forall (v_i, v_j) \in A. \end{array}\right\} \quad \text{or} \quad \begin{cases} (c^0 + c^t)^T x \to \min \\ Ax = b, \\ 0 \le x \le U. \end{cases} \tag{3}$$

Usually it is assumed that traveling costs are exactly known. However, these costs depend in reality on many factors, e.g., on the traveling time, which

depends also on weather, on the present traffic, on the current situation on this part of the road (traffic jams, road works, etc.). This means that in many situations such costs can not be exactly known, but they are estimated [5]. Hence, they are better considered as fuzzy numbers. The membership function of these numbers is assumed to have trapezoidal form given by four parameters

$$\widetilde{c}^0 = (\underline{c}^0; \overline{c}^0; \alpha^0; \beta^0)_{L-L}, \tag{4}$$

where $\underline{c}^0, \overline{c}^0$–are the left and right borders of the fuzzy number $\widetilde{c}^0$ corresponding to the λ-cut, where $\lambda = 1$, and α^0 and β^0 are non-negative real numbers [10].

We consider model (3) and assume now that the traveling costs are described as fuzzy numbers. Then we obtain a model with fuzzy coefficients in the objective function:

$$\left.\begin{array}{l} (\widetilde{c}^0 + c^t)^T x \to \min \\ Ax = b, \\ 0 \le x \le U. \end{array}\right\} \tag{5}$$

In (5), the traveling costs $\widetilde{c}_{ij}$ are given as fuzzy numbers and the toll charges c_{ij}^t are crisp, but unknown.

3 The Toll Finding Problem and Its Reformulation

Now, we are in a position where it is possible to formulate the problem of determining best toll charges in a fuzzy environment:

$$\begin{array}{l} f(x, c^t) \longrightarrow \min \\ x \text{ solves problem (5) for fixed } c^t, \\ c^t \in C. \end{array} \tag{6}$$

Here C is a set of feasible toll charges and the real valued function $f(x, c^t)$ gives the profit resulting from the flow x and the toll charges c^t. Clearly, the flow depends on the toll charges and the profit is earned, e.g., by the government.

Problem (6) is a so called bilevel programming problem [6]. In the crisp situation this problem has widely been investigated, e.g., in [8, 12]. In our opinion, the more realistic formulation yet is using a fuzzy environment. To the best of our knowledge fuzzy bilevel programming problems have hardly ever been investigated, see [15] for a heuristic approach.

Denote an optimal solution of (5) by $x(c^t)$. Then, substituting this function for x in the objective function of (6) we derive

$$\min\{f(x(c^t), c^t) : c^t \in C\}. \tag{7}$$

Recall, that since (5) is a fuzzy linear programming problem, its solution should rather be considered as fuzzy set $\widetilde{X}(c^t) = \{(x(c^t), \mu_{\widetilde{X}}(x(c^t)))\}$. Hence, also the function $c^t \mapsto f(\widetilde{X}(c^t), c^t)$ has fuzzy sets as values. Problem (7) is

then a realization of the problem

$$\min\{f(\widetilde{X}(c^t), c^t) : c^t \in C\}. \tag{8}$$

To solve this fuzzy optimization problem (8), approaches in the papers [4,14] can be used. This means that we may use realizations (7) for attacking (8).

Assume that we take a best realization in the sense of [14]. To formulate the lower level problem replacing (5) in the sense of [14] let there be given a set Λ for determining the λ-cuts of the membership functions of the fuzzy cost coefficients. The selection of these cuts is beyond the topic of this paper. Clearly the obtained result is sensitive with respect to the selected cuts. We suggest to check usefulness of the obtained solution against applicability and to modify the values of the cuts if this fails. Assume that $y^{*\lambda}_{\min}$ and $y^{*\lambda}_{\max}$ are lower bounds for the objective function value of a crisp linear programming problems $(LP^{\lambda}_{\min})$ and $(LP^{\lambda}_{\max})$ obtained if the fuzzy cost coefficients $\widetilde{c}^0_{ij}$ in (5) are replaced by the left-hand $\underline{c}^{0\lambda}_{ij}$, respectively, the right-hand side $\overline{c}^{0\lambda}_{ij}$ of their λ-cuts. Note that these bounds are assumed to be valid for all c^t which can be obtained if the objective function in (5) is minimized both with respect to x and $c^t \in C$. If, as it is reasonable from the practical point-of-view, the set C is a box constraint $C = \{c^t : \underline{c}^t_{ij} \le c^t_{ij} \le \overline{c}^t_{ij}\}$ these bounds can easily be computed if c^t is replaced componentwise by the lower bounds.

Also assume that there are given values $\overline{y}^{\lambda}_{\min}$ and $\overline{y}^{\lambda}_{\max}$ representing the maximal acceptable objective function value of the problems $(LP^{\lambda}_{\min})$ and $(LP^{\lambda}_{\max})$. Such values can e.g. be generated if the optimal solutions of the problems $(LP^{\lambda}_{\min})$ and $(LP^{\lambda}_{\max})$ are interchanged as in the paper [14] and the values of c^t are replaced by $\overline{c}^t$.

Then, as in [14] we can compute membership functions

$$\mu^{\lambda}_{y\,\min} = \begin{cases} \dfrac{\overline{y}^{\lambda}_{\min} - (\underline{c}^{0\lambda} + c^t)^T x}{\overline{y}^{\lambda}_{\min} - y^{*\lambda}_{\min}} & \text{if } y^{*\lambda}_{\min} \le (\underline{c}^{0\lambda} + c^t)^T x \le \overline{y}^{\lambda}_{\min} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu^{\lambda}_{y\,\max} = \begin{cases} \dfrac{\overline{y}^{\lambda}_{\max} - (\overline{c}^{0\lambda} + c^t)^T x}{\overline{y}^{\lambda}_{\max} - y^{*\lambda}_{\max}} & \text{if } y^{*\lambda}_{\max} \le (\overline{c}^{0\lambda} + c^t)^T x \le \overline{y}^{\lambda}_{\max} \\ 0 & \text{otherwise} \end{cases}$$

Now, if (5) is replaced by the crisp problem of maximizing the smallest value of all the membership functions we obtain the optimization problem

$$\begin{aligned} \omega &\longrightarrow \max \\ (\overline{y}^{\lambda}_{\min} - y^{*\lambda}_{\min})\omega + (\underline{c}^{0\lambda} + c^t)^T x &\le \overline{y}^{\lambda}_{\min},\ \forall \lambda \in \Lambda \\ (\overline{y}^{\lambda}_{\max} - y^{*\lambda}_{\max})\omega + (\overline{c}^{0\lambda} + c^t)^T x &\le \overline{y}^{\lambda}_{\max},\ \forall \lambda \in \Lambda \\ Ax &= b, \\ 0 \le x &\le U. \end{aligned} \tag{9}$$

Then, the realization (7) of (8) with this selection reduces to

$$\begin{aligned} & f(x(c^t), c^t) \longrightarrow \min \\ & x(c^t) \text{ solves (9) for fixed } c^t, \\ & c^t \in C. \end{aligned} \tag{10}$$

4 Solution Algorithm for the Bilevel Programming Problem

Problem (10) is a crisp bilevel programming problem [6]. To be precise, the formulation (10) is correct only in the case when the lower level programming (9) has a uniquely determined optimal solution for all parameter values $c^t \in C$. If this is not the case there is an ambiguity in the definition of (10) in the sense that the objective function value of this problem can be evaluated only after publication of the choice of an optimal solution by the lower level decision maker, i.e. after an optimal solution $x(c^t)$ of (9) has been selected. This means the upper level decision maker has to wait for the lower level decision maker's selection. This, of course, is a difficult situation for an optimization process.

In bilevel programming theory [6] there are at least two ways out of this unpleasant situation. The first one is the optimistic approach where the upper level decision maker supposes that the lower level one supports him and allows him to select such an optimal solution which is a best one from the upper level point-of-view, i.e., which minimizes the upper level objective function with respect to $x(c^t)$ over the solution set of (9). Sometimes this approach is justified by the possibility of the upper level decision maker to share a part of his revenue $f(x(c^t), c^t)$ with the lower level decision maker [2].

If this is not possible, the other approach often used is the pessimistic one which rests on bounding the damage resulting from a "bad" solution selected by the lower level decision maker to the upper level objective function value. This means that the upper level decision maker has to maximize the upper level objective function with respect to $x(c^t)$ over the solution set of (9).

In both cases a new function

$$\varphi_o(c^t) := \min_{x(c^t)} \{ f(x(c^t), c^t) : x(c^t) \text{ solves (9) for fixed } c^t \}$$

in the optimistic and

$$\varphi_p(c^t) := \max_{x(c^t)} \{ f(x(c^t), c^t) : x(c^t) \text{ solves (9) for fixed } c^t \}$$

in the pessimistic approach arises which is then to be minimized for $c^t \in C$.

The situation here is a little bit different. The optimal solution of (5) is a fuzzy set of feasible points and (9) is used to compute elements of this set. According to [4] elements with the largest membership function values should be selected. Prospective candidates for this are basic feasible solutions of the

vector optimization problems solved in [4] since nonbasic solutions do not remain Pareto optimal under slight perturbations of the problem, see [9, 16].

For numerical reasons we do not suggest to apply the ideas in [4, 9, 16] to compute the lower level solutions in problem (10) but the ideas in [14]. But we adopt the above idea in the sense that we assume that a basic optimal solution of (9) is selected in (10) and that the characteristics of this basic optimal solution remain valid under perturbations of c^t. This seems to be reasonable due to the following facts. The crisp optimization problem solved in [14] realizes a method to compute one Pareto optimal solution for a linear optimization problem maximizing the membership function values of the objective function if certain different λ-cuts of the objective function coefficients are used. This Pareto optimal solution is computed by solving (9). The computed optimal solution of this problem can be found at the boundary of the feasible set. These membership function values are sensitive with respect to the selected λ values. Hence, if these values change, also the obtained solution of the solved (9) changes. In this situation optimal solutions not being basic solutions most probably will not maintain optimality. But optimal basic solutions are relatively stable with respect to perturbations of the values of λ. As the characteristics of these solutions we can take the basic matrix.

To keep the notation simple we will use the following more formal description of (9) together with an upper bound strategy in the simplex algorithm. Slack variables will not be added to the basic variables.

$$\begin{aligned} \omega &\longrightarrow \max \\ a\omega + (C^0 + c^t)^T x &\leq d \\ Ax &= b, \\ 0 \leq x &\leq U. \end{aligned} \tag{11}$$

Then, some of the inequality constraints are satisfied as equations in the optimal solution and nonbasic variables are zero or u_{ij}. We assume that the first inequality constraints are satisfied strictly followed by the active ones and that the set of basic variables is followed by the set of nonbasic variables. This means that the coefficient matrix is decomposed into a basic and a nonbasic matrix as follows:

$$\begin{pmatrix} a & (C^0 + c^t)^T \\ 0 & A \end{pmatrix} = \begin{pmatrix} a_1 & (C^0_{11} + c^t_{11})^T & (C^0_{12} + c^t_{12})^T \\ a_2 & (C^0_{21} + c^t_{21})^T & (C^0_{22} + c^t_{22})^T \\ 0 & A_1 & A_2 \end{pmatrix}$$

Note that ω is a basic variable for all c^t. To compute the basic solution we now have to solve the system of equations

$$\begin{pmatrix} a_2 & (C^0_{21} + c^t_{21})^T \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \omega \\ x_B \end{pmatrix} = \begin{pmatrix} d_2 \\ b \end{pmatrix}. \tag{12}$$

From the theory of linear programming, the matrix

$$B = \begin{pmatrix} a_2 & (C_{21}^0 + c_{21}^t)^T \\ 0 & A_1 \end{pmatrix}$$

is regular and (12) has a unique solution. To prove optimality of this solution for (11) both feasibility (i.e., validity of the first set of inequalities) together with boundedness of the reduced costs in sign has to be verified. These conditions are all linear in (ω, x_B). But note that the solution (ω, x_B) of system (12) is nonlinear in c^t which appears in the coefficient matrix of a linear programming problem.

For solving (10) we now suggest the following procedure:

Step 0: select $c^t \in C$ and a set Λ of cuts. Select $\varepsilon > 0$.
Step 1: solve (9) to get an optimal basic solution $(\omega(c^t), x(c^t))$. Fix the basic matrix B.
Step 2: compute a feasible descent direction r for the problem

$$\begin{aligned} & f(x(c^t), c^t) \longrightarrow \min_{c^t} \\ & \begin{pmatrix} a_2 & (C_{21}^0 + c_{21}^t)^T \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \omega \\ x_B \end{pmatrix} = \begin{pmatrix} d_2 \\ b \end{pmatrix} \\ & c^t \in C. \end{aligned} \tag{13}$$

If there is no feasible descent direction, go to Step 4.
Step 3: compute a step size γ such that:

1. The solution of the system (12) for $c^t := c^t + \gamma r$ is an optimal solution of (11)
2. $c^t + \gamma r \in C$
3. $f(x(c^t + \gamma r), c^t + \gamma r) \leq f(x(^t), c^t) + \varepsilon\gamma s$

where s is the optimal objective function value of (13). If this was successful, set $c^t := c^t + \gamma r$ and go to Step 1
Step 4: if either no feasible descent direction in Step 2 or no step size in Step 3 can be determined, try to replace the basic matrix B by another one and repeat Step 2. If the basic matrix is uniquely determined or no other basic matrix leads to a smaller objective function value in (10), respectively, a successful selection in Step 3, terminate.

This algorithm is an application of the descent algorithm in Sect. 6.1 of [6] to our problem, see also [7].

For the computation of a feasible descent direction in Step 2 we can use that the optimal solution $(\omega(c^t), x(c^t))$ of (12) is differentiable since B is a basic matrix. For this, use the famous implicit function theorem. The gradient $(\nabla\omega(c^t), \nabla x_B(c^t), \nabla x_N(c^t))$ with $\nabla x_N(c^t) = 0$ of $(\omega(c^t), x(c^t))$ is a solution of the equation

$$\begin{pmatrix} a_2 & (C_{21}^0 + c_{21}^t)^T \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \nabla\omega \\ \nabla x_B \end{pmatrix} + \begin{pmatrix} x_B(c^t) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{14}$$

Using this and assuming that the set C is a convex polyhedron with tangent cone

$$T_C(c^t) = \{r : c^t + \gamma r \in C \text{ for all sufficiently small } \gamma > 0\}$$

at c^t, the computation of the feasible descent direction in Step 2 reduces to solving

$$\nabla f(x(c^t), c^t) \begin{pmatrix} \nabla x_B(c^t) \\ 0 \\ 1 \end{pmatrix} r \longrightarrow \min_{r \in T_C(c^t)}.$$

This is a linear programming problem.

A remark seems to be appropriate with respect to the existence of an optimal solution. In the case of multiple optimal solutions of a crisp lower level problem, the optimistic approach of bilevel programming has an optimal solution under weak assumptions. We adopt this idea in the above algorithm under the assumption that the follower will take a basic optimal solution of his problem in any case.

5 Conclusion

In this paper we have modeled the problem of determining a best toll charge as a bilevel fuzzy optimization problem. The fuzzy optimization problem is the lower level problem in which it is assumed that the cost of traveling through a street is the sum of the toll charge and a fuzzy value. To attack the fuzzy bilevel programming problem we transform it into a crisp one by using the approach by Rommelfanger et al. [14] and assuming that the lower level decision maker will select a basic optimal solution of the resulting lower level problem for all values of the toll charge. This assumption seems to be reasonable for the fuzzy lower level optimization problem. Then, the resulting problem can be reformulated as a one level programming (7) with a piecewise differentiable objective function. Using the properties of the lower level basic optimal solutions it was possible to formulate a first solution algorithm for this problem.

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Modified Interval Global Weights in AHP

Tomoe Entani and Hideo Tanaka

Summary. The decision problem in Analytic Hierarchy Process (AHP) is structured hierarchically as several criteria and alternatives. It is proposed to determine the global weights of alternatives considering the referenced priority weights of criteria and local weights of alternatives. We assume them as intervals, since they are obtained from the corresponding pairwise comparison matrices given by a decision maker based on his/her intuition. The width represents the possibility of each weight reflecting the inconsistency of the given comparisons. Then, the global weights calculated with them should be also intervals and such intervals tend to contain redundant parts. We propose the models to modify the intervals so as to be normalized keeping their possibilities. Instead of crisp normalization, the interval probability fills the role of interval normalization. The modified interval global weights reflect a decision maker's uncertain judgments as intervals without redundancy.

Key words: Analytic hierarchy process, Interval probability, Interval normalization, Interval global weight.

1 Introduction

AHP (Analytic Hierarchy Process) is a useful method in multicriteria decision making problems [1, 2]. It is structured hierarchically as criteria and alternatives and proposed to determine the priority weights of alternatives which are the global weights as decision. In the decision problem of AHP, it is possible to construct several layers of criteria as in Fig. 1, where C_1 consists of m_1 criteria, $\mathrm{C}_{11}, \mathrm{C}_{12}, \ldots, \mathrm{C}_{1m_1}$.

From the pairwise comparison matrix for criteria, the referenced priority weights are obtained by eigenvector method [1]. In the same way, from the pairwise comparison matrix for alternatives under each criterion, the local weights for the criterion are obtained. The elements of the two types of comparison matrices are relative measurements given by a decision maker. The obtained weights from the matrix can reflect his/her attitude in the actual

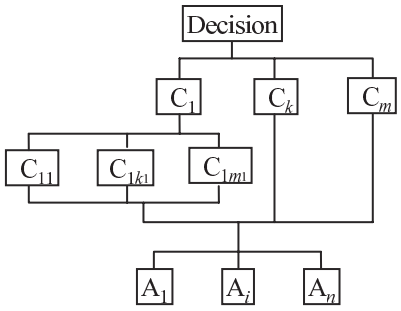

Fig. 1. Structure of decision problem in AHP

decision problem. In this paper, it is assumed that the estimated referenced priority and local weights are normalized intervals to reflect inconsistency of the given pairwise comparisons. The normalized intervals satisfy the conditions of interval probabilities in [3] so that the redundancy can be reduced.

With the obtained interval referenced priority and local weights, the global weights of all alternatives are also obtained as intervals. Because of interval arithmetic, the widths of the obtained intervals seem to be large. They have some redundancy and are not normalized. Therefore, the model to obtain normalized interval global weights has already been proposed [4]. In the model the crisp referenced priority weights such that their sum becomes one are selected and the interval global weights are calculated by them. However, the normalized interval global weights depart from those by interval arithmetic. The possibilities denoted as the widths of intervals are lessened. This paper focuses on obtaining the interval global weights which are normalized and reflect the original possibility as precisely as possible. We propose the models to modify the obtained interval global weights so as to be interval probabilities. The normalized interval global weights are useful information for a final decision in the sense that they reflect a decision maker's uncertain judgements as intervals without redundancy.

2 Interval Priority Weights from Crisp Pairwise Comparisons

AHP is a method to deal with the weights with respect to many alternatives and to determine the priority weight of each alternative [1]. When there are n alternatives, a decision maker compares a pair of alternatives for all possible pairs to obtain a pairwise comparison matrix A as follows.

$$A = [a_{ij}] = \begin{pmatrix} 1 & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & 1 \end{pmatrix}$$

where a_{ij} shows the priority ratio of alternative i comparing to alternative j.

The pairwise comparison matrix satisfies the following relations so that the decision maker gives $n(n-1)/2$ comparisons.

$$\begin{aligned} &\text{Diagonal elements} \quad a_{ii} = 1 \\ &\text{Reciprocal elements } a_{ij} = 1/a_{ji} \end{aligned} \tag{1}$$

As for the consistent comparison matrix, the following relation is satisfied.

$$a_{ij} = a_{ik}a_{jk} \ \ \forall(i,j) \tag{2}$$

The given pairwise comparison a_{ij} is approximated by the ratio of priority weights, w_i and w_j, symbolically written as $a_{ij} \approx w_i/w_j$. Since this is the ratio relation, we normalize the priority weights in order to obtain ones. When the decision maker gives consistent comparisons, it holds $a_{ij} = w_i/w_j$. The comparisons are given one after another based on decision maker's intuitive judgements so that they scarcely ever consistent. In this paper the priority weight w_i is assumed as an interval and the interval priority weights denoted as $W_i = [\underline{w}_i, \overline{w}_i]$. We consider interval probabilities proposed in [3] so as to normalize interval weights. Their conditions are defined as follows.

Definition 1 Interval weights $(W_1, ..., W_n)$ are called interval probabilities if and only if

$$\begin{aligned} &\textstyle\sum_{i\neq j} \overline{w}_i + \underline{w}_j \geq 1 \ \ \forall j \\ &\textstyle\sum_{i\neq j} \underline{w}_i + \overline{w}_j \leq 1 \ \ \forall j \end{aligned} \tag{3}$$

where $W_i = [\underline{w}_i, \overline{w}_i]$.

It can be said that the conventional normalization is extended to the interval normalization by using the above conditions. This is effective to reduce redundancy under the condition that the sum of crisp weights in the interval weights is equal to one.

The approximated pairwise comparison with the interval weights is defined as the following interval.

$$\frac{W_i}{W_j} = \left[\frac{\underline{w}_i}{\overline{w}_j}, \frac{\overline{w}_i}{\underline{w}_j}\right]$$

where the upper and lower bounds of the approximated comparison are defined as the maximum range.

The interval priority weights are obtained so as to include the given interval by the approximation model [5, 6]. The obtained interval weights satisfy the following inclusion relations.

$$a_{ij} \in \frac{W_i}{W_j} = \left[\frac{\underline{w}_i}{\overline{w}_j}, \frac{\overline{w}_i}{\underline{w}_j}\right] \ \ \forall(i,j) \ \ \leftrightarrow \ \ \frac{\underline{w}_i}{\overline{w}_j} \leq a_{ij} \leq \frac{\overline{w}_i}{\underline{w}_j} \ \ \forall(i,j) \tag{4}$$

In order to obtain the least upper approximation, the width of each weight must be minimized. Simply the sum of widths of all weights is minimized as the following LP problem.

$$\begin{aligned} & \min \quad \textstyle\sum_i (\overline{w}_i - \underline{w}_i) \\ \text{s.t.} \; & \textstyle\sum_{i \neq j} \overline{w}_i + \underline{w}_j \geq 1 \;\; \forall j \\ & \textstyle\sum_{i \neq j} \underline{w}_i + \overline{w}_j \leq 1 \;\; \forall j \\ & \dfrac{\underline{w}_i}{\overline{w}_j} \leq a_{ij} \leq \dfrac{\overline{w}_i}{\underline{w}_j} \;\; \forall (i,j) \\ & \overline{w}_i \geq \underline{w}_i \geq 0 \quad \forall i \end{aligned} \tag{5}$$

where constraint conditions consist of the interval normalization (3) and inclusion relations (4).

The interval weights by (5) include the given inconsistent comparisons and the least uncertain weights are obtained. The width of the obtained interval weight represents the possibility of the priority of the alternative i. The possibility is needed to include the inconsistency among the given comparisons. The widths can be regarded as the index of inconsistency. If the consistent comparisons are given, we can obtain crisp weights by the proposed approach and they are the same as ones obtained by eigenvector method.

2.1 Interval Global Weights

The decision problem in AHP is structured hierarchically as criteria and alternatives as in Fig. 1. A decision maker gives pairwise comparison matrix for alternatives $\mathrm{A}_i (i = 1, \ldots, n)$ under each criterion and also comparison matrix for criteria $\mathrm{C}_k (k = 1, \ldots, m)$ comparing alternatives and criteria importance, respectively. When the pairwise comparison matrices for criteria and alternatives under each criterion are given by a decision maker, the interval referenced priority and local weights are obtained, respectively, by (5). In Table 1, the local weight of alternative A_i under criterion C_k is denoted as $W_{ik} = [\underline{w}_{ik}, \overline{w}_{ik}]$ and the referenced priority weight of criterion C_k is denoted as $P_k = [\underline{p}_k, \overline{p}_k]$.

With the obtained interval referenced priority and local weights, the global weight of alternative A_i is obtained as follows by interval arithmetic.

$$W_i = \textstyle\sum_k P_k W_{ik} = \left[\sum_k \underline{p}_k \underline{w}_{ik}, \sum_k \overline{p}_k \overline{w}_{ik} \right] \tag{6}$$

Table 1. Weights at each layer of decision problem in AHP

	Criterion			
Alternative	C_1	$\cdots$ C_k $\cdots$	C_m	Decision
A_1	W_{11}	W_{1k}	W_{1m}	W_1
$\vdots$	$\vdots$	*Local weight*		$\vdots$
A_i	W_{i1}	*under criterion*		W_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	*Global weight*
A_n	W_{n1}	W_{nk}	W_{nm}	W_n
	P_1	$\cdots$ P_k $\cdots$	P_m	
	Referenced priority weight			

By multiplying the upper and lower bounds of intervals, their widths seem to be large. Although W_{ik} and P_k are all normalized intervals, W_i may have some redundancy and is not normalized. The conditions of interval probabilities in (3) are written as follows.

$$\begin{aligned} &\forall j \ \textstyle\sum_{i\neq j}\sum_k \overline{w}_{ik}\overline{p}_k + \sum_k \underline{w}_{jk}\underline{p}_k \\ &\quad = \textstyle\sum_k(\sum_{i\neq j}\overline{w}_{ik} + \underline{w}_{jk})\overline{p}_k - \sum_k \underline{w}_{jk}(\overline{p}_k - \underline{p}_k) \end{aligned} \tag{7}$$

$$\begin{aligned} &\forall j \ \textstyle\sum_{i\neq j}\sum_k \underline{w}_{ik}\underline{p}_k + \sum_k \overline{w}_{jk}\overline{p}_k \\ &\quad = \textstyle\sum_k(\sum_{i\neq j}\underline{w}_{ik} + \overline{w}_{jk})\underline{p}_k + \sum_k \overline{w}_{jk}(\overline{p}_k - \underline{p}_k) \end{aligned} \tag{8}$$

In case of $(7) \geq 1$ and $(8) \leq 1$, the interval global weights can be interval probabilities. Let consider the case where the referenced priority weights are crisp. We assume the referenced priority weights are crisp, that is $p_k = \overline{p}_k = \underline{p}_k$ and $\sum_k p_k = 1$.

$$\begin{aligned} (7) &= \textstyle\sum_j(\sum_{i\neq j}\overline{w}_{ik} + \underline{w}_{jk})p_k \geq 1 \\ (8) &= \textstyle\sum_j(\sum_{i\neq j}\underline{w}_{ik} + \overline{w}_{jk})p_k \leq 1 \end{aligned} \tag{9}$$

From (9) it is noticed that the two conditions in (3) are satisfied so that the interval global weights are obtained as interval probabilities in case of crisp referenced priorities. Although, in other cases we can not tell they satisfy the conditions in (3).

Then, the problem to obtain the normalized interval global weights has been proposed by selecting crisp referenced priority weights in the obtained interval referenced priority weights [4]. It is formulated as follows.

$$\begin{aligned} \forall i \quad & \max \textstyle\sum_k(p_k^{i*}\overline{w}_{ik} - p_{k*}^{i}\underline{w}_{ik}) \\ s.t. \ & \textstyle\sum_k p_k^{i*} = 1 \\ & \textstyle\sum_k p_{k*}^{i} = 1 \\ & \underline{p}_k \leq p_k^{i*} \leq \overline{p}_k \quad \forall k \\ & \underline{p}_k \leq p_{k*}^{i} \leq \overline{p}_k \quad \forall k \end{aligned} \tag{10}$$

where $p_k^{i*}, p_{k*}^{i}(k = 1, ..., m)$ are variables and each represents the crisp referenced priority weight of criterion C_k under determining the upper and lower bounds of the interval global weight of alternative A_i.

In (10) the crisp referenced priority weights such that their sum becomes one are selected for each alternative and the interval global weight is determined as $W_i^* = \left[\sum_k p_{k*}^{i}\underline{w}_{ik}, \sum_k p_k^{i*}\overline{w}_{ik}\right] \subseteq W_i$. The interval global weights of all alternatives are obtained by solving n problems. For the upper and lower bounds of the interval global weights, they are determined by maximizing and minimizing the global weights for each alternative, respectively. The selected crisp weights depend on the bounds of interval global weights and the local weights of each alternative. Although the normalized interval global weights are included in those by interval arithmetic (6), they depart from each other.

The possibilities of the global weights denoted as the widths of intervals are lessened.

As in Fig. 1, the local weights of alternatives under C_1 can be calculated by the same way as (6) with the local weights under C_{11},...,C_{1m_1}, and they are not always obtained as interval probabilities. They also need to be modified, since the local weights obtained from the given comparison matrices by (5) are interval probabilities.

The purpose of this paper is to reduce redundancy of the obtained interval global weights by interval arithmetic. The interval global weights which are normalized and reflect the possibilities of the original intervals by (6) as precisely as possible are obtained. We propose the models to modify the interval global weights based on the concept of interval approximation. Intervals can be approximated so as to be interval probabilities by three approaches, the least upper, greatest lower, and closest approximations.

The least upper approximation model is formulated as follows.

$$\begin{aligned} & \min \textstyle\sum_i (\sum_j \overline{p}_k \underline{w}_{ik} - \overline{g}_i)^2 + (\sum_j \underline{p}_k \underline{w}_{ik} - \underline{g}_i)^2 \\ s.t.\ & \underline{g}_i \le \textstyle\sum_k \underline{p}_k \underline{w}_{ik} \quad \forall i \\ & \textstyle\sum_k \overline{p}_k \overline{w}_{ik} \le \overline{g}_i \quad \forall i \\ & \textstyle\sum_{i \ne j} \overline{g}_i + \underline{g}_j \ge 1 \quad \forall j \\ & \textstyle\sum_{i \ne j} \underline{g}_i + \overline{g}_j \le 1 \quad \forall j \end{aligned} \tag{11}$$

The greatest lower approximation model is formulated as follows.

$$\begin{aligned} & \min \textstyle\sum_i (\sum_j \overline{p}_k \underline{w}_{ik} - \overline{g}_i)^2 + (\sum_j \underline{p}_k \underline{w}_{ik} - \underline{g}_i)^2 \\ s.t.\ & \textstyle\sum_k \underline{p}_k \underline{w}_{ik} \le \underline{g}_i \quad \forall i \\ & \overline{g}_i \le \textstyle\sum_k \overline{p}_k \overline{w}_{ik} \quad \forall i \\ & \textstyle\sum_{i \ne j} \overline{g}_i + \underline{g}_j \ge 1 \quad \forall j \\ & \textstyle\sum_{i \ne j} \underline{g}_i + \overline{g}_j \le 1 \quad \forall j \end{aligned} \tag{12}$$

In ordinal interval regression analysis where linear system is assumed, the sum of widths or squared widths of approximations is maximized or minimized. Instead, in these two models where linear systems are not assumed, the sum of squared deviations between the original and modified interval global weights, W_i and G_i, is minimized. The modified interval global weights by (11) and (12) are included in and include (6), respectively. The second and third constraints are the conditions of interval probabilities. The intervals W_i are not interval probabilities because of several upper and/or lower bounds. If (7)< 1 for j', the lower bound of j' and/or the upper bounds of others should be increased to be interval probabilities. Because of inclusion relations of (11) and (12), the upper bound of j' is increased by (11) and the lower bound of others are increased by (12). By the upper or lower approximation model, the upper bound of j' and lower bounds of others are not increased at the same time, because of inclusion relation.

We propose the other approximation model without considering inclusion relation. The modified interval global weights are obtained as the following problem.

$$\begin{aligned} & \min \textstyle\sum_i \{(\sum_j \underline{p}_k \underline{w}_{ik} - \underline{g}_i)^2 + (\sum_j \overline{p}_k \overline{w}_{ik} - \overline{g}_i)^2\} \\ s.t.\ & \underline{g}_i \le \overline{g}_i \quad \forall i \\ & \textstyle\sum_{i \ne j} \overline{g}_i + \underline{g}_j \ge 1 \quad \forall j \\ & \textstyle\sum_{i \ne j} \underline{g}_i + \overline{g}_j \le 1 \quad \forall j \end{aligned} \tag{13}$$

The upper and lower bounds of the modified interval global weights can be greater or smaller than those of the original ones and G_i is not always included in and includes W_i. G_i can be considered as the closest approximations of W_i. By minimizing the deviations in (13), the modified intervals G_i are close to the original ones W_i by interval arithmetic and keep their possibilities as precisely as possible. The excess and deficiency of being interval probabilities are distributed to most of alternatives by QP problem (13). The intervals of most of alternatives are modified to some extent.

Although we need to solve n LP problems by (10) in order to determine the interval global weights of all alternatives, they are obtained by solving (11), (12), or (13). In AHP, such modified normalized interval global weights are useful information for a final decision in the sense that they reflect a decision maker's uncertain judgements as intervals without redundancy. In case of more than two layers of criteria, it is necessary to modify interval global weights so as to be interval probabilities with keeping the possibilities in a sense of reflecting the uncertainty of the given information.

3 Numerical Example

There are six alternatives (A1, A2, A3, A4, A5, A6) and six criteria (C1, C2, C3, C4, C5, C6) in the same layer. A decision maker gives pairwise comparison matrices for six criteria and six alternatives under each criterion. Then, the global weights of all alternatives are calculated and modified by the proposed models.

The decision maker compares all the pairs of criteria and gives the comparisons. The pairwise comparison matrix with six criteria followed by (1) is as follows.

$$\text{Pairwise comparison matrix with criteria} = \begin{pmatrix} 1 & 2 & 1/2 & 2 & 3 & 2 \\ 1/2 & 1 & 3 & 2 & 1 & 1 \\ 2 & 1/3 & 1 & 1 & 2 & 1/2 \\ 1/2 & 1/2 & 1 & 1 & 3 & 2 \\ 1/3 & 1 & 1/2 & 1/3 & 1 & 1/2 \\ 1/2 & 1 & 2 & 1/2 & 2 & 1 \end{pmatrix}$$

Since the given comparisons are inconsistent each other, they do not satisfy (2). The interval referenced priority weights by the approximation model (5) are obtained from the above matrix.

$$\text{Interval referenced priority weights } \boldsymbol{P} = \begin{pmatrix} [0.099, 0.222] \\ [0.111, 0.222] \\ [0.074, 0.198] \\ [0.111, 0.222] \\ [0.074, 0.111] \\ [0.111, 0.148] \end{pmatrix} \tag{14}$$

In the same way, six pairwise comparison matrices for alternatives under six criteria and the obtained interval local weights are shown in Table 2.

With the interval referenced priority weights (14) and interval local weights in Table 2, the interval global weights W_i are obtained by interval arithmetic (6) as follows.

$$\text{Interval global weight } \boldsymbol{W} \text{ by (6)} = \begin{pmatrix} [0.293, 0.575] \\ [0.052, 0.156] \\ [0.049, 0.124] \\ [0.032, 0.124] \\ [0.030, 0.120] \\ [0.030, 0.118] \end{pmatrix}$$

Assuming $j = 1$ in (7), that is for A_1, $(7) = 0.935 < 1$, therefore, they do not satisfy the 1st condition in (3). The modified interval global weights by the proposed models (11), (12), and (13) and the ones by the selected crisp referenced priority weights by (10) are shown in Table 3.

Table 2. Interval local weights of alternatives under each criterion

	Interval local weight		
Alternative	C1	C2	C3
A1	0.536	0.530	0.517
A2	[0.107,0.153]	[0.076,0.136]	[0.103,0.164]
A3	[0.051,0.107]	[0.091,0.106]	[0.082,0.086]
A4	[0.077,0.089]	[0.068,0.091]	[0.055,0.086]
A5	[0.027,0.077]	[0.045,0.091]	[0.043,0.103]
A6	[0.038,0.107]	[0.045,0.106]	[0.043,0.103]
	Interval local weight		
Alternative	C4	C5	C6
A1	0.536	0.405	0.482
A2	[0.107,0.107]	[0.068,0.162]	[0.080,0.120]
A3	[0.089,0.107]	[0.054,0.162]	[0.120,0.120]
A4	[0.036,0.134]	[0.054,0.203]	[0.040,0.096]
A5	[0.054,0.107]	[0.081,0.135]	[0.060,0.161]
A6	[0.054,0.107]	[0.081,0.081]	[0.054,0.120]

Table 3. Modified interval global weights

Alternative	W_i by (6)	G_i by (11)	G_i by (12)	G_i by (13)	W_i^* by (10)
A1	[0.293,0.575]	[0.293,0.575]	[0.357,0.575]	[0.304,0.575]	[0.508,0.516]
A2	[0.052,0.156]	[0.052,0.169]	[0.052,0.156]	[0.052,0.167]	[0.091,0.143]
A3	[0.049,0.124]	[0.049,0.137]	[0.049,0.124]	[0.043,0.135]	[0.079,0.113]
A4	[0.032,0.124]	[0.032,0.136]	[0.032,0.124]	[0.031,0.134]	[0.053,0.113]
A5	[0.030,0.120]	[0.030,0.133]	[0.030,0.120]	[0.025,0.131]	[0.047,0.111]
A6	[0.030,0.118]	[0.030,0.131]	[0.030,0.118]	[0.026,0.129]	[0.048,0.107]

The interval global weight W_i^* by (10) is included in W_i by (6). As for A_1, the width of W_1 by interval arithmetic is greatest in six alternatives, its lower parts are ignored in W_1^* by (10). This does not happen for the modified interval global weights G_1 by (11), (12), and (13). The proposed models keep the possibilities denoted as widths of W_i by (6) more precisely than (10). The interval global weights W_i and the modified intervals G_i^* by the proposed models have the following relations, as $[G_i \text{ by (12)}] \subseteq [W_i \text{ by (6)}] \subseteq [G_i \text{ by (11)}]$. By the upper approximation (11), the upper bounds of A_2,A_3,A_4,A_5, and A_6 are increased, while by the lower approximation (12) only the lower bound of A_1 is decreased. On the other hand, by (13) most of the upper and lower bounds of all alternatives are modified. The modified intervals G_i is not always included in and includes W_i. In this example, the global weights by interval arithmetic are almost interval probabilities since the amounts of modified are very small. The modified interval global weights include inconsistency in the given comparisons without redundancy.

4 Conclusion

In the decision problem in AHP, the global weights of alternatives considering criteria are obtained as the final decision. In each aspect, such as comparing criteria or alternatives under each criterion, a decision maker gives intuitive judgements as pairwise comparisons. The given comparisons in each aspect might be inconsistent each other. The interval weights, which are suitable for representing uncertain information, are obtained from the given crisp pairwise comparison matrix. The interval referenced and local weights are obtained from the respective comparison matrices even if their elements are crisp. The widths of the obtained interval weights reflect inconsistency in the given pairwise comparison matrix by a decision maker's intuitive judgements. In possibility analysis, the widths are as important as the upper and lower bounds.

Then, the global weight of the alternative is obtained by multiplying the interval referenced priority weight by the interval local weights under the criterion and adding them over all criteria. Since the global weights are calculated with intervals, they should be also denoted as intervals to maintain

the uncertainty of all the given information. By multiplying the upper bounds and lower bounds of intervals, their widths tend to be too large or small and contain redundant parts. We propose the models to modify such intervals to be normalized ones keeping their possibilities. In this paper, the interval global weights reflecting the possibilities of the referenced and local weights are obtained. The least upper, greatest lower, and closest approximations of the original intervals are obtained by QP problems. By the least upper and greatest lower approximation models, the modified ones are obtained so that they include and are included in the original intervals by interval arithmetic. In order to deal with redundant parts, their widths are widened and reduced in the upper and lower approximation models, respectively. By the closest approximation model, their widths are widened and reduced at the same time. The excess and deficiency of being interval probabilities are distributed to most of alternatives. The modified intervals by the proposed models keep the possibilities of the referenced and local weights as precisely as possible. Therefore, they reflect the uncertainty of the given information as their widths. The proposed models by which intervals are normalized can be used not only for determining the interval global weights, but also for determining interval local weights in case of several layers of criteria. The normalized interval global weights are useful information for a final decision in the sense that they reflect a decision maker's uncertain judgements without redundancy. These models modifying interval global weights so as to be interval probabilities might be extended to the model by which a set of intervals based on human judgements is normalized. It is natural that the given information is inconsistent and contains redundant parts. The proposed models help to deal with such redundant parts keeping the possibilities of the given intervals.

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Fuzzy Approaches to Trust Management

Elizabeth Chang, Ernesto Damiani, and Tharam Dillon

Summary. We start from an overview of the most significant approaches to managing and negotiating trust, introducing the basic concepts on which trust and reputation management systems are built. Then, we illustrate opportunities for fuzzy research in this area.

Key words: Trust management, Reputation-based systems, Fuzzy aggregators.

1 Introduction

Automatic trust management over the global information infrastructure has become an essential requirement in modern economy, where unknown parties can interact, at different levels of anonymity for the purpose of acquiring or offering services [29].

The e-business research community has proposed several trust and reputation models for open environments such as electronic marketplaces as a measure of the reliability of participants (for an early survey, see [21]). This notion of trust intuitively incorporates a certain degree of uncertainty. In this paper we adopt this community's notation conventions as put forward in [18] and [16], where a consistent terminology is proposed. Namely, we shall use term *trust* to denote a user's willingness to start a transaction with a given partner p, while the term *reputation* will be used to quantitatively express a user's judgment about previous transactions with p.[1] The security research community has developed a distinct, though closely related, notion of trust. Namely, it considers trust management as a development of existing *access control* (AC) models, whose policies express authorizations granted to subjects for accessing resources. In trust systems, policies specify which *credentials* signed by a trusted third party are needed to access a given resource.

[1] Note that in general trust T and reputation R do not coincide; at the very least, one should take into account *reputation aging*, e.g., by writing $T(t) = R(t_0)e^{-\alpha(t-t_0)}$, for $t > t_0$ [4].

Being closely related to AC policies relying on Boolean conditions on user attributes, these trust systems initially included few provisions for dealing with uncertainty.

However, some of them supported a notion of degree, e.g., by linking trust values to the length of *chains-of-trust* connecting third parties signing credentials. Recently, the availability of new profile credentials linked to inherently uncertain use attributes (e.g., location-based ones) has triggered a new wave of research [3]. Another distinction can be drawn from an architectural point of view: depending on the underlying network architecture, trust and reputation management systems can be classified as *centralized* or *distributed*. In both cases, alternative methods for computing reputation and trust have been proposed [18]. Centralized models include the classical eBay one [25], where the difference between the sum of positive scores and the sum of negative ones represents the final reputation value of a resource. Distributed models, suitable for peer-to-peer (P2P) environments, were initially proposed by Karl Aberer and Zoran Despotovic [1]. Recently, a considerable amount of research has been done on trust and reputations models for distributed multiagent environments, where agents act on behalf of their stakeholders [7, 27]. Also, secure trust computation has been researched for anonymous distributed environments such as P2P networks [4]. The purpose of this paper is to give an overview of existing and proposed approaches to trust management, identifying open issues and highlighting opportunities for fuzzy research. The remainder of the paper is organized as follows. Section 2 describes traditional rule-based approaches to trust management and gives a brief overview of recent reputation-based approaches. Section 3 discusses credential-based trust negotiation protocols, while Sect. 4 presents reputation-based ones, focusing on the recent P2PRep proposal. Section 5 deals with the general problem of aggregating reputations, introducing fuzzy techniques; it also contains a worked-out example of aggregation of fuzzy trust values. Section 6 discusses the use of use of *linguistic fuzzy rules* [35] to define a rule-based trust evaluation model capable of taking into account trust uncertainty and degrees of trust. Finally, in Sect. 7 we draw the conclusions and give an outlook to our future work.

2 Trust Management Systems

Security research's approach to trust management relies on the notion of *security credentials*, i.e., signed assertions made by third parties, whose signature must be verified before the credential may be used. Such credentials can be used to establish *trust relationships* based on specific policies. Early trust management systems include KeyNote [6] and rule-controlled environment for evaluation of rules, and everything else (REFEREE) [11]. In these systems, a user becomes trusted (according to a policy) by presenting a credential signed by a third party who is already trusted by the server. This notion of trust

management does provide an interesting framework for reasoning about trust between unknown parties, by posing two major problems:

(i) The *cold start* problem: how can a party become trusted in the first place?
(ii) The *trust propagation* problem: how much should I trust someone who is trusted by someone I trust?

In a credential-based environment, the former problem can be dealt with by means of *trust authorities*, while the second has been tackled by formalizing trust function behavior along *trust-chains* and, more in general, *trust-graphs*, i.e., directed graphs whose nodes are potential transaction participants, and whose arcs connect each *trustor* to the corresponding *trustee* [16]. Unfortunately, trust-graph-based solutions do not always scale well on an global open networks, where some degree of anonymity must be preserved. Intuitively, reputation can be used in lieu of trust-chains to determine the extent to which a party can be trusted (e.g., to possess an attribute). Architecturally, research approaches [17] distinguish between two main types of reputation-based trust management systems: *centralized* and *distributed reputation systems*. In centralized systems, reputation information is collected from community members in the form of *ratings*. A central authority collects all ratings and derives a trust value for each resource. In a distributed reputation system there is no central location for submitting reputation ratings and obtaining resources' trust scores; instead, ratings can be submitted at distributed points. Inn the last few years, reputation-based trust management systems have been applied to many different contexts such as P2P networks [22]. P2P reputation models allow the expression and reasoning about putting *trust* in a peer based on its past behavior [24].

3 Trust Negotiation Protocols

Trust negotiation protocols govern interactions between partners that need to establish a trust relationship. At first sight, the notion of a trust negotiation protocol may seem straightforward: when a client requests an access to a service, the server checks if the client has provided the credentials (signed by a suitable trusted third-party) required by its policy. If this is the case, access is granted; otherwise, the server specifies the missing credentials, giving the client the opportunity to gather and present them.[2] If possible, the client gathers the requested credentials and sends them to the server, gaining access to the service. This naive protocol has however several drawbacks. At the first step, the client releases all credentials it possesses, even if some of them may be invalid or redundant. At the second step, by explicitly requesting all the required credentials, the server is disclosing (part of) its policy to the client.

[2]Note that here the term "missing" may refer to credentials being physically missing or lacking the required level of trust.

Advanced trust negotiation protocols are based on the notion of *gradual trust establishment*. At the initial step, the client only discloses the (possibly empty) set of credentials that it is willing to show to the server. If the server is itself not known or not trusted by the client, this set may well be empty. In turn, the server replies asking some of the credentials needed to gain access.[3] The client, according to its local policy, decides if it is willing to disclose the requested credentials. Theoretical research has been conducted aimed at proving minimality of disclosed information for given classes of negotiation algorithms [8].

4 Reputation-Based Protocols

The basic idea behind reputation-aware trust management is avoiding chains-of-trust on credentials; rather, reputation-based systems let remote parties rate each other, for example after the completion of a transaction, and use the aggregated reputation ratings about a given party to derive a reputation score.[4] Reputation can then be used by other parties to compute trust when deciding whether or not to transact with that party in the future. This technique is more suitable to P2P environments; in credential-based settings, linking reputations to parties and/or to their attributes would heavily impact on the trust framework inasmuch it poses additional requirements on credentials production and management.

4.1 The Example of P2PRep

P2PRep is a reputation-based protocol proposed by the research group on security at the Department of Information Technology of the University of Milan, including one of the authors of the present paper [12]. P2PRep was designed for *unstructured* P2P environments like Gnutella, where data is distributed randomly over the peers and broadcasting mechanisms are used for searching. However, it is readily extendable to *structured* P2P environments like Chord [31], including data access structures to route search requests. P2PRep runs in a fully anonymous P2P environment, where peers are identified using self-assigned *opaque identifiers*. Protocol P2PRep consists of four steps. In step 1, an initiator i locates available services sending a `Query` broadcast message. Other peers answer with a `QueryHit` message notifying i that they are interested the requested service. Upon receiving a set of `QueryHit`

[3] Alternatively, a proof of possession of the credential (including the third party signature) may be accepted. This avoids having to the disclose the actual credential content.

[4] An important underlying assumption is that interactions outcomes can be evaluated on a globally agreed scale.

messages, initiator i selects a potential service provider s and polls the community for any available reputation information on s.[5] When a peer receives a `Poll` message, it checks its local *experience repository* of its previous experiences with other peers. If it has some information to offer and wants to express an opinion on service provider s, it generates a vote based on its experience, and returns a `PollReply` message to i. As a result of step 2, i receives a set V of votes. In step 3, i evaluates the votes to collapse any set of votes that may belong to a clique and explicitly selects a random set of votes for verifying their trustworthiness [12]. In step 4, an aggregated community-wide reputation value is computed from the set of reputations collected in step 3.[6]

Based on this reputation value, i can take a decision on whether using the service provided by s or not (step 5). Finally, i will update its local trust on s, based on the service quality.

5 Aggregation-Based Methods for Computing Trust

In the original presentation of P2PRep [12] reputation and trust were represented as single values in the interval $[0, 1]$. Regardless of the protocol used to collect reputation ratings, however, the problems of i) representing them and ii)) synthesizing them into a single trust value must be tackled. As far as the former problem is concerned, singleton values can be readily extended to more complex array-based representations taking into account multiple features [4]. For the latter problem, ratings may have not to be aggregated at all. *Game-theoretic reputation models* take a different approach: if the reputation system is designed properly, trustworthy behavior emerges as the most convenient one. Several game-theoretical approaches to trust management have been proposed by economic systems research [23]. Unfortunately, game theoretical approaches need a relatively high number of transactions to reach equilibrium, making them less suitable than direct aggregation for many applications. Other probabilistic approaches [33] use *Bayesian network*s, taking binary ratings as inputs and computing trust scores by statistically updating probability density functions (PDF). An evolution of these techniques is *network-based* reputation aggregation. This class of approaches normally implies the aggregation of all reputation information available on a (previously established) trust graph. This process requires checking all paths on the trust graph from the computation initiator to the candidate partner and aggregating reputation values along them; finally, path reputations are merged into a network-wide value. Network-based aggregation of reputation is at the basis of several recent proposals, including the *Eigentrust* system by

[5] Polling is performed by sending a `Poll` message, broadcasted in the same way as `Query` messages.

[6] Note that, unlike the underlying protocol, the aggregation operator depend on the reputation and vote semantics.

Hector Garcia-Molina et al. [19]. However, its complexity is high and its overhead in terms of messages is not negligible.

In [2], Karl Aberer and Zoran Despotovic show that a simple probabilistic technique, *maximum likelihood estimation*, can substantially reduce overhead when employed as the feedback aggregation strategy. As we shall see below, the same line of reasoning in favor of straightforward aggregations can be applied to "lightweight" fuzzy aggregations [5].

5.1 Fuzzy Aggregations

We now focus on fuzzy research, where several lines of research on trust have been proposed. In particular Castelfranchi et al. [9] used *Fuzzy Cognitive Maps* (FCM) [20] to model the relevance of the system inputs before their aggregation. A distinct approach was taken by the REGRET system [26], where fuzzy concepts are integrated into the analysis of social networks in electronic marketplaces.

In [4] fuzzy aggregators have been used for the synthesis of opinions expressed by peers in a P2P distributed reputation system. The behavior of fuzzy aggregation was assessed by comparison with other approaches like Eigentrust.

5.2 A Worked-Out Example

The mathematical properties of fuzzy aggregators used for computing trust as a reputation aggregation have been discussed in several papers. Papers [13] and [10] describe a method based on the WOWA aggregator [32], taking into consideration both the relevance of the sources and the one of the values. Here, for the sake of simplicity, we shall use the simpler *Ordered Weighted Averaging* operator [34], whose behavior has been analyzed in detail in the fuzzy literature [14,15]. OWA permits weighting the values in relation to their ordering, because it weights reputation values in relation to their size, without taking into account which sources have expressed them.

Let w be a weigh vector of dimension $n (w = [w_1 w_2 \dots w_n])$ such that (a) $w_i \in [0, 1]$; (b) $\sum_i w_i = 1$.

In this case a mapping $f_{OWA} : R^n \to R$ is an *Ordered Weighted Averaging* (OWA) operator of dimension n if

$$f_{OWA}(a_1, a_2, \dots, a_n) = \sum_i \omega_i a_{\sigma(i)} \tag{1}$$

where $\{\sigma(1), \sigma(2), \dots, \sigma(n)\}$ is a permutation of $\{1, 2, \dots, n\}$ such that $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i = 2, \dots, n$.

We now briefly illustrate the use of the OWA operator in the computation of several trust values on different sources $\mathbf{S}, S1, \dots, S10$ providing a service. Here, we shall focus on the aggregation step only. At an initial time $t = 0$ there are four users, $\mathbf{U_r 1}, \dots, \mathbf{U_r 4}$, voting randomly on the resources. For

Table 1. Trust values at $t = 0$

V	S	U_r1	U_r2	U_r3	U_r4	T
T	$S1$	0.2	0.5	0.5	0.8	0.59
T	$S2$	0.5	0.5	0.8	0.2	0.59
F	$S3$	0.5	0.8	0.8	0.2	0.68
T	$S4$	0.2	0.2	0.2	0.5	0.32
F	$S5$	0.8	0.5	0.5	0.5	0.62
F	$S6$	0.2	0.5	0.5	0.8	0.59
F	$S7$	0.5	0.8	0.2	0.8	0.68
T	$S8$	0.5	0.8	0.2	0.2	0.53
F	$S9$	0.8	0.2	0.2	0.8	0.62
F	$S10$	0.2	0.5	0.2	0.8	0.53

Table 2. Trust values at $t = 1$

V	S	U_r1	U_r2	U_r3	U_r4	U1	T
F	$S3$	0.8	0.5	0.5	0.2	0.2	0.54
F	$S7$	0.2	0.8	0.5	0.5	0.2	0.54
F	$S5$	0.2	0.8	0.8	0.5	0.2	0.62
F	$S9$	0.5	0.2	0.2	0.2	0.2	0.3
T	$S1$	0.5	0.5	0.5	0.8	0.8	0.68
T	$S2$	0.8	0.5	0.5	0.2	0.8	0.66
F	$S6$	0.8	0.8	0.8	0.5	0.2	0.72
T	$S8$	0.2	0.2	0.8	0.5	0.8	0.62
F	$S10$	0.8	0.2	0.2	0.8	0.2	0.56
T	$S4$	0.8	0.2	0.5	0.2	0.8	0.62

the sake of simplicity, we assume that these users use only three fuzzy values to represent their evaluation of trustworthiness: 0.8 (*very trustworthy*), 0.5 (*trustworthy*), and 0.2 (*untrustworthy*).

Also, we assume that, at different times, four new users join in to vote on all resources. In our example these users, $\mathbf{U1}, \ldots, \mathbf{U4}$, are assumed to know resources' trustworthiness, and again express it using the fuzzy values 0.8 (*very trustworthy*) and 0.2 (*untrustworthy*).

It is easy to see that after $\mathbf{U1}, \ldots, \mathbf{U4}$ joined in, resources providing a good service tend to assume high trustworthiness values even if users $\mathbf{U_r1}, \ldots, \mathbf{U_r4}$ keep on voting randomly. The following tables illustrate, at different times, the evolution of trustworthiness values related to different partners (Tables 1–4).[7]

At time $t = 1$ a single user **U1** votes correctly on services provided by partners $S1, \ldots, S10$.

A new user **U2** joins in at time $t = 2$.

At time $t = 3$ three users ($\mathbf{U1}, \mathbf{U2}, \mathbf{U3}$) vote correctly and partners providing a good service rapidly earn first positions in the table.

[7] At time $t = 0$ partners are ordered casually and voters express random votes.

Table 3. Trust values at $t = 2$

V	S	U_r1	U_r2	U_r3	U_r4	U1	U2	T
F	$S6$	0.2	0.5	0.5	0.8	0.2	0.2	0.5
T	$S1$	0.5	0.5	0.8	0.2	0.8	0.8	0.7
T	$S2$	0.5	0.8	0.8	0.2	0.8	0.8	0.74
F	$S5$	0.2	0.2	0.2	0.5	0.2	0.2	0.29
T	$S8$	0.8	0.5	0.5	0.5	0.8	0.8	0.71
T	$S4$	0.2	0.5	0.2	0.8	0.8	0.8	0.67
F	$S10$	0.5	0.8	0.8	0.8	0.2	0.2	0.67
F	$S3$	0.5	0.8	0.2	0.2	0.2	0.2	0.44
F	$S7$	0.8	0.5	0.2	0.8	0.2	0.2	0.57
F	$S9$	0.8	0.8	0.8	0.8	0.2	0.2	0.71

Table 4. Trust values at $t = 3$

V	S	U_r1	U_r2	U_r3	U_r4	...	U3	T
T	$S2$	0.8	0.2	0.5	0.2	0.8	0.8	0.7
T	$S8$	0.2	0.8	0.5	0.5	0.8	0.8	0.73
F	$S9$	0.2	0.8	0.8	0.5	0.2	0.2	0.53
T	$S1$	0.5	0.8	0.2	0.2	0.8	0.8	0.7
T	$S4$	0.5	0.5	0.5	0.8	0.8	0.8	0.74
F	$S10$	0.8	0.5	0.5	0.2	0.2	0.2	0.47
F	$S7$	0.8	0.2	0.8	0.5	0.2	0.2	0.53
F	$S6$	0.2	0.2	0.8	0.5	0.2	0.2	0.41
F	$S3$	0.8	0.2	0.5	0.8	0.2	0.2	0.53
F	$S5$	0.8	0.8	0.8	0.8	0.2	0.2	0.67

Table 5. Trust values at $t = 4$

V	S	U_r1	U_r2	U_r3	U_r4	...	U4	T
T	$S4$	0.5	0.8	0.8	0.2	0.8	0.8	0.77
T	$S8$	0.5	0.8	0.2	0.2	0.8	0.8	0.73
T	$S2$	0.8	0.8	0.2	0.2	0.2	0.2	0.75
T	$S1$	0.2	0.5	0.5	0.2	0.8	0.8	0.69
F	$S5$	0.5	0.5	0.5	0.5	0.8	0.8	0.42
F	$S9$	0.5	0.5	0.8	0.5	0.2	0.2	0.48
F	$S7$	0.8	0.5	0.8	0.8	0.2	0.2	0.59
F	$S3$	0.8	0.2	0.2	0.8	0.2	0.2	0.45
F	$S10$	0.5	0.2	0.8	0.8	0.2	0.2	0.5
F	$S6$	0.8	0.2	0.8	0.8	0.2	0.2	0.55

At time $t = 4$, with four user expressing a correct evaluation on partners, the fake ones are discarded.

We now order partners based on column **T** of Table 5:

$S4(T)(0.77)$, $S2(T)(0.75)$, $S8(T)(0.73)$,
$S1(T)(0.69)$, $S7(F)(0.59)$, $S6(F)(0.55)$,

$S10(F)(0.5)$, $S9(F)(0.48)$, $S3(F)(0.45)$,
$S5(F)(0.42)$.

As expected, "good" partners tend to bubble upwards. More importantly, should the appropriate protocol be adopted, this would happen even if $\mathbf{U_r1}, \ldots, \mathbf{U_r4}$ showed a consistently malicious behavior instead of voting randomly as in this example. For a complete experimentation, see [4].

6 Fuzzy Rule-Based Methods for Trust Management

While directly aggregating reputation values provides a highly efficient solution for trust computation, approaches based on fuzzy reasoning look very promising inasmuch they provide a high-level symbolic representation of trust computation as an inference process, potentially supporting full human understanding of trust-based decisions. Aggregated reputation values become one of the (potentially many) context representation inputs that can be used by a fuzzy rule-based system to assess the trustworthiness of a potential partner in a transaction. In [30], the authors present a P2P reputation system based on fuzzy logic inferences, aimed at handling uncertainty and incomplete information in peer trust reports. This system aggregates peers' reputations with an affordable message overhead. Research by one of the authors of the present paper [27] discusses the choice of fuzzy inference. Recent analysis [28] has shown that Mamdani-type and Sugeno-type fuzzy inference methods have their merits in different situations. For example, the more compact Sugeno-type fuzzy inference approach reduces the computational burden, resulting in faster selection of partners. However, to measure the *quality of service* (QoS) after a business interaction, a Mamdani-type fuzzy inference represents a better choice, since it allows higher flexibility and a more intuitive approach.

7 Conclusions

We described discussed recent fuzzy approaches for establishing trust between unknown parties. Aggregation and rule-based techniques for trust computation show a very good potential for integration. For instance, automatic generation and tuning of fuzzy rules based on voting behavior is a well-known task based either on the construction of a fuzzy decision tree or on the direct generation of fuzzy rules (e.g. association rules, gradual rules, or fuzzy summaries). In all these approaches, linguistic terms definitions are tuned online to improve fuzzy reasoning. In our framework, adaptation of rules could be performed based on the output of vote aggregation. Another promising direction of research is expressing reputation values as *fuzzy numbers* to be

(fuzzily) incremented whenever the client successfully interacts with the service and decremented when an interaction is a failure or, automatically, after a (crisp) time-to-live.

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Proposal of Holographic 3D-Movie Generation Using Coherent Neural-Network Interpolation

Akira Hirose and Tomoaki Higo

1 Introduction

Computer-generated hologram (CGH) widens the application fields rapidly in particular in recent three-dimensional movies. For example, many optics researchers investigate flexible optical tweezers to manipulate small particles and biological cells in water [1]. Phase CGH is most suitable for this purpose because of the extremely low optical-power loss [2].

However, the generation of even a single CGH requires a large calculation cost. To generate a movie, which is a stream of still CGH images, spends huge amount of time. Realistically meaningful movie streams, however, including the optical tweezers, present continuous deformation of object. Therefore, if we can interpolate CGH images generated sparsely in time, the calculation cost will greatly be reduced.

We propose a small-calculation-cost method to generate a CGH stream on the basis of coherent neural networks that deal with phase information with generalization ability in the carrier-frequency domain.

2 Hologram Interpolation Utilizing Generalization

Neural networks possess generalization ability. Coherent neural networks, having a carrier frequency f, can change the learning and processing behavior according to f. Then the frequency-domain generalization will bring a smoothly-changing time-sequential phase signal [3].

Figure 1 shows a schematic diagram of the proposed system. First, we calculate a certain number of CGH images sparsely in the stream, and map them at frequency points f_p in the carrier-frequency domain in the coherent neural network, whose neuron is shown in Fig. 2. Then we use them as teacher signals of learning [4]. For generating a smooth three-dimensional movie, we sweep f to change continuously the CGH images time-sequentially.

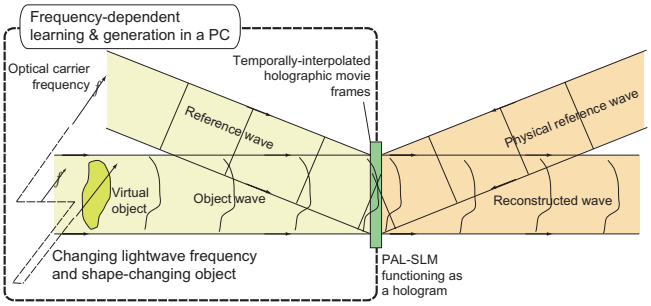

Fig. 1. Experimental setup of the neurally-interpolated phase-holographic generation of three-dimensional movie

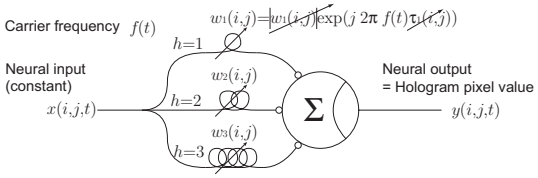

Fig. 2. Coherent neuron with multiple connections for a single pixel at position (i, j) on the Spatial light modulator (SLM) functioning as a hologram

3 Simulation Experiment

Figure 3 shows a simulation result. Teacher images exist every four steps. Calculation time becomes 1/7 of conventional one for three-dimensional images.

4 Summary

We have proposed a small-calculation-cost method to generate a three-dimensional movie based on coherent neural networks.

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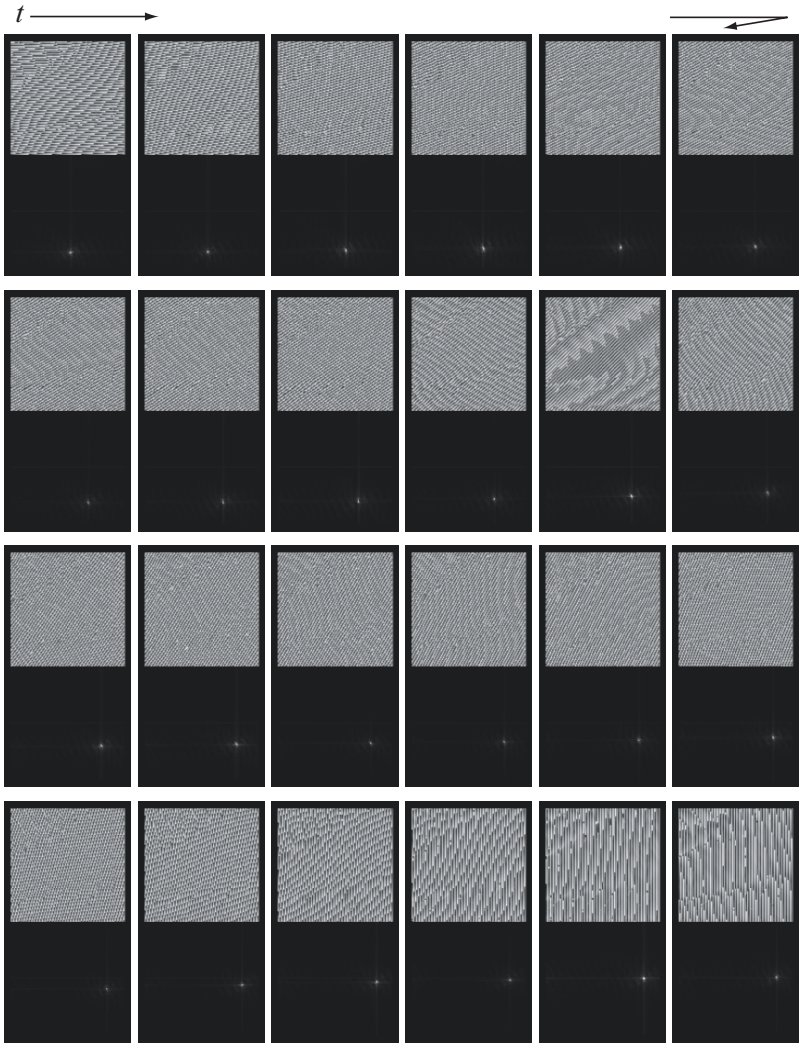

Fig. 3. Interpolated time-sequential hologram (upper, phase images in gray scale) and reconstructed images (lower, brightness on screen in gray scale) as the optical carrier frequency f is swept in the interpolation calculation: Single point moving on an arc

Blur Identification Using Neural Network for Image Restoration

Igor Aizenberg, Dmitriy Paliy, Claudio Moraga and Jaakko Astola

Summary. A prior knowledge about the distorting operator and its parameters is of crucial importance in blurred image restoration. In this paper the continuous-valued multilayer neural network based on multivalued neurons (MLMVN) is exploited for identification of a type of blur among six trained blurs and of its parameters. This network has a number of specific properties and advantages. Its backpropagation learning algorithm does not require differentiability of the activation function. The functionality of the MLMVN is higher than the ones of the traditional feedforward neural networks and a variety of kernel-based networks. Its higher flexibility and faster adaptation to the mapping implemented make possible an accomplishment of complex problems using a simpler network. Therefore, the MLMVN can be used to solve those nonstandard recognition and classification problems that cannot be solved using other techniques.

Key words: Derivative free backpropagation learning, Complex-valued neural network, Image restoration.

1 Introduction

A multilayer neural network based on multivalued neurons (MLMVN) has been introduced in [1] and then investigated and developed further in [2]. This network consists of multivalued neurons (MVN). That is a neuron with complex-valued weights and an activation function, defined as a function of the argument of a weighted sum. This activation function was proposed in 1971 in the pioneer paper of Aizenberg et al. [3].

The multivalued neuron was introduced in [4]. It is based on the principles of multiple-valued threshold logic over the field of complex numbers formulated in [5] and then developed in [6]. A comprehensive observation of the discrete-valued MVN, its properties and learning is presented in [6]. A continuous-valued MVN and its learning are considered in [1,2]. In this paper we consider the continuous-valued MVN (further simply MVN) only.

The most important properties of MVN are: the complex-valued weights, inputs and output lying on the unit circle, and the activation function, which

maps the complex plane into the unit circle. It is important that MVN learning is reduced to the movement along the unit circle. The MVN learning algorithm is based on a simple linear error correction rule and it does not require differentiability of the activation function.

Different applications of MVN have been considered during recent years, e.g., MVN as a basic neuron in the cellular neural networks [6], as the basic neuron of the neural-based associative memories [6–10], as the basic neuron in a variety of pattern recognition systems [10–12], and as a basic neuron of the MLMVN [1, 2]. The MLMVN outperforms a classical multilayer feedforward network and different kernel-based networks in the terms of learning speed, network complexity, and classification/prediction rate tested for such popular benchmarks problems as the parity n, the two spirals, the sonar, and the Mackey–Glass time series prediction [1, 2]. These properties of MLMVN show that it is more flexible and adapts faster in comparison with other solutions. In this paper we apply MLMVN to identify blur and its parameters, which is a key problem in image restoration.

Usually blur refers to the low-pass distortions introduced into an image. It can be caused, e.g., by the relative motion between the camera and the original scene, by the optical system which is out of focus, by atmospheric turbulence (optical satellite imaging), aberrations in the optical system, etc. [13]. Any type of blur, which is spatially invariant, can be expressed by the convolution kernel in the integral equation [14, 15]. Hence, deblurring (restoration) of a blurred image is an ill-posed inverse problem [16], and regularization is commonly used when solving this problem [16].

There is a variety of sophisticated and efficient deblurring techniques such as deconvolution based on the Wiener filter [13, 17], nonparametric image deblurring using local polynomial approximation with spatially-adaptive scale selection based on the intersection of confidence intervals rule [17], Fourier-wavelet regularized deconvolution [18], expectation-maximization algorithm for wavelet-based image deconvolution [19], etc. All these techniques assume a prior knowledge of the blurring kernel or its point spread function (PSF) and its parameter.

When the blurring operator is unknown, the image restoration becomes a blind deconvolution problem [20–22]. Most of the methods to solve it are iterative, and, therefore, they are computationally costly. Due to the presence of noise they suffer from the stability and convergence problems [23].

The original solutions of blur identification problem that are based on the use of MVN-based neural networks were proposed in [12, 24, 25]. Two different single-layer MVN-based networks have been used to identify blur and its parameter (e.g., variation for the Gaussian blur, extent for motion blur, etc.) in [24]. The results were good, but this approach had some disadvantages. For instance, the networks used have specific architecture with no universal learning algorithm, thus each neuron was trained separately. Another disadvantage is the use of too many spectral coefficients as features (quarter of image size). Thus the learning process was heavy.

Significant improvement was obtained in [25] comparing to [24]. A single neural network (the discrete-valued MLMVN) with the original backpropagation training scheme was used to identify both smoothing operator and its parameter on a single observed noisy image. However, the discrete-valued MLMVN in [25] had such a drawback as discrete inputs which results in quantization error of pattern vectors. This reduces its applicability.

In this paper, we propose to use the continuous-valued MLMVN (further simply MLMVN) to solve both the blur and its parameters identification problems in order to overcome the disadvantages mentioned above. The modification of the MLMVN results in significant improvement of the functionality.

2 Image Restoration Problem

Mathematically, a variety of image capturing principles can be modeled by the Fredholm integral of the first kind in $\mathbb{R}^2$ space $z(t) = \int_x v(t,l)y(l)dl$, where $t, l \in X \subset \mathbb{R}^2$, v is a point-spread function (PSF) of a system, y is an image intensity function, and $z(t)$ is an observed image [15]. A natural simplification is that the PSF v is shift-invariant which leads to a convolution operation in the observation model. We assume that the convolution is discrete and noise is present. Hence, the observed image z given in the following form:

$$z(t) = (v \otimes y)(t) + \varepsilon(t), \tag{1}$$

where "$\otimes$" denotes the convolution, t is defined on the regular $L_1 \times L_2$ lattice, $t \in X = \{(t_1, t_2);\ t_i = 0, 1, \dots L_i - 1, \quad i = 1, 2\}$, and $\varepsilon(t)$ is a noise. It is assumed that the noise is white Gaussian with zero-mean and variance σ^2, $\varepsilon(t) \sim N(0, \sigma^2)$. In the 2D frequency domain the model (1) takes the form:

$$Z(\omega) = V(\omega)Y(\omega) + \varepsilon(\omega), \tag{2}$$

where $Z(\omega) = F\{z(t)\}$ is a representation of a signal z in a Fourier domain and $F\{\cdot\}$ is a discrete Fourier transform, $V(\omega) = F\{v(t)\}$, $Y(\omega) = F\{y(t)\}$, $\varepsilon(\omega) = F\{\varepsilon(t)\}$, and $\omega \in W$, $W = \{(\omega_1, \omega_2);\ \omega_i = 2\pi k_i / L_i, k_i = 0, 1, \dots, L_i - 1, \quad i = 1, 2\}$ is the normalized 2D frequency.

The removal of the degradation caused by a PSF is an inverse problem, widely referred as a deconvolution. Usually this problem is ill-posed which results in the instability of a solution, i.e., it is highly sensitive to the noise. The stability can be provided by constraints imposed on the solution. A general approach to this kind of problems refers to the methods of Lagrange multipliers and the Tikhonov regularization [16]. The regularized inverse filter can be obtained as a solution of the least square problem with a penalty term:

$$J = \|Z - VY\|_2^2 + \alpha \|Y\|_2^2, \tag{3}$$

where $\alpha \geq 0$ is a regularization parameter and $\|\cdot\|_2$ denotes l^2-norm. Here, the first term $\|Z - VY\|_2^2$ gives the fidelity to the available data Z and the second

term bounds the power of this estimate by means of the regularization parameter α. In (3), and further, we omit the argument ω in the Fourier transform variables. We obtain the solution in the following form by minimizing (3):

$$\widehat{Y} = \frac{\overline{V}}{|V|^2 + \alpha} Z, \widehat{y}_\alpha(x) = F^{-1}\{\widehat{Y}\}. \tag{4}$$

where $\widehat{Y}$ is an estimate of Y, and $\overline{V}$ denotes complex-conjugate value of V.

In this paper we consider Gaussian, motion, rectangular (boxcar) and diagonal blurs. We aim to identify both blur, which is characterized by PSF, and its parameter using a single network.

The PSF v describes how the point source of light is spread over the image plane. It is one of the main characteristics of the optical system. For a variety of devices, like photo or video camera, microscope, telescope, etc., PSFs are often approximated by the Gaussian function:

$$v(t) = \frac{1}{2\pi\tau^2} \exp\left(-\frac{t_1^2 + t_2^2}{\tau^2}\right) \tag{5}$$

where τ^2 is a parameter of the PSF (the variance of the Gaussian function) (Fig. 1a). Its Fourier transform V is also a Gaussian function and its absolute values $|V|$ are shown in Fig. 1d.

Another example of blur is a uniform linear motion which happens while taking a picture of a moving object relatively to the camera:

$$v(t) = \begin{cases} \frac{1}{h}, \sqrt{t_1^2 + t_2^2} < h/2, t_1 \cos\phi = t_2 \sin\phi, \\ 0, \qquad\qquad \text{otherwise}, \end{cases} \tag{6}$$

where h is a parameter which depends on the velocity of the moving object and describes the lenght of motion in pixel, and ϕ is the angle between the motion orientation and the horizontal axis. Any unifrom function like (6) is characterized by the number of SLOPES in the frequency domain (Fig. 1b,e). The uniform rectangular blur is described by the following function (Fig. 1c):

$$v(t) = \begin{cases} \frac{1}{h^2} \ |t_1| < \frac{h}{2}, |t_2| < \frac{h}{2}, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{7}$$

where parameter h defines the size of smoothing area. The frequency characteristics of (7) are shown in Fig. 1f.

In order to solve (4) one should know PSF V. In this paper we propose to use a neural network to recognize type and parameter of V from the noisy observation Z.

3 Multilayer Neural Network Based On Multi-Valued Neurons

A continuous-valued MVN has been introduced in [1,2]. It performs a mapping between n inputs and a single output using n+1 complex-valued weights

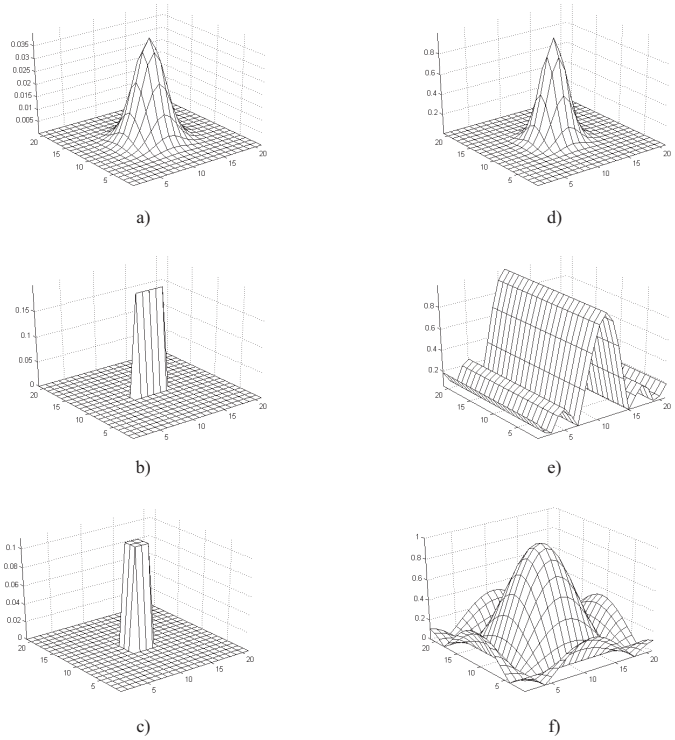

Fig. 1. Types of PSF used: (**a**) Gaussian PSF with $\tau = 2$ and size 21×21; (**b**) Linear uniform motion blur of the length 5; (**c**) Boxcar blur of the size 3×3; (**d**) frequency characteristics of (**a**); (**e**) frequency characteristics of (**b**); (**f**) frequency characteristics of (**c**)

$$f(x_1, \ldots, x_n) = P(w_0 + w_1 x_1 + \ldots + w_n x_n), \tag{8}$$

where $X = (x_1, \ldots, x_n)$ is a vector of complex-valued inputs (a pattern vector) and $W = (w_0, w_1, \ldots, w_n)$ is a weighting vector. P is the activation function of the neuron:

$$P(z) = \exp(i(\arg \mathrm{z})) = e^{iArg\, z} = \frac{z}{|z|}, \tag{9}$$

where $z = w_0 + w_1 x_1 + \ldots + w_n x_n$ is a weighted sum, arg z is an argument of the complex number z, Arg z is a main value of the argument of the complex number z and $|z|$ is its modulo. The function (9) maps the complex plane into a whole unit circle.

The MVN learning is reduced to the movement along the unit circle. This movement does not require differentiability of the activation function. Any direction along the circle always leads to the target. The shortest way of this movement is completely determined by an error that is a difference between the desired and actual outputs. The corresponding learning rule is [1, 2]:

$$W_{r+1} = W_r + \frac{C_r}{(n+1)}(\varepsilon^q - e^{iArg\, z})\bar{X} = W_r + \frac{C_r}{(n+1)}\left(\varepsilon^q - \frac{z}{|z|}\right)\bar{X}, \quad (10)$$

where $\bar{X}$ denotes vector with the complex-conjugated elements to input pattern vector X, W_r is a current weighting vector, W_{r+1} is a weighting vector after correction, C_r is a learning rate. A modified learning rule is [1, 2]:

$$W_{r+1} = W_r + \frac{C_r}{(n+1)\,|z_r|}\left(\varepsilon^q - \frac{z}{|z|}\right)\bar{X}, \quad (11)$$

where z_r is a current value of the weighted sum.

A multilayer feedforward neural network based on multivalued neurons (MLMVN) has been proposed in [1, 2]. It refers to the basic principles of the network with a feedforward dataflow through nodes proposed in [26] by Rumelhart and McClelland. The most important is that there is a full connection between the consecutive layers (the outputs of neurons from the preceding layer are connected with the corresponding inputs of neurons from the following layer). The network contains one input layer, m-1 hidden layers and one output layer. Let us use here the following notations. Let T_{km} be a desired output of the kth neuron from the mth (output) layer; Y_{km} be an actual output of the kth neuron from the mth (output) layer. Then the global error of the network taken from the kth neuron of the mth (output) layer is calculated as follows:

$$\delta^*_{km} = T_{km} - Y_{km}. \quad (12)$$

The square error functional for the s^{th} pattern $X_s = (x_1, \ldots, x_n)$ is as follows:

$$E_s = \sum_k (\delta^*_{km})^2(W), \quad (13)$$

where δ^*_{km} is a global error taken from the kth neuron of the mth (output) layer, E_s is a square error of the network for the sth pattern, and W denotes all the weighting vectors of all the neurons of the network. The mean square error functional for the network is defined as follows:

$$E = \frac{1}{N}\sum_{s=1}^{N} E_s, \quad (14)$$

where N is a total number of patterns in the training set.

Following the backpropagation learning algorithm for the MLMVN proposed in [1, 2], the errors of all the neurons from the network are determined by the global errors of the network (12). Finally, the MLMVN learning is based on the minimization of the error functional (14). It is fundamental that the global error of the network consists not only of the output neurons errors, but of the local errors of the output neurons and hidden neurons. It means that in order to obtain the local errors for all neurons, the global error must be shared among these neurons.

Let us use the following notations. Let w_i^{kj} be the weight corresponding to the ith input of the kjth neuron (kth neuron of the jth level), Y_{ij} be the actual output of the ith neuron from the jth layer ($j = 1, \ldots, m$), and N_j be the number of the neurons in the jth layer (it means that the neurons from the j+1st layer have exactly N_j inputs). Let $x_1, \ldots, x_n$ be the network inputs.

Hence, the local errors are represented in the following way. The errors of the mth (output) layer neurons are:

$$\delta_{km} = \frac{1}{s_m}\delta^*_{km}, \tag{15}$$

where km is a kth neuron of the mth layer; $s_m = N_{m-1}+1$, i.e., the number of all neurons on the previous layer (layer m-1 which the error is backpropagated to) incremented by one. The errors of the hidden layers neurons are computed as follows:

$$\delta_{kj} = \frac{1}{s_j}\sum_{i=1}^{N_{j+1}} \delta_{ij+1}(w_k^{ij+1})^{-1}, \tag{16}$$

where kj specifies the kth neuron of the jth layer ($j = 1, \ldots, m-1$); $s_j = N_{j-1}+1, j = 2, \ldots, m, s_1 = 1$ is the number of all neurons on the layer j-1 incremented by one. Thus, (15),(16) determine the error backpropagation for MLMVN. It is worth to stress on its principal distinction from the classical error backpropagation: (15),(16) do not contain a derivative of the activation function!

A factor $1/s_j$ in (15),(16) ensures sharing of the particular neuron error among all the neurons on which this error depends. It should be mentioned that for the just hidden layer the parameter $s_1 = 1$ because there is no previous hidden layer, and there are no neurons the error may be shared with.

The weights for all neurons of the network are corrected after calculation of the errors. In order to do that, we use the learning rule (10) (for the output layer) and (11) (for the hidden layers). Hence, the following correction rules are used for the weights [1, 2]:

$$\begin{aligned} \tilde{w}_i^{kj} &= w_i^{km} + \frac{C_{km}}{(N_m+1)}\delta_{km}\bar{\bar{Y}}_{im-1}, i = 1, \ldots, n, \\ \tilde{w}_0^{km} &= w_0^{km} + \frac{C_{km}}{(N_m+1)}\delta_{km}, \end{aligned} \tag{17}$$

for the neurons from the mth (output) layer (kth neuron of the mth layer),

$$\begin{aligned} \tilde{w}_i^{kj} &= w_i^{kj} + \frac{C_{kj}}{(N_j+1)|z_{kj}|}\delta_{kj}\bar{\bar{Y}}_{ij-1}, i = 1, \ldots, n, \\ \tilde{w}_0^{kj} &= w_0^{kj} + \frac{C_{kj}}{(N_j+1)|z_{kj}|}\delta_{kj}, \end{aligned} \tag{18}$$

for the neurons from the second till m-1st hidden layers (kth neuron of the jth layer ($j = 2, \ldots, m-1$), and

$$\begin{aligned}\tilde{w}_i^{k1} &= w_i^{k1} + \frac{C_{k1}}{(n+1)|z_{k1}|}\delta_{k1}\bar{x}_i, i = 1, \ldots, n,\\ \tilde{w}_0^{k1} &= w_0^{k1} + \frac{C_{k1}}{(n+1)|z_{k1}|}\delta_{k1},\end{aligned} \tag{19}$$

for the neurons of the first hidden layer.

It should be mentioned that in our simulations we used $C_{kj} = 1$ in (17)–(19).

In general, the learning process should continue until the following condition is satisfied:

$$E = \frac{1}{N}\sum_{s=1}^{N}\sum_{k}(\delta_{kms}^*)^2(W) = \frac{1}{N}\sum_{s=1}^{N}E_s \le \lambda, \tag{20}$$

where λ determines the precision of learning. In particular, in the case when $\lambda = 0$, (20) is transformed to $\forall k, \forall s\ \delta_{kms}^* = 0$.

4 Simulations

4.1 Training Set Formation

The observed image $z(t)$ is modeled as the output of a linear shift-invariant system (1) which is characterized by the PSF v. Since in the frequency domain this model is a product of the true object function Y and V we state the problem as a recognition of the shape of V and its parameters from the power-spectral density (PSD) of the observation Z, i.e., from $|Z|^2 = Z \cdot \overline{Z}$. In terms of statistical expectation we can rewrite that as follows:

$$E\left\{|Z|^2\right\} = E\left\{|YV + n|^2\right\} = |Y|^2|V|^2 + \sigma^2, \tag{21}$$

where σ^2 is the variance of noise in (2).

Examples of $\log|Z|$ values are shown in Fig. 2. The distortions of PSD for the test image Cameraman (Fig. 2a) that are typical for each type of blur (Fig. 2b,c) are clearly visible in Fig. 2e,f.

For the sake of simplicity we consider the image $z(t)$ with the equal sizes, i.e., $L = L_1 = L_2$ in (1),(2). In order to obtain the training vector $X = (x_1, \ldots, x_n)$ in (19) as an input data of the network, and taking into account that the PSF v is symmetrical, PSD of $z(t)$ (21) is used as follows:

$$x_j = \exp\left(2\pi i \cdot (K-1)\frac{\log\left(|Z\left(\omega_{k_1,k_2}\right)|\right) - \log\left(|Z_{\min}|\right)}{\log\left(|Z_{\max}|\right) - \log\left(|Z_{\min}|\right)}\right), \tag{22}$$

where

$$\begin{cases} j = 1, \ldots, L/2-1, & \text{for } k_1 = k_2, k_2 = 1, \ldots, L/2-1,\\ j = L/2, \ldots, L-2, & \text{for } k_1 = 1, k_2 = 1, \ldots, L/2-1,\\ j = L-1, \ldots, 3L/2-3, & \text{for } k_2 = 1, k_1 = 1, \ldots, L/2-1, \end{cases} \tag{23}$$

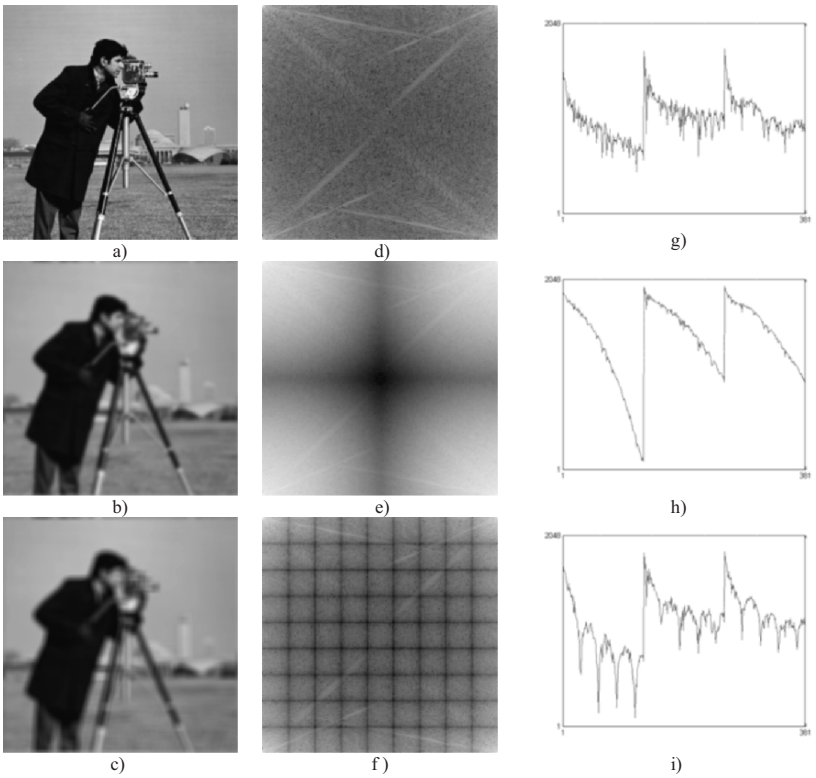

Fig. 2. True test *Cameraman* image (**a**) blurred by: (**b**) Gaussian blur with $\tau = 2$; (**c**) boxcar blur of the size 9×9. Logarithm of the PSD of the true test *Cameraman* image (**d**) blurred by: (**e**) Gaussian blur with $\tau = 2$; (**f**) rectangular blur of the size 9×9. The normalized multiplied by K-1 logarithm values of PSD of Z used as arguments to generate training vectors in (22),(23) obtained from the true test *Cameraman* image (**g**) blurred by: (**h**) Gaussian blur with $\tau = 2$; (**i**) boxcar blur of the size 9×9

and $Z_{\max} = \max_{k_1,k_2}(Z(\omega_{k_1,k_2}))$, $Z_{\min} = \min_{k_1,k_2}(Z(\omega_{k_1,k_2}))$, and K is a number of sectors in (9). Eventually, the length of the pattern vector is $n = 3L/2 - 3$.

Some examples of vectors of PSD log values multiplied by K-1 used in (22),(23) to obtain the input training vector X are shown in Fig. 2g,i.

4.2 Neural Network Structure

We provide two experiments in order to test performance of the neural network. In the first experiment (Experiment 1) we consider six types of blur with the following parameters. The Gaussian blur is considered with $\tau \in \{1, 1.33, 1.66, 2, 2.33, 2.66, 3\}$ in (5); the linear uniform horizontal $\phi = 0$

motion blur of the lengths 3, 5, 7, 9, in (6); the data corrupted by the linear uniform vertical $\phi = 90$ motion blur of the length 3, 5, 7, 9, in (6); the linear uniform diagonal motion from South-West to North-East blur ($\phi = 45$ in (6)) of the lengths 3, 5, 7, 9, in (6); the linear uniform diagonal motion from South-East to North-West blur ($\phi = 135$) of the lengths 3, 5, 7, 9, in (6); rectangular has sizes 3×3, 5×5, 7×7, 9×9, in (7). The MLMVN has two hidden layers consisting of 5 and 35 neurons, respectively, and the output layer which consists of the same number of neurons as the number of classes, i.e., types of blur. Since we consider six types of blur (Gaussian, rectangular, and the four motion ones: linear uniform horizontal, $\phi = 0$ in (6), vertical, $\phi = 90$ in (6), and two diagonal $\phi = 45$ and $\phi = 135$ in (6) the output layer contains six neurons. Therefore, the structure of network is 5→35→6. Each neural element of the output layer has to classify a parameter of the corresponding type of blur, and reject other blurs (as well, as an unblurred image). The MVN activation function (9) for the output layer neurons has a specific form (Fig. 3): the equal subdomains (nonoverlapping sectors) of the complex plane are reserved to classify a particular blur and its parameters and to reject other blurs and unblurred images. For instance, the first neuron is used to identify the Gaussian blur and to reject the non Gaussian ones. If the weighted sum for the first neuron at the output (third) layer hits jth group, $j \in \{1, \ldots, 7\}$, then the input vector $X = (x_1, \ldots, x_n)$ corresponds to the Gaussian blur and its parameter is τ_j.

In the second experiment (Experiment 2) we are targeting classification of a single Gaussian blur type, but with much higher precision. The grid of the blur's parameters is finer with significantly larger number of them on the same interval $\tau \in \{1 + 0.15\Delta : \Delta = 0, 1, \ldots, 14\}$ in (5), which makes the problem of classification more difficult. The output layer of the network contains in this case a single neuron, and the network structure is 5→ 35→1.

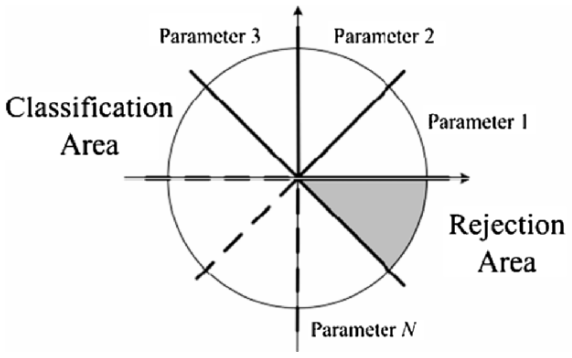

Fig. 3. Structure of the neural element on the output layer of MLMVN

4.3 Results

We have used a database which consists of 150 grayscale images with sizes 256×256 to generate the training and testing sets. Hundred images are used to generate the training set and 50 for the testing set. The images with no blur and no noise are also included in both the training and testing set. Eventually, the training set consists of 2,700 pattern vectors, and the testing set consists of 1,350 vectors for the Experiment 1, and 1,600 and 800 for the Experiment 2, correspondingly. The level of noise in (1) is selected to satisfy the blurred signal-to-noise ratio (BSNR) [17, 18] to be equal to 40 dB.

When the training set is generated the backpropagation training algorithm (15)–(19) is exploited to train MLMVN.

The trained network is used to make classification on the testing set. The classification rate is used as an objective criterion of classification. It is computed as the number of correct classifications in terms of percentage (%) for each type of blur.

The results are presented in Table 1. The first row corresponds to the recognition of the original nonblurred images. All the output layer neurons should classify them as those that are not distorted by any of the considered types of blur. Finally, the classification rate for images which are not blurred is computed as average among all rejections.

Other rows present the results for blurred images classification and identification a parameter of a blurring function. The results for six types of blur (Experiment 1) are better or comparative with those presented in [12, 24, 25]. The best ones are highlighted by the bold font. It was succeeded for the first time to classify six blurs (compare to three in [12, 24] and four in [25]).

The results of using the MLMVN for image reconstruction are shown in Fig. 4 for the test Cameraman image. The adaptive deconvolution technique proposed in [17] has been used after the blur and its parameter identified. This technique is available following the link http://www.cs.tut.fi/∼lasip/.

Table 1. Classification rate for Blur identification

Blur	[12],[24]	Discrete-valued MLMVN [25]	Continuous-valued MLMVN	
			Exp. 1	Exp. 2[1]
No blur	n/a	**100%**	96.0%	90.0%
Gaussian	93.5%	98.7%	**99.0%**	85.0%
Rectangular	95.6%	97.9%	**98.0%**	n/a
Motion Horizontal	98.1%	97.8%	**98.5%**	n/a
Motion Vertical	n/a	97.2%	**98.3%**	n/a
Motion North-East Diagonal	n/a	n/a	**97.9%**	n/a
Motion North-West Diagonal	n/a	n/a	**97.2%**	n/a

[1]Experiment 1 and Experiment 2 are not comparable to each other because they simulate the different problems (see Sect. 4).

The image was blurred by the Gaussian PSF (5) with $\tau = 2$. It is seen that if classified PSF coincides with the true PSF then the value of improved signal-to-noise ratio (ISNR) [17] criterion is 3.88 dB. If the image is blurred using $\tau = 1.835$ or $\tau = 2.165$ then the network classifies them as blurred with $\tau = 2$ and reconstruction is applied using the recognized value. Then, the error of reconstruction is approximately 0.6 dB lower, comparing to the accurate value.

In order to reduce this error we propose to consider Experiment 2. Results are given for the Gaussian blurring function with a great denser than in Experiment 1 grid consisting of 15 parameters on the same interval. It is evident that the error of classification is formally higher (see Table 1). Nevertheless, it is very important that the error of reconstruction for the similar experiment as shown in Fig. 4 does not exceed 0.1 dB, which is a minor value in practice. During the reconstruction simulation we assumed that the images are blurred

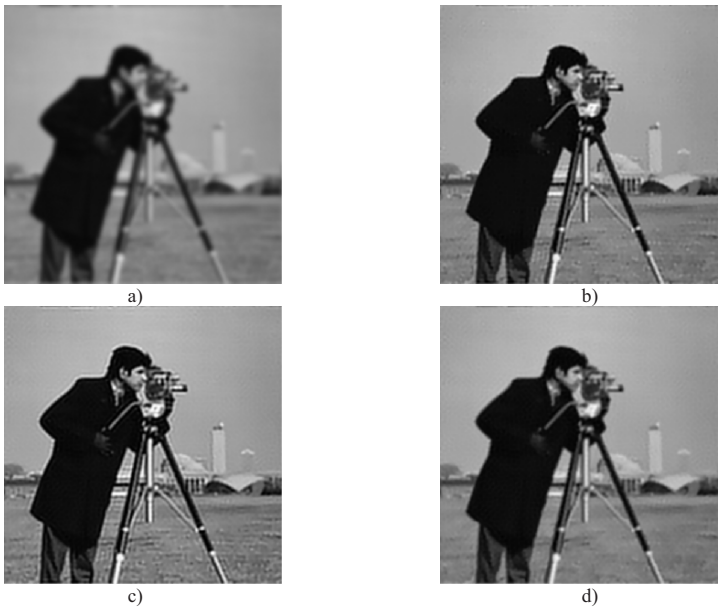

Fig. 4. Test noisy blurred *Cameraman* image with Gaussian PSF $\tau = 2$ (**a**) reconstructed using the regularization technique [17] after the blur and its parameter has been identified as Gaussian PSF with $\tau = 2$ (ISNR = 3.88 dB) (**b**);. the original Cameraman image blurred by the Gaussian PSF with $\tau = 1.835$[2] and then reconstructed using the regularization technique [17] after the blur and its parameter has been identified as Gaussian PSF with $\tau = 2$ (ISNR = 3.20 dB) (**c**); the original Cameraman image blurred by Gaussian PSF with $\tau = 2.165$[2] and then reconstructed using the regularization technique [17] after the blur and its parameter has been identified as Gaussian PSF with $\tau = 2$ (ISNR = 3.22 dB) (**d**)

[2]This blurred image does not differ visually from the one in Fig. 4a.

with $\tau = 1.925$ and $\tau = 2.075$, while the reconstruction has been done as for $\tau = 2$.

The number of features used for classification in this paper is 381 while in [12, 24] it is equal to 16384. It is worth to note that time spent on the network training for Experiment 1 was about 24 hours on a computer with Pentium 4, 3.2 GHz CPU and 45 minutes for Experiment 2 on the same computer.

5 Conclusions

In this paper we propose a novel technique for blur identification using a single observed image. The technique employs a continuous-valued feedforward MLMVN which is trained for a database of images. Then this network is used to identify both type and parameters of the blur. This identification procedure is computationally fast and cheap. The obtained results show the high efficiency of the proposed approach. It is shown by simulations that this network can be used as an efficient estimator of PSF, whose precise identification is of crucial importance for the image deblurring.

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Solving the Parity n Problem and Other Nonlinearly Separable Problems Using a Single Universal Binary Neuron

Igor Aizenberg

Summary. A universal binary neuron (UBN) operates with the complex-valued weights and the complex-valued activation function, which is the function of the argument of the weighted sum. This makes possible the implementation of the nonlinearly separable (nonthreshold) Boolean functions on the single neuron. Hence the functionality of the UBN is incompatibly higher than the functionality of the traditional perceptron, because this neuron can implement the nonthreshold Boolean functions. The UBN is closely connected with the discrete-valued multi-valued neuron (MVN). This is also a neuron with the complex-valued weights and the complex-valued activation function, which is the function of the argument of the weighted sum. A close relation of the MVN and UBN and of the multiple-valued threshold functions and P-realizable Boolean functions is considered in this paper. A modified learning algorithm for the UBN is presented. It is shown that such classical nonlinearly separable problems as the XOR and Parity n can be easily solved using a single UBN, without any network.

Key words: XOR problem, Parity problem, Complex-valued neuron.

1 Introduction

A well-known limitation of the classical Rosenblat's perceptron [1], which cannot implement the nonthreshold (nonlinearly separable) Boolean functions, formed a common view on the implementation of the nonthreshold Boolean functions. This view became a classical "axiom" that states: a nonlinearly separable Boolean function cannot be implemented using a single neuron, and to implement it, a network has to be designed [2]. It was formulated and repeated many times in many later issues (see, e.g., [3]).

The classical examples of the nonlinearly separable problem that considered unsolvable using a single neuron are the XOR problem (mod 2 addition of the two Boolean variables) and the Parity n problem, which is a generalization of the XOR problem for n variables [2,3]. These two examples one can find in any modern fundamental book on neural networks (see, e.g., [3]). The number of linearly separable Boolean functions of n variables is very small in

comparison with the number of all Boolean functions of n variables for $n > 3$. Really, if for $n = 2$ there are 14 linearly separable functions of 16 (only the XOR function and its negation are nonlinearly separable), for $n = 3$ there are 104 linearly separable functions of 256, then for $n = 4$ there are only around 2,000 linearly separable functions of 65,536 [4].

Is it possible to design such a neuron, which will make it possible to overcome a limitation of the classical perceptron?

A direction for solving this problem has been proposed in [5, 6]. The proposed idea was concentrated on the use of the complex-valued weights and an activation function, which is a function of the argument of the weighted sum. This idea was developed in [7, 8] and deeply considered in [9]. Actually, this idea can be considered as a generalization of an approach that was used for development of a theory of multiple-valued threshold logic over the field of complex numbers [10,11], where the use of the complex-valued weights and of a complex activation function has been proposed for the first time.

These ideas form a theoretical background behind a multi-valued neuron (MVN [12]) and a universal binary neuron (UBN [8]). The XOR problem has been easily solved using a single UBN in [9].

In this paper we consider how the P-realizable Boolean functions are connected with the multiple-valued threshold functions, and how the UBN is connected with the MVN, respectively. We will consider how the close relationships among the UBN and MVN are used for simplification of the UBN learning, which can be reduced to the MVN learning. The corresponding modified learning algorithm for the UBN will be considered. Then this algorithm will be used for solving the Parity n problem (for $3 \leq n \leq 14$) using a single UBN. We will also consider how the edge detecting Boolean functions that are used for edge detection by narrow direction and that are nonlinearly separable may be implemented using a single UBN and the same modified learning algorithm.

2 UBN and MVN

Let us recall the most important basic notions related to the UBN and MVN.

A key point behind a UBN is a notion of a P-realizable Boolean function.

The Boolean function $f(x_1, \ldots, x_n)$ is called a *P-realizable* function over the field of complex numbers $\mathbb{C}$, if the following correspondence holds for all the values of the variables $x_1, \ldots, x_n$ from the domain of the function f [6, 9]

$$f(x_1, \ldots, x_n) = P_B(w_0 + w_1 x_1 + \cdots + w_n x_n), \tag{1}$$

where $W = (w_0, w_1, \ldots, w_n), w_j \in \mathbb{C}, j = 0, 1, \ldots, n$ is a complex-valued weighting vector and P_B is a binary predicate, which is defined as follows:

$$P_B(z) = (-1)^j, \text{ if } 2\pi j/m \leq \arg(z) < 2\pi(j+1)/m, m = 2t, t \in N, \tag{2}$$

where m is some even positive integer ($m > n$), and j is nonnegative: $0 \leq j < m$. Definition of the predicate P_B (see 2) is illustrated in Fig. 1. The predicate

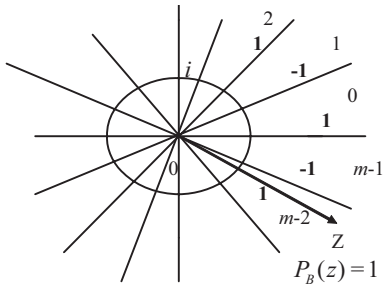

Fig. 1. Definition of the function P_B (see 2)

P_B separates the complex plane on $m = 2t$ equal sectors, and the function P_B is equal to 1 for the complex numbers from the even sectors 0, 2, 4,. . . , $m-2$, and it is equal to -1 for the numbers from the odd sectors 1, 3, 5, . . . , $m-1$.

We use here a Boolean alphabet 1, -1 instead of a classical alphabet {0, 1}. If $y \in \{0,1\}$ then $x \in \{1,-1\}$ is obtained by a linear transformation $x = 1-2y$. Thus, $0 \in \{0,1\} \longrightarrow 1 \in \{1,-1\}$ and $1 \in \{0,1\} \longrightarrow -1 \in \{1,-1\}$.

A UBN over the field of complex numbers is a neural element with the activation function (2), which performs according to (1) for a given input/output mapping described by a Boolean function of n variables [6,8,9].

For $t = 1, m = 2$ in (2) the functionality of the UBN coincides with the functionality of a traditional perceptron: it can implement only the threshold Boolean functions. However, for $t > 1$ and $m > 2$ in (2) the functionality of the UBN with the activation function (2) is always higher than the functionality of the classical perceptron. For example, for $t = 2, m = 4$ the UBN has the complete functionality for $n = 2$ [6,9] (which means that all Boolean functions of two variables are P-realizable and therefore they can be implemented using a single UBN). There are only two nonlinearly separable Boolean functions of two variables: XOR and its negation. As we mentioned above, the XOR problem was considerred as a classical problem that cannot be solved using a single neuron [1–3]. However, this is not true, because the XOR problem was solved using a single UBN in [9]. Indeed, let $t = 2, m = 4$ in (2). This means that the activation function (2) separates the complex plane onto four equal sectors: in two of them $P_B = 1$, in other two of them $P_B = -1$ (see Fig. 2). Table 1 illustrates that the UBN solves the XOR problem using the weighting vector (0, i, 1).

It is important to mention that all 256 Boolean functions of three variables are P-realizable and it is possible to implement them using a single UBN with $m = 6$ in (2); all 65,536 Boolean functions of four variables are P-realizable and it is possible to implement them using a single UBN with $m = 8$ in (2); all $2^{2^5} = 2^{32}$ Boolean functions of five variables are P-realizable and it is possible to implement them using a single UBN with $m = 10$ in (2).

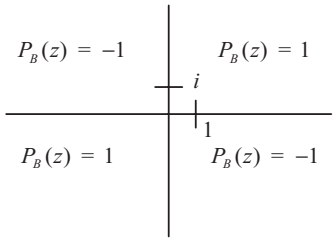

Fig. 2. Activation function (2) with m = 4

Table 1. Solving the *XOR* problem on the single UBN using the weighting vector (0, i, 1)

No.	x_1	x_2	$z = w_0 + w_1x_1 + w_2x_2$	$P_B(z)$	XOR $= x_1 \underset{\text{mod } 2}{\oplus} x_2$
(1)	**1**	**1**	$1 + i$	**1**	**1**
(2)	**1**	**−1**	$-1 + i$	**−1**	**−1**
(3)	**−1**	**1**	$1 - i$	**−1**	**−1**
(4)	**−1**	**−1**	$-1 - i$	**1**	**1**

A MVN was introduced in [12] as a neural element based on the principles of multiple-valued threshold logic over the field of complex numbers. These principles have been proposed in [11] and then deeply considered in [9]. A discrete-valued MVN performs a mapping between n inputs and a single output [9,12]. This mapping is described by a multiple-valued (k-valued) function of n variables $f(x_1, \ldots, x_n)$ using the $n+1$ complex-valued weights as follows

$$f(x_1, \ldots, x_n) = P(w_0 + w_1x_1 + \cdots + w_nx_n), \tag{3}$$

where $x_1, \ldots, x_n$ are the variables, on which the performed function depends and $w_0, w_1, \ldots, w_n$ are the weights. The values of the function and variables are complex. They are the kth roots of unity: $\varepsilon^j = \exp(i2\pi j/k)$, $j \in 0, k-1$, i is an imaginary unity. P is the activation function of the neuron

$$P(z) = \exp(i2\pi j/k), \text{ if } 2\pi j/k \leq \arg z < 2\pi(j+1)/k, \tag{4}$$

where $j = 0, 1, \ldots, k-1$ are values of the k-valued logic, $z = w_0 + w_1x_1 + \cdots + w_nx_n$ is the weighted sum, arg z is the argument of the complex number z. Equation 4 is illustrated in Fig. 3. Function (4) divides a complex plane onto k equal sectors and maps the whole complex plane into a subset of points belonging to the unit circle. This is exactly a set of the kth roots of unity.

Multi-vauled neuron learning is reduced to the movement along the unit circle. This movement does not require differentiability of the activation function. The shortest way of this movement is completely determined by the error that is a difference between the desired and actual output. Thus, the learning algorithm for the discrete-valued MVN is based on the error correction learning rule. It was proposed in [9] and recently it was modified in [14]:

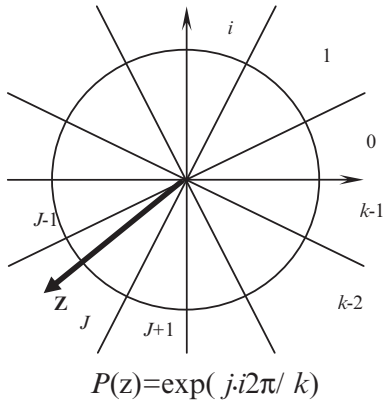

Fig. 3. Geometrical interpretation of the discrete-valued MVN activation function

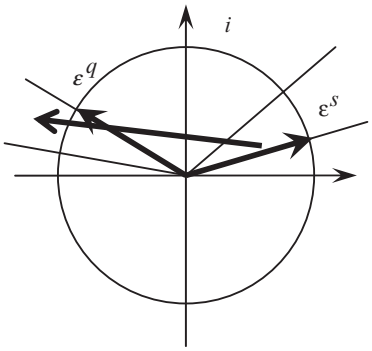

Fig. 4. Geometrical interpretation of the MVN learning rule

$$W^{r+1} = W^r + \frac{C_r}{(n+1)|z_r|}(\varepsilon^q - \varepsilon^s)\bar{X}, \tag{5}$$

where X is an input vector, n is the number of neuron inputs, $\bar{X}$ is a vector with the components complex conjugated to the components of vector X, ε^q is a desired output of the neuron, $\varepsilon^s = P(z)$ is an actual output of the neuron (see Fig. 4), r is the number of the learning iteration, W^r is a current weighting vector (to be corrected), W^{r+1} is the following weighting vector (after correction), C_r is a constant part of the learning rate, and $|z_r|$ is an absolute value of the weighted sum obtained on the rth iteration, which is a variable part of the learning rate. The use of a factor $1/|z_r|$ can be important for learning the nonlinear functions with many jumps. However, it should not be used for learning the smoothed functions without many jumps. The rule (5) ensures such a correction of the weights that a weighted sum is moving from the sector s to the sector q (see Fig. 4). The direction of this movement is completely determined by the error $\delta = \varepsilon^q - \varepsilon^s$. The convergence of the learning algorithm based on the rule (5) is proven in [9].

It is easy to see that the activation functions (2) and (4) (Figs. 2 and 3, respectively) are very similar to each other: both of them separate the complex plane on equal sectors, and both are the functions of the argument of a weighted sum. Hence if a Boolean function $f(x_1, \ldots, x_n)$ is P-realizable with the weighting vector $(w_0, w_1, \ldots, w_n)$, and the predicate P_B is defined by (2) then a partially defined only on the Boolean sets of variables m-valued function $\tilde{f}(x_1, \ldots, x_n)$ is the m-valued threshold function with the same weighting vector $(w_0, w_1, \ldots, w_n)$. It is known from [9] that any P-realizable Boolean function can be implemented on a single UBN, and any m-valued threshold function can be implemented on a single MVN. This means that the UBN learning can be reduced to the MVN learning.

In fact, the learning algorithm for the UBN can be based on the same learning rule (5) as the learning algorithm for the MVN. An incorrect output of the UBN for some input vector X from the learning set means that a weighted sum has fallen into an "incorrect" sector. Thus, the weights should be corrected to direct the weighted sum into one of the neighboring sectors (see Fig. 5). A natural choice of the "correct" sector (left or right) is based on the closeness of the current value of the weighted sum to them. Let a current weighted sum z is laying in the sth sector, and this sector is "incorrect". Thus to apply the learning rule (5) for the UBN learning, we can choose

$$\begin{aligned} q = \quad & s-1 (\mathrm{mod}\ m), \text{ if } z \text{ is closer to } (s-1)\text{st sector}, \\ q = \quad & s+1 (\mathrm{mod}\ m), \text{ if } z \text{ is closer to } (s+1)\text{st sector}. \end{aligned} \tag{6}$$

Let us illustrate how the XOR problem can be solved using learning rule (5) and (6). We will omit in this example a factor $1/|z_r|$ in (5). Let $t = 2$, $m = 4$ in (2) (see Fig. 2) and $C_r = 1$ in (5). Let us start the learning process from the weighting vector $W^0 = (1, 1, 1)$

Iteration 1.

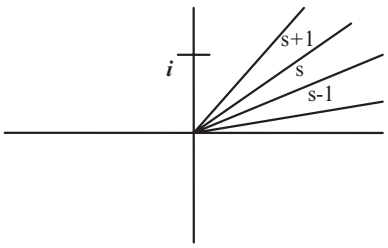

Fig. 5. Geometrical interpretation of the UBN learning rule (5)–(6)

(1) Inputs (1, 1). The weighted sum is equal to $z = 1+1\cdot 1+1\cdot 1 = 3; P_B(z) = P_B(3) = 1$. Since $f(1,1) = 1$, no further correction of the weights is needed.

(2) Inputs (1, −1). The weighted sum is equal to $z = 1 + 1 \cdot 1 + 1 \cdot (-1) = 1$; $P_B(z) = P_B(1) = 1$. Since $f(1, -1) = -1$, we have to correct the weights. According to (6) $\varepsilon^q = \varepsilon^3 = i$. $\delta = -i - 1$. Then we have to correct the weights according to (5):
$\tilde{w}_0^2 = 1 + \frac{1}{3}(-i-1) = \frac{2}{3} - \frac{1}{3}i; \tilde{w}_1^2 = 1 + \frac{1}{3}(-i-1)\cdot 1 = \frac{2}{3} - \frac{1}{3}i; \tilde{w}_2^2 = 1 + \frac{1}{3}(-i-1)\cdot(-1) = \frac{4}{3} + \frac{1}{3}i$. Thus $\tilde{W} = \left(\frac{2}{3} - \frac{1}{3}i,\ \frac{2}{3} - \frac{1}{3}i,\ \frac{4}{3} + \frac{1}{3}i\right)$.
The weighted sum after the correction is equal to $z = \frac{2}{3} - \frac{1}{3}i + \left(\frac{2}{3} - \frac{1}{3}i\right)\cdot 1 + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot(-1) = -i$; $P_B(z) = P_B(-i) = -1$.
Since $f(1, -1) = -1$, no further correction of the weights is needed.

(3) Inputs (−1, 1). The weighted sum is equal to $z = \frac{2}{3} - \frac{1}{3}i + \left(\frac{2}{3} - \frac{1}{3}i\right)\cdot(-1) + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot 1 = \frac{4}{3} + \frac{1}{3}i$; $P_B(z) = P_B\left(\frac{4}{3} + \frac{1}{3}i\right) = 1$. Since $f(-1,1) = -1$, we have to correct the weights. According to (6) $\varepsilon^q = \varepsilon^3 = i$. $\delta = -i - 1$. Then we have to correct the weights according to (5):

$$\tilde{w}_0^3 = \frac{2}{3} - \frac{1}{3}i + \frac{1}{3}(-i-1) = \frac{1}{3} - \frac{2}{3}i;$$
$$\tilde{w}_1^3 = \frac{2}{3} - \frac{1}{3}i + \frac{1}{3}(-i-1)\cdot(-1) = 1;$$
$$\tilde{w}_2^3 = \frac{4}{3} + \frac{1}{3}i + \frac{1}{3}(-i-1)\cdot(1) = 1.$$

Thus $\tilde{W} = \left(\frac{1}{3} - \frac{2}{3}i, 1, 1\right)$. The weighted sum after the correction is equal to $z = \frac{1}{3} - \frac{2}{3}i + 1\cdot(-1) + 1\cdot 1 = \frac{1}{3} - \frac{2}{3}i$; $P_B(z) = P_B\left(\frac{1}{3} - \frac{2}{3}i\right) = -1$.
Since $f(-1,1) = -1$, no further correction of the weights is needed.

(4) Inputs (−1, −1). The weighted sum is equal to

$$z = \frac{1}{3} - \frac{2}{3}i + 1\cdot(-1) + 1\cdot(-1) = -\frac{5}{3} - \frac{2}{3}i;$$
$$P_B(z) = P_B\left(-\frac{5}{3} - \frac{2}{3}i\right) = 1.$$

Since $f(-1,-1) = 1$, no correction of the weights is needed.

Iteration 2.

(1) Inputs (1, 1). The weighted sum is equal to $z = \frac{1}{3} - \frac{2}{3}i + 1\cdot 1 + 1\cdot 1 = \frac{7}{3} - \frac{2}{3}i$; $P_B(z) = P_B\left(\frac{7}{3} - \frac{2}{3}i\right) = -1$. Since $f(1, 1) = 1$, we have to correct the weights. According to (6) $\varepsilon^q = \varepsilon^0 = 1$. $\delta = 1 - (-i) = 1 + i$. Then we have to correct the weights according to (5):

$$\tilde{w}_0^1 = \frac{1}{3} - \frac{2}{3}i + \frac{1}{3}(1+i) = \frac{2}{3} - \frac{1}{3}i;$$
$$\tilde{w}_1^1 = 1 + \frac{1}{3}(1+i)\cdot 1 = \frac{4}{3} + \frac{1}{3}i;$$
$$\tilde{w}_2^1 = 1 + \frac{1}{3}(1+i)\cdot 1 = \frac{4}{3} + \frac{1}{3}i.$$

Thus $\tilde{W} = \left(\frac{2}{3} - \frac{1}{3}i, \frac{4}{3} + \frac{1}{3}i, \frac{4}{3} + \frac{1}{3}i\right)$.
The weighted sum after the correction is equal to

$$z = \frac{2}{3} - \frac{1}{3}i + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot 1 + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot 1 = \frac{10}{3} + \frac{2}{3}i;$$
$$P_B(z) = P_B\left(\frac{10}{3} + \frac{2}{3}i\right) = 1.$$

Since $f(1,1) = 1$, no correction of the weights is needed.

(2) Inputs (1, −1). The weighted sum is equal to

$$z = \frac{2}{3} - \frac{1}{3}i + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot 1 + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot(-1) = \frac{2}{3} - \frac{1}{3}i;$$
$$P_B(z) = P_B\left(\frac{2}{3} - \frac{1}{3}i\right) = -1.$$

Since $f(1,-1) = -1$, no correction of the weights is needed.

(3) Inputs (−1, 1). The weighted sum is equal to

$$z = \frac{2}{3} - \frac{1}{3}i + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot(-1) + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot 1 = \frac{2}{3} - \frac{1}{3}i;$$
$$P_B(z) = P_B\left(\frac{2}{3} - \frac{1}{3}i\right) = -1.$$

Since $f(-1,1) = -1$, no correction of the weights is needed.

(4) Inputs (−1, −1). The weighted sum is equal to

$$z = \frac{2}{3} - \frac{1}{3}i + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot(-1) + \left(\frac{4}{3} + \frac{1}{3}i\right)\cdot(-1) = -2 - i;$$
$$P_B(z) = P_B(2-i) = 1.$$

Since $f(-1,-1) = 1$, no correction of the weights is needed.

This means that the iteration process converged, and the XOR function is implemented on the *single* UBN using the weighting vector $\tilde{W} = \left(\frac{2}{3} - \frac{1}{3}i,\ \frac{4}{3} + \frac{1}{3}i,\ \frac{4}{3} + \frac{1}{3}i\right)$ obtained as the result of the learning process.

3 Solving the Parity n Problem Using a Single UBN

Let us consider an example for $n = 3$. Let $t = 3,\ m = 6$ in (2). Thus the activation function (2) separates a complex plane onto six sectors (see Fig. 6).

The Parity 3 function can be implemented on a single UBN with the activation function (2) ($m = 6$) using the weighting vector $(0, \varepsilon_6, 1, 1)$, where $\varepsilon_6 = \exp(i2\pi/6)$ is a primitive sixth root of unity (see Table 2). This example rules out a common view that the Parity problem cannot be solved on a single neuron: it can be solved on a single UBN.

Actually the Parity n problem for $3 \le n \le 12$ can be easily solved on a single UBN using the learning algorithm based on the rule (5)–(6). However, for $n > 10$ this learning algorithm requires thousands of iterations for its convergence. Although its principal convergence is based in [9], it would be useful and attractive to modify the learning algorithm in such a way that it will converge faster.

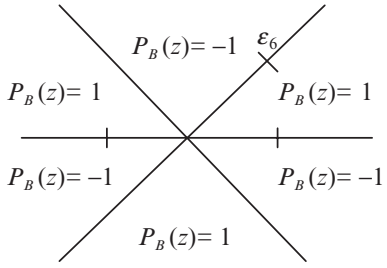

Fig. 6. Activation function (2) with $m = 6, t = 3$

Table 2. Solving the *Parity 3* problem using a single UBN with the weighting vector $(0, \varepsilon, 1, 1)$

x_1	x_2	x_3	$z = w_0 + w_1x_1 + w_2x_2 + w_3x_3$	Number of sector	$P_B(z)$	$f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$
1	1	1	$\varepsilon_6 + 2$	0	1	1
1	1	−1	ε_6	1	−1	−1
1	−1	1	ε_6	1	−1	−1
1	−1	−1	$\varepsilon_6 - 2$	2	1	1
−1	1	1	$-\varepsilon_6 + 2$	5	−1	−1
−1	1	−1	$-\varepsilon_6$	4	1	1
−1	−1	1	$-\varepsilon_6$	4	1	1
−1	−1	−1	$-\varepsilon_6 - 2$	3	−1	−1

Let us recall that a *characteristic vector* of the Boolean function $f(x_1,\ldots,x_n)$ is a vector $\mathbf{b}=(b_0,b_1,\ldots,b_n)$, which is defined as follows [4]: $b_j=(\mathbf{f},\mathbf{x}_j)$, $j=0,1,\ldots,n$, where $(,)$ is a scalar product of two vectors, $\mathbf{f}$ is a 2^n-dimensional vector of all the function values, $\mathbf{x}_j, j=1,\ldots,n$, is a 2^n-dimensional vector of all the values of the Boolean variable x_j, and $\mathbf{x}_0=(1,\ldots,1)^T$ is a 2^n-dimensional vector, whose all components are equal to 1. A *characteristic vector* of the multiple-valued function $f(x_1,\ldots,x_n)$ partially defined (or fully defined) on s sets of input variables is a vector $\mathbf{b}=(b_0,b_1,\ldots,b_n)$, whose components are equal to [9]: $b_j=(\mathbf{f},\mathbf{x}_j)$, $j=0,1,\ldots,n$, where $(,)$ is a scalar product of two vectors, $\mathbf{f}$ is an s-dimensional vector of all the function values, $\mathbf{x}_j, j=1,\ldots,n$, is an s-dimensional vector of all the values of the variable x_j, and $\mathbf{x}_0=(1,\ldots,1)^T$ is an s-dimensional vector, whose all components are equal to 1.

As it is mentioned in [4], a characteristic vector can be used as an initial approximation for the weighting vector of the threshold Boolean function. The same fact was experimentally confirmed for the multiple-valued threshold functions in [12]. Actually, a characteristic vector is nothing else than a weighting vector obtained using a Hebb rule [3,13]. It was shown experimentally that the MVN learning algorithm converges incompatibly faster starting from a normalized characteristic vector $\tilde{\mathbf{b}}$ of a function (mapping) to be implemented than starting from the random vector [12], where a normalized characteristic vector is obtained as follows:

$$\tilde{b}_j=(\mathbf{f},\mathbf{x}_j)/s=b_j/s \;\; j=0,1,\ldots,n. \tag{7}$$

Among other approaches to the UBN learning and synthesis, it was proposed in [9] to reduce the UBN learning to the MVN learning replacing a Boolean function to be implemented by a multiple-valued one not "virtually"-like in the algorithm (5)–(6), but "physically," creating a partially defined multiple-valued function, whose parity coincides with a parity of the initial Boolean function. However, this approach can be used only, if it is a prior confidence that a multiple-valued function, which is created, is a multiple-valued threshold function, so it will be possible to implement it on a single MVN.

We would like to combine here the last approach with the learning algorithm (5)–(6). This can be done in the following way. Let $X_j=\left(x_1^j,\ldots,x_n^j\right)$ is the jth set of the input variables.

Let us call a multiple-valued function $f(x_1,\ldots,x_n)$ as a *minimal-monotonic multiple-valued function*, if the following condition holds for a whole domain of the function: if $X_j \prec X_k$ (X_j precedes to X_k, i.e., $x_i^j \le x_i^k, i=1,\ldots,n$) then $\mathbf{f}_j \le \mathbf{f}_k+1$, i.e., $\mathbf{f}_j=\mathbf{f}_k$ or $\mathbf{f}_j=\mathbf{f}_k+1$.

A very interesting experimental fact is that a partially defined minimal-monotonic multiple-valued function is a multiple-valued threshold function, thus it can be implemented using a single MVN. Mathematically this is still an open problem, but experimentally no refuting example was found.

Let us consider the Parity n function: $f(y_1, \ldots, y_n) = y_1 \oplus \cdots \oplus y_n; y_i \in \{0,1\}, i = 1, \ldots, n$ ("$\oplus$" is a mod 2 addition).

Let us create a multiple-valued function from $f(y_1, \ \ldots, \ y_n)$ as follows

$$\tilde{f}(y_1, \ldots, y_n) = y_1 + \cdots + y_n; y_i \in \{0,1\}, i = 1, \ldots, n, \tag{8}$$

where "+" is a regular addition. It is easy to check that $\tilde{f}(y_1, \ldots, y_n)$ is a minimal-monotonic multiple-valued function partially defined only on the Boolean sets of variables. For example, for $n = 3$ $\tilde{\mathbf{f}} = (0,1,1,2,1,2,2,3)^T$, for $n = 4$ $\tilde{\mathbf{f}} = (0,1,1,2,1,2,2,3,1,2,2,3,2,3,3,4)^T$, etc. It is important for us to translate these representations into the language of the multiple-valued logic over the field of complex numbers. Let $\varepsilon = \exp(i2\pi/m)$, where m is the number of sectors in (2), is a primitive mth root of unity. Then for $n = 3$ $\tilde{\mathbf{f}} = (\varepsilon^0, \varepsilon^1, \varepsilon^1, \varepsilon^2, \varepsilon^1, \varepsilon^2, \varepsilon^2, \varepsilon^3)^T$, for $n = 4$ $\tilde{\mathbf{f}} = (\varepsilon^0, \varepsilon^1, \varepsilon^1, \varepsilon^2, \varepsilon^1, \varepsilon^2, \varepsilon^2, \varepsilon^3, \varepsilon^1, \varepsilon^2, \varepsilon^2, \varepsilon^3, \varepsilon^2, \varepsilon^3, \varepsilon^3, \varepsilon^4)^T$, and finally for any n

$$\begin{aligned}\tilde{\mathbf{f}} = (&\varepsilon^0, \varepsilon^1, \varepsilon^1, \varepsilon^2, \varepsilon^1, \varepsilon^2, \varepsilon^2, \varepsilon^3, \ldots, \varepsilon^{n-3}, \varepsilon^{n-2}, \varepsilon^{n-2}, \varepsilon^{n-1}, \ldots, \\ &\ldots, \varepsilon^{n-2}, \varepsilon^{n-1}, \varepsilon^{n-1}, \varepsilon^n)^T.\end{aligned} \tag{9}$$

The Parity n function is a symmetric function for any n: it is an even function for even n and a self-dual (odd function) for odd n. It would be natural to assume that the most effective implementation like (1)–(2) for a symmetric function will be such that the weighted sums corresponding to the opposite sets of inputs will get into the opposite sectors. In this way it would be reasonable to transform the complex-valued multiple-valued functions corresponding to the Parity n functions in the following way: for $n = 3$ $\tilde{\mathbf{f}} = (\varepsilon^0, \varepsilon^1, \varepsilon^1, \varepsilon^2, -\varepsilon^2, -\varepsilon^1, -\varepsilon^1, -\varepsilon^0)^T$, for n=4 $\tilde{\mathbf{f}} = (\varepsilon^0, \varepsilon^1, \varepsilon^1, \varepsilon^2, \varepsilon^1, \varepsilon^2, \varepsilon^2, \varepsilon^3, -\varepsilon^3, -\varepsilon^2, -\varepsilon^2, -\varepsilon^1, -\varepsilon^2, -\varepsilon^1, -\varepsilon^1, -\varepsilon^0)^T$, etc. This means that for even n we have to take $m = 2t$ (the number of sectors in (2)) such that t is even, while for odd n we have to take $m = 2t$ such that t is odd. For the even values of t the activation function P_B takes the same values in the opposite sectors, while for the odd values of t the activation function P_B takes the opposite values in the opposite sectors. Hence, for any n a multiple-valued function created form the Parity n function can be expressed as follows:

$$\begin{aligned}&\tilde{\mathbf{f}} = (\varepsilon^0, \varepsilon^1, \varepsilon^1, \varepsilon^2, \ldots, \varepsilon^{n-1}, -\varepsilon^{n-1}, \ldots, -\varepsilon^2, -\varepsilon^1, -\varepsilon^1, -\varepsilon^0)^T, \\ &\varepsilon = \exp(i2\pi/m).\end{aligned} \tag{10}$$

Let us now use the following modified algorithm for solving the Parity n problem on a single UBN.

(1) Create a multiple-valued function corresponding to the Parity n function according to (8).

(2) Chose even t in (2) for even n or odd t in (2) for odd n such that $2n \leq m = 2t \leq\sim 3n$ ($\sim 3n$ means the number of order $3n$ such that this number must be equal to $2t$, where either t is odd for odd n or t is even for even n) and represent the function (8) obtained on Step 1 as a multiple-valued function over the field of complex numbers like (9).
(3) Recode the function (9) in such a way that it will take the opposite values on the opposite input sets. Thus we obtain a function like (10).
(4) Obtain a normalized characteristic vector (7) for the multiple-valued function (10) obtained on Step 3.
(5) Apply either the learning algorithm (5)–(6) to the initial Parity n function or the learning algorithm (5) to the function (10) obtained on Step 3 using the normalized characteristic vector (7) obtained on the Step 4 as a starting weighting vector.

This algorithm shows incompatibly better results than either direct application of the learning algorithm (5)–(6) to the initial Parity n function or the learning algorithm (5) to the function (10) starting from the random initial weighting vector. This is illustrated in Table 3.

Table 3. Solving the Parity n problem for $3 \leq n \leq 14$ using a single UBN with the activation function (2)

No. of variables, n	3	4	5	6	7	8	9	10	11	12	13	14
No. of sectors[a] (m in (2))	6	8	10	12	14	16	22	28	30	36	38	44
Number of iterations for the modified learning algorithm	3	3	7	8	12	17,744	30	181	819	127	595	1,143
The use of factor $1/\|z_r\|$ in (5)	-	-	-	-	-	-	-	-	-	-	-	-
Number of iterations for the learning algorithm (5) and (6) (median of five runs)	8	23	37	52	55	24,312	57	428	1,383	1,525	–[b]	–[b]

[a]The smallest number of sectors (m in (2)), for which the convergence of the learning process may be gotten in the reasonable time was used
[b]–means that for $n = 13$ and $n = 14$ there was still no convergence of the learning algorithm (5)–(6) after 2,000 iterations if only one time for five independent experiments

4 Implementation of the Edge Detecting Boolean Functions

It is important to mention that the modified learning algorithm described here can be used not only for solving the Parity n problem, but also for implementation of any Boolean function on a single UBN. It is especially effective for the symmetric functions (self-dual functions of odd number of variables and even functions of even number of variables). However, this is not a restriction, because any Boolean function of n variables can be easily supplemented to the self-dual function (if n is even) or to the even function (if n is odd) of $n+1$ variables.

Let us consider how the proposed modified UBN learning algorithm can be effectively used for implementation the edge detecting Boolean functions (that are nonlinearly separable and that cannot be learned by the learning algorithm (5)–(6)) using a single UBN.

A problem of edge detection using Boolean functions has been considered in [9]. It was proposed to split a gray-scale image (or the color channels of a color image) onto the binary planes, to detect the edges on them using one of the edge detecting Boolean functions of nine variables, and then to merge the resulting binary planes into the resulting image. A Boolean function of nine variables is applied to analyze a local 3×3 image window, which contains exactly nine pixels. There are a number of different Boolean functions that detect the global edges and the edges by narrow direction corresponding to the upward and downward brightness jumps. Let us consider the following four nonlinearly separable Boolean functions that are used for the edge detection by narrow direction corresponding to the upward brightness jumps.
Direction West $\leftrightarrow$ East

$$\begin{aligned} f(x_1,\ldots,x_9) &= (x_5 \,\&\, x_4) \,\&\, ((\bar{x}_1 \vee \bar{x}_2) \vee (\bar{x}_7 \,\&\, \bar{x}_8)) \\ &\vee (x_5 \,\&\, x_6) \,\&\, ((\bar{x}_2 \vee \bar{x}_3) \vee (\bar{x}_8 \,\&\, \bar{x}_9)), \end{aligned} \tag{11}$$

where $\bar{x}$ is a negation of the Boolean variable x.
Direction North-West $\leftrightarrow$ South-East:

$$f(x_1,\ldots,x_9) = (x_5 \,\&\, x_1) \,\&(\bar{x}_2 \vee \bar{x}_4) \vee (x_5 \,\&\, x_9) \,\&(\bar{x}_6 \vee \bar{x}_8). \tag{12}$$

Direction South-West $\leftrightarrow$ North-East:

$$f(x_1,\ldots,x_9) = (x_5 \,\&\, x_7) \,\&(\bar{x}_4 \vee \bar{x}_8) \vee (x_5 \,\&\, x_3) \,\&(\bar{x}_2 \vee \bar{x}_6). \tag{13}$$

Direction North $\leftrightarrow$ South:

$$\begin{aligned} f(x_1,\ldots,x_9) &= (x_5 \,\&\, x_2) \,\&\, ((\bar{x}_1 \vee \bar{x}_4) \vee (\bar{x}_3 \,\&\, \bar{x}_6)) \\ &\vee (x_5 \,\&\, x_8) \,\&\, ((\bar{x}_4 \vee \bar{x}_7) \vee (\bar{x}_6 \,\&\, \bar{x}_9)) \end{aligned} \tag{14}$$

Table 4. Implementation of the nonlinearly separable Boolean functions (11)–(14) of nine variables using a single UBN with the activation function (2)

Function	(11)	(12)	(13)	(14)
Number of sectors (m in (2))	18	18	18	18
Number of learning iterations for the modified learning algorithm	24,824	1	54,465	17,611,996
The use of factor $1/\|z_r\|$ in (5)	+	−	+	+

The Boolean functions (11)–(14) are nonlinearly separable and moreover, they can not be implemented using a single UBN by the learning algorithm (5)–(6). There is no convergence after at least 25,000,000 iterations for all of them. However, these functions can be successfully implemented using a modified learning algorithm presented above. The results are summarized in Table 4.

It is interesting that function (12) is implemented by the normalized characteristic vector (obtained according to (7)) of the minimal monotonic Boolean function corresponding to the function (12) and obtained by (10).

4 Conclusions

We have presented in this paper, how the XOR problem and Parity n problem for $3 \leq n \leq 14$ can be solved using the single UBN. The modified learning algorithm for the UBN has been proposed. This algorithm is especially effective for the symmetric (self-dual and even) Boolean functions. The key points of this algorithm are the creation of the minimal-monotonic multiple-valued function over the field of complex numbers, its transformation to the symmetric multiple-valued function over the field of complex numbers and the use of its normalized characteristic vector as a starting one for the learning process. It was shown that this modified algorithm converges much faster than a traditional learning algorithm for the UBN. It was also shown that other important nonlinearly separable problems can be solved using a single UBN and the same modified learning algorithm.

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Some Novel Real/Complex-Valued Neural Network Models

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Summary. Traditional models of neurons are based on the assumption that a synapse is a lumped element represented by a scalar synaptic weight. But to faithfully model biological neurons, synapse is considered as a linear filter. Thus, a new model of continuous time neuron is discussed. It is described how such model leads to interesting neural networks. Also continuous time, complex-valued neuron is discussed. It is also described, how a synapse can be modeled as an FIR filter. Such a model of neuron leads to practically useful neural networks. A novel, continuous time associative memory is proposed. An approach to resolve the convergence of state of such an associative memory is discussed. Various interesting generalizations of neural networks are described.

Key words: Synapse model, Continuous time perception, Biological neural networks.

1 Introduction

Artificial neural networks are innovated to provide models of biological neural networks. The currently available models of neurons are utilized to build single layer (e.g., single layer perceptron) as well as multilayer neural networks (e.g., multilayer perceptron). These neural networks were utilized successfully in several applications. Also various paradigms of neural networks such as radial basis functions, self-organizing memory are innovated and utilized in applications.

In the case of conventional real-valued neural networks, the inputs, outputs belong to the Euclidean space. In these neural networks, a synapse is represented/modeled by a single synaptic weight which is lumped at one point. These synaptic weights are updated in the training phase using one of the learning laws (for example, Perceptron learning law, gradient rule, etc.). In the case of supervised training, these learning laws enable one to classify the input patterns into finitely many classes (based on the training samples).

1.1 Motivation for a Better Model of Neurons

- By reflecting on modeling biological neurons, we are naturally led to making the realistic assumption that synapses constitute distributed elements rather than lumped elements. Thus, a realistic model of a synapse is a linear system (characterized by impulse response) while at the same time maintaining tractability.
- In conventional neuronal models, the input at each synapse is a constant and is acted on by the scalar synaptic weight. But in biological neurons, it is most natural to consider that the input signal samples are not scalar values, but are functions defined over a finite support. The synapses (characterized by impulse response) act on these input signals which are defined on the domain (restricted to a support) $[0, T]$. Thus the class of input signals belong to a function space (defined on $[0, T]$). For the sake of notational convenience, let the synaptic weight functions be also defined on $[0, T]$.

In summary, a continuous time, real-valued neuron has input signals (which are real-valued functions of time) defined over a finite support. The input signals are fed to synapses acting as linear systems/filters and sum of responses is operated on by an activation function. Using this model of a neuron, various feed-forward/recurrent networks of neurons are designed and studied.

This research paper is organized as follows. In Sect. 2, Sect. 3, continuous time perceptron model is discussed. Also in this section, the continuous time perceptron learning law is discussed. In Sect. 4, abstract mathematical structure of neuronal models is discussed. In Sect. 5, neuronal model based on finite impulse response filter model of synapse is discussed. Also the associated neural networks are proposed. In Sect. 6, a novel continuous time associative memory is proposed and the convergence theorem is discussed. In Sect. 7, various multi-dimensional neural network generalizations are discussed. In Sect. 8, complex-valued neural networks based on the continuous time neuronal model are discussed. The research paper concludes in Sect. 9.

2 Continuous Time Perceptron and Generalizations

The area of artificial neural networks was pioneered by the efforts of McCulloch and Pitts to provide a model of neuron. Soon, it was realized by Minsky et al. that such a model of neuron has no training of the synaptic weights. Thus they proposed the model of single perceptron as well as single layer of perceptrons. Further they provided the perceptron learning law. This law was proved to converge when the input patterns are linearly separable. Later it was shown that a multilayer perceptron, a feed-forward network

can be trained (using the back-propagation algorithm) to classify nonlinearly separable patterns. In the following (as discussed in Sect. 1), we propose a more accurate (biologically) model of neuron and use it to construct various artificial neural networks.

3 A New Mathematical Model of Neuron/Single Perceptron

Consider finitely many (say M) input signals which are defined on a bounded support $[0, T]$. Let each of these signals be input to synapses which are characterized by synaptic weight functions (that are defined on support $[0, T]$). Since each of the synapses act as a linear filter, the output of each synapse is a convolution of the input function with the synaptic weight function. Mathematically, let $a_i(t), W_i(t)\ for\ 1 \leq i \leq M$ be the input functions, synaptic weight functions, respectively. Let the signum function be the activation function of the neuron. Thus the output of the neuron is given by

$$y(t) = sign\left(\sum_{i=1}^{M} a_i(t) \otimes W_i(t) - T\right), \tag{1}$$

where $\otimes$ denotes the convolution operation between two time functions (and T is the threshold at the neuron. Without loss of generality, T can be assumed to be zero). More explicitly,

$$y(t) = sign\left(\left(\sum_{i=1}^{M} \int_0^T a_i(t) W_i(t-\tau) d\tau\right) - T\right). \tag{2}$$

The successive input functions are defined over the interval [0,T]. They are fed as inputs to the continuous time neurons at successive SLOTS (Fig. 1).

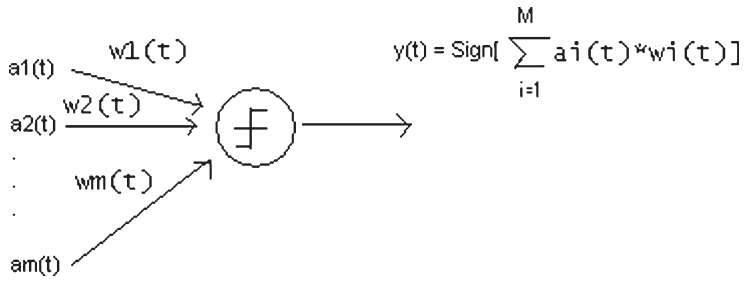

Fig. 1. A novel model of continuous time neuron (in the figure $*$ denotes the convolution operator)

3.1 Continuous Time Perceptron Learning Law: Proof

As in the case of "conventional perceptron," a continuous time perceptron learning law is given by

$$W_i^{n+1}(t) = W_i^n(t) + \eta(S(t) - g(t))a_i(t), \tag{3}$$

where $S(t)$ is the target output for the current training example, g(t) is the output generated by the perceptron, and η is a positive constant called the learning rate.

Proof. In this model of continuous time perceptron, the weights are functions of time defined on the interval [0, T]. Thus, since the synaptic weights are functions of time, we are led to investigating the type of convergence: (1) pointwise or (2) uniform.

Suppose we fix the time point, t. The convergence of synaptic weights in 3 is assured by the proof of convergence in the case of conventional perceptron. Since the choice of time point is arbitrary, we are assured of pointwise convergence of synaptic weights based on training sample input functions.

It is interesting to know under what conditions, the sequence of synaptic weight functions converge uniformly. □

Continuous Time Multilayer Perceptron

Using the above approach to model a neuron, it is straightforward to arrive at a multilayer feed forward network. In such a multilayer perceptron, the activation function at each neuron is changed from being a signum function to a sigmoid function, i.e.

$$y(t) = \frac{1}{1 + e^{-z(t)}}, \quad \text{where } y(t) \text{ is output of neuron,} \tag{4}$$

$$z(t) = \sum_{i=1}^{M} a_i(t) \otimes W_i(t). \tag{5}$$

The generalization of back-propagation algorithm (based on conventional model of neuron) follows essentially in a one-to-one manner. The details are avoided for brevity. Also various recurrent networks based on the continuous time neuron can be designed and implemented.

It is possible to consider a model of neuron in which the input functions are defined over the function space $[0, \infty]$. It is possible to consider neural networks based on such inputs. The inputs are divided into testing and training classes.

4 Abstract Mathematical Structure of Neuronal Models

Consider the inputs to a continuous time neuron which are defined on a finite support $[0, T]$. Let the impulse responses of synapses modeled as linear filters

be defined on the finite support $[0, T]$. Thus, the inputs as well as synaptic weight functions belong to the function space defined over the finite support $[0, T]$. We answer the following question.

Q: Under reasonable assumptions, what is the mathematical structure of the function space defined over $[0, T]$?

Let F be the set (function space) on which the following operations are well defined: addition, convolution (these operations are like addition, multiplication defined on the sets: real numbers, complex numbers).

Lemma *Let the identically zero function be the additive identity element and Delta function ($\delta(t) = 1$ for $t = 0$ and $\delta(t) = 0$ for $t \neq 0$) be the multiplicative identity. Then, the set F on which addition, multiplication (of functions defined on [0,T]) operations are defined constitutes a field.*

Proof. Involves routine verification of axioms of the field (closure under addition, convolution operations between the members of F, i.e., functions) and are avoided for brevity.

Now define a vector space. G over the field. The set of input functions incorporated into a vector belongs to G. The usual "multiplication" operation is replaced by "convolution".

1. *Hyperplane*: In the vector space defined above, a "hyperplane" defined by a "vector" (specified by synaptic weight functions $(W_i(t), 1 \leq i \leq M)$ is given by
$$\sum_{i=1}^{M} a_i(t) \otimes W_i(t) = L(t). \tag{6}$$
2. *Linear Separability*: Consider the "field". F of functions defined over $[0, T]$. Let G be the vector space defined over F. A class of functions is separable into two classes, if there exists a hyperplane such that the two regions are defined by
$$\sum_{i=1}^{M} a_i(t) \otimes W_i(t) \leq L(t) \quad \text{and} \quad \sum_{i=1}^{M} a_i(t) \otimes W_i(t) > L(t). \tag{7}$$

Similarly, it is straightforward to define the class of functions which are classifiable into "N" classes.

4.1 Fourier Transform: Associated Field

It is well known that the Fourier transform of the convolution of two functions is the product of Fourier transforms of the individual functions. It is found that processing the functions (by applying the activation function) has advantages in the transform domain [12].

The function space being operated on by the activation function is now the field of rational functions over $[0, T]$. Thus there is a natural mapping between the two fields (associated with continuous time neuron).

With the above discussion summarizing the abstract mathematical structure of neuronal modeling (being considered), we arrive at the following conclusions:

- Consider a single layer of continuous time perceptrons being trained by input function samples. As long as the input samples are "linearly separable," the set of synaptic weight functions converge (to an equilibrium vector).
- Consider a continuous time multilayer perceptron being trained by input function samples. The back-propagation algorithm utilized to train synaptic weight functions converges even when the input function samples are nonlinearly separable (provided there are sufficient number of continuous time neurons in the hidden layer).

5 Finite Impulse Response Model of Synapses: Neural Networks

- So far we have considered continuous time neural networks in which the synaptic weight function corresponds to an analog linear filter. A natural question arises whether it is possible to conceive a synapse whose impulse response corresponds to that of a digital filter, i.e., a finite impulse response filter (FIR). In the following, we consider neural network with such a model of synapse.
- Typically, let the discrete time input signals be considered over the finite horizon $[0, 1, 2, \ldots, S]$. For the sake of simplicity, let the length of all FIR filters modeling the synapses be the same, say T (the generalization to the case where the FIR filters have different lengths is straightforward). Thus, the impulse response sequences (associated with different synapses) extend over the duration $(0, 1, 2, \ldots, T)$.
- The output of the synapse (described by an FIR filter) depends on the input signal values over a finite horizon (depending on the length of the impulse response). Typically the length of filter is smaller than the support of a distinct input sequence, i.e., $T \ll S$. It should be noted that the successive input sequences are of same length.

$$y(n) = sign(\sum_{i=1}^{M} C^i(n) \otimes a^i(n)) \tag{8}$$

$$= sign(\sum_{i=1}^{M} \sum_{k=0}^{T} C^i(k) a^i(n-k)) \tag{9}$$

where $C^i(k)\ for\ k = 1, 2, \ldots, T$ is the impulse response sequence of ith synapse and $a^i(k)$ is the ith input sequence to the neuron

– Thus the synaptic weight sequence values (impulse response of FIR filters) can be trained according to the following perceptron learning law

$$C_i^{n+1}(k) = C_i^n(k) + \eta(S(k) - g(k)), \tag{10}$$

where S(k) is the target output for the current training example, g(k) is the output generated by the perceptron at time k and η is a positive constant called the learning rate.

This update rule converges when the input patterns are linearly separable. Using the same model of neuron, a multilayer perceptron is trained using a modified version of back-propagation algorithm.

It is possible to consider neuronal models in which the synapse acts as an infinite impulse response filter. Furthermore, based on such a model of neuron (synapse acting as an FIR filter), it is possible to discuss a novel associative memory.

Currently, the models of neurons discussed (in Sect. 2, 5) are being compared with traditional models of neurons [12].

6 Novel Continuous Time Associative Memory

In addressing, the problem of signal design for magnetic/optical recording channels, Wyner formulated an open research problem [3]. The problem statement is provided below.

Open research problem: Consider a single input, single output linear time invariant filter modeling a magnetic/optical recording channel. Let the class of inputs (to the linear filter) defined on bounded support $[0, T]$ be bounded in magnitude by unity (1). Determine the optimal signals such that the total output over finite horizon $\int_0^T y^2(t)dt$ (where y(t) is the output of linear filter) is maximized.

The author [10] as well as Honig and Stieglitz [5] independently solved the problem. The solution in [10] is more general in the sense that we considered multi-input, multi-output (MIMO) linear time varying filters and derived the optimal input vector. Let $\overline{Y}(t)$ be an optimal input vector. Then it satisfies the following signed integral equation

$$\overline{Y}(t) = sign(\int_0^T R(t,u)\overline{Y}(u)du), \tag{11}$$

where R(t,u) is the energy density matrix of the MIMO, linear time varying system. In the case of MIMO, linear time invariant system, the optimal input vector satisfies the following equation:

$$\overline{Y}(t) = sign(\int_0^T R(t-u)\overline{Y}(u)du). \tag{12}$$

In the following paragraph, we consider a successive approximation procedure for computing the optimal control vector starting with an arbitrary binary vector defined on the support $[0, T]$. Consider a vector of binary-valued functions $Y^{(n)}(t)$ ($+1$ or -1 valued) defined on the finite support $[0, T]$. Let R(t) be the energy density matrix of a MIMO linear system representing the time varying synaptic weight matrix. The following successive approximation scheme is used to compute the local optimum stable function starting with a initial binary vector $Y^{(0)}(t)$

$$Y^{(n+1)}(t) = sign(\int_0^T R(t-\tau)Y^{(n)}(\tau)d\tau). \tag{13}$$

From practical considerations, it is necessary to know whether the above successive approximation scheme converges or not. This problem is converted into an equivalent problem by discretizing the continuous time linear system into a discrete time system. Such discretization can always be done for some types of systems (satisfying some regularity conditions) without fear of approximating the system dynamics. The standard procedure of discretizing a continuous time system is summarized in many text books including Gopal's book ([4], pp 185–187), With the discrete time system equivalent to the continuous time system, the argument technique adopted for convergence is once again the energy function hill climbing in successive iterations.

Theorem 1. *Consider a MIMO, linear time-invariant system described by the dynamics*

$$\dot{X}(t) = AX(t) + CY(t) \tag{14}$$
$$Z(t) = CX(t) \tag{15}$$

The discrete time simulation (of the above continuous time system) of the following form

$$X(k+1) = FX(k) + GY(k) \tag{16}$$
$$Z(k) = HX(k) \tag{17}$$

can always be done. The discrete simulation is almost exact except for the error introduced by sampling the input and that caused by the iterative procedure for evaluating the matrices.

Proof. Follows from the procedure described in Gopal ([4], pp 185–187). □

With such a discrete time system corresponding to a continuous time system, we have the following recursion (successive approximation scheme)

$$Y^{(n+1)}(k) = sign(WY^{(n)}(k) \text{ for } n \geq 0, \tag{18}$$

where Y(k) is the optimal control vector associated with the discrete time linear system (obtained by discretizing a continuous time system) and W is the energy density tensor (associated with the discrete time system). Thus we have a Hopfield network with W as the synaptic weight matrix. Hence starting with an initial vector $Y^{(0)}(k)$, the above recursion converges to a stable state (local optimum vector) or atmost a cycle of length 2 (by invoking the convergence theorem associated with Hopfield neural network whose Connection matrix is W).

Thus, the above approach converts the problem of determining the convergence of scheme in 13, to that associated with a discrete time linear system. The iteration reminds of L^{∞} version of Neumann series. The energy function (Lyapunov function) optimized over the state trajectory of continuous time linear system is a quadratic form [8].

In [2], various possible generalized neural networks are discussed. These neural networks are associated with an energy function which is a higher order form than a quadratic form (associated with a Hopfield neural network). It is very natural to formalize associative memories which are generalizations of those discussed in this paper.

Several generalizations of the results are documented in the technical report [12]. For instance, the complex-valued, continuous time associative memory is discussed in detail in the technical report [12, 13]. For such a complex-valued associative memory, a convergence theorem is stated and proved.

7 Multi-Dimensional Generalizations

- In this research paper, so far, we have considered single/multilayer continuous time neural networks, whose input as well as output are vectors. It is straightforward to generalize the results to the case where the input/output is a three-dimensional/multi-dimensional array [8, 9]. Tensor products are utilized to determine the output of each neuron in the network. Such three/multi-dimensional neural networks arise in the biological neural network in human/animal brain.
- In the case of a human/animal brain, the associative memory operates on three-dimensional input patterns. Thus, the state of the associative memory is not a vector (one-dimensional array) but a three-dimensional array. An appropriate model of such memory is a three-dimensional, continuous time associative memory. It is easy to see that the model described in Sect. 5 can easily be generalized along the lines of the work in [9].

8 Generalization to Complex-Valued Neural Networks

Activation Functions: consider a complex-valued, continuous time neuron whose inputs as well as synaptic weight functions (defined on support $[0, T]$) and thresholds are complex-valued functions. In such a model of neuron, it is possible to utilize various activation functions.

Let $z(t) = (c(t) + jd(t))$ be the net contribution (after convolving the input functions with the synaptic weight functions) at a neuron. The following activation functions can be utilized

1. *Complex Signum Function*

$$Sign(c(t) + jd(t)) = Sign(c(t)) + j\ Sign(d(t)). \tag{19}$$

   With such an activation function, the continuous time perceptron convergence law described in 3 for real valued neurons is easily generalized to continuous time, complex-valued perceptrons.
   In the case of conventional complex-valued perceptron (with the above activation function), it is well known that the perceptron training law is easily generalized [1]. Using the similar proof technique, in the case of complex-valued, continuous time neurons, the convergence proof utilized by Aizenberg et al. is generalized.
   Also, in the case of conventional, complex-valued neuron, the above activation function is utilized in [13] for arriving at an associative memory.
2. *Complex Sigmoid Function*

$$g(z(t)) = \frac{1}{1 + e^{-z(t)}} \text{ or alternatively} \tag{20}$$

$$g(z(t)) = \tanh(z(t)) \tag{21}$$

   In the case of complex-valued, continuous time multilayer perceptron, we utilize the above complex-valued sigmoidal function as the activation function at each (complex-valued) neuron. With such a model of neuron, the back-propagation algorithm in Nitta and Furuya [6], and Nitta [7] is generalized to the case of continuous time neural networks.

Utilizing traditional model of a neuron, unified theory of control, communication, and communication is discovered and formalized [11]. This unified theory is generalized using the models of neurons discussed in this paper [8].

9 Conclusions

In this research paper, novel models of neurons are proposed. The synapses are considered as distributed elements rather than lumped elements. Thus, synapses are modeled as linear filters in continuous time as well as discrete time. Using these novel models of neurons, associated neural networks are

proposed. Also, a novel model of associative memory is proposed. Using such a model, convergence aspects of various modes of operation are discussed. Multi-dimensional generalizations of neural networks are discussed. Also associated complex-valued neural networks are discussed.

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Extending the Fuzzy Rule Interpolation "FIVE" by Fuzzy Observation

Szilveszter Kovács

Summary. The chapter introduces a way for extending the "FIVE" (Fuzzy Interpolation based on Vague Environment) fuzzy rule interpolation (FRI) method by the ability of handling fuzzy observations. The proposed extension is based on the concept of "vague environment merging", the unification of the fuzzy observation vague environments to the vague environments of the antecedent universes. The original "FIVE" FRI method together with an example is also introduced briefly in the chapter.

Key words: Fuzzy Rule Interpolation (FRI), Interpolation-based fuzzy reasoning, Vague environment.

Some difficulties emerging during the construction of fuzzy rule bases are inherited from the type of the applied fuzzy reasoning. In fuzzy systems, when classical methods (e.g. the Compositional Rule of Inference) are applied, the completeness of the fuzzy rule base is required to generate meaningful output. This means, that the fuzzy rule base has to cover all possible inputs. The way of building a complete rule base is not always straightforward. One simple solution to handle sparse fuzzy rule bases and to make infer reasonable output is the application of fuzzy rule interpolation (FRI) methods. On the other hand most of the FRI methods share the burden of high computational demand. However there is a method "FIVE" (Fuzzy Interpolation based on Vague Environment, originally introduced in [8–10]) which is simple and quick enough to fit even the requirements of direct control, where the conclusions are applied as real-time control actions, too. Beyond the simplicity and therefore the high reasoning speed, "FIVE" has two obvious drawbacks, the lack of the fuzziness on the observation and conclusion side. The main contribution of this paper is the introduction of a way for handling fuzzy observations by extending the original "FIVE" concept with the ability of merging vague environments.

1 Introduction

Since the classical fuzzy reasoning methods (e.g. compositional rule of inference) are demanding complete rule bases, the classical rule base construction claims a special care of filling all the possible rules. In case if there are some rules missing, observations may exist which hit no rule in the rule base and therefore no conclusion is obtained. Having no conclusion in a fuzzy control structure is hard to explain. E.g. one solution could be to keep the last real conclusion instead of the missing one, but applying historical data automatically to fill undeliberately missing rules could cause unpredictable side effects. Another solution for the same problem is the application of the fuzzy rule interpolation (FRI) methods, where the derivable rules are deliberately missing. Since the rule base of an FRI controller is not necessarily complete, it could contain the most significant fuzzy rules only without risking the chance of having no conclusion for some of the observations. On the other hand most of the FRI methods are sharing the burden of high computational demand, e.g. the task of searching for the two closest surrounding rules to the observation, and calculating the conclusion at least in some characteristic α-cuts. Moreover in some methods the interpretability of the fuzzy conclusion gained is also not straightforward [7]. There have been a lot of efforts to rectify the interpretability of the interpolated fuzzy conclusion [14]. In [1] Baranyi et al. give a comprehensive overview of the recent existing FRI methods (namely the α-cut, modified α-cut and generalised fuzzy interpolation methods). Beyond these problems, some of the FRI methods are originally defined for one dimensional input space, and need special extension for the multidimensional case (e.g. [3, 4]). In [17] Wong et al. gave a comparative overview of the recent multidimensional input space capable FRI methods. In [3] Jenei introduced a way for axiomatic treatment of the FRI methods. In [11] Perfilieva studies the solvability of fuzzy relation equations as the solvability of interpolating and approximating fuzzy functions with respect to a given set of fuzzy rules (e.g. fuzzy data as ordered pairs of fuzzy sets).

The high computational demand, mainly the search for the two closest surrounding rules to an arbitrary observation in the multidimensional antecedent space makes many of these methods hardly suitable for real-time applications. Some FRI, e.g. the method introduced by Jenei et al. in [4], eliminate the search for the two closest surrounding rules by taking all the rules into consideration, and therefore speed up the reasoning process. On the other hand, keeping the goal of constructing fuzzy conclusion, and not simply speeding up the reasoning process, they still require some additional (or repeated) computational steps for the elements of the level set (or at least some relevant α levels) to get the fuzzy conclusion.

A rather different application oriented aspect of the FRI emerges in the concept of "FIVE". The fuzzy reasoning method "FIVE" (*Fuzzy Interpolation based on Vague Environment*, originally introduced in [8–10]) was developed to fit the speed requirements of direct fuzzy control, where the conclusions

of the fuzzy controller are applied directly as control actions in a real-time system (see e.g. a downloadable and runable code of a real-time vehicle path tracking and collision avoidance control at [18]).

Beyond the simplicity and therefore the high reasoning speed, the FIVE has two obvious drawbacks, the lack of the fuzziness on the observation and conclusion side. The reason is this deficiency is inherited from the nature of the applied vague environment, which describes the indistinguishability of two points and therefore the similarity of a fuzzy set and a singleton only. The lack of the fuzziness on the conclusion side has a small influence on common applications where the next step after the fuzzy reasoning is the defuzzification. On the other hand, the lack of the fuzziness on the observation side can restrict applicability of the method.

In the followings, a way of merging vague environments and therefore the extension of the original FIVE concept with the ability of handling fuzzy observations will be introduced.

2 The concept of Vague Environment

The FIVE FRI method is based on the concept of the vague environment [5]. Applying the idea of the vague environment the linguistic terms of the fuzzy partitions can be described by scaling functions [5] and the fuzzy reasoning itself can be replaced by classical interpolation. The concept of a vague environment is based on the similarity or indistinguishability of the considered elements. Two values in a vague environment are ϵ-distinguishable if their distance is greater than ϵ. The distances in a vague environment are weighted distances. The weighting factor or function is called *scaling function (factor)* [5].

Two values in the vague environment X are *ϵ-indistinguishable* if

$$\varepsilon \geq \delta_s(x_1, x_2) = \left| \int_{x_2}^{x_1} s(x)\, dx \right|, \tag{1}$$

where $\delta_s(x_1, x_2)$ is the scaled distance of the values x_1, x_2 and $s(x)$ is the scaling function on X.

For finding connections between fuzzy sets and a vague environment the membership function $\mu_A(x)$ can be introduced as indicating level of similarity of x to a specific element a that is a representative or prototypical element of the fuzzy set $\mu_A(x)$, or, equivalently, as the degree to which x is indistinguishable from a (2) [5]. The α-cuts of the fuzzy set $\mu_A(x)$ are the sets which contain the elements that are $(1-\alpha)$-indistinguishable from a (see Fig. 1 also):

$$\begin{aligned} &1-\alpha \geq \delta_s(a,b), \\ &\mu_A(x) = 1 - \min\{\delta_s(a,b), 1\} = 1 - \min\left\{ \left| \int_a^b s(x)\, dx \right|, 1 \right\}. \end{aligned} \tag{2}$$

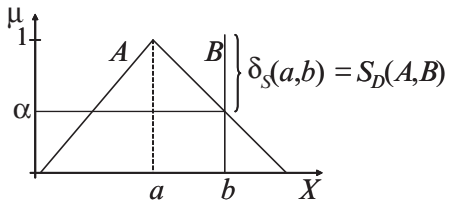

Fig. 1. The α-cuts of $\mu_A(x)$ contain the elements that are $(1-\alpha)$-indistinguishable from a

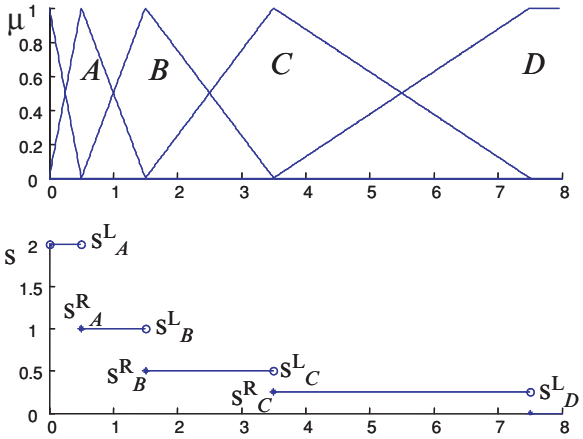

Fig. 2. A Ruspini fuzzy partition and its scaling function

In this case (see Fig. 1), the scaled distance of points a and b ($\delta_s(a,b)$) is the *Disconsistency Measure* (S_D) (mentioned and studied among other distance measures in [16] by Turksen et al.) of the fuzzy sets A and B (where B is a singleton):

$$S_D(a,b) = 1 - \sup_{x \in X} \mu_{A \cap B}(x) = \delta_s(a,b) \quad if \quad \delta_s(a,b) \in [0,1], \tag{3}$$

where $A \cap B$ notes the min t-norm, $\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)]$, $\forall x \in X$.

Taking into account the most common way of building a traditional fuzzy logic controller, where the first step is defining the fuzzy partitions on the antecedent and consequent universes by setting up the linguistic terms and then based on these terms building up the fuzzy rule base, the concept of vague environment [5] is straightforward. The goal of the fuzzy partitions is to define indistinguishability, or vagueness on the different regions of the input, output universes. This situation is clearer, if intentionally Ruspini partitions are chosen and the cores of the linguistic terms are set only (see e.g. Fig. 2). The designer has no intention to specify particular fuzzy sets, but the vagueness of

the terms and therefore the vagueness of the rules build from them. The vague environment is characterised by its scaling function. For generating a vague environment of a fuzzy partition an appropriate scaling function is needed, which describes the shapes of all the terms in the fuzzy partition. A fuzzy partition can be characterised by a single vague environment if and only if the membership functions of the terms fulfil the following requirement [5]:

$$\begin{aligned} &s(x) = |\mu'(x)| = \left|\tfrac{d\mu}{dx}\right| \ \ \textit{exists iff} \\ &\min\{\mu_i(x), \mu_j(x)\} > 0 \ \Rightarrow \ |\mu_i'(x)| = \left|\mu_j'(x)\right|, \end{aligned} \tag{4}$$

$\forall i, j \in I$, where $s(x)$ is the scaling function of the vague environment (see e.g. on Fig. 2).

3 Approximate Scaling Function

Generally condition (4) is not fulfilled, so the question is how to describe all fuzzy sets of the fuzzy partition with one "universal" scaling function. For this task the concept of an *approximate scaling function*, as an approximation of the scaling functions describing the terms of the fuzzy partition separately is proposed in [8–10].

The concept of an approximate scaling function is based on the assumption that the original goal of setting up a fuzzy partition was to characterise a scaling on a universe by some given points (member sets of the fuzzy partition), where the scaling factor of the universe is known. This case, as a general way of describing scaling on a universe, the member sets of the fuzzy partition can be restricted to triangular (trapezoidal) shaped terms. Supposing that the fuzzy terms are triangles, each fuzzy term can be characterised by three values (by a triple), by the values of the left and the right scaling factors and the value of its core point (see e.g. on Fig. 3). Having these cardinal points, as an approximate scaling function, the scaling function can be simply interpolated. In [8–10] the

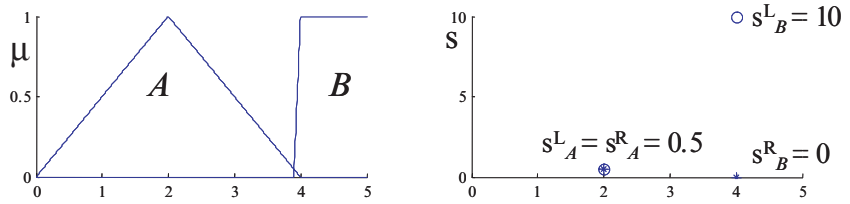

Fig. 3. Fuzzy partitions consisting of triangular fuzzy sets can be characterised by triples, by the values of the left s^L and the right s^R scaling factors and the cores

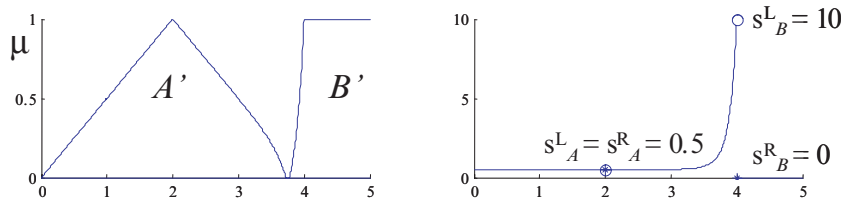

Fig. 4. Approximate scaling function generated by nonlinear interpolation (5) ($k = 1$) of the fuzzy partition shown on Fig. 3, and the partition as the approximate scaling function describes it (A', B')

following nonlinear formula was suggested for interpolation of the corresponding scaling factors between the neighbouring terms (see e.g. on Fig. 4):

$$s(x) = \begin{cases} \frac{w_i}{(d_i+1)^{k \cdot w_i}} \cdot \left(\frac{(d_i+1)^{k \cdot w_i}}{(x-x_i+1)^{k \cdot w_i}} - 1 \right) + s_{i+1}^L \,|\, s_i^R \geq s_{i+1}^L, \\ \frac{w_i}{(d_i+1)^{k \cdot w_i}} \cdot \left(\frac{(d_i+1)^{k \cdot w_i}}{(x_{i+1}-x+1)^{k \cdot w_i}} - 1 \right) + s_i^R \,|\, s_i^R < s_{i+1}^L, \end{cases} \tag{5}$$

$$x \in [x_i, x_{i+1}), w_i = \left| s_{i+1}^L - s_i^R \right|, d_i = x_{i+1} - x_i, \forall i \in [1, n-1],$$

where $s(x)$ is the approximate scaling function; x_i is the core of the ith term of the approximated fuzzy partition; s_i^L, s_i^R are the left and right side scaling factors of the ith triangle shaped term, n is the number of the terms in the approximated fuzzy partition; and k, $k > 0$ is the sensitivity factor for neighbouring scaling factor differences.

For a detailed discussion of questions related to approximate scaling functions see [8–10].

4 Shepard Interpolation for Fuzzy Reasoning: "FIVE"

The main idea of the FRI method "FIVE" (Fuzzy Interpolation based on Vague Environment) can be summarised in the followings:

1. If the vague environment of a fuzzy partition (the scaling function or at least the approximate scaling function) exists, the member sets of the fuzzy partition can be characterised by points in that vague environment. (These points are indicating the cores of the fuzzy terms, while the membership functions are described by the scaling function itself.)
2. If all the vague environments of the antecedent and consequent universes of the fuzzy rule base exist, all the primary fuzzy sets (linguistic terms) compounding the fuzzy rule base can be characterised by points in their vague environment. Therefore the fuzzy rules (built-up from the primary

fuzzy sets) can be characterised by points in the vague environment of the fuzzy rule base too. In this case the approximate fuzzy reasoning can be handled as a classical interpolation task.

3. Applying the concept of vague environments (the distances of points are weighted distances), any crisp interpolation, extrapolation, or regression method can be adapted very simply for approximate fuzzy reasoning [8–10].

Because of its simple multidimensional applicability, for interpolation-based fuzzy reasoning in this paper the adaptation of the *Shepard operator* based interpolation (first introduced in [12]) is suggested. Beside the existing deep application oriented investigation of the Shepard operator e.g. [2], it is also successfully applied in the *Kóczy-Hirota fuzzy interpolation* [6]. (The stability and the approximation rate of the Shepard operator based Kóczy-Hirota fuzzy interpolation is thoroughly studied in [13, 15].) The Shepard interpolation method for arbitrarily placed bivariate data was introduced as follows [12]:

$$S_0(f,x,y) = \begin{cases} f_k & \text{if } (x,y) = (x_k,y_k) \text{ for some } k, \\ \dfrac{\sum\limits_{k=0}^{n} f(x_k,y_k)/d_k^{\lambda}}{\sum\limits_{k=0}^{n} 1/d_k^{\lambda}} & \text{otherwise}, \end{cases} \tag{6}$$

where measurement points (x_k, y_k) $(k \in [0,n])$ are irregularly spaced on the domain of $f \in \Re^2 \to \Re$, $\lambda > 0$, and $d_k = \left[(x-x_k)^2 + (y-y_k)^2\right]^{1/2}$. This function can be typically used when a surface model is required to interpolate scattered spatial measurements.

The adaptation of the Shepard interpolation method for interpolation-based fuzzy reasoning in the vague environment of the fuzzy rule base is straightforward by substituting the Euclidian distances d_k by the scaled distances $\delta_{s,k}$:

$$\delta_{s,k} = \delta_s(\mathbf{a}_k, \mathbf{x}) = \left[\sum_{i=1}^{m} \left(\int_{a_{k,i}}^{x_i} s_{x_i}(x_i)\, dx_i \right)^2 \right]^{1/2}, \tag{7}$$

where s_{x_i} is the ith scaling function of the m dimensional antecedent universe, $\mathbf{x}$ is the m dimensional crisp observation and $\mathbf{a}_k$ are the cores of the m dimensional fuzzy rule antecedents $\boldsymbol{A}_k$.

Thus in case of singleton rule consequents the fuzzy rules $\mathtt{R}_k$ has the following form:

$$\texttt{If } x_1 = A_{k,1} \texttt{ And } x_2 = A_{k,2} \texttt{ And ... And } x_m = A_{k,m} \texttt{ Then } y = c_k \tag{8}$$

by substituting (7) to (6) the conclusion of the interpolative fuzzy reasoning can be obtained as:

$$y(x) = \begin{cases} c_k & if\ \mathbf{x} = \mathbf{a}_k\ for\ some\ k, \\ \dfrac{\sum\limits_{k=1}^{r} c_k / \delta_{s,k}^{\lambda}}{\sum\limits_{k=1}^{r} 1/\delta_{s,k}^{\lambda}} & otherwise. \end{cases} \tag{9}$$

The interpolative fuzzy reasoning (9) can be extend simply to be able to handle fuzzy conclusions by introducing the vague environment (scaling function) of the consequence universe. In this case the fuzzy rules $\mathtt{R}_k$ has the following form:

$$\texttt{If}\ \ x_1 = A_{k,1}\ \texttt{And}\ x_2 = A_{k,2}\ \texttt{And}\ \ldots\ \texttt{And}\ x_m = A_{k,m}\ \texttt{Then}\ y = B_k \tag{10}$$

By introducing scaled distances on the consequence universe:

$$\delta_s(b_0, b_k) = \int_{b_0}^{b_k} s_y(y)\, dy, \tag{11}$$

where s_y is the ith scaling function of the one dimensional consequent universe, b_k are the cores of the one dimensional fuzzy rule consequents B_k.

Introducing the first element of the one dimensional consequence universe $\boldsymbol{b}_k$ (Y: $b_0 \leq y$, $\forall y \in Y$), based on (9) and (11), the requested one-dimensional conclusion $y(\mathbf{x})$ can be obtained from the following formula [8–10]:

$$\delta_s(b_0, y(\mathbf{x})) = \begin{cases} \delta_s(b_0, b_k) & if\ \mathbf{x} = \mathbf{a}_k\ for\ some\ k, \\ \dfrac{\sum\limits_{k=1}^{r} \delta_s(b_0, b_k) / \delta_{s,k}^{\lambda}}{\sum\limits_{k=1}^{r} 1/\delta_{s,k}^{\lambda}} & otherwise. \end{cases} \tag{12}$$

5 Fuzzy Observation by Merging Vague Environments

The lack of the fuzziness on the observation side in FIVE is inherited from the nature of the vague environment (see Sect. 5), which describes the indistinguishability of two points and hence the Disconsistency Measure of a fuzzy set and a singleton only. For introducing fuzzy observation in FIVE, the concept of vague environment is needed to be extended to the observation too.

One possible solution for this task is an obvious one. If the observation is a fuzzy set, it can be also characterised by a vague environment in the same manner as it was done with the corresponding antecedent fuzzy partitions. This case the question of introducing fuzzy observation turns to be the question of merging two vague environments, the vague environment of the fuzzy observation and the corresponding antecedent fuzzy partition.

For merging two vague environments, the concept of *equal Disconsistency Measures* is applied. According to (3) the Disconsistency Measure of fuzzy sets A and B is the following:

$$S_D(A,B) = 1 - \sup_{x \in X} \mu_{A \cap B}(x), \tag{13}$$

where $A \cap B$ notes the min t-norm, $\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)]$, $\forall x \in X$.

Reconsidering the relation of the Disconsistency Measure of a fuzzy set and a singleton to the scaled distance of two values in a vague environment (according to (3)), the merged vague environment can be defined as the vague environment, where the scaled distance of two values is equal to the Disconsistency Measure of the two corresponding fuzzy sets (see e.g. on Fig. 5):

$$\begin{aligned} S_D(A,B) &= \left| \int_a^{x_0} s_A(x)\,dx \right| = \left| \int_{x_0}^{b} s_B(x)\,dx \right| \\ &= \left| \int_a^{b} s_{A'}(x)\,dx \right| = \delta_s(a,b) = S_D(A',B'), \end{aligned} \tag{14}$$

where $s_A(x)$ is the scaling function of fuzzy set A, $s_B(x)$ is the scaling function of fuzzy set B, $s_{A'}(x)$ is the merged scaling function on X and $\delta_s(a,b)$ is the scaled distance of the values a, b in $s_{A'}(x)$.

Solving (14) in case of constant scaling functions (according to the notation of Fig. 5), the following *merged scaling function* ($s_{A'}$) can be obtained:

$$\begin{aligned} S_D(A,B) &= s_A \cdot (x_0 - a) = s_B \cdot (b - x_0) = S_D(A',B'), \\ S_D(A',B') &= s_{A'} \cdot (b-a) = \frac{s_A \cdot s_B}{s_A + s_B} \cdot (b-a). \end{aligned} \tag{15}$$

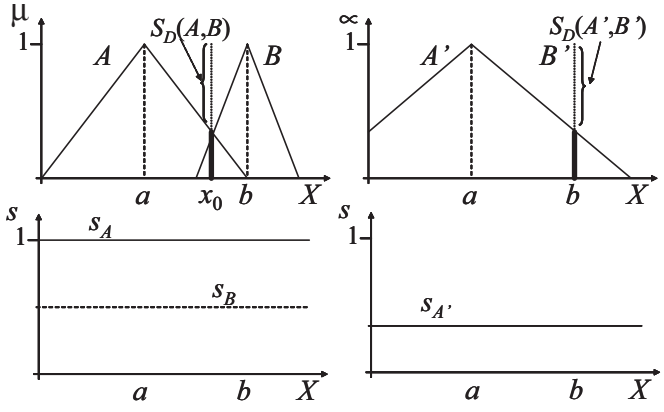

Fig. 5. In the *merged scaling function* $s_{A'}$ (16), the scaled distance of two values a, b is equal to the Disconsistency Measure of the two corresponding fuzzy sets A, B

$$s_{A'}(x) = \frac{s_A(x) \cdot s_B(x)}{s_A(x) + s_B(x)}, \quad \forall x \in X. \tag{16}$$

It is obvious, that generally (16) is not fulfilling the requirements of equal Disconsistency Measures for merging arbitrary vague environments, but it can serve as a kind of "approximation" for the merged scaling function.

Applying the concept of merged scaling function, the method FIVE can be simply completed by fuzzy observation. There is only one additional step required for the original method, the merging of the fuzzy observation vague environments to the vague environments of the corresponding antecedent fuzzy partitions. In the merged vague environment, the fuzzy observation turns to be a singleton, and hence the original FIVE method can be continued in the ordinary way.

Unfortunately, the vague environment merging of the fuzzy observations to the corresponding antecedent fuzzy partitions needed to be repeated in every reasoning step if the scaling function of the fuzzy observation is changing. On the other hand, in some cases, when all the observations can be characterised by the same scaling function (e.g. if all the fuzzy observations have the same isosceles triangle shaped membership function) the merging step needed to be completed only once for all the reasoning steps (see e.g. on Fig. 6).

6 Example

Simple one-dimensional example for the fuzzy observation extended (16) FIVE method (12) is introduced in Fig. 7. For comparing the crisp conclusions of FIVE to the classical methods, the conclusions generated by the max–min compositional rule of inference (CRI) and the centre of gravity defuzzification for the same rule base is also noted in the figure.

In Fig. 7 the label "*fuzzy*" notes the case of fuzzy observation. For comparison, the figure also contains the conclusions of the crisp observations (label

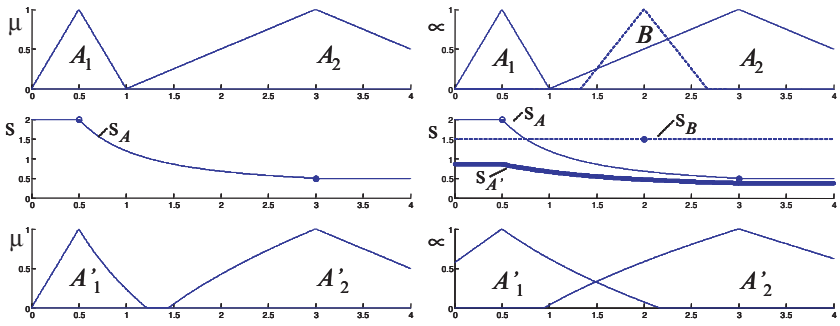

Fig. 6. Fuzzy partition A_i described by scaling function s_A and the *merged scaling function* $s_{A'}$ constructed from s_A and s_B according to (16)

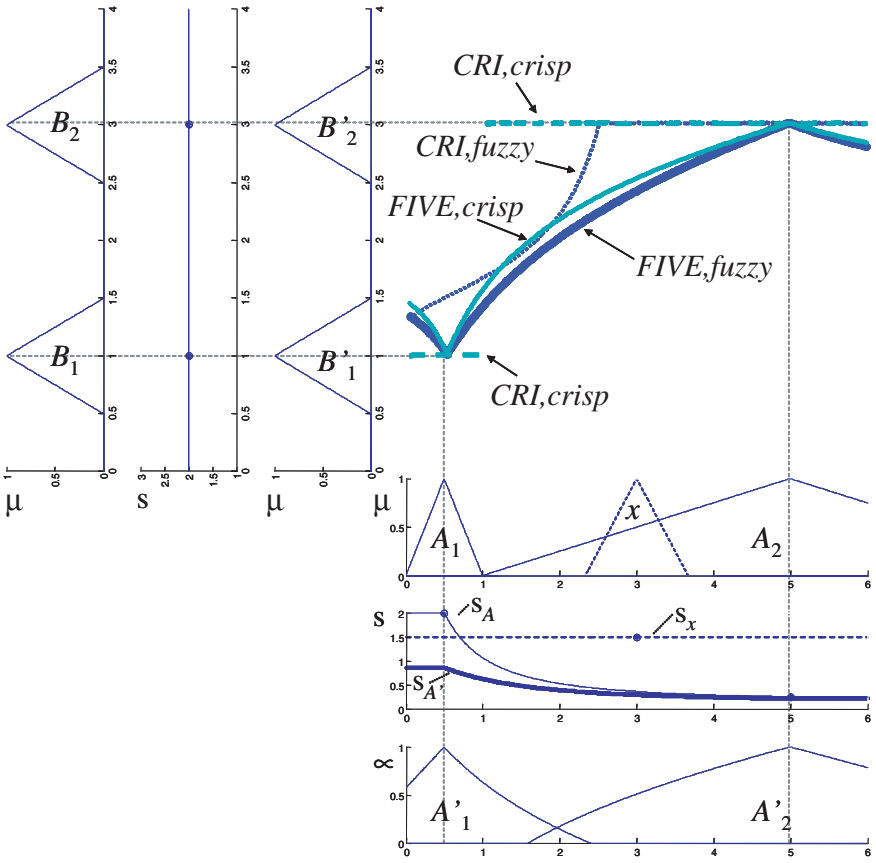

Fig. 7. Interpolation of two fuzzy rules ($A_1 \rightarrow B_1$, $A_2 \rightarrow B_2$) applying the fuzzy observation extended (16) FIVE fuzzy rule interpolation (12), $\lambda = 1$

"*crisp*") for the same rule base. In the example it was assumed, that the running observation has the same isosceles triangle shaped membership function (see x on Fig. 7) everywhere in the observation universe X. For the notation of the scaling function merging in Fig. 7, see Fig. 6.

7 Conclusions

The goal of this paper was to introduce a way for extending the "FIVE" FRI method to be able to handle fuzzy observations. The proposed extension, the "*vague environment merging*", unifies the vague environments of the fuzzy observations to the vague environments of the antecedent universes, and hence introduce the ability of handling fuzzy observation in FIVE.

The main drawback of the proposed extension is the additional step required for merging the observation and antecedent vague environments. This vague environment merging step is needed to be repeated in every reasoning case if the scaling function of the fuzzy observation is changing. On the other hand, when all the fuzzy observations can be characterised by the same scaling function, this merging step is needed to be done only once.

The merged vague environment is introduced as the vague environment, where the scaled distance of two values is equal to the Disconsistency Measure of the two corresponding fuzzy sets characterised by the two separate vague environments intended to be merged (see Sect. 6 for more details). The function proposed for vague environment merging in this paper (16) is only a kind of approximation. Generally the requirement of equal Disconsistency Measure is not fulfilled, save the case when the scaling functions are constants. In spite of this drawback, the proposed merging function (16) is simple enough to keep the simplicity and reasoning speed of the fuzzy observation extended FIVE method. (Since the main goal of developing FIVE was to construct an FRI method, which could be implemented to be simple and quick enough to fit the requirements of real-time direct fuzzy logic control systems.)

A freely applicable code of the extended FIVE introduced in this paper, together with some application examples can be downloaded from [18].

Acknowledgement

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Fuzzy Rule Interpolation Based on Polar Cuts

Zsolt Csaba Johanyák and Szilveszter Kovács

Summary. Fuzzy logic systems applying sparse rule bases should use inference techniques that can produce acceptable output in cases when there no rules for some observations. This paper presents a new fuzzy rule interpolation technique called FRIPOC that is applicable in such cases. Main advantages of the method are its extrapolation capability, its applicability even in subnormal cases and its comprehensibility.

Key words: Fuzzy rule interpolation, Polar cut.

Systems applying fuzzy logic are rule based ones. The collection of the rules the so-called rule base can be characterized as dense or sparse depending on whether there exist rules for all the possible observations. In the sparse case for some observations there are no rules whose antecedent part would overlap the observation at least partially. Therefore the classical compositional reasoning methods can not produce an acceptable conclusion. The inference techniques based on fuzzy rule interpolation are developed for especially this purpose.

This paper proposes a new fuzzy rule interpolation based inference technique applying the concept of linguistic term shifting and polar cut. It is called FRIPOC (Fuzzy Rule Interpolation based in POlar Cuts) and it is applicable in the case of sparse and dense rule bases, too. Its main advantages are its comprehensibility, extrapolation capability and its applicability even if the height of one or more fuzzy sets is smaller than one.

The rest of this paper is organized as follows. Sect. 1 gives a brief overview on the relevant fuzzy rule interpolation techniques grouping them depending on the main steps they are following. Sect. 2 presents the main structure and the steps and stages that characterize the method FRIPOC. Sect. 3 introduces the concept of the polar cut and a fuzzy set interpolation technique called FEAT-p based on it as a possible implementation for the first and third stage of the first step. In Sect. 4 the authors propose a technique for the determination of the position of the consequent sets that is an extension and adaptation of the Shepard 2D interpolation [14]. Sect. 5 introduces a new polar

cut based single rule inference method for the determination of the conclusion. In Sect. 6 some relevant features of the new method are outlined through some numerical examples.

1 A Brief Overview of Fuzzy Rule Interpolation Methods

The fuzzy rule interpolation (FRI) based inference techniques have been used for several years in order to alleviate the problems arising from the information gaps in sparse rule bases. They can be divided into two groups depending on whether they are producing the approximated conclusion directly or a new intermediate rule is interpolated first.

Relevant members of the first group are among others the α-cut based interpolation (KH) [10] proposed by Kóczy and Hirota, which was the first developed technique, the modified α-cut based interpolation (MACI) [16] introduced by Tikk and Baranyi, the fuzzy interpolation based on vague environment (FIVE) [11] developed by Kovács and Kóczy, the improved fuzzy interpolation technique for multi-dimensional input spaces (IMUL) [19] proposed by Wong, Gedeon and Tikk, the interpolative reasoning based on graduality (IRG) [2] introduced by Bouchon-Meunier, Marsala and Rifqi, the interpolation by the conservation of fuzziness (GK) [4] developed by Gedeon and Kóczy, the method based on the conservation of the relative fuzziness (CRF) proposed by Hirota, Kóczy and Gedeon, and the VKK method [18] introduced by Vass, Kalmár and Kóczy.

The structure of the methods belonging to the second group can be described best by the generalized methodology of the fuzzy rule interpolation introduced by Baranyi, Kóczy and Gedeon in [1]. As other typical members of this group can be mentioned the ST method [20] introduced by Yan, Mizumoto and Qiao, the interpolation with generalized representative values (IGRV) [5] developed by Huang and Shen, the technique proposed by Jenei in [6], and the method being presented in this paper.

The solvability and approximate solvability of fuzzy relation equations and the approximation quality of approximate solutions was studied by Perfilieva and Gottwald in [12].

2 The Structure of the Proposed Method

The method FRIPOC (Fuzzy Rule Interpolation based on POlar Cuts) essentially follows the concepts of the generalized methodology of fuzzy rule interpolation (GM) introduced by Baranyi et al. in [1]. The position of the fuzzy sets is characterized by a reference point during the calculations. For example the centre of the core, the centre of gravity, the centre of the support or the projection of the centre of the core to the horizontal axis can play this role (Fig. 1.). In the case of the polar cut based set interpolation and single rule reasoning methods the latter choice offers the most advantages. Besides its information content about the middle one from the most relevant (having the maximal

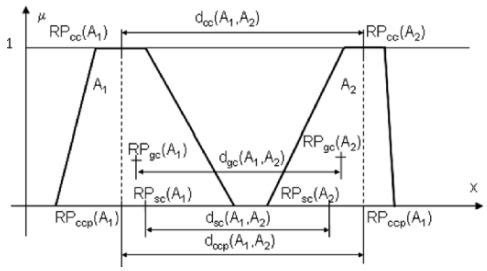

Fig. 1. Options for the reference point and the related set distances

membership value) elements of the set it also reduces the need for calculations due to the fact that its ordinate value is always zero. Further on this type of reference point is used during the calculations. The distance of the fuzzy sets is measured by the horizontal distance between the reference points of the sets.

The method consists of two steps. First a new intermediate rule is interpolated, of which antecedent part contains fuzzy sets whose position is identical with the position of the sets describing the observation in each dimension. This task is done in three stages. First the antecedent part of the new rule is determined through a set interpolation method. The application of the technique FEAT-p introduced in Sect. 3 is proposed by the authors for this purpose. Next the position of the fuzzy sets belonging to the consequent part of the new rule is calculated. The method suggested by the authors for this task is presented in Sect. 4. Thirdly the shape of the consequent sets is determined using the same technique as in the case of the antecedent sets.

The conclusion is determined in the second step by firing the interpolated rule. A special single rule reasoning technique called SURE-p, which is based also on polar cuts, is introduced for this task in Sect. 5.

3 Fuzzy Set Interpolation Based on Linguistic Term Shifting and Polar Cuts

The task of the fuzzy set interpolation is to determine the antecedent and consequent sets that belong to the new rule. The method is the same in the case of each linguistic term regardless of it belongs to an antecedent or consequent universe of discourse. The calculations are done separately for each input and output dimension. The starting point is a fuzzy partition with the reference points of the sets determined in advance and the reference point of the observation (conclusion) in the actual dimension/partition. All the sets in the partition belong to the antecedent (consequent) part of one or more rules.

The reference point of the new set is identical with the the reference point of the observation (conclusion) in the actual dimension. The method goes out from the assumption that a better set approximation can be attained by taking into consideration not only the two sets flanking the observation/conclusion but all the available linguistic terms in the partition. First all sets are shifted

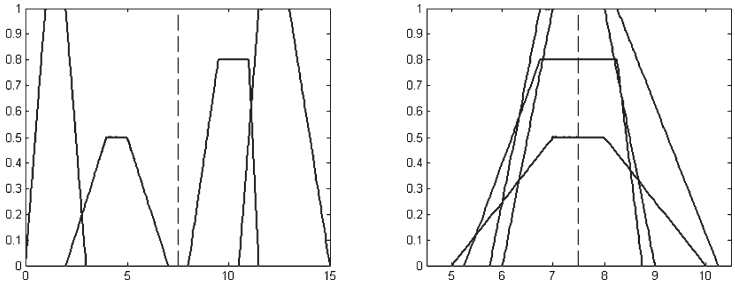

Fig. 2. The original partition and the result of the shifting

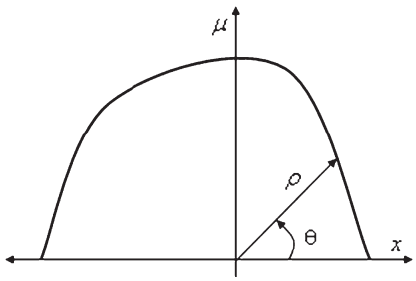

Fig. 3. Polar cut

horizontally in order to reach the coincidence of the horizontal position of their reference points with the position of the interpolation (see Fig. 2). This idea is similar to the concept in [2], but that method uses and translates only the two flanking sets into the location of the observation. Next the shape of the new set is determined from the collection of the overlapped sets.

There are several solutions for this task. Similar to the choice of the reference point the selection of the calculation mode of the shape is also a tuning point. In [8] the authors present a solution with low computational complexity called FEAT-α (Fuzzy SEt interpolAtion Technique based on α-cuts). It is based on α-cuts and its application area is however, restricted to the most popular case of the convex and normal fuzzy (CNF) sets.

Further on the concept of the polar cut is introduced and next based on it a solution called FEAT-p (Fuzzy SEt interpolAtion Technique based on polar cuts) is proposed. Its main advantage is that it can also be applied in cases when the normality condition is not satisfied for all the sets participating in the interpolation process, i.e. the height of one or more sets is smaller than one.

The concept of the polar cut is strong related to the application of a polar co-ordinate system whose origin coincides with the abscissa of the reference point of the observation. A polar cut is defined by a value pair $\{\rho, \theta\}$ that determines a point on the shape of the linguistic term. The value ρ denotes the polar distance at the angle θ (Fig. 3). The authors are going out from the

assumption that an extension and a resolution principle of the fuzzy sets can be defined for polar cuts, too. This extension principle states that the solution of a problem for fuzzy sets can be found in the form of solving it first for its polar cuts and then extending the solution to the fuzzy case. The resolution principle states in this case that a fuzzy set can be decomposed into polar cuts.

The shape calculation technique FEAT-p is based on the above defined extension principle. For each polar cut of the interpolated set the value ρ is calculated as weighted average of the polar distances ρ of the shifted sets for the same θ angle using (1).

$$\rho\left(A^i_{j\theta}\right) = \begin{cases} \dfrac{\sum\limits_{k=1}^{n_j} w_{jk} \cdot \rho(A_{jk\theta})}{\sum\limits_{k=1}^{n_j} w_{jk}} & d\left(A^*_j, A_{jk}\right) > 0 \\ \rho\left(A_{jk\theta}\right) & d\left(A^*_j, A_{jk}\right) = 0, k = 1..n_j \end{cases} \tag{1}$$

where ρ denotes the length of a polar cut, j is the actual antecedent (consequent) dimension, θ is the angle of the actual cut, n_j is the number of the sets in the partition, $A_{jk\theta}$ is the polar cut of the kth set, w_{jk} is the weighting factor of the kth set, $A^i_{j\theta}$ is the interpolated polar cut and the superscript i denotes that the set is an interpolated one. The collection of the angles , the so called polar levels, for which the calculations are done, should be set-up in such mode to include the values 0, $\pi/2$ and π.

It seems to be natural that the sets whose original position were in the neighbourhood of the reference point of the observation to exercise higher influence as those ones situated in farther regions of the universe of discourse. Therefore the weighting factor should be dependent on distance. The simplest weighting factor is the reciprocal value of the distance, which can be expressed by (2) with $p = 1$, but there are several recommendations in the literature for more or less analogue cases. For example in [10] the square of the reciprocal value of the distance is suggested ($p = 2$). The authors of [17] propose the use of the reciprocal value of the distance on the mth power ($p = m$), where m is the number of the antecedent dimensions.

$$w_{jk} = \frac{1}{d\left(A^*_j, A_{jk}\right)^p} \tag{2}$$

The formula (1) separates the case when the position of the interpolation coincides with the actual set of the partition ($d\left(A^*_j, A_{jk}\right) = 0$). Its reason is that the weighting factor (2) contains the distance in the denominator. Thus if the reference point of the observation (conclusion) is the same as one of the original sets of the partition the interpolated set will be the same as that linguistic term. This feature ensures the fulfilment of the condition four from [7], namely the compatibility with the rule base, for the rule interpolation method based on the above mentioned method.

4 The Position of the Consequent Sets

The position of the fuzzy sets belonging to the consequent part of the new rule is determined independently in each output dimension. The task can be defined as a problem of finding a point on a hyper-surface defined by the reference points of the antecedent sets (sets belonging to the antecedent parts of the existing rules) and the consequent sets in the actual output dimension. Due to the sparse character of the rule base an n_a dimensional interpolation has to be done for irregularly spaced data, where n_a is the number of the output dimensions. It can be expressed in general by the formula (3).

$$RP\left(B_l^i\right) = f\left(RP\left(A_1^i\right), RP\left(A_2^i\right), ..., RP\left(A_j^i\right), ..., RP\left(A_{n_a}^i\right)\right) \tag{3}$$

where $RP\left(B_l^i\right)$ is the reference point of the interpolated consequent set in the lth output dimension and $RP\left(A_j^i\right)$ is the reference point of the interpolated antecedent set in the jth input dimension. The function should pass through the known points of the hyper-surface and it should be smooth, i.e. continuous and once differentiable.

There are several applicable linear or non-linear functions that take into consideration either only the points (rules) situated in the closest neighbourhood of the interpolation or all the known points. The authors suggest the use of an interpolation function that is an extension and adaptation of the Shepard interpolator [14] for the case of arbitrary number of antecedent dimensions.

The antecedent part of each rule can be thought of as a point in the antecedent hyper-space. Its co-ordinates are given by the reference points of the sets belonging to it. The point corresponding to the antecedent of the interpolated rule is at the same time also the representing point of the observation. Further on the Euclidean distance between these points is used as the measure of the closeness of the antecedents and by this means also the closeness of the rules. The proposed interpolation function (4) determines the reference point of the conclusion as a weighted average of the reference points of the consequent sets of the known rules in the actual output dimension.

$$RP\left(B_l^i\right) = \frac{\sum_{j=1}^{N} RP\left(B_{lj}\right) \cdot s_j}{\sum_{j=1}^{N} s_j} \tag{4}$$

where $RP\left(B_l^i\right)$ is the reference point of the interpolated consequent set in the lth dimension, N is the number of the rules, j denotes the actual rule, s_j is the weight attached to the jth rule.

The rules whose antecedent part is in the closer neighbourhood of this point should exercise higher influence than those situated farther. Therefore the weighting factor is a distance function. Shepard proposed in [14] several variants of the weighting factors for its interpolation function. The first of them, which applies the inverse of the square of the distance, was chosen by

the authors to be applied considering it as the one having the lowest computational complexity. Its adapted version, the formula (5) is the inverse of the square of the distance between the antecedent of the interpolated rule and the antecedent of the jth rule. It is actually the sum of the squares of distances measured along each antecedent dimension.

$$s_j = \frac{1}{d\left(RA^i, RA_j\right)^2} = \frac{1}{\sum\limits_{k=1}^{n_a} \left(RP\left(A_k^i\right) - RP\left(A_{jk}\right)\right)^2} \tag{5}$$

where RA^i is the antecedent of the interpolated rule, RA_j is the antecedent of the jth rule, $RP\left(A_k^i\right)$ is the reference point of the interpolated antecedent in the kth dimension (identical with the reference point of the observation in the kth dimension), $RP\left(A_{jk}\right)$ is the reference point of the reference point of the antecedent set of the jth rule in the kth dimension and n_a is the number of the antecedent dimensions. Generally the fuzzy sets are identified in the different formulas by two indexes (e.g. in (1) and (2)) the first indicating the dimension and the second indicating the ordinal number of the set. Contrary to this in the last two formulas ((4) and (5)) the second subscript gives the number of the rule of which antecedent part the set belongs to. It is because this notation mode simplifies the formulas.

Shepard suggested in [14] for the 2D case the use of maximum 10 closest points in order to reduce the computational needs. However, when the number of the dimensions is much more than two and the rule base is sparse it seems to be easier to take into consideration all the rules than to seek those ones that are in a special proximity of the observation.

5 Single Rule Reasoning Based on Polar Cuts

In the second step of the inference the conclusion is generated by firing the new rule. The reference point of the interpolated conclusion in the current dimension will be the same as the reference point of the consequent set of the new rule in the current dimension. Usually the antecedent part of the rule does not fit perfectly the observation. Therefore a special single rule reasoning technique is needed. There are some methods for this task in the literature, but their common drawback is that their applicability is restricted to some regular cases. For example the similarity transfer method introduced in [15] requires the normality of the sets. Beside this the revision principle based FPL and SRM techniques presented in [13] also demand the coincidence between the support of the antecedent set and the support of the observation. Generally these conditions are not fulfilled. Therefore some transformations of the fuzzy relation are needed when one decides for their application.

The technique SURE-p (Single rUle REasoning based on polar cuts) being presented alleviates this problem. In addition its advantage is its applicability in multi-dimensional cases. SURE-p is based on the concept of polar cut. Although it determines the conclusion sets in each consequent dimension independently, there are some common calculations that have to be done only

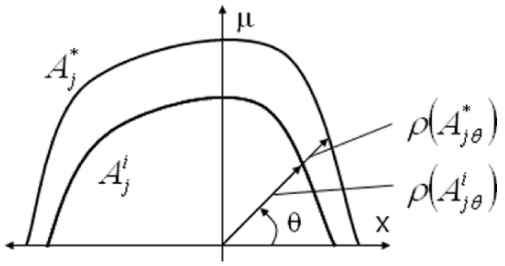

Fig. 4. Polar distances used for the calculation of the relative difference

once at the beginning. Thus for each polar cut firstly the difference between the polar distance of the antecedent set and the polar distance of the observation in each dimension is calculated, and the result is divided by the range of the linguistic variable (6).

$$r_{j\theta} = \frac{\rho\left(A_{j\theta}^i\right) - \rho\left(A_{j\theta}^*\right)}{range_{aj}} \tag{6}$$

where $r_{j\theta}$ is the relative difference at the θ level in the jth antecedent dimension, $\rho\left(A_{j\theta}^i\right)$ the polar distance of the antecedent set (see fig. 4), $\rho\left(A_{j\theta}^*\right)$ is the polar distance of the observation and $range_{aj}$ is the range of the antecedent linguistic variable in the jth dimension. Next an average relative difference is calculated taking into consideration the relative differences in all antecedent dimensions (7).

$$\overline{r}_\theta = \frac{\sum_{j=1}^{n_a} r_{j\theta}}{n_a} \tag{7}$$

where $\overline{r}_\theta$ is the average relative difference at the θ level, n_a is the number of the antecedent dimensions. In each consequent dimension the corresponding polar cut is calculated supposing that the relative difference at θ level between the polar distances of the rule consequent and the conclusion is equal to $\overline{r}_\theta$ as expressed in formula (8).

$$\frac{\rho\left(B_{l\theta}^i\right) - \rho\left(B_{l\theta}^*\right)}{range_{cl}} = \overline{r}_\theta \tag{8}$$

where $\rho\left(B_{l\theta}^i\right)$ the polar distance of the interpolated consequent set, $\rho\left(B_{l\theta}^*\right)$ is the polar distance of the conclusion, $range_{cl}$ is the range of the consequent linguistic variable in the lth output dimension and θ is the polar angle. Due to the nature of the fuzzy sets the resulting height of the conclusion has to be maximized to one. Thus arises the formula (9).

$$\rho\left(B_{l\theta}^*\right) = \begin{cases} Min\left[\rho\left(B_{l\theta}^i\right) - \overline{r}_\theta \cdot range_{cl}, \frac{1}{sin(\theta)}\right] & sin\left(\theta\right) > 0 \\ \\ rho\left(B_{l\theta}^i\right) - \overline{r}_\theta \cdot range_{cl} & sin\left(\theta\right) = 0 \end{cases} \tag{9}$$

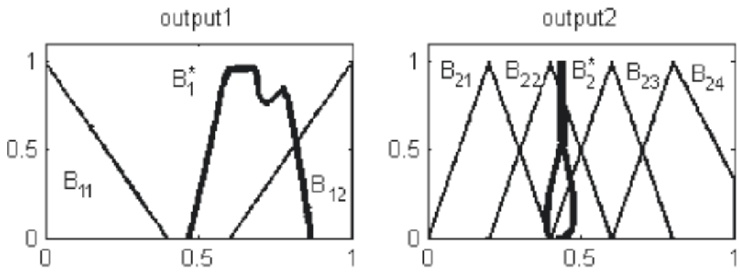

Fig. 5. Non-convex conclusions obtained by the formula (9)

Due to the revision of the interpolated consequent sets based on the average relative antecedent difference the formula (9) can easily lead to a non-convex fuzzy set. As an example Fig. 5 presents the consequent partitions of a system with two output dimensions (*output*1 and *output*2). The interpolated conclusion sets (B_1^* and B_2^*) obtained by (9) are drawn with bold lines. In order to alleviate this problem the calculations should start at polar level $\pi/2$ in top-down direction, they should be done separately for the right and left flanks of the linguistic terms as well a control and correction algorithm should be included. Further on the basic ideas and the steps that have to be done are presented only for the case of the right flank of the set. The calculation of the left flank is similar.

The convexity requirement is satisfied if and only if the horizontal distance to the centre of the polar co-ordinate system of each point is not smaller than the same distance calculated for the previous point and if the vertical distance to the centre of the polar co-ordinate system of each point is not greater than the same distance calculated for the previous point. This condition can be expressed by the formula (10).

$$\rho\left(B^{*c}_{l\theta(k)}\right) = \begin{cases} \rho\left(B^{*}_{l\theta(k)}\right) & k = 1 \\ \rho\left(B^{*}_{l\theta(k-1)}\right) \cdot \frac{cos(\theta(k-1))}{cos(\theta(k))} & k > 1 \; and \\ & \frac{\rho\left(B^{*}_{l\theta(k-1)}\right) \cdot cos(\theta(k-1))}{\rho\left(B^{*}_{l\theta(k)}\right) \cdot cos(\theta(k))} > 1 \\ \rho\left(B^{*}_{l\theta(k-1)}\right) \cdot \frac{sin(\theta(k-1))}{sin(\theta(k))} & k > 1 \; and \\ & \frac{\rho\left(B^{*}_{l\theta(k)}\right) \cdot sin(\theta(k))}{\rho\left(B^{*}_{l\theta(k-1)}\right) \cdot sin(\theta(k-1))} > 1 \\ \rho\left(B^{*}_{l\theta(k)}\right) & otherwise \end{cases} \tag{10}$$

where θ is an array containing the polar angles necessary for the calculation of the right flank from $\pi/2$ to 0 in descending order, $\rho\left(B^{*}_{l\theta(k)}\right)$ is the polar

distance calculated by the formula (9) and $\rho\left(B^{*c}_{l\theta(k)}\right)$ is the corrected polar distance.

6 Numerical Examples

In the followings the sensitivity of the method FRIPOC to the value of its parameter p will be studied through some numerical examples. Figure 6 presents a fuzzy system having two antecedent (*input*1 and *input*2) dimensions and one consequent dimension (*output*1). There are triangular, trapezoidal and rectangular (crisp) set shapes and four of the sets are subnormal. For the sake of simplicity each original antecedent partition contains only two sets that are surrounding the observation drawn by bold line. Based on the same consideration the original consequent partition also contains two fuzzy sets. The rule base consist of two rules according to (11).

$$\begin{aligned} R_1: &\ if\ A_1^* = A_{11}\ and\ A_2^* = A_{21}\ then\ B_1^* = B_{11} \\ R_2: &\ if\ A_1^* = A_{12}\ and\ A_2^* = A_{22}\ then\ B_1^* = B_{12} \end{aligned} \tag{11}$$

The observation is trapezoid shaped in both antecedent dimensions (A_1^* and A_2^*). The third axes (frame) contains the final interpolated conclusion marked by bold line and obtained for the value $p = 0.001$. Figure 7 contains three further results obtained for the values 1, 2 and 10 of the parameter p. One can clearly observe that increasing the value of p the second rule (R_2), which is visibly the nearest one to the observation keeps getting more dominant and the corrected interpolated conclusion (B_1^{*c}) becomes more and more similar to the set B_{12}.

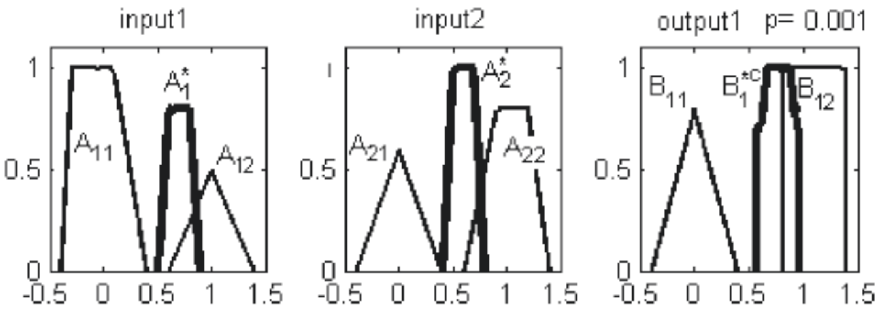

Fig. 6. FRIPOC applied with $p = 0.001$ to a system with two input and one output dimension and two rules

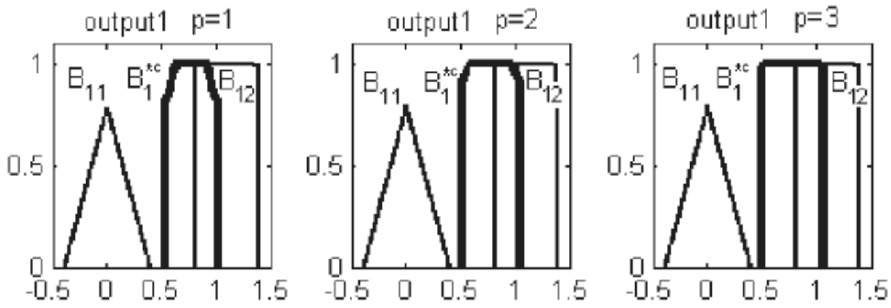

Fig. 7. Conclusions obtained for $p = 1$, $p = 2$ and $p = 3$

7 Conclusions

The interpolation based fuzzy reasoning methods ensure an acceptable conclusion even in cases when there are no rules whose antecedent part would overlap the observation.

In this paper a new technique called FRIPOC is presented that introduces the concept of polar cuts and linguistic term shifting for fuzzy rule interpolation. It determines the conclusion in two steps following the concept of GM [1]. First an intermediate rule is interpolated whose antecedent part is in the same position as the observation in each antecedent dimension and next the result is determined by firing the new rule. The authors suggest the application of a new method called FEAT-p for the set interpolation tasks and the use of an adapted version of the Shepard interpolation for the determination of the position of the consequent part of the rule in the first step. A new technique called SURE-p is suggested as single rule reasoning method for the second step. The main advantages of the method FRIPOC are its comprehensibility, extrapolation capability and its applicability even in subnormal cases. The sensitivity of the method to the value of the parameter p is outlined through some numerical examples. The method is implemented in Matlab and can be downloaded from [3]. This website is dedicated to a fuzzy rule interpolation Matlab toolbox development project (introduced in [9]) aiming the implementation of various FRI techniques.

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Approximate Reasoning Using Fodor's Implication

Adrian Giurca and Ion Iancu

Summary. Using generalized modus ponens reasoning, we examine the values of the inferred conclusion using the Fodor's implication in order to interpret a *fuzzy if–then rule* with a single input single output and the T-norms $t_1(x, y) = \min(x, y)$, $t_2(x, y) = xy$, and $t_3(x, y) = \max(0, x + y - 1)$ for composition operation. These are the very used T-norms in generalized modus ponens reasoning.

Key words: Fuzzy sets, Fuzzy implication, Generalized modus ponens, T-norm, T-conorm.

1 Introduction

An investigation of inference processes in the fuzzy if–then rules is still a subject of many papers in literature: [1, 5–23]. The principal difficulty in the utilization of these rules appears when the observed facts do not match the condition expressed in the premise. These problems led Zadeh [23] to outline the theory of approximate reasoning that is the deduction of imprecise conclusion from a set of imprecise premises. He extends the traditional modus ponens rule in order to deduce an imprecise conclusion from imprecise premises; thus he obtained generalized modus ponens rule. This inference mechanism states that from the propositions "If X is A then Y is B" and "X is A'" we can deduce "Y is B'" where A, B, A', and B' are modeled by fuzzy sets, X and Y are variables whose domains are U and V, respectively; evidently, $A, A' \subseteq U$ and $B, B' \subseteq V$. The proposition "X is A" can be understood as "the quantity X satisfies the predicate A" or "the variable X takes its values in the set A." Its semantic content can be represented by $\pi_X = \mu_A$, where π_X is the possibility distribution restricting the possible value of X and μ_A is the membership function of the set A.

A causal link from X to Y is represented as conditional possibility distribution [22,23], $\pi_{Y/X}$ which restricts the possible values of Y for a given value of X. For the rule $If\ X\ is\ A\ then\ Y\ is\ B$ this is

$$\forall u \in U,\ \forall v \in V,\ I(u,v) \equiv \pi_{Y/X}(v,u) = \mu_A(u) \rightarrow \mu_B(v),$$

where $\rightarrow$ is an implication operator and μ_A and μ_B are the possibility distributions of the propositions "X is A" and "Y is B," respectively.

If $\mu_{A'}$ is the possibility distribution of the proposition "X is A'" then the possibility distribution $\mu_{B'}$ of the conclusion Y *is* B' is computed as $\mu_{B'}(v) = \sup\left\{t\left(\mu_{A'}(u), \pi_{Y/X}(v,u)\right)/u \in U\right\}$ where t is a T-norm.

The membership function μ_A can be expressed using parametric representation, which is achieved by the five-tuple $(L_A, l_A, R_A, r_A, \theta)$ [2,3]:

$$\mu_A(x) = \begin{cases} \theta & if\ x \le L_A - l_A \text{ or } x \ge R_A + r_A \\ 1 & if\ L_A \le x \le R_A \\ \Psi_1(x)\ if\ L_A - l_A \le x \le L_A \\ \Psi_2(x)\ if\ R_A \le x \le R_A + r_A \end{cases}$$

where $\theta \in [0,1]$ describes the uncertainty that accompanies the piece of information, Ψ_1 is a nondecreasing function, and Ψ_2 is a nonincreasing function. Besides, the continuity conditions are necessary:

$$\Psi_1(L_A) = \Psi_2(R_A) = 1 \text{ and } \Psi_1(L_A - l_A) = \Psi_2(R_A + r_A) = \theta.$$

For $\theta = 0$ the piece of information is certain. For practical reasons we shall work with trapezoidal distributions: Ψ_1 and Ψ_2 are linear functions and $\theta = 0$.

2 Basic Concepts

We recall the definitions of basic concept used in generalized modus ponens reasoning.

Definition 1. *A T-norm, $T : [0,1]^2 \rightarrow [0,1]$ is a commutative, associative, nondecreasing function, and $T(x,1) = x\ \forall x \in [0,1]$.*

A T-conorm, $S : [0,1]^2 \rightarrow [0,1]$ is a commutative, associative, nondecreasing function, and $S(x,0) = x\ \forall x \in [0,1]$.

A strong negation, $N : [0,1] \rightarrow [0,1]$ is an involutive and continuous decreasing function from $[0,1]$ to itself.

Definition 2. *A* fuzzy implication *is a function $I : [0,1]^2 \rightarrow [0,1]$ satisfying the following conditions:*

(I1) If $x \le z$ then $I(x,y) \ge I(z,y)$ for all $x, y, z \in [0,1]$.
(I2) If $y \le z$ then $I(x,y) \le I(x,z)$ for all $x, y, z \in [0,1]$.
(I3) $I(0,y) = 1$ (falsity implies anything) for all $y \in [0,1]$.
(I4) $I(x,1) = 1$ (anything implies tautology) for all $x \in [0,1]$.
(I5) I(1,0)=0 (Booleanity).

The following properties could be useful in some applications:

(I6) $I(1,x) = x$ (tautology cannot justify anything) for all $x \in [0,1]$.
(I7) $I(x, I(y,z)) = I(y, I(x,z))$ (exchange principle) for all $x, y, z \in [0,1]$.

(I8) $x \leq y$ if and only if $I(x, y) = 1$ (implication defines ordering) for all $x, y \in [0, 1]$.
(I9) $I(x, 0) = N(x)$ for all $x \in [0, 1]$ is a strong negation.
(I10) $I(x, y) \geq y$ for all $x, y \in [0, 1]$.
(I11) $I(x, x) = 1$ (identity principle) for all $x \in [0, 1]$.
(I12) $I(x, y) = I(N(y), N(x))$ for all $x, y \in [0, 1]$ and a strong negation N.
(I13) I is a continuous function.

The most important families of implications [4] are given by

Definition 3. *A R-implication associated with a T-norm T is defined by*

$$I_T(x, y) = \sup\{z \in [0, 1]/T(x, z) \leq y\}, \ \forall x, y \in [0, 1].$$

A S-implication associated with a T-conorm S and a strong negation N is defined by

$$I_{S,N}(x, y) = S(N(x), y) \ \ \forall x, y \in [0, 1].$$

A QL-implication is defined by

$$I_{T,S,N}(x, y) = S(N(x), T(x, y)), \ \forall x, y \in [0, 1].$$

We shall work with Fodor's implication

$$I_F(x, y) = \begin{cases} 1 & if\ x \leq y \\ max(1 - x, y) & otherwise \end{cases}$$

which is a R-implication for $T = \min_0$, a S-implication for $S = \max_0$ and a QL-implication for $T = \min$ and $S = \max_0$, where

$$min_0(x, y) = \begin{cases} 0 & if\ x + y \leq 1 \\ min(x, y) & if\ x + y > 1 \end{cases}$$

and

$$max_0(x, y) = \begin{cases} 1 & if\ x + y \geq 1 \\ max(x, y) & if\ x + y < 1 \end{cases}$$

and $N(x) = 1 - x$. Besides, the Fodor's implication verifies the properties I1–I12.

3 Generalized Modus Ponens with Fodor's Implication

Taking into account the properties verified by Fodor's implication it results that it is one of the most important implication operators. That is why, we shall investigate the generalized modus ponens reasoning using the Fodor's implication and the T-norms:

(T1) $t_1(x, y) = \min(x, y)$,
(T2) $t_2(x, y) = xy$, and
(T3) $t_3(x, y) = \max(0, x + y - 1)$.

Our aim is to obtain the conclusion Y is B' from the rule

$$If\ X\ is\ A\ then\ Y\ is\ B$$

and the fact

$$X\ is\ A'$$

when A, A', B, and B' are represented by trapezoidal possibility distribution. The set B' is computed as

$$\mu_{B'}(v) = \sup_{u \in U} \{t(\mu_{A'}(u), I_F(\mu_A(u), \mu_B(v)))\}.$$

We shall analyze five cases, depending on the relation between μ_A and $\mu_{A'}$ and we will give the proof only for T-norm t_1; for t_2 and t_3 one repeats the reasoning used for t_1.

Theorem 1. *If the premise contains the observation:* $\mu_{A'}(u) \leq \mu_A(u),\ \forall u \in U$, *then*

1. *for T-norm* t_1*:*
   $\mu_{B'}(v) \geq \mu_B(v)$ *if* $\mu_B(v) \leq 0.5$
   $\mu_{B'}(v) = \mu_B(v)$ *if* $\mu_B(v) > 0.5$
2. *for T-norm* t_2*:*
   $\mu_{B'}(v) = \mu_B(v)$ *if* $\mu_B(v) \geq 0.5$ *or* $(0.25 \leq \mu_B(v) < 0.5)$
   $\mu_{B'}(v) < 0.25$ *if* $\mu_B(v) < 0.25$
3. *for T-norm* t_3*:*
   $\mu_{B'}(v) = \mu_B(v)$.

Proof. (I1) Value on the set $U_1 = \{u \in U / \mu_A(u) \leq \mu_B(v)\}$
Because $I_F(\mu_A(u), \mu_B(v)) = 1$, we have

$$\mu_{B'}(v) = \sup_{u \in U_1} \min(\mu_{A'}(u), 1) = \sup_{u \in U_1} \mu_{A'}(u) \leq \sup_{u \in U_1} \mu_A(u) \leq \mu_B(v).$$

(I2) Value on the set
$U_2 = \{u \in U / \mu_A(u) > \mu_B(v) \geq 0.5\} \cup \{u \in U / \mu_A(u) > 1 - \mu_B(v) > 0.5\}$.
We have $I_F(\mu_A(u), \mu_B(v)) = \mu_B(v)$ and therefore

$$\mu_{B'}(v) = \sup_{u \in U_2} \min(\mu_{A'}(u), \mu_B(v)) = \mu_B(v).$$

(I3) Value on the set $U_3 = \{u \in U / \mu_B(v) < \mu_A(u) \leq 1 - \mu_B(v)\}$.
In this case $I_F(\mu_A(u), \mu_B(v)) = 1 - \mu_A(u)$ and therefore

$$\mu_{B'}(v) = \sup_{u \in U_3} \min(\mu_{A'}(u), 1 - \mu_A(u)) < 1 - \mu_B(v).$$

Taking into account that, in this case, $1 - \mu_B(v) > \mu_B(v)$, we obtain the final conclusion. □

If the observation is more precise than the premise of the rule, then it gives more information than the premise. However, it does not seem reasonable to think that the generalized modus ponens allows to obtain a conclusion more precise than that of the rule. The result of the inference is valid if $\mu_{B'}(v) = \mu_B(v),\ \forall v \in V$. This case is illustrated in the following example from [11]:

Example 1.
Rule: if the result in the mathematics is good then my recommendation for the choice of a university faculty is mathematics.
Observation: the result in mathematics is very good.
Consequence: my recommendation is mathematics.

Sometimes, the deduction operation allows the reinforcement of the conclusion (see also [15]), as in the example below:

Example 2.
Rule: if the tomato is red then the tomato is ripe.
Observation: this tomato is very red.
If we know that the maturity degree increases with respect to color, we can infer.
Consequence: this tomato is very ripe.

On the other hand, in the next example:

Example 3.
Rule: if the melon is ripe then it is sweet.
Observation: the melon is very ripe.
we do not infer that "the melon is very sweet because it can be so ripe that it can be rotten.

This examples show that the conclusion depends on the knowledge base. If the expert has not supplementary information about the connection between the variation of the premise and the conclusion, he must be satisfied with the conclusion $\mu_{B'}(v) = \mu_B(v)$. Theorem 1 says that for this we can choose the T-norm t_3.

Theorem 2. *If the premise and the observation coincide, i.e.,* $\mu_A(u) = \mu_{A'}(u),\ \forall u \in U$, *then*

1. $\mu_{B'}(v) = \max(\mu_B(v), 0.5)$, *for T-norm* t_1.
2. $\mu_{B'}(v) = \max(\mu_B(v), 0.25)$, *for T-norm* t_2.
3. $\mu_{B'}(v) = \mu_B(v)$, *for T-norm* t_3.

Proof. In this case all inequalities (generated by inequality $\mu_{A'}(u) \leq \mu_A(u)$) from the proof of the Theorem 1 become equalities. □

When the observation and the premise of the rule coincide the convenient behavior of the fuzzy deduction is to obtain an identical conclusion. But, the

T-norms t_1 and t_2 give a different conclusion. This fact indicates the appearance of an uncertainty in the conclusion that is totally unreasonable. In order to avoid this possibility we suggests to use the T-norm t_3.

Theorem 3. *If the observation contains the premise, i.e.,* $\mu_A(u) \leq \mu_{A'}(u)$, $\forall u \in U$, *then* $\mu_{B'}(v) \geq \mu_B(v)$ $\forall v \in V$.

Proof. (I1) Value on the set $U_1 = \{u \in U / \mu_A(u) \leq \mu_B(v)\}$
We have $I_F(\mu_A(u), \mu_B(v)) = 1$ and therefore

$$\mu_{B'}(v) = \sup_{u \in U_1} min(\mu_{A'}(u), 1) = \sup_{u \in U_1} \mu_{A'}(u) \geq \mu_B(v).$$

(I2) Value on the set
$U_2 = \{u \in U / 0.5 \leq \mu_B(v) < \mu_A(u)\} \cup \{u \in U / \mu_A(u) > 1 - \mu_B(v) > 0.5\}$.
We have $I_F(\mu_A(u), \mu_B(v)) = \mu_B(v)$ and therefore

$$\mu_{B'}(v) = \sup_{u \in U_2} \min(\mu_{A'}(u), \mu_B(v)) = \mu_B(v).$$

(I3) Value on the set $U_3 = \{u \in U / \mu_B(v) < \mu_A(u) \leq 1 - \mu_B(v)\}$.
We have $I_F(\mu_A(u), \mu_B(v)) = 1 - \mu_A(u)$ and therefore

$$\mu_{B'}(v) = \sup_{u \in U_3} \min(\mu_{A'}(u), 1 - \mu_A(u)) \geq \mu_B(v). \qquad \square$$

Theorem 4. *If there is a partial overlapping between the sets* A *and* A', *then*

1. $\mu_{B'}(v) = 1$ *if* $core(A') \cap (U - A_{\mu_B(v)}) \neq \emptyset$ *and*
2. $\mu_{B'}(v) \geq \mu_B(v)$ *otherwise*

where A_α *denotes the* α*-cut of* A.

Proof. (I1) The case $core(A') \cap (U - A_{\mu_B(v)}) \neq \emptyset$.
On the set $U_1 = \{u \in U / \mu_A(u) \leq \mu_B(v)\}$ we have

$$I_F(\mu_A(u), \mu_B(v)) = 1$$

and therefore

$$\mu_{B'}(v) = \sup_{u \in U_1} \min(\mu_{A'}(u), 1) = \sup_{u \in U_1} \mu_{A'}(u) = 1.$$

(I2) The case $core(A') \cap (U - A_{\mu_B(v)}) = \emptyset$.
For $\mu_B(v) \geq 0.5$, on the set

$$U_2 = \{u \in U / \mu_A(u) > \mu_B(v) \geq 0.5\}$$

we have

$$I_F(\mu_A(u), \mu_B(v)) = \mu_B(v)$$

and

$$\mu_{B'}(v) = \sup_{u \in U_2} \min(\mu_{A'}(u), \mu_B(v)) = \mu_B(v).$$

For $\mu_B(v) < 0.5$, on the set

$$U_3 = \{u \in U / \mu_A(u) > 1 - \mu_B(v) > 0.5\}$$

we obtain the same result, $\mu_{B'}(v) = \mu_B(v)$.
It results that the value of $\mu_{B'}(v)$ is at least $\mu_B(v)$. □

Theorem 5. *If the premise and the observation are contradictory, i.e.,* $\forall u \in U\ \mu_{A'}(u) = 1 - \mu_A(u)$*, then* $\mu_{B'}(v) = 1\ \forall v \in V$.

Proof. On the set $U_1 = \{u \in U / \mu_A(u) \leq \mu_B(v)\}$ we have

$$I_F(\mu_A(u), \mu_B(v)) = 1$$

and therefore

$$\mu_{B'}(v) = \sup_{u \in U_1} \min(\mu_{A'}(u), 1) = \sup_{u \in U_1} \min(1 - \mu_A(u), 1) = 1$$

because there is $u_0 \in U_1$ with $\mu_A(u_0) = 0$. □

The value $\mu_{B'}(v) = 1$ from the Theorems 4 and 5 represents an indeterminate conclusion, all elements v $\in V$ having a possibility equal to 1. The result from the last theorem is valid for every T-norm t.

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Brain-, Gene-, and Quantum-Inspired Computational Intelligence: Challenges and Opportunities

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Summary. This paper discusses opportunities and challenges for the creation of artificial neural network (ANN) and more general – computational intelligence (CI) models inspired by principles at different levels of information processing in the brain – neuronal, genetic, and quantum, and mainly – the issues related to the integration of these principles into more powerful and accurate ANN models. A particular type of ANN, evolving connectionist systems (ECOS), is used to illustrate this approach. ECOS evolve their structure and functionality through continuous learning from data and facilitate data and knowledge integration and knowledge elucidation. ECOS gain inspiration from the evolving processes in the brain. Evolving fuzzy neural networks and evolving spiking neural networks are presented as examples. With more genetic information becoming available now, it becomes possible to integrate the gene and the neuronal information into neurogenetic models and to use them for a better understanding of complex brain processes. Further down in the information-processing hierarchy in the brain, are the quantum processes. Quantum-inspired ANN may help to solve efficiently the hardest computational problems. It may be possible to integrated quantum principles into brain–gene-inspired ANN models for a faster and more accurate modeling. All the topics above are illustrated with some contemporary solutions, but many more open questions and challenges are raised and directions for further research outlined.

Key words: Artificial neural networks, Computational intelligence, Neuroinformatics, Bioinformatics, Evolving connectionist systems, Gene regulatory networks, Computational neurogenetic modeling, Quantum information processing.

1 Introduction: Brain, Gene, and Quantum Levels of Information Processing in the Brain as Inspirations for ANN and CI Models

The brain is an evolving information-processing system that evolves its structure and functionality in time through information processing at different levels (Fig. 1).

At the quantum level, particles (e.g., atoms, electrons, ions, photons, etc.) are in a complex evolving state all the time [35]. The atoms are the material that everything is made of. They can change their characteristics due to the frequency of external signals [11, 22].

At a molecular level, RNA and protein molecules evolve in a cell and interact in a continuous way, based on the stored information in the DNA and on external factors, and affect the functioning of a cell (neuron) under certain conditions (Crick 1970).

At the level of a neuron, the internal information processes and the external stimuli cause the neuron to produce a signal that carries information to be transferred to other neurons.

At the level of neural ensembles, all neurons operate in a "concert," defining the function of the ensemble, for instance perception of sound.

At the level of the whole brain, cognitive processes take place, such as language and reasoning, and global information processes are manifested, such as consciousness.

At the level of a population of individuals, species evolve through evolution changing the genetic DNA code for a better adaptation.

The principles of each of the above processes have inspired the creation of different artificial neural network (ANN) models with the goals of understanding the brain; creating powerful methods and systems of computational intelligence (CI) for solving complex problems in all areas of science and the humanity.

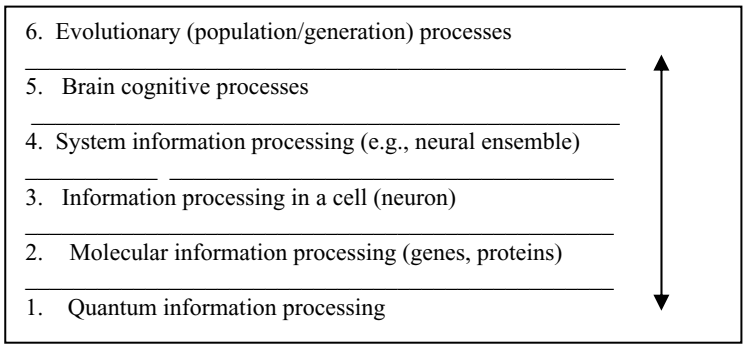

Fig. 1. Levels of information processing in the brain and the interaction between the levels

ANN models, that are brain-inspired (using some principles from the brain) or brain-like (more biologically plausible models, usually developed to model a brain function) have already been proposed (for references, see [2, 3]). Examples are: models of single neurons and neural network ensembles [1, 12, 28, 29, 48, 51, 67, 68, 83] cognitive ANN models [4, 5, 74]. Levine and Aparicio; etc.

The information processes at each level from Fig. 1 are very complex and difficult to understand, but much more difficult is to understand the interaction between the different levels. It may be that understanding the interaction through its modeling would help to understand better each level of information processing in the brain and perhaps the brain as a whole. Using principles from different levels in one ANN and modeling their relationship can lead to a next generation of ANN – more powerful tools to understand the brain and to solve complex problems.

Some examples of ANN that combine principles from different levels in Fig. 1 are:

Computational neurogenetic models (CNGM; [43,53,54] Benuskova 2006);

Quantum-inspired ANN [11, 20, 62, 64, 73];

Evolutionary ANN models [23, 84].

Suggestions are made also that modeling higher cognitive functions and consciousness can be achieved only if the principles of quantum information processing are considered [60, 61].

There are many issues and open questions to be addressed when creating ANN CI models that integrate principles from different levels. Here we will focus on the issues related to a class of ANN models called evolving connectionist systems (ECOS); [39,41]. ECOS are ANN that develop their structure and functionality over time through incremental learning from incoming information and through interaction.

The paper discusses in Sect. 2 two particular models inspired by the principles of evolving neuronal information processes – local learning ECOS and evolving spiking neural networks (SNN). In Sect. 3, the issue of combining neuronal with genetic information processing is discussed and one particular CNGM is presented for illustration, along with a list of open questions. Sect. 4 presents some ideas behind the quantum-inspired ANN models and offers further open questions about the integration of principles from quantum, genetic, and neuronal information processing.

2 Some Brain-Inspired ECOS Models

Many evolving ANN models have been suggested so far, where the structure and the functionality of the models evolve through incremental, continuous learning from incoming data, some times in an on-line mode, and through interaction with other models and the environment. Examples are: growing

neural gas [63], RAN (Platt 1991), cascade-correlation ANN [21], on-line learning ANN [9, 10, 24, 32, 34, 36, 70, 77], FuzzyARTMAP [12], EFuNN [39, 40], DENFIS [42], and others. Here two models are presented only for illustration, mainly due to the author's personal involvement in their development, modification and applications, but more general open questions are raised at the end of this section.

2.1 Local, Knowledge-Based Learning: EFuNN, DENFIS, and TWNFI

Incremental, local learning from a stream of input data and specialization of an ensemble of neurons to perform a certain function as part of a more global goal is a principle of the human brain [5, 25].

Local learning ECOS are connectionist systems that evolve their nodes (neurons) and connections between them through incremental learning from data vectors where the nodes capture local information from the data in a supervised or unsupervised mode [41]. One of the ECOS models, the evolving fuzzy neural network EFuNN [40], is shown in a simplified version in Fig. 2. It consists of five layers: input nodes, representing input variables; fuzzy input nodes, representing the degree to which input values belong to fuzzy membership functions that are used to define concepts such as low value or high value for a variable; rule nodes, representing cluster centers of samples in the problem space and their associated local output functions; fuzzy output nodes, representing membership degrees of the output values to predefined output membership functions; and output nodes that represent output variables. The fuzzy representation nodes are optional.

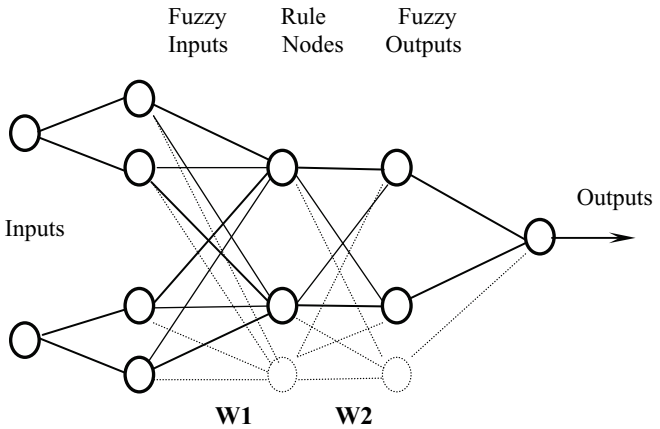

Fig. 2. A simplified version of an evolving fuzzy neural network EFuNN (from Kasabov 2001)

ECOS evolve incrementally rule nodes to represent cluster centers of the input data, where the first layer W1 of connection weights of these nodes represent their coordinates in the input space, and the second layer of connections W2 represents the local models (functions) allocated to each of the clusters.

Data samples are allocated to rule nodes based on their similarity, measured either in the input space – this is the case in some of the ECOS models, e.g., the dynamic neuro-fuzzy inference system DENFIS [42] and the zero instruction set computers – ZISC, or in the input and the output space – this is the case in the evolving fuzzy neural networks EFuNN (Fig. 2). Samples that have a distance to an existing cluster center (rule node) N of less than a threshold Rmax (for the EfuNN models the output vectors of these samples have to be different from the output value associated with this cluster center in not more than an error tolerance E) are allocated in the same cluster Nc. Samples that do not fit into existing clusters form new clusters. Cluster centers are continuously adapted to new data samples, or new cluster centers are created. The distance between samples and nodes can be measured in different ways. The most popular measurement is the normalized Euclidean distance as it is in the self-organized maps SOM [48].

In case of missing values for some of the input variables, a partial normalized Euclidean distance can be used which means that only the existing values for the variables in a current sample S (x,y) are used for the distance measure between this sample and an existing node N

$$\mathrm{d(S,N)} = \mathrm{sqrt}(\Sigma\ \mathrm{i} = 1, .., \mathrm{n}\ (\mathrm{xi} - \mathrm{W1N(i)})2)/\mathrm{n}, \tag{1}$$

for all n input variables xi that have a defined value in the sample S and an already established connection WN(i).

At any time of the EFuNN or DENFIS continuous and incremental learning, rules can be derived from the ANN structure that represent the local functions. Each rule associates a cluster area from the input variable space with a local output function applied to the data in this cluster, e.g.:

IF <data are in cluster Ncj, defined by a cluster center Nj, a cluster radius Rj, and a number of examples Njex in this cluster>

$$\text{THEN <the output function is Fc>} \tag{2}$$

In case of DENFIS, first-order local fuzzy rule models are derived incrementally from data, for example:

IF <the value of x1 is in the area defined by Gaussian membership function with a center at 0.1 and a standard deviation of 0.05, AND the value of x2 is in the area defined by a Gaussian function with parameters (0.25,0.1), respectively>

$$\text{THEN <the output y is calculated by the formula: y} = 0.01{+}0.7\mathrm{x1}{+}0.12\mathrm{x2} > \tag{3}$$

In case of EFuNN, local simple fuzzy rule models are derived, for example:
IF: IF x1 is (Medium 0.8) and x2 is (Low 0.6)

$$\text{THEN y is (High 0.7), radius R} = 0.24; \text{ Nexamp} = 6, \tag{4}$$

where low, medium, and high are fuzzy membership functions defined for the range of each of the variables x1, x2, and y; the number and the type of the membership functions can either be deduced from the data through learning algorithms, or it can be predefined based on human knowledge [14, 83, 85]; R is the radius of the cluster and Nexamp is the number of examples in the cluster.

Further development of the EFuNN and the DENFIS local ECOS models is the transductive weighted neuro-fuzzy inference engine (TWNFI); [72]. In this approach, for every new vector (sample S/example) a "personalized" model is developed from existing nearest samples, where each of the variables is normalized in a different subrange of [0,1] so that they have a different influence on the Euclidean distance, therefore they are ranked in terms of their importance to the output calculated for any new sample individually. Samples are also weighted in the model based on their distance to the new sample, where in the Euclidean distance formula variables are also weighted. Each personalized model can be represented as a rule (or a set of rules) that represents the personalized profile for the new input vector. The TWNFI model is evolving as new data samples, added to a data set, can be used in any further personalized model development. That includes using different sets of variables, features.

2.2 Incremental Feature Selection for ECOS

The brain has the ability to incrementally improve and optimize the set of features while learning continuously to recognize patterns. In many CI problems data samples arrive in chunks and sometimes – new class samples are presented – see for illustration Fig. 3. Inspired by the brain ability to select features incrementally, several methods have been proposed.

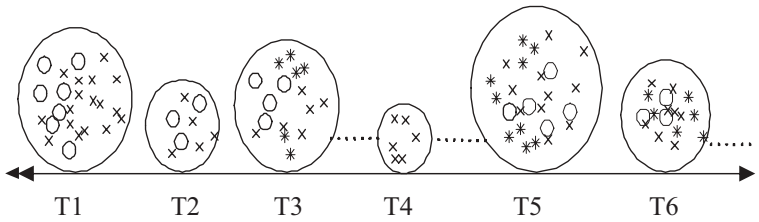

Fig. 3. Incremental presentation of chunks of data over time periods T1, T2,..., having samples of initially two classes (time T1), but introducing at a time T3 a third class samples (from Ozawa et al. 2005)

In [57, 58] a method for incremental PCA learning from a stream of data is presented. In [59] a method for incremental LDA feature selection is proposed. While the structure of an ECOS is evolving incrementally, the set of the input variables (features) in the model can also be evolving, changing over time.

2.3 Evolving SNN

Spiking models of a neuron and neural networks – SNN have been inspired and developed to mimic more biologically the spiking activity of neurons in the brain when processing information [51].

One model – the spike response model (SRM) of a neuron; [27, 52] – is described below and extended in Sect. 3 to a CNGM.

A neuron i receives input spikes from presynaptic neurons $i \in \Gamma_i$, where Γ_i is a pool of all neurons presynaptic to neuron i. The state of the neuron i is described by the state variable $u_i(t)$ that can be interpreted as a total postsynaptic potential (PSP) at the membrane of soma. When $u_i(t)$ reaches the firing threshold $\vartheta_i(t)$, neuron i fires, i.e., emits a spike (Fig. 4, 5). The moment of $\vartheta_i(t)$ crossing defines a firing time t_i of an output spike. The value of the state variable $u_i(t)$ is the sum of all PSPs, i.e.,

$$u_i(t) = \sum_{j \in \Gamma_i} \sum_{t_j \in F_j} J_{ij} \varepsilon_{ij}(t - t_j - \Delta_{ij}^{ax}). \tag{5}$$

The weight of synaptic connection from neuron j to neuron i is denoted by J_{ij}. It takes positive (negative) values for excitatory (inhibitory) connections, respectively. Depending on the sign of J_{ij}, a presynaptic spike generated at time t_j increases (or decreases) $u_i(t)$ by an amount $\varepsilon_{ij}(t - t_j - \Delta_{ij}^{ax})$. Δ_{ij}^{ax} is an axonal delay between neurons i and j which increases with Euclidean distance between neurons.

The positive kernel $\varepsilon_{ij}(t - t_j - \Delta_{ij}^{ax}) = \varepsilon_{ij}(s)$ expresses an individual PSP evoked by a presynaptic neuron j on neuron i. A double exponential formula can be used

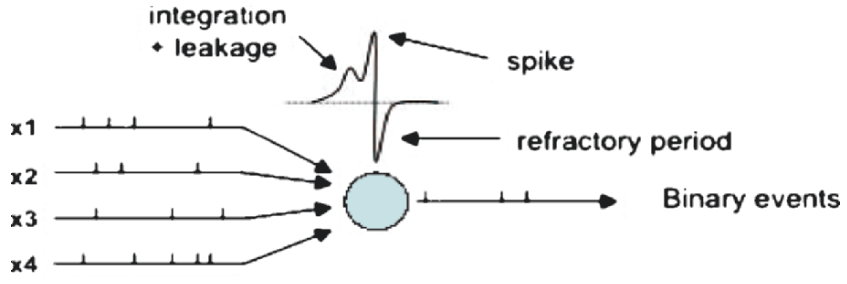

Fig. 4. A general representation of a spiking neuron model (from Kasabov, et al. 2005)

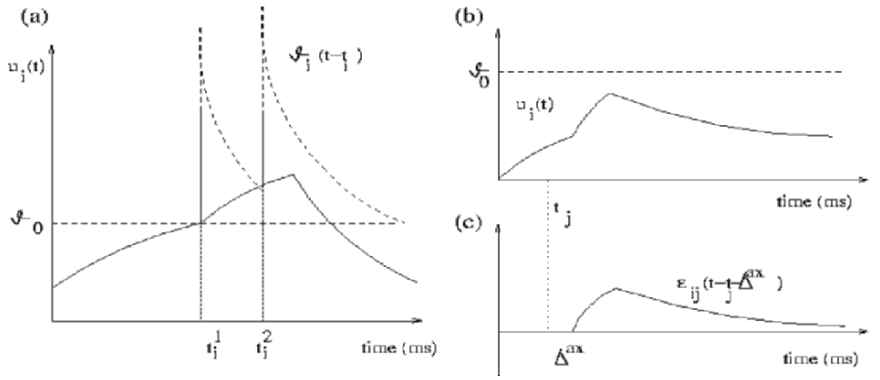

Fig. 5. Spiking behavior of a neuron – the spiking threshold increases after the first spike and then goes back to a normal state [45]

$$\varepsilon_{ij}^{synapse}(s) = A^{synapse}\left(\exp\left(-\frac{s}{\tau_{decay}^{synapse}}\right) - \exp\left(-\frac{s}{\tau_{rise}^{synapse}}\right)\right), \qquad (6)$$

where $\tau_{decay/rise}^{synapse}$ are time constants of the rise and fall of an individual PSP, A is the PSP's amplitude, and synapse = fast_excitation, fast_inhibition, slow_excitation, and slow_inhibition, respectively. These types of PSPs are based on neurobiological data ([19], Semyanov 2002).

Immediately after firing an output spike at t_i, neuron's firing threshold ϑ_i(t) increases m times and then returns to its initial value ϑ_0 in an exponential fashion:

$$\vartheta_i(t - t_i) = m \times \vartheta_0 \exp\left(-\frac{t - t_i}{\tau_{decay}^{\tau}}\right), \qquad (7)$$

where $\vartheta_{decay}^{\vartheta}$ is the time constant of the threshold decay. In such a way, absolute and relative refractory periods are modeled (Fig. 5).

External inputs from the input layer are added at each time step, thus incorporating the background noise and/or the background oscillations. Each external input has its own weight $J_{ik}^{ext\_input}$ and $\varepsilon_k(t)$, such that

$$u_i^{ext\_inpu}(t) = J_{ik}^{ext\_input}\,\varepsilon_{ik}(t), \qquad (8)$$

It is optional to add some degree of Gaussian noise to the right-hand side of the equation above to obtain a stochastic neuron model instead of a deterministic one.

Spiking neurons within a SNN can be either excitatory or inhibitory. Lateral connections between neurons in a SNN may have weights that decrease in value with distance from neuron i for instance according to a Gaussian formula while the connections between neurons themselves can be established at random.

SNN can be used to build biologically plausible models of brain sections as illustrated in, Semyanov (2002), and [19, 43].

In *evolving SNN* new neurons and connections can be created incrementally to accommodate new data samples over time. For example, in [86] a new submodule of several spiking neurons and connections is evolved when a new class of objects (e.g., a new face, in case of face recognition problem) is presented to the system for learning at any time of this process. This work extends the work in [18].

Developing new methods for learning in evolving SNN is a challenging direction for future research with a potential for applications in multimodal information processing (e.g., speech, image, odor, gestures).

2.4 Some Open Questions

Further development of brain-like or brain-inspired ANN requires some questions to be addressed:

- How much should an ANN mimic the brain in order to be an efficient CI model?
- How is a balance between structure definition and learning achieved in ANN?
- How can ANN evolve and optimize their parameters and input features over time in an efficient way?
- How incremental learning in ANN can be achieved without a presentation of an input signal ("sleep" learning)?
- Can ANN have "dreams" and how that can affect their evolving learning and structure?

3 Brain–Gene-Inspired CNGM

3.1 General Notions

With the advancement of molecular and brain research technologies more and more data and information are being made available about the genetic basis of some neuronal functions (see for example: the brain–gene map of mouse at http://alleninstitute.org; the brain–gene ontology BGO at http://www.kedri.info).

This information can be utilized to create biologically plausible ANN models of brain functions and diseases that include models of gene interaction. This area integrates knowledge from computer and information science, brain science, molecular genetics, and it is called here computational neuro-genetic modeling (CNGM; [43]).

Several CNGM models have been developed so far varying from modeling a single gene in a biologically realistic ANN model, to modeling a set of genes

forming an interaction gene regulatory network (GRN; [43,53,54]; Benuskova et al. 2006). In this section we give an example of a CNGM that combines SNN and GRN into one model.

3.2 A CNGM that Integrate GRN within a SNN Model

The main idea behind the model proposed in [43] is that interaction of genes in neurons affect the dynamics of the whole ANN through neuronal parameters, which are no longer constant, but change as a function of gene/protein expression. Through optimization of the GRN, the initial gene/protein expression values, and the ANN parameters, particular target states of the ANN can be achieved, so that the ANN can be tuned to model real brain data in particular.

This idea is illustrated here by means of a simple neurogenetic model of a SNN. The behavior of the SNN is evaluated by means of the local field potential (LFP), thus making it possible to attempt modeling the role of genes in different brain states, where EEG data are available to test the model. A standard FFT signal-processing technique is used to evaluate the SNN output and to compare it with real human EEG data. Broader theoretical and biological background of CNGM construction is given in [43]. A simple linear version of an internal GRN with preliminary results on epilepsy modeling can be found in [7]. In Benuskova et al. (2006) a more realistic nonlinear model of GRN is proposed with a list of real proteins/genes that are involved in CNGM. The model performance is compared to real human EEG data using the same signal-processing technique, where an optimization procedure is proposed to obtain a CNGM with parameters leading to modeling of the real EEG signal.

In general, we consider two sets of genes – a set $\mathrm{G_{gen}}$ that relates to general cell functions, and a set $\mathrm{G_{spec}}$ that defines specific neuronal information-processing functions (receptors, ion channels, etc.). The two sets form together a set $\mathbf{G}=\{G_1, G_2, \ldots, G_n\}$. We assume that the expression level of each gene is a nonlinear function of expression levels of all the genes in $\mathbf{G}$, inspired by discrete models:

$$g_j(t+\Delta t') = \sigma\left(\sum_{k=1}^{n} w_{jk} g_k(t)\right). \tag{9}$$

It is assumed here that: (1) one protein is coded by one gene; (2) relationship between the protein level and the gene expression level is nonlinear; (3) protein levels lie between the minimal and maximal values. Thus, the protein level is expressed by

$$p_j(t+\Delta t) = (p_j^{\max} - p_j^{\min})\sigma\left(\sum_{k=1}^{n} w_{jk} g_k(t)\right) + p_j^{\min}. \tag{10}$$

The delay constant introduced in the formula corresponds to the delay caused by the gene transcription, mRNA translation into proteins and

posttranslational protein modifications, and also the delay caused by gene transcription regulation by transcription factors.

Some proteins and genes are known to be affecting the spiking activity of a neuron represented in a SNN model by neuronal parameters. Some neuronal parameters and their correspondence to particular proteins are summarized in Table 1.

Relevant protein expression levels are directly related to neuronal parameter values P_j such that

$$P_j(t) = P_j(0)p_j(t), \tag{11}$$

where $P_j(0)$ is the initial value of the neuronal parameter at time $t = 0$.

Besides the genes, coding for the proteins mentioned above and directly affecting the spiking dynamics of a neuron, a GRN model can include other genes relevant to a problem in hand, e.g., modeling a brain function or a brain disease. In [7] these genes/proteins are c-jun, mGLuR3, Jerky, BDNF, FGF-2, IGF-I, GALR1, NOS, S100beta.

An example of a CNGM is given in Fig. 6 for the purpose of modeling inputs from the thalamus to the cortex. It uses the SRM [27], with excitation and inhibition having both fast and slow components, both expressed as double exponentials with amplitudes and the rise and decay time constants.

The goal of the CNGM is to achieve a desired SNN output through optimization of the model parameters. The LFP of the SNN, defined as LFP $= (1/\mathrm{N})\Sigma u_i(t)$, by means of FFT is evaluated in order to compare the SNN output with the EEG signal analyzed in the same way. It has been shown that brain LFPs in principle have the same spectral characteristics as EEG [46].

In order to find an optimal GRN within the SNN model, so that the frequency characteristics of the LFP of the SNN model are similar to the

Table 1. Neuronal parameter and their related proteins

Neuronal parameter amplitude and time constants of	Protein
Fast excitation PSP	AMPAR
Slow excitation PSP	NMDAR
Fast inhibition PSP	GABRA
Slow inhibition PSP	GABRB
Firing threshold	SCN, KCN, CLC

Abbreviations: PSP = postsynaptic potential AMPAR = (amino-methylisoxazole-propionic acid) AMPA receptor, NMDAR = (N-methyl-D-aspartate acid) NMDA receptor, GABRA = (gamma-aminobutyric acid) $GABA_A$ receptor, GABRB = $GABA_B$ receptor, SCN = sodium voltage-gated channel, KCN = kalium (potassium) voltage-gated channel, CLC = chloride channel

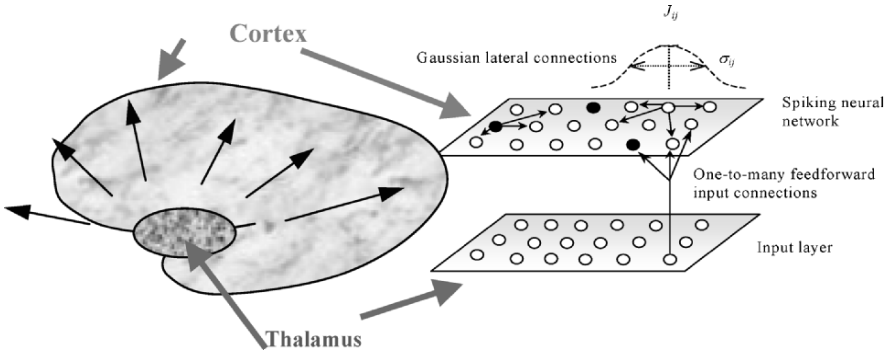

Fig. 6. An example of an SNN model used in a CNGM. About 10 to 20% of $N = 120$ neurons are inhibitory neurons that are randomly positioned on the grid (**filled circles**). External input is random with a defined average frequency (e.g., between 10 and 20 Hz; from Benuskova et al. 2006)

brain EEG characteristics, the following evolutionary computation procedure is used:

1. Generate a population of CNGMs, each having randomly generated values of coefficients for the GRN matrix W, initial gene expression values $g(0)$, initial values of SNN parameters $P(0)$, and different connectivity.
2. Run each SNN over a period of time T and record the LFP.
3. Calculate the spectral characteristics of the LFP using FFT.
4. Compare the spectral characteristics of SNN LFP to the characteristics of the target EEG signal. Evaluate the closeness of the LFP signal for each SNN to the target EEG signal characteristics. Proceed further according to the standard GA algorithm to find a SNN model that matches the EEG spectral characteristics better than previous solutions.
5. Repeat steps 1–4 until the desired GRN and SNN model behavior is obtained.
6. Analyze the GRN and the SNN parameters for significant gene patterns that ause the SNN model to manifest similar spectral characteristics as the real data.

In [47] some preliminary results of analysis performed on real human interictal EEG data are presented. The model performance and the real EEG data are compared for the following relevant to the problem subbands: delta (0.5–3.5 Hz), theta (3.5–7.5 Hz), alpha (7.5–12.5 Hz), beta 1 (12.5–18 Hz), beta 2 (18–30 Hz), gamma (above 30 Hz). This particular SNN had an evolved GRN with only five genes out of 16 (s100beta, GABRB, GABRA, mGLuR3, c-jun) and all other genes having constant expression values. A GRN is obtained that has a meaningful interpretation and can be used to model what will happen if a gene/protein is suppressed by administering a drug, for example.

In *evolving CNGM* new genes can be added to the GRN model at a certain time, in addition to the new spiking neurons and connections created

incrementally, as in the *evolving SNN*. Developing new evolving CNGM to model brain functions and brain diseases, such as epilepsy, Alzheimer, Parkinson disease, Schizophrenia, mental retardation, and others, is a challenging problem for a future research.

There are some technical questions that emerged from the first CNGM experiments, such as:

1. How many different GRNs would lead to similar LFPs and what do they have in common?
2. What neuronal parameters to include in the ANN model and how to link them to activities of genes/proteins?
3. What genes/proteins to include in the model and how to represent the gene interaction over time within each neuron?
4. How to integrate in time the output activity of the ANN and the genes as it is known that neurons spike in millisecond intervals and the process of gene transcription and translation into proteins takes minutes?
5. How to create and validate a CNG model in a situation of insufficient data?
6. How to measure brain activity and the CNGM activity in order to validate the model?
7. What useful information (knowledge) can be derived from a CNG model?
8. How to adapt incrementally a CNGM model in a situation of new incoming data about brain functions and genes related to them?

3.3 Open Questions

Integrating principles from gene and neuronal information processing in a single ANN model raises many general questions that need to be addressed in the future, for example:

1. Is it possible to create a truly adequate CNGM of the whole brain? Would gene–brain maps help in this respect (see http://alleninstitute.org)?
2. How can dynamic CNGM be used to trace over time and predict the progression of a brain diseases, such as epilepsy and Parkinson's?
3. How to use CNGM to model gene mutation effects?
4. How to use CNGM to predict drug effects?
5. How CNGM can help to understand better brain functions, such as memory and learning?
6. What problems of CI can be efficiently solved with the use of a brain–gene-inspired ANN?

4 Quantum-Inspired Evolving Connectionist Models

4.1 Why Quantum-Inspired Models and Systems?

Quantum computation is based upon physical principles from the theory of quantum mechanics [22].

One of the basic principles is the *linear superposition* of states. At a macroscopic or classical level a system exists only in a single basis state as energy, momentum, position, spin, and etc. However, at microscopic or quantum level a quantum particle (e.g., atom, electron, positron, ion) or a quantum system is in a superposition of all possible basis states. At the microscopic level any particle can assume different positions in the same time, can have different values of energy, can have many values of the spins, and etc. This *superposition* principle is counterintuitive because in the classical physics one particle has only one position, energy, spin, and etc.

If a quantum system interacts in any way with its environment, the superposition is destroyed and the system *collapses* into one single real state as in the classical physics (Heisenberg). This process is governed by a probability amplitude. The square of the intensity for the probability amplitude is the quantum probability to observe the state.

Another quantum mechanics principle is the *entanglement* – two or more particles, regardless of their location, are in the same state with the same probability function. The two particles can be viewed as "correlated," undistinguishable, "synchronized," coherent. An example is a laser beam consisting of millions of photons having same characteristics and states.

Quantum systems are described by a probability density ψ that exists in a Hilbert space. The Hilbert space has a set of states $|\phi\mathrm{i}\rangle$ forming a basis. A system can exist in a certain quantum state $|\psi\rangle$ which is defined as

$$|\psi\rangle = \sum ci|\phi i\rangle, \quad |a|^2 + |b|^2 = 1, \tag{12}$$

where the coefficients ci may be complex. $|\psi\rangle$ is said to be in a superposition of the basis states $|\phi\mathrm{i}\rangle$. For example the quantum-inspired analogue of a single bit in classical computers can be represented as a qubit in a quantum computer

$$|x\rangle = a|0\rangle + b|1\rangle, \tag{13}$$

where $|0\rangle$ and $|1\rangle$ represent the states 0 and 1. The qubit is not a single value entity, but it is a function of parameters which values are complex numbers. After the loss of coherence the qubit will collapse into one of the states $|0\rangle$ or $|1\rangle$ with the probability a^2 for the state $|0\rangle$ and the probability b^2 for the state $|1\rangle$. So, in quantum mechanics and in any scientific domain, where we use the superposition, the introduction of the qubit to measure information states change radically any interpretation of the information processes and also of any computation.

The state of a qubit can be changed by an operation called a *quantum gate*. A quantum gate is a reversible gate and can be represented as a unitary operator U acting on the qubit basis states. The defining property of an unitary matrix is that its conjugate transpose is equal to its inverse. There are several quantum gates already introduced, such as the NOT gate, controlled NOT gate, rotation gate, Hadamard gate, etc. [15, 62].

Quantum mechanical computers and quantum algorithms try to exploit the massive quantum parallelism which is expressed in the principle of *superposition.* The principle of superposition can be applied to many existing methods of CI, where instead of a single state (e.g., a parameter value, or a finite automata state, or a connection weight, etc.) a superposition of states will be used, described by a wave probability function, so that all these states will be computed in parallel increasing the speed of computation by many orders of magnitude.

Quantum mechanical computers have been proposed in the early 1980s and a description was formalized in the late 1980s (Benioff 1980). This kind of computers proved to be superior to classical computers in various specialized problems. Many efforts were undertaken to extend the principal ideas of quantum mechanics to other fields of interest. There are well-known quantum algorithms such as Shor's quantum factoring algorithm [71] and Grover's database search algorithm [30]. Hogg extended the work of Grover in order to demonstrate the application of quantum algorithms in the context of combinatorial search [37]

The advantage of quantum computing is that, while a system is uncollapsed, it can carry out more computing than a collapsed system, because, in a sense, it is computing in many universes at once. The above quantum principles have inspired research in both computational methods and brain study.

It is widely accepted now that NP-hard problems (e.g., time complexity grows exponentially with the size of the problem) can be solved by a quantum computer. Penrose [61] argues that solving the quantum measurement problem is prerequisite for understanding the mind as consciousness emerges as a macroscopic quantum state due to a coherence of quantum-level events within neurons.

4.2 Quantum-Inspired Evolutionary and Connectionist Models

Quantum-inspired methods of evolutionary computation (QIEC) have already been discussed in Han and Kim (2002), Jang et al. (2003), that include: genetic programming (Spector 2004), particle swarm optimizers (Liu et al. 2005), finite automata, and Turing machines (Benioff, 1980). In QIEC, the population of Q-bit individuals at time t can be represented as

$$Q(t) = \{q_1^t, q_2^t, \ldots, q_n^t\}, \tag{14}$$

where n is the size of the population.

Evolutionary computing with Q-bit representation has a better characteristic of population diversity than other representations, since it can represent linear superposition of states probabilistically. The Q-bit representation leads to a quantum parallelism in the system as it is able to evaluate the function on a superposition of possible inputs. The output obtained is also in the form of superposition which needs to be collapsed to get the actual solution.

Recent research activities focus on using quantum principles for ANN [20, 56, 65, 66, 78, 80]. Considering quantum ANN seems to be important for at least two reasons. There is evidence for the essential role that quantum processes may play in realizing information processing in the living brain. [60] argued that a new physics binding quantum phenomena with general relativity can explain such mental abilities as *understanding*, *awareness*, and *consciousness* [11]. The second motivation is the possibility that the field of classical ANN could be generalized to the promising new field of quantum computation (Brooks 1999). Both considerations suggest a new understanding of mind and brain function as well as new unprecedented abilities in information processing. Ezhov and Ventura [20] are considering the quantum neural networks as the next natural step in the evolution of neurocomputing systems. Several quantum-inspired ANN models have been proposed and illustrated on small examples. In [78] a QIEA is used to train a MLP ANN.

Naraynan and Meneer [56] simulated classical and various types of quantum-inspired ANN and compared their performance. Their work suggests that there are indeed certain types of problems for which quantum neural networks will prove superior to classical ones.

Other relevant work includes quantum decision making [17], quantum learning models [49], quantum networks for signal recognition (Tsai et al. 2005), and quantum associative memory [75, 79]. There are also recent approaches to quantum competitive learning where the quantum system's potential for excellent performance is demonstrated on real-world data sets [80, 82].

The quantum-inspired neural network (QUINN) proposed by Narayanan and Meneer (2000) interprets each input pattern Sp (p = 1,2,. . . ,k) as a particle, being learned in a separate NNp model in a separate universe Up, the superposition of all ANN constituting the ANN model. The structure of all ANN is the same, so that a connection weight between neuron Ni and neuron Nj in the total model is a superposition of all connection weights Wij (k) of all k ANNs. When an input pattern S is presented, the ANN model "collapses" into a particular NN–S that recognizes this pattern. Each pattern needs to be presented only once in order an NN model to be created for this pattern and become part of the superposition of all NN models.

In *evolving* quantum-inspired ANN, presenting a new pattern S_{k+1} (a new particle) would cause the creation of a new ANN model that becomes part of the superposition of connection weights and states of the whole system.

Quantum-inspired SNN would have a smaller number of neurons and a much larger number of states due to the superposition principle. A challenge would be to represent the spikes as superposition of trains of signals across many QI-SNN.

4.3 Quantum-Inspired CNGM: Some Preliminary Thoughts

QI-CNGM would open new possibilities for modeling gene–neuron interactions. In Sect. 3 a CNGM was presented that combines principles of information processing in gene/protein molecules with neuronal spiking activity, and

then – to the information processing of a neuronal ensemble that is measured as LFPs. How the quantum information processes in the atoms and particles (ions, electrons, etc.) that make the large gene/protein molecules, relate to the spiking activity of a neuron and to the activity of a neuronal ensemble, is not known yet and it is a challenging question for the future.

What is known at present, is that the spiking activity of a neuron relates to the transmission of thousands of ions and neurotransmitter molecules across the synaptic clefts and to the emission of spikes. Spikes, as carriers of information, are electrical signals made of particles that are emitted in one neuron and transmitted along the nerves to many other neurons. These particles are characterized by their quantum properties. So, quantum properties may influence, under certain conditions, the spiking activity of neurons and of the whole brain, as brains obey the laws of quantum mechanics as everything else does.

Similarly to a chemical effect of a drug to the protein and gene expression levels in the brain, that may affect the spiking activity and the functioning of the whole brain (modeling of these effects is subject of the computational neurogenetic modeling CNGM), external factors like radiation, high frequency signals, etc. may influence the quantum properties of the particles in the brain through gate operators. According to Penrose [61], icrotubules in the neurons are associated with quantum gates.

So, the question is: *Is it possible to create a CNGM that incorporates some quantum principles – a QI-CNGM?*

We can represent the above problem as a set of hypothetical functions as follows. A future state Q' of a particle or a group of particles (e.g., ions, electrons, etc.) depends on the current state Q and on the frequency spectrum Eq of an external signal, according to the Max Planck constant:

$$\mathrm{Q}' = \mathrm{Fq}(\mathrm{Q}, \mathrm{Eq}). \tag{15}$$

A future state of a molecule M' or a group of molecules (e.g., genes, proteins) depends on its current state M, on the quantum state Q of the particles, and on an external signal Em:

$$\mathrm{M}' = \mathrm{Fm}(\mathrm{Q}, \mathrm{M}, \mathrm{Em}). \tag{16}$$

A future state N' of a spiking neuron, or an ensemble of neurons will depend on its current state N, on the state of the molecules M, on the state of the particles Q, and on external signals En

$$\mathrm{N}' = \mathrm{Fn}(\mathrm{N}, \mathrm{M}, \mathrm{Q}, \mathrm{En}). \tag{17}$$

A future cognitive state C' of the brain will depend on its current state C and also on the neuronal – N, on the molecular – M, and on the quantum – Q states of the brain:

$$\mathrm{C}' = \mathrm{Fc}(\mathrm{C}, \mathrm{N}, \mathrm{M}, \mathrm{Q}, \mathrm{Ec}). \tag{18}$$

Some support for the above hypothetical model of integrated function representations comes from the following [5, 25, 60, 61, 74]

1. A large amount of atoms are characterized by the same quantum properties, possibly related to the same gene/protein expression profile of a large amount of neurons characterized by spiking activity;
2. A large neuronal ensemble can be represented by a single LFP;
3. A cognitive process can be represented perhaps as a complex function Fc that depends on all previous levels.

4.4 Some Open Questions

Several reasons can be given in support to the research in integrating principles from quantum, molecular, brain information processing into future ANN model.

1. This would lead to a better understanding of both molecular and quantum information processing;
2. Modeling molecular processes are needed for progress in many areas of biology, chemistry, and physics;
3. At the nano-level of microelectronic devices, quantum processes may have a significant impact;
4. Using these processes as inspiration for new computer devices – million times faster and more accurate

Many open questions need to be answered in this respect. Some of them are listed below:

How quantum processes affect the functioning of a living system in general?

1. How quantum processes affect cognitive and mental functions?
2. Is it true that the brain is a quantum machine – working in a probabilistic space with many states (e.g., thoughts) being in a superposition all the time and only when we formulate our thought through speech or writing, then the brain "collapses" in a single state?
3. Is fast pattern recognition in the brain, involving far away segments, a result of both parallel spike transmissions and particle entanglement?
4. Is communication between people and between living organisms in general, is a result of entanglement processes?
5. How does the energy in the atoms relates to the energy of the proteins, the cells and the whole brain?
6. Would it be beneficial to develop different quantum-inspired (QI) computational intelligence techniques, such as: QI-SVM, QI-GA, QI-decision trees, QI-logistic regression, QI-cellular automata, QI-ALife?
7. How do we implement the QI computational intelligence algorithms in order to benefit from their high speed and accuracy? Should we wait

for the quantum computers to be realized many years from now, or we can implement them efficiently on specialized computing devices based on classical principles of physics?

5 Conclusions and Directions for Further Research

This paper presents some CI methods and in particular – evolving ANN models, inspired by principles from different levels of information processing in the brain – including higher cognitive level, gene/protein level, and quantum level, and argues that ANN models that integrate principles from different levels of information processing would be beneficial for a better understanding of brain functions and for the creation of more powerful methods and systems of computational intelligence in general.

Further directions in this research are:

1. Building large ontology systems that integrate facts, information, ANN models, and other CI models of the three levels of information processing in the brain and their interaction, such as brain–gene–quantum ontology systems;
2. Building novel brain-, gene-, and quantum-inspired ANN and CI models, such as: new ECOS, evolving SNN, evolving CNGM, QI-CNGM, QI-SVM, etc.;
3. Studying the characteristics of the above models and interpreting the results;
4. Applying the new methods to solving complex problems in neuroinformatics, such as modeling learning and memory, understanding brain diseases, etc.;
5. Applying the new methods to solve complex problems in bioinformatics, such as selecting dynamically genes and proteins related to cancer, modeling cellular processes, modeling GRNs and metabolic pathways;
6. Applying the new methods for multimodal information processing, biometric tasks, robotics, and other practical tasks of computational intelligence.

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Incremental Learning for E-mail Classification

Sigita Misina

Summary. Machine learning algorithms can be divided into two categories – statistic learning and incremental learning algorithms. Incremental learning training set examples distribute over time, maintaining examples one by one. This paper describes partial instance memory incremental learning algorithm *FLORA2* - algorithm for supervised learning of concepts that are subject to concept drift. The algorithm *FLORA2* uses a special feature for keeping a subset of examples in memory – a window. In length of time new examples are being added to the window while others are considered outdated and are forgotten. In this paper incremental learning algorithm *FLORA2* with dynamic window size is applied in e-mail classification task. An e-mail classification task is the junk e-mail problem solving task. Active e-mail user receives thousands of messages each month. Separating the good messages from the junk using artificial intelligence algorithm decreases the time what users spending for managing their inboxes.

Key words: Incremental learning, Window, *FLORA2*.

1 Introduction

Most of earlier machine learning algorithms are conservative. They assume the object class is stable and does not change with time. But in many real-world applications, the object concepts may change. For example, in financial manipulations, in medical diagnosis, in text classification in accordance with their importance, the object concept does not stay stable. This paper reports an incremental learning for object classification in cases of changeable contexts. In incremental learning, training examples are classified one by one. Incremental learning systems must have a memory model that dictates how to treat the previous training examples. There are three possibilities [1]:

– *Full instance memory*, in which the learner retains all previous training examples.

- *Partial instance memory*, in which only some of the previous training examples are retained.
- *No instance memory*, in which none of the previous examples are retained.

This paper describes partial instance memory incremental learning algorithm *FLORA2* [2]. Partial instance memory means that part of the instances are kept in memory because effective learning in environments with hidden context and class drift demand the algorithm that can detect context changes without explicit information, regenerate after context changes and use previous experience in situations where old contexts recur.

The paper provides practical experiments with the *FLORA2* algorithm that consider e-mail messages classification task. Experiments are done to study working principles of the *FLORA2* algorithm, to compare algorithm *FLORA2* results with the results of previous researches where multilayer incremental inference algorithm (*MLII*) [3] was used and results the gained by interface agent in e-mail messages filtering task.

2 Incremental Learning

There are two kind of machine learning algorithms – static learning and incremental learning algorithms. An algorithm for incremental learning can be executed in five steps [1]:

1. Learn rules from training example set;
2. Store new rules and discard the oldest examples;
3. Use the rules learned to predict and navigate;
4. When new examples arrive, learn new rules using old rules and new instances;
5. Go on to step 2.

As the context in incremental learning varies in time, the learner trusts newest examples more – this example set is called the *window* [4]. As time passes new examples are being added to the window while the others are considered outdated and are forgotten (deleted from it).

In simple cases, the window will be of fixed size, and the oldest example will be dropped whenever a new one comes in (see Fig. 1). For a window of the fixed size, the choice of "good" window size is a compromise between fast adaptability (small window) and good and stable learning results without or with little concept change (large window). An adaptive size window is most effective. The dynamic window size is calculated heuristically to avoid concept drift [4].

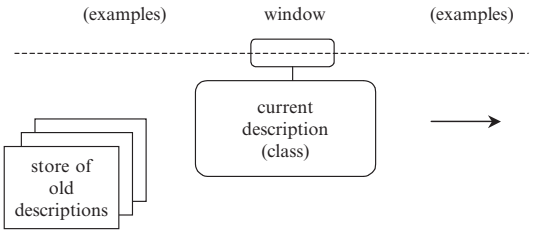

Fig. 1. The window moving along the stream of examples

3 Partial Memory Incremental Learning Algorithm FLORA2

FLORA2 is the algorithm for supervised learning of concepts that are subject to concept drift [2]. A typical use of that algorithm is for modeling the aging of knowledge. The learning process is incremental in that the examples are processed one by one. There is a special feature in algorithm *FLORA2* for keeping in memory a subset of examples – a *window*. In the simplest case [2], this means that every time when a new example appears, it is added to the window (*learn*) and one of the oldest examples will be deleted from the window (*forget*). In line with concept drift, the system keeps in memory not only valid descriptions of the concepts derived from the objects currently present in the window, but also "candidate descriptions" [4] that may turn into valid ones in the future.

The basic idea of algorithm *FLORA2* is to keep in memory all possible concept descriptions. Let us forget, for the moment, about a combinatorial "explosion" that could later be avoided by suitable heuristics. In *FLORA2* the following description sets are used [2]:

- *ADES* set, that have been derived from the positive examples and have not been connected with any negative examples.
- *PDES* – set of "candidate descriptions" that cover both positive and negative examples and which is worth supporting.
- *NDES* set of descriptions that cover only negative examples from all the example sets.

Since the possibility of a concept drift is assumed, we have more trust in more recent examples, than in older ones. *FLORA* therefore takes into account only a subset of examples called "window". The window thus moves along the stream of examples (see Fig. 1) and the changes in its contents induce changes in those of *ADES, PDES,* and *NDES.* The window size is calculated by size adjustment heuristics (see Table 1):

Table 1. Window size adjustment heuristics

Let N = examples (positive) count covered by *ADES* and S = *ADES* size
IF $N/S < 1.2$ (coverage is low) and *PDES* is not empty (exists alternative hypothesis) THEN decrease window size on 10% and forget oldest examples ELSE IF $N/S > 5$ (coverage is high), THEN keep the window size, ELSE increase window size on 1 (add new example without old one forgetting).

3.1 Incremental Update of Description Sets

Let us assume that description sets *ADES*, *PDES*, and *NDES* already exist (in the beginning, they might be empty). *ADES* is a set of all descriptions that are consistent (they match only positive examples). *PDES* is a set of candidate descriptions that if taken together are complete, but not consistent (it matches all positive examples, but also some negative ones). In its turn *NDES* is a consistent description of the negative instances seen so far (it matches no positive instances).

Each description set can be interpreted as a disjunctive normal form (*DNF*) expression. *DNF* allows detecting if an expression is or is not inconsistent. *DNF* is disjunction of conjunctions. Members of those conjunctions are positional variables (with negation or without it).

Thus the three description sets [4] have the following form:

$$ADES = \{ADes_1/AP_1, ADes_2/AP_2, \ldots\}, \tag{1}$$

$$PDES = \{PDes_1/PP_1/PN_1, PDes_2/PP_2/PN_2, \ldots\}, \tag{2}$$

$$NDES = \{NDes_1/NN_1, NDes_2/NN_2, \ldots\}, \tag{3}$$

where

$ADes_i\,(PDes_i,\ NDes_i)-$ description items (conjuncts or descriptors),

$\{A,P\}P_i\ldots$ the number of positive examples matching description $\{A,P\}Des_i$,

$\{P,N\}N_i\ldots$ the number of negative examples matching description $\{P,N\}Des_i$.

An important thing is that the system keeps counts of the number of instances matched. These numbers concern only instances that are in the current window. They are used to decide when to move a description to a different description set or when to drop it altogether.

These description sets are stored in minimal power, in order to prevent a combinatorial "explosion". In *FLORA2* this is achieved by exploiting the subsumption ordering in the description space to see that $ADES$ contains only

the most general descriptions consistent with positive examples (if two description items $ADes_i$ and $ADes_j$ both are consistent with positive instances and $ADes_i$ subsumes $ADes_j$,only $ADes_i$ is kept in $ADES$). These conditions are checked whenever one of description sets is modified. Instances transactions among the description sets are represented in Fig. 2.

3.2 Incremental Learning

Let us assume that a new training instance is presented to the system, with given classification $C \in \{positive,\ negative\}$. If the classification C is positive, all three description sets are updated (see Table 2).

Corrections done in description sets if classification C is negative are similar, only then description $ADes_i$ is deleted from description set $ADES$, if a new negative example matches that description [2].

3.3 Incremental Forgetting

Let us assume that the system decides "deliberately" to forget old training examples with known classification C. This happens when an old example

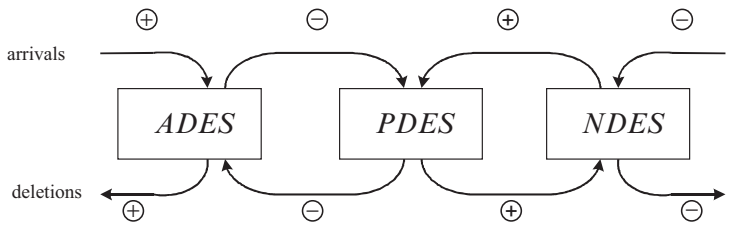

Fig. 2. Transactions among the description sets

Table 2. Incremental learning for a positive instance

For all $ADes_i$ in $ADES$:
if match (instance, $ADes_i$), then $AP_i := AP_i + 1$
For all $PDes_i$ in $PDES$:
if match (instance, $PDes_i$), then $PP_i := PP_i + 1$
For all $NDes_i$ in $NDES$:
if match instance $Ndes_i$, then remove $NDes_i$ from $NDES$ and include it in $PDES$ as triple $NDes_i/1/NN_i$ and check the updated $PDES$ for subsumptions;
If there is no $ADes_i$ in $ADES$ that matches the new instance; then include in $ADES$ all possible generalizations of the new positive instance with all $ADes_j$ present in $ADES$, such that the resulting expressions are maximally general and do not subsume any descriptions in $PDES$ or $NDES$; as an extreme case, the description of the instance itself may be added to $ADES$; then check $ADES$ for subsumptions.

Table 3. Incremental forgetting for a positive instance

For all $ADes_i$ in $ADES$:
if match (instance, $ADes_i$), then $AP_i := AP_i - 1$;
if $AP_i := 0$, then remove $ADes_i$ from $ADES$;
For all $PDes_i$ in $PDES$:
if match (instance, $PDes_i$) then $PP_i := PP_i - 1$;
if $PP_i := 0$ then remove $PDes_i$ from $PDES$ and include it in $NDES$ as a pair $PDes_i/PN_i$ and check the updated $NDES$ for subsumptions.

is dropped from the current window. If the forgotten old example is positive, the description sets are updated [2] by the algorithm demonstrated in Table 3.

If an example that is forgotten matches the negative class, description sets *PDES* and *NDES* are updated in the same way.

4 Experiments

Three experiments based on e-mail messages classification task are performed to observe *FLORA2* working principles and compare the results with previous task – the *MLII – Multi-Layer Incremental Learning* [3] algorithm based on the *CN2* algorithm [6].

Basic data set consists of e-mail correspondence from mail server about usage, restrictions and specific functions of application *Lotus Notes*. These electronic messages have been described with four attributes: *From*; *Subject*; *Body*; *Category* (message category, one of the following: question, answer, suggestion or information) and *Classes* (action: "forward" or "delete" new incoming message).

As the e-mail messages attributes *Subject* and *Body* contain sufficient amount of words and lot of them are not significant, it is necessary to process them using some method. To get most common occurred words from *Subject* and *Body*, the *Levenshtein Distance* [7] algorithm is used. Instead of simple frequency count *Levenshtein Distance* algorithm save time and prevent human factor error.

A software *levenstain.jsp* that is based on *Levenshtein Distance* algorithm is utilized written in *Java* language [7] and adapted for this particular task solution. From field *Subject* three words with smallest distance (distance is a measure of the similarity between two strings – smaller the distance, the more closer word are) are chosen, but from *Body* – four words are chosen.

The learning examples are constructed by combining one word from each e-mail description field (Fig. 3 shows an example). After processing basic 40 e-mail messages, 410 examples are gained, 160 of them belonging to class "delete" and 250 belonging to class "forward".

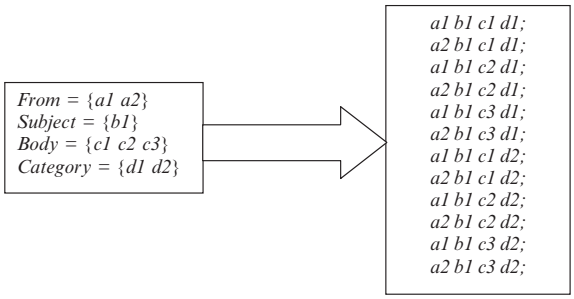

Fig. 3. Learning examples construction mechanism

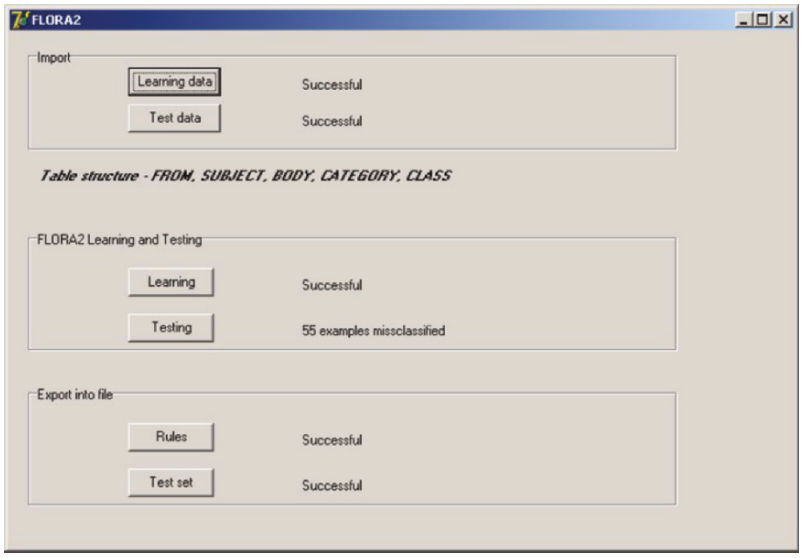

Fig. 4. FLORA2 learning and testing software user interface

4.1 Results of experiments

The learning set examples are randomly shuffled and three experiments proceed to see *FLORA2* working principles.

The initial learning data set size is chosen as 30 examples and the size of conjunctions set as 2, the incremental learning and incremental forgetting done using dynamic window size calculation. Learning and testing processes are done using software written in *InterBase* in *IBManager 3* in the form of stored procedures, while user interface (see Fig. 4) is written in *Delphi* version 7.0.

Table 4. Summary of experiments

	Experiment 1	Experiment 2	Experiment 3
Number of training examples	266	290	220
Number of test examples	144	123	25
Correct classification	89	74	19
Misclassifications	55	49	7
No of rules	446	449	548
Accuracy of rules	61,81%	60,16%	72%

The final rule set is tested on test set examples by supervisor (in this time without subsumption process), the summary of experiments and test results – misclassifications are provided in Table 4.

4.2 Analysis of Experiments

As a result of FLORA2 learning process, the rule set is generated; the size of rule sets in all three experiments is impressive. The reason for such rule count could be the FLORA2 possibility to store "candidate" descriptions, which can be useful in future learning or subsumption necessity.

As compared to previous e-mail filtering task practical solution by *MLII* [5] algorithm, *MLII* shows better results (the accuracy is between 75% and 93%) and provides smaller count of misclassified examples, but the *FLORA2* algorithm gives much more larger rule sets, where in case of *MLII* rule sets size is between 6 and 49.

5 Conclusions

In conclusion it can be stated that, the strength of the incremental learning algorithm *FLORA2* lays in the explicit representation of the three description sets *ADES, PDES* and *NDES*. These three sets together see summarize the important information in the training examples. There is no need to re-examine all the instances at every learning step. Once the learning process is well on the way, there is little need for the construction of new descriptions. Most of the action is migration of descriptions between the three sets. All this contributes significantly to the efficiency of the algorithm.

To research the *FLORA2* incremental learning and forgetting principles, a practical e-mail classification task is performed. Three experiments of supervised inductive learning and testing are done. It can be concluded that *FLORA2* generates a large amount rules in the form of conjunctions, also the "candidate" descriptions, which belong to both classes and could be useful for further learning process. Comparing with previous experiments [5], it can be concluded that in e-mail classification task algorithm *MLII* shows better results in testing process, but *FLORA2* generates more rules.

There is the description subsumption usage and the next *FLORA* family algorithms *FLORA3, FLORA4* left for further experiments.

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Reduction of Search Space for Instance-Based Classifier Combination

Anatoly Sukov and Arkady Borisov

Summary. To combine classifiers on the basis of the local distribution of performance quality of separate algorithms, searching for the nearest neighbours of the record under classification is necessary. This study proposes to reduce the space of the mentioned search by using the methodology of data condensation. An algorithm is developed that enables one to perform searching in two stages. At the first stage only a small and the most representative part of the data, the condensed set, is examined. At the second stage, clarification of neighbours takes place in the local part of the complete data set. Experimental results confirm the hypothesis about a higher accuracy level of classifier combination as compared to separate models.

Key words: Classifier combination, Data condensation, Data mining.

1 Introduction

In solving classification tasks, one of practical problems of data mining [1, 2] consists in choosing an optimal classification algorithm, or classifier. Instead of choosing a particular classifier, generation of classifier ensemble [3] or their combination [4, 5] is often possible. The first approach employs the so-called base classifier, trains it a certain number of times and shapes a final hypothesis as a result of ensemble voting [3]. The well-known classification performance improvement algorithm AdaBoost [6] exemplifies an approach of that kind. In its turn, the second approach can assume several different by nature classifiers as a basis and combine them thus adapting itself to the local performance quality distribution of those algorithms [4]. Practical application of classifier ensembles and classifications confirms an idea that hypothesis voting might describe the real function separating the classes more precisely [3].

This paper deals with an approach that is aimed at a more effective application of classifier combination. With that end in view, the solving of two basic problems related to local evaluation of classifiers is suggested [5]. Since the methodology is based on finding nearest neighbours of a new object, this

is limitations of that algorithm that present the essence of the aforementioned difficulties. First, a search of that kind might require significant increase in computational resources, when the amount of the data available for learning increases. Second, the application of k-nearest neighbours is only adequate when relevant metric for the evaluation of the distance between the objects (records), is available.

Previously, the problem of lessening computational efforts for instance-based learning was solved by reducing the number of instances used for learning, or by changing data representation [7]. The approach employed in this paper is based on the data pre-processing called data condensation [8]. Such a pre-processing enables one to operate first with the reduced set of data and then with the necessary part of the total set, which as a whole reduces calculations aimed to find the nearest neighbours. The application of data condensation to the reduction of search space is considered in detail in Sect. 2 of this paper.

To raise the adequacy of the metric employed, it is proposed to use those distance evaluation functions in classifier combination which are effective for specific data kind. Principles of instance-based classifier combination as well as the used distance evaluation metric are described in Sect. 3.

One of application examples of the adaptive combination of classifiers is the task of credit scoring [9]. In the course of the study, a number of experiments aimed at combining classifiers on the basis of the suggested methodology, were conducted. The experiments performed are discussed in Sect. 4, which is followed by Conclusion.

2 Reduction of Search Space via Data Condensation

Data condensation means initial data set based generation of a set containing only those data which are most representative under the given number of neighbours, k (control parameter). The representativeness manifests itself in that the chosen object is able to cover the largest number of the remaining objects in some radius (depending on k and local data distribution density). Thus, high-density regions will be represented by a larger number of objects whereas sparser regions will have significantly less representatives [8].

After the pre-processing, neighbour searching for the new object is suggested first in the condensed set (Stage 1 of the search), which differs from the original set in sufficiently smaller number of records (the ratio is called condensation ratio, CR), and then in the original set. It should be noted, however, that only that part of the data from the original set is used, which – according to condensation results – is covered by the objects found at Stage 1. Since the number of the data in the condensed set, NCR, is considerably less than that in the original one even at small values of k [8], owing to the first stage, a quick localisation of the area of searched neighbours occurs which

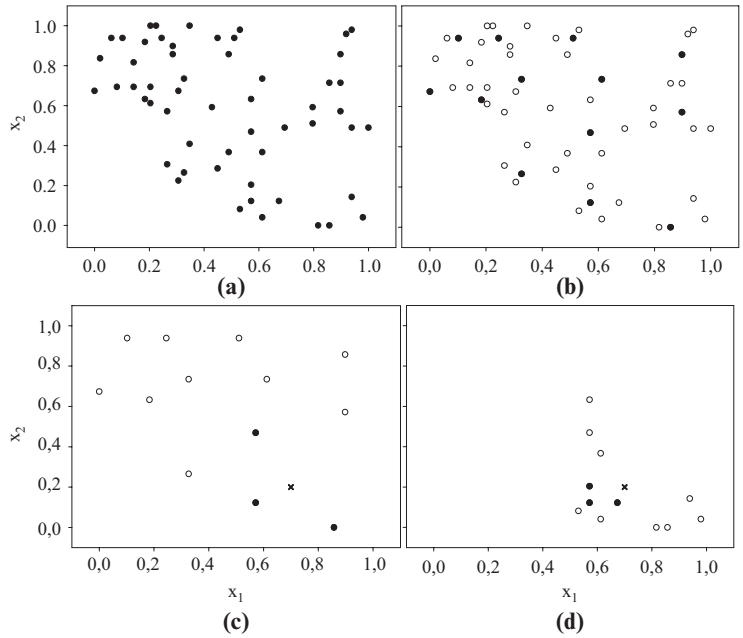

Fig. 1. Illustration of execution stages of the algorithm for search space reduction. Active at the given time objects (records) are marked as black: (**a**) initial set of data – $x^i = \left(x_1^i, x_2^i\right) \in [0,1]^2$, $N = 55$; (**b**) condensed set, $k = 3$, $N_{CR} = 13$ ($CR \approx 23,64\%$); (**c**) searching for three nearest neighbours for a new object (marked with a cross) first in the condensed set; (**d**) finding the neighbours searched among the data of the initial set; in the course of search only those records were used that fall under the condensed data found as neighbours at stage (**c**)

are then defined more accurately during the second stage. Besides, a minimal collection of the data necessary for finding neighbours of the new object participates in the second stage. An example of the aforementioned algorithm execution is shown in Fig. 1.

3 Adaptive Classifier Combination

The concept of evaluation of the local accuracy of class recognition was suggested in [4]. For each new record under classification it is necessary to evaluate how the classification of that record's neighbours (from the learning set) was effected by different classifiers. Using that information, it is possible to make a conclusion about the most probable class of a new record. As a continuation of that approach, special methods of calculating the similarity between the objects were used as well as a parameter controlling neutrality or predomination of one class over the other was introduced [5]. If we consider classification from

the point of view of choosing the most probable class, the probability of class i, is the evaluation of that class, divided by the sum of evaluations of all classes:

$$p_{C=i} = \frac{\alpha_i}{\sum_i \alpha_i}. \tag{1}$$

To control the neutrality or predomination of one class over the other, threshold is introduced such that the target class is considered winning if it has reached (or is greater than):

$$p_{C=i} \geq \varepsilon. \tag{2}$$

Expression (2) foresees neutrality provided that in case of two classes, $\varepsilon = 0.5$. In other cases one class will prevail over the others, which is frequently necessary due to the cost of incorrect information [9].

A topical aspect of combination is selection of similarity measure among the objects (records). In particular, for the mixed data – that exemplify the most widespread kind of the data – this paper employs the interpolated value difference metric (IVDM) [10]. As opposed to the methodology of distance computation described in [5], using IVDM provides a more system and effective – from the viewpoint of classification – character to difference evaluation of the objects described by the mixed type of attributes.

4 Case Study: Credit Scoring Data Set

The experiments were performed using credit granting statistics available in one of German banks [11]. The set consists of 1,000 records described by 20 attributes, three of which are continuous but the other 17 are categories.

To compare the effect of parameter k on condensation ratio, CR, two simulated sets with two continuous attributes were also used. The first set contained 55 records but the second one consisted of 486 records. Besides, the aforementioned scoring data set was condensed with category attributes only. For the simulated sets the Euclidean metric was employed, for the complete scoring data set the IVDM was used and for the set of only category attributes VDM was used. Condensation results obtained at $k \in [1, 20]$ are shown in Fig. 2.

It is apparent that the general condensation trend on the whole does not depend on the character of the data set; it only slightly varies for each of the sets (see also Fig. 2). As regards search space reduction for classifier combination, using condensation, say, at $k = 5$, at the first stage allows one to calculate the distance to less than 15% of the initial records.

In the second part of the experiments classifier combination using two approaches to testing was implemented. Hold-out at the separation 80% for training and 20% for testing as well as tenfold stratified cross-validation were employed. The base classifiers were the following: naïve Bayes [2], logistic model trees, LMT [12], and classification tree J4.8, which is one of the versions of popular machine learning algorithm C4.5 [13]. All the mentioned models are available in the Weka environment [14].

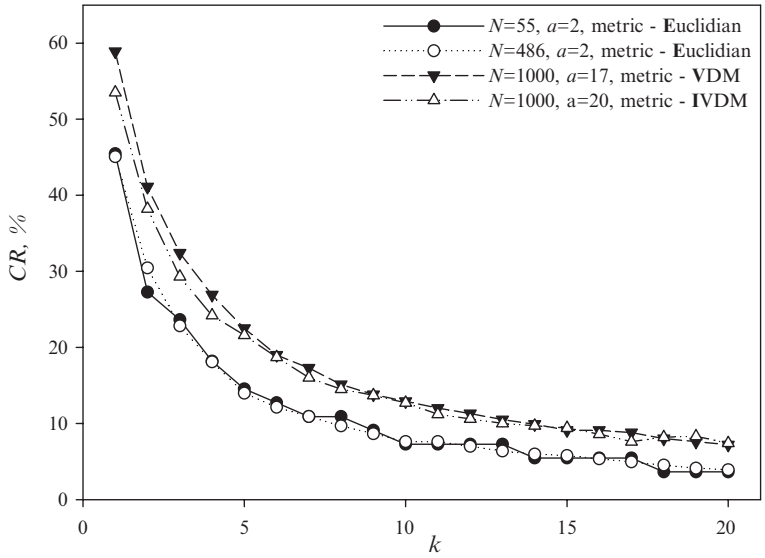

Fig. 2. Dependence of condensation ratio (CR) on the number k

In case of hold-out, the error rate for separate classifiers was: NB – 28%, LMT – 26.5%, and J4.8 – 28%. Combination error was 25% ($\varepsilon = 0.5$). In case of tenfold stratified cross-validation, the error rate for separate classifiers was: NB – 26.2%, LMT – 24.8%, and J4.8 – 26.4%. In its turn, when these classifiers were combined, the error was 23.8% ($\varepsilon = 0.5$). Thus it can be concluded that adaptive combination of classifiers excels separate base algorithms in performance. Moreover, further classification improving is possible owing to the adjusting of parameter to the needs of a particular task.

5 Conclusions

The paper examined the problem of search space reduction for the instance-based classifier combination. Using the principle of data condensation, an algorithm can be constructed which enables one to effectively reduce the above space. As a result, instead of exhaustive searching for all possible nearest neighbours, it is necessary to find them first within the condensed set of the data and then – in the part which is covered by the records found at the first stage.

The experiments conducted have demonstrated stable combination results which excel separately taken base classifiers. Besides, the application of the algorithm of nearest neighbour search space reduction considerably cuts down combination time.

Acknowledgement

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Linguistic Matrix Aggregation Operators: Extensions of the Borda Rule

José Luis García-Lapresta, Bonifacio Llamazares and Miguel Martínez-Panero

Summary. In this paper we consider a decision making situation where agents pairwise compare all the alternatives and show their levels of preference through previously established linguistic labels. Then, from these non-numerical inputs, we introduce two extensions of the Borda rule, called broad and narrow approaches. The difference between them arise when individual assessments are aggregated through matrix operators: the first one takes into account all the labels which compare each alternative and all others, while the second one only consider those favorable. Once these linguistic Borda rules designed, the fulfillment of some properties within the Social Choice framework is verified.

1 Introduction

Decision Theory was initially based on utility functions (see, for instance, von Neumann and Morgenstern [30]). Under this approach, if an alternative provides greater utility than another one, then the first alternative ought to be preferred to the second one. Therefore, a *complete preorder* (or *weak order*) is associated to each individual, and both the preference and the indifference relations are transitive.

But this cardinal approach has received serious criticisms, so that the ordinal one appears to be a more appropriate tool for dealing with human preferences (see Arrow [1]). In this framework, individuals show their preferences among feasible alternatives through crisp binary relations satisfying some suitable properties as irreflexivity, asymmetry, acyclicity, transitivity or negative transitivity. Taking into account different concepts of rationality, weaker models than complete preorders have been considered in the literature: semiorders, interval orders, quasitransitivity, acyclicity (see, for instance, Roubens and Vincke [27] and García-Lapresta and Rodríguez-Palmero [16]).

However, all the above mentioned models require individuals to show their preferences in a dichotomomic way: given two alternatives, either one of them

is preferred to the other or both are indifferent. There is no possibility of declaring preferences with different degrees. In order to deal with intensities of preference, fuzzy preferences allow agents to show their opinions within the unit interval: being 0 the null preference and 1 the absolute preference, individuals have the possibility of declaring intermediate preferences between these two extreme values. On this, see Nurmi [26], Tanino [29], Fodor and Roubens [9], De Baets and Fodor [6] and García-Lapresta and Llamazares [12], among others.

Although fuzzy preferences generalize the classical model in a fruitful way, it is also true that individuals usually have difficulties for declaring their preferences by means of exact numerical assessments. In order to capture in a more faithful manner the lack of precision in human behavior —vague and imprecise—, linguistic preferences ("highly preferred", "slightly preferred", etc.) seem to be more realistic than fuzzy preferences. In this sense, Zadeh [32–34] introduced the program *computing with words* which has produced a vast literature within the Decision Theory framework (see Delgado, Verdegay and Vila [7, 8], Yager [31], Herrera, Herrera-Viedma and Verdegay [19–21], Bordogna, Fedrizzi and Pasi [4], Herrera and Herrera-Viedma [18], among others).

The Borda rule, a well known and appropriate decision making procedure, was originally proposed for linear orders in Borda [3] —in what follows, we will call it *classic Borda rule*—. It has been widespread analyzed and extended to more general orders from its initial design (see Black [2] and Gärdenfors [17], among others). In the last years it has also been considered in a fuzzy framework (see Marchant [22, 23] and García-Lapresta and Martínez-Panero [13, 14]), and in a linguistic context (see García-Lapresta, Lazzari and Martínez-Panero [11] and García-Lapresta, Martínez-Panero and Meneses [15]).

In the last paper, two extensions of the Borda rule were introduced by using linguistic labels represented through trapezoidal fuzzy numbers. One of them, the *broad Borda rule*, corresponds to the sum of all linguistic labels which compare each alternative and all others. The other possibility, the *narrow Borda rule*, takes into account for each alternative only those labels corresponding to worse alternatives than that considered to be valued. In this paper we follow an abstract point of view of the linguistic Borda rules which essentially takes into account that of García-Lapresta [10] for the simple majority, and we focus on verifying the fulfillment of several desirable properties in the Social Choice framework.

The paper is organized as follows: In Sect. 2 we present as preliminaries the classic conception of the Borda rule and the linguistic preference relations which will be the informational basis for our extensions of such procedure. As a tool in our future design, this section also introduces the concept of ordered monoid of linguistic labels. The aforementioned linguistic approaches to the Borda rule (broad and narrow) are presented through linguistic matrix aggregation operators in Sect. 3, where some examples are also considered. In Sect. 4

some properties such as anonymity, neutrality, consistency, monotonicity and strong monotonicity are defined in a linguistic context and verified for our extensions. Finally, in Sect. 5 we include some further extensions and concluding remarks.

2 Extending the Classic Borda Rule to a Linguistic Framework

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, where $n \geq 2$, and m voters, with $m \geq 2$. We consider $P^1, P^2, \ldots, P^m$ to be *ordinary preference relations* (i.e., asymmetric binary relations on X) of the voters and $I^1, I^2, \ldots, I^m$ the associated indifference relations, respectively. The preference/indifference opinions of individual k can be expressed through the following matrix representation:

$$M_k = \begin{pmatrix} r_{11}^k & r_{12}^k & \cdots & r_{1n}^k \\ r_{21}^k & r_{22}^k & \cdots & r_{2n}^k \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1}^k & r_{n2}^k & \cdots & r_{nn}^k \end{pmatrix},$$

where

$$r_{ij}^k = \begin{cases} 1, & \text{if } x_i \, P^k \, x_j, \\ \frac{1}{2}, & \text{if } x_i \, I^k \, x_j, \\ 0, & \text{if } x_j \, P^k \, x_i. \end{cases}$$

For the classic Borda rule to be implemented, each P^k must be a linear order (i.e., the individuals ought to arrange the alternatives from best to worst in a linear manner). There are two main ways to define individual Borda counts, as follows. The first one assigns to each alternative an individual score consisting in adding up all the numerical values which compare such alternative with all others, itself included, namely:

$$\overline{r}_k(x_i) = \sum_{j=1}^{n} r_{ij}^k.$$

The second way gives to each alternative an individual score which coincides with the number of alternatives worse than that considered:

$$\widehat{r}_k(x_i) = \sum_{x_i P^k x_j} r_{ij}^k.$$

In order to obtain a score for the alternative x_i —from the point of view of the individual preference matrices— the first count adds up all the entries in the ith row in the matrix M_k, while the second one only take into account those greater than $\frac{1}{2}$.

By adding the individuals scores up, collective scores are defined for each alternative as

$$\overline{\overrightarrow{r}}(x_i) = \sum_{k=1}^{m} \overline{r}_k(x_i),$$
$$\widehat{\overrightarrow{r}}(x_i) = \sum_{k=1}^{m} \widehat{r}_k(x_i).$$

The most collective scored alternatives become the winners. In fact, collective preference relations (negatively transitive) can be obtained from these collective scores as

$$x_i P^{\overline{B}} x_j \;\Leftrightarrow\; \overline{\overrightarrow{r}}(x_i) > \overline{\overrightarrow{r}}(x_j),$$
$$x_i P^{\widehat{B}} x_j \;\Leftrightarrow\; \widehat{\overrightarrow{r}}(x_i) > \widehat{\overrightarrow{r}}(x_j).$$

Notice that being each P^k a linear order (and hence each alternative is only indifferent to itself), the two aforementioned individual counts verify $\overline{r}_k(x_i) = \widehat{r}_k(x_i) + \frac{1}{2}$. Thus, their respective collective counts are equivalent in the sense that they provide the same collective preference relations. However, if we consider the introduced individual Borda counts for more general orders where voters can express indifference among different alternatives, then the collective counts are not equivalent at all.

Since ordinary preference relations are too rough for capturing individual opinions, we consider another possible approach, more flexible and closer to agent's way of thinking than the above mentioned model (see Zadeh [32–34]). In this way, individuals will be allowed to declare their preferences between each pair of alternatives in a linguistic manner, namely: one alternative is preferred to the other (*absolutely* or *somewhat* —in several allowed degrees—), or both alternatives are *indifferent*. Along the paper we consider a set of *linguistic labels* $L = \{l_0, l_1, \ldots, l_s\}$, with $s \geq 2$, ranked by a linear order: $l_0 < l_1 < \cdots < l_s$. Suppose that there is an intermediate label representing indifference, and the rest of labels are defined around it symmetrically. The number of labels, $s+1$, will be odd and, consequently, $l_{s/2}$ is the central label.

The symmetric structure of the set of linguistic labels can be captured through the classic negation operator $\mathcal{N} : L \longrightarrow L$ defined by $\mathcal{N}(l_i) = l_{s-i}$ for every $i \in \{0, 1, \ldots, s\}$.

Definition 1. *A* linguistic preference relation on X based on L *is a* L*-valued binary relation* $R : X \times X \longrightarrow L$ *satisfying* $r_{ji} = \mathcal{N}(r_{ij})$ *for all* $x_i, x_j \in X$*, where* $r_{ij} = R(x_i, x_j)$.

We denote by $\mathcal{R}_L(X)$ the set of linguistic preference relations on X based on L. Notice that $r_{ii} = l_{s/2}$ for every $i \in \{1, \ldots, n\}$ whenever $R \in \mathcal{R}_L(X)$. Moreover, it is easy to check that $r_{ij} > l_{s/2}$ if and only if $r_{ij} > r_{ji}$.

It is worth emphasizing that the notion of linguistic preference relation is related to that of reciprocal fuzzy preference relation (see, for instance, Nurmi [26], Tanino [29] and García-Lapresta and Llamazares [12]), where $L = [0,1]$ and $\mathcal{N}(x) = 1 - x$.

The linguistic preference relation for each agent k, R^k, can be represented by means of the matrix M_k whose entries are linguistic labels, defined as follows:

$$M_k = \begin{pmatrix} r_{11}^k & r_{12}^k & \cdots & r_{1n}^k \\ r_{21}^k & r_{22}^k & \cdots & r_{2n}^k \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1}^k & r_{n2}^k & \cdots & r_{nn}^k \end{pmatrix}.$$

Obviously, M_k is *antisymmetric* in the sense that $r_{ji}^k = \mathcal{N}(r_{ij}^k)$ for all $i, j \in \{1, \ldots, n\}$. We denote by $\mathcal{M}_n(L)$ the set of antisymmetric matrices of size n and coefficients in L.

According to the Borda rule conception, individual opinions expressed through linguistic labels must be aggregated, and the winner must be determined as the best alternative according to a previously established ordering. So, labels have to be added, and the outcomes ought to be compared. This is the reason why we consider the commutative monoid $\big(\langle L\rangle, +\big)$ generated by L through all possible sums of labels of L with an associative and commutative binary operation $+$ on L, where l_0 is the neutral element:

1. $L \subset \langle L\rangle$
2. $l + l' \in \langle L\rangle$, for all $l, l' \in \langle L\rangle$
3. $l + (l' + l'') = (l + l') + l''$, for all $l, l', l'' \in \langle L\rangle$
4. $l + l' = l' + l$, for all $l, l' \in \langle L\rangle$
5. $l + l_0 = l$, for all $l \in \langle L\rangle$.

In addition to this, $\langle L\rangle$ is considered to be endowed with a total order $\leq$ which is compatible with the former order on L:

6. $l \leq l$, for all $l \in \langle L\rangle$
7. $(l \leq l' \text{ and } l' \leq l) \Rightarrow l = l'$, for all $l, l' \in \langle L\rangle$
8. $(l \leq l' \text{ and } l' \leq l'') \Rightarrow l \leq l''$, for all $l, l', l'' \in \langle L\rangle$
9. $l \leq l'$ or $l' \leq l$, for all $l, l' \in \langle L\rangle$
10. $l_0 < l_1 < \cdots < l_s$, where $<$ is the strict order associated with $\leq$ ($l < l'$ if $l \leq l'$ and $l \neq l'$, for all $l, l' \in \langle L\rangle$).

We also assume the following compatibility property:

11. $l < l' \Rightarrow l + l'' < l' + l''$, for all $l, l', l'' \in \langle L\rangle$.

So, $\big(\langle L\rangle, +, \leq\big)$ is a totally ordered monoid (see García-Lapresta [10]). We denote by $\mathcal{M}_n\big(\langle L\rangle\big)$ the set of matrices of size n and coefficients in $\langle L\rangle$.

3 Linguistic Matrix Aggregation Operators and Decision Rules

From now on, we deal with the matrix representation M_k associated with the linguistic preference relation of voter k, R^k. In order to introduce our linguistic extensions of the Borda rule, an equivalent procedure to the classic one is used as follows. Firstly, we will componentwise add all the individual preference matrices up —either adding all the values up (broad case), or only taking into account those favorable (narrow case)—. Then, we will consider a collective value for the alternative x_i by adding all the entries up in the i-th row in the aggregated matrix, which could be obtained through linguistic matrix aggregation operators mapped to matrices with coefficients in $\langle L\rangle$ instead of L.

Definition 2. *A* linguistic m-matrix aggregation operator *is a mapping*

$$A^{(m)} : \big(\mathcal{M}_n(L)\big)^m \longrightarrow \mathcal{M}_n\big(\langle L\rangle\big).$$

Definition 3. *A* linguistic matrix aggregation operator *is a mapping*

$$\overrightarrow{A} : \bigcup_{m=2}^{\infty} \big(\mathcal{M}_n(L)\big)^m \longrightarrow \mathcal{M}_n\big(\langle L\rangle\big).$$

In this paper we focus in two possibilities taking into account all the opinions or only those favorable ones when we aggregate them:

1. $\overrightarrow{\overline{A}}(M_1,\ldots,M_m) = \overline{M} = (\overline{r}_{ij})$, where

$$\overline{r}_{ij} = \sum_{k=1}^{m} r_{ij}^k.$$

2. $\overrightarrow{\widehat{A}}(M_1,\ldots,M_m) = \widehat{M} = (\widehat{r}_{ij})$, where

$$\widehat{r}_{ij} = \begin{cases} \displaystyle\sum_{k\in K_{ij}} r_{ij}^k, & \text{if } K_{ij} \neq \emptyset, \\ l_0, & \text{otherwise,} \end{cases}$$

and $K_{ij} = \{k \in \{1,\ldots,m\} \mid r_{ij}^k > l_{s/2}\}$.

From these matrices with entries in $\langle L\rangle$, different ordinary preference relations could be obtained. If we denote by $\mathcal{P}(X)$ the set of ordinary preference relations on X, this association can be formally given by means of a mapping

$$O : \mathcal{M}_n\big(\langle L\rangle\big) \longrightarrow \mathcal{P}(X).$$

In order to extend the classic Borda rule to the linguistic framework, we will consider the mapping

$$O_B : \mathcal{M}_n\big(\langle L\rangle\big) \longrightarrow \mathcal{P}(X).$$

which assigns $P_M \in \mathcal{P}(X)$ to $M = (r_{ij}) \in \mathcal{M}_n\big(\langle L\rangle\big)$ in the following way:

$$x_i \, P_M \, x_j \;\Leftrightarrow\; \sum_{u=1}^{n} r_{iu} > \sum_{u=1}^{n} r_{ju}.$$

As usual, we denote by I_M the indifference relation associated with P_M, i.e.,

$$x_i \, I_M \, x_j \;\Leftrightarrow\; \sum_{u=1}^{n} r_{iu} = \sum_{u=1}^{n} r_{ju}.$$

Definition 4. *Given a linguistic matrix aggregation operator* $\overrightarrow{A}$ *and* $O : \mathcal{M}_n\big(\langle L\rangle\big) \longrightarrow \mathcal{P}(X)$*, the composition*

$$F = O \circ \overrightarrow{A} : \bigcup_{m=2}^{\infty} \big(\mathcal{M}_n(L)\big)^m \longrightarrow \mathcal{P}(X).$$

is called a linguistic-based decision rule.

In this way, the linguistic-based decision rule $F_{\overline{B}} = O_B \circ \overrightarrow{\overrightarrow{A}}$ will be called *broad Borda rule* and the linguistic-based decision rule $F_{\widehat{B}} = O_B \circ \overrightarrow{\widehat{A}}$ will be called *narrow Borda rule*. In both cases, the obtained collective preference relations are negatively transitive, in a similar way to the classic model for $P^{\overline{B}}$ and $P^{\widehat{B}}$.

It is worth mentioning that Sen [28] also defines broad and narrow Borda counts in connection with the fulfillment of the independence of irrelevant alternatives principle. However, these variants, although sharing names, are different of ours. Sen takes into account the amplitude of the referential set of alternatives in order to define a choice function, while we consider opinions (all of them or only the favorable ones) in pairwise comparisons of alternatives.

We note that when $L = \{l_0, l_1, l_2\}$ and the linguistic labels are identified with 0, $\frac{1}{2}$ and 1, respectively (i.e., agents show their opinion through ordinary preferences), both broad and narrow Borda rules coincide with the classic Borda count, supposed the preferences to be linear orders. However, in general, these extended procedures are different as shown in the following example.

Example 1. Consider three individuals who express their preferences over $X = \{x_1, x_2, x_3\}$ by means of linguistic labels $L = \{l_0, l_1, l_2, l_3, l_4, l_5, l_6\}$ whose meaning is given in Table 1.

Suppose that they have associated the following matrix representations of their linguistic preferences:

$$M_1 = \begin{pmatrix} l_3 & l_4 & l_6 \\ l_2 & l_3 & l_5 \\ l_0 & l_1 & l_3 \end{pmatrix} \quad M_2 = \begin{pmatrix} l_3 & l_0 & l_3 \\ l_6 & l_3 & l_6 \\ l_3 & l_0 & l_3 \end{pmatrix} \quad M_3 = \begin{pmatrix} l_3 & l_4 & l_3 \\ l_2 & l_3 & l_1 \\ l_3 & l_5 & l_3 \end{pmatrix}$$

Table 1. Meaning of linguistic labels

Label	Meaning
l_0	x_j is totally preferred to x_i
l_1	x_j is highly preferred to x_i
l_2	x_j is slightly preferred to x_i
l_3	x_i is indifferent to x_j
l_4	x_i is slightly preferred to x_j
l_5	x_i is highly preferred to x_j
l_6	x_i is totally preferred to x_j

When the broad Borda rule is used we obtain the following aggregated matrix:

$$\overrightarrow{\overline{A}}(M_1, M_2, M_3) = \overline{M} = \begin{pmatrix} 3l_3 & 2l_4 & l_6 + 2l_3 \\ l_6 + 2l_2 & 3l_3 & l_6 + l_5 + l_1 \\ 2l_3 & l_5 + l_1 & 3l_3 \end{pmatrix}.$$

Since $l_6+2l_4+5l_3 > l_5+5l_3+l_1$ and $2l_6+l_5+3l_3+2l_2+l_1 > l_5+5l_3+l_1$ we have $x_1 \, P_{\overline{M}} \, x_3$ and $x_2 \, P_{\overline{M}} \, x_3$. However, the relationship between x_1 and x_2, which would determine the effective winner, might depend on the considered totally ordered monoid.

On the other hand, under the narrow Borda rule the following aggregated matrix is obtained:

$$\overrightarrow{\widehat{A}}(M_1, M_2, M_3) = \widehat{M} = \begin{pmatrix} l_0 & 2l_4 & l_6 \\ l_6 & l_0 & l_6 + l_5 \\ l_0 & l_5 & l_0 \end{pmatrix}.$$

Now, from $2l_6 + l_5 > l_6 + 2l_4 > l_5$ we have $x_2 \, P_{\widehat{M}} \, x_1 \, P_{\widehat{M}} \, x_3$. Notice that in this case x_2 is the winner and the collective outcome is independent of the considered totally ordered monoid. It is worth to mention that this fact does not always hold under the narrow approach. For instance, in the following example we show how the collective order may depend on the established totally ordered monoid.

Example 2. Consider three voters who express their preferences over the set of alternatives $X = \{x_1, x_2, x_3\}$ by means of the same linguistic labels and their meaning as in the previous example, whose matrix representations of linguistic preferences are the following:

$$M_1 = \begin{pmatrix} l_3 & l_5 & l_6 \\ l_1 & l_3 & l_4 \\ l_0 & l_2 & l_3 \end{pmatrix} \quad M_2 = \begin{pmatrix} l_3 & l_2 & l_3 \\ l_4 & l_3 & l_4 \\ l_3 & l_2 & l_3 \end{pmatrix} \quad M_3 = \begin{pmatrix} l_3 & l_3 & l_6 \\ l_3 & l_3 & l_4 \\ l_0 & l_2 & l_3 \end{pmatrix}$$

When the narrow Borda rule is used we obtain the following aggregated matrix:

$$\widehat{\overrightarrow{A}}(M_1, M_2, M_3) = \widehat{M} = \begin{pmatrix} l_0 & l_5 & 2l_6 \\ l_4 & l_0 & 3l_4 \\ l_0 & l_0 & l_0 \end{pmatrix}$$

Since $2l_6 + l_5 > l_0$ and $4l_4 > l_0$ we have $x_1 \, P_{\widehat{M}} \, x_3$ and $x_2 \, P_{\widehat{M}} \, x_3$. Again the relationship between x_1 and x_2 depends on the representation of the labels, and then on the way of defining the additive and order structure of the monoid. Following García-Lapresta [10], we consider three ways of representing the established linguistic labels through real numbers, intervals and triangular fuzzy numbers (TFNs), as appearing in Table 2.

There is no discussion about how to define the addition of real numbers, intervals and TFNs, as well as the way of defining an order of real numbers. However, there appear in the literature several possibilities for defining orders of intervals and TFNs. In this example we will consider the following order of intervals:

$$[a, b] > [a', b'] \Leftrightarrow \begin{cases} a + b > a' + b' \\ \text{or} \\ a + b = a' + b' \text{ and } b - a < b' - a'. \end{cases}$$

Concerning the TFNs, the following order already considered in García-Lapresta, Lazzari and Martínez-Panero [11] will be used:

$$(a, b, c) > (a', b', c') \Leftrightarrow \begin{cases} a + 2b + c > a' + 2b' + c' \\ \text{or} \\ a + 2b + c = a' + 2b' + c' \text{ and } c > c' \\ \text{or} \\ a + 2b + c = a' + 2b' + c', \;\; c = c' \text{ and } a > a'. \end{cases}$$

We now show that the collective preference between x_1 and x_2 depends on the chosen representation.

– If real numbers are considered, then $x_1 \, I_{\widehat{M}} \, x_2$ because

$$2l_6 + l_5 = 2.8 = 4l_4.$$

Table 2. Representation of linguistic labels

Label	RN	Interval	TFN
l_0	0	$[0, 0]$	$(0, 0, 0)$
l_1	0.2	$[0, 0.2]$	$(0, 0.2, 0.4)$
l_2	0.3	$[0.2, 0.4]$	$(0.1, 0.3, 0.5)$
l_3	0.5	$[0.4, 0.6]$	$(0.4, 0.5, 0.6)$
l_4	0.7	$[0.6, 0.8]$	$(0.5, 0.7, 0.9)$
l_5	0.8	$[0.8, 1]$	$(0.6, 0.8, 1)$
l_6	1	$[1, 1]$	$(1, 1, 1)$

– If the interval representation is taken into account, then $x_1 \, P_{\widehat{M}} \, x_2$ because

$$2l_6 + l_5 = [2.8, 3] > [2.4, 3.2] = 4l_4.$$

– If TFNs are used, then $x_2 \, P_{\widehat{M}} \, x_1$ because

$$4l_4 = (2, 2.8, 3.6) > (2.6, 2.8, 3) = 2l_6 + l_5.$$

4 Social Choice Type Properties

Within the Social Choice framework, we now propose several properties of the introduced decision rules which extend to a linguistic context those usual when ordinary preferences are considered.

Definition 5. *Let F be a linguistic-based decision rule.*

1. *F is* anonymous *if for every $(M_1, \ldots, M_m) \in \big(\mathcal{M}_n(L)\big)^m$ and every bijection $\sigma : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$:*

$$F\left(M_{\sigma(1)}, \ldots, M_{\sigma(m)}\right) = F(M_1, \ldots, M_m).$$

2. *F is* neutral *if for every bijection $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ and all $(M_1, \ldots, M_m)$, $(N_1, \ldots, N_m) \in \big(\mathcal{M}_n(L)\big)^m$, such that $M_k = (r_{ij}^k)$, $N_k = (s_{ij}^k)$, $F(M_1, \ldots, M_m) = P_M$ and $F(N_1, \ldots, N_m) = P_N$:*

   If $r_{ij}^k = s_{\sigma(i)\sigma(j)}^k$ for all $i, j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then $x_i \, P_M \, x_j \Leftrightarrow x_{\sigma(i)} \, P_N \, x_{\sigma(j)}$, for all $i, j \in \{1, \ldots, n\}$.

3. *F is* consistent *if for all $x_i, x_j \in X$ and all $(M_1, \ldots, M_m) \in \big(\mathcal{M}_n(L)\big)^m$, $(N_1, \ldots, N_{m'}) \in \big(\mathcal{M}_n(L)\big)^{m'}$, such that $F(M_1, \ldots, M_m) = P_M$, $F(N_1, \ldots, N_{m'}) = P_N$ and $F(M_1, \ldots, M_m, N_1, \ldots, N_{m'}) = P_{M+N}$:*

   If $x_i \, P_M \, x_j$ and $x_i \, P_N \, x_j$, then $x_i \, P_{M+N} \, x_j$.

4. *F is* monotonic *if for all $(M_1, \ldots, M_m)$, $(N_1, \ldots, N_m) \in \big(\mathcal{M}_n(L)\big)^m$, with $M_k = (r_{ij}^k)$, $N_k = (s_{ij}^k)$, $F(M_1, \ldots, M_m) = P_M$, $F(N_1, \ldots, N_m) = P_N$, such that there exist $h \in \{1, \ldots, m\}$ and $i, j \in \{1, \ldots, n\}$ with $r_{ij}^h < s_{ij}^h$, $r_{pq}^h = s_{pq}^h$ whenever $\{p, q\} \neq \{i, j\}$ and $M_k = N_k$ for all $k \neq h$:*

   i) If $x_i \, P_M \, x_j$, then $x_i \, P_N \, x_j$.
   ii) If $x_i \, I_M \, x_j$, then $x_i \, (P_N \cup I_N) \, x_j$.

5. *F is* strongly monotonic *if for all $(M_1, \ldots, M_m)$, $(N_1, \ldots, N_m) \in \big(\mathcal{M}_n(L)\big)^m$, with $M_k = (r_{ij}^k)$, $N_k = (s_{ij}^k)$, $F(M_1, \ldots, M_m) = P_M$, $F(N_1, \ldots, N_m) = P_N$, such that there exist $h \in \{1, \ldots, m\}$ and $i, j \in \{1, \ldots, n\}$ with $r_{ij}^h < s_{ij}^h$, $r_{pq}^h = s_{pq}^h$ whenever $\{p, q\} \neq \{i, j\}$ and $M_k = N_k$ for all $k \neq h$:*

   If $x_i \, (P_M \cup I_M) \, x_j$, then $x_i \, P_N \, x_j$.

Strong monotonicity is related to a property within the classical approach of Social Choice called *positive responsiveness* (see May [24]). An extension of this property within the linguistic approach has been considered in García-Lapresta [10]. We note that every linguistic decision rule satisfying strong monotonicity is also monotonic.

The following theorem shows that the broad Borda rule satisfies all the previous properties while the narrow Borda rule only does not satisfy the strong monotonicity. Consequently, they essentially inherit those features of the classic Borda rule.

Theorem 1. *For every totally ordered monoid* $\big(\langle L\rangle, +, \leq\big)$ *based on a set of linguistic labels* L*, the associated linguistic Borda rules (both in the broad and narrow cases) are anonymous, neutral, consistent and monotonic. Moreover, the broad Borda rule is also strongly monotonic.*

Proof. We only prove the result for the broad Borda rule (the proof for the narrow Borda rule can be obtained in a similar way).

1. Anonymity. It is obvious because the linguistic matrix aggregation operator $\overrightarrow{\overrightarrow{A}}$ satisfies $\overrightarrow{\overrightarrow{A}}\big(M_{\sigma(1)},\dots,M_{\sigma(m)}\big) = \overrightarrow{\overrightarrow{A}}(M_1,\dots,M_m)$, for all bijection $\sigma:\{1,\dots,m\}\longrightarrow\{1,\dots,m\}$ and $(M_1,\dots,M_m)\in\big(\mathcal{M}_n(L)\big)^m$.
2. Neutrality. Let $\sigma:\{1,\dots,n\}\longrightarrow\{1,\dots,n\}$ be a bijection and $(M_1,\dots,M_m)$, $(N_1,\dots,N_m)\in\big(\mathcal{M}_n(L)\big)^m$ such that $M_k=(r_{ij}^k)$, $N_k=(s_{ij}^k)$, $F_{\overline{B}}(M_1,\dots,M_m)=P_{\overline{M}}$ and $F_{\overline{B}}(N_1,\dots,N_m)=P_{\overline{N}}$. If $r_{ij}^k=s_{\sigma(i)\sigma(j)}^k$ for all $i,j\in\{1,\dots,n\}$ and $k\in\{1,\dots,m\}$, then

$$\begin{aligned} x_i\,P_{\overline{M}}\,x_j &\Leftrightarrow \sum_{u=1}^{n}\overline{r}_{iu}>\sum_{u=1}^{n}\overline{r}_{ju} \;\Leftrightarrow\; \sum_{u=1}^{n}\sum_{k=1}^{m}r_{iu}^k>\sum_{u=1}^{n}\sum_{k=1}^{m}r_{ju}^k\\ &\Leftrightarrow \sum_{u=1}^{n}\sum_{k=1}^{m}s_{\sigma(i)\sigma(u)}^k>\sum_{u=1}^{n}\sum_{k=1}^{m}s_{\sigma(j)\sigma(u)}^k\\ &\Leftrightarrow \sum_{u=1}^{n}\overline{s}_{\sigma(i)\sigma(u)}>\sum_{u=1}^{n}\overline{s}_{\sigma(j)\sigma(u)} \;\Leftrightarrow\; x_{\sigma(i)}\,P_{\overline{N}}\,x_{\sigma(j)}, \end{aligned}$$

   for all $i,j\in\{1,\dots,n\}$.
3. Consistency. Let $(M_1,\dots,M_m)\in\big(\mathcal{M}_n(L)\big)^m$, $(N_1,\dots,N_{m'})\in\big(\mathcal{M}_n(L)\big)^{m'}$ such that $M_k=(r_{ij}^k)$, $N_k=(s_{ij}^k)$, $F_{\overline{B}}(M_1,\dots,M_m)=P_{\overline{M}}$, $F_{\overline{B}}(N_1,\dots,N_{m'})=P_{\overline{N}}$ and $F_{\overline{B}}(M_1,\dots,M_m,N_1,\dots,N_{m'})=P_{\overline{M+N}}$. Given $x_i,x_j\in X$, if $x_i\,P_{\overline{M}}\,x_j$ and $x_i\,P_{\overline{N}}\,x_j$, then

$$\sum_{u=1}^{n}\sum_{k=1}^{m}r_{iu}^k>\sum_{u=1}^{n}\sum_{k=1}^{m}r_{ju}^k \quad\text{and}\quad \sum_{u=1}^{n}\sum_{k=1}^{m'}s_{iu}^k>\sum_{u=1}^{n}\sum_{k=1}^{m'}s_{ju}^k.$$

Therefore,

$$\sum_{u=1}^{n}\left(\sum_{k=1}^{m} r_{iu}^{k}+\sum_{k=1}^{m'} s_{iu}^{k}\right)>\sum_{u=1}^{n}\left(\sum_{k=1}^{m} r_{ju}^{k}+\sum_{k=1}^{m'} s_{ju}^{k}\right),$$

i.e., $x_i\, P_{\overline{M+N}}\, x_j$.

4. Strong monotonicity. Let $(M_1,\ldots,M_m)$, $(N_1,\ldots,N_m) \in \big(\mathcal{M}_n(L)\big)^m$, with $M_k=(r_{ij}^k)$, $N_k=(s_{ij}^k)$, $F_{\overline{B}}(M_1,\ldots,M_m)=P_{\overline{M}}$, $F_{\overline{B}}(N_1,\ldots,N_m)=P_{\overline{N}}$, such that there exist $h\in\{1,\ldots,m\}$ and $i,j\in\{1,\ldots,n\}$ with $r_{ij}^h<s_{ij}^h$, $r_{pq}^h=s_{pq}^h$ whenever $\{p,q\}\neq\{i,j\}$ and $M_k=N_k$ for all $k\neq h$. Since $r_{ij}^h<s_{ij}^h$, we also have $s_{ji}^h<r_{ji}^h$. Therefore,

$$\overline{r}_{ij}=\sum_{k=1}^{m} r_{ij}^{k}<\sum_{k=1}^{m} s_{ij}^{k}=\overline{s}_{ij} \quad \text{and} \quad \overline{r}_{ji}=\sum_{k=1}^{m} r_{ji}^{k}>\sum_{k=1}^{m} s_{ji}^{k}=\overline{s}_{ji},$$

while $\overline{r}_{iu}=\overline{s}_{iu}$ if $u\neq j$ and $\overline{r}_{ju}=\overline{s}_{ju}$ if $u\neq i$. Consequently,

$$\sum_{u=1}^{n}\overline{r}_{iu}<\sum_{u=1}^{n}\overline{s}_{iu} \quad \text{and} \quad \sum_{u=1}^{n}\overline{r}_{ju}>\sum_{u=1}^{n}\overline{s}_{ju}.$$

Now, if $x_i\,(P_{\overline{M}}\cup I_{\overline{M}})\,x_j$, then

$$\sum_{u=1}^{n}\overline{s}_{iu}>\sum_{u=1}^{n}\overline{r}_{iu}\geq\sum_{u=1}^{n}\overline{r}_{ju}>\sum_{u=1}^{n}\overline{s}_{ju},$$

i.e., $x_i\, P_{\overline{N}}\, x_j$. □

We note that in García-Lapresta, Martínez-Panero and Meneses [15] other properties, namely representativity and Condorcet-type properties, have been analyzed. The fulfillment of representativity (which states that Borda qualifications must agree with linguistic pairwise comparisons of alternatives for each agent) is granted under rationality requirements for the linguistic narrow Borda rule weaker than those for the broad one. However, a Condorcet loser (an alternative which is defeated by simple majority when opposed to each other in pairwise tournaments) might became a winner under the narrow approach. Such undesirable fact will never happen under the broad one —supposed the linguistic labels to be represented through symmetric trapezoidal fuzzy numbers— and this is a compelling argument for the broad linguistic Borda rule when confronted to the narrow one.

5 Some Further Extensions and Concluding Remarks

The way of generalizing the Borda rule to a linguistic context also allows us to extend other decision making procedures. In this sense we can obtain a linguistic extension of simple majority when we consider the mapping

$$O_S : \mathcal{M}_n\big(\langle L\rangle\big) \longrightarrow \mathcal{P}(X)$$

which assigns P_M to $M = (r_{ij}) \in \mathcal{M}_n\big(\langle L\rangle\big)$ in the following way:

$$x_i \, P_M \, x_j \;\Leftrightarrow\; r_{ij} > r_{ji}.$$

Analogously, we also obtain a linguistic extension of the Copeland rule ([5]) when we consider the mapping

$$O_C : \mathcal{M}_n\big(\langle L\rangle\big) \longrightarrow \mathcal{P}(X)$$

which assigns Q_M to $M = (r_{ij}) \in \mathcal{M}_n\big(\langle L\rangle\big)$ in the following way:

$$x_i \, Q_M \, x_j \;\Leftrightarrow\; \sum_{u=1}^{n} c(i,u) > \sum_{u=1}^{n} c(j,u),$$

where $c : \{1,\ldots,n\}^2 \longrightarrow \{-1,0,1\}$ is the function given by:

$$c(i,j) = \begin{cases} 1, & \text{if } r_{ij} > r_{ji}, \\ 0, & \text{if } r_{ij} = r_{ji}, \\ -1, & \text{if } r_{ij} < r_{ji}. \end{cases}$$

In conclusion, linguistic matrix aggregation operators provide a versatile and appropriate tool for dealing with several decision making procedures, as shown for the Borda rule and pointed out for other well-known methods.

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An Evolutionary Algorithm for the Biobjective QAP

Istvan Borgulya

Summary. In this paper we present a new method for the biobjective quadratic assignment problem. This method is a modified version of an earlier multi-objective evolutionary algorithm. It uses a special truncation selection, and the descendents are derived from the parents by mutation based on an EC-memory method. This EC-memory method is an extended version of an earlier method, and we can use on the more value discrete space. The quality of the results of our algorithm is better than the results of some stochastic local search, or ACO algorithms.

Key words: Multi-objective optimization, EC-memory method, QAP, Evolutionary algorithm.

1 Introduction

Knowles and Corne [7] presented a QAP variation considering several flows and distances. This multi-objective QAP problem has a number of potential applications. For example, in hospital layout problem we may be concerned with simultaneously minimizing the flows of doctors of their rounds, of patients, of hospital visitors, and of pharmaceuticals and other equipment [7].

The mathematical expression is then

$$\min_{\pi \in S_n} F(\pi) = \{f_1(\pi), f_2(\pi), \ldots, f_m(\pi)\}$$

$$\text{where } f_k(\pi) = \sum_{i,j=1}^{n} fw_{ij}^k d_{\pi(i)\pi(j)} 1 \leq k \leq m.$$

n is the number of facilities and locations, fw_{ij}^k denotes the kth flow between i- and j-facilities, S_n is the set of all permutations with n elements and $\pi \in S_n$, d_{ij} is the distance between location i and location j and π_i gives the location of facility i in permutation π.

In the last years Knowles and Corne [8] presented instance generators for the biobjective QAP (bQAP) and some methods were developed for bQAP.

E.g. an ACO algorithm was designed [9], stochastic local search algorithms were developed [11], and a parallel evolutionary technique for bQAP was present [4].

In this paper we present a new evolutionary algorithm (EA) for the bQAP. This algorithm is based on an earlier multi-objective EA, named MOSCA2 [2]. For the bQAP we modified MOSCA2, we use other selection, mutation operator and we use a local search procedure. To improve the quality of the results we use a modified version of an EC-memory method by mutation, instead of recombination operator.

We compared our algorithm with other algorithms (e.g. robust tabu search (RoTS), ACO, stochastic local search algorithms). The quality of the results of our algorithm became better and we got this result after similar or longer running times.

In addition to this introduction section, this paper is organized into the following sections. The new, extended version of an EC-memory method is described in Sect. 2. Section 3 includes the new version of the MOSCA2. In Sect. 4, we present our computational experience with the new version and compare our results with other heuristics results. Section 5 contains concluding remarks.

2 The EC-Memory Method

There are many variants of explicit collective memory (EC-memory) methods that memorises the past events and/or past successes of the evolution process in the EA. With the help of this method we can chose e.g. appropriate evolutionary operators during the evolutionary process, we can drive the offspring generate process, or we can select the individuals (e.g. the PIBS of Baluja et al. [1], the "Virtual loser" of Sebag et al. [14], ant colony algorithm [3], cultural algorithm [13]).

We choose to adopt the method of Sebag et al. [14] that memorises the past failures of evolution through a virtual individual, the virtual loser (VL). We can use the VL in the binary space, and its memory is a numeric vector that gives the average values of the worst individual by every bit position (variable). With the help of the VL we can give the probability of mutating a bit in an individual: the probability of mutating bit i in individual X should reflect how much it discriminates X_i from VL, that is, it should increase with $p_i = 1 - |VL_i - X_i|$. We can use this technique e.g. by continuous function optimization, or by combinatorial problem, discretized through a binary or a Gray coding.

In this paper we modify and extend the VL. We made to be able the VL more discrete values handling. The discrete values can be integer, or real number, but different objects, e.g. values of permutations too. Generally we

can not compute the average of the discrete values (e.g. by permutation) as by the VL, but we can compute the relative frequency of every discrete values by the variables one by one.

The principle of the new *VL* version, named *EVL* (*E*xtended *V*irtual *L*oser) is the following. Let be m different values of the variables. (We see only this simple version. If the numbers of the discrete values or the discrete values are not the same by every variable we can easily modify the next formulas.) Let us notice ECM an nxm matrix that stores the relative frequency of the different values of the variables. This matrix is updated through the search procedure using a few of the worst performing individuals.

Let ECM_{ij}^{gen} be the relative frequency of the ith values on the jth position (variable) in the genth generation. We can update the elements of the ECM matrix similar way as by the *VL*:

$$ECM_{ij}^{gen+1} = (1-\alpha)ECM_{ij}^{gen} + \alpha\ dECM_{ij} \quad (e.g.\ \alpha = 0.2)$$

where $dECM_{ij}$ is the relative frequency of the ith value on the position jth based on the worse individuals of the genth generation and α denotes some relaxation factor.

For the probability of mutating the jth variable in individual X we can use the

$$q_j = \frac{ECM_{X_j,j}}{\sum_{k=1}^{n} ECM_{kj}}$$

formula. We get the highest q_j values by the worst values of the variable X_j based on the worst individuals. Consequently we can chose a better value for X_j with higher probability than $1-q_j$. To improve the probability we can use the best individual too. Let be B one of the best individuals. If $X_j = B_j$ we change evidently the value of X_j with low probability. With the help of B_j the probability of mutating the jth variable in individual X is the following:

$$p_j = 1 - |q_j - a_j|$$

where if $X_j = B_j$ then $a_j = 1$ else $a_j = 0$.

Mutation based on the EVL. Let be X a descendant. We rank the variables X_i decreasing based on q_i, and select the first (e.g. max $n/2$) elements of the queue. Let U notice the set of the selected variables. By every variable $X_j \in U$ we search an other $X_z \in U$ such a way, that the probability

$$p_j = 1 - |\frac{ECM_{X_z,j}}{\sum_{k=1}^{n} ECM_{kj}} - a_i|$$

is maximal (where if $X_z = B_j$ then $a_i = 1$ else $a_i = 0$). After that we write the value of X_z into the ith position and we delete the variables X_z from U.

3 The New Algorithm

3.1 The MOSCA2

The fundamental principles of MOSCA2 [2] are the following: Let us segregate the members of the population into t subpopulations, each subpopulation will approximate an other part of the Pareto front sought. Each subpopulation is storing only non-dominated individuals of the possible members of the subpopulation (at a limited amount). The dominance of a new descendant getting into the subpopulation is determined by comparing it to a designated non-dominated individual, the prototype. If it founds a non-dominated descendant superior to the previous prototype, it deletes the former members of the subpopulation, and replaces the prototype by the new descendant.

During the evolution process the new potential Pareto optimal solutions are periodically stored in a separate archive, and we get the result in this archive. If this separate archive ($SARC$) is full, the algorithm deletes a given percentage (10%) of its elements. We select first the most dominated individuals for deletion, after that continuously one of the individuals close to each other.

The MOSCA2 uses a 2-stage algorithm structure where every stage is a steady-state EA. The first stage is a quick "preparatory" stage that is designated to improve the quality of the initial population. The second stage is an evolutionary strategy with some special operators (more details in [2]).

3.2 MOSCA2b for bQAP

The new version of the MOSCA2, named MOSCA2b, uses some special operators. So, the selection is a special version of the truncation selection, the descendents are derived from the parents by mutation based on the EVL and the algorithm uses the 2-opt local search procedure with weighted objective.

By solving a multi-objective problem there are two important tasks generally: to reach a good convergence to the Pareto optimal front and to cover all points of this front with different solutions. MOSCA2b solves these tasks only with the help of the truncation selection and the EVL method.

The main steps of MOSCA2b:

Procedure MOSCA2b (*t, subt, arcn, itend, ddp*)
  $it = 0, SUBP_i = \emptyset$ $(i = 1, 2, ..., t)$ /* The initial values.
  Let $p_i \in SUBP_i$ $(i = 1, 2, ..., t) : SARC = \emptyset$
  $itt = 400, kn = 100$
  /* First stage *
  Fitness evaluation: ranking of P
  Do *itt* times
    $it = it + 1$
    A descendant is generated randomly. Reinsertion.
  od
  /* Second stage *

```
  Ranking of P. Initial values of ECM
  Repeat
    Do kn times
      it = it + 1
      Truncation selection, mutation based on EVL, local search.
      Reinsertion.
    od
    Ranking of P, Update of SARC, Deleting. Update ECM.
  until it > itend
end
```

The parameters of the algorithm:
t – the number of the subpopulation
subt – the maximum size of each subpopulation
arcn – the maximum size of *SARC*
itend – the maximal number of the generation
ddp – parameter of the *Deleting* procedure

The main function and characteristics of the two EAs are as follows:

- The first individuals are randomly generated.
- The P population is built from subpopulations: $SUB_1, SUB_2, \ldots, SUB_t$ ($P = \cup SUB_i$). There is a designated non-dominated individual in every subpopulation, the prototype.
- The value of the fitness function is a rank number determined according to the Pareto ranking method by Goldberg [6].
- In the first stage the descendants are randomly selected from S. In the second stage the algorithm randomly selects a parent with rank number 1 from P (this is a special truncation selection).
- *Mutation based on the EVL.* Let be X a descendant. We rank the variables X_i decreasing based on q_i, and select the first, maximum n/2 elements of the queue for mutation and we change the value of the variables according in Sect. 2 (By the bQAP, B is the prototype of the subpopulation of X).
- *ECM.* It is the matrix in the *EVL* method. *ECM* is defined after the termination of the first stage. It is periodically updated by using the weakest individuals. In the updating procedure we use 20% of the population.
- As local search the algorithm uses the 2-opt local search with weighted objective [5].
- On reinsertion in a subpopulation the algorithm do not use the rank number, it is enough to examine the Pareto dominance between a prototype and the descendant. In the first stage, the algorithm compares the descendant with the most similar prototype. In the second stage the descendant is compared with the prototype of the subpopulation of the parent.
- *Deleting.* A given percent of the most dominated individuals in P will be deleted based on the rank number. This percent decreases as the number of iteration increases (The deleted subpopulations will be replaced with new one).

- *Function computation.* Based on observation in [12] on the transformation of asymmetric flow or distance matrix into symmetric one, we compute in a simple form the change of the function values in the local search procedure similar way as in [11].
- *Stopping criteria.* The algorithm is terminated if a pre-determined number of iterations have been performed.

4 Experimental Results

Test Problems

We used the test problem of [11] that was generated with the instance generator of [8]. These instances and the reference solutions of RoTS are available at http://www.intellektik.informatik.tu-darmstadt.de/~lpaquete/QAP. The instances were generated with $n \in \{25, 50, 75\}$ locations and with correlations between the flow matrices of $\rho \in \{-0.75, -0.50, -0.25, 0.0, 0.25, 0.50, 0.75\}$.

Parameter Selection

Our experience with the earlier version of MOSCA2b helped easier to chose the values of the parameters. Only with a little difference we could use the same values. So the used parameters were the following: $t = 100$, $subt = 10$, $arcn = 1{,}000$ and $ddp = 30\%$. The maximal number of the generation (or fitness evaluations) was 1,500 or 2,300 depending from the problem (The MOSCA2b was implemented in Visual Basic and ran on a Pentium 4 1.8 GHz with 256 MB RAM).

Comparative Results

As performance measure we used the binary ϵ-indicator from [11]. The binary ϵ-indicator gives the factor by which an approximation set is worse than another with respect to all objective [15]. In practice, the binary ϵ-indicator is calculated as

$$I_\epsilon(A, B) = \max_{b \in B} \min_{a \in A} \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$$

where A and B are non-dominated objective value vectors of a problem with two objectives. With the help of this measure we can compare two solutions: if $I_\epsilon(A, B) > 1$ and $I_\epsilon(B, A) \leq 1$, then the set B completely dominates the set A.

To compare the results of the MOSCA2b we chose the RoTS algorithm. With the help of the reference solutions of the RoTS we compared the performance of the MOSCA2b and RoTS, used shorter running times as the RoTS running time. Every test problem was run 10 times, and Table 1 shows the average comparative results of MOSCA2b. In Table we see the results

Table 1. The average comparative results of MOSCA2b

ρ	n	ϵ_1	ϵ_2	avgt	avgn
0.75	25	1.096	0.923	18.8	2
	50	1.041	0.967	86.0	3
	75	1.032	0.978	249.9	2
0.50	25	1.094	0.908	18.1	5
	50	1.021	0.989	86.5	7
	75	1.014	0.985	289.4	5
0.25	25	1.099	0.933	19.4	8
	50	1.045	0.971	87.6	7
	75	1.009	0.985	1464.2	10
0.00	25	1.051	0.947	18.7	12
	50	1.009	0.975	88.1	11
	75	1.012	0.993	1465.5	12
−0.25	25	1.073	0.928	18.8	19
	50	1.020	0.968	144.5	16
	75	1.010	0.986	1454.1	22
−0.50	25	1.084	0.952	19.3	28
	50	1.022	0.962	157.4	20
	75	1.031	0.978	1231.2	36
−0.75	25	1.086	0.990	19.5	64
	50	1.007	0.986	153.9	71
	75	1.001	0.993	1036.4	82

by different values of correlation (ρ) and size (n), ϵ_1 gives $I_\epsilon(B, A)$, ϵ_2 gives $I_\epsilon(A, B)$ (where A is the outcomes of the MOSCA2b and B is the outcomes of RoTS), *avgt* is the average computation times in second and *avgn* is the average number of solutions.

Analyzing the results, we can conclude that the outcomes of MOSCA2b are better by all test problems based on the performance measure than the outcomes of RoTS. That shows Fig. 1 too. The plot gives the non-dominated results of the methods and every MOSCA2b' results dominate the RoTS' results. Only the numbers of the non-dominated solution of the RoTS are more that by MOSCA2b, but the quality of the MOSCA2b'solutions are better.

We can compare MOSCA2b with other methods too. In [9] an ACO algorithm, and in [11] two stochastic local search (Pareto Local Search and Two-Phase Local Search) were developed for the bQAP. In both papers the methods were compared based on the binary ϵ-indicator with RoTS, so we can compare MOSCA2b with these methods too. Because the RoTS has better performance as the ACO and the two stochastic local search algorithms (see [9,11]), MOSCA2b has also better performance by the given problems as these methods. Only the running time is different: MOSCA2b has similar or shorter running times as the ACO, and has longer running times as the two local search algorithms. (We considered by the comparison that the methods run on different computers.)

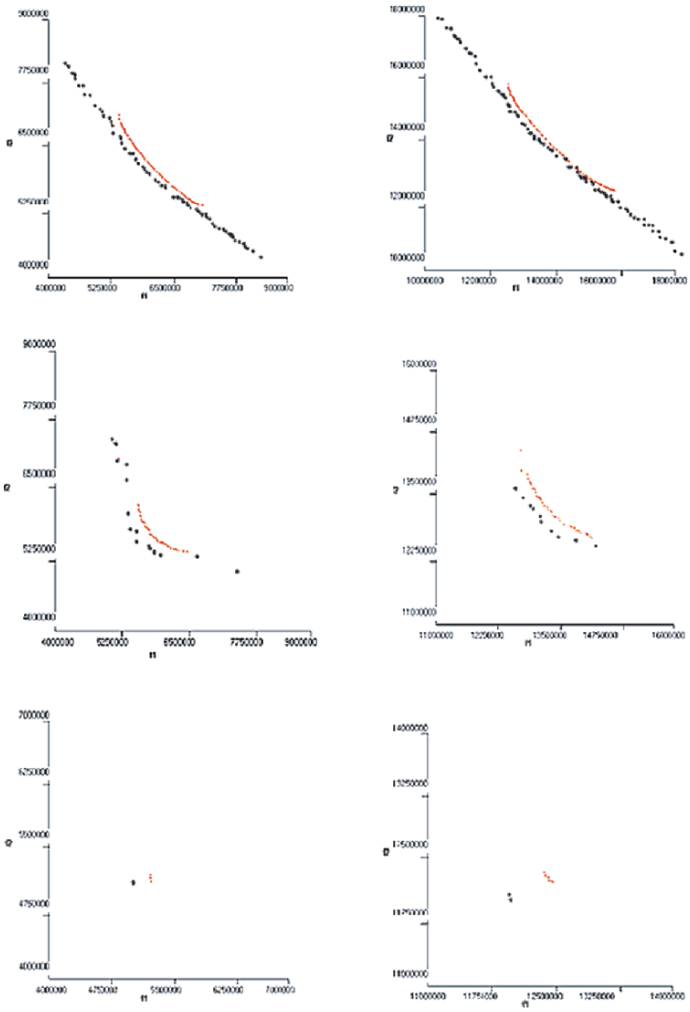

Fig. 1. The plot gives the non-dominated results of MOSCA2b (*thicker points*) and the results of RoTS (*thinner points*). The results correspond to three–three instances with size 50 (left column) and with size 75 (right column) and with correlation $\rho = -0.75$ (line top), $\rho = 0$ (line center), $\rho = 0.75$ (line bottom)

We can observe that the solution quality of MOSCA2b (and the other methods too) depends from the correlation (ρ) between the flow matrices. Depending on this correlation there are significant differences in the results. For high positive correlations the search is very hard and by every method the average number of solutions is low. With decreasing correlation the methods found the solutions easier and the average number of solutions increases continuously.

5 Summary

With the modification of an earlier multi-objective EA, we have developed a method for bQAP. The new version, named MOSCA2b, uses some special operators: the selection is a special version of the truncation selection, and the descendents are derived from the parents by mutation based on an EC-memory method, named *EVL*. The *EVL* is an extended version of the "*virtual loser*" method of Sebag et al. [14].

With the help of the truncation selection and the *EVL* method our algorithm can reach a good convergence to the Pareto optimal front and cover all points of this front with different solutions. Other EA methods can solve these important tasks only with special plus techniques, operators.

The quality of the results our algorithm is better than the results of some stochastic local search, or ACO algorithms, but the running times are generally similar, or longer. As future work we can improve the speed of our algorithm, and we can try to use our algorithm by other multi-objective optimization problems.

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On a Hill-Climbing Algorithm with Adaptive Step Size: Towards a Control Parameter-Less Black-Box Optimisation Algorithm

Lars Nolle

1 Introduction

Many scientific and engineering problems can be viewed as search or optimisation problems, where an optimum input parameter vector for a given system has to be found in order to maximise or to minimise the system response to that input vector. Often, auxiliary information about the system, like its transfer function and derivatives, etc., is not known and the measures might be incomplete and distorted by noise. This makes such problems difficult to be solved by traditional mathematical methods. Here, heuristic optimisation algorithms, like Genetic Algorithms (GA) [1] or Simulated Annealing (SA) [2], can offer a solution. But because of the lack of a standard methodology for matching a problem with a suitable algorithm, and for setting the control parameters for the algorithm, practitioners often seem not to consider heuristic optimisation.

The main reason for this is that a practitioner, who wants to apply an algorithm to a specific problem, and who has no experience with heuristic search algorithms, would need to become an expert in optimisation algorithms before being able to choose a suitable algorithm for the problem at hand. Also, finding suitable control parameter settings would require carrying out a large number of experiments. This might not be an option for a scientist or engineer, who simply wants to use heuristic search as a tool.

For such practitioners, an optimisation algorithm that would have no control parameters to choose, while still being effective and efficient, would clearly be of benefit. The aim of this project was to develop a search algorithm with as few control parameters as possible that is still effective for solving black-box optimisation tasks. In this paper, a novel optimisation algorithm is introduced, which has only one control parameter. Experiments presented in this paper demonstrate that the algorithm, which is a population based hill-climbing algorithm with self-adapting step size, is very effective and also very efficient.

2 Self-Adaptive Step-size Search (SASS)

For heuristic search algorithms, like Hill-Climbing (HC) [3], it was previously shown that the definition of the neighbourhood, and in particular the chosen step size, is crucial to the success of the algorithm [4], not only for continuous parameter search, but also for discrete parameters, when the search space is too large to consider direct neighbours of a candidate solution for performance reasons. It was shown that selection schemes with random step sizes with an upper limit (maximum step size s_{max}) outperform neighbourhood selection schemes with a constant step length. It was also demonstrated that using a scaling function for reducing s_{max} over time could again increase the performance of Hill-Climbing algorithms.

However, it would clearly be of benefit if the maximum step length would be more adaptive to the search progress itself. Therefore, a new population-based adaptation scheme with a self-adaptive step size, referred to as Self-Adaptive Step-size Search (SASS) throughout this article, has been developed for HC, where the temporary neighbourhood of a particle p_i is determined by the distance between itself and a randomly selected sample particle s_i of the population during each iteration.

At the beginning of a search this distance is likely to be large, because the initial population is uniformly distributed over the search space and the chances are high that s_i is drawn from a different region within the input space. When the search is progressing, each particle is attracted by a local optimum and hence the population is clustered around a number of optima. If both, p_i and s_i are located in different clusters, p_i has the chance to escape its local optimum if it samples from a region with a higher fitness, i.e. lower costs. Towards the end of the search, most particles have reached the region of the global optimum and hence their mean distance is much smaller than in the initial population. As a result, the maximum step size s_{max} is sufficiently small to yield the global optimum. Figure 1 shows pseudocode of the algorithm. The main advantage of SASS is, that it only has one control parameter that has to be chosen in advance, which is the number of particles n in the population.

3 Experiments

In order to evaluate the effectiveness and the efficiency of the new algorithm, a set of experiments was conducted using two well-established standard test functions, the inverted Schwefel function and the inverted Griewank function [5]. The first test function, the n-dimensional inverted Schwefel function (1), was chosen because of its interesting characteristic that the second best optimum is located far away from the global optimum, which can cause an optimisation algorithm to converge towards one of the local optima. The global optimum of zero is located at the coordinate $x = (420.969, 420.969, \ldots)$.

```
Procedure selfAdaptiveStepSizeSearch
Begin
  initialise population of n particles
  While stopping criterion not met
  Begin
    For every particle p in population
    Begin
      select random particle s  ≠  p
      For every component p_i in particle p
      Begin
        S_max  ←  |  p_i − s_i  |
        generate random value r  ∈ [−s_max; +s_max]
        p'_i  ←  p_i + r
      End
      If f(p') better than f(p) then p  ←  p'_i
    End
  End
  Return best result
End
```

Fig. 1. Pseudocode of the SASS algorithm

$$f(x) = 418.98288n + \sum_{i=1}^{n} -x_i \sin(\sqrt{|x_i|}) \tag{1}$$

where $-500 \leq \quad x_i \quad \leq +500$.

The second test function, the n-dimensional inverted Griewank function (2), was chosen because the product term introduces a correlation between the function variables and hence a high degree of epistasis [6]. This can disrupt optimisation techniques that work on one function's variable at a time

$$f(x) = 1 + \sum_{i=1}^{n} \frac{x_i^2}{4000} - \prod_{i=1}^{n} \cos(\frac{x_i}{\sqrt{i}}) \tag{2}$$

where $-30 \leq x_i \leq +30$.

The global optimum of zero is located at the point $\mathrm{x} = (0, 0, \ldots)$. There are many local optima in the landscape of this function. An increase in the number of variables decreases the number of local optima since it makes the function surface flat. The 2, 5, and 10 dimensional versions of both test functions have been used. The population sizes were varied from 3 to 30 particles and every experiment was repeated 100 times in order to prove reproducibility.

4 Experimental Results

Figure 2 shows the average costs of the 100 experiments per population size for the different versions of the Schwefel function, whereas Fig. 3 shows the same for the different versions of the Griewank function.

Figures 4 and 5 document the percentage of experiments that have converged towards the global optimum for the Schwefel, respectively, the Griewank function, for the different population sizes.

As it can be seen, SASS was able to find the global optima with a very high reproducibility of about 99%, provided that the population sizes used exceeded a certain minimum. For example, for the two dimensional Schwefel function, the algorithm proved to be very robust if the population size was greater than 12, whereas for the 10 dimensional version, it was necessary to use population sizes greater than 20. Similar behaviour was observed for the different versions of the Griewank function.

5 Analysis of the SASS Algorithm

In this section, a typical search run of SASS for the 2 dimensional Schwefel function is analysed. Figure 6 shows the development of the average costs of the population and the lowest costs found in a population over time, i.e. iterations. It can be seen that the population converged after approximately 800 iterations.

Figure 7 shows the actual development of the average s_{max} in the population over time for the same search run. It can be seen that it starts off with

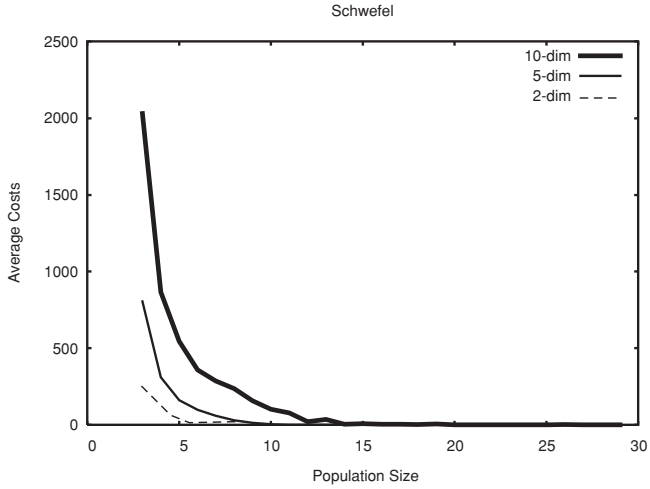

Fig. 2. Average costs for the 2, 5, and 10 dimensional versions of the Schwefel function

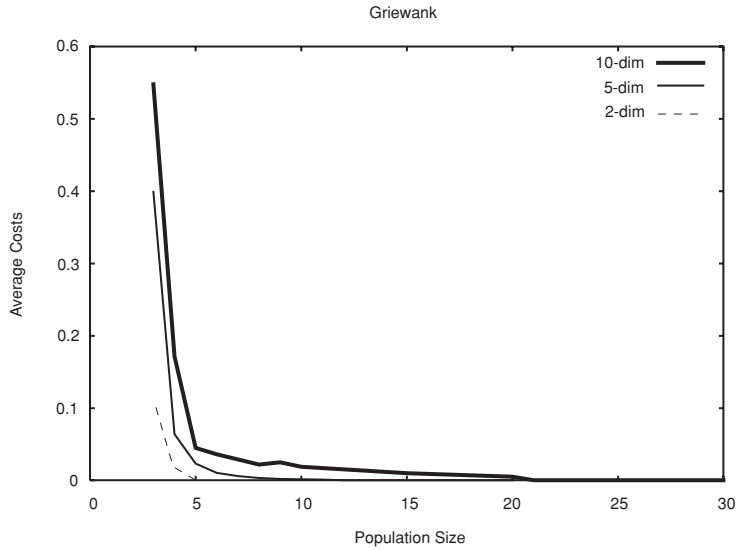

Fig. 3. Average costs for the 2, 5, and 10 dimensional versions of the Griewank function

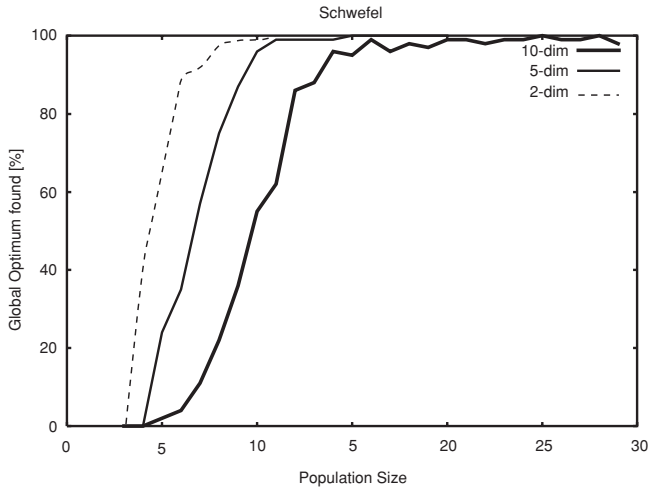

Fig. 4. Percentage of experiments that converged towards the global optimum for the 2, 5, and 10 dimensional versions of the Schwefel function

relatively large values compared to the size of the input space and that it finally settles on very small numbers after approximately 800 iterations. This is in line with Fig. 6, which shows that the population has converged after approximately 800 iterations. It also confirms the initial assumptions given in Sect. 2.

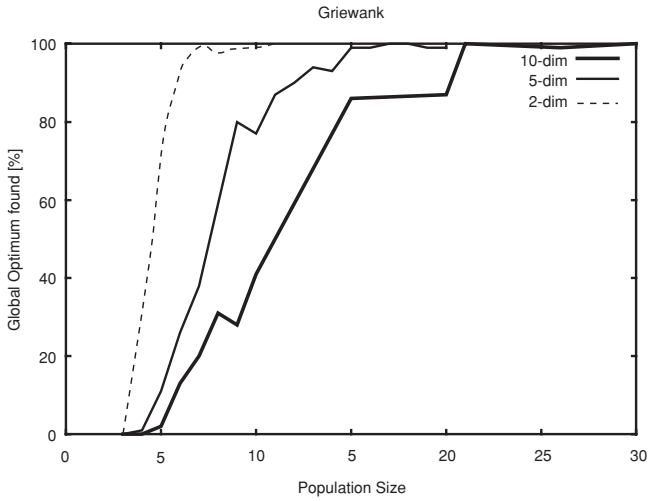

Fig. 5. Percentage of experiments that converged towards the global optimum for the 2, 5, and 10 dimensional versions of the Griewank function

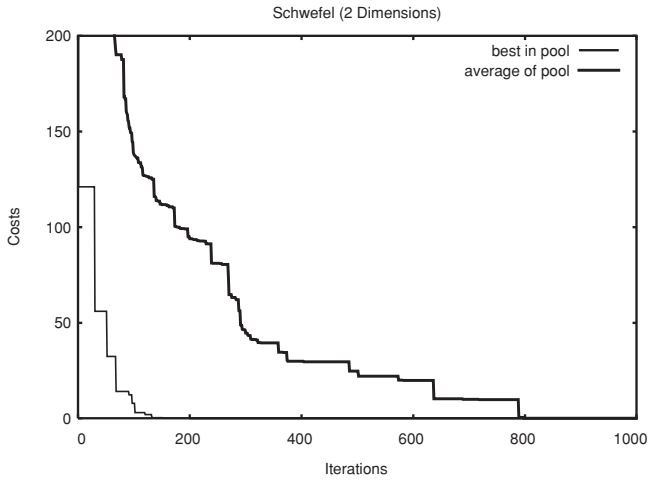

Fig. 6. Typical run of the SASS algorithm for the 2 dimensional Schwefel function

Figure 8 shows a contour plot of the initial population for the same search run. It can be seen that the population is more or less uniformly distributed over the input space, although no particle starts near the global optimum at (420.969, 420.969). After 600 iterations, all particles have settled into three different clusters (Fig. 9), which are located relatively far away from each other. Algorithms like GA, SA, and HC would be very unlikely to escape from the local optima, while SASS was able to achieve this and to converge towards the global optimum after about 800 iterations (Fig. 10).

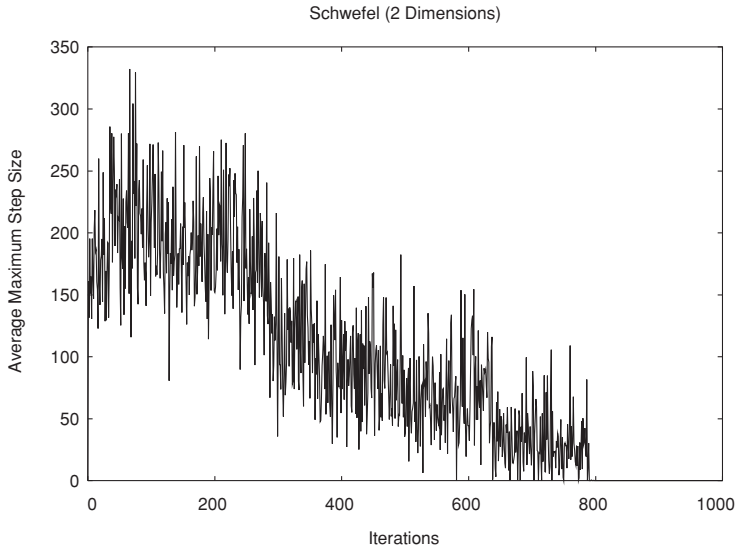

Fig. 7. Development of s_{max} over time

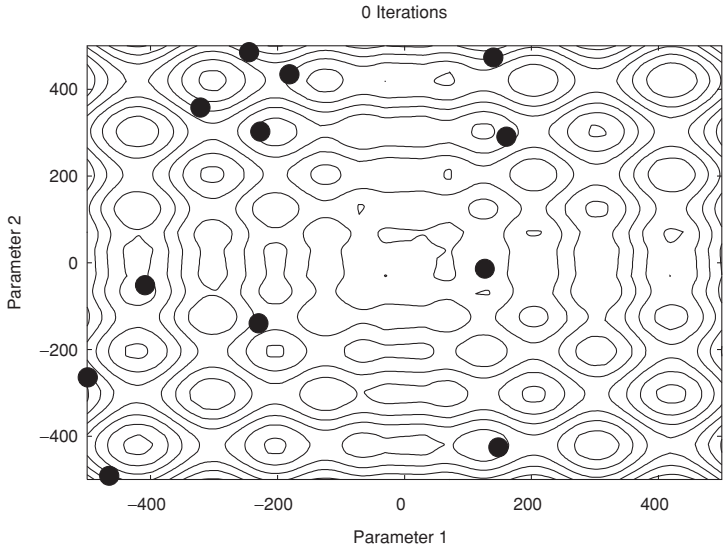

Fig. 8. Initial distribution of the population

In other words, whereas these large distances between the clusters would make it impossible for SA and HC to converge, for SASS they result into large values for s_{max} for particles from different clusters, and hence enabled the particles to escape the local optima and hence enable the algorithm to converge.

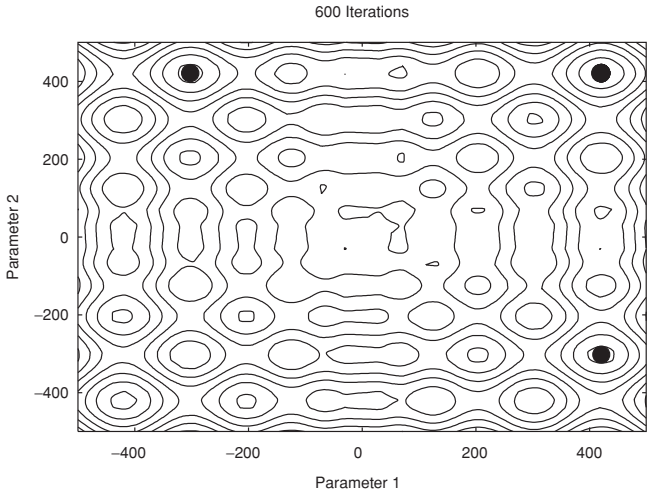

Fig. 9. Distribution of population after 600 iterations

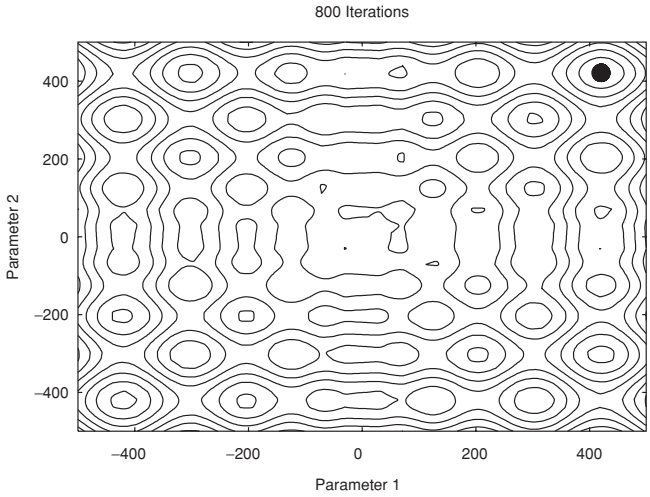

Fig. 10. Distribution of population after 800 iterations

6 Conclusions and Future Work

This paper introduced SASS, a novel population based Hill-Climbing algorithm with a self-adaptive step size. SASS has only one control parameter to be chosen in advance, which is the population size. Although the algorithm is very simple, the results presented in this paper have demonstrated that SASS is capable of finding the global optimum with a very high reproducibility, provided that there are enough particles in the population. This number depends on problem difficulty and ranged from 12 for the 2 dimensional test functions to more than 20 for the 10 dimensional versions. The next step will

be to apply SASS to more test problems and to develop a heuristic that can help to determine the minimum number of particles needed for a particular problem at hand. This would make SASS a truly control parameter-less search algorithm.

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Self-Adaptive Baldwinian Search in Hybrid Genetic Algorithms

Tarek A. El-Mihoub, Adrian A. Hopgood, Lars Nolle and Alan Battersby

Summary. The problem of proper utilization of the search time to adapt a hybrid to a given problem can be viewed as a problem of finding optimal control parameter settings. The algorithm's time utilization can be optimized through adapting the local search duration. Evolving this control parameter via genetic operations is one possible way to achieve this adaptation. However, the hindering effect can obstruct the self-adaptive ability of the Baldwinian search. Local search methods with narrow steps and the use of the local search duration to discriminate between solutions can help to alleviate this problem.

Key words: Self-adaptaion, Hybrid genetic algorithms, Baldwinian search, Hindering effect.

1 Introduction

A genetic algorithm is usually combined with a domain-specific method to solve a real-world problem [8]. The success of such a hybrid algorithm in solving a given problem efficiently depends on its success in achieving a balance between exploration and exploitation [3, 4, 8]. Among the factors that affect this balance is the duration of local search [5], which is defined as the number of the consecutive local search iterations that is performed on a solution before terminating a local search procedure. This control parameter can be used to adapt the hybrid on-line to a specific problem.

The interactions between local search duration, learning strategy, fitness topology, and other genetic components have a great impact on search time utilization [4, 5]. The idea of evolutionary self-adaptation [6] can be applied to adapt the local search duration in order to optimise the performance of a hybrid on a particular problem without the need for external control.

The impact of the hindering effect [9] on obscuring genetic differences can obstruct the Baldwinian [7] search's self-adapting ability to a given problem. The genotypes cannot be effectively discriminated according to their fitness

without considering the learning cost and hence the evolution of effective solutions can be hindered.

The ability of genetic search to find favourable parameter settings for pure genetic algorithms has been proven [6]. However, its ability within a hybrid to self-adapt the control parameters, especially those related to incorporating a local method, may require further investigation. In this paper, we analyse the influence on the behaviour of the Baldwinian hybrid of simultaneously exploring both the problem search space and the control space of local search duration. This analysis can help to gain some insight into the factors that may affect the search performance in order to find ways to improve it.

2 Evolutionary Self-Adaptation and Duration of Local Search

In evolutionary self-adaptive algorithms, the fitness of the individual associated with a specific control parameter value is used as feedback to assess the suitability of the control parameter values for solving a given problem. The link between the duration-of-local-search control parameter and the individual's fitness depends on the fitness function topology, the details of the local search method and the genetic algorithm's setup. By allowing the duration of the local search to evolve by means of genetic operations, the link between favourable duration of the local search and the fitness can be exploited. Genetic operations can adaptively control the duration of the local search method to optimise the individuals fitnesses. In this way, this link can be defined, which is essential for the adaptation of control parameters [11].

However, it may be difficult to define this link when the genetic algorithm is combined with Baldwinian search. The acquired fitness is the sum of the improvements introduced by applying a local search method for the encoded duration and the innate fitness. The hindering effect can direct the search towards individuals with long durations and a small innate fitness. The search process, in this case, is degraded from optimising the fitness function to optimising a single control parameter. The possibility of leading the search in this direction increases as the dimension of the fitness function increases, since it may be easier for the algorithm to optimise a single control parameter than to optimise a large number of variables. It can also waste its resources as it can direct the individuals towards performing useless local search iterations. The use of the acquired fitness as a metric to assess the quality of solutions in the Baldwinian search can produce an algorithm with poor performance.

The use of a local search method, which takes narrow steps in the search space while restricting the values of the duration of local search to very small numbers, can help to combat the hindering effect problem. In this way, the problems consequences on the ability of the algorithm to define a link between this control parameter and the fitness in the direction of optimising solutions quality can be alleviated.

However, the ultimate solution for the hindering effect problem is to rely on innate fitness to decide between solutions of equal acquired fitness values. Since the number of local iterations, which is a good indication of the cost of learning, is already encoded into the individual, it can be used together with the acquired fitness to direct the search towards solutions of high quality.

It may be beneficial to allow the local search method to cooperate with the global genetic algorithm to explore the search space in the early stages of the search by allowing wide local steps. However, as the Baldwinian search reaches the fitness-convergence-state, taking narrow local steps can be more helpful. By adapting the local step size according to the standard deviation of the population fitness, the search performance may be improved.

3 Experiments

A set of experiments was conducted to gain some insight into the evolutionary self-adaptive behaviour of the Baldwinian search using three different hybrids. Hybrid-A, which uses a local search method with a predefined maximum local step size and discriminates between solutions based on the acquired fitness only, was used to study the effect of local search step size on the performance. Hybrid-B, which is identical to Hybrid-A except that it uses local search iterations to discriminate between solutions of an equal acquired fitness, was used to investigate the effectiveness of using the local search duration to discriminate between solutions. The possibility of improving the hybrid performance by employing an adaptive local step was examined through Hybrid-C, which uses an adaptive local step size and utilizes the local search duration to discriminate between effective solutions.

In these hybrids, the number of local search iterations that should be performed by an individual was encoded into its chromosome. At each iteration, the local search method tries to find the smallest possible step in the allowed range of a randomly selected variable space that improves the fitness. Starting from the least significant bit of a randomly chosen variable and moving towards its most significant bit, the local search method keeps flipping the bits until an improvement in the fitness produced or a specified number of bits are flipped. In the case of no improvement in the fitness, the process is repeated for another randomly chosen variable until an improvement is produced. By controlling the maximum number of bits that can be scanned for fitness improvement of each variable before randomly selecting another variable, the algorithm controls the size of the local search step.

The generalized Ellipsoidal [2], Ackley [1], Schwefel [10], Rastrigin [10], and Griewank [10] functions were selected as a test suite. The hybrids used the simple elitist genetic algorithm with binary tournament selection, two-point crossover, and simple mutation. The values of the duration of local search parameter were restricted to very small values in the range 0–3. For all

experiments, the rate of crossover and mutation were set to 0.7 and (population size)$^{-1}$, respectively. The population sizes for the 2- and 10-dimensional functions were set to 50 and 100, respectively. Each variable was represented by a 10-bit string. The stopping criterion for all experiments was a maximum number of function evaluations. The number of bits that were exposed to modification was limited to a specific percentage of the length of the variables string. Each experiment was repeated 100 times.

4 Discussion

The results of the first two hybrids clearly show that as the size of the local search step decreases, the ability of the evolutionary self-adaptive Baldwinian hybrid to find a global optimum increases. This is depicted in Fig. 1 for the 10-dimensional Ellipsoidal, Ackley, and Schwefel functions. The algorithms were unable to find the global optimum for the 10-dimensional Griewank and Rastrigin functions. However, the curves of the best fitness of these functions show a similar trend. The curves of the percentage of experiments that found a global optimum of the 10-dimensional functions, as expected, have a steeper slope than the 2-dimensional functions. As shown in Fig. 1, Hybrid-B outperformed Hybrid-A in terms of the percentage that converged.

The experiments also show that using small local steps improves the speed of the algorithms in finding the global optimum (Fig. 2). Figure 2 also illustrates that Hybrid-B significantly outperforms Hybrid-A in terms of the search speed of the 10-dimensional Ackley and Schwefel functions.

Hybrid-C produced a near optimal performance in terms of the percentages that converged and an optimal performance in terms of convergence speed for the 10-dimensional Schwefel and Ellipsoidal functions, as illustrated by the

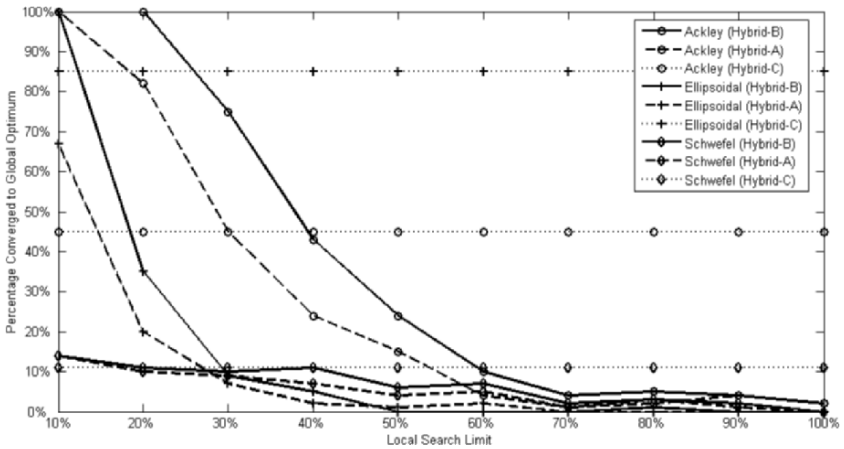

Fig. 1. The effect on convergence ability

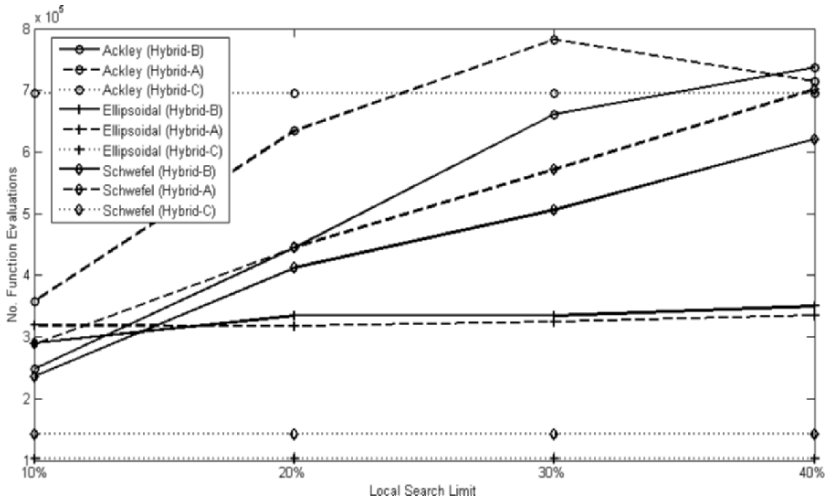

Fig. 2. The effect on convergence speed

dotted lines in Figs. 1 and 2. Hybrid-C also improved the best fitness and the search speed of the Rastrigin and Griewank functions. However, the algorithm produced a poor performance for the Ackley function.

5 Conclusions

The hindering effect can obstruct the ability of Baldwinian search to self-adapt the duration-of-local-search control parameter. The possibility of obstructing this ability increases as the dimension of the fitness function increases. The results presented in this paper also show that the use of a local search method with narrow steps in the search space can help to alleviate this problem and hence improve the performance of the Baldwinian search in terms of solution quality and convergence speed. The performance of the Baldwinian search can be further improved when the local search duration is used alongside the acquired fitness to discriminate between effective solutions. The use of an adaptive local search step can improve the performance of the Baldwinian search on some of the tested problems.

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Intragenerational Mutation Shape Adaptation

Stefan Berlik and Bernd Reusch

Summary. The aim of this work is to discuss questions that arise when porting the directed mutation principle from the uncorrelated setting in classical evolution strategies to the correlated case as given in covariance matrix adaptation-evolution strategies. The main problem to be addressed here is the shape vector update. The shape vector controls the distribution's skewness and has to be updated either intergenerational or intragenerational. Starting with an analogue to the intergenerational parameter update mechanics used in CMA-ES, we argue that an intragenerational update is of greater benefit. Different heuristics will be presented and compared, additionally some experimental data of several test functions is provided.

Key words: Intragenerational adaptation, Directed covariance matrix adaptation, Directed mutation, Mutation operator, Evolutionary algorithm, Evolution strategy, DCMA-ES, CMA-ES.

1 Introduction

Since the early work of Rechenberg [13] and Schwefel [15] the design of mutation operators turned out to be one of the most critical points in Evolution Strategies (ESs). These early works relied on just one single mutation strength, i.e. step-size, for all problem dimensions (isotropic mutation) and were concerned mainly with determining the optimal step-size for a faster search. To put it in a more general light, the covariance matrix of the mutation operator's distribution was considered to be the identity matrix. Soon Schwefel extended this approach and proposed to self-adapt one step-size per variable, i.e. to use a diagonal covariance matrix with positive entries. Consequently, as the most general case, he later suggested self-adapting of the whole covariance matrix (correlated mutation). A more detailed review of the field's history is given e.g. by Bck et al. [5, 6].

However, all of these methods rely on normally distributed mutations and relatively little effort has been put into examining different distributions as

mutation operators. One such example is the so-called Fast Evolution Strategy by Yao [18], where a Cauchy distribution is proposed as mutation operator. Nevertheless, Rudolph [14] later proofed that the order of local convergence is identical to that of normal mutations. Just to exchange the mutation distribution seems in general to be a questionable idea. The intention of directed mutation on the other hand is to introduce a different mutation *principle*. It will abandon the *random mutation hypothesis* – a fundamental tenet postulating that mutations occur at random, regardless of fitness consequences to the resulting offspring. This seems to be justified by the fact that the ES knows its optimization history and is thus able to extrapolate the evolution path to some extent. Under the assumption of a local similar objective function it is obviously reasonable to generate a bigger portion of offspring along the successful path.

2 The DCMA-ES Algorithm

So far, directed mutation was realized with uncorrelated mutation models only. However, its usefulness has been shown for both, for test function optimization [7,8] as well as in real-world scenarios [9]. Now, as already mentioned in the introduction, there are several even more powerful ES approaches that rely on the flexibility of correlated mutations. The performance of these EA depends obviously highly on the choice of the covariance matrix C, which has to be adjusted not only to the problem at hand, but also to the current state of the evolution process. Several methods have been proposed, from the self-adaptation of the mutation parameters in ES (SA-ES) [16] to the Covariance Matrix Adaptation-ES (CMA-ES) [12]. While the first removes the need to manually adjust the covariance matrix, the latter takes into account the history of evolution and deterministically adapts the covariance matrix from the last moves of the algorithm, thereby directing the search to use the most recent descent direction. In [11] an advanced version of the CMA-ES is presented, that is computationally more efficient.

All these approaches use symmetric normally distributed random numbers. The aim of the sequel is therefore to accommodate the CMA-ES with a multivariate skew-normal distribution, yielding the Directed Covariance Matrix Adaptation-ES (DCMA-ES). Especially the update of the distribution's shape vector will be investigated. While in classical ES with multi-recombination of less than the whole population an intergenerational update is inevitable, the CMA-ES setting allows to establish an intragenerational tuning of the shape vector. Here one distinct mean, i.e. the center of the parental population, is calculated and used subsequently to generate all descendants. This fixed point can be treated as reference for the whole offspring generation cycle. Recent studies have shown remarkable results. However, much further research is necessary and the results are in that sense preliminary.

A conceptually related approach, called LS-CMA-ES, was presented by Auger et al. [2]. It is based on quasi-Newton techniques, i.e. relying on local curvature information to find out the next points to sample. Therefore it aims at learning the local Hessian matrix by solving a linear least-square minimization problem. The solution is then found by evaluating the pseudo-inverse of this linear system. The cost of the direct computation of this pseudo-inverse by standard numerical methods is scaling as n^6, indicating also a high effort of the approximative solution. In contrast the DCMA-ES is computationally by far less expensive.

The rest of this section is organized as follows: First we recall the univariate skew-normal distribution, present then a multivariate version, and give a hint how to generate corresponding random vectors. Afterwards different update strategies of the shape vector are discussed. Finally, some first experimental data is provided.

2.1 Univariate Skew-Normal Distribution

The class of distributions that is used to build the following directed mutation operators is called skew-normal (SN) distribution and was introduced by Azzalini [3]. A detailed presentation of the SN distribution, some extensions, and a small historical review are given by Arnold and Beaver [1].

The univariate SN density function is defined by

$$f_{SN}(z;\lambda) = 2\varphi(z)\Phi(\lambda z) \tag{1}$$

where φ and Φ represents the probability density function and the cumulative distribution function of the standard normal density, respectively. λ is a real parameter that controls the skewness, i.e. the shape. Positive (negative) values indicate positive (negative) skewness. In the case $\lambda = 0$ the SN density gets back to the normal density (see Fig. 1). With $Z \sim SN(\lambda)$ we denote a random variable that has density (1).

2.2 Multivariate Skew-Normal Distribution

An extension of the skew-normal distribution to the multivariate setting was proposed by Azzalini and Dalla Valle [4]. An n-dimensional random vector X is said to have a multivariate SN distribution, denoted by $\mathrm{SN_n}(\boldsymbol{\mu}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$, if it is continuous with probability density function

$$f_{\mathrm{SN_n}}(\boldsymbol{z};\boldsymbol{\mu},\boldsymbol{\Omega},\boldsymbol{\alpha}) = 2\varphi_n(\boldsymbol{z};\boldsymbol{\mu},\boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}^T(\boldsymbol{z}-\boldsymbol{\mu})) \tag{2}$$

where $\varphi_n(\boldsymbol{z};\boldsymbol{\mu},\boldsymbol{\Omega})$ is the n-dimensional probability density function with mean $\boldsymbol{\mu}$ and correlation matrix $\boldsymbol{\Omega}$. $\Phi(\cdot)$ is the standard normal distribution function $\mathrm{N}(0,1)$ and $\boldsymbol{\alpha}$ is a n-dimensional shape vector.

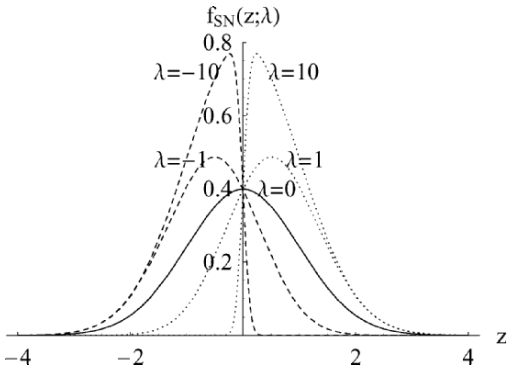


Fig. 1. The density functions SN(−10), SN(−1), SN(0), SN(1), and SN(10)

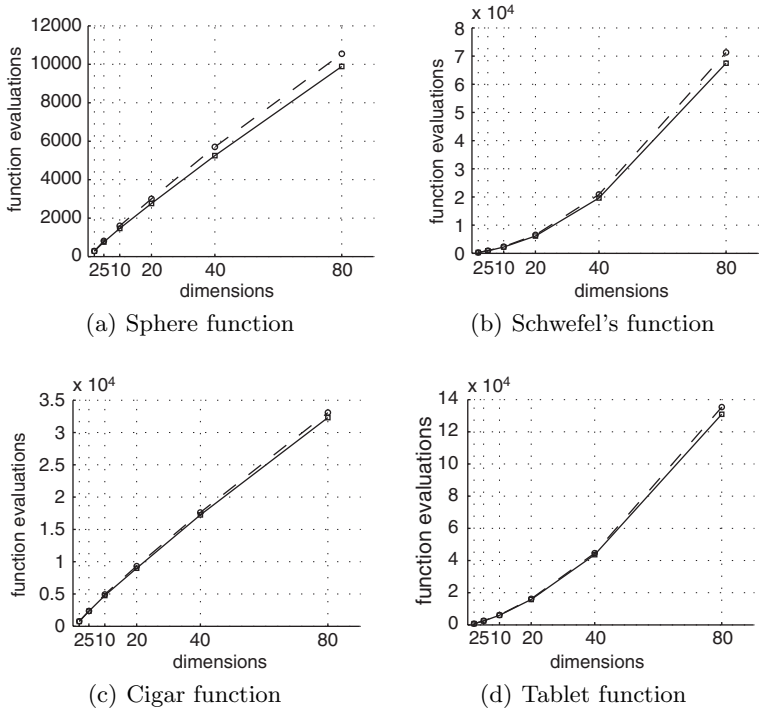
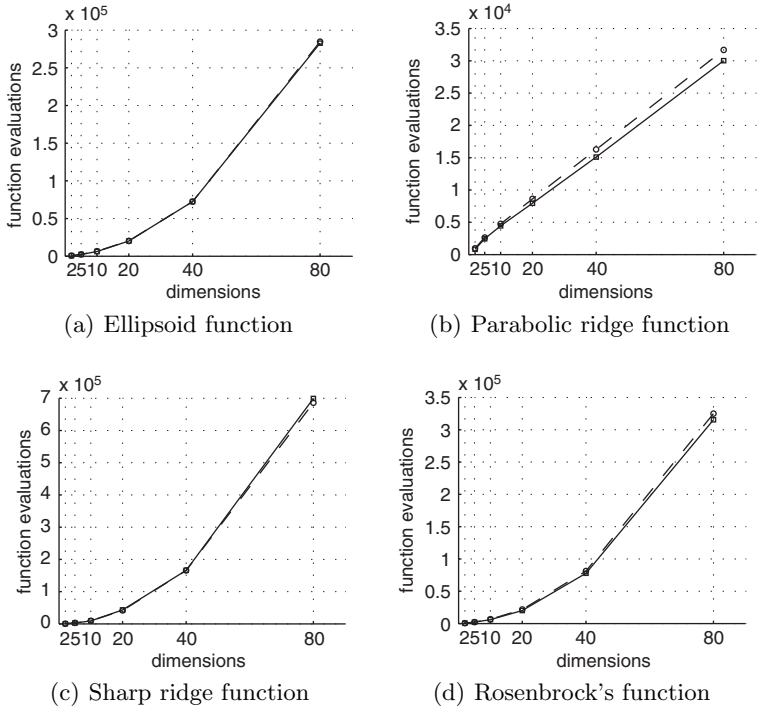


Fig. 2. Number of function evaluations versus the problem dimension for the functions f_{sphere}, f_{Schwefel}, f_{cigar}, and f_{tablet}. The Directed-CMA is plotted with a solid line, the original CMA with a dashed one



(a) Ellipsoid function (b) Parabolic ridge function

(c) Sharp ridge function (d) Rosenbrock's function

Fig. 3. Number of function evaluations versus the problem dimension for the functions f_{elli}, f_{parabR}, f_{sharpR}, and f_{Rosen}. The Directed-CMA is plotted with a solid line, the original CMA with a dashed one

To generate SN_n distributed random vectors their stochastic representation is used. Let Y have the probability density function $\varphi_n(\boldsymbol{z}; \boldsymbol{\mu}, \boldsymbol{\Omega})$ and W be a $\text{N}(0,1)$ distributed random variable. If

$$Z = \begin{cases} Y + \boldsymbol{\mu} & \text{if } W < \boldsymbol{\alpha}^T Y \\ -Y + \boldsymbol{\mu} & \text{otherwise,} \end{cases} \tag{3}$$

then $Z \sim \text{SN}_\text{n}(\boldsymbol{\mu}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$, see e.g. Wang et al. [17].

2.3 Shape Vector Control

As already mentioned, integration of the multivariate SN distribution into the CMA-ES framework is at a very early stage. As an ad hoc implementation we first used the mechanics of step-size adaptation to adjust the shape vector. Shape control then reads

$$\begin{aligned} \boldsymbol{p}_\alpha^{(g+1)} = {} & (1 - c_\alpha)\boldsymbol{p}_\alpha^{(g)} \\ & + \sqrt{c_\alpha(2 - c_\alpha)\mu_\text{eff}}\boldsymbol{C}^{(g)^{-\frac{1}{2}}} \frac{\boldsymbol{m}^{(g+1)} - \boldsymbol{m}^{(g)}}{\alpha^{(g)}} \end{aligned} \tag{4}$$

with learning rate

$$c_\alpha = \frac{\mu_{\text{eff}} + 2}{n + \mu_{\text{eff}} + 3} \tag{5}$$

and all other constants as given by Hansen and Kern [10]. Although the learning rate was altered over the whole $[0, \ldots, 1]$ range, no satisfying results were obtained during the test runs. Therefore the *intergenerational* shape update was replaced by an *intragenerational* one. Thus, we track the fitness of the generated offspring within every generation and adapt the shape vector accordingly.

One appropriate heuristic is given as follows: calculate the normalized direction vector from the mean of the current distribution to the actual offspring. If the fitness of this offspring is better than the mean fitness, then factor the direction vector into the shape vector. Otherwise take the opposite direction. Additionally, the fitness ratio is weighted exponentially and with the dimension. The definition of the update vector thus reads

$$\boldsymbol{u} = n \exp\left(\frac{\text{fit}\,\boldsymbol{x}}{\text{fit}\,\boldsymbol{x}_{\text{mean}}}\right) \frac{\boldsymbol{x} - \boldsymbol{x}_{\text{mean}}}{|\boldsymbol{x} - \boldsymbol{x}_{\text{mean}}|} \tag{6}$$

where n is the dimension, $\boldsymbol{x}$ and $\boldsymbol{x}_{\text{mean}}$ are the actual individual and the mean used to generate the offspring, respectively, and the function $\text{fit}(\cdot)$ gives the fitness of a sample. The intragenerational update of the shape vector $\boldsymbol{p}_\alpha^{[l]}$ depends on the actual individual's fitness.

$$\boldsymbol{p}_\alpha^{[l]} = \begin{cases} \boldsymbol{p}_\alpha^{[l-1]} + \boldsymbol{u} & \text{if } \text{fit}\,\boldsymbol{x} \succ \text{fit}\,\boldsymbol{x}_{\text{mean}} \\ \boldsymbol{p}_\alpha^{[l-1]} - \boldsymbol{u} & \text{else} \end{cases} \tag{7}$$

with $l \in [1, \ldots, \lambda]$ and $\boldsymbol{p}_\alpha^{[0]} = \boldsymbol{0}$.

Until now, neither the shape control nor the learning rate has been adapted satisfactorily to the special demands of shape vector control. However, even with this crude treatment of the shape vector some very promising results have been obtained.

3 Experimental Results

Two different CMA-ESs are experimentally investigated: the original variant as described by Hansen and Kern [10], using $\text{N}(0, \boldsymbol{C})$ distributed random vectors and the DCMA-ES, using instead $\text{SN}_\text{n}(\boldsymbol{\mu}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$ distributed random vectors with intragenerational shape update. For the comparison of the two functions, a test suite consisting of the eight well-known functions shown in Table 1 is used. Initial values are set to $\boldsymbol{x}^{(0)} \in [-1, 1]^n$, $\sigma^{(0)} = 1$, and $\alpha^{(0)} = \boldsymbol{0}$ for all functions except for Rosenbrock's case where $\boldsymbol{x}^{(0)} = \boldsymbol{0}$, $\sigma^{(0)} = 0.1$, and again $\alpha^{(0)} = \boldsymbol{0}$. As stopping criterion for all functions but f_{parabR} and f_{sharpR}

Table 1. Test functions

Name	Function	$\boldsymbol{x}^{(0)}$	$\sigma^{(0)}$	$\boldsymbol{\alpha}^{(0)}$	f^{stop}
Sphere	$f_{\text{sphere}}(\boldsymbol{x}) = \sum_{i=1}^{n} x_i^2$	$[-1,1]^n$	1	$\mathbf{0}$	10^{-10}
Schwefel	$f_{\text{Schwefel}}(\boldsymbol{x}) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} x_j\right)^2$	$[-1,1]^n$	1	$\mathbf{0}$	10^{-10}
Cigar	$f_{\text{cigar}}(\boldsymbol{x}) = x_1^2 + \sum_{i=2}^{n} (1000 x_i)^2$	$[-1,1]^n$	1	$\mathbf{0}$	10^{-10}
Tablet	$f_{\text{tablet}}(\boldsymbol{x}) = (1000 x_1)^2 + \sum_{i=2}^{n} x_i^2$	$[-1,1]^n$	1	$\mathbf{0}$	10^{-10}
Ellipsoid	$f_{\text{elli}}(\boldsymbol{x}) = \sum_{i=1}^{n} \left(1000^{\frac{i-1}{n-1}} x_i\right)^2$	$[-1,1]^n$	1	$\mathbf{0}$	10^{-10}
Parabolic ridge	$f_{\text{parabR}}(\boldsymbol{x}) = -x_1 + 100 \sum_{i=2}^{n} x_i^2$	$[-1,1]^n$	1	$\mathbf{0}$	-10^{10}
Sharp ridge	$f_{\text{sharpR}}(\boldsymbol{x}) = -x_1 + 100 \sqrt{\sum_{i=2}^{n} x_i^2}$	$[-1,1]^n$	1	$\mathbf{0}$	-10^{10}
Rosenbrock	$f_{\text{Rosen}}(\boldsymbol{x}) = \sum_{i=2}^{n-1} \left(100(x_i^2 - x_{i+1})^2 + (x_i - 1)^2\right)$	$\mathbf{0}$	0.1	$\mathbf{0}$	10^{-10}

Table 2. Sphere function

Dim	DCMA			CMA			Comparison	
	$\bar{x}$	σ	m	$\bar{x}$	σ	m	Δ	η
2	283.68	28.01	276	286.08	36.53	288	2.4	1.0085
5	756.48	48.94	760	820.8	68.20	808	64.32	1.085
10	1466.4	81.13	1460	1598.8	84.87	1600	132.4	1.0903
20	2762.88	72.27	2760	3005.28	127.28	3024	242.4	1.0877
40	5250.6	154.1	5265	5703.6	131	5730	453	1.0863
80	9888.56	176.92	9843	10545.44	192.25	10540	656.88	1.0664

fitness better than 10^{-10} is demanded, for the two others fitness has to be less than -10^{10}. All functions, except for f_{sphere}, are highly nonseparable. Tests are carried out in $n = [2, 5, 10, 20, 40, 80]$ dimensions and for offspring numbers $\lambda = 4 + \lfloor 3 \log n \rfloor$ with parent numbers $\mu = \lceil \lambda/2 \rceil$. For each combination 25 runs are done. Depicted in figures are the results of each case, the corresponding figures are given in the Tables 2–9. Reported is the mean $\bar{x}$ of necessary function evaluations, the standard deviation σ, and the median m for each constellation. In the comparison $\Delta = \bar{x}_{\text{CMA}} - \bar{x}_{\text{DCMA}}$ gives the difference of the means, where positive values indicate better performance of the DCMA. $\eta = \bar{x}_{\text{CMA}}/\bar{x}_{\text{DCMA}}$ represents the ratio of the means, i.e. the factor the DCMA performs better.

3.1 Discussion of the Results

The runs on f_{sphere} show a DCMA-ES that outperforms the CMA-ES relative constantly at about 8% for all but $n = 2$. On f_{Schwefel} it performs approximately 6% better for all dimensions, while the gain on f_{cigar} and f_{tablet} is only

Table 3. Schwefel's function

Dim	DCMA			CMA			Comparison	
	$\bar{x}$	σ	m	$\bar{x}$	σ	m	Δ	η
2	290.4	31.18	288	309.12	31.94	312	18.72	1.0645
5	907.2	63.2	912	961.6	90.3	936	54.4	1.06
10	2216	88.88	2230	2378	119.44	2350	162	1.0731
20	6136.8	226.07	6156	6538.08	236.29	6516	401.28	1.0654
40	19629	566.82	19650	20872.2	505.01	20835	1243.2	1.0633
80	67476.4	941.4	67286	71267.4	1020.08	71400	3791	1.0562

Table 4. Cigar function

Dim	DCMA			CMA			Comparison	
	$\bar{x}$	σ	m	$\bar{x}$	σ	m	Δ	η
2	753.36	57.47	762	778.8	66.27	786	25.44	1.0338
5	2335.68	79.48	2344	2380.48	108.37	2384	44.8	1.0192
10	4765.2	134.79	4780	4948.4	163.27	4940	183.2	1.0384
20	8992.8	184.51	9000	9328.8	186.74	9336	336	1.0374
40	17245.8	241.34	17295	17634	320.4	17535	388.2	1.0225
80	32305.44	263.17	32300	33073.16	308.83	33167	767.72	1.0238

Table 5. Tablet function

Dim	DCMA			CMA			Comparison	
	$\bar{x}$	σ	m	$\bar{x}$	σ	m	Δ	η
2	747.36	56.82	762	778.08	78.95	780	30.72	1.0411
5	2452.16	128.87	2400	2501.44	129.09	2504	49.28	1.0201
10	5971.2	216.4	5960	6108.4	199.6	6090	137.2	1.023
20	15768	297.87	15720	16211.04	230.48	16236	443.04	1.0281
40	43626.6	476.93	43620	44574	528.52	44760	947.4	1.0217
80	130870.08	1176.82	130764	135356.72	1039.39	135252	4486.64	1.0343

Table 6. Ellipsoid function

Dim	DCMA			CMA			Comparison	
	$\bar{x}$	σ	m	$\bar{x}$	σ	m	Δ	η
2	748.56	55.51	762	750.72	66.69	750	2.16	1.0029
5	2318.72	97.7	2312	2456.32	92.25	2472	137.6	1.0593
10	6298	176.28	6310	6522	269.61	6570	224	1.0356
20	20229.6	347.31	20280	20405.76	442.88	20484	176.16	1.0087
40	72435.6	704.93	72495	72492.6	557.66	72420	57	1.0008
80	282727	4438.34	285039	284641.2	3501.13	285736	1914.2	1.0068

Table 7. Parabolic ridge

Dim	DCMA			CMA			Comparison	
	$\bar{x}$	σ	m	$\bar{x}$	σ	m	Δ	η
2	824.88	58.61	822	983.28	86.28	978	158.4	1.1920
5	2444.8	131.13	2440	2654.4	161	2672	209.6	1.0857
10	4442.4	205.41	4390	4756	235.67	4780	313.6	1.0706
20	7949.28	246.66	7884	8619.36	273.10	8568	670.08	1.0843
40	15095.4	233.46	15030	16271.4	182.14	16215	1176	1.0779
80	30016.0	261.38	29971	31681.88	269.24	31654	1665.88	1.0555

Table 8. Sharp ridge

Dim	DCMA			CMA			Comparison	
	$\bar{x}$	σ	m	$\bar{x}$	σ	m	Δ	η
2	617.28	87.93	624	771.6	101.1	780	154.32	1.25
5	2596.16	365.47	2584	2856.96	249.11	2808	260.8	1.1005
10	9275.2	1544.72	9090	9644.4	1385.66	9280	369.2	1.0398
20	43298.4	3752.47	42504	41407.68	2681.13	41328	−1890.72	0.9563
40	166173	8355.13	168225	165607.8	8854.61	165900	−565.20	0.9966
80	699454.8	25741.55	701658	685775.24	22478.54	682329	−13679.56	0.9804

Table 9. Rosenbrock

Dim	DCMA			CMA			Comparison	
	$\bar{x}$	σ	m	$\bar{x}$	σ	m	Δ	η
2	585.84	64.04	582	644.16	59.67	654	58.32	1.0995
5	2136.96	151.21	2136	2318.08	177.45	2320	181.12	1.0848
10	6116.8	247.94	6120	6655.6	315.52	6630	538.8	1.0881
20	20182.56	597.03	20148	21740.16	519.10	21576	1557.6	1.0772
40	77881.2	1008.21	77760	81233.4	1221.92	81285	3352.2	1.0430
80	315456.08	2525.97	315418	325051.56	3497.75	324037	9595.48	1.0304

about 3%, and greater than 2%, repectively. On the function f_{elli} the outcome is somewhat irregular. The result is significant only for 5 and 10 dimensions, where the DCMA-ES is slightly superior. Interesting is the 2-dimensional case on f_{parabR}, where the DCMA-ES is 19% better, besides about 8%. The high performance of the DCMA-ES in low dimensions is also true on f_{sharpR}. But here its performance is rapidly decreasing with increasing dimensions. For $n > 10$ it is outperformed by the CMA-ES. The same tendency can be seen on. Here the gain of the DCMA-ES decreases from 10% for $n = 2$–3% for $n = 80$. In general, the DCMA-ES performs better on all functions except on f_{sharpR}. On average, there is a gain of a few percentage points. This must be seen against the background of the CMA-ES considered already as state-of-the-art in parameter optimization, the preliminary design of shape vector control, and the very small overhead caused by directed mutation. In fact, all

that has to be done is to calculate one n-dimensional scalar product, generate one univariate random number, and do one comparison. Compared to a function evaluation in a real world application this can rather be neglected.

4 Conclusions

With directed mutation a promising new mutation principle for uncorrelated EAs had been presented that has now been ported to the correlated setting in a CMA-ES context. First results with the DCMA-ES left us optimistic about the potential of this approach. It is a zeroth-order algorithm causing only very small overhead. Regarding the presented results, it has to be kept in mind that intergenerational adaptation of the shape vector is still left open, intragenerational adaptation itself is in an inadequate state, and the learning rate has to be further investigated. Thus, much work is left to be done to tune the DCMA-ES. Finally, comprehensive experimental studies which should also include multi-objective problems are undone.

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The Choquet-Integral as an Aggregation Operator in Case-Based Learning

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Summary. In case-based learning, various types of aggregation problems have to be solved. In particular, proceeding from an attribute-value representation of cases, a (global) similarity measure for (pairs of) cases is commonly derived from local similarity functions pertaining to individual attributes. Moreover, in connection with the nearest neighbor estimation principle, an aggregation operator is needed in order to combine the votes coming from the query's neighbors. In this paper, we argue that considering potential *interdependencies* between individual pieces of information (like similarity degrees or votes) is quite important in the context of case-based learning. In this connection, we advocate the Choquet integral as a suitable aggregation operator. In particular, we introduce a method called Cho-k-NN, which generalizes the weighted nearest neighbor estimation by taking the mutual similarities between the query's neighbors into account. Besides, we suggest using the Choquet integral in order to combine local similarity functions into global measures.

1 Introduction

The case-based learning (CBL) paradigm relies upon memorizing cases in the form of successful problem solving experiences. When it comes to solving a new problem, each of these cases serves as an individual piece of evidence that gives an indication of the solution to that problem. Thus, rather than inducing a global model (theory) from the data and using this model for further reasoning, as inductive, model-based machine learning methods typically do, CBL systems simply store the data itself. The processing of the data is deferred until a prediction (or some other type of query) is actually requested, a property which qualifies CBL as a *lazy* learning method [1]. Predictions are then derived by combining the information provided by the stored cases, primarily by those which are *similar* to the new query. Thus, the concept of similarity is of central importance in CBL.

Case-based *aka* instance-based learning algorithms have been applied successfully in diverse fields, including machine learning and pattern recognition [2, 4]. Besides, the case-based learning paradigm is also at the heart of

case-based reasoning (CBR), a problem solving methodology which goes beyond standard prediction problems like classification and regression [10, 14].

The need to *aggregate* different pieces of information arises in case-based learning in several respects. In particular, proceeding from an attribute-value representation of cases, the (global) similarity between two cases is typically defined as an aggregation of local similarity degrees referring, respectively, to the individual attributes. Moreover, the basic estimation principle in case-based prediction, the nearest neighbor (NN) rule, requires combining the votes coming from the query's nearest neighbors.

A standard aggregation operator which is commonly employed in CBL is the weighted arithmetic mean. This operator implicitly assumes the *independence* of the pieces of information to be aggregated. In this paper, we advocate the Choquet integral as an alternative aggregation operator, mainly due to its ability to take interdependencies between information sources into account. In fact, we argue that this ability is quite useful in the context of CBL, especially for the two types of aggregation problems mentioned above.

By way of background, Sect. 2 gives a concise review of the NN principle, which constitutes the core of case-based learning algorithms. In Sect. 3, we discuss the problem of interaction between cases in CBL and propose a new NN inference principle which takes such interactions into account. This method, called Cho-k-NN, is evaluated empirically in section 4. In Sect. 5, we discuss the use of the Choquet integral as an operator for combining several local similarity measures into a global one.

2 Nearest Neighbor Estimation

The well-known nearest neighbor (NN) estimation principle is applicable to both classification problems (prediction of discrete class labels) and regression (prediction of numeric values). Consider a setting in which an instance space $\mathcal{X}$ is endowed with a similarity measure $\text{sim} : \mathcal{X} \times \mathcal{X} \to [0, 1]$. An instance corresponds to the description x of an object (usually in attribute-value form, cf. Sect. 5). In the standard classification framework, each instance x is assumed to have a (unique) label $y \in \mathcal{L}$. Here, $\mathcal{L}$ is a finite (typically small) set comprised of m class labels $\{\ell_1 \dots \ell_m\}$, and $\langle x, y\rangle \in \mathcal{X} \times \mathcal{L}$ is a labeled instance (case).

The NN principle originated in the field of pattern recognition [4]. Given a sample S consisting of n labeled instances $\langle x_\imath, y_\imath\rangle$, $1 \leq \imath \leq n$, and a novel instance $x_0 \in \mathcal{X}$ (a query), this principle prescribes to estimate the label y_0 of the yet unclassified query x_0 by the label of the nearest (most similar) sample instance. The k-nearest neighbor (k-NN) approach is a slight generalization, which takes the $k \geq 1$ nearest neighbors of x_0 into account. That is, an estimation y_0^{est} of y_0 is derived from the set $\mathcal{N}_k(x_0)$ of the k nearest neighbors of x_0, usually by means of a *majority vote*. Besides, further conceptual extensions of the (k-)NN principle have been devised, such as distance weighing [5]:

$$y_0^{est} = \arg\max_{\ell \in \mathcal{L}} \sum_{\langle x,y\rangle \in \mathcal{N}_k(x_0)} \omega_x \cdot \mathbb{I}(y = \ell) \tag{1}$$

where ω_x is the weight of the instance x and $\mathbb{I}(\cdot)$ the standard $\{\text{true}, \text{false}\} \to \{0, 1\}$ mapping. (Throughout the paper, we assume the weights to be given by $\omega_x = \text{sim}(x, x_0)$.)

The NN principle can also be used for regression problems, i.e., for realizing a (locally weighted) approximation of real-valued target functions $x \mapsto y = f(x)$ (in this case, $\mathcal{L} = \mathbb{R}$). To this end, one reasonably computes the (weighted) mean of the k nearest neighbors of a new query point:

$$y_0^{est} = \frac{\sum_{\langle x,y\rangle \in \mathcal{N}_k(x_0)} \omega_x \cdot y}{\sum_{\langle x,y\rangle \in \mathcal{N}_k(x_0)} \omega_x} \tag{2}$$

3 The Cho-k-NN Method

In k-NN estimation, the cases in a query's neighborhood are basically considered as *independent* information sources: In classification, the evidences in favor of a certain class label are simply added up (see (1)). Likewise, in regression the estimation is a simple linear combination of the observed outcomes (see (2)). This assumption of independence between case-based evidence can thoroughly be called into question [8]. Indeed, it is not even in agreement with the key assumption underlying CBL, namely the "similarity hypothesis" suggesting that similar problems (instances) have similar solutions (outputs). In fact, if this hypothesis is true, then two neighbored cases that are not only similar to the query case *but also similar among each other* will probably provide similar information regarding the query. In other words, when taking the similarity hypothesis for granted, the information coming from the neighbored cases is at least not independent. In particular, from a problem solving perspective, one should realize that a set of cases can be *complementary* in the sense that the experiences represented by the individual cases complement or reinforce each other. On the other hand, cases can also be *redundant* in the sense that much of the information is already represented by a smaller subset among them.

To illustrate this point by an example, consider a simple estimation problem, namely to predict the yearly rainfall at a certain location (city). For

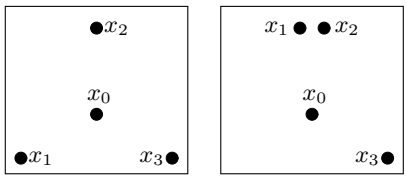

Fig. 1. Different configurations of locations in two-dimensional space

instance, given the rainfall $y_\imath$ at location $x_\imath$ ($\imath = 1, 2, 3$), what about the rainfall at location x_0 in the two scenarios shown in Fig. 1? The important point to notice is that even though the individual distances between x_0 and the $x_\imath$ are the same in both scenarios, the $y_\imath$ should not be combined in the same way. In this example, this is due to the different *arrangements* of the neighbors [19]: Simply predicting the arithmetic mean $(y_1 + y_2 + y_3)/3$ seems to be reasonable in the left scenario, while the same prediction appears questionable in the scenario shown in the right picture. In fact, since x_1 and x_2 are closely neighbored in the latter case, information about the rainfall at these locations will be partly redundant. Consequently, the weight of the (joint) evidence that comes from the observations $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$ should not be twice as high as the weight of the evidence that comes from $\langle x_3, y_3 \rangle$.

The above example shows the need for taking interdependencies between observed cases into account and, hence, provides a motivation for the method that will be proposed below. Before proceeding, let us make two further remarks: First, the above type of interaction between cases seems to be less important if the sample size is large and even becomes negligible in asymptotic analyses of NN principles. In fact, strong results on the performance of NN estimation can be derived [4], but these are valid only under idealized statistical assumptions and arbitrarily large sample sizes. Roughly speaking, if the sample size n tends to infinity, the distance between the query and its nearest neighbors becomes arbitrarily small (with high probability). This holds true even if the size k of the neighborhood is increased too, as a function $k(n)$ of n, provided that $k(n)/n \to 0$ for $n \to \infty$. Moreover, if the individual observations are independent and identically distributed in a statistical sense, the neighborhood becomes "well distributed". Under these assumptions, it is intuitively clear that interdependencies between observations will hardly play any role. On the other hand, it is also clear that statistical assumptions of such kind will almost never be satisfied in practice.

The second remark concerns related work. In fact, there are a few methods that fit into the CBL framework and that allow for taking certain types of interaction between observations into account. Particularly, these are methods that make assumptions on the statistical correlation between observations, depending on their distance [11]. For example, in our rainfall example one could employ a method called *kriging*, which is well-known in geostatistics [12]. Usually, however, such methods are specialized on a particular type of application and, moreover, make rather restrictive assumptions on the mathematical (metric) structure of the instance space. Our approach, to be detailed below, is much more general in the sense that it only requires a similarity measure $\mathrm{sim}(\cdot)$ to be given. We do not make any particular assumptions on this measure (such as symmetry or any kind of transitivity), apart from the fact that it should be normalized to the range $[0, 1]$. From an application point of view, this seems to be an important point. In CBR, for example, cases are typically complex objects that cannot easily be embedded into a metric space.

3.1 Modeling Interaction in Case-Based Learning

Let X be a finite set and $\nu(\cdot)$ a nonadditive measure (fuzzy measure) $2^X \to [0,1]$. That is, $\nu(\cdot)$ is normalized ($\nu(\emptyset) = 0$, $\nu(X) = 1$) and monotone ($\nu(A) \leq \nu(B)$ for $A \subseteq B$) but not necessarily additive [16]. For any $A \subseteq X$, we interpret $\nu(A)$ as the *weight* or, say, the *degree of relevance* of the set of elements A.

In connection with the problem of combining evidence in NN estimation, the set X of elements corresponds to the neighbors of the query case x_0:

$$X = \mathcal{N}_k(x_0) = \{x_1, x_2 \ldots x_k\} \tag{3}$$

It is well-known that fuzzy measures (as opposed to additive measures) can principally be used for modeling interaction between elements (subsets) of X. The basic question that we have to address in the CBL context is the following: What is the *evidence weight* or simply the *weight*, $\nu(A)$, of a subset A of the neighborhood (3)?

First, the boundary conditions $\nu(\emptyset) = 0$ and $\nu(X) = 1$ should of course be satisfied, expressing that the full evidence is provided by the complete neighborhood X. Moreover, according to our comments above, the evidence coming from a set of cases $A \subseteq X$ should be discounted if these cases are similar among themselves. Likewise, the weight of A should be increased if the cases are "diverse" (hence complementary) in a certain sense. To express this idea in a more rigorous way, we define the *diversity* of a set of cases A by the sum of pairwise dissimilarities:

$$\mathrm{div}(A) \stackrel{\mathrm{df}}{=} \sum_{x_i \neq x_j \in A} 1 - \mathrm{sim}(x_i, x_j)$$

(By definition, the diversity is 0 for singletons and the empty set.) Now, the idea is to modify the basic (additive) measure

$$\mu(A) \stackrel{\mathrm{df}}{=} \sum_{x_i \in A} \mathrm{sim}(x_0, x_i) \tag{4}$$

by taking the diversity of A into account. This can of course be done in different ways. Here, we use the following approach:

$$\bar{\nu}(A) \stackrel{\mathrm{df}}{=} \mu(A) + \alpha\, \mathrm{div}(A) \tag{5}$$

As can be seen, the higher the diversity of a set of cases A, the more the original measure $\mu(A)$ is increased. The parameter $\alpha \geq 0$ controls the extent to which interactions between cases are taken into consideration. For $\alpha = 0$, interactions are completely ignored and the original measure $\mu(\cdot)$ is recovered.

The measure $\bar{\nu}(\cdot)$ in (5) is obviously monotone. To guarantee the boundary condition, we finally define

$$\nu(A) \stackrel{\mathrm{df}}{=} \bar{\nu}(A)/\bar{\nu}(X). \tag{6}$$

Before going on, let us comment on the derivation of the measure $\nu(\cdot)$ from the similarity function $\mathrm{sim}(\cdot)$. Firstly, even though the measure (6) captures our intuitive idea of decreasing (increasing) the evidence weight of cases that are (dis)similar by themselves, we admit that it remains ad hoc to some extent, and by no means we exclude the existence of alternative formalizations. In fact, in [9], where we first introduced the Cho-k-NN method, we used a different measure. Secondly, even though the assumption that similar cases provide redundant information is supported by the similarity hypothesis, one might of course argue that the similarity between the predictive parts of two cases, $x_\imath$ and $x_\jmath$, is not sufficient to call them redundant. Rather, the associated output values $y_\imath$ and $y_\jmath$ should be similar as well. Indeed, if y_1 differs drastically from y_2, the first two measurements in our rainfall example might better be considered as nonredundant. (In that case, the two measurements in conjunction suggest that there is something amiss ...) This conception of redundancy can easily be represented by deriving $\nu(\cdot)$ from an extended similarity measure $\mathrm{sim}'(\cdot)$ which is defined over $\mathcal{X} \times \mathcal{L}$.[1]

3.2 Aggregation of Interacting Pieces of Evidence

So far, we have a tool for modeling the interaction between different pieces of evidence in case-based learning. The next question that we have to address is how to *combine* these pieces of evidence, i.e., how to aggregate them in agreement with the evidence measure $\nu(\cdot)$.

For the time being we focus on the problem of regression. Recall that in the standard approach to NN estimation, an aggregation of the output values $f(x_\imath) = y_\imath$ is realized by means of a simple weighted average:

$$y_0^{est} \;=\; \sum_{x_\imath \in X} \mu(\{x_\imath\}) \cdot f(x_\imath), \tag{7}$$

where $\mu(\{x_\imath\}) = \mathrm{sim}(x_0, x_\imath) \left(\sum_{x_\imath \in X} \mathrm{sim}(x_0, x_\imath)\right)^{-1}$. Interestingly, (7) is nothing else than the standard Lebesgue integral of the function $f : X \to \Re$ with respect to the additive measure $\mu(\cdot)$ (i.e., the additive extension of the $\mu(\{x_\imath\})$ in (7)):

$$y_0^{est} \;=\; \int_X f \, d\mu$$

In order to generalize this estimation, an integral with respect to the nonadditive measure $\nu(\cdot)$ is needed: the Choquet integral, a concept that originated in capacity theory [3].

Let $\nu(\cdot)$ be a fuzzy measure and $f(\cdot)$ a non-negative function.[2] The Choquet integral of $f(\cdot)$ with respect to $\nu(\cdot)$ is then defined by

[1] We did not explore this alternative in detail so far.

[2] The Choquet integral can be extended to any real-valued function through decomposition into a positive and negative part.

$$\int^{ch} f \, d\nu \stackrel{\text{df}}{=} \int_0^{\infty} \eta([f > t]) \, dt$$

where $[f > t] = \{x \,|\, f(x) > t\}$. The integral on the right-hand side is the standard Lebesgue integral (with respect to the Borel measure on $[0, \infty)$). In our case, where X is a finite set, we can refer to the *discrete* Choquet integral which can be expressed in a rather simple form:

$$y_0^{est} = \sum_{\imath=1}^{k} f(x_{\pi(\imath)}) \cdot \big(\nu(A_\imath) - \nu(A_{\imath-1})\big) \,, \tag{8}$$

where $\pi(\cdot)$ is a permutation of $\{1 \dots k\}$ such that $0 \leq f(x_{\pi(1)}) \leq \ldots \leq f(x_{\pi(k)})$, and $A_\imath = \{x_{\pi(1)} \dots x_{\pi(\imath)}\}$.

The discrete Choquet integral (8) can be seen as a special type of aggregation operator, namely a generalized arithmetic mean. Indeed, (8) coincides with (7) if $\nu(\cdot)$ is an additive measure. Otherwise, it is a proper generalization of the standard (weighted) NN estimation.

So far, we have focused on the problem of regression. In the case of classification, the Choquet integral cannot be applied immediately, since an averaging of class labels $y_\imath$ does not make sense. Instead, the Choquet integral can be derived for each of the indicator functions $f_\ell : y \mapsto \mathbb{I}(y = \ell)$, $\ell = \ell_1 \dots \ell_m$. As in (1), the evidence in favor of each class label is thus accumulated separately. Now, however, the interaction between cases is taken into account. As usual, the estimation is then given by the label with the highest degree of accumulated evidence.

4 Empirical Validation

In order to validate the extension of NN estimation as proposed in the previous section, called Cho-k-NN, we have performed several experimental studies using benchmark data sets from the UCI repository[3] and the StatLib archive.[4] Experiments were performed in the following way: A data set is randomly split into a training and a test set of the same size. For each example in the test set, a prediction is derived using the training set in combination with weighted k-NN resp. Cho-k-NN. In the case of regression, an estimation y_0^{est} is evaluated by the relative estimation error $|y_0^{est} - y_0| \cdot |y_0|^{-1}$, and the overall performance of a method by the mean of this error over all test examples. In the case of classification, we simply took the misclassification rate as a performance index. Moreover, we derived statistical estimations of the *expected* performance of a method by repeating each experiment 100 times.

For the purpose of similarity computation, all numeric attributes have first been normalized to the unit interval by linear scaling. The similarity was then defined by 1-distance for numeric variables and by the standard

[3] `http://www.ics.uci.edu/~mlearn`

[4] `http://stat.cmu.edu/`

0/1-measure in the case of categorical attributes. The overall similarity sim(·) was finally obtained by the average over all attributes. As the purpose of our study was to *compare* – under equal conditions – weighted k-NN with Cho-k-NN in order to verify whether or not taking interactions into account is useful, we refrained from tuning both methods, e.g., by including feature selection or feature weighing (even though it is well-known that such techniques can greatly improve performance [17]). Results have been derived for neighborhood sizes of $k = 5$ and $k = 7$; the parameter α in (5) has always been set to 1/3.

The application of Cho-k-NN for regression has shown that it consistently improves weighted k-NN, sometimes only slightly but often even considerably. Some results are shown in Table 1. In particular, it seems that the smaller the size of the data set, the higher the gain in performance. This finding is intuitively plausible, since for large data sets the neighborhoods of a query tend to be more "balanced"; as already said, the neglect of interaction is likely to be less harmful under such circumstances.

For classification problems, it is also true that Cho-k-NN consistently outperforms weighted k-NN; see Table 2. Usually, however, the gain in classification accuracy is only small, in many cases not even statistically significant. Again, this is especially true for large data sets, and all the more if the classification error is already low for standard k-NN. Nevertheless, one should bear in mind that, in the case of classification, the final prediction is largely insensitive toward modifications of the estimated evidences in favor of the potential labels. In fact, in this study we only checked whether the final prediction is correct or not and, hence, used a rather crude quality measure. More subtle improvements of an estimation such as, e.g., the enlargement of an example's margin [15], are not honored by this measure.

Table 1. Estimation of expected relative estimation error and its standard deviation

Data set	k	Weighted k-NN	Cho-k-NN
auto-mpg	5	12.21 (0.05)	11.36 (0.04)
	7	12.18 (0.06)	11.21 (0.05)
bolts	5	47.07 (1.19)	38.73 (0.69)
	7	51.36 (1.25)	39.92 (0.80)
housing	5	14.83 (0.08)	14.41 (0.06)
	7	14.99 (0.09)	14.55 (0.06)
detroit	5	16.02 (0.55)	14.90 (0.45)
	7	15.93 (0.55)	14.66 (0.51)
echomonths	5	97.77 (7.17)	72.87 (3.87)
	7	99.03 (8.98)	74.80 (7.55)
pollution	5	4.12 (0.05)	4.01 (0.05)
	7	4.22 (0.05)	4.16 (0.04)

Table 2. Estimation of expected classification error and its standard deviation

Data set	k	Weighted k-NN	Cho-k-NN
glass	5	33.58 (0.34)	33.16 (0.31)
	7	34.54 (0.37)	33.65 (0.32)
wine	5	3.52 (0.20)	3.41 (0.18)
	7	3.27 (0.20)	3.24 (0.19)
zoo	5	10.43 (0.60)	10.09 (0.61)
	7	11.65 (0.51)	11.37 (0.53)
ecoli	5	16.30 (0.25)	16.25 (0.22)
	7	16.17 (0.24)	16.07 (0.21)
balance	5	15.62 (0.14)	15.24 (0.17)
	7	13.28 (0.16)	13.16 (0.18)
derma	5	3.75 (0.13)	3.47 (0.11)
	7	3.57 (0.12)	3.39 (0.10)

5 Specification of Similarity Measures

Instances (cases) are usually formalized in terms of an attribute-value representation, i.e., an instance x is characterized as a vector $x = (a_1 \dots a_n)$ of attribute values. Denote by $\mathcal{A}_\imath$ the domain of the $\imath$th attribute and let $\mathcal{X} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$. Moreover, suppose a *global similarity measure* $\mathrm{sim} : \mathcal{X} \times \mathcal{X} \to \mathfrak{R}_{\geq 0}$ to be given. The so-called *local-global principle* makes the following assumption [13]: There are *local similarity measures* $\mathrm{sim}_\imath : \mathcal{A}_\imath \times \mathcal{A}_\imath \to \mathfrak{R}_{\geq 0}$ and a *composition function* $f : (\mathfrak{R}_{\geq 0})^n \to \mathfrak{R}_{\geq 0}$ such that

$$\mathrm{sim}(x, x') = f\left(\mathrm{sim}_1(a_1, a'_1) \dots \mathrm{sim}_n(a_n, a'_n)\right) \tag{9}$$

for all $x = (a_1 \dots a_n), x' = (a'_1 \dots a'_n) \in \mathcal{X}$. There are some reasonable properties that might be assumed in connection with the representation (9). For example, the *global monotonicity axiom* states that, for all $a, a', a'' \in \mathcal{A}$:

$$\mathrm{sim}(a, a') < \mathrm{sim}(a, a'') \Rightarrow \exists\, \imath \in \{1 \dots n\} \,:\, \mathrm{sim}_\imath(a_\imath, a'_\imath) < \mathrm{sim}_\imath(a_\imath, a''_\imath).$$

According to the local-global principle for similarities, deriving a global similarity relation from a set of individual relations comes down to defining an adequate aggregation operator. Ideally, such an operator should preserve certain properties of the individual relations. Most aggregation operators do preserve reflexivity and symmetry, but not necessarily transitivity. This remark does already apply to the weighted arithmetic mean. On the other hand, when using a t-norm $\otimes$ as an aggregation operator, it is not difficult to show that the global measure

$$\mathrm{sim} : (a, a') \mapsto \mathrm{sim}_1(a_1, a'_1) \otimes \mathrm{sim}_2(a_2, a'_2) \otimes \dots \otimes \mathrm{sim}_n(a_n, a'_n)$$

is $\otimes$-transitive whenever the same holds true for the local measures $\mathrm{sim}_\imath$ ($1 \leq \imath \leq n$). In fact, several aggregation operators which have been studied extensively in the literature on fuzzy sets are quite interesting in the context of case-based reasoning. For example, *weighted* aggregations [6] are of particular interest since they allow for assigning a level of importance to the attributes. Ordered weighted averaging (OWA) operators, which generalize several well-known operators such as, e.g., the mean and the minimum, have also been proposed as aggregation operators in CBR [18].

Another interesting aggregation operator is again the Choquet integral, as it does not only allow for weighing individual attributes, but also to take interdependencies between attributes into account. By making use of the (discrete) Choquet integral as an aggregation operator, the following global measure is obtained:

$$\mathrm{sim} : (x, x') \mapsto \int^{ch} h \, d\eta, \tag{10}$$

where $h : \{1 \ldots n\} \to [0, 1]$ is given by the mapping $\imath \mapsto \mathrm{sim}_\imath(a_\imath, a'_\imath)$. That is, $h(\imath)$ denotes the similarity between the attribute values $a_\imath$ and $a'_\imath$. Moreover, $\eta : 2^{\{1 \ldots n\}} \to [0, 1]$ is again a normalized and inclusion-monotone measure.

Let π denote a permutation of $\{1 \ldots n\}$ such that $h(\pi(\imath)) \leq h(\pi(\imath + 1))$ for $1 \leq \imath < n$. That is, π arranges the attributes according to the degree of similarity. The similarity function (10) can then be written as follows:

$$\mathrm{sim} : (x, x') \mapsto \sum_{\imath=1}^{n} h(\pi(\imath)) \cdot (\eta(\{\pi(1) \ldots \pi(\imath)\}) - \eta(\{\pi(1) \ldots \pi(\imath - 1)\})), \tag{11}$$

where $\eta(\emptyset) \stackrel{\mathrm{df}}{=} 0$. Note that (11) includes several known aggregation operators as special cases. For instance, with η being the counting measure $X \mapsto 1/n \cdot |X|$ we obtain the arithmetic mean. More generally, let η be the additive measure with $\eta(\{\imath\}) = \alpha_\imath$ for all $1 \leq \imath \leq n$, where $0 \leq \alpha_\imath \leq 1$ and $\alpha_1 + \ldots + \alpha_n = 1$. The global measure (11) is then given by the weighted arithmetic mean

$$\mathrm{sim} : (x, x') \mapsto \sum_{\imath=1}^{n} \alpha_\imath \cdot \mathrm{sim}_\imath(a_\imath, a'_\imath).$$

OWA operators are recovered if $\eta(\cdot)$ is symmetric (commutative), i.e., if $\eta(X)$ only depends on the cardinality of X. For example, a kind of threshold similarity can be modeled by letting, for a fixed $k \in \{1 \ldots n\}$, $\eta(X) = 1$ if $|X| \geq n - k + 1$ and $\eta(X) = 0$ otherwise. The similarity between two objects is then given by the k-th highest among the similarity degrees, that is,

$$\mathrm{sim}(a, a') = h(\pi(n - k + 1)), \tag{12}$$

expressing that the objects must resemble each other according to "at least k out of n" criteria. The special case $k = n$ yields the minimum operator as an aggregation function:

$$\mathrm{sim} : (a, a') \mapsto \min_{1 \leq \imath \leq n} \mathrm{sim}_\imath(a_\imath, a'_\imath). \tag{13}$$

As already mentioned above, an interesting aspect in connection with the Choquet functional as an aggregation operator is its capability to take interdependencies between different attributes into account. In fact, in many applications the global similarity between two objects does not simply correspond to the (weighted) sum of the local similarities. Suppose, for example, that the $\imath$th and the $\jmath$th attribute are complementary in a certain sense. In order to call two objects similar, it might hence be required that *both*, $a_\imath$ is similar to $a'_\imath$ and $a_\jmath$ is similar to $a'_\jmath$. The minimum in (13), for instance, might be seen as an adequate aggregation operator if *all* attributes are complementary in this sense. The measure (12) combines this type of complementarity of attributes with a compensation effect, since the similarity with regard to one attribute can compensate for the dissimilarity with respect to another one.

6 Concluding Remarks

In this paper, we advocated the use of the Choquet integral for two types of aggregation problems that arise in case-based learning: combining predictions in nearest neighbor estimation, and combining local similarity functions into global measures.

Our CHO-k-NN method is a direct extension of the standard weighted NN estimation. As opposed to the latter, however, it allows one to take the mutual dependencies between neighbored cases into account. Our experimental results have shown that CHO-k-NN consistently outperforms standard (weighted) k-NN on publicly available benchmark data. Nevertheless, there is scope for further development. For example, one might think of alternative strategies for deriving the nonadditive measure $\nu(\cdot)$ from the similarity function $\mathrm{sim}(\cdot)$ (cf. Sect. 3). In particular, the degree to which a set of cases is complementary resp. redundant might not only depend on their mutual similarity but also on other aspects.

Regarding the specification of similarity measures, Sect. 5 has only given a first idea of how to make use of generalized measures and integrals. Of course, there are questions of practical importance which call for further investigation. In particular, this concerns the definition (elicitation) of the measure $\eta(\cdot)$ [7]. How should an expert determine η to depict his view of similarity in an optimal way? Besides, it would be interesting to solve the *inverse problem*: Given a set of examples in the form of global similarity evaluations provided by some expert, induce (or approximate) the measure η this expert has used in order to derive these evaluations. Likewise, given a set of training examples, one might try to adapt the measure $\eta(\cdot)$ so as to maximize the predictive accuracy of a case-based learner.

A further topic of future work, relevant to both types of aggregation problems addressed in this paper, concerns the aggregation of *qualitative*

information, especially the combination of values from an *ordinal* scale. The latter problem arises, for example, if the NN estimator is applied to ordinal classification problems. Regarding the specification of similarity measures, the use of ordinal similarity scales might be appealing if precise numerical similarity degrees are difficult to obtain. The Sugeno integral [16] might be an interesting aggregation operator for these types of problems.

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Fuzzy Sets and Multicriteria Decision Making

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Summary. We discuss and propose some fuzzy set based methods to build a preference structure in multicriteria decision making. First we discuss fuzzy measures/aggregation operators based approach, including its finite refinements. Next preference structure construction is based on residual implications adjoint to left-continuous t-norms. Finally, we transform score vectors into fuzzy quantities and apply orderings over fuzzy quantities.

Key words: Aggregation operator, Fuzzy measure, Multicriteria decision making, Preference relation, Triangular norm.

1 Introduction

In multicriteria decision making procedures, for each alternative $\mathbf{a} \in \mathcal{A}$ (the set of all discussed alternatives), a finite number n of criteria is applied for its evaluation. These criteria reflect several qualitative and quantitative aspects and can be related one with another. In this contribution we will deal with a simple model, when to each alternative $\mathbf{a} \in \mathcal{A}$ can be assigned a degree of satisfaction $a_i \in [0,1]$ for each criterion $i \in \{1,\ldots,n\}$ (here $a_i = 1$ means that the alternative $\mathbf{a}$ fully satisfies the criterion i, while $a_i = 0$ means complete failure of $\mathbf{a}$ in criterion i). Thus $\mathbf{a} \in \mathcal{A}$ can be identified with a score vector $(a_1,\ldots,a_n) \in [0,1]^n$, or, equivalently, with a fuzzy subset of $\{1,\ldots,n\}$ with the relevant membership function. In what follows, we will not distinguish the alternative $\mathbf{a} \in \mathcal{A}$ and the corresponding score vector $(a_1,\ldots,a_n) \in [0,1]^n$.

The aim of this contribution is a discussion and proposal of some methods based on fuzzy sets allowing to construct a crisp preference relation $R : \mathcal{A} \times \mathcal{A} \to \{0,1\}$ (i.e., $R \subset \mathcal{A} \times \mathcal{A}$) defined on the set of all alternatives (such preference structure is crucial for a decision maker to help him in choosing the best alternative). Here $R(\mathbf{a},\mathbf{b})$, or, equivalently, $\mathbf{a} \geq \mathbf{b}$, means that $\mathbf{a}$ is weakly preferred to $\mathbf{b}$ (i.e., $\mathbf{a}$ is better or nondistinguishable from $\mathbf{b}$). Note that the basic results and terminology for this topic can be found in [7].

2 Fuzzy Measures/Aggregation Approach

One typical way how to construct R is linked with the idea of utility function $u : \mathcal{A} \to \mathbb{R}$. Without any loss of generality we may suppose that $\mathrm{Ran}(u) \subset [0,1]$. Moreover, we expect that utility function u reflects the weak Pareto principle, i.e., $u(\mathbf{a}) \geq u(\mathbf{b})$ whenever $a_i \geq b_i, i = 1, \ldots, n$. However, then u can be understood as (possibly a restriction of) an aggregation operator Q [2,11,12].

Definition 1. *Let $n \in \mathbb{N}$, $n > 2$, be given. A function $Q : [0,1]^n \to [0,1]$ is called an aggregation operator whenever it is nondecreasing in each coordinate and $Q(0, \ldots, 0) = 0$, $Q(1, \ldots, 1) = 1$.*

Proposition 1. *Let $Q : [0,1]^n \to [0,1]$ be an aggregation operator. Then the relation $R_Q \subset \mathcal{A} \times \mathcal{A}$ (i.e., $\succsim_Q$) given by $(\mathbf{a}, \mathbf{b}) \in R_G$ (i.e., $\mathbf{a} \succsim_Q \mathbf{b}$) if and only if $Q(a_1, \ldots, a_n) \geq Q(b_1, \ldots, b_n)$ is a transitive complete weak preference relation.*

Observe that Q can be understood also as a fuzzy measure defined on fuzzy events from $[0,1]^n$. Zadeh's fuzzy probability measures [21] yield in this case the class of weighted arithmetic means, $W(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. Klement's $T_{\mathbf{M}}$-measures [9] generalize fuzzy probability measures and they yield in our case operators $Q(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i F_i(x_i)$, where $F_i : [0,1] \to [0,1]$ are (restrictions of) distribution function of random variables with range in $[0,1]$. For several other types of aggregation operators (fuzzy measures on fuzzy events) we recommend [1,2,11].

The problem of too many ties in approach described in Proposition 1 can be solved by limit approach refining the preference relation R_Q.

Proposition 2. *Let $\alpha = (Q_k)_{k \in \mathbb{N}}$ be a system of aggregation operators, $Q_k : [0,1]^n \to [0,1]$, with pointwise limit $\lim_{k \to \infty} Q_k = Q$. Then $R_\alpha = \liminf_{Q_k \in \mathcal{A}} R_{Q_k}$ is a refinement of R_Q.*

Note that the preference relation R_α preserves R_Q in the case of strict preference, i.e., $Q(\mathbf{a}) > Q(\mathbf{b})$ implies $R_\alpha(\mathbf{a}, \mathbf{b})$, however, some R_Q-ties can be broken by R_α.

Example 1. (i) Let $\alpha = (T_k^{AA})_{k \in \mathbb{A}}$, be a system of Aczél-Alsina t-norms [10]. Then $\lim_{k \to \infty} T_k^{AA} = T_{\mathbf{M}}$ (min-operator) and $\mathbf{a} \geq_\alpha \mathbf{b}$ if and only if either $\min a_i = \min b_i = 0$ or $(a_1, \ldots, a_n) \geq_{\text{leximin}} (b_1, \ldots, b_n)$ (for leximin order see, for example, [4]).

(ii) Let $\alpha = (P_k)_{k \in \mathbb{N}}$ be a system of power-root operators [6],

$$P_k(a_1, \ldots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^k \right)^{\frac{1}{k}}.$$

Then $\lim_{k\to\infty} P_k = S_{\mathbf{M}}$ (max-operator) and $\mathbf{a} \geq_\alpha \mathbf{b}$ if and only if

$$(a_1, \ldots, a_n) \geq_{\text{leximax}} (b_1, \ldots, b_n).$$

Observe that Propositions 1 and 2 can be formulated with different dimensions (e.g., when the information about the score for some criteria is missing), simply dealing with extended aggregation operators $Q : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$. Then different systems α in Proposition 2 may lead to different extensions of the leximin or leximax orders. Namely, for α from Example 1 (i), the extended leximin ordering (i.e., lexicographical ordering over permuted score vectors rearranging them in the nondecreasing order) means that any score vector can be extended by adding new score coordinates with full satisfaction $a_{n+1} = \ldots = a_m = 1$, so that for $n < m$, n-dimensional score vector $\mathbf{a}$ and m-dimensional score vector $\mathbf{b}$ are compared after completion of $\mathbf{a} = (a_1, \ldots, a_n)$ into m-dimensional vector $\mathbf{a}_{(m)} = (a_1, \ldots, a_n, 1, \ldots, 1)$. Similarly in the case of leximax we can add new score coordinates $a_{n+1} = \ldots = a_m = 0$. Completely different is the leximin (leximax) extension related to the system α from Example 1 (ii). Then to compare n-dimensional score vector $\mathbf{a}$ and m-dimensional score vector $\mathbf{b}$ with $n \neq m$ we introduce and compare two new $n \cdot m$-dimensional vectors $\mathbf{a}^{(m)} = (\mathbf{a}, \ldots, \mathbf{a})$ (m copies of $\mathbf{a}$) and $\mathbf{b}^{(n)} = (\mathbf{b}, \ldots, \mathbf{b})$ (n copies of $\mathbf{b}$). For some details we recommend [5, 13].

The following proposition generalizes the Lorenz ordering.

Proposition 3. *Let $\beta = (Q_k)_{k\in\mathcal{K}}$ be a (finite or infinite) system of (n-ary, extended) aggregation operators. Define $\mathbf{a} \geq_\beta \mathbf{b}$ if and only if $Q_k(\mathbf{a}) \geq Q_k(\mathbf{b})$ for all $k \in \mathcal{K}$. Then $\geq_\beta$ is a transitive weak preference relation.*

Note that the preference relation $\geq_\beta$ on $[0,1]^n$ is the Lorenz ordering if, for $\mathbf{x} \in [0,1]^n$, $\mathcal{K} = \{1, 2, \ldots, n\}$, and $Q_k(\mathbf{x}) = \sum_{i=k}^{n} x'_i$, where x'_i is the i-th order statistics from the sample $(x_1, \ldots, x_n)$. Preference relation $\geq_\beta$ is not complete, in general. To avoid this inconvenience, we introduce a lexiβ-ordering $\geq_{\text{lexi}\beta}$ as follows: $\mathbf{a} \geq_{\text{lexi}\beta} \mathbf{b}$ if and only if $\mathcal{K}$ is an interval in $\mathbb{N}$ and $(Q_k(\mathbf{a}))_{k\in\mathcal{K}} \geq_{\text{lex}} (Q_k(\mathbf{b}))_{k\in\mathcal{K}}$ ($\geq_{\text{lex}}$ is the standard lexicographical linear order on $[0,1]^{\mathcal{K}}$).

3 Fuzzy Logic Based Construction of Preference Relations

Let $T : [0,1]^2 \to [0,1]$ be a left-continuous t-norm [10] and let

$$I_T : [0,1]^2 \to [0,1]$$

be the adjoint residual implication, $I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}$. Then $x \leq y$ if and only if $I_T(x,y) \geq I_T(y,x)$. This fact allows to introduce preference relations on $\mathcal{A}$ as follows:

Definition 2. *Let* $T : [0,1]^2 \to [0,1]$ *be a left-continuous t-norm and let* $H : [0,1]^n \to [0,1]$ *be an aggregation operator. Then the preference relation* $R_{T,H} \subseteq \mathcal{A}^2$ *is given by* $(\mathbf{a}, \mathbf{b}) \in R_{T,H}$, *i.e.,* $\mathbf{a} \succsim_{T,H} \mathbf{b}$ *if and only if*

$$H(I_T(a_1, b_1), \ldots, I_T(a_n, b_n)) \leq H(I_T(b_1, a_1), \ldots, I_T(b_n, a_n)). \tag{1}$$

Note that if H has neutral element 1 or if H is cancellative then the decision about relation of $\mathbf{a}$ and $\mathbf{b}$ depends only on those score for which $a_i \neq b_i$, i.e., the discriminative approach to decision making as discussed in [14] is applied. Observe that though in some cases $\succsim_{T,H}$ can be represented in the form $\succsim_Q$, see Proposition 1, and thus the preference relation $R_{T,H}$ is transitive, in general this is not true. Note also that H need not be symmetric.

Example 2.

(i) For any nilpotent t-norm T with an additive generator $t : [0,1] \to [0,\infty]$, see [10], and the quasiarithmetic mean M_t generated by t, see [2], $\succsim_{T,M_t}$ is exactly $\succsim_{M_t}$, and $\mathbf{a} = (a_1, \ldots, a_n) \succsim_{T,M_t} \mathbf{b} = (b_1, \ldots, b_n)$ if and only if $\sum_{i=1}^{n} t(a_i) \leq \sum_{i=1}^{n} t(b_i)$. Thus the transitivity of $\succsim_{T,M_t}$ is obvious.

(ii) Similarly for any strict t-norm T with an additive generator $t : [0,1] \to [0,\infty]$ (recall that now $t(0) = \infty$, while in the nilpotent case we have $t(0) < \infty$), $\succsim_{T,M_t}$ is transitive, but there is no aggregation operator Q such that $\succsim_{T,M_t} \equiv \succsim_Q$. Observe that now $\mathbf{a} \succsim_{T,M_t} \mathbf{b}$ if and only if $\sum_{a_i^2+b_i^2>0} t(a_i) \leq \sum_{a_i^2+b_i^2>0} t(b_i)$.

(iii) For any nilpotent t-norm T with an additive generator t, $\succsim_{T,T_{\mathbf{M}}}$ is not transitive. Observe that $\mathbf{a} \succsim_{T,T_{\mathbf{M}}} \mathbf{b}$ if and only if $\min(t^{-1}(\max(t(b_1) - t(a_1), 0), \ldots, t^{-1}(\max(t(b_n) - t(a_n), 0)) \leq \min(t^{-1}(\max(t(a_1) - t(b_1), 0), \ldots, t^{-1}(\max(t(a_n) - t(b_n), 0))$.

4 Preference Relations Based on Orderings of Fuzzy Quantities

In the criterion i, the dissimilarity $D_i(x, y)$ of a score x and another score y, with $x, y \in [0,1]$, is described by the dissimilarity function $D_i : [0,1]^2 \to \mathbb{R}$, such that $D_i(x, y) = K_i(f_i(x) - f_i(y))$, where $K_i : \mathbb{R} \to \mathbb{R}$ is a convex function with the unique minimum $K_i(0) = 0$ (shape function), and $f_i : [0,1] \to \mathbb{R}$ is a strictly monotone continuous function (scale transformation). Evidently, each D_i is then continuous. Observe that this approach to dissimilarity is based on the ideas of verbal fuzzy quantities as proposed and discussed in [15–17]. Note also that the concept of dissimilarity functions is closely related to the penalty functions proposed by Yager and Rybalov [19], compare also [3]. Finally, remark that the dissimilarity function D is related to some standard

metric on the interval $[0,1]$ whenever it is symmetric, i.e., if K is an even function.

The dissimilarity of score $(x_1,\ldots,x_n)$ and the unanimous score $(r,\ldots,r)$ is described by the real vector $(D_1(x_1,r),\ldots,D_n(x_n,r))$. The fuzzy utility function U, compare [8,20], assigns to each alternative $\mathbf{a}$ (with score $(a_1,\ldots,a_n)$) the fuzzy quantity $U(\mathbf{a})$ with membership function $\mu_{\mathbf{a}}:[0,1]\to[0,1]$,

$$\mu_{\mathbf{a}}(r)=\frac{1}{1+\sum_{i=1}^{n}D_i(a_i,r)} \tag{2}$$

Proposition 4. *For each alternative $\mathbf{a}\in\mathcal{A}$, the fuzzy utility function value $U(\mathbf{a})$ with membership function given by* (2) *is a continuous quasiconvex fuzzy quantity.*

Now we are ready to define a (weak) preference relation R_U ($\precsim_U$) on $\mathcal{A}$.

Definition 3. *Let $\mathcal{A}$ be a set of alternatives and let $U:\mathcal{A}\to\mathcal{F}(\mathbb{R})$ be a fuzzy utility function given by (2). Let $\leq$ be a crisp ordering on the set of all continuous quasiconvex fuzzy quantities. Then we define a weak preference relation $\preceq$ on $\mathcal{A}$ as follows: $\mathbf{a}\preceq\mathbf{b}$ whenever $U(\mathbf{a})<U(\mathbf{b})$ or $U(\mathbf{a})=U(\mathbf{b})$ and $\mathrm{hgt}(U(\mathbf{a}))\geq\mathrm{hgt}(U(\mathbf{b}))$.*

More details about these ideas can be found in [18]. Note only that each defuzzification method DEF on fuzzy quantities compatible with the fuzzy maximum yields a nondecreasing operator $Q_{\mathrm{DEF}}:[0,1]^n\to\mathbb{R}$, $Q_{\mathrm{DEF}}(\mathbf{x})=\mathrm{DEF}(U(\mathbf{x}))$. In the case of mean of maxima defuzzification method MOM, Q_{MOM} is even an idempotent aggregation operator and thus our approach allows to construct idempotent aggregation operators.

Example 3.

(i) Let $D_1=\ldots=D_n=D$ with $K(x)=x^2$ and arbitrary f. Then $Q_{\mathrm{MOM}}=M_f$ is the quasigeometric mean generated by f.
(ii) Let $D_1=\ldots=D_n=D$ with $K(x)=|x|$ and arbitrary f. Then $Q_{\mathrm{MOM}}=\mathrm{Med}$(median operator).
(iii) Let $f(x)=x$ and $f_1=f_2=f$, and $K_1(x)=c|x|$, $K_2(x)=dx^2$ for some $c>0,d>0$. Then $Q_{\mathrm{MOM}}(x_1,x_2)=\mathrm{Med}(x_1,x_2-\frac{c}{2d},x_2+\frac{c}{2d})$.

5 Conclusion

We have introduced and discussed several methods based on fuzzy set theory to build preference relations as a support for multicriteria decision making. As a byproduct we have obtained also a construction method for idempotent aggregation operators. This method allows easy incorporation of weights (importances) of single criteria simply multiplying the dissimilarity functions D_i by the corresponding weights w_i. Note that this method allows to incorporate weights also into the nonsymmetric aggregation, compare Example 3 (iii).

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Fuzzy Reinforcement Learning for Routing in Wireless Sensor Networks

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Summary. The wireless sensor networks (WSNs) are composed of a large number of sensing devices. They are used for such tasks as environmental monitoring, sensing and actuating faculties. The sensors gather the data and transfer the information to the sink. All sensor nodes are battery-powered. It is thus to devise efficient power usage, communication and message routing schemes. In this paper, we introduce a new fuzzy reinforcement learning algorithm which learns the best routing with a high reduction of energy consumption in WSNs. We also show that our routing algorithm is able to select the best routing strategy with regards to all the parameters such as battery power, distances between sensors, etc.

Key words: Q-learning, Reinforcement learning, Multiagent systems, Wireless sensor networks.

1 Introduction

Wireless Sensor Networks (WSNs) are ad hoc wireless networks of small, low cost sensors. They have enabled a new revolution of distributed embedded computing, where micro-miniaturized low-power versions of processors, memory, and the radio modems, responsible for wireless communication, are integrated into a single chip [1, 2]. Wireless sensor networks have many applications in the commercial, medical, scientific, industrial and military domains. They are often used in intelligent transportation systems, monitoring systems, the sensing of many parameters such as temperature, humidity, pressure, ozone, pH value, etc. A widely employed energy-saving technique is to place sensor nodes in a sleep mode. A low-power consumption is also used for the data transfer in the WSNs.

The best routing strategy in WSNs is a compromise between the shortest path and the minimum consumption of the energy power. This approach reduces the load of WSNs and increases the system lifetime. In order to find the best routing scheme in WSNs the method based on reinforcement learning [3–5] can be used.

As a popular reinforcement learning method a Q-learning algorithm is often applied. At first it was adapted for packet routing in computer networks by Boyan and Littman [6]. The given method was very useful for finding the shortest path without a prior knowledge on network topology. A routing algorithm called a Q-routing was developed by Kumar [7]. In this algorithm the quality of exploration was improved by attacking confidence measures to each the Q-values in the network. These confidence values are used in determining the learning rate for the Q-values. Nevertheless, this method cannot take into consideration the energy consumption by sensors.

Currently, in the paper by Dowling et al. [8] the so-called collaborative reinforcement learning for routing optimalisation in mobile ad hoc networks (MANETs) was proposed. Analogously as in the previously mentioned paper, the problem of energy consumption was not undertaken.

In this paper, we introduce a new routing scheme for wireless sensor networks which is based on fuzzy reinforcement technique. Each sensor is treated here as an agent in Multi-Agent Systems (MAS). Each agent learns which action should be made by applying a fuzzy Q-learning algorithm. With the help of standard simulation techniques, we show that our algorithm is able to choose the best routing strategy with regard to the energy power of sensor batteries and the size of the transfered data.

The rest of this paper is structured as follows. In the next section we give the problem formulation. Section 3 gives the fuzzy reinforcement learning approach for routing in WSNs. In Sect. 4 we discuss the results of the simulation experiments. In Sect. 5, we conclude the paper and give future research directions.

2 Assumptions and Problem Formulation

In this section we describe the problem and explain in detail the activity of WSNs.

A wireless sensor network can be defined as a directional communication graph $G = (V, E)$, where V corresponds to the set of the subsets of sensor nodes V_i, $i = 1, \ldots, n$ such that $\cup_{i=1,\ldots,n} V_i = V$ and E represents the wireless connections between the sensor nodes. For each set V_i there is a unique node c_i, the clusterhead, that represents the set and can take on various tasks (data aggregation, communication with other clusterheads, etc.).

The Q-learning algorithm was introduced by Watkins [3], Watkins and Dayan [4]. It is based on scalar rewards given by the environment. It was proved by Tsitsiklis [9] that under certain conditions the Q-learning algorithm guarantees the convergence to the optimal solution of a Markov decision problem. According to the Q-learning algorithm at moment t the agent in MAS chooses action a_t and reward r_t. The function Q, denoted by $Q_t(s_t, a_t)$, is denoted as

$$Q_t(s_t, a_t) = (1 - \alpha)Q_t(s_t, a_t) + \alpha(r_t + \gamma \max_{a_t \in A} Q_t(s'_t, a'_t)) \tag{1}$$

initialization $t = 0$, $r_T = (s_t, a_t) = 0$;
begin
for $\forall\ s_t \in S$ **and** $a_t \in A$ **do**
begin
$t := t + 1$;
access the current state s_t;
$a_t \leftarrow$ *choose_action*(s_t, Q_t);
perform_action a_t;
compute: $r_t(s_t, a_t)$, s_{t+1};
update
$\Delta Q_t \leftarrow (r_t + \gamma \max_{a_t}(Q_t(s_{t+1}, a_t)) - Q_t(s_t, a_t)$;
$Q_t(s_t, a_t) \leftarrow (1 - \alpha)Q_t(s_t, a_t) + \alpha \Delta Q_t$;
end;
end;

Fig. 1. Q-learning algorithm estimates new state obtained by performing the chosen action at each time step

where A is the set of all the possible actions, α $(0 \leq \alpha < 1)$ and γ $(0 \leq \gamma \leq 1)$ denote the learning rate and the discount parameter, $Q_t(s'_t, a'_t)$ is the value of the Q function after the execution of action a'_t.

Figure 1 shows the raw form of the Q-learning algorithm (without possible improvements). It can be seen that Q-learning algorithm is an incremental reinforcement learning method. The choice of the action did not shows how to obtain it. Therefore, the Q-learning can use other strategies that it learns. It is irrespective of the strategy. It means that it does not need such actions which maximize the reward function r_t.

Additionally, in certain situations the Q-learning algorithm is ineffective. For instance, if some actions are constantly repeated, the complexity of the algorithm is increased. Moreover, it is not effective in all the situations while it estimates another Q-function belonging to another agent in the environment. One of the methods to overcome some of the problems with the use of the Q-learning algorithm depends on the use of the fuzzy reinforcement learning.

3 Fuzzy Reinforcement Learning for Routing in Wireless Sensor Networks

In this section we formulate our model of fuzzy reinforcement learning (FRL) for routing in WSNs.

Let us assume a model of a WSN system in which each agent (sensor) can observe all other agents and the possible possessions (clusterheads, sinks, etc.). Therefore, we assume that each agent has a sufficient number of receptors needed for the observation all other agents and possessions within its reach within radius R.

We assume that a single receptor $p_i^{(j)}$, $(i = 1, \ldots, n)$ of agent j can collect the information about only one other sensor or possession. For the sake of

routing in WSN this information must be used to appreciate the usefulness of all the neighbouring agents for agent coalition formation. The coalition of agents will be intermediated in the data transferring from the source to the sink. Each agent from the coalition seeks also the possessions that are helpful in packet delivery.

For a two-dimensional environment all the information obtained by the jth agent about another kth agent is defined by the membership functions: $\mu_x^{(j)}(agent^{(k)})$, $\mu_y^{(j)}(agent^{(k)})$ (see Fig. 2a,b). Analogously, the information obtained by agent j about possession l is defined in a two-dimensional environment by two membership functions $\mu_x^{(j)}(possession^{(l)})$ and $\mu_y^{(j)}(possession^{(l)})$ - see Fig. 2c,d. The usefulness of another agent defined on the basis of the battery energy and the distance between agent j and agent k are described for a two-dimensional environment by the membership functions: $\mu_x^{(j)}(usefulness^{(k)})$ and $\mu_y^{(j)}(usefulness^{(k)})$. Model membership functions for a sensor with a visual depth equal to 3 are shown in Fig. 2d,e.

A membership value defining the fuzzy state of agent k with respect to agent j is

$$\mu_{state}^{(j)}(agent^{(k)} = \mu_x^{(j)}(agent^{(k)}) \cdot \mu_y^{(j)}(agent^{(k)}) \tag{2}$$

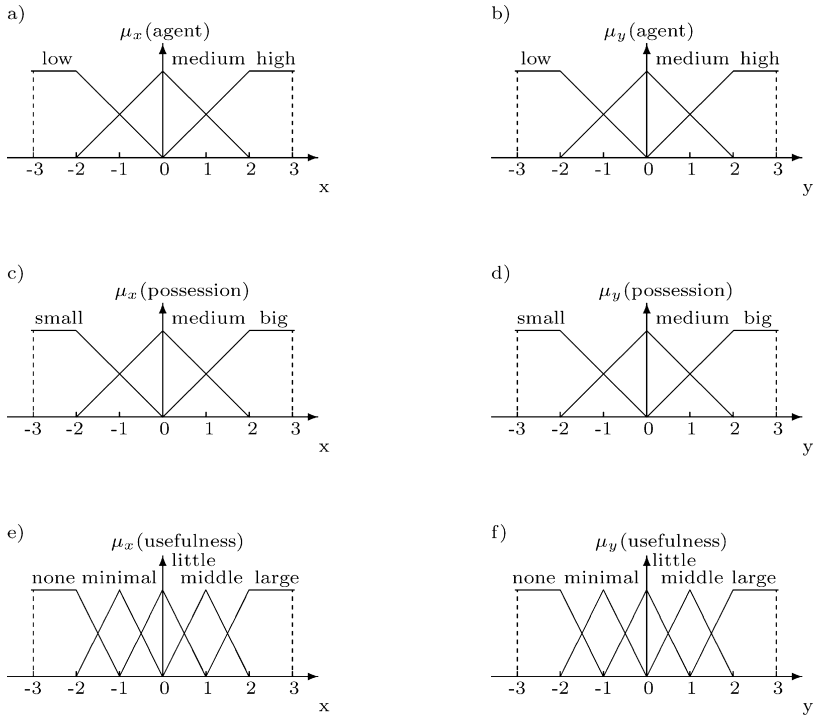

Fig. 2. Fuzzy sets for two-dimensional environment defining: other agent (**a, b**); possession (**c, d**); usefulness of other agent (**e, f**)

A membership function defining the fuzzy state of the jth agent in respect of the lth possession for a two-dimensional environment is as follows:

$$\mu^{(j)}_{state}(possession) = \mu^{(j)}_{x}(possession^{(j)}) \cdot \mu^{(j)}_{y}(possession^{(j)}) \tag{3}$$

Similarly, the usefulness of agent k for agent j which also defines the fuzzy state of agent j for a two-dimensional environment is computed as:

$$\mu^{(j)}_{state}(usefulness) = \mu^{(j)}_{x}(usefulness^{(j)} \cdot \mu^{(j)}_{y}(usefulness^{(j)}) \tag{4}$$

The model of the system described in this way consists of a multidimensional membership function which can be treated as a multidimensional hypercube. We can use an aggregation of the fuzzy state for the jth agent described by the fuzzy pair (s_n, a_n) for the nth fuzzy variable. It is given by

$$Q^{(j)}_{state}(s,a) \leftarrow Q^{(j)}_{state}(s,a) + \sum_{n=1}^{N} \alpha^{(j)}_{n} \cdot \mu^{(j)}_{state}(s_n, a_n) \tag{5}$$

where N is the total number of fuzzy variables.

For the three exemplary fuzzy variables we have the Q-function for jth agent, namely

$$\begin{aligned} Q^{(j)}_{state} \leftarrow Q^{(j)}_{state}(s,a) &+ \sum_{k=1}^{K} (\alpha^{(j)}_{k} \mu^{(j)}_{state}(agent^{(k)}) + \alpha^{(j)}_{k} \mu^{(j)}_{state}(usefulness^{(k)})) \\ &+ \sum_{l=1}^{L} \alpha^{(j)}_{l} \mu^{(l)}_{state}(possession^{(l)}) \end{aligned} \tag{6}$$

where $\alpha^{(j)}_{n}$ is the learning rate for agent j with respect to nth fuzzy variable, K is the total number of agents, L is the total number of possessions.

Assuming that the range of the agent observation has the radius equal to R, we can again define the Q-function as follows

$$Q^{(j)}_{state}(s_{t+1}, a_{t+1}) \leftarrow \begin{cases} 0 & \text{if } \; j \notin \{J\} \\ Q^{(j)}_{state}(s_t, a_t) + \alpha^{(j)}_{state}(s_t, a_t) & \text{if } \; j \in \{J_{0<r\leq 0.5R}\} \\ Q^{(j)}_{state}(s_t, a_t) + \beta^{(j)} Q^{(j)}_{state}(s_t, a_t) & \text{if } \; j \in \{J_{0.5R<r\leq R}\} \end{cases} \tag{7}$$

where $\{J\}$ is the set of agents and possessions in the range of the agent observation with the radius equal to R, $\{J_{0<r\leq 0.5\cdot R}\}$ and $\{J_{0.5\cdot R<r\leq R}\}$ are the sets of agents and possessions in the range of the agent observation with the radius equal to $0 < r \leq 0.5 \cdot R$ and $0.5 \cdot R < r \leq R$, respectively. $\beta^{(j)}$ are learning rate factors.

It should be noted that the Q-values are usually stored in a lookup table [10]. The state space in a reinforcement learning can be treated as stochastic problems. The standard approach for dealing with this problem

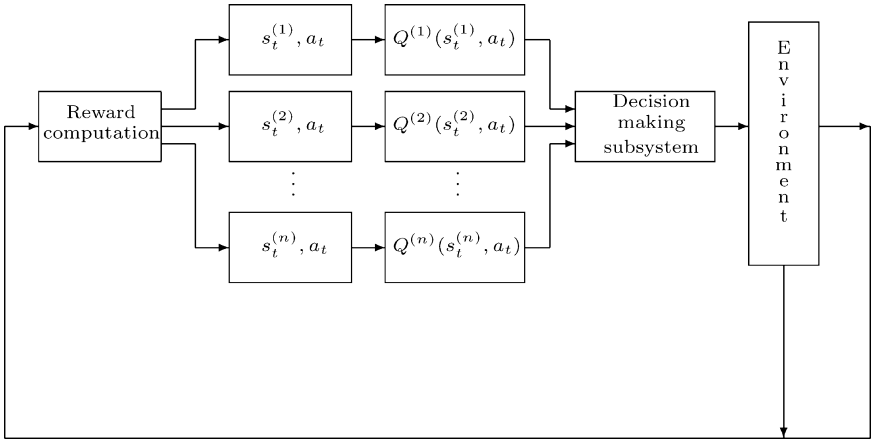

Fig. 3. A block diagram for an agent system architecture in case of data mining

is to generalize the Q-values across states by using a function approximation $Q(s, a, r)$ for approximating $Q(s, a)$, where r is the set of all learned parameters. The decisions are undertaken by use the specially prepared fuzzy logic mechanism [11, 12]. Here, we apply data mining approach to handle the incomplete information received from environment.

The system architecture proposed for the data mining process of single agent in the environment is shown in Fig. 3. The data mining process with reference to a single agent can be given by following procedure:

Procedure 1

1. The agent by use its receptors fixes the current values of all the membership functions. Further, it defines the actual value of state-action pair.
2. The agent computes the learning rate α, which for the jth agent is given as follows:
$$\alpha^{(j)} = \frac{1}{\sum_{n=1}^{N} \mu_{state}^{(n)}} \tag{8}$$
where N is the total number of fuzzy variables. Above equation shows that by increase of the number of fuzzy variables the learning rate becomes smaller.
3. The agent computes the Q-function for each fuzzy parameter. We applied the selection procedure based on Kóczy-Hirot method [13] here. This method computes a conclusion as a weighted sum of vague consequent values b_n, which is given by
$$C(b_h) = \frac{\sum_{h=1}^{H} w_h \cdot dist(y_0, b_h)}{\sum_{h=1}^{H} w_h} \tag{9}$$

where w_h is the weight inverse proportional to the vague distance of the observation x from action a. For the hth rule the weight is defined as

$$w_h = \frac{1}{dist(x, a)} \tag{10}$$

4. After the choice of the action by the agent the reward function $r_t(s_t, a_t)$ is computed. Further, it upgrades the ΔQ_t and computes $Q_t(s_t, a_t)$.
5. The agent goes to step 1. □

4 Experimental Results

To test the effectiveness of the learning process of our algorithm and demonstrate its possibility, we conducted some experiments. Our MAS system consists of some number of agents and possessions. In our approach all the possessions are identified as clusterheads of WSNs. All agents are randomly placed in the environment.

Nine rules were defined for the decision subsystem of each agent, namely: *do nothing*, *connect with another agent*, *disconnect from another agent*, *connect with a possession*, *disconnect from a possession*, *connect with the agent and possession*. Each rule has a weight. The rule concerning both connections (with another agent and possession) has weight equal 3. The weight equal 2 is assigned to all the rules which are concerned with one connection, either the agent or the possession. The weight equal 1 is used only for one connection and one disconnection. A lack of an action or a double disconnection is associated with the weight equal 0. In our approach all the weights are multiplied by the rule in the subsystem decision.

Initially, all the values of the Q-function had an identical Q-value equal 0.5. The learning rates are $\alpha = 0.1$ and $\beta = 0.06$. The radius of the agent observation equals 3 and 2.

We present three sets of simulation results (see Fig. 4) in which we compare our two FRL routing scheme with the *Destination-Sequenced Distance Vector* (DSDV) protocol [14]. In the first two sets, we study mean energy consumption per time steps for two values of radius in FRL. In the third set, we show only the energy consumption against the time steps. The results have shown that the decrease of radius R increases the growth of the mean energy consumption in time steps.

5 Conclusion

In this paper, we introduce a fuzzy reinforcement learning for the routing in wireless sensor networks. We take into consideration all the important parameters of sensors such as the battery energy and the distances between sensors and clusterheads. The obtained results have shown that our method works independently of the sensor placements and the energy consumption.

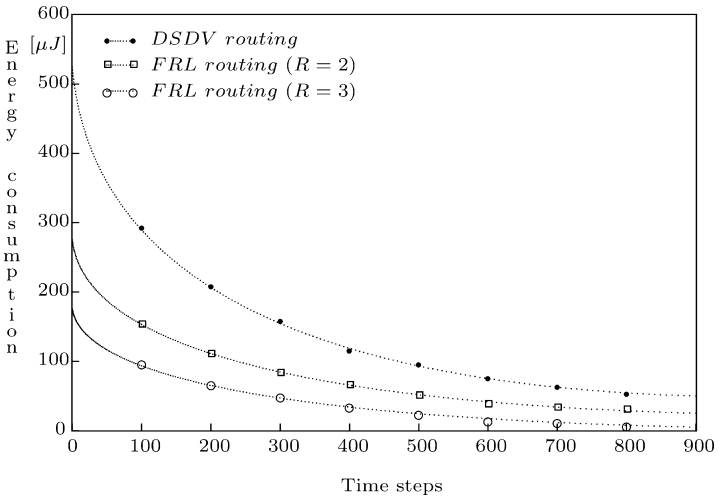

Fig. 4. Mean energy consumption versus time steps

In the future, we will compare our method of routing in WSN with other routing algorithms used in WSNs.

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Outlier Resistant Recursive Fuzzy Clustering Algorithms

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Vitaliy Kolodyazhniy

Summary. The problem of fuzzy clustering on the basis of the probabilistic and possibilistic approaches under the presence of outliers in data is considered. Robust recursive fuzzy clustering algorithms are proposed, which optimize the objective function suitable for clustering data with heavy-tailed distribution density. Advantages of the proposed algorithms in comparison with the well-known fuzzy c-means algorithm are demonstrated in an experiment in clustering and classification of data with outliers. The robustness property results in finding correct cluster prototypes whose locations are not affected by anomalous observations, and in achieving thus higher classification accuracy.

Key words: Fuzzy clustering, Heavy-tailed distribution, Objective function, Robustness, Recursive algorithm, Classification.

1 Introduction

Clustering and classification of large datasets are key problems of data mining, and effective solving of these problems is important for knowledge acquisition by analysis of observations.

Generally, cluster analysis is the algorithmic basis of data classification by means of separation of the available data into a number of classes (clusters) without *a priori* defined membership of any observation sample to one of the class (unsupervised learning). In the traditional (crisp) approach it is assumed that every observation belongs to only one class. The k-means algorithm [1] and the nearest-neighbor rule [2] are the most popular examples of this approach. However, it is much more natural to assume that every observation may belong to several clusters at the same time with certain degrees of membership. This assumption is the basis of fuzzy cluster analysis [3,4]. At present time many fuzzy clustering algorithms are widely used, such as Bezdek's fuzzy c-means [3], the Gustafson-Kessel algorithm [5], fuzzy k-nearest neighbors [6], fuzzy shell cluster analysis by Klawonn-Kruse-Timm [7], mountain clustering by Yager and Filev [8]. The approaches mentioned above are capable

of efficient data clustering when the clusters are overlapping, but only with the assumption that the clusters are compact, i.e., they do not have abrupt (anomalous) outliers. Whereas real datasets usually contain up to 20% of outliers [9–11], the assumption of cluster compactness may sometimes become inadequate.

Thus, the problem of cluster analysis of data with heavy-tailed distributions has received more and more attention in recent years. Various modifications of clustering methods mentioned above were proposed and designed to process data containing outliers [12–14]. At the same time, most of the proposed robust fuzzy clustering algorithms are not suitable for real-time or sequential operation. So it is advisable to develop recursive algorithms for robust fuzzy clustering, having adaptive properties and suitable for the sequential processing of incoming data.

The source information for all the mentioned algorithms is the data set of N $n-$dimensional feature vectors $X = \{x(1), x(2), \ldots, x(N)\}$, $x(k) \in R^n$, $k = 1, 2, \ldots, N$. The output of the algorithms is the separation of the source data into m clusters with some degree of membership $w_j(k)$ of the kth feature vector to the jth cluster.

2 Recursive Fuzzy Clustering Algorithm

In this paper, we make an attempt to derive an adaptive computationally simple outlier resistant robust fuzzy clustering algorithm for recursive data processing in online mode as more and more data become available. The recursive probabilistic and possibilistic fuzzy clustering approaches, stable to outliers, are considered. They belong to the class of the objective function based algorithms designed to solve the clustering problem via the optimization of a certain predetermined clustering criterion [3].

For such a criterion, we use the objective function

$$E^R(w_j(k), c_j) = \sum_{k=1}^{N} \sum_{j=1}^{m} w_j^{\beta} \sum_{i=1}^{n} \left(1 - \operatorname{sech}^2(x_i(k) - c_{ji})\right) (x_i(k) - c_{ji})^{\frac{2}{5}}, \quad (1)$$

which is an everywhere differentiable even function, close to the quadratic near the extremum, and close to the linear as the distance from it increases.

Here $w_j(k) \in [0,\ 1]$ is the degree of membership of the vector $x(k)$ to the jth cluster, c_j is the prototype (center) of the jth cluster, β is a non-negative parameter, referred to as "fuzzifier" (usually $\beta = 2$).

It is the form of the objective function, whose derivative decreases with the distance from the extremum point, that provides the robust properties of the clustering procedure, weakening the influence of anomalous observations.

Introducing the system of standard constraints for probabilistic and possibilistic [15–17] approaches to clustering, the respective Lagrangians and

Kuhn–Tucker equations, and applying the procedure of Arrow–Hurwitz–Uzawa for finding the saddle point [15], we obtain the combined recursive clustering algorithm

$$\begin{cases} w_j^{prob}(k) &= \frac{(D^R(x(k),c_j(k)))^{\frac{1}{1-\beta}}}{\sum_{l=1}^{m}(D^R(x(k),c_l(k)))^{\frac{1}{1-\beta}}}, \\ \mu_j(k) &= \frac{\sum_{p=1}^{k}(w_j^{prob}(p))^{\beta} D^R(x(p),c_j(k))}{\sum_{p=1}^{N}(w_j^{prob}(p))^{\beta}}, \\ w_j^{pos}(k) &= \left(1+\left(\frac{D^R(x(k),c_j(k))}{\mu_j(k)}\right)^{\frac{1}{\beta-1}}\right)^{-1}, \\ c_{ji}(k+1) &= c_{ji}(k) - \eta(k)(w_j^{pos}(p))^{\beta}[2\ \mathrm{sech}^2(x_i(k)-c_{ji}(k)) \\ &\quad \cdot \tanh(x_i(k)-c_{ji}(k))|x_i(k)-c_{ji}(k)|^{\frac{2}{5}} \\ &\quad +0.4(1-\mathrm{sech}^2(x_i(k)-c_{ji}(k))) \\ &\quad \cdot|x_i(k)-c_{ji}(k)|^{-\frac{3}{5}}\mathrm{sign}(x_i(k)-c_{ji}(k))], \end{cases} \tag{2}$$

where $D^R(x(k),c_j(k)) = \sum_{i=1}^{n}\left(1-\mathrm{sech}^2(x_i(k)-c_{ji}(k))\right)(x_i(k)-c_{ji}(k))^{\frac{2}{5}}$ is the adopted metrics, that determines the distance from the feature vector $x(k)$ to the jth prototype at the kth step; $\mu_j(k)$ is the scalar parameter that determines the distance at which the degree of membership equals 0.5; $\eta(k)$ is the learning rate parameter.

Thus, the proposed procedure computes the probabilistic memberships $w_j^{prob}(k)$, then uses them to compute the parameter $\mu_j(k)$ which is further used to compute the possibilistic levels of membership $w_j^{pos}(k)$, and, finally, corrects the cluster prototypes $c_j(k)$.

3 Experiments

We used the proposed algorithms in the problem of data classification on a specially generated artificial data set containing three two-dimensional data clusters with samples labeled as 'o', 'x', and '+' (see Fig. 1). The distance from each sample to its cluster center is distributed according to the heavy-tailed Laplacian density

$$p(x_i) = \sigma(1+x_i^2)^{-1}, \tag{3}$$

where σ is the width of a cluster.

The data set contains 9,000 samples (3,000 for each cluster), divided into the training (7,200 samples) and checking (1,800 samples) sets. Note that there are some data points that are very distant from the cluster prototypes (Fig. 1a). The prototypes are to be found somewhere in the central part of the data shown in Fig. 1b. In order to find correct prototypes, the clustering algorithm should be insensitive to the outliers. For each of the compared algorithms, the procedure was as follows. First, the training set was clustered

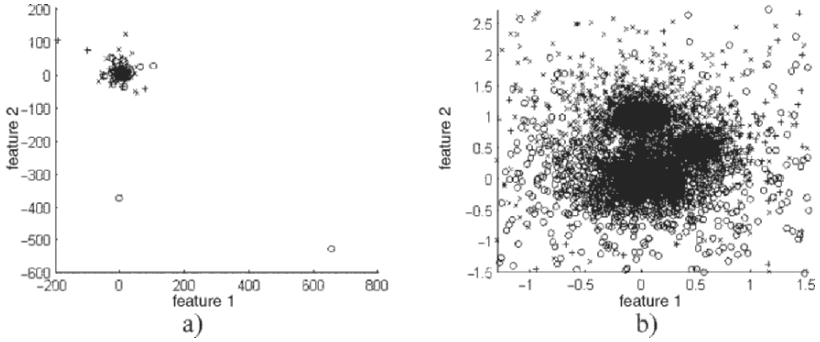

Fig. 1. Complete data set (**a**) and its central part (**b**)

Table 1. Classification results for the checking data sets

	Classification error rate		
Algorithm	Min.	Max.	Avg.
Bezdek's fuzzy c-means	8.3% (150)	59.8% (1076)	12.5% (224)
Robust probabilistic clustering	8.9% (161)	12.9% (232)	10.7% (192)
Robust possibilistic clustering	7.6% (137)	11.2% (202)	9.3% (168)

using the respective algorithm and the prototypes of the clusters were found. Then, the checking set was classified using the cluster prototypes obtained for the training data. The membership of a sample to a certain cluster during the process of classification was calculated according to the type of the clustering algorithm that was used. The cluster to which the given sample belongs with maximum degree of membership determined the class of that sample.

Training and classification were performed assuming $\beta = 2$ and $\eta(k) = 0.01$. To avoid the influence of random initialization of cluster prototypes and to get a cross-validation effect, the test was performed 500 times. Each time the dataset was shuffled and then divided into the training and checking sets. The results for the checking sets are shown in Table 1.

The drawback of fuzzy clustering methods based on the quadratic objective function could be visually shown by plotting the obtained prototypes over the data set. From Fig. 2 it could be easily seen that the cluster centers (prototypes) obtained using Bezdek's fuzzy c-means algorithm are displaced from the visual cluster centers because of the heavy-tailed distribution density of observations, in contrast to the robust objective function-based methods which found the cluster prototypes more accurately. This is confirmed by lower classification error rates as shown in Table 1. It is also easy to see that the classification errors, obtained with the robust algorithms, have significantly smaller variance than those of the fuzzy c-means. This fact confirms the stability of the proposed procedures, and their low sensitivity to initial conditions.

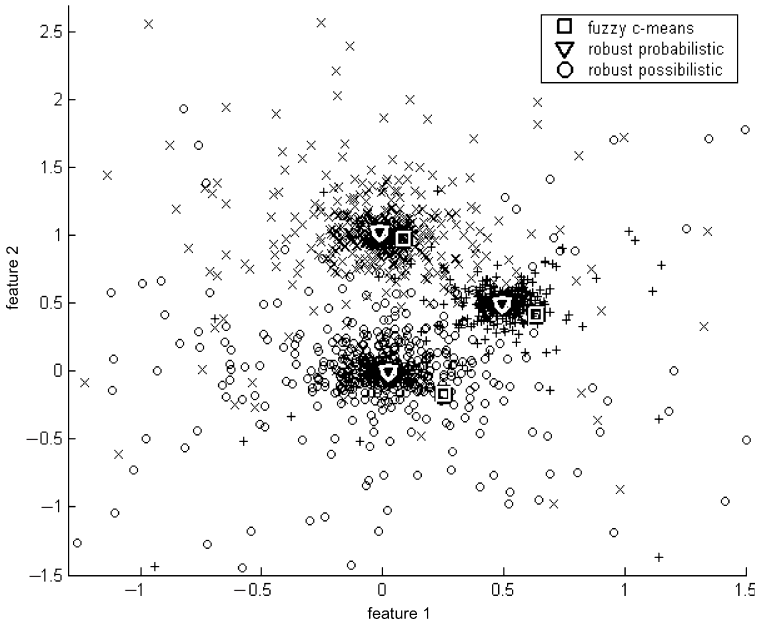

Fig. 2. Example of cluster prototypes obtained with different clustering algorithms, plotted over a checking data set

4 Conclusion

In the paper, robust recursive probabilistic and possibilistic fuzzy clustering algorithms based on the objective function of a special form suitable for processing of data with heavy-tailed distributions or contaminated with outliers were proposed. The robustness property resulted in significantly higher accuracy and stability of clustering and classification compared to those obtained with a non-robust algorithm. The proposed algorithms could be used in a wide range of applications, such as fault detection, data mining, pattern recognition in self-organizing mode when the data contain outliers, the size of the data set is not known *a priori*, and the data must be processed in real time.

The algorithms can be modified by the use of other robust distance metrics or objective functions.

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Fuzzy Set Theory – 40 Years of Foundational Discussions

Siegfried Gottwald

Summary. For classical sets one has the cumulative hierarchy of sets, and also the category **SET** of all sets and mappings as standard approaches toward the universe of all sets. Both of them discussed within the realm of classical logic.

We discuss the corresponding situation for fuzzy set theory, and the suitable formal logics for it, and give is a (concise) survey of important such approaches which have been offered since Zadeh published his seminal paper.

Key words: Fuzzy set theory, Universes of fuzzy sets, Categories of fuzzy sets, Axiomatic fuzzy set theory, Foundations for fuzzy sets, Graded equalities, History, 20s century mathematics.

1 Introduction

For classical set theory one has a quite satisfactory naive foundation in the idea of the cumulative hierarchy of sets which forms *the* standard model for the common axiomatizations of set theory, like the systems ZFC or NBG.

The paradigmatic situation for a category theoretic characterization of classical sets is the category **SET** of (crisp) sets and mappings [38].

To have a similar situation in the fuzzy field, one is interested to find either some kind of cumulative universe of fuzzy sets which should be closed under the formation of fuzzy (sub-)sets, and which can serve as a kind of standard model for axiomatizations of fuzzy set theory, or one is interested to have a category of fuzzy sets and mappings.

So one immediately faces at least the following problems:

1. What are fuzzy sets of higher level?
2. Is there a kind of *standard* cumulative universe of fuzzy sets? or
3. Is there a kind of *standard* category of fuzzy sets?

But there is still more in the background here: the intuition of graduation of membership calls for a background logic which is a many-valued one – with the structure of the membership degree as its structure of truth degrees. Even

more is intuitively convincing: as the basic set theoretic notion of membership is a graded one, other set theoretic notions should become graded too. One very basic such notion seems to be a graded identity between fuzzy sets.

These problems have been approached in quite different ways. If one tries to classify the approaches then the following types can be recognized:

1. Constructions of some cumulative universe of fuzzy sets, either in a "naive" or in a model based manner
2. Axiomatic approaches
3. Category theoretic approaches

In the following we give a concise survey of important developments in this area. A much more detailed presentation shall be given in [19, 20].

2 Model Oriented Constructions

The consideration of model oriented approaches started with two versions for a cumulative hierarchy of *many-valued* sets introduced in 1965 by the German mathematician D. Klaua independent of the seminal paper [53]. As membership degrees this author had chosen a finite subset of the unit interval $[0,1]$ understood as set of truth degrees of a finitely valued Łukasiewicz logic. But this difference to the standard fuzzy sets is an inessential, purely technical point.

In general, all the cumulative universes V have been constructed by induction through (a subclass of) the ordinals. Each such construction yields an expanding hierarchy of partial universes V_α with V as the global union, and with unions over the preceding partial universes at limit stages. So there is a natural notion of *rank*, and constructions at successor stages become the crucial steps.

2.1 The Very First Universe

For the first one of these hierarchies, D. Klaua [31, 34] started from some infinite (crisp) set U of urelements with a *graded* identity relation $\equiv$, i.e. a relation which is reflexive, symmetric and $\&_{\mathsf{L}}$-transitive for the Łukasiewicz t-norm $\&_{\mathsf{L}}$ and its associated implication $\longrightarrow_{\mathsf{L}}$. Then he forms, with reference to the standard (crisp) power set operation $\mathbb{P}$, the hierarchy

$$\begin{aligned} V^*(0) &= U \times \{0\}\,, \\ V^*(n+1) &= \mathbb{P}(V^*(n)) \times \{1\}\,, \end{aligned}$$

and introduces a *graded identity* $=_w$, a graded *membership* $\in_w$, and a graded inclusion $\subseteq_w$ as follows:

$$x =_w y = \begin{cases} \mathsf{pr}_1(x) \equiv \mathsf{pr}_1(y)\,, \\ 0\,, \\ x \subseteq_w y \wedge y \subseteq_w x\,, \end{cases}$$

depending whether x, y are urelements, of different rank, or of equal rank, and with

$$x \in_w y = \sup_{v \in \mathsf{pr}_1(y)} (x =_w v)\,, \qquad x \subseteqq_w y = \inf_{u \in \mathsf{pr}_1(x)} u \in_w y\,.$$

The authors main results are (1) There is a natural embedding of the ωth level of the standard hierarchy of crisp sets into V^*. (2) Suitable (graded) versions of the axioms of *extensionality* and of *comprehension* hold true in this universe. and (3) Some elementary set algebra is developed.

2.2 A Naive Fuzzy Power Set Iteration

The same author D. Klaua almost immediately modified his first approach and considered in [32] for some crisp set U a transfinite hierarchy starting from $V_0 = U$ with the particular extension steps determined by the sets of all functions $f : V_\alpha \longrightarrow [0, 1]$ which satisfy the condition that their support is rank-cofinal in V_α.

This condition intends to avoid the "doubling" of objects – in the sense that it forbids to add to V_α membership functions which have the same support as membership functions in V_α, but only a larger domain.

The graded membership predicate ε, for some object a, is defined as the value of the membership function at a. Contrary to the first approach, there is no graded identity inside the set of urelements. But there is again a graded inclusion and a graded identity:

$$x \subseteqq_w y = \forall z(z\,\varepsilon\,x \longrightarrow_{\mathsf{L}} z\,\varepsilon\,y)\,, \tag{1}$$

$$x =_w y = x \subseteqq_w y \ \wedge\ y \subseteqq_w x\,. \tag{2}$$

As indicated here, the author refers to a language of many-valued logic. Its main connectives are min-conjunction, max-disjunction, $(1-\ldots)$-negation, and Łukasiewicz implication.

The considerations, continued in [33,35,36], have as main results (1) that the graded inclusion and identity have suitable properties; (2) that suitable (many-valued) versions of the axioms of extensionality, of comprehension and of separation hold true; and (3) that a lot of set algebra can be developed, essentially only in an elementary way, up to the basic notions of cardinals and integers.

2.3 Taking Graded Identity More Seriously

The idea that for a graded identity $\equiv$ the fuzzy sets of the hierarchy should satisfy an *extensionality* condition like

$$\models\ x \equiv y\ \&\ y\,\varepsilon\,z \longrightarrow x\,\varepsilon\,z \tag{3}$$

was the starting point for the modification of that approach [13, 14] by the present author. The idea was to save the simple graded inclusion and identity (1), (2) and additionally to satisfy condition (3).

So the hierarchy of fuzzy sets was determined by extension steps determined by the sets of all functions $f : V_\alpha^* \longrightarrow [0,1]$ with their support rank-cofinal in V_α^*, satisfying the additional condition

$$\forall x \forall y (f(x) \,\&_{\mathsf{L}}\, [\![y \equiv^* x]\!] \leq f(y)) \tag{4}$$

which uses the identity (2), and a fuzzified rank equality $\Box^*$, to give $\equiv^*$ by

$$y \equiv^* x = x =_w y \wedge y \,\Box^*\, x \,.$$

The main results of this approach are again (1) that within this cumulative universe of fuzzy sets suitable versions of all the basic ZF axioms are satisfied; (2) that natural many-valued generalizations of the basic laws of set and relation algebra hold true; and (3) that one can even define the uniqueness of relations in some argument, and thus extend this approach up to suitable notions of fuzzy functions and of equipotency, and thus of cardinality, cf. [15].

2.4 A Type Theoretic Approach

Quite recently L. Běhounek/P. Cintula [1] offer a new approach. Their background (fuzzy) logic is the first-order system ŁΠ with identity which is a combination of the infinite valued Łukasiewicz logic L_∞ with the product logic Π (both understood as t-norm based residuated logics). Thus their background logic has a high expressive power. For technical reasons ŁΠ is treated as a two-sorted system, and Baaz's $\triangle$-operator is added.[1]

These authors primarily intend to give a formalized theory of fuzzy subsets of a given universe, i.e., of fuzzy sets of first level. However, this is understood as the starting point for a full set theory – or type theory – over a given set of urelements.

The basic axioms here are the axiom of extensionality in the form

$$\forall x \triangle (x \in X \leftrightarrow x \in Y) \rightarrow X = Y \,,$$

and the schema of comprehension axioms in the form

$$\exists X \triangle \forall x (x \in X \leftrightarrow \varphi(x))$$

for each formula φ not containing the set variable X.

This theory is consistent, because the "standard" class of all fuzzy subsets over a given universe of discourse provides a model. And the authors develop a considerable part of elementary set algebra for fuzzy sets in this theory.

[1] For these systems of many-valued logic the interested reader may consult [18] for the propositional case, and [23] for the first-order case.

Additionally these authors sketch the extension of first-order ŁΠ to higher order logics, particularly second and third order. This is more or less a routine matter and allows them to treat in this more general context also the theory of fuzzy relations with graded relation properties, a topic previously discussed in a less formalized setting in [16], as well as the well known extension principle of Zadeh, cf. e.g. again [16].

2.5 Strongly Model Theoretic Constructions

The basic background idea comes from the Boolean valued models for ZF set theory of Scott/Solovay, cf. e.g. [2, 17]. For these Boolean valued models $V^{\mathbf{B}}$ a given Boolean algebra $\mathbf{B}$ acts as set of truth degrees.

The first such model theoretic approach toward the construction of a universe of fuzzy sets was sketched by Zhang Jin-Wen [54, 55], and given in more detail in [56]. But this paper restricts the degree structure $\mathbf{G}$ to the standard structure $\mathbf{G}_Z = \langle [0,1], \max, \min, 1 - \ldots, 0, 1\rangle$ enriched with the Gödel implication. The main results are that this universe of fuzzy set is a model of essentially all the ZF-axioms, with the exception that in the axiom schema of replacement only negation-free formulas are allowed.

A bit later G. Takeuti/S. Titani [46,47] started their construction of a similar type of universe (without urelements) from a complete Heyting algebra (cHA) H.

Their main argument in favor of this intuitionistic case is that

$$\varphi \wedge (\varphi \longrightarrow_{\mathsf{L}} \psi) \longrightarrow_{\mathsf{L}} \psi \tag{5}$$

does not hold true for min-conjunction and Łukasiewicz implication (the original Zadeh case), which would mean not to get extensionality for the resulting (fuzzy) sets. But this is a very weak argument, because the exchange of the min-conjunction $\wedge$ in (5) by the Łukasiewicz arithmetic conjunction gives a logically valid formula.

As usual in the intuitionistic context, implication is residuation, i.e., the relative pseudo-complement.

As main result the authors give an axiomatic set theory, based on a sequent calculus, which has besides axioms for *equality* suitable versions of all the ZF axioms, together with two technical axioms concerning the embedding of the ZF universe into the actual cHA valued universe.

The case of intuitionistic logic is extended by these authors in [48]. In that paper they include, over the set $[0, 1]$ of truth degrees, also the connectives for Łukasiewicz's negation and conjunction, for product conjunction, and a truth degree constant for $\frac{1}{2}$.

2.6 A BL-Algebra Valued Universe

A very recent approach along these lines is by P. Hájek/Z. Haniková [24] and based upon the *basic t-norm logic* BL of Hájek enriched with the "globalization" operator Δ, denoted by BL$\forall\Delta$, as explained in [21].

In a language with primitive predicates $\in$, $\subseteqq, =$ their axioms are suitable versions of the standard ZF axioms together with an axiom stating the existence of the *support* of each fuzzy set.

The "standard" model for this theory is formed w.r.t. some complete BL-chain $\mathbf{L}$ and given by a hierarchy which starts from the empty set and extends the partial universes V_α by all partial functions from V_α into $\mathbf{L}$.

The primitive predicates are interpreted as

$$\begin{aligned}
[\![x \in y]\!] &= \bigcup_{u \in \mathsf{dom}\,(y)} ([\![u = x]\!] * y(u))\,, \\
[\![x \subseteqq y]\!] &= \bigcup_{u \in \mathsf{dom}\,(x)} (x(u) \Rightarrow [\![u \in y]\!])\,, \\
[\![x = y]\!] &= \Delta [\![x \subseteqq y]\!] * \Delta [\![y \subseteqq x]\!]\,.
\end{aligned}$$

The last condition forces the equality to be crisp, and makes the authors standard form of the axiom of extensionality trivially true in the model.

The authors main result is that the structure $V^{\mathbf{L}} = \bigcup_{\alpha \in \mathsf{On}} V_\alpha^{\mathbf{L}}$ is a model of all the axioms given by the authors.

3 Axiomatizations

There is no clear division between model based approaches which discuss the forms of standard set theoretic axioms which are satisfied in the particular models under considerations, and those approaches which intend to give some – hopefully convincing – axiomatization for fuzzy set theory.

In this section we shall consider only such axiomatizations, let us call them "pure", which are not combined with the idea of some (preferably cumulative) universe of fuzzy sets.

Over the years a lot of such proposals have been offered. None of them has really convinced the fuzzy set community. Therefore we shall mention only the most interesting ones of them which offer some non-standard points of view, but we shall not go into details.

These approaches started with an axiomatization by Chapin [5, 6] who chose for the membership degrees fuzzy sets themselves. Semi-lattices as structures of the membership degrees was the choice of Weidner [49]. A broader notion of fuzzy objects, with fuzzy sets as a subclass of these fuzzy objects, was the background for an axiomatization by Prati [39]; however there is a lack of intuition about what fuzzy objects are to mean.

An interesting unification of fuzzy sets and multisets, both understood as particular cases of objects with positive reals as membership degrees, was offered by Lake [37] and Blizard [3].

And most recently again P. Hájek [22] was going back to an older approach and considered a *Cantorian set theory* over L_∞.

That older approach toward a consistency proof of naive set theory, i.e., set theory with *comprehension and extensionality* only, in the realm of Łukasiewicz logic was initiated by Skolem [43] and resulted – after a series of intermediate steps mentioned e.g., in [17] – in a proof theoretic proof (in the realm of L_∞) of the consistency of naive set theory with *comprehension only* by White [50].

Two equality predicates come into consideration here – Leibniz equality $=_l$ and extensional equality $=_e$ with definitions

$$x =_l y =_{def} \forall z(x \in z \leftrightarrow y \in z)\,,$$
$$x =_e y =_{def} \forall z(z \in x \leftrightarrow z \in y)\,.$$

Leibniz equality is shown to be a *crisp* predicate, but extensional equality is *not*.

The whole system becomes *inconsistent* by the coincidence assumption

$$x =_l y \quad \leftrightarrow \quad x =_e y\,.$$

A (crisp) set of natural numbers can be added. But again the adjunction of some simple and sufficiently strong *induction principle* makes the system *inconsistent*.

It seems thus that one gets in this kind of approach only a rather weak set theory.

4 Category Theoretic Approaches

With the development of the notion of an *elementary topos* through Lawvere and Tierney, cf. [9], and the understanding that topoi describe generalized set theories, the situation changed and the paradigm became the Higgs topos $\mathsf{SET}(H)$ of cHA valued sets, which was introduced in [25] and used essentially also in [10].

Since H-valued maps admit an internalization as characteristic morphism in $\mathsf{SET}(H)$, some authors claimed that the Higgs topos would give *the* category of all fuzzy sets too, i.e., that fuzzy set theory had to be nothing more but a kind of intuitionistic, or at least cHA valued, set theory: a point of view never accepted by those people which had a closer relationship to (and better knowledge of) fuzzy set theory.

There are two core points for the rejection of this "solution" by the fuzzy people (1) that the Łukasiewicz negation cannot be internalized as a truth arrow in the Higgs topos $\mathsf{SET}([0,1])$, and (2) that the internal logic of topoi is the intuitionistic one.

4.1 The First Approaches

The first who introduced a category of fuzzy sets was J.A. Goguen [11]. He considered a category $\mathbf{S}(L)$ of L-fuzzy sets and L-fuzzy relations where L is

a *closg*, i.e., a complete lattice ordered semi-group $\langle L, \wedge, \vee, * \rangle$ satisfying the complete distributive law

$$a * \bigvee_i b_i = \bigvee_i (a * b_i) \,.$$

There are no particular results about this category in Goguen's paper. The category, however, is a *quasitopos*, cf. [45]. And it can e.g., even become extended to describe a kind of graded inclusion relation together with a notion of fuzzy power set, as explained in [41].

A bit later Goguen [12] gave a categorical *characterization* of fuzzy sets. This time he considers another category $\mathbf{Set}(L)$ of L-fuzzy sets, L some completely distributive lattice. And the main result is that this category $\mathbf{Set}(L)$ is *characterized* (up to categorical equivalence) by the properties that it:

- has initial and terminal objects
- has associative images
- is disjointedly cdl-ordered
- has coproducts as disjoint unions, and satisfies that each disjoint union is the coproduct of its summands
- has an atomic monic projective generator P, and $P + P$ is not isomorphic to P

The fuzzy sets background is rather standard, particularly the author does neither know a graded inclusion nor a graded identity in these considerations.

Rather similar to these approaches by Goguen is the approach of M. Eytan [8]. Starting point is a cHA H. M. Eytan's category $\mathbf{Fuz}(H)$ of H-valued sets can be understood as the category of subobjects of constant objects in the Higgs topos. This category was claimed in [8] to be a topos, but it is not.

4.2 Further Approaches

A unifying survey of such categorical approaches gives O. Wyler [51, 52], extending [44]. The basic logic remains the intuitionistic one, i.e. reference is only to cHA's H as value structures. But *graded identities* come into consideration, as is the case in the Higgs topos, and e.g. also in [40].

H-valued fuzzy sets are pairs $A = (|A|, \varepsilon_A)$ of a crisp set $|A|$ and a membership function $\varepsilon_A : |A| \longrightarrow H$.

H-valued totally fuzzy sets (or H-sets, for short) are pairs $A = (|A|, \delta_A)$ of a set $|A|$ and a map $\delta_A : |A| \times |A| \longrightarrow H$ subjected to the conditions of symmetry and transitivity.

H-valued fuzzy set $A = (|A|, \varepsilon_A)$ determine H-sets $A_\varepsilon = (|A|, \delta_{\varepsilon_A})$ by

$$\delta_{\varepsilon_A}(x, y) = \begin{cases} \varepsilon_A(x) \,, & \text{if } x = y \\ \bot & \text{otherwise.} \end{cases}$$

Thus, H-sets form a refinement of fuzzy set, and the H-valued fuzzy sets are the *discrete H-sets*.

After having fixed objects describing the intuitive idea of fuzzy sets, one has to specifying morphisms. There exist at least two fundamentally different types of morphisms: crisp maps and certain binary H-valued relations.

We begin with Wyler's category $\mathbf{Set}_{tc}(H)$: *Objects* are just the H-sets with non-empty support and *morphisms* $f : A \longrightarrow B$ are ordinary maps $f : |A| \longrightarrow |B|$ satisfying

$$\delta_A(x, x') \leq \delta_B(f(x), f(x'))\,. \tag{6}$$

The subcategory of all discrete H-sets (i.e. of all H-valued fuzzy sets) coincides with Goguen's category $\mathbf{Set}(L)$ $(= \mathbf{Set}_{dc}(H))$.

A trouble with $\mathbf{Set}_{tc}(H)$-morphisms is that maps between H-sets are not necessarily extensional. This leads to a crisp equivalence relation $\cong$ in the hom-sets of $\mathbf{Set}_{tc}(H)$:

$$f \cong g \Leftrightarrow \delta_A(x, x) \leq \delta_B(f(x), g(x))\,.$$

f and g are *extensionally equal* iff $f \cong g$.

Now the category $\mathbf{Set}_{te}(H)$ of extensional morphisms can be defined as follows: Objects are again H-sets with non empty support set and morphisms are $\cong$-equivalence classes.

There exists also an alternative to express morphisms by binary H-valued relations; e.g. the Higgs topos – i.e. the category $\mathbf{Set}_{tf}(H)$ – consists of the following data: *Objects* are again H-sets with non-empty support sets, but *morphisms* are H-valued *functional* relations.

Now the subcategory of all discrete H-sets of $\mathbf{Set}_{tf}(H)$ coincides with Eytan's category $\mathbf{Fuz}(H)$ $(= \mathbf{Set}_{df}(H)$ in Wyler's notation).

Some interesting *results* are:

- $\mathbf{Set}_{dc}(H)$ is a topological quasitopos over sets, and hence has "crisp" internal logic.
- $\mathbf{Set}_{tc}(H)$ is cartesian closed, but not a quasitopos.
- $\mathbf{Set}_{tf}(H)$ is a topos with H-valued internal logic, and equivalent to the category $\mathsf{sh}(H)$ of *sheaves* over H, cf. [10, p. 363].
- $\mathbf{Set}_{te}(H)$ is a quasitopos with H-valued internal logic, and equivalent to the category $\mathsf{spsh}(H)$ of separated presheaves over H, cf. [7].

The other categories are, in general, neither topoi nor quasitopoi.

But this structural deficiency never was the main point of criticism from fuzzy people: their core objection always was that the intuitionistic context is too restricted by not allowing to discuss non-idempotent conjunctions, cf. [4, 30, 44].

4.3 Categories of Monoidal Sets

A more general non-intuitionistic, particularly a monoidal context together with a graded notion of identity for fuzzy sets is the core problem for the approach of U. Höhle as explained mainly in [26–29].

Instead of totally fuzzy sets, i.e. instead of H-sets, he considers M-sets, with $M = (L, \leq, *)$ an integral, divisible, commutative completely lattice-ordered monoid with zero, i.e. a complete divisible residuated lattice – called *GL-monoid* by this author.

The point is to consider *global M-sets* $A = (|A|, \delta_A)$ which are, like the H-sets of the Higgs topos, characterized by an M-valued global equality relation δ_A satisfying the conditions of reflexivity, symmetry, and $*$-transitivity. They are called *separated* iff they additionally satisfy the condition

$$\delta_A(x, y) = 1 \quad \Rightarrow \quad x = y\,. \tag{7}$$

The separated, global M-sets are natural generalizations of fuzzy sets (of higher level), because the most natural, naive understanding of a graded identity between fuzzy sets seems to involve the ideas that (1) each fuzzy set is identical with itself to the highest possible degree, and (2) that two fuzzy sets which are identical to the highest possible degree are truly identical.

The separated, global M-sets become the objects of a category if one takes as *morphisms* the "structure preserving" maps $f : |A| \longrightarrow |B|$ which have to satisfy the *preservation of equality* condition

$$\delta_A(x, y) \;\leq\; \delta_B(f(x), f(y))\,. \tag{8}$$

For these separated, global M-sets this choice of morphisms gives a category with interesting properties. Particularly this category is complete and cocomplete, i.e. has all limits and colimits. However, it does not allow for a unique classification of (extremal) subobjects, as shown in [29].

The problem is to find a finitely complete category $\mathcal{C}$ of M-sets, or of some other (suitably related) objects, which

- Has a subobject classifier Ω and a truth arrow t
- Allows the unique classification of the (Ω, t)-classifiable subobjects
- Internalizes M-valued maps as $\mathcal{C}$-morphisms with codomain Ω
- Is equivalent with the Higgs topos in the case that the underlying GL-monoid is a cHA

Höhle has in mind that for each cHA H the Higgs topos $\mathsf{SET}(H)$ of H-valued sets (and H-set morphisms) is categorically equivalent to the category $\mathsf{sh}(H)$ of sheaves over H. Furthermore one knows that Fourman/Scott [10] have shown that every presheaf determines an H-set, and that the sheaves correspond to the complete H-sets (which have all their singleton subsets determined by a single element).

His generalization thus starts with the search for a notion of presheaf in the monoidal context by looking for suitable singletons. And this happens not for global M-sets, but for M-sets, which generalize the global M-sets in that they refer to a notion of *local existence* as was done for the cHA valued case in [42].

The details are too complex to be explained here in detail. But the approach is successful.

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Fuzzy Control – Expectations, Current State, and Perspectives

Mirko Navara and Milan Petrík

Summary. We summarize the history of fuzzy sets. We try to find the reasons why fuzzy control has been so successful in applications. This is mainly explained by the fact that fuzzy logic created an alternative to exact computation and it better fits to the human way of reasoning.

We point out some aspects in which current fuzzy systems are not completely satisfactory and directions in which they should develop in the future.

Key words: Fuzzy set, Fuzzy control, Computational complexity, Fuzzy arithmetic, Stability.

The idea of partial truth and partial membership is old and it has been rediscovered many times (e.g., [4, 7, 13]). However, the seminal paper [28] has opened a new epoch of its rapid development.

Our first question is why exactly this work initiated a revolution if many theoretical results (see [4, 24]) have been derived before and remained almost unnoticed.

One reason is that Zadeh expressed this idea in a way accepted by experts in many fields, not only theoretical, but also applied, even by engineers. The preceding papers were recognized only by a limited community of mathematicians. Now the principle was expressed in a way understandable to everybody and in a context drawing new horizons and capabilities of the new technology based on it. It might have been crucial that the applications in control theory followed very soon [14, 26, 29]. Their success ensures permanent interest of industrial partners and financial support of this field.[1]

The second reason of success of fuzzy logic in Zadeh's approach is the state of control theory in the sixties. Preceding development of computers and cybernetics has brought ambitious expectations which have been satisfied only partially. The rapid development of control theory, as initiated by Wiener, has slowed down. It solved successfully some problems, in particular in control of linear systems, but it has encountered difficulties in control of

[1] In some countries.

systems with high non-linearity. These were partially solved by the developing non-linear control theory and by adaptive control, but this effort has brought much more complex questions without a clear trend to their satisfactory solutions. We bring arguments that in some sense the same happened to fuzzy control a few decades later.

The third reason is a disillusion from the limits of computational power. At the first moment, people were fascinated by the newly open possibility of cheap high-precision computations offered by computers. However, they recognized soon that some solutions are far from satisfactory. Simplified models failed to describe important features of real systems and the solutions did not perform well on some real-world systems. Then it was found out that supreme precision is not as important. Instead of that, we need to describe (at least roughly) the complexity of the surrounding world. This requires a representation of numerous relations which are not precisely known, but whose effect is at least intuitively understood by humans. Fuzzy logic offered a tool allowing to implement these ideas easily. This returned the technology closer to the human way of reasoning.

1 Success of Fuzzy Control

Fuzzy control celebrated a rapid success. If we look for its reasons, we may emphasize [6]:

- Easy design and tuning
- Interpretability of rules
- Possibility of combination of human knowledge with optimization and adaptive methods, neural networks, and genetic algorithms
- Description similar to our understanding and human way of reasoning

1.1 Through Simplification to Higher Complexity

Generally, we may say that fuzzy control represents a return from exact computing to computing with words or approximate quantities. The lost precision is compensated by the possibility to describe complex relations easily. It is not by chance that the development of fuzzy techniques followed (with some delay) the boom of computers. People were first fascinated by the possibility of fast and cheap computations with supreme precision. This was followed by a disillusion when users recognized that this technology does not produce satisfactory solutions to many problems which humans can solve easily. Fuzzy logic has brought a new hope. The ease of its use allows to include more rules describing many situations. What is also important, the antecedents (premises) of rules ensure that they will not be applied in inadequate situations, thus they will not influence rules applied in different modes. As an example, a cookbook is full of fuzzy algorithms. An experienced cook can work without

precise measurements, but with numerous feedback loops which compensate the uncertainty. If we replace a cook by an automaton, the recipe is strictly followed, but the result is not much better, often even worse.

In control, fuzzy logic is mostly used for *approximation*. In general, information used for approximation can be divided to data and a program (procedural knowledge, procedural information) working on them. A very general complexity criterion is the *minimal description length* (MDL). It simply counts all information needed, the total length of the data and the program. Current solutions usually suffer from a triviality of the procedural knowledge with respect to the high complexity of data. This is partially caused by technical means: The rapidly developing hardware allows to handle large data files; their allowed size increases exponentially. In contrast to this, the design of procedural information requires more advanced techniques (often human intervention) and thus it grows slowly. We usually know that the MDL of our solution is not optimal because a longer program could shorten the necessary data significantly. Nevertheless, we lack ideas and time for development of more complex programs, an extensive use of large data is a cheaper way. Fuzzy logic tries to break this trend by an offer of easy design of procedural information. Precision is not needed, because it can be improved by the additional data. (This is a typical situation in information compression, here considered as a specific approximation task, see [3].)

Remark 1. It would be interesting to classify the genetic information from this point of view. Current understanding of DNA suggests that it is rather a procedural information. Only small pieces (e.g., telomeres) are recognized as pure data which only control the procedural part. Maybe that nature discovered a more efficient encoding based mainly on procedural information. However, our understanding of information encoded in the proteins accompanying the DNA is not sufficient and it may change this conclusion.

A similar effect can be found in computer games, mainly chess. Current chess programs reach the level of top human players; however, this is achieved by extensive computation and enormous speed, not by deeper understanding [18]. In this aspect, people are still much ahead in the classification of alternatives which are worth attention.

1.2 Incompletely Described Tasks

Practice shows that people can control systems which they do not fully understand. E.g., a child can learn to drive a bike without any knowledge of partial differential equations used in classical solutions of this task. The sufficient principles can be described even without precise measurement of forces, moments, etc. Fuzzy logic takes the advantage of this phenomenon very often. What is even more important, the mutual relations between variables need not be precisely known. Otherwise, they cause problems also in the classical methods. E.g., the description of a joint distribution of several random

variables may be very complex and it might be difficult to collect sufficient statistical data for its determination. In this situation, classical methods usually introduce drastically simplifying assumptions (e.g., independence of variables which are known to be dependent) or they work with many probabilities which are fictitious, not rigorously determined. Fuzzy logic overcomes this difficulty due to its *functionality* (in the logical sense). This is often criticized as an incorrect tool, but alternative approaches also use drastic simplifications which only are not as apparent. On the other hand, fuzzy logic imitates a human approach to the solution and this appeared satisfactory in many situations.

1.3 Fuzzy Logic as Human–Machine Interface

The development of technology before fuzzy logic has led to many useful but very abstract notions. E.g., we know that the stability of a linear dynamic system depends on its eigenvalues. However, if the system is instable, it is difficult to find which change of parameters allows to stabilize it. This is sometimes a necessary approach, but there are also many situations which can be solved differently. Humans succeed to stabilize many systems without an exact knowledge of their parameters, applying only simple rules. Thus there must be an easy solution to these problems.

Remark 2. The study of insects, their perception, recognition, and orientation, shows that a satisfactory solution can often be achieved by surprisingly simple principles, see e.g., [22]. During evolution, such solutions appeared sufficient and efficient.

Why should we care to imitate humans in these "easy" control tasks? There are several reasons:

- Fast design
- The controller is easily understandable; this facilitates the tuning, debugging, and repairs
- The human-like solutions usually offer also a high level of robustness
- Sometimes we only want to substitute a human who does the task satisfactorily, but needs a rest, adequate working conditions, etc.

Typical successful stories of this kind are the control of a cement kiln [9] and an unmanned helicopter [25].

1.4 Intelligent Database Search and Antispam

Intelligent database search has been an intended area of applications of fuzzy logic. Instead of crisp search criteria, fuzzy sets can better describe the desired goal. The (combined) membership degree can be used as a criterion of relevance. Recently these principles were particularly successful in Google. We may hope that similar approach will improve also antispam filters. Current

solutions (e.g., SpamAssassin) combine numerous criteria in a rather naive way which ignores their dependence. More advanced classification techniques could improve the performance. However, there is a problem with a permanently changing situation. Once learned parameters become obsolete within several weeks of months and additional rules should be applied. Thus professional antispam service requires continuous updating of the rule base. New rules should not destroy the effect of other rules; we need a soft tuning during the run of the program. Fuzzy logic offers an adequate tool for formulation and implementation of new rules. However, problems may occur if we add too many new rules; this problem will be discussed in Sect. 3.3.

2 General Problems

2.1 From Global to Local

Current technology has the following specific problem which distinguishes it from human perception: For a computer, it is easy to recognize small items – words in a text, details of images, individual items in a database. However, global features require more advanced and extensive computing; it is difficult to recognize large objects in the image (with occlusions, shadows, etc.), prevailing features of objects in a database, etc. So far a computer cannot summarize a book. Even if it recognizes at least the type of a text document, this classification is based on separate statistical features and not on general understanding. In any case, this computation requires much more time than recognition of details. Human perception goes in the opposite direction—we first see a person and only later a freckle on it. This shows much difference between the two approaches.

Fuzzy logic might be expected to contribute here because it allows to describe the objects (characterized in fuzzy terms) that should be recognized. However, this did not lead to much faster algorithms, and in the current state of technology, it even cannot for the reasons that will be mentioned in Sect. 3.1.

2.2 What can be Solved Automatically?

Computers opened new perspectives, but there are many human activities which failed to be solved by classical (hard) computing. E.g., it is possible to recognize the cover of any CD and play it from a database, without the CD record. On the other hand, the task of recognition of natural objects is by far too difficult. Consider, e.g., a field guide of plants. At the first sight, it seems ready for programming. Besides, an implementation could overcome the uncomfortable tree structure. However, the field guide is full of imprecise terms difficult to explain to the computer. Fuzzy logic offers an excellent tool for their description and further manipulation. Nevertheless, the recognition is still very difficult and trained human experts can perform much better in

this task; for them, a single leaf can be a sufficient clue. Another problem causing difficulties here was described in Sect. 2.1.

2.3 Cheap Solutions

Fuzzy logic is often criticized as an offer of cheap solutions to difficult tasks. This need not be correct. Even a simple solution can be optimal if we take into account the speed; sometimes this admits to work with a fast approximation rather than a precise time-consuming solution. In control, the computation time is often important and requires to predict the state at the time when the output signal is ready. This may be difficult. A simple algorithm does not need a long prediction and thus may perform satisfactorily.

Sometimes simple solutions are required by hardware restrictions. Common processors have dimensions in centimetres and power consumption in tens of watts or more. Applications which require small size and consumption cannot be based on this standard hardware. Fuzzy control suggests a reasonable alternative; some tasks may be solved by microchips with a very low consumption, e.g., [1].

Also the speed of design may play a decisive role. E.g., the manipulator for the repair of the Hubble telescope has been equipped with a fuzzy controller simply because this was the only one which had been designed in time [19].

Finally, some solutions based on fuzzy control can be hardly called cheap and simple; also this technique contributed by rather advanced solutions.

The objection remains that fuzzy systems are often designed without a good understanding of the relations. Thus the reliability is doubtful. Therefore investigation of the guaranteed properties of fuzzy controllers is crucial. Nevertheless, the classical approach need not be more reliable; it offers exact solutions to well described systems which, however, are only very simplified approximations of reality, thus a serious error also cannot be excluded.

So far, the boom of fuzzy technologies is mainly concentrated in areas where reliability is not a supreme requirement. Fuzzy controlled vacuum cleaners, shavers, rice cookers, washing machines, etc. do not risk very much if a failure occurs. Also the intelligent database search is a useful tool which *helps* people in orientation in large data files, but does not take responsibility for the decision. Without any doubt, fuzzy systems proved to be *useful*. However, for other applications they need to prove to be *at least as reliable as other approaches*.

3 Specific Problems

3.1 Paradox of Computation with Uncertainty

The human way of computing gives rough estimates quickly, more exact results require additional effort and time. In contrast to this, conventional computers

give fast results with the machine precision (which is limited, but high enough for most applications). Any uncertain quantity (fuzzy number, as well as a random variable) requires more information for its representation and much longer time for arithmetical operations. Besides the *standard fuzzy arithmetic* (see, e.g., [11]), alternatives were suggested, in particular the *constrained fuzzy arithmetic* [10]; surprisingly, its computational complexity is even higher [17].

This is a paradox – a less precise value requires more computation than the "precise" one. This might be caused by our hardware and encoding. In fact, *imprecision is described very precisely*, using more information, in contrast to the representation of quantities in a human brain. There are some common aspects of his paradox and the problem of global/local features discussed in Sect. 2.1.

3.2 Defuzzification

Defuzzification is a necessary part of most fuzzy systems. It seems to be underdeveloped in comparison with the other elements. There are many defuzzification methods offering different advantages and disadvantages. Little is known about the selection of defuzzification methods. This is usually quietly left to the designer. This is a weak point of fuzzy control which is rightly criticized: From the user point of view, there is no specific requirement on the defuzzification. We deal thoroughly with the rule base, but not with defuzzification. Trying to compute with words or fuzzy quantities, we have no natural terms for specification of the method of defuzzification; we simply rely that it is performed in a way sufficient for interpretation of our rule base. This is treated only sometimes as a part of the inference mechanism.

3.3 Adding a New Rule

Many fuzzy systems automatically add new rules when the current rule base appears insufficient. It is possible to modify the whole rule base after this change, but this is usually not done. It may be even undesirable because of a risk of a change of behavior in other situations which were handled properly before. Typically, the new rule is merely added to those already used. As shown by H. Prade, this approach may have an undesirable side effect, especially if it is used repeatedly and many new rules are added.

If we add a new rule to a Mamdani–Assilian controller [14], the control surface, as well as the output, increase. Thus the output membership degrees become closer to 1 and they may carry less information.

In the residuum-based controller [23], the opposite situation occurs: The control surface and the output decrease. This monotonic development may result in output membership degrees closer to 0.

In both cases, the information about a desirable output becomes less specific. Besides, there are more problems with solvability of the respective system of fuzzy relational equations [2, 8, 21].

One attempt to overcome this drawback has been suggested by D. Peri [20]; he considers not only positive, but also negative rules. When their antecedent is satisfied, the consequent of a negative rule says which output values are *not* desirable.

Another solution is the controller with conditionally firing rules proposed in [15]. The newly added rule does not change the *form* of other rules, but it modifies their *effect* by attenuation of their influence within the domain of (the antecedent of) the new rule. This increases the membership degrees in points of the new consequent and decreases the membership degrees of other points. Thus the controller with conditionally firing rules may be recommended in systems where we expect many new rules added during the phase of tuning.

In [16], the latter two methods were tested on the automatic generation of fuzzy rules for approximation in the Fuzzy Rule Learner according to [27] and implemented in a system for medical diagnostics.

4 Conclusions and Perspectives

The above arguments lead us to a conclusion that fuzzy control reached a critical point. The following decade will decide its future role. There are many difficult control problems which combine high non-linearity with supreme requirements on reliability. If fuzzy control succeeds to solve them, it may expect a second boom and a wide spectrum of applications. If it fails, it will remain only one alternative method of cheap design of cheap controllers for applications where reliability is not the highest preference. Applications in vacuum cleaners, washing machines, etc. are nice, but not satisfactory as the only applications of an ambitious theory. Therefore the study of guaranteed properties of fuzzy controllers (in particular stability) is crucial for their future.

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Fuzzy Sets in Categories of Sets with Similarity Relations*

Jiří Močkoř

Summary. Several examples of categories **K** of sets with similarity relations are investigated. Objects of **K** are pairs (A, δ), where A is a set and δ is a similarity relation on A with values in an MV-algebra Ω. A fuzzy set t in (A, δ) in a category **K** (in symbol $t \subseteq_{\sim \mathbf{K}} (A, \delta)$) is then defined as a morphism $t : (A, \delta) \to (\Omega, \leftrightarrow)$ in a category **K**. Some properties of these fuzzy sets are investigated in some special categories **K** .

Key words: Fuzzy sets, Sets with similarity relation, Categories of sets with similarity relations.

1 Introduction

There are several categories of sets with similarity relations defined over lattice structures which are of importance [1, 3]. Let us mention at least three examples. The first one is a category $\mathbf{SetF}(\Omega)$ defined over a complete MV-algebra $\Omega = (L, \wedge, \vee, \otimes, \to, 1_\Omega, 0_\Omega)$. This category consists of objects (A, δ) (called Ω-sets), where A is a set and δ is a similarity relation, i.e. a map $\delta : A \times A \to \Omega$ such that

(a) $\delta(x, x) = 1 = 1_\Omega$,
(b) $\delta(x, y) = \delta(y, x)$,
(c) $\delta(x, y) \otimes \delta(y, z) \leq \delta(x, z)$.

A morphism $f : (A, \delta) \to (B, \gamma)$ in $\mathbf{SetF}(\Omega)$ is a map $f : A \to B$ such that $\gamma(f(x), f(y)) \geq \delta(x, y)$ for all $x, y \in A$. A category $\mathbf{SetR}(\Omega)$ which is an analogy of a category of sets with relations between sets as morphisms is also of importance. Objects of this category $\mathbf{SetR}(\Omega)$ are the same as in the

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category $\mathbf{SetF}(\Omega)$ and morphisms $f : (A, \delta) \to (B, \gamma)$ are maps $f : A \times B \to \Omega$ such that

(a) $(\forall x, z \in A)(\forall y \in B) \quad \delta(x, z) \otimes f(x, y) \leq f(z, y)$,
(b) $(\forall x \in A)(\forall y, z \in B) \quad \gamma(y, z) \otimes f(x, y) \leq f(x, z)$,
(c) $(\forall x \in A) \quad 1 = \bigvee\{f(x, y) : y \in B\}$.

Finally let $\mathbf{Set}(\Omega)$ be a category with the same objects as in the category $\mathbf{SetF}(\Omega)$ and with morphisms $f : (A, \delta) \to (B, \gamma)$ which satisfy conditions for morphisms in $\mathbf{SetR}(\Omega)$ and the condition

(a) $(\forall x \in A)(\forall y, z \in B) \quad f(x, y) \otimes f(x, z) \leq \gamma(y, z)$ (functionality of f).

This category is an analogy of a category of classical sets with relations which are functions as morphisms.

It is clear that all these categories of sets with similarity relations are generalizations of a classical category **Set** of sets with maps as morphisms. Hence for any category $\mathbf{K}$ of Ω-sets it is than natural to investigate objects which could be understood as a generalizations of fuzzy sets in a category **Set**. In this paper we want to introduce these fuzzy sets and investigate some of their properties. We will be specially interested in fuzzy set which are derived from classical subsets.

2 Fuzzy Sets in Ω-sets

We introduce fuzzy sets objects in Ω-sets by the following definition.

Definition 1. *Let* $\mathbf{K}$ *be a category with* Ω*-sets as objects. Then a fuzzy set* s *in an object* (A, δ) *in a category* $\mathbf{K}$ *(in symbol* $s \subsetsim_{\mathbf{K}} (A, \delta)$*) is a morphism*

$$s : (A, \delta) \to (\Omega, \leftrightarrow)$$

in $\mathbf{K}$*, where* $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$.

The following are some examples of fuzzy sets in categories defined above.

Example 1.

(a) A fuzzy set $s \subsetsim_{\mathbf{SetF}(\Omega)} (A, \delta)$ is a map $s : A \to \Omega$ such that $s(x) \otimes \delta(x, y) \leq s(y)$ for all $x, y \in A$ (These objects are called *extensional objects* (see [2]).
(b) A fuzzy set $s \subsetsim_{\mathbf{SetR}(\Omega)} (A, \delta)$ is a map $s : A \times \Omega \to \Omega$ such that
  (i) $s(a, \alpha) \otimes \delta(a, a') \leq s(a', \alpha)$,
  (ii) $s(a, \alpha) \otimes (\alpha \leftrightarrow \beta) \leq s(a, \beta)$,
  (iii) $1 = \bigvee_{\alpha \in \Omega} s(x, \alpha)$.
(c) A fuzzy set $s \subsetsim_{\mathbf{Set}(\Omega)} (A, \delta)$ is a map $s : A \times \Omega \to \Omega$ such that $s \subsetsim_{\mathbf{SetR}(\Omega)} (A, \delta)$ and $s(a, \alpha) \otimes s(a, \beta) \leq \alpha \leftrightarrow \beta$.

Let $\mathbf{K}$ be a category with objects Ω-sets (A, δ). We set

$$\mathcal{F}_{\mathbf{K}}(A, \delta) = \{s : s \subset_{\sim_{\mathbf{K}}} (A, \delta)\}$$

(the set of all fuzzy sets in (A, δ) in $\mathbf{K}$).

Theorem 1. *For* $\mathbf{K} = \mathbf{SetF}(\Omega), \mathbf{SetR}(\Omega)$ *or* $\mathbf{Set}(\Omega)$, $\mathcal{F}_{\mathbf{K}} : \mathbf{K} \to \mathbf{Set}$ *is a contravariant functor.*

It is well known that there exists a functor $F : \mathbf{SetF}(\Omega) \to \mathbf{Set}(\Omega)$ which is an identity function on objects and for a morphism $f : (A, \delta) \to (B\,\gamma)$, $F(f)(a, b) = \gamma(f(a), b)$ holds for all $a \in A, b \in B$.

Proposition 1. *There exists a natural transformation*

$$\sigma : \mathcal{F}_{\mathbf{SetF}(\Omega)} \to \mathcal{F}_{\mathbf{Set}(\Omega)} \circ F.$$

For an object (A, δ) we define a map $\sigma_{(A,\delta)} : \mathcal{F}_{\mathbf{SetF}(\Omega)}(A, \delta) \to \mathcal{F}_{\mathbf{Set}(\Omega)}(A, \delta)$ such that for $s \in \mathcal{F}_{\mathbf{SetF}(\Omega)}(A, \delta)$, $\sigma_{(A,\delta)}(s)(a, \alpha) = s(a) \leftrightarrow \alpha$ for all $a \in A$ and $\alpha \in \Omega$. Then it can be proved that σ is a natural transformation.

Let S be a subset in an Ω-set (A, δ), $S \subseteq A$. Analogously as for classical sets any subset defines a fuzzy set in a corresponding Ω-set for all categories defined above.

Proposition 2. *Let* (A, δ) *be an* Ω*-set and let* $S \subseteq A$.

(i) Let $\mathbf{K} = \mathbf{SetF}(\Omega)$ *and let* $\in_S (x) = \bigvee_{s \in S} \delta(x, s)$. *Then* $\in_S \subset_{\sim_{\mathbf{SetF}(\Omega)}} (A, \delta)$.

(ii) Let $\mathbf{K} = \mathbf{SetR}(\Omega)$ *and let* $\in_S (x, \alpha) = \alpha \to \bigvee_{s \in S} \delta(x, s)$. *Then* $\in_S \subset_{\sim_{\mathbf{SetR}(\Omega)}} (A, \delta)$.

(iii) Let $\mathbf{K} = \mathbf{Set}(\Omega)$ *and let* $\in_S (x, \alpha) = \alpha \leftrightarrow \bigvee_{s \in S} \delta(x, s)$. *Then* $\in_S \subset_{\sim_{\mathbf{Set}(\Omega)}} (A, \delta)$.

It is clear that a value $\alpha \to \in_S (a)$ then represents in some sense a truth value $\|a \in_\alpha S\|_{\mathbf{K}}^{(A,\delta)}$ of an interpretation of a statement $a \in_\alpha S =$ "$a \in S$ in a degree at least α" in an object (A, δ) in a category $\mathbf{K} = \mathbf{SetF}(\Omega)$. Analogously $\in_S (a, \alpha)$ represents the same truth value in a category $\mathbf{SetR}(\Omega)$ or $\mathbf{Set}(\Omega)$.

There is even a more closer relationship between subsets and fuzzy set. Recall that a subset $S \subseteq A$ is called *complete* in (A, δ) if $\overline{S} = \{a \in A : \bigvee_{x \in S} \delta(a, x) = 1\} = S$.

Proposition 3. *Let* (A, δ) *be an* Ω*-set and let* $S \subseteq A$.

(i) $\in_S = \in_{\overline{S}}$ *for any category* $\mathbf{K} = \mathbf{Set}(\Omega), \mathbf{SetF}(\Omega)$ *or* $\mathbf{SetR}(\Omega)$.

(ii) Let $p \subset_{\sim_{\mathbf{K}}} (A, \delta)$, *where* $\mathbf{K} = \mathbf{Set}(\Omega), \mathbf{SetR}(\Omega)$, *respectively. Then* $S_p = \{a \in A : p(a, 1) = 1\}$ *is a complete set.*

(iii) Let $\mathbf{K} = \mathbf{Set}(\Omega), \mathbf{SetR}(\Omega)$, *respectively. Then* $S_{\in_S} = S$ *if and only if* S *is a complete set in* (A, δ).

Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in a category $\mathbf{K}$ of Ω-sets and let $T \subseteq B$. Depending on a category $\mathbf{K}$ we can define various types of inverse images of T under a morphism f. For example, if $\mathbf{K} = \mathbf{SetF}(\Omega)$ then $f^{-1}(T) = \{a \in A : f(a) \in T\}$ and for $\mathbf{K} = \mathbf{SetR}(\Omega)$, $f^{-1}(T) = \{a \in A : \bigvee_{y\in T} f(a,y) = 1\}$.

Proposition 4. *Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{K}$, $T \subseteq B$ and let $a \in A$, $\alpha \in \Omega$. Then*

$$\begin{aligned} &\textit{(i)}\ \|a \in_\alpha f^{-1}(T)\|_{\mathbf{K}}^{(A,\delta)} \quad \leq \quad \|f(a) \in_\alpha T\|_{\mathbf{K}}^{(B,\gamma)}, \\ &\textit{(ii)}\ \mathcal{F}_{\mathbf{K}}(f)(\in_T)(a,\alpha) \leq \quad \|f(a) \in_\alpha T\|_{\mathbf{K}}^{(B,\gamma)} \end{aligned}$$

for $\mathbf{K} = \mathbf{SetF}(\Omega)$ and $\mathbf{SetR}(\Omega)$.

It can be proves that the equality relation does not hold in previous proposition, in general. On the other hand we have the following partial result.

Proposition 5. *Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{K} = \mathbf{Set}(\Omega)$.*

(a) If the operation $\otimes$ is idempotent in MV-algebra Ω then the equality relation holds in a statement (b) from Proposition 2.4.
(b) If for any $a \in A$ there exits $b \in B$ such that $f(a,b) = 1$ then the equality relation holds in a statement (b) from Proposition 2.4.

An inverse image of a set $T \subseteq B$ in a morphism $f : (A,\delta) \to (B,\gamma)$ in a category $\mathbf{Set}(\Omega)$ can be also introduced in a little different form as follows:

$$f^{(-1)}(T) = \{a \in A : \bigvee_{x\in T} \bigvee_{b\in B} f(a,b) \otimes \gamma(b,x) = 1\}.$$

In the following proposition we present some properties of these sets.

Proposition 6. *Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in a category $\mathbf{Set}(\Omega)$ and let $T \subseteq B$.*

(a) $f^{(-1)}(T)$ is a complete set in (A,δ),
(b) $f^{(-1)}(\overline{T}) = f^{(-1)}(T)$,

Finally, let $f : (A,\delta) \to (B,\gamma)$ be a morphism in a category $\mathbf{SetF}(\Omega)$. By using a functor F we obtain a morphism $F(f)$ in a category $\mathbf{Set}(\Omega)$, namely $F(f)(a,b) = \gamma(f(a),b)$ for any $a \in A, b \in B$. By the following proposition we can simply calculate the inverse image of this morphism $F(f)$.

Proposition 7. *Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in a category $\mathbf{SetF}(\Omega)$ and let $T \subseteq B$. Then we have*

$$F(f)^{-1}(T) = f^{-1}(\overline{T}) = \{a \in A : f(a) \in \overline{T}\}.$$

3 Properties of functors $\mathcal{F}_{\mathbf{K}}$

In this section we present some principal properties of functors $\mathcal{F}_{\mathbf{K}}$ for some categories $\mathbf{K}$ of sets with similarity relations. In the first theorem we show that fuzzy sets in an object (A,δ) in a category $\mathbf{SetF}(\Omega)$ can be identified with some *characteristic morphism* $(A,\delta)\to(\Omega^*,\mu)$, where

$$\Omega^* = (\{(\alpha,\beta)\in L\times L \mid \alpha\geq\beta\},\mu),$$
$$\mu((\alpha_1,\beta_1),(\alpha_2,\beta_2)) = \alpha_1\otimes(\beta_1\to\beta_2)\wedge\alpha_2\otimes(\beta_2\to\beta_1).$$

Theorem 2. *There exists a natural equivalence*

$$\zeta : \mathcal{F}_{\mathbf{SetF}(\Omega)}(-)\cong Hom_{\mathbf{SetF}(\Omega)}(-,(\Omega^*,\mu)).$$

Instead of a contravariant functor $\mathcal{F}_{\mathbf{SetF}(\Omega)} : \mathbf{SetF}(\Omega)^{op}\to\mathbf{Set}$ we can introduce a covariant version $\mathcal{F}^{cov}_{\mathbf{SetF}(\Omega)} : \mathbf{SetF}(\Omega)\to\mathbf{Set}$ such that $\mathcal{F}^{cov}_{\mathbf{SetF}(\Omega)}(A,\delta) = \mathcal{F}_{\mathbf{SetF}(\Omega)}(A,\delta)$ and for a morphism $f:(A,\delta)\to(B,\beta)$ in $\mathbf{SetF}(\Omega)$ let $\mathcal{F}^{cov}_{\mathbf{SetF}(\Omega)}(f)$ be defined such that for $s\in\mathcal{F}_{\mathbf{SetF}(\Omega)}(A,\delta)$ and $b\in B$ we have

$$\mathcal{F}^{cov}_{\mathbf{SetF}(\Omega)}(f)(s)(b) = \bigvee_{x\in A} s(x)\otimes\beta(b,f(x)).$$

Analogously we can introduce a covariant version of a functor Hom. Let $hom : \mathbf{SetF}(\Omega)\to\mathbf{Set}$ be a covariant functor such that

$$hom(A,\delta) = Hom_{\mathbf{SetF}(\Omega)}((A,\delta),(\Omega^*,\mu))$$

and for a morphism $f:(A,\delta)\to(B,\beta)$, $u\in hom(A,\delta), b\in B$,

$$hom(f)(u)(b) = (1_\Omega, \mathcal{F}^{cov}_{\mathbf{SetF}(\Omega)}(f)(pr_2.u)(b)).$$

Theorem 3. *The natural equivalence ζ is also a natural equivalence*

$$\zeta : \mathcal{F}^{cov}_{\mathbf{SetF}(\Omega)}(-)\cong hom(-).$$

If $\mathbf{K}$ is a category of sets with similarity relationss then a set of fuzzy sets $\mathcal{F}_{\mathbf{K}}(A,\delta)$ of an object (A,δ) can be transformed onto a set with a similarity relation. For example if $\mathbf{K}=\mathbf{SetF}(\Omega)$ then on a set $\mathcal{F}_{\mathbf{SetF}(\Omega)}(A,\delta)$ two similarity relations σ,τ can be defined such that

$$\sigma(s,t) = \sigma_{(A,\delta)}(s,t) = \bigwedge_{x\in A} s(x)\leftrightarrow t(x),$$

$$\tau(s,t) = \tau_{(A,\delta)}(s,t) = \begin{cases} \bigvee_{x\in A} s(x)\otimes t(x), & \text{if } s\neq t \\ 1, & \text{if } s=t \end{cases},$$

If $\mathbf{K} = \mathbf{Set}(\Omega)$ then on a set $\mathcal{F}_{\mathbf{Set}(\Omega)}(A,\delta)$ a similarity relation ρ can be defined such that

$$\rho(p,q) = \rho_{(A,\delta)}(p,q) = \bigwedge_{x\in A} p(x,1) \leftrightarrow q(x,1).$$

Recall that a fuzzy set $t \subsetsim_{\mathbf{Set}(\Omega)} (A,\delta)$ is a *weak singleton* if $\bigvee_{a\in A} t(a,1) = 1$ and $t(a,1) \otimes t(b,1) \le \delta(a,b)$ for all $a,b \in A$. Let $\mathcal{F}^{ws}_{\mathbf{Set}(\Omega)}(A,\delta) = \{t : t \text{ is a weak singleton in } (A,\delta)\}$.

Definition 2. *(a) Let* $\mathcal{F}^{\leftrightarrow}(\mathbf{SetF}(\Omega)) \hookrightarrow \mathbf{SetF}(\Omega)$ *be a full subcategory of* $\mathbf{SetF}(\Omega)$ *with objects* $(\mathcal{F}_{\mathbf{SetF}(\Omega)}(A,\delta),\sigma)$,
(b) Let $\mathcal{F}^{\otimes}(\mathbf{SetF}(\Omega)) \hookrightarrow \mathbf{SetF}(\Omega)$ *be a full subcategory of* $\mathbf{SetF}(\Omega)$ *with objects* $(\mathcal{F}_{\mathbf{SetF}(\Omega)}(A,\delta),\tau)$,
(c) Let $\mathcal{F}_{ws}(\mathbf{Set}(\Omega)) \hookrightarrow \mathbf{Set}(\Omega)$ *be a full subcategory of* $\mathbf{Set}(\Omega)$ *with objects* $(\mathcal{F}^{ws}_{\mathbf{Set}(\Omega)}(A,\delta),\rho)$.

Theorem 4. *Any of the above defined subcategories is a weak reflective subcategory in a corresponding category.*

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Fuzzy Sets as a Special Mathematical Model of Vagueness Phenomenon*

Vilém Novák

Summary. In the paper, the indeterminacy phenomenon is discussed, that is, a phenomenon having two facets: uncertainty and vagueness. We argue that fuzzy sets are a reasonable mathematical tool for modeling of the latter. The necessary sound foundations of their theory can now be more easily established because of significant progress reached in the formal theory of fuzzy logic. Further direction in the development of fuzzy set theory is also briefly discussed.

Key words: Fuzzy set theory, Fuzzy logic, Vagueness, Uncertainty, Indeterminacy.

1 Introduction

Forty-one years have already passed since Zadeh published his first paper [23]. It attracted a lot of researchers and also philosophers by turning their interest into vagueness phenomenon which, in the same way as randomness, has also been recognized as an unavoidable feature of the surrounding world. Moreover, Zadeh has shown in a sequence of papers that his ideas can be used in many kinds of real applications. We claim that fuzzy set theory is a reasonable mathematical theory providing a working model of vagueness phenomenon in a way similar to probability theory which provides the same for uncertainty.

The history of fuzzy sets is analogous to the history of classical set theory. In the beginning (see [23]), it was formulated purely intuitively as a theory of groupings with unsharp boundaries. During the time, several attempts to formulate their theory as a formal axiomatic theory appeared. But only recently, thanks to the rapid development of the formal fuzzy logic, sound foundations of fuzzy set theory have been established [1, 5, 11].

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In this paper, we will briefly characterize the vagueness phenomenon and the way, how it can be encompassed by the fuzzy set theory. We will also briefly discuss the distinction between potentiality and actuality and show that these phenomena lay in the core of understanding to the distinction between truth and possibility.

2 Uncertainty and Vagueness

Two phenomena whose importance in science raised especially in 20th century are *uncertainty* and *vagueness* [2,19]. Both of them characterize situations in which the amount and extent of knowledge we have at disposal is crucial. It is important to stress that both uncertainty as well as vagueness form two complementary facets of a more general phenomenon called *indeterminacy*. In reality, we often meet indeterminacy *with both its facets* present, i.e., *vague* phenomena are at the same time *uncertain*.

The *uncertainty* phenomenon emerges when there is a *lack of knowledge* about *occurrence* of some *event*. This means that it is encountered when an experiment (process, test, etc.) is to proceed, the result of which is not known to us; it may also refer to variety of potential outcomes, ways of solution, choices, etc. We will speak about events also in this case. Its specific form is *randomness* which is uncertainty raising in connection with time. There is no randomness (uncertainty) after the experiment was realized and the result is known to us. From this point of view, uncertainty is epistemological concept. Note that it is connected with the question whether a given event may be regarded within some time period, or not. This becomes apparent on the typical example with tossing a player's cube. The phenomenon to occur is *the number of dots on the cube* and it occurs after the experiment (i.e., tossing the cube one times) has been realized. Thus, we refer here to the future. However, the variety of potential events may raise even a more abstract uncertainty that is less dependent on time. We may, for example, analyze uncertainty in potentiality (that is, lack of knowledge) without necessary reference to time, or with reference to the past (such as a posterior Bayesian probability). This supports the ideas presented in the next section. It should be stressed, however, that when speaking about the pure uncertainty we refrain from the character of events in concern so that they can be both crisply as well as vaguely delineated.

The mathematical model (i.e., quantified characterization) of the uncertainty phenomenon is provided especially by *probability theory*. In everyday terminology, probability can be thought of as a numerical measure of the likelihood that a particular event will occur. There are also other mathematical theories addressing the mentioned abstract uncertainty, for example possibility theory, belief measures and others.

The *vagueness phenomenon* raises when trying to *group* together objects that have a certain property φ. The result is an *actualized* grouping of objects

$$X = \{o \mid o \text{ is an object having the property } \varphi\}.$$

Note that X, in general, it *cannot be taken as a set* since the property φ may be *vague*, i.e., it may not be possible to characterize the grouping X precisely and unambiguously; there can exist *borderline* elements o for which it is unclear whether they have the property φ (and thus, whether they belong to X), or not. On the other hand, it is always possible to characterize, at least some *typical objects* (prototypes), i.e., objects having typically the property in concern. For example, everybody can show a "blue sweater" or "huge building" but it is impossible to show "all huge buildings".

Vagueness is opposite to exactness and we argue that it cannot be avoided in the human way of regarding the world. Any attempt to explain an extensive detailed description necessarily leads to using vague concepts since precise description contains abundant number of details (see the incompatibility principle formulated by Zadeh in [24]). To understand it, we must group them together — and this can hardly be done precisely. A nonsubstitutable role is here played by natural language. However, the problem lays deeper, in the way how people regard the phenomena around them.

Vagueness should also be distinguished from *generality* and from *ambiguity*. "More general" means that more (various) groupings of objects are taken into account, while ambiguity occurs in the language when more alternative meanings are assigned to the same word or expression. Unlike uncertainty where we always have to consider whether some phenomenon *occurs or not*, vagueness concerns the way how the *phenomenon itself* is delineated, no matter whether it will occur or not.

A typical feature of vagueness is its continuity: a small difference between objects cannot lead to abrupt change in the decision of whether either of them has, or has not a vague property (cf. Black's "museum of applied logic" in [2]). The transition from having a (vague) property to not having it is smooth.

We believe that it is now clear that the vagueness phenomenon cannot be avoided. In the most striking way it is exhibited in the semantics of natural language. Let us stress that this is not its weakness but vice-versa, its great strength and it is the main reason why we are able to communicate in natural language about everything we see in the surrounding world. Therefore, when used for description of some process, the theory harnessing its semantics enables us to develop methods for harnessing the process itself. This is the secret of success, for example, of fuzzy control.

3 Actuality and Potentiality

When shall we use the degree of truth and when some other degree, such as possibility or probability? We argue that this difference follows from the difference between *actuality* and *potentiality*.

Due to the deep analysis of Vopěnka in [21,22], every set in classical (Cantor) set theory is understood as *actual*, i.e., we take all its elements as already existing and at our disposal in one moment. Therefore, our reasoning about any set stems from the assumption that it is at our disposal as a whole. Of

course, when a set is infinite then only God is able to see it as a whole while we can see only a part of it. On the other hand, most events around us are only *potential*, i.e., they may, but need not, occur or happen. Thus, to create a grouping of objects, we may have only a method how a new element can be created but all of them will never exist together. For example, if a tailor is given a piece of cloth, many various dresses can be sewed off it but only one will actually be finished. It is even impossible to imagine all dresses that can be sewed from one piece of cloth.

We conclude that the difference between actuality and potentiality corresponds to the difference between vagueness and uncertainty: vagueness applies to an *actualized* non-sharply delineated groupings while uncertainty is encountered when dealing with still *nonactualized* ranges of objects: we speculate about the whole X, but only part of it indeed exists. Once an actualized, i.e., already existing grouping of objects is at our disposal, we may speak about *truth* of the fact that some element belongs to it. Indeed, let an object y be created (at least in our mind). If we learn that it has a property φ, we know that it falls into (the existing part of) X, i.e., we *know the truth of* $y \in X$. If, however, we *do not know* whether y *will be* created (will exist) or not, it has no sense to speak about the truth of the fact that $y \in X$ since there is no such fact.

The *possibility* raises when we temporarily actualize a potential grouping, i.e., we imagine all (or, some) still not existing elements as existing. Then the "added" part may be, or may be not, possible. For example, given a tossing cube, we can imagine all dots that can be tossed, i.e., 1–6 as already existing (though they will never exist together). Hence, any of the numbers 1–6 *is possible*. On the other hand, since there is no number 7 on the cube, it has no sense to imagine it as a member of this temporarily actualized grouping. Consequently, 7 is, in this case, *impossible*. At the same time, we may try to guess whether a given number will indeed be tossed (i.e., the given element of this grouping will be created). This situation is usually modeled using probability. Note that only possible events can indeed occur with various probability and so, probability is majorized by possibility. Has that the temporarily actualized grouping can be at the same time vague and so, possibility needs not be crisp.

Finally, let us remark that the difference between actuality and potentiality has been implicitly considered by Zadeh as *conjunctive* and *disjunctive* view on fuzzy sets. The latter appears in his possibility theory [25].

4 Fuzzy Sets Naturally Emerge as a Graded Model of Vagueness

It is clear that a working mathematical theory of the vagueness phenomenon is necessary. There are not many theories aiming at this goal. Let us mention supervaluation [9, 20], alternative set theory [21, 22] and fuzzy logic (fuzzy set theory). The latter has a privileged position among all of them especially

because it is the best elaborated theory. Its main idea applies a principle called *graded approach* (sometimes also *fuzzy approach*) that is, a relation between object and its property is characterized using a scale. This is a general principle of the human mind for which it is natural to introduce a scale whenever a vague property is encountered. For example, we often say "almost white dress", "very strong engine", "too unpleasant situation", etc. In all these examples, we introduce some degrees of intensity of the property in concern. The degrees are taken from a scale that must have certain necessary properties: it must be an ordered set and it must have potential to capture the continuity feature of vagueness, i.e., to be uncountable. Furthermore, it must enable us to represent various kinds of operations with the properties. The result is an *algebra of truth values* (see [4, 7, 8, 16]).

Let us stress that fuzzy sets take the role of *approximation* of vagueness (this has been mathematically formulated by Novák first in [10] and later also in [12]). We argue that such approximation is a consequence of an *indiscernibility*, i.e., of our inability to discern objects (cf. Vopěnka [21, 22], and also Novák [12]). For example, a movie is a sequence of pictures. When projected at a sufficient speed, we are unable to distinguish them one from another and the result is a vague phenomenon that we regard as a continuous movement. Similarly, a shape of a heap of stones is also vague and when adding or removing one stone, its shape indiscernibly changes. This is the core of the *sorites paradox* (for the mathematical model of sorites using indiscernibility relation see [14]). Just another example is the so-called *ostensive definition*, that is, learning by examples: for example, mother shows her child a given plant and says "this is a tree". After several repetitions with different trees, the child captures this idea and he/she is able to point out correctly various trees that may be significantly different from those originally shown. This means that he/she captured certain indiscernibility relation enabling him/her to classify trees.

The indiscernibility is in fuzzy set theory modeled via fuzzy equality (fuzzy equivalence). Note this idea is contained in the works of various authors (e.g., [6, 7]) and it fully conforms with the original idea that fuzzy sets deal with gradedness or graduality.

With respect to the discussion in the previous section, it becomes clear that when temporarily actualizing a potential and, at the same time, vague grouping and we use fuzzy sets for this task, we naturally arrive at *degrees of possibility*. From this point of view, Zadeh's possibility postulate [25] stating that the possibility distribution comes out of a membership function of a corresponding fuzzy set is quite natural.

Stated as a conclusion of this section, fuzzy sets characterize actualized vague groupings while possibility distributions characterize potential ranges of objects and we imagine certain part of them (sometimes even whole of them) as temporarily actualized. In the former case, the membership degree is a *degree of truth* that a given element has a property in concern, while in the latter case, it is a *possibility degree* that an object can be actualized (i.e., created).

5 Future Development of Fuzzy Set Theory

We see the main tract of further development of fuzzy set theory and its applications in the following:

1. Development of complete formal logical and philosophical backgrounds of fuzzy set theory.
2. Seeking new problems stemming from display of the vagueness phenomenon and offering solutions using fuzzy sets (and fuzzy logic).
3. Development of new methods based on fuzzy sets thanks to which it is possible to solve classical problems in a novel way, with less effort and more transparent results.

Ad 1. Clearly, the development of fuzzy set theory is closely related to that of fuzzy logic, which achieved a significant progress especially during recent 15 years. There is still a lot of work before us to establish good mathematical foundations of fuzzy set theory. However, various attempts have been accomplished (see [3] and the citations contained therein). A very promising work on foundations based on the formal theory of fuzzy logic, both predicate as well as higher order, is contained in the recent papers [1, 5] (cf. also the fuzzy type theory [13] whose model comprises fuzzy sets of all orders).

It seems also important to clarify the relation of fuzzy sets to supervaluation theory ([9,20]). The main idea of the latter is that vague phenomena can be made precise in a variety of different ways. The truth that an element has a vague property is its supervaluation, which is a function of the tentative ordinary (classical) truth valuations of this proposition. For each way of making it precise, we get a new tentative classical valuation indicating whether the proposition, as thus made precise, is true or false. If every way of making the proposition precise makes it (classically) true, all of its tentative valuations will be true. If every precise version of the proposition is false, all of the tentative classical valuations are false. Otherwise, we get a mixture of tentative valuations. The result of supervaluation of the vague proposition is true if all the tentative valuations are true and false if they are all false; otherwise it is undefined. We argue that this situation can be embedded into fuzzy logic and thus, in fuzzy set theory. The details, however, have to be elaborated.

Ad 2. One of the problems of fuzzy set theory can be seen in the fact that people often confine to, sometimes rather cheap, generalization of the known results obtained in classical mathematics. We cannot completely cast this work aside since it may often be a starting point to something deeper. However, it is insufficient as a general paradigm. It should be noted that Běhounek and Cintula have shown in [1] that a great deal of results that have been developed for special models of fuzzy sets are more or less direct consequence of the very general syntactic theory of fuzzy classes. Consequently, a lot of effort can be spared for seeking and solving questions and problems specific for the vagueness phenomenon. We may be successful only when we carefully realize, what is fuzzy set theory about, what it can, or cannot, offer and at which moment the vagueness phenomenon really prevents classical mathematical methods

(i.e., those ignoring vagueness) from providing a good solution. Example of such problem is the theory of *evaluating linguistic expressions* (see, e.g., [14]) that may hardly be formed using classical set theory and that, in our opinion, plays a very important role in many applications of fuzzy sets. Note that this theory has also been initiated by Zadeh and it is closely related to his concepts of "computing with words" and "precisiated natural language" (cf. [26]).

We argue that fuzzy sets can offer means for solution of nonstandard problems in the newly emerging theories, e.g., in artificial intelligence, robotics, computer science, when developing human-like robots, modeling of swarms, and elsewhere.

Ad 3. This interesting possibility emerged not long ago. A typical example of this idea is the concept of *fuzzy approximation* which considers a problem of approximation of ordinary functions in other than classical spaces of functions equipped with similarities or fuzzy partitions.

One of already existing results of fuzzy approximation theory is a surprisingly powerful technique of *fuzzy transforms* developed by I. Perfilieva (see, e.g., [17, 18]). It can be demonstrated that this technique can be used for solution of many conventional problems such as numerical integration, solution of differential equations (including partial ones), or compression of pictures. Technique of fuzzy transforms is in many respects simpler than the technique of other known transforms (Fourier, Laplace, etc.) and less sensitive to setting of initial conditions.

6 Conclusion

Fuzzy set theory is a well established mathematical theory that is far from being trivial but still has a great potential for further development. It is reasonably philosophically substantiated and provides a wide field both for further theoretical investigation as well as for the development of new methods and applications not only in itself but also as a part of many other theories.

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Fuzzy IF-THEN Rules from Logical Point of View*

Irina Perfilieva

Summary. The theory of IF-THEN rules proposed by Lotfi A. Zadeh attracted many researchers and practitioners because of its simplicity and elegance. This contribution is an attempt to create a comprehensive logical theory of fuzzy IF-THEN rules based on Hájek's predicate BL-fuzzy logic. The formal logical theory presented in this contribution emphasizes that:

- A system of fuzzy IF-THEN rules with the Compositional Rule of Inference characterizes a partially given fuzzy function.
- In any model of the theory of IF-THEN rules the respective system of fuzzy IF-THEN rules is solvable.

Key words: Fuzzy logic, Fuzzy IF-THEN rules.

1 Introduction

The theory of fuzzy IF-THEN rules proposed by Lotfi A. Zadeh (see e.g., [9]) attracted many researchers and practitioners because of its simplicity and elegance. The main idea is to show that deduction can proceed via computation. With this purpose, he proposed two rules of entailment:

$$\textbf{CRI:}\quad \frac{\begin{array}{l} x \text{ is } A \\ (x,y) \text{ is } R \end{array}}{y \text{ is } (A \circ R)}$$

and

$$\textbf{GMP:}\quad \frac{\begin{array}{l} \mathsf{IF}\ x \text{ is } A\ \mathsf{THEN}\ y \text{ is } B \\ x \text{ is } A' \end{array}}{y \text{ is } B'}$$

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where **CRI** abbreviates the Compositional Rule of Inference and **GMP** stands for the Generalized Modus Ponens. Without going to specific details which will be discussed later, let us comment that in **CRI**, $\circ$ denotes a composition between A and R and in **GMP**, A' and B' denotes restrictions related to A and B. Both rules can be understood as generalization of classical Modus Ponens rule

$$\textbf{MP:}\quad \begin{array}{l} \mathsf{IF}\ x \text{ is } A\ \mathsf{THEN}\ y \text{ is } B \\ x \text{ is } A \\ \hline y \text{ is } B \end{array}$$

when considering its semantical interpretation extended to the set $[0,1]$ of truth values.

The following semantical interpretation of **CRI** and **GMP** rules has been proposed by Zadeh:

- Choose $[0,1]$ as a set of truth values.
- For interpretation of logical connectives choose $\min(a,b)$ for conjunction, $\max(a,b)$ for disjunction, $\min(a,b)$ for implication and $1-a$ for negation.
- For interpretation of R in **CRI** choose a fuzzy relation $R : X \times Y \longrightarrow [0,1]$.
- For interpretation of $\mathsf{IF}\ x$ is $A\mathsf{THEN}\ y$ is B in **GMP** choose a fuzzy relation $(A \to B)(x,y) = \min(A(x), B(y))$.
- To compute $(A \circ R)$ in **CRI** choose the sup $-\wedge$-composition

$$(A \circ R)(y) = \bigvee_{x \in X} A(x) \wedge R(x,y), \qquad y \in Y.$$

- Compute B' in **GMP** as follows:

$$B'(y) = (A' \circ (A \to B))(y) = \bigvee_{x \in X} (A'(x) \wedge A(x) \wedge B(y)), \qquad y \in Y. \quad (1)$$

This means that the original intention of Zadeh was to consider both rules semantically. Therefore, he did not put any restriction on interpretation of A, A', B, R.

The problem arises if we try to apply the proposed interpretation to **MP**. This leads to verification of the equality

$$B(y) = (A \circ (A \to B))(y) = \bigvee_{x \in X} (A(x) \wedge B(y)). \quad (2)$$

It turns out that (2) does not always hold. To show this, it is sufficient to choose universes X, Y and put $B(y_0) > 0$ for some $y_0 \in Y$ and put $A(x) \leq \frac{B(y_0)}{2}$ for all $x \in X$. The right hand side of (2) gives $(A \circ (A \to B))(y)$ which is not equal to $B(y)$ at y_0 and so, the chosen interpretation violates **MP**.

The situation becomes even more complicated if we consider a set of fuzzy IF-THEN rules and extend the conditional premise of **GMP** to

$$\mathsf{IF}\ x \text{ is } A_i\mathsf{THEN}\ y \text{ is } B_i, \qquad i = 1, \ldots n.$$

It is desirable that particularization of extended **GMP** to *extended* **MP** of the form

$$\textbf{MP:}\quad \begin{array}{l} \mathsf{IF}\ x \text{ is } A_i \ \mathsf{THEN}\ y \text{ is } B_i, \qquad i = 1, \dots n \\ x \text{ is } A_j \\ \hline y \text{ is } B_j \end{array}$$

should be valid for each $j = 1, \dots, n$.

1.1 Semantical Interpretation of Fuzzy IF-THEN Rules

Much effort has been made to find a proper semantics and keep **MP** valid as a particular case of **CRI** or **GMP**. Let us list the most recognizable ones:

- Interpretation of $\rightarrow$ which together with max and min would allow to keep (2) valid.
- Interpretation of the main logical connectives by the corresponding operations from some residuated lattice.
- Interpretation of $A_i|_{i=1,\dots,n}, B_i|_{i=1,\dots,n}, R$ in such a way that R solves the *corresponding system of fuzzy relation equations*:

$$(A_i \circ R)(y) = B_i(y), i = 1, \dots, n. \tag{3}$$

Although many results have been obtained in this direction and gave rise to corresponding theories (cf. [1, 3, 4, 6]), they did not result in a formal logical theory that would explain reasoning based on fuzzy IF-THEN rules.

1.2 Formal Logical Theories of Fuzzy IF-THEN Rules

Two approaches are worth to be mentioned in a correspondence with the topic of this subsection: Novák's fuzzy logic in broader sense based on fuzzy logic with evaluated syntax [7] and Hájek's theory of approximate reasoning as a special theory of predicate BL-fuzzy logic [5].

Hájek actually created three special theories in predicate fuzzy logic: *Comp*, $Comp_{MP}$ and $Comp_{CR}$, each with the respective special axiom related to **CRI** or **GMP**. In each theory he used a trick of splitting an axiom and creating by this a rule of inference. However, he did not consider a relationship between **CRI** and **MP** or **GMP** and **MP**. Therefore, his special theories have no interconnections.

1.3 Synergy of Semantics and Syntax of Fuzzy IF-THEN Rules

Logical deductions via computations and *computing with words* — these paradigms are still actual and attractive (see e.g., [2]). They motivate creation of formal logical theories capable of producing and explaining deduction based

on fuzzy IF-THEN rules, answering queries and, at the same time, proving that the produced answers are correct. In other words, they require both semantics as well as syntax of fuzzy IF-THEN rules and deduction over them.

This contribution is an attempt to create a new comprehensive logical theory of fuzzy IF-THEN rules based on Hájek's predicate BL-fuzzy logic (see also [8]). The formal logical theory which will be presented here emphasizes the following features:

- A system of fuzzy IF-THEN rules with the Compositional Rule of Inference characterizes a partially given fuzzy function.
- A corresponding system of fuzzy relation equations (3) is solvable in any model of the theory of fuzzy IF-THEN rules.
- The extended **MP** is provable in the theory of fuzzy IF-THEN rules.
- A particular form of **GMP** is provable in the theory of fuzzy IF-THEN rules.
- A proper extension of the theory of IF-THEN rules by additional axioms (rules) can be modeled by an extended model which preserves all parameters computed before.

2 Special Theory of Fuzzy IF-THEN Rules

2.1 Language of BL∀ and Its Structure

Let us recall [5] that the language J of the basic predicate fuzzy logic (BL∀) consists of predicate symbols $P, Q, S, \ldots$, object variables $\{x, y, \ldots\}$, object constants, $\{a, b, \ldots\}$, connectives $\{\&, \rightarrow\}$, quantifiers $\{\forall, \exists\}$ and truth constants $\{\bar{0}, \bar{1}\}$. Terms and formulas are defined as in classical predicate logic. Further defined connectives are

$$p \wedge q = p\&(p \rightarrow q), \qquad \neg p = p \rightarrow \bar{0},$$
$$p \vee q = \neg(\neg p \wedge \neg q), \qquad p \equiv q = (p \rightarrow q) \wedge (q \rightarrow p)$$

Evaluation of formulas is determined by a linearly ordered BL-algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$$

and an $\mathcal{L}$-structure for the language J:

$$\mathbf{M} = \langle M, \{r_P\}, \{m_a\}\rangle.$$

It is supposed that $\mathbf{M}$ is safe, i.e., all the necessary suprema/infima in truth evaluation of formulas exist. We refer to [5] for other details of predicate calculus BL∀.

2.2 A Theory $\mathcal{R}_n$ as a Conservative Extension of BL∀

Let the language J be extended to J_n, $n \geq 1$, by:

- Unary predicate symbols $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$.
- Binary predicate symbol R.
- Symbol $\circ$ and a new formula $(\varphi \circ \psi)(y)$ as an abbreviation for the formula $(\exists x)(\varphi(x) \,\&\, \psi(x, y))$.

A theory $\mathcal{R}_n$ as a special theory of BL∀ extends the latter by a set of special axioms:

(S1) $R(x, y) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))$,
(S2$_i$) $B_i(y) \to (A_i \circ R)(y)$, $\quad i = 1, \ldots n$.

The following theorems can be proved in $\mathcal{R}_n$:

- Properties of $\circ$

$$\mathcal{R}_n \vdash \varphi \to (\psi \to \varphi \circ \psi)$$
$$\mathcal{R}_n \vdash \varphi \circ (\varphi \to \psi)(y) \to \psi(y)$$
$$\mathcal{R}_n \vdash (\varphi_1 \equiv \varphi_2) \to (\varphi_1 \circ \psi \equiv \varphi_2 \circ \psi)$$
$$\mathcal{R}_n \vdash (\varphi_1 \vee \varphi_2) \circ \psi \to (\varphi_1 \circ \psi \vee \varphi_2 \circ \psi)$$

- Properties of a relation model of fuzzy IF-THEN rules

$$\mathcal{R}_n \vdash (\forall y)(B_i(y) \equiv (A_i \circ \bigwedge_{i=1}^{n}(A_i \to B_i))(y)) \qquad \text{Extended } \mathbf{MP}$$
$$\mathcal{R}_n \vdash R(x, y) \to (A(x) \to (A \circ R)(y)) \qquad \text{Compositional Rule}$$
$$\mathcal{R}_n \vdash (\forall x)(A(x) \equiv A_i(x)) \to (\forall y)((A \circ R)(y) \equiv B_i) \quad \text{Generalized MP}$$

An $\mathbf{L}$-structure for J_n

$$\mathbf{M}_n = \langle M, \{r_P\}, \{r_{A_i}|_{i=1,\ldots,n}, r_{B_i}|_{i=1,\ldots,n}, r_R\}, \{m_a\}\rangle$$

is an expansion of the $\mathbf{L}$-structure $\mathbf{M}$ for J.

The following propositions characterize conditions on a structure $\mathbf{M}_n$ to be a model of $\mathcal{R}_n$:

- $\mathbf{M}_n$ is a model of $\mathcal{R}_n$ if and only if the system of fuzzy relation equations

$$r_{A_i} \circ r_R = r_{B_i}, \qquad i = 1, \ldots, n,$$

  is solvable and the fuzzy relation r_R gives its solution.
- each model $\mathbf{M}$ of BL∀ has an expansion to a model $\mathbf{M}_n$ of $\mathcal{R}_n$.

Moreover, the following two statements can be proved:

Proposition 1. *Let* $\mathbf{M}_n$ *be a model for* $\mathcal{R}_n$*. Then* $\hat{\mathbf{M}}_n$ *is also a model for* $\mathcal{R}_n$ *where* $\hat{\mathbf{M}}_n$ *differs from* $\mathbf{M}_n$ *in the interpretation* $\hat{r}^n_R$ *of* R*:*

$$\hat{r}^n_R(x,y) = \bigwedge_{i=1}^{n} (r_{A_i}(x) \to r_{B_i}(y)).$$

Proposition 2. $\mathcal{R}_n$ *is a conservative extension of* $BL\forall$*.*

2.3 Extension of the Special Theory for Fuzzy IF-THEN Rules by New Rules

A crucial question for making our theory closer to reality is a possibility to extend it by new rules. We will show that $\mathcal{R}_n$ can indeed be properly extended.

First, we extend the language J_n to J_{n+1} by putting

$$J_{n+1} = J_n \cup \{A_{n+1}, B_{n+1}\}$$

where A_{n+1}, B_{n+1} are new unary predicate symbols. Furthermore, we extend the theory $\mathcal{R}_n$ to $\mathcal{R}_{n+1}$ by putting

$$\mathcal{R}_{n+1} = \mathcal{R}_n \cup \{\mathrm{S1}_{n+1}, \mathrm{S2}_{n+1}\}$$

where

$(\mathrm{S1}_{n+1})$ $\quad R(x,y) \to (A_{n+1}(x) \to B_{n+1}(y)),$

$(\mathrm{S2}_{n+1})$ $\quad B_{n+1}(y) \to (A_{n+1} \circ R)(y).$

Proposition 3. *Let* $\hat{\mathbf{M}}_n$ *from Proposition 1 be a model of* $\mathcal{R}_n$*. Assume that unary predicates* $r_{A_{n+1}}$ *and* $r_{B_{n+1}}$ *are chosen in such a way that the system*

$$r_{A_i} \circ r_R = r_{B_i}, \qquad i = 1, \ldots, n+1,$$

is solvable with respect to r_R*. Then the structure*

$$\hat{\mathbf{M}}_{n+1} = \hat{\mathbf{M}}_n \setminus \{\hat{r}^n_R\} \cup \{r_{A_{n+1}}, r_{B_{n+1}}, \hat{r}^{n+1}_R\},$$

where $\hat{r}^{n+1}_R = \hat{r}^n_R \wedge (r_{A_{n+1}} \to r_{B_{n+1}})$*, is a model of* $\mathcal{R}_{n+1}$*.*

Corollary 1. $\mathcal{R}_{n+1}$ *is a conservative extension of* $\mathcal{R}_n$*.*

Proposition 3 shows how a certain model of theory $\mathcal{R}_n$ can be extended to a model of theory $\mathcal{R}_{n+1}$ and not computed from the very beginning again.

3 Conclusion

In this paper, a comprehensive logical theory of fuzzy IF-THEN rules based on Hájek's predicate BL-fuzzy logic has been formulated. Its main features are the following:

- A system of fuzzy IF-THEN rules with the Compositional Rule of Inference characterizes a partially given fuzzy function.
- In any model of the theory of IF-THEN rules the respective system of fuzzy IF-THEN rules is solvable.
- A particular form of the Generalized Modus Ponens is provable in the theory of IF-THEN rules.
- A proper extension of the theory of IF-THEN rules by additional axioms (rules) can be interpreted in an extended model which preserves all the properties of the original model.

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Synthesizing Adaptive Navigational Robot Behaviours Using a Hybrid Fuzzy A* Approach

Antony P. Gerdelan and Napoleon H. Reyes

Summary. Previously, we have devised a novel Hybrid Fuzzy A* algorithm that seamlessly integrates the forward planning feature of A* and the refined reactionary robot maneuvering capabilities of Fuzzy Logic in a real-time simulation environment. This paper further explores the uncharted domain of synthesizing three primary robot maneuvering behaviours, namely target pursuit, obstacle avoidance and opponent evasion in an adaptive compact Hybrid Fuzzy A* navigation system. In addition, this work sheds some light onto the dark pits of the previous Fuzzy A* architecture proposed, as the former Hybrid approach did not account for the necessity of evasive behavior, and so modifications to the forward planning layer are deemed to be necessary. In light of this, this chapter presents a new undesirability component that is injected into the A* algorithm, as well as optimisations to the cascade of fuzzy systems architecture that calculates the robot speed and angle adaptively. Empirical results are also presented that attest to the algorithm's robustness when faced with a formidable army of moving obstacles while in pursuit of a target, as well as evading multiple opponents.

Key words: Autonomous navigational systems, Path planning, Fuzzy logic, The A* algorithm, Robot soccer.

1 Introduction

The evasion algorithm presented here is designed to augment the path-planning layer of the previously developed Hybrid Fuzzy A* Robot Navigation System [1], and relies on receiving reliable threat information from the sensors of the robot system. After receiving information concerning threats, a series of maps (conceptual grids) is created to represent the robot's environment and a modified version of the A* algorithm is used to construct a shortest path to a goal, avoiding obstacles and evading potential threats to the robot. The system presented in this chapter has been created to extend navigation systems specifically for the game of robot soccer, but is applicable to problem domains

with the same key features. The problem domain for which this algorithm is intended consists of:

- A complex, three-dimensional environment
- Multiple hostile agents to evade
- A dynamic, moving target to pursue
- A static goal location
- Static and dynamic obstacles to avoid

The evasion algorithm is based on the premise that whilst seeking a target, a robot must also avoid obstacles and evade competing robots. The game of robot soccer operates in real time at very high speeds, and therefore demands extremely fast processing. The window of time available for calculation is approximately 33 ms and must be shared with machine vision processing and other system components. Navigation systems must be fast enough to remain synchronous with the state of the robots, and therefore a balance between speed of calculation and optimality must be struck.

2 General System Architecture (Fig. 1)

Environment information is collected by the sensors and analysed for key environment features. Environment features required for the planning layer with the evasion algorithm are:

- Hostile agent locations and headings
- Obstacle locations
- A goal location

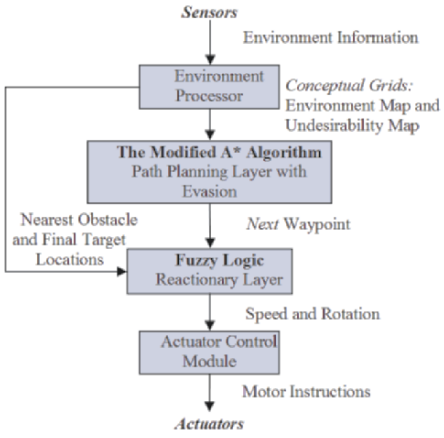

Fig. 1. Architecture of the augmented hybrid fuzzy A* system

These information are then used to create conceptual grids (maps) of the environment which are then passed to the planning layer. The environment processor must create two conceptual grids:

- An environment map
- An undesirability map

An environment map marks obstructed areas of the environment to be excluded from the search domain of the path-finding algorithm. An undesirability map awards undesirability ratings to all areas of the environment based on the level of perceived threat from hostile agents to that area. The planning layer draws on both maps to plan a path toward its goal location, avoiding obstacles and undesirable areas where possible. The waypoints of the path are passed down to the Fuzzy Logic Control layer, which refines robot movement and reacts to avoid immediate obstacles. Defuzzified outputs for speed and rotation of the robot are sent to the actuator control module, where speed and rotation outputs are decomposed into specific motor instructions.

3 The Evasion Algorithm

The evasion algorithm presented in this chapter is an extension of a dynamic A* path-finding algorithm [1]. Navigation systems employing dynamic A* path-finding operate on a 2D grid [2–4], representational of the real environment, and cells of the grid correspond to nodes in the search domain of the A* algorithm.

3.1 Dynamic A* Path Finding

For the A* path finding to operate in a dynamic environment, the conceptual grid must be continually regenerated, keeping up-to-date information on the locations of moving obstacles. This allows robots to recalculate paths on the fly when dynamic obstacles move and obstruct the intended path. Cells containing obstacles are excluded from the search process. Figure 2 illustrates a path calculated on such a grid, where obstructed nodes have been excluded from path calculation. Other nodes are awarded an f* score from the formula:

$$f^* = h^* + g^* \tag{1}$$

where h* represents a heuristic distance from the examined node to the goal node, and where g* represents the sum of distances between nodes; from the initial node to the examined node. The A* algorithm finds the shortest path by searching the domain of nodes and connecting nodes with the lowest f* scores from the start node to the goal. As can be viewed in Fig. 3, the h* values for nodes in 2D conceptual grid can be represented in three dimensions, where z-axis represents the h* value for each node.

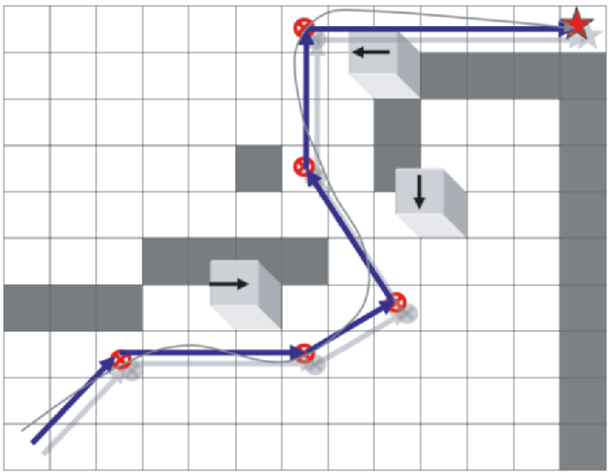

Fig. 2. A Sample A* path finding result *(directed arrows)* vs. Hybrid Fuzzy A* Path Finding *(curve)*

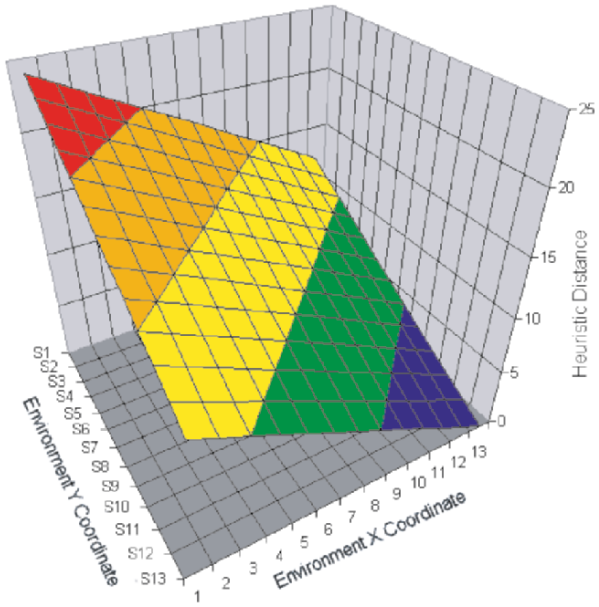

Fig. 3. Example h* values for cells in a conceptual grid

Figure 3 illustrates the h* values for cells in a conceptual grid, where a start node is at grid position (1, S1) and a goal node at grid position (13, S13). We can clearly see that, excluding obstacles, the shortest path is in a straight line; from the highest point to the lowest.

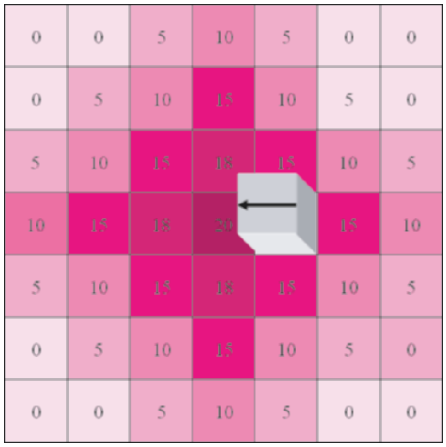

Fig. 4. Undesirability values awarded to grid cells surrounding a hostile agent

3.2 Undesirability Maps

For the path-planning layer of the navigation system to evade opponents, it must be provided with information detailing the level of threat to areas of the environment. To this end, a second conceptual grid of the environment is created, and each cell is awarded a value based on the undesirability of the area. A function is required to determine the relative undesirability of each cell. The valuing function must be tailored to the specific problem domain and environment.

For the game of robot soccer levels of undesirability centre on each hostile agent; a teardrop-shaped field of undesirability extends away from the agent (refer to Fig. 4). Areas of highest undesirability are immediately in front of the agent. Areas beside the agent are less undesirable to reflect the reduced threat to those areas, as the agent must rotate before directly threatening those areas. Areas behind the agent are awarded a lower undesirability value also; reflecting the reduced threat of the agent reversing or turning around.

Figure 5 illustrates undesirability values in three dimensions for the same environment as illustrated in Fig. 3, where the z-axis represents the undesirability value of each cell. This effect can be compared to the implementation of navigation systems employing potential field methods [5–8], but extends the potential field concept by preempting the movements of hostile agents and their direction of travel.

3.3 An Evasive Path-Finding Algorithm

Undesirability values and heuristic distance values can be combined to form a new f* score, balancing the weight of heuristic distance with that of undesirability:

$$f^* = h^* + u^* + g^* \tag{2}$$

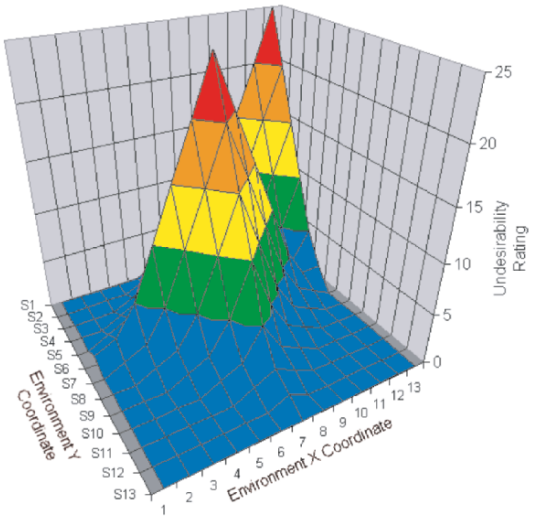

Fig. 5. Example undesirability values for cells in a conceptual grid

Equation (2) presents a new formula; undesirability has been factored into the heuristic component of the A* algorithm, where h* represents the heuristic distance from the examined node to the goal node, where u* represents the undesirability value of the examined node, and where g* represents the cumulative distance from the start node to the examined node.

The combined heuristic distance and undesirability values can be represented in three dimensions for a "hills-and-valleys" effect, where the z-axis represents the combined values for each cell in the conceptual grid. Figure 6 illustrates a combination of the heuristic distance values represented in Fig. 3 and the undesirability values represented in Fig. 5. We can see that, as the algorithm will try to construct a path with lower h* + u* values, it will no longer plan a path directly to the goal through the centre of the grid, but will circumnavigate the undesirable areas en route to the goal. The cumulative g* value ensures that a shortest possible path is created, and not simply the most "downhill" path.

3.4 Considerations for Evasion

Experimentation with simulation has shown that the weighting given to undesirability values is of great importance to the effectiveness of the evasion system. Values that are too low will effectively produce a system that has no evasive behavior undesirable regions will be outweighed by the path-finding components of the f* score formula. Referring to Fig. 6, undesirability values that are too low would be observed as the peaks created by u* disappearing below the sloping area created by h*. Experiments have shown that if

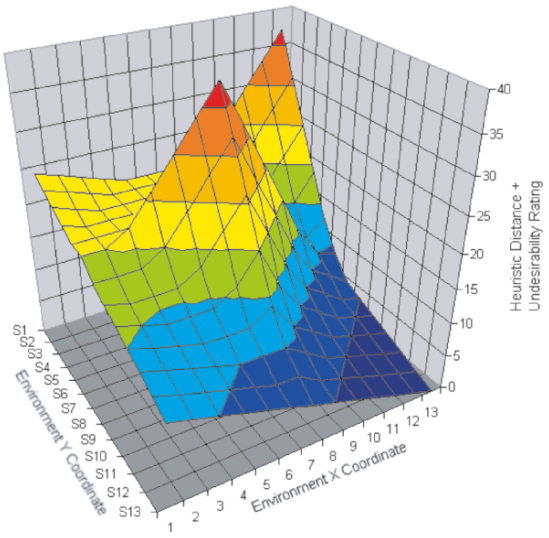

Fig. 6. h* + u* values for cells in a conceptual grid

undesirability values are too high, the undesirability component u* of the formula will overcome the path-finding components, compromising the reliability of the algorithm to produce consistent paths. A balance must be struck by choosing an appropriate range of undesirability values. Undesirability values that are too high would be observed by the total dominance of the u* peaks over the h* slope. The operation taken to determine the undesirability values for nodes where areas of threat overlap will result in subtle changes to robot behavior. If, in the problem domain of the robot, areas in between multiple threats are even more undesirable than areas near one threat alone, it may make sense to award the area affected the sum of all of the overlapping undesirability values. Otherwise, the maximum of the values may be the most effective choice.In the game of robot soccer, robots attempting to evade other robots are often crushed against the walls. In this problem domain, the undesirability of areas near static obstacles (the walls) can be increased to good effect. For broader applicability, areas where a robot has more room to evade to maneuver and change course should be less undesirable than those areas where a robot is more confined. The direction, speed and other details of hostile agents can be incorporated into the undesirability rating function, so that a very accurate representation of threat can be created in the threat map. Areas further in front of fast-moving hostile agents are awarded high undesirability ratings, whilst ratings for areas to the sides and behind those agents are reduced, as it takes those agents longer to turn around. This additional information makes more accurate threat maps, but requires additional calculation and is of diminishing importance to the robot as hostile agents are further away from it.

4 Cascade of Fuzzy Systems

Robot navigation systems employing Fuzzy Logic [9–14] require input information about a robot's target; the distance between the robot and the target, and the angle between the robot's heading and the target. Utilising the same inputs, the proposed Fuzzy system architecture embodies multiple fuzzy systems that collectively perform angle and speed refinements for the tasks of target pursuit and obstacle avoidance. As can be seen in the diagram of the Reactionary Layer (Fig. 7), the Path Planning Layer feeds the next intermediary waypoint to a cascade of Fuzzy systems which paves the way for a smooth robot maneuvering towards the target.

Such refined robot movements are made precisely to suit the prevailing circumstances. Target pursuit is carried out by the system whenever it is safe to do so, and obstacle avoidance is instantaneously engaged with the onset of any opponent interference. On the top layer is the Fuzzy system for target pursuit (Fuzzy System 1) that reacts on two inputs, namely the robot's distance from the target, and the difference between the robot's heading angle and target. The main task of such system is to calculate the correct turning

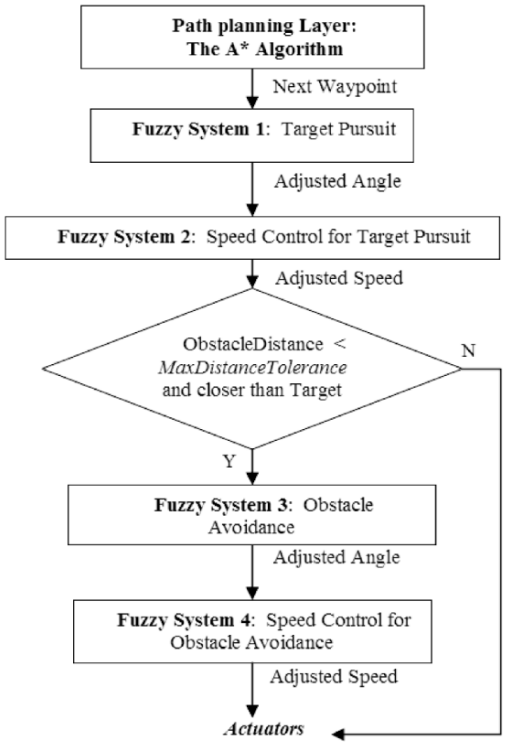

Fig. 7. Reactionary layer: cascade of fuzzy systems

Table 1. Fuzzy associative memory matrix for angle control: target pursuit

	NEAR	FAR	VERY FAR
SMALL	Mild Turn	Mild Turn	Zero Turn
MEDIUM	Med Turn	Mild Turn	Mild Turn
LARGE	Sharp Turn	Med Turn	Med Turn

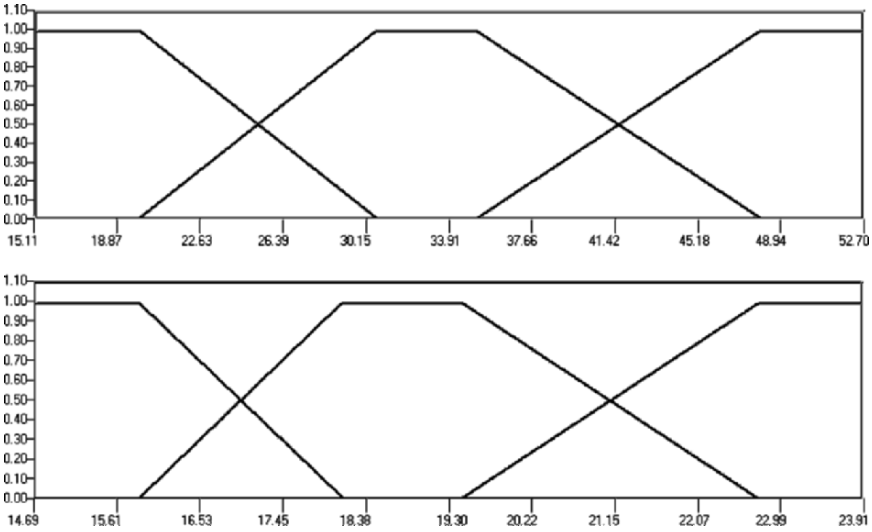

Fig. 8. Fuzzy membership functions for speed control: target pursuit – angle *(top)* and Distance *(bottom)*

angle towards the target relative to its current orientation. Table 1 depicts the collection of rules that dictate the correct turning angle for a combination of distance and angle conditions. Such fuzzy sets are defined using trapezoidal membership functions (Fig. 8). As an example, one of the rules states that:

> If the ***Distance*** from the Target is *NEAR* and the ***Angle*** from the Target is *SMALL* Then the robot should make a ***Mild Turn***.

4.1 Taking Advantage of Angle Symmetry

It is worth mentioning that the design for the fuzzy associative memory matrix takes advantage of the angle symmetry; thereby, considering only the right-half of the angles involved, from [0, 90] and [270, 360]. As can be viewed in Fig. 9, the angles were partitioned only into three overlapping parts, each with its own corresponding fuzzy set. Using this simplified approach, the size of the FAMM was considerably reduced, since both left and right cases were

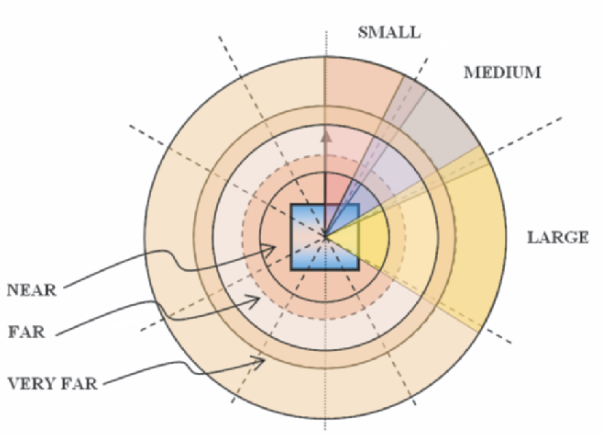

Fig. 9. Fuzzy sets for angle and distance

Table 2. Fuzzy associative memory matrix for speed control: target pursuit

	NEAR	FAR	VERY FAR
SMALL	Med Speed	Fast Speed	Very Fast
MEDIUM	Slow Speed	Med Speed	Fast Speed
LARGE	Very Slow Speed	Slow Speed	Slow Speed

accounted for using the same generic FAMM. In particular, for cases where the obstacle or target is found on the left-hand side, considering angles from [90, 270] counter-clockwise, relative to the robot's orientation, the fuzzy output is simply negated. Moreover, the Fuzzy systems were designed to respond by taking the minimum turning angle towards the desired robot orientation. This reactionary robot pursuit movement is further enhanced by yet another Fuzzy system that handles speed control based on the same inputs fed into Fuzzy System 1. As an example, a fuzzy rule for speed control comes in the following form: If the Distance from the Target is VERY FAR and the Angle from the Target is SMALL Then the robot should move Very Fast. Finally, the two Fuzzy Systems at the bottom were designed to perform course corrections to account for cases where obstacles are close to the robot. Similar to Fuzzy System 1, except that it is considering an obstacle instead of the target, Fuzzy System 3 adjusts the robot's steering angle based on the robot's distance from the obstacle, and difference between the robot's heading angle and the angle to the obstacle. In conjunction with Fuzzy System 3, Fuzzy System 4 deals with speed adjustment, which is also similar to Fuzzy System 2 (Table 2), except that its rule base is designed to avoid collisions.

4.2 Limits of the Reactionary Fuzzy Systems

Despite the system's ability to perform refined course corrections to pursue the target and avoid the obstacles, there are cases however where forward planning is necessary to prevent the robot from taking routes that could lead to it getting trapped. Since the fuzzy systems do not take into account the directions of the moving obstacles, using the fuzzy system solely is not enough to prevent collisions completely. Thus, the A* algorithm is used to guide the cascade of fuzzy systems.

5 Conclusions

This chapter has extended our previous paper on a novel Hybrid Fuzzy A* navigational system, inculcating a predictive quality into autonomous robot navigation. Robot path planning is now able to second guess the movements of hostile agents in order to evade the onslaught of potential threats in real time.

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Fuzzy Impulse Noise Reduction Methods for Color Images

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Summary. The reduction or removal of noise in a color image is an essential part of image processing, whether the final information is used for human perception or for an automatic inspection and analysis. In addition to all the classical based filters for noise reduction, many fuzzy inspired filters have been developed during the past years [3–26]. However, it is very difficult to judge the quality of all these different filters. For which noise types are they designed? How do they perform compared to each other? Are there some filters that clearly outperform the others? Do the numerical results correspond with the visual results? In this paper we answer these questions for color images that are corrupted with impulse noise. We also have developed a Java Applet (http://www.fuzzy.ugent.be/Dortmund.html). The Java Applet is used to compare all the mentioned filters with each other. It illustrates the numerical and visual performance of all these filters. Users have the possibility to load and corrupt an image from a predefined list.

1 Introduction

Noise can be systematically introduced into digital images, e.g., due to the circumstances of recording (e.g., dust on a lens, electronic noise in cameras and sensors, ...), transmission (e.g., interaction with satellite images, transmission over a channel, ...), scanning, etc. A fundamental problem of image processing is to reduce noise effectively from a digital image while keeping its features intact. Therefore, it is not surprising that different algorithms are developed for different noise types. During the past years, also a lot of fuzzy logic based filters have been introduced. In this article we will present a comparative study for color images (there already exist studies for grayscale images [1]). Besides this comparative study, we illustrate the shortcomings of the common filter techniques in order to stimulate researchers to design noise reduction techniques that reduce noise on the one hand and that preserve colors and image structures on the other hand. We also discuss some recent solutions that are especially designed for color images. These methods try to preserve the color component differences while performing efficient noise removal.

In this paper we discuss the case of impulse noise in digital color images. A digital color image (denoted as I) can be modeled in a certain color space. The most common color space is the RGB color space. By mixing red, green and blue light in different proportions it is possible to obtain a wide range of colors. For that reason colors in the RGB space are represented by a three-dimensional vector, with as first component the red, second the green and third the blue pigment. These pigments are called the three primary components, each quantized to the range $[0, 2^m - 1]$ (mostly with $m = 8$). Practically a digital color image I can be represented by a two-dimensional array of vectors, where an address (i, j) defines a position in I, called a pixel or picture element. The color components for a certain pixel are denoted by $I(i, j, 1)$, $I(i, j, 2)$, and $I(i, j, 3)$ for the red, green, and blue component, respectively.

One of the common impulse noise types is salt and pepper noise; for other types we refer to [7, 8]. In an image that is corrupted with salt and pepper noise, two things can happen to each of the components of a pixel: it remains unchanged, it gets a value n_1 or it changes to a value n_2. Generally these two values are $n_1 = 0$ and $n_2 = 2^m - 1$. An important parameter is the noise density δ, which expresses the fraction of the image pixels that are corrupted.

Besides the RGB color space, we also take into account other well known color spaces [2]: YIQ, HSV, HSL, CMY, XYZ, XYZ2.

2 Filters for Noise Reduction

The wide range of filters found in the literature can be divided in three subclasses (1) classical filters; (2) fuzzy-classical filters, i.e., fuzzy logic based filters that are extensions or modifications of classical filters; and (3) fuzzy filters, i.e., filters that are completely based on fuzzy logic and have no connection with classical filters. This classification will be used during the paper. Our comparative study concerns 42 different algorithms. The fuzzy-classical and fuzzy filters are accompanied by a reference for those readers who are interested in more background information.

2.1 Classical Filters

The classical filters pass step-by-step over the whole image and process each pixel by some filter depending operator (e.g., the median). Thus, all the pixels are changed independently whether the pixel was or was not distorted. In that manner, usually the fine details of the image are blurred, because the fine texture elements are not taken into account. The shortcomings of many classical filters is the inability of expressing pixels into several degrees of noise. Most classical filters are not able to distinguish noisy pixels from uncorrupted ones. Some very popular classical filters are:

- MF: the Median Filter, which takes the median from a certain neighborhood around the filtered pixel.
- WF: the Weighted Filter, which averages all the pixel values from the neighborhood around the filtered pixel.
- AWF: the Adaptive Weighted Filter, which is an extension of the weighted filter.
- WIENER: Wiener Filter.
- GAUS: Gaussian Filter, which assumes a Gaussian distribution of the noise.
- VMF1: Vector Median Filter based on the Euclidian distance [25].
- VMF2: Vector Median Filter based on the Minimum Angle distance [26].

2.2 Fuzzy-Classical Filters

When images are corrupted with noise it will be difficult to make the difference between noise and texture elements and distinguish degrees of contamination. This illustrates the added value of fuzzy set theory that is used to model such kind of uncertainties. This allows us to improve the quality of noise reduction methods. In general, a fuzzy filter for noise reduction uses both numerical information and linguistic information (modeled by fuzzy set theory, e.g., "small" differences, "similar" pixels, etc.) to filter the noise. The fuzzy extensions of the classical filters are:

- FMF: Fuzzy Median Filter [3, 4]
- TMED: Symmetrical Triangle Fuzzy Filter with median center [5, 6]
- ATMED: Asymmetrical Triangle Fuzzy Filter with median center [5, 6]
- GMED: Gaussian Filter with Median Center [5, 6]
- WFM: Weighted Fuzzy Mean Filter [10, 11]
- FWM: Fuzzy Weighted Mean [4]
- AWFM: first Adaptive Weighted Fuzzy Mean Filter [10]
- AWFM2: second Adaptive Weighted Fuzzy Mean Filter [11]
- CK: Choi & Krishnapuram Filter [12]
- FDDF: Fuzzy Decision Directed Filter [13]
- TMAV: Symmetrical Triangle Fuzzy Filter with Moving Average Center [5, 6]
- ATMAV: Asymmetrical Triangle Fuzzy Filter with Moving Average Center [5, 6]
- DWMAV: Decreasing Weight Fuzzy Filter with Moving Average Center [5, 6]
- GMAV: Gaussian Fuzzy Filter with Moving Average Center [5, 6]
- MPASS: Multipass Fuzzy Filter [14, 15]
- FMMF: Fuzzy Multilevel Median Filter [15]
- FVRF: Fuzzy Vector Rank Filter based on the Euclidian distance [25]
- FCWVMF: Fuzzy Center Weighted Vector Median Filter based on the Minimum Angle distance [26]

2.3 Fuzzy Filters

Fuzzy logic based filters that have no connections with classical filters are:

- FIRE: Fuzzy Inference Ruled by Else-action Filter [16]
- DSFIRE: Dual Step FIRE Filter [17]
- PWLFIRE1: first (nonadaptive) Piecewise Linear Fuzzy Inference Ruled by Else-action Filter [18]
- PWLFIRE2: second (adaptive) Piecewise Linear Fuzzy Inference Ruled by Else-action Filter [18]
- IFCF: Iterative Fuzzy Control based Filter [19]
- MIFCF: Modified IFCF Filter [19]
- EIFCF: Extended IFCF Filter [19]
- SFCF: Smoothing Fuzzy Control based Filter [20]
- SSFCF: Sharpening SFCF Filter [19]
- GOA: Gaussian Noise Reduction Filter [21]
- HAF: Histogram Adaptive Filter [22]
- FSB1: first Fuzzy-Similarity-Based Noise Reduction Filter [23, 24]
- FSB2: second Fuzzy-Similarity-Based Noise Reduction Filter [23, 24]
- FSB1R: first Recursive Fuzzy-Similarity-Based Noise Reduction Filter [23, 24]
- FSB2R: second Recursive Fuzzy-Similarity-Based Noise Reduction Filter [23, 24]
- FIDRM: Fuzzy Impulse noise Detection and Reduction Method [7, 8]
- FIDRMC: Fuzzy Impulse noise Detection and Reducing Method for Color images [9]

3 Comparative Study

As a measure of objective similarity between a filtered image and the original one, we use the peak signal to noise ratio. The peak signal to noise ratio, often abbreviated as PSNR, is an engineering term for the ratio between the maximum value of a signal and the magnitude of the background noise. Because many signals have a very wide dynamic range, PSNRs are usually expressed in terms of the logarithmic decibel scale. The PSNR is used to measure the quality of reconstruction. Reconstructed images with higher metrics are judged better. This similarity measure is based on the dissimilarity measure called mean square error (MSE). The MSE and PSNR are defined as:

$$MSE(A,B) = \frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{c=1}^{3} \Big[A(i,j,c) - B(i,j,c) \Big]^2,$$

$$PSNR(A,B) = 10 \cdot \log_{10} \frac{255^2}{MSE(A,B)}$$

where A is the original color image, B the filtered color image of size NM.

The evaluation is carried out on two levels: numerical (based on the PSNR values) and visual (based on visual inspection). In order to get a clear idea of the performance with respect to the level of impulse noise, experiments have been carried out for several impulse noise levels. The results can be seen on the website http://www.fuzzy.ugent.be/Dortmund.html. During the experiments we have used several different color images, such as the Lena image (256×256), the Baboon image (512×512), the Hill image (768×1024), the Tree image (258×350) etc. The test images are also available at http://www.fuzzy.ugent.be/Dortmund.html. Some of these numerical results for the Lena and the Hill image are summarized in Table 1 with $\delta = 0.1$, 0.2, 0.3, 0.4, and 0.5.

For this conference we have developed a Java Applet (http://www.fuzzy.Ugent.be/Dortmund.html). The Java Applet is used to compare all the mentioned filters with each other. It illustrates the numerical and visual performance of all these filters. Users have the possibility to load and corrupt an image from a predefined list. We implemented three noise types: impulse noise, white Gaussian noise and Speckle noise. After the corruption (the mixture of these noise types is possible too), the users can apply one of these filters so that the numerical and visual results appear. Besides the RGB color space, it is possible to work in other color models as well.

3.1 Numerical Results

In Table 1 the numerical performance in terms of the PSNR is pictured for a (256×256) colored Lena image (a) and a (768×1024) colored Hill image (b). The images are corrupted with salt and pepper noise [8] for different δ's. We can summarize our conclusions w.r.t. the numerical results based on our experiments with the Lena, Baboon, Hill and Tree images, as follows:

- The FIDRMC filter performs best for all levels of impulse noise. In case of the Lena image it generally increases the PSNR value by a factor 3.68 for low levels ($\delta = 0.1$) as well for high levels ($\delta \geq 0.5$) w.r.t. the noisy image. For the other filters these factors range between 1.75 and 1.95. These are very satisfying results.
- The FIDRM filter is the second best performing filter. In case of the Lena image it generally increases the PSNR value by a factor 2.50 for low levels ($\delta = 0.1$) but for high levels ($\delta \geq 0.5$) it almost performs as good as the FIDRMC.
- For low noise levels ($\delta = 0.1$ and $\delta = 0.2$) there are several filters that return very good results: the DSFIRE filter (it is the third best filter for noise levels $\delta \leq 0.15$), the FMF filter, the AWFM2 filter (whose performance increases when the noise rate gets higher), the AWFM filter (which always has smaller PSNR values than AWFM2), GMED and MED.
- For higher noise levels ($\delta \geq 0.3$) the top five of the best performing filters remains always the same: the FIDRMC together with the FIDRM filter are the best ones, the AWFM2 filter which is a little bit better than the

Table 1. Numerical results (in PSNR) for images corrupted with impulse noise for $\delta = 0.1$, 0.2, 0.3, 0.4 and 0.5, (a) colored (256×256) Lena image and (b) colored (768×1024) Hill image

	(a)					(b)				
	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
Noisy	15.33	12.35	10.55	9.28	8.35	15.25	12.39	10.63	9.25	7.62
MF	30.23	27.58	23.14	18.59	15.05	28.89	26.28	21.78	17.68	15.2
VMF1	18.51	12.62	10.18	8.98	7.83	18.57	12.53	11.13	8.79	7.2
VMF2	20.31	14.83	11.67	9.59	8.23	20.40	14.56	13.35	9.32	7.4
WF	23.10	20.26	18.18	16.61	15.41	23.18	20.65	18.48	17.04	15.91
AWF	15.31	12.36	10.57	9.27	8.37	15.35	12.38	10.63	9.39	7.63
WIENER	14.88	14.79	14.35	14.02	13.27	14.15	13.96	13.83	13.73	12.67
GAUS	15.37	12.39	10.63	9.31	8.39	15.44	12.54	10.66	9.42	8.46
FMF	33.14	28.56	23.84	18.62	16.45	32.62	27.41	22.40	18.59	16.37
TMED	30.13	27.20	22.94	17.75	15.35	28.83	25.96	21.68	17.84	15.48
ATMED	29.20	28.10	26.15	22.36	20.29	26.22	24.78	23.35	21.54	20.06
GMED	30.19	27.55	23.13	17.53	15.05	28.86	26.27	21.78	17.68	15.20
WFM	27.31	27.00	26.73	25.70	23.23	27.15	26.99	26.67	25.72	20.60
FWM	20.72	18.58	16.99	14.55	13.27	21.23	18.39	16.41	14.83	12.50
AWFM	28.07	27.55	27.04	25.61	24.98	27.23	26.62	26.18	22.7	21.30
AWFM2	31.39	30.51	29.72	27.67	26.91	30.07	29.11	28.48	20.66	22.62
CK	21.81	18.10	15.42	13.30	11.55	22.30	18.60	15.86	13.68	10.41
FDDF	18.54	14.22	11.61	10.20	8.73	18.38	13.98	11.50	9.86	8.80
TMAV	29.87	26.83	22.77	18.26	16.03	28.75	25.70	21.66	18.22	16.04
ATMAV	28.00	27.47	27.00	26.19	23.67	21.64	20.76	20.84	21.13	19.40
DWMAV	23.10	20.26	18.18	16.51	15.41	23.18	20.38	18.48	17.04	15.91
GMAV	11.29	9.16	8.08	10.56	6.65	17.54	14.37	12.34	10.83	8.24
MPASS	27.80	24.91	20.90	16.31	14.19	27.46	23.91	19.88	16.47	14.33
FMMF	17.59	14.27	12.06	10.13	9.24	18.34	14.95	12.83	11.26	10.26
FVRF	24.47	21.09	18.79	16.89	15.43	24.49	21.30	19.05	17.35	14.80
FCWVMF	27.32	24.98	21.21	17.15	14.21	27.27	25.05	21.27	17.32	11.78
FIRE	29.31	23.62	19.11	15.47	13.37	29.46	23.19	18.81	15.62	13.52
DSFIRE	34.33	30.64	25.81	20.25	17.22	34.28	30.17	25.23	20.66	17.57
PWLFIRE1	27.99	21.83	17.42	12.93	11.94	23.94	19.09	15.77	13.27	12.06
PWLFIRE2	30.69	23.00	18.11	14.15	12.28	29.95	22.06	17.49	14.34	12.42
IFCF	29.36	27.08	24.08	20.67	18.24	28.94	21.28	24.19	21.17	18.61
MIFCF	28.64	24.12	22.42	18.67	16.33	28.86	25.98	22.37	19.00	16.55
EIFCF	29.32	27.01	23.91	20.41	17.98	28.84	26.82	23.93	20.84	18.26
SFCF	27.76	24.12	20.31	16.57	14.57	27.38	23.71	19.84	16.75	14.68
SSFCF	28.27	24.61	20.49	16.31	14.20	27.63	23.94	19.81	16.49	14.31
GOA	23.32	21.33	19.57	17.62	17.00	23.91	21.93	20.39	19.11	18.22
HAF	29.46	29.01	28.59	26.98	26.18	28.92	28.29	26.88	24.90	23.82
FSB1	29.79	27.25	22.88	17.18	14.89	28.62	25.88	21.34	17.28	11.67
FSB2	29.84	27.11	22.63	17.18	14.76	28.58	25.87	21.37	17.34	11.72
FSB1R	29.05	27.67	25.92	23.02	20.42	28.28	27.35	25.93	23.90	18.28
FSB2R	29.08	27.88	26.05	21.73	20.06	28.12	27.21	25.19	22.21	15.96
FIDRM	38.82	34.80	33.14	31.10	30.08	37.00	33.44	31.14	30.35	27.84
FIDRMC	52.81	45.16	40.81	34.46	31.23	52.56	46.06	40.29	34.65	30.12

HAF filter (this filter removes the impulse noise but causes some kind of blur) followed by AWFM, ATMAV, WFM, ATMED, FSB1R and FSB2R filters.
- The numerical results for vector based approaches (VMF, FVRF and FCWVMF) are the worst of all filters. When the noise level increases, the performance decreases. A second disadvantage of those methods is the computational complexity.

3.2 Visual Results

Visual results are presented in Fig. 1 for a part of the Hill image, which is contaminated with 30% salt and pepper noise. Additionally, we presented visual and numerical results for other test images on the website http://www.fuzzy.ugent.be/Dortmund.html. We can summarize our conclusions with respect to the visual observations as follows:

- The FIDRMC filter yields the best visual performance. Especially for low impulse noise this filter outperforms the others. But for images contaminated with very high impulse noise it does not reduce all noise. But in comparison to other filters it remains the best.
- The FIDRM filter filters out the impulse noise very well but introduces many artifacts. These artifacts are mostly situated at the contour pixels, which cause lower PSNR values for lower impulse noise levels. The reason for these artifacts is situated in the way FIDRM handles color. FIDRMC tries to keep the color differences while FIDRM performs on each color component separately.
- For all noise levels the HAF, FSBR1, ATMED, and MF filter produce a more blurry picture than the FIDRMC, FIDRM, and AWFM2 filters. In other words, the FIDRMC, FIDRM, and AWFM2 filter have the property that they keep the sharpness of the image.
- For the very higher noise levels ($\delta = 0.5$–0.9) the third and fourth best performing filters show an increasing number of dots or impulse noise pixels that are not filtered out, which does not occur with the FIDRMC filter and FIDRM filter.
- As seen in the Java Applet the chosen color space has not much influence on the performance of each filter.
- Many filters perform better for grayscale images than for color images. This can also be concluded from the Java Applet available at the website. The reason is almost the same: by not taking into account the original color differences it occurs that pixels mostly situated on edges are flipped over so that dots appear. This extra information (e.g., the color differences) should be taken into account in future work in order to improve noise reduction methods for color images.
- It is also noticeable that the numerical and visual experiments confirm each other.

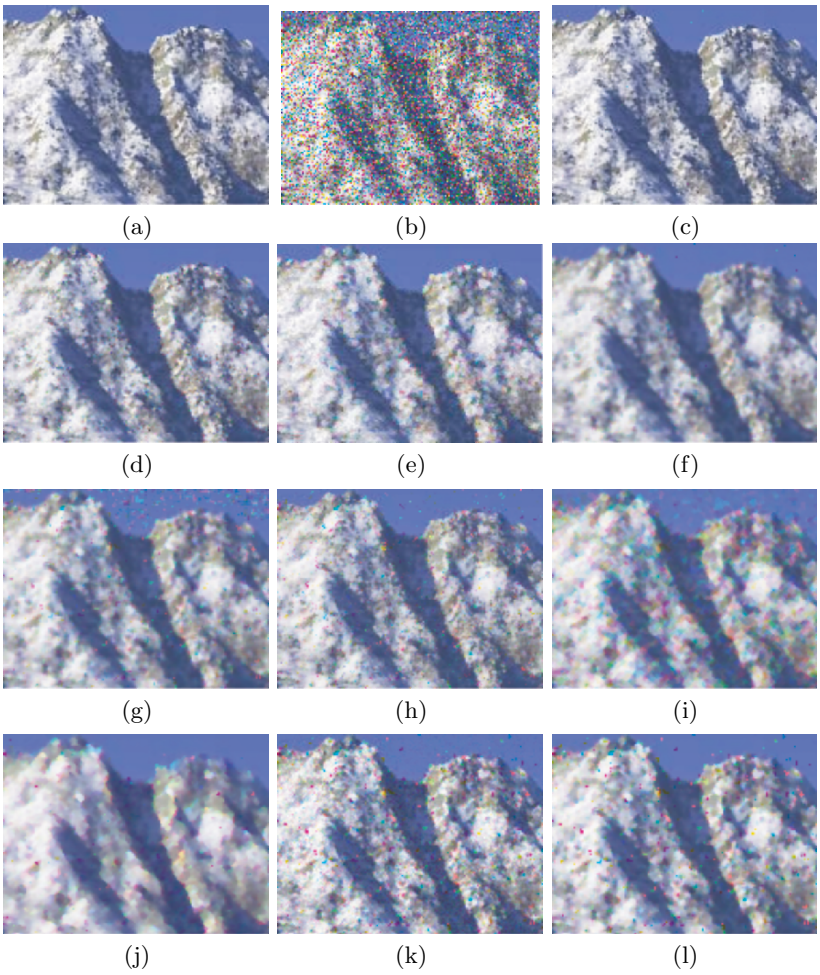

Fig. 1. The restoration of a magnified part of a colored Hill image (768×1024): (**a**) a noise free part, (**b**) the same part contaminated with 30% impulse noise (i.e., salt & pepper noise with $\delta = 0.3$), (**c**) FIDRMC, (**d**) FIDRM, (**e**) AWFM2, (**f**) HAF, (**g**) ATMED, (**h**) DSFIRE, (**i**) IFCF, (**j**) FSB1R, (**k**) FMF, (**l**) MF

From both numerical and visual observations we see that color images should be treated in a different way as grayscale images. Filtering each component separately from each other will cause new impulse noise like artifacts especially on the edges of the image. Vector based approaches were developed for this reason but the numerical and visual results have shown that the filtering capacity decreases in comparison to the other filters. The first satisfactory filter that treated this problem were presented in our earlier work [9]. In this work we have incorporated the information about the color component

differences. Instead of filtering each component we filtered each color component difference. This comparative study has illustrated that future work should be spent on this issue to improve the current methods for color images.

4 Conclusion

The numerical and visual experiments are in accordance with each other: the FIDRMC filter returns the best for all noise levels, followed by the FIDRM filter which even performs equally well as the FIDRMC filter for extremely high noise levels. Other well filtering methods for low noise levels are: DSFIRE, FMF, AWFM2, GMED, and MED. And for high noise levels we can also advise: AWFM2, HAF, ATMAV, FSB1R, and FSB2R. These results show that the use of fuzzy techniques in image processing can have an added value. Indeed, all best performing filters belong to the class of fuzzy-classical or even purely fuzzy filters. But especially the visual results suggest that future work must be done in order to develop more sophisticated color methods, which also incorporate the original color differences.

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Use of Variable Fuzzy Sets Methods for Desertification Evaluation

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Summary. Many factors affect the desertification and the factors generally are not in the same scope, it also brings about a great difficulty for evaluating desertification. Under global view of system that the variable fuzzy sets method is presented to set up comprehensive evaluating model for desertification degree and transform the qualitative assessment into quantitative one. The method can scientifically and reasonably determine and relative membership functions of disquisitive indexes at level interval that relating to desertification, also it can fully use one's experience and knowledge, qualitative and quantitative information of index system to obtain weights of indexes for operating comprehensive evaluation. The numerical example of the dry and hot valley of Jinsha River has shown that the proposed method is feasible and effective, and it provides a new theory for desertification study.

Key words: Desertification evaluation, Variable fuzzy sets, Difference function, Rank feature values.

1 Introduction

Desertification is one of the most serious ecological and environmental problems in the world. It directly influences regional economic development and social stability. As to people affected by desertification, Asia is influenced by desertification most seriously on the earth, and China, with a territory of 9.6 million km^2, is one of the most severely deserted countries in Asia [1]. Desertification is threatening the lives of close to 400 million people and has affected about 3.3 million km^2 of land [2]. The rest of the land is degraded land with sparse vegetation covering less than 5%, which mainly consists of desert and gobi (CCICCD, 1997). It was estimated that the direct economic loss of sandy desertification in China is about 54.1billion RMB per year [3, 4], at present, development of western China has become an important national task for the next 10–20 years and environment improvement is considered the most important issue. Therefore, study desertification is of extreme importance for

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dealing desertification to China with respect to the conservation of natural resources and sustainable development.

Desertification, which often conceived of as a "global" problem, is multicausal and highly dependent on cultural, economic and biophysical factors [5]. And desertification evaluation is the key step of controlling desertification, which not only can reveal the actual situations of ecological and social development, but more important is that it can quantitatively analyze the fruit of desertification prevention and provide accessorial decision for the task of future desertification observation [6,7]. At present, there are many methods in fields of researching desertification, yet most of which are single qualitative or quantitative evaluation (pure mathematic model). And that the geological investigation must synthesize qualitative picture with quantitative analysis, and no exceptions of desertification evaluation. Comprehensive evaluation of variable fuzzy sets (VFS) can effectively solve influence of border fuzzy and monitor error of estimation standard to assessment result. The method can scientifically and reasonably determine membership degrees and relative membership functions of disquisitive objectives (or indexes) at level interval that relating to desertification, also it can fully use one's experience and knowledge, qualitative and quantitative information of index system to obtain weights of objectives (or indexes) for operating comprehensive evaluation of desertification.

2 Principle of VFS

2.1 Definition of VFS

In defining the concept, let us suppose that U is a fuzzy concept (alternative or phenomenon) $\underset{\sim}{A}$, and to any elements $u(u \in U)$, $\mu_{\underset{\sim}{A}}(u)$ and $\mu_{\underset{\sim}{A}^c}(u)$ are relative membership degree (RMD) function that express degrees of attractability and repellency respectively [8]. Let

$$D_{\underset{\sim}{A}}(u) = \mu_{\underset{\sim}{A}}(u) - \mu_{\underset{\sim}{A}^c}(u) \tag{1}$$

Where $D_{\underset{\sim}{A}}(u)$ is defined as relative difference degree of u to $\underset{\sim}{A}$. Mapping

$$\begin{aligned} D_{\underset{\sim}{A}} : D &\longrightarrow [-1, 1] \\ u| &\longrightarrow D_{\underset{\sim}{A}}(u) \in [-1, 1] \end{aligned} \tag{2}$$

is defined as relative difference function of u to $\underset{\sim}{A}$. And we have

$$\mu_{\underset{\sim}{A}}(u) + \mu_{\underset{\sim}{A}^c}(u) = 1 \tag{3}$$

Then

$$D_{\underset{\sim}{A}}(u) = 2\mu_{\underset{\sim}{A}}(u) - 1 \tag{4}$$

Or

$$\mu_{\underset{\sim}{A}}(u) = 1 + D_{\underset{\sim}{A}}(u)/2 \tag{5}$$

Where $0 \le \mu_{\underset{\sim}{A}}(u) \le 1, 0 \le \mu_{\underset{\sim}{A}^c}(u) \le 1$. Let

$$\underset{\sim}{V} = \left\{ (u, D) |\, u \in U, D_{\underset{\sim}{A}}(u) = \mu_{\underset{\sim}{A}}(u) - \mu_{\underset{\sim}{A}}(u), D \in [-1, 1] \right\} \tag{6}$$

$$A_+ = \left\{ u |\, u \in U, \mu_{\underset{\sim}{A}}(u) > \mu_{\underset{\sim}{A}^c}(u) \right\} \tag{7}$$

$$A_- = \left\{ u |\, u \in U, \mu_{\underset{\sim}{A}}(u) < \mu_{\underset{\sim}{A}^c}(u) \right\} \tag{8}$$

$$A_0 = \left\{ u |\, u \in U, \mu_{\underset{\sim}{A}}(u) = \mu_{\underset{\sim}{A}^c}(u) \right\} \tag{9}$$

Here $\underset{\sim}{V}$ is just defined as VFS of U; A_+,A_- and $A_0$are defined as attracting (as priority) sets, repelling (as priority) sets and balance boundary or qualitative change boundary of VFS $\underset{\sim}{V}$, respectively. Assume that C is variable factors sets of $\underset{\sim}{V}$

$$C = \{C_A, C_B, C_C\} \tag{10}$$

Here C_A are variable model sets, C_B are variable model parameters sets and C_C are variable other factors sets except model and its parameters. Let

$$A^+ = C(A_-) = \{\, u |\, u \in U,\ \mu_{\underset{\sim}{A}}(u) < \mu_{\underset{\sim}{A}^c}(u), \quad \mu_{\underset{\sim}{A}}(C(u)) > \mu_{\underset{\sim}{A}^c}(C(u)) \,\} \tag{11}$$

$$A^- = C(A_+) = \{\, u |\, u \in U,\ \mu_{\underset{\sim}{A}}(u) > \mu_{\underset{\sim}{A}^c}(u), \quad \mu_{\underset{\sim}{A}}(C(u)) < \mu_{\underset{\sim}{A}^c}(C(u)) \,\} \tag{12}$$

We generally define these two subsets as qualitative change sets of VFS $\underset{\sim}{V}$ to variable elements sets C. Let

$$A^{(+)} = C(A_+) = \{\, u |\, u \in U,\ \mu_{\underset{\sim}{A}}(u) > \mu_{\underset{\sim}{A}^c}(u), \\ \mu_{\underset{\sim}{A}}(C(u)) > \mu_{\underset{\sim}{A}^c}(C(u)) \,\} \tag{13}$$

$$A^{(-)} = C(A_-) = \{\, u |\, u \in U,\ \mu_{\underset{\sim}{A}}(u) < \mu_{\underset{\sim}{A}^c}(u), \\ \mu_{\underset{\sim}{A}}(C(u)) < \mu_{\underset{\sim}{A}^c}(C(u)) \,\} \tag{14}$$

We generally define these two subsets as quantitative change sets of VFS $\underset{\sim}{V}$ to variable elements sets C.

VFS models include fuzzy optimization model, fuzzy pattern recognition model and fuzzy clustering iteration model etc. [8]. Variable parameters sets of model include indexes weights, standard indexes values and other important parameters. We will illustrate changeability of model and parameters in application of desertification evaluation.

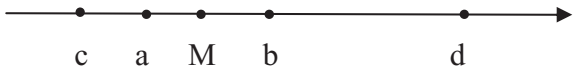

Fig. 1. Relationship between points$x$, M and internals $[a, b]$, $[c, d]$

2.2 Methods of Relative Difference Function

We suppose that $X_0 = [a, b]$ are attracting (as priority) sets of VFS $\underset{\sim}{V}$ on real axis, i.e. interval of $\mu_{\underset{\sim}{A}}(u) > \mu_{\underset{\sim}{A}^c}(u)$, $X = [c, d]$ is a certain interval containing X_0, i.e. $X_0 \subset X$ (see Fig. 1).

According to definition of VFS we know that interval $[c, a]$and $[b, d]$ all are repelling (as priority) sets of VFS, i.e. interval of $\mu_{\underset{\sim}{A}}(u) < \mu_{\underset{\sim}{A}^c}(u)$. Suppose that $M$is point value of $\mu_{\underset{\sim}{A}}(u) = 1$ in attracting (as priority) sets $[a, b]$, and M can be determined by actual problem or selected as midpoint value of interval $[a, b]$. x is value of random point in interval X, then if x locates at left side of M, its difference function is

$$\begin{cases} D_{\underset{\sim}{A}}(u) = \left(\frac{x-a}{M-a}\right)^{\beta} & x \in [a, M] \\ D_{\underset{\sim}{A}}(u) = -\left(\frac{x-a}{c-a}\right)^{\beta} & x \in [c, a] \end{cases} \tag{15}$$

And if x locates at right side of M, its difference function is

$$\begin{cases} D_{\underset{\sim}{A}}(u) = \left(\frac{x-b}{M-b}\right)^{\beta} & x \in [M, b] \\ D_{\underset{\sim}{A}}(u) = -\left(\frac{x-b}{d-b}\right)^{\beta} & x \in [b, d] \end{cases} \tag{16}$$

Where $\beta$is index than bigger than 0, usually we take it as β=1, viz. (15) and (16) become linear functions. Equations (15) and (16) satisfy: (i) $x = a$,$x = b$,$D_{\underset{\sim}{A}}(u) = 0$ or $\mu_{\underset{\sim}{A}}(u) = \mu_{\underset{\sim}{A}^c}(u) = 0.5$; (ii) $x = M$, $D_{\underset{\sim}{A}}(u) = 1$ or $\mu_{\underset{\sim}{A}}(u) = 1$; (iii) $x = c$, $x = d$,$D_{\underset{\sim}{A}}(u) = -1$ or $\mu_{\underset{\sim}{A}}(u)$=0. Then according to (15) or (16) and (5) we can obtain values of difference function $\mu_{\underset{\sim}{A}}(u)$ of disquisitive indexes.

3 VFS for Comprehensive Evaluation of the Desertification Degree

In this paper we take the case of desertification degree to the dry and hot valley of Jinsha River as example, and use data in [9] to show application of VFS method for desertification evaluation.

Desertification is affected by many complex factors, in addition to natural conditions of climate, vegetation and soil, it also involves many human economic activities, such as agriculture and animal husbandry, which directly or indirectly destroy the vegetation, affect the soil configuration, degrade the land and decrease arable area, and all these deteriorate environment and leads to desertification. As the elevation indexes should reflect reduced infield or deteriorative land, here we choose Yuanmou county, the dry and hot valley of Jinsha River, as study area, illustrate the factors of desertification with single component schematics and analyze its weight of indexes, establish evaluation indexes system of land degradation (see Table 1). we also select 23 samples for this research under actual status (Table 2).

According to Table 1 and [10], we set up values matrix of parameters (a, b, c, d, m) for calculating difference function of VFS:

$$\mathrm{I}_{[\mathrm{a,b}]} = \begin{bmatrix} [20,5] & [5,4] & [4,3] & [3,2] & [2,0] \\ [90,35] & [35,25] & [25,15] & [15,7] & [7,0] \\ [0,5] & [5,10] & [10,15] & [15,25] & [25,100] \\ [0,5] & [5,15] & [15,25] & [25,33] & [33,200] \\ [0,110] & [110,125] & [125,132] & [132,150] & [150,250] \end{bmatrix}$$

$$\mathrm{I}_{[\mathrm{c,d}]} = \begin{bmatrix} [20,4] & [20,3] & [5,2] & [4,0] & [3,0] \\ [90,25] & [90,15] & [35,7] & [25,0] & [15,0] \\ [0,10] & [0,15] & [5,25] & [10,100] & [15,100] \\ [0,15] & [0,25] & [5,33] & [15,200] & [25,200] \\ [0,125] & [0,132] & [110,150] & [125,250] & [132,250] \end{bmatrix}$$

$$\overrightarrow{M} = \begin{bmatrix} 20 & 4.5 & 3.5 & 2.5 & 0 \\ 90 & 30 & 20 & 11 & 0 \\ 0 & 7.5 & 12.5 & 20 & 100 \\ 0 & 10 & 20 & 29 & 200 \\ 0 & 117.5 & 128.5 & 141 & 250 \end{bmatrix}$$

Based on matrixes $I_{[a,b]}$,$I_{[c.d]}$ and $\overrightarrow{M}$, we judge that evaluating index x locates at left side or right side of point M, and according these to select (15) or

Table 1. Evaluation indexes system of land degradation in the Dry-hot Valley of Jinsha River, Yunnan province

	physiognomy		soil		
Degree of degradation	Density of incision (km/km^2)	Slope degree ($\circ$)	thickness (cm)	organism (g/kg)	Vegetation index
Special 1	>5	>35	<5	<5	<110
Strong 2	54	3525	510	515	110125
Intermediate 3	43	2515	1015	1525	125132
Light 4	32	157	1525	2533	132150
Tiny 5	≤ 2	<7	>25	>33	>150

Table 2. Measured values of desertification indexes on the 23 sample lands (with permission)

Sample	physiognomy		soil		Vegetation index
	Density of incision (km/km^2)	Slope degree (∘)	Thickness (cm)	organism (g/kg)	
1	0	5	2.5	100	144
2	0	5	0.6	60	140
3	4	12	0.3	15	131
4	0	0	0.66	10	134
5	0	12	0.7	20	140
6	0	3	0.55	15	135
7	5.5	0	0.8	100	131
9	3.5	15	0.45	35	135
10	2.5	23	0.7	10	139
11	2	23	0.7	10	165
12	2.3	30	0.7	15	138
14	2.8	15	2.5	50	160
15	2.8	17	1.5	40	149
16	3.5	15	0.7	30	140
17	4.3	0	0.4	20	122
19	4.1	3	0.65	100	122
20	4	20	0.41	20	131
21	2.8	26	0.45	25	138
22	4.5	15	0.4	10	134
23	0	24	1.2	15	162
25	0	0	0.6	50	131
26	0	3	1.8	100	143
27	0	6	0	0	100

(16) for calculating difference function $\mu_h(u_{ij})$ of indexes to standards. Here h is grade number and $h = 1, 2, 3, 4, 5$; i is indexes number and $i = 1, 2, 3, 4, 5$.

From Table 2, for sample 1, when $h = 1$, its attracting (as priority) matrix $[a, b]$, interval matrix $[c, d]$ and point values matrix $\overrightarrow{M}$ respectively are

$$[a, b] = (\,[0, 5]\ [5, 10]\ [10, 15]\ [15, 25]\ [25, 100]\,)$$

$$[c, d] = (\,[0, 10]\ [0, 15]\ [5, 25]\ [10, 100]\ [15, 100]\,)$$

$$\overrightarrow{M} = (\,0\ 7.5\ 12.5\ 20\ 100\,)$$

When $i = 3$, density of physiognomy incision $_{31}$=2.5, and that $c_{31} = 0$, $a_{31} = 0$, $b_{31} = 5$, d_{31}=10, $M_{31} = 0$, then we can see that index value (2.5) locates at right of M_{11} and belongs to interval $[\mathrm{M}_{31}, \mathrm{b}_{31}]$, so we select equation $D_{\mathrm{A}}(u_{31}) = -(x_{31} - b_{31})^{\beta}/(\mathrm{M}_{31} - b_{31})^{\beta}$ in (16). Substituting β=1 and other relevant parameters into this equation then we obtain $D_{\underset{\sim}{A}}(u_{31}) = 0.5$; according to (5) we obtain $\mu_{\underset{\sim}{A}}(u_{31}) = 0.75$. Analogously, we get relative

membership function $\mu_{\underset{\sim}{A}}(u_{ih})$ of each single index under $i = 1, 2, 3, 4, 5$ to degrees $h = 1, 2, 3, 4, 5$ as:

$$\mu_{\underset{\sim}{A}}(u_{ih})_{5\times 5} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0.3572 & 0.6429 \\ 0.75 & 0.25 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.2994 & 1 \\ 0.5 & 0.5 & 0.1667 & 0.8334 & 0.3333 \end{bmatrix}$$

To get synthetic RMD of each index, we use variable fuzzy recognition model presented by [11]

$$u_{ih} = \frac{1}{1 + \left\{ \dfrac{\sum\limits_{i=1}^{m} [w_i(1-\mu_{\underset{\sim}{A}}(u_{ih}))]^p}{\sum\limits_{i=1}^{m} (w_i \mu_{\underset{\sim}{A}}(u_{ih}))^p} \right\}^{\alpha/p}} \tag{17}$$

Through it we obtain synthetic RMD of each index for desertification by using (17), after normalizing them that we get normalized synthetic RMD of each index. Here w_i is index weight; m is number of recognition indexes; α is rule parameter of model optimization, $\alpha = 1$ is least single method and $\alpha = 2$ is least square method; p is distance parameter, $p = 1$ is hamming distance and $p = 2$ is Euclidean distance.

To determine weights of five indexes to five standards, we use consistency theorem of taxis on importance of determining indexes weights [11] and get qualitative scribe of four indexes by their influence to comparison between elements:

Taxis

$$F = \begin{bmatrix} 0.5 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 1 & 0 \\ 1 & 1 & 0 & 0.5 & 0 \\ 1 & 1 & 1 & 1 & 0.5 \end{bmatrix} \begin{matrix} (4) \\ (5) \\ (2) \\ (3) \\ (1) \end{matrix}$$

According to taxis F and experience, we take vegetation index, whose ranking is first, as comparison standard and get under consideration:

Vegetation index is on way from "rather" to "obvious" important than soil thickness; vegetation index is on way from "obvious" to "remarkable" important than soil organism; vegetation index is on way from "very" to "extra" important than physiognomy density of incision; vegetation index is on the way from "exceeding" to "extreme" important than land slope degree (see Table 3)

Table 3. Relationships between mood operator relative membership degrees of quantitative scale

Mood operator	Equal		Slight		Somewhat		Rather	
Quantitative scale	0.50	0.525	0.55	0.575	0.60	0.625	0.65	0.675
RMD	1.0	0.905	0.818	0.729	0.667	0.60	0.538	0.481
Mood operator	Obvious		Remarkable		Very		Extra	
Quantitative scale	0.70	0.725	0.75	0.775	0.80	0.825	0.85	0.875
RMD	0.429	0.379	0.333	0.290	0.250	0.212	0.176	0.143
Mood operator	Exceeding		Extreme		Incomparable			
Quantitative scale	0.90	0.925	0.95	0.975	1			
RMD	0.111	0.081	0.053	0.026	0			

And according to Table 3 [11] we obtain weights of five evaluation indexes as:

$$w' = (\,0.212 \quad 0.081 \quad 0.481 \quad 0.379 \quad 1\,) = (w'_i)$$

Then normalized weights vector of indexes is:

$$w = (\,0.0985 \quad 0.0376 \quad 0.2234 \quad 0.176 \quad 0.4645\,) = (w_i)$$

Therefore we may use variable fuzzy recognition model (17) to calculate synthetic RMD of sample 1. When taking rule parameter of model optimization $\alpha = 2$ distance parameter $p = 1$ and substituting relative data into model (17) we get synthetic RMD as

$$u' = (\,0.6103 \quad 0.3897 \quad 0.2175 \quad 0.6277 \quad 0.4337\,)$$

After normalized it is:

$$u = (\,0.2678 \quad 0.1710 \quad 0.0954 \quad 0.2755 \quad 0.1903\,)$$

Using rank feature values (RFV) [11] and we get RFV of sample 1 as

$H = (1, 2, 3, 4, 5)\cdot(\,0.2678 \quad 0.1710 \quad 0.0954 \quad 0.2755 \quad 0.1903\,)^T = 2.9494$

For sample (as disquisitive objective), due to its standard is five grades, so we have:

(a) If $1.0 \leq H \leq 1.5$,then desertification degree belongs to tiny (1 grade).
(b) If $1.5 < H \leq 2.5$,then it belongs to slight (2 grade).
(c) If $2.5 < H \leq 3.5$, then it belongs to Intermediate(3 grade).
(d) If $3.5 < H \leq 4.5$, then it belongs to strong(4 grade).
(e) If $4.5 < H \leq 5.0$, it belongs to special(5 grade).

Table 4. Synthetic relative membership degree and evaluation results of 23 sample lands

sample	U1(p)	U2(p)	U3(p)	U4(p)	U5(p)	H(p)	Judgment results	Investigation data
1	0.2678	0.1710	0.0954	0.2755	0.1903	2.9494	3	D3
2	0.2586	0.1166	0.2619	0.2836	0.0794	2.8087	3	D4
3	0.2142	0.1196	0.3032	0.1058	0.2573	3.07243	3	D3
4	0.3031	0.2416	0.1820	0.2341	0.0392	2.4647	2	D2
5	0.0017	0.2899	0.0193	0.4727	0.2164	3.6120	4	D4
6	0.0018	0.4486	0.0198	0.3082	0.2216	3.2993	3	D3
7	0.0000	0.3471	0.1542	0.3560	0.1427	3.2942	3	D2
9	0.0020	0.1684	0.0226	0.5536	0.2534	3.8879	4	D4
10	0.0013	0.2908	0.0068	0.4384	0.2627	3.6703	4	D4
11	0.0059	0.0296	0.0283	0.1294	0.8068	4.7016	5	D5
12	0.0026	0.0134	0.0128	0.6073	0.3639	4.3165	4	D4
14	0.0064	0.0324	0.0256	0.0534	0.8822	4.7725	5	D5
15	0.3356	0.1358	0.0544	0.2787	0.1954	2.8625	3	D3
16	0.0018	0.0093	0.3122	0.4231	0.2535	3.9172	4	D4
17	0.8625	0.0591	0.0021	0.0217	0.0547	1.3469	1	D1
19	0.8447	0.0579	0.0020	0.0212	0.0742	1.4224	1	D1
20	0.2472	0.0835	0.1729	0.0686	0.4277	3.3461	3	D3
21	0.3316	0.0001	0.0024	0.0921	0.5739	3.5764	4	D4
22	0.1608	0.6948	0.0005	0.0199	0.1240	2.2513	2	D2
23	0.0005	0.0066	0.1915	0.0265	0.7749	4.5687	5	D5
25	0.0006	0.0076	0.0687	0.0305	0.8927	4.8070	5	D5
26	0.0005	0.0072	0.1243	0.0287	0.8393	4.6991	5	D5
27	0.7392	0.0009	0.1485	0.0037	0.1077	1.7396	1	D1

Notes: D1- extreme desertification D2- strong desertification D3- remarkable desertification D4- slight desertification D5- tiny desertification

Hence we judge that comprehensive desertification evaluation (3.0219) belongs to 3 grade, the rest can be obtained in the same way. The results are showed in Table 4 (the RMD has been normalized).

Compared with actual instances, exception for 2#, 7#, 23#, that the rest 20 evaluation results are all tallied with the investigative results, and the accurate rate is 87%. When quantifying appraisal target, and considering the influence of indeterminacy in estimation, we find that the method and the process are quite perfect for comprehensive evaluation of desertification. Yet VFS use RFV to operate evaluation, it can intuitively reflect desertification degree partial to another rank, so the conclusion of VFS are more reasonable and appropriate.

4 Conclusion

1. The difference function, which describes the process that includes the change of objective things movement and development, the transition of quantitative and qualitative, and the essence of transform-attract and

distract, conforms to natural dialectics. Concepts, theory and method of VFS is the further advancement of relative membership degree of engineering fuzzy set, and it will be verified, improved and developed in application.
2. The RMD of VFS can scientifically and reasonably determine membership degrees and relative membership functions of disquisitive objectives at level interval that relating to desertification, also it can fully use one's experience and knowledge, qualitative and quantitative information of index system to obtain weights of objectives (or indexes) for comprehensive evaluation. The numerical example has shown that the proposed method is feasible and effective.
3. Though great progress has been obtained in VFS study, there are still exist much issue need to be solved, such as selecting of objective and subjective factors, the difference function design, research territory expansion and so on.

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A Fuzzy Ultrasonic System for Estimating Degradation of Insulating Oil

Yutaka Hata, Kensuke Iseri, Syoji Kobashi, Katsuya Kondo, and Kazuhiko Taniguchi

Summary. This paper proposes an ultrasonic estimation system for the degradation of the insulating oil of electric power supply aided by fuzzy inference. In general, it is known that the viscosity of insulating oil depends on the acidity. We employ the viscosity as characteristic value for estimating the degradation. First, we show a strong positive correlation between the acidity and the viscosity. Second, we construct a fuzzy estimation system. Third, we infer the acidity of insulating oil by using the system. Our experimental results show that the fuzzy system can identify the acidity with the accuracy of 74% on 50 samples. Thus, this system can successfully estimate the degradation of the oil.

Key words: Fuzzy logic, Estimation, Ultrasound, Oil, Acidity, Viscosity.

1 Introduction

The stability of the power supply is very important in human life. Especially, since the short time failure of electric power supply can significantly cause large outage that affects information technology society, nonstop power supply is required. In this paper, we consider the failure of transformer in electric supply system. The transformer includes insulating oil. This oil has currently been changed every 10–20 years because of the degradation of the oil. The proper periodic inspection is therefore essential to supply continuously and to save this change cost. Current inspection methods [1–4] such as gas-in-oil analysis and dielectric test need large measurement system, and they require the sampling of insulating oil from the transformer. These samplings require stopping the electric supply. To solve this problem, we employ an ultrasonic testing system that enables us to inspect the oil without stopping the supply. This ultrasonic system is feasible to a portable low cost system. It analyzes the oil by the propagation characteristics of the ultrasonic wave from the outside of the transformer. In this paper, we propose a fuzzy system for estimating the degradation of insulating oil by the ultrasonic system. It is known that the

degradation of the oil can be evaluated by the acidity. Moreover, it is shown that the viscosity of insulating oil depends on the acidity [5]. First, we show a strong positive correlation between the acidity and the viscosity in our system. We employ the viscosity as characteristic value for estimating the degradation. In our method, ultrasonic burst wave (2 MHz) is used to calculate a viscosity value. We employ fuzzy min–max-center-of-gravity method [6,7] for estimating the degradation of the insulating oil. As a result, we can successfully identify the acidity with the accuracy of 74% on 50 samples. Thus, this fuzzy ultrasonic system can successfully estimate the degradation of the insulating oil.

2 Preliminaries

The ultrasonic system used is shown in Fig. 1. The pulsar receiver provides burst ultrasonic wave (max. amplitude 15 V, 2.0 MHz, wave number 10) by a probe, and the other probes (2 MHz) receive the wave, as shown in Fig. 2. The oscilloscope (DL 1720CL, Yokogawa Co.) receives the wave (data sampling 500 MS s^{-1}). The received waves are transmitted to personal computer.

We used five oils with different acidity of 0.11, 0.20, 0.27, 0.32, and 0.35 mg KOH per g. Each oil is in a 20 l oil box. We sample the oil (about 20 cm^3) from each oil box.

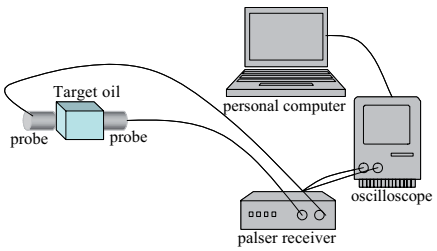


Fig. 1. The ultrasonic system

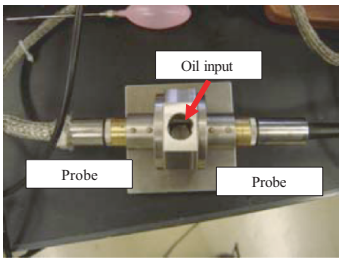


Fig. 2. The probe system

3 Fuzzy Ultrasonic Estimation System

3.1 Relationship Between Viscosity and Acidity

Figure 3a shows the received wave. The wave includes four echoes and these echoes are used for the estimation. The first wave is first received wave, and the second wave is the reflected wave of the first wave. The third and forth waves are received in a similar way, as shown in Fig. 3b.

First, we show a calculation method of the viscosity.

The viscosity, v, is calculated by (1):

$$\mathrm{v} = \frac{2\mathrm{a}\mathrm{c}^3\omega^2}{(\omega^2 + \mathrm{a}^2\mathrm{c}^2)^2}, \tag{1}$$

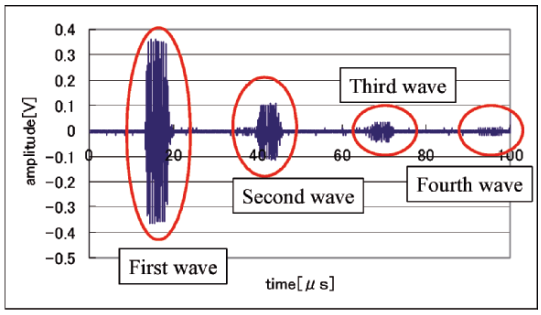
where notation a denotes attenuation rate, ω does angular frequency, and c does the ultrasonic speed. These values are calculated from the received wave below.

Attenuation

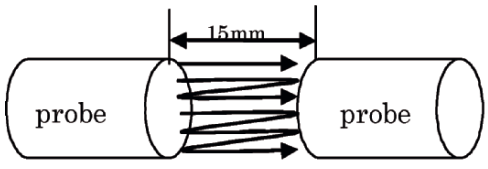
The attenuation, a, is calculated by (2):

$$\mathrm{a} = 20log\frac{\mathrm{A}_1}{\mathrm{A}_4}, \tag{2}$$

where notation A_1 denotes the amplitude of the first wave and A_4 denotes the amplitude of the fourth wave. The amplitudes are derived the integration of



(a) The received wave



(b) Measurement system

Fig. 3. The received wave and measurement system

the wave between −2 wave lengths and +2 wave lengths (total 4 wavelengths) from the central point of the echo.

Speed of ultrasound

The speed of ultrasound is derived from the time between the start point and the first echo. The speed is calculated by dividing the distance between the pulser and receiver probes (10 mm) by the calculated time.

Frequency, ω

The frequency is calculated by Fourier transform of the received wave.

We investigate the relationship between the acidity and the viscosity for known data. As the known data, we sampled the oils (10 samples per each oil box, total 50 samples). We did experiment for every sample. Their viscosities are calculated. Table 1 tabulated the mean and the standard deviation, SD, of viscosity values for the corresponding acidity values. Figure 4 shows the graph between the viscosity and the acidity. This graph shows a strong positive correlation between our calculated viscosity values and the acidity values. This correlation coefficient is 0.99.

Table 1. Acidity and viscosity on 50 oil samples

acidity	viscosity ($mm^2\,s^{-1}$)	
	mean	standard deviation
0.11	16.94686	0.50
0.20	17.10941	0.45
0.27	17.32979	0.67
0.32	17.35831	0.55
0.35	17.45735	0.54

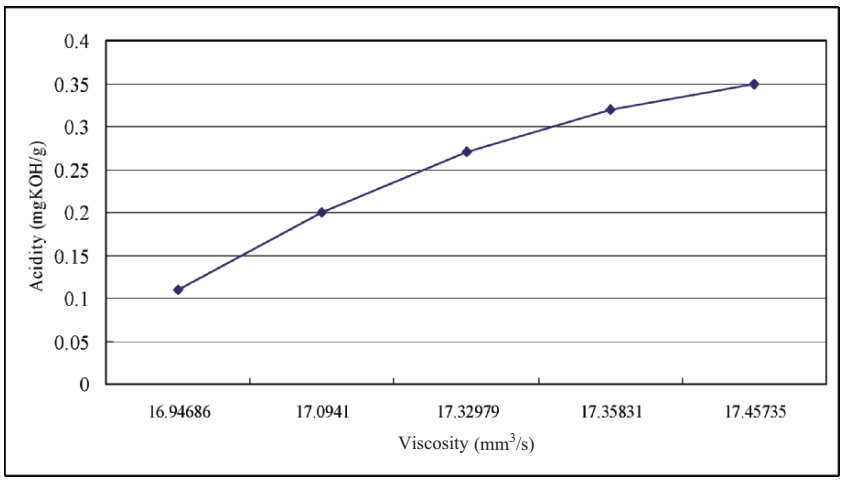


Fig. 4. Relationship between the acidity and the viscosity

3.2 Fuzzy Inference for Estimating Degradation

The overview of our method is shown in Fig. 5. As shown in this figure, we calculate the viscosity, which is derived from attenuation ratio, speed of ultrasound, and the frequency of the oil. We form fuzzy membership functions from the known viscosity values in the construction part. These membership functions are employed for the fuzzy estimation system. In the estimation, this fuzzy system estimates the degradation for unknown oil.

We describe this fuzzy inference system. In it, fuzzy min–max-center-of-gravity fuzzy inference method is employed.

Figure 4 means that we can calculate the acidity, A(v), from the viscosity v. We define the function f: v $\longrightarrow$ A(v). We consider the following fuzzy rule by considering this function.

For input viscosity x, IF x is viscosity v, THEN the acidity is A(v).

We made four fuzzy membership functions with respect to the viscosity and the acidity. First, we determine fuzzy membership functions, MF_{Vi}(v), with respect to the viscosity from Table 1.

The notation V_i denotes viscosity of i = 16.94686, 17.10941, 17.32979, 17.35831, and 17.457350. The notation m_{Vi} does the mean of ten viscosity values, and SD_{Vi} does the standard deviation of them for i. First, we form four fuzzy membership functions, MF_{Vi}(v) as shown in Fig. 6. Second, we made four membership functions, $MF_{A0.11}(y)$, $MF_{A0.20}(y)$, $MF_{A0.27}(y)$, $MF_{A0.32}(y)$, and $MF_{A0.35}(y)$ with respect to the acidity as follows:

$$MF_{A0.11}(y) = \begin{cases} 1 - \frac{|y-0.11|}{0.09} & if\ 0.02 \leq y \leq 0.2 \\ 0 & otherwise \end{cases}$$

$$MF_{A0.20}(y) = \begin{cases} 1 - \frac{|y-0.20|}{0.09} & if\ 0.11 \leq y \leq 0.20 \\ 1 - \frac{y-0.20}{0.07} & if\ 0.20 \leq y \leq 0.27 \\ 0 & otherwise \end{cases}$$

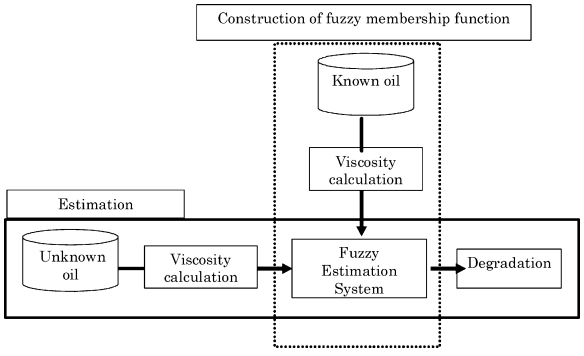


Fig. 5. Overview of our procedure

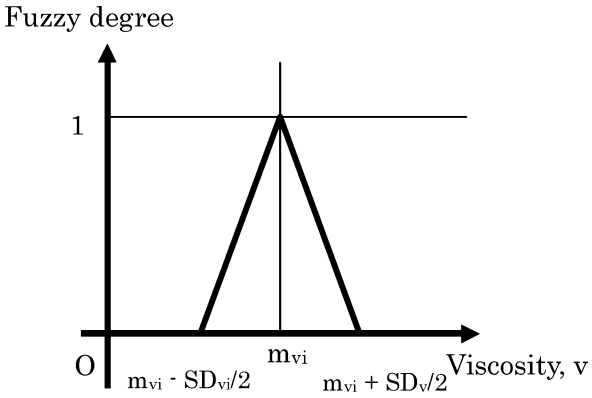


Fig. 6. A fuzzy membership function $MF_{Vi}(\mathrm{v})$

$$
\begin{aligned}
MF_{A0.27}(y) &= \begin{cases} 1 - \frac{|y-0.27|}{0.07} & if\ 0.20 \le y \le 0.27 \\ 1 - \frac{y-0.27}{0.05}\ if\ 0.27 \le y \le 0.32 \\ 0 \qquad\qquad otherwise \end{cases} \\
MF_{A0.32}(y) &= \begin{cases} 1 - \frac{|y-0.32|}{0.05} & if\ 0.27 \le y \le 0.32 \\ 1 - \frac{y-0.32}{0.03}\ if\ 0.32 \le y \le 0.35 \\ 0 \qquad\qquad otherwise \end{cases} \\
MF_{A0.35}(y) &= \begin{cases} 1 - \frac{|y-0.35|}{0.03}\ if\ 0.32 \le y \le 0.38 \\ 0 \qquad\qquad otherwise \end{cases}
\end{aligned}
\tag{3}
$$

The fuzzy if–then rule consisting of these eight membership functions can represent as follows:

For input viscosity $\mathrm{v_x}$,

IF $\mathrm{v_x}$ is $MF_{V1}(\mathrm{v})$ THEN acidity $\mathrm{A}(\mathrm{v_x})$ is $MF_{A0.11}(\mathrm{y})$,
ELSE IF $\mathrm{v_x}$ is $MF_{V2}(\mathrm{v})$ THEN acidity $\mathrm{A}(\mathrm{v_x})$ is $MF_{A0.20}(\mathrm{y})$,
ELSE IF $\mathrm{v_x}$ is $MF_{V3}(\mathrm{v})$ THEN acidity $\mathrm{A}(\mathrm{v_x})$ is $MF_{A0.27}(\mathrm{y})$,
ELSE IF $\mathrm{v_x}$ is $MF_{V4}(\mathrm{v})$ THEN acidity $\mathrm{A}(\mathrm{v_x})$ is $MF_{A0.32}(\mathrm{y})$.
ELSE $\mathrm{v_x}$ is $MF_{V5}(\mathrm{v})$ THEN acidity $\mathrm{A}(\mathrm{v_x})$ is $MF_{A0.35}(\mathrm{y})$.

These fuzzy rules can be calculated by the min–max rule:

$$
\begin{aligned}
\mu(y) = {} & \max(\min(\min(MF_{V1}(\mathrm{v}), \mathrm{S_{vx}}(\mathrm{v})), MF_{A0.11}(\mathrm{y})), \\
& \min(\min(MF_{V2}(\mathrm{v}), \mathrm{S_{vx}}(\mathrm{v})), MF_{A0.20}(\mathrm{y})), \\
& \min(\min(MF_{V3}(\mathrm{v}), \mathrm{S_{vx}}(\mathrm{v})), MF_{A0.27}(\mathrm{y})), \\
& \min(\min(MF_{V4}(\mathrm{v}), \mathrm{S_{vx}}(\mathrm{v})), MF_{A0.32}(\mathrm{y})), \\
& \min(\min(MF_{V5}(\mathrm{v}), \mathrm{S_{vx}}(\mathrm{x})), MF_{A0.35}(\mathrm{y})) \\
= {} & \sum_{A=A0.11,v=v1}^{A0.35,v5} \vee(MF_v(v) \wedge S_{vx}(v) \wedge MF_A(y))
\end{aligned}
\tag{4}
$$

Table 2. Estimation results of min–max-center-of-gravity method (50 samples)

acidity (degradation)	estimation result (correctly estimated sample #)
0.11	7
0.20	8
0.27	6
0.32	7
0.35	9

Table 3. Examples of the result

input viscosity	17.01	17.14	17.27	17.37	17.50
acidity	0.171	0.228	0.282	0.324	0.362
input viscosity	16.69	17.09	17.15	17.32	17.47
acidity	0.098	0.184	0.232	0.303	0.360

In (4), the fuzzy singleton function, $S_v(x)$, is defined as $S_v(x) = 1$ if $x = v$; $= 0$ otherwise, and the notation $\vee$ denotes maximum and $\wedge$ denotes minimum. Finally, we calculated the acidity, $A(v_x)$, as the center of gravity by

$$A(v_x) = \frac{\int y \cdot \mu(y)}{\int \mu(y)}. \tag{5}$$

4 Experimental Results

As the unknown data, we newly sampled the oil (10 samples per each oil box, total 50 samples). We calculate the viscosity and inferred the acidity from all data. The results are tabulated in Table 2 on 50 samples. In it, we consider that if the inferred value is the closest to one truth value of five acidity values, the value is correct. From the result, we can identify the acidity with the accuracy of 74%. Several examples of the experimental results are shown in Table 3. Thus, this system can calculate the acidity values for various viscosities by this fuzzy inference. From the inferred acidity, we can successfully estimate the degradation of the oil.

5 Conclusions

This paper describes a fuzzy estimation system of degradation of oil by identifying the acidity on an ultrasonic system. In this estimation, we employ the min–max-center-of-gravity method for identifying the acidity. There are many

methods to identify them [8]. We need to compare it with the other methods. This ultrasonic system can be easily installed in fabrication stage of the electric transformer. This ultrasonic system can estimate the degradation of the insulating oil by identifying the acidity in the running transformer. Thus, ultrasonic testing is the good choice to identify the degradation of the oil. In this chapter, first, we describe that our calculated viscosity and the acidity shows a strong positive correlation; the correlation coefficient is 0.99. Second, we construct the fuzzy estimation system by standard fuzzy inference method, and test the fuzzy system on 50 samples for five kinds of acidity. Then, we were able to estimate the degradation of unknown oils. Consequently, both a fuzzy inference and an ultrasonic equipment can realize the estimation system of the oil degradation. In our experiments, we sampled from five kinds of the oil for both design and test processes. It therefore remains as the future study to evaluate the system for different kinds of oils.

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A Genetic Algorithm-Based Fuzzy Inference System in Prediction of Wave Parameters

M. Zanganeh, S.J. Mousavi, A. Etemad-Shahidi

Summary. An important issue in application of fuzzy inference systems (FISs) to a class of systems identification problems such as forecasting problems is to extract the structure and type of fuzzy if-then rules from an input–output data set available. Given a FIS whose number and structure of fuzzy rules are known, artificial neural networks (ANNs) may be used to tune the shape of membership functions of fuzzy variables or other parameters of the fuzzy rule base. Adaptive-Network-Based Fuzzy Inference System (ANFIS) is an example of models in which the shape parameters of the membership functions of fuzzy premise variables as well as the linear parameters of the consequent part of fuzzy rules in a Takagi-Sugeno (TKS) FIS are tuned using ANNs. Genetic algorithms (GAs) may also be used for optimizing the parameter values of the subtractive clustering method by which the number and structure of an initial FIS is determined before it is tuned by ANNs. In this paper, a hybrid Genetic Algorithm-ANFIS (GA-ANFIS) model has been developed in which both clustering and rule base parameters are simultaneously optimized using GAs and ANNs. The model has been applied in prediction of wave parameters (wave significant height and peak spectral period) in Lake Michigan. The data set of year 2001 was used as training set and that of year 2004 as testing data. The results obtained by the hybrid GA-ANFIS model proposed are presented and analyzed.

Key words: Fuzzy inference systems, Prediction, Wave parameters, Genetic algorithms.

1 Introduction

Prediction of wave parameters plays an important role in ocean activities and costal engineering. Wave parameters are required for design of offshore and onshore structures, sediment transports estimation, design of submarine pipelines, etc. Several methods have been developed for prediction of significant wave parameters of which is numerical models working based on the solution of energy balance equations. Numerical models are time consuming and may not be justified from practical and economic point of view for small

projects. Therefore many engineers tend to use simplified wave prediction methods such as SPM (US. Army Corps of Engineers, 1984) and Coastal Engineering Manual, CEM (US Army Corps of Engineers, 2003) methods. Empirical methods often estimate significant wave parameters based on three main parameters which are wind speed, wind duration and fetch length while considering wave generation conditions (fetch-limited or duration-limited). Soft computing-based models have been recently used for prediction of significant wave parameters using the same input or premise variables used in empirical methods. Tsai et al. (2002) developed ANN-based models for predicting the wave parameters. Kazeminezhad et al. (2005) used ANFIS in prediction of wave parameters in fetch-limited conditions. In this study a hybrid GA-ANFIS model has been presented wherein the parameters of the subtractive clustering method, by which the number of fuzzy rules is controlled, are varied by a GA model within which ANFIS (Jang, 1993) is called for tuning fuzzy rule base parameters.

2 Fuzzy Inference Systems (FISs)

FISs may be used as tools for approximating ill-defined nonlinear functions. They can import qualitative aspects of human knowledge and reasoning processes by data sets without employing precise quantitative analyses using the following five functional components as shown in Fig. 1:

- A rule base containing a number of fuzzy if-then rules.
- A database defining the membership functions of fuzzy sets.
- Decision making unit as the inference engine.
- A fuzzification interface which transforms crisp inputs to linguistic variables.
- A defuzzification interface converting fuzzy outputs to crisp outputs.

ANFIS is an architecture which is functionally equivalent to a TSK fuzzy rule base whose parameters are tuned by using a learning algorithm in existence of input–output data. Assume a simple TSK fuzzy inference system with two inputs x and y and one output f and a rule base with two fuzzy if-then rules as follows:

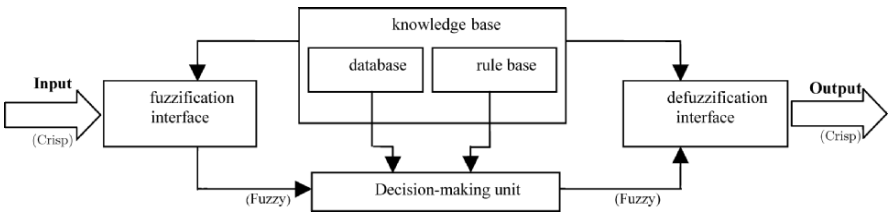

Fig. 1. Components of a FIS

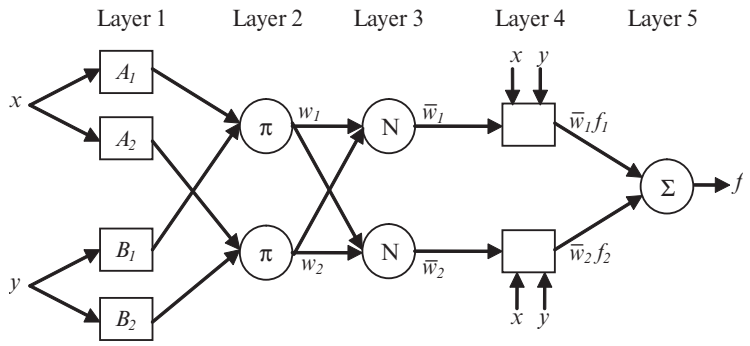

Fig. 2. ANFIS structure

Rule 1: If x is A_1 and y is B_1 then $f_1 = p_1 x + q_1 y + r_1$,

Rule 2: If x is A_2 and y is B_2 then $f_2 = p_2 x + q_2 y + r_2$,

where A_1, A_2 and B_1, B_2 are respectively fuzzy sets of input premise variables x and y; and p_1, q_1, r_1 and p_2, q_2, r_2 are parameters of the consequent or output variable.

The general structure of ANFIS is presented in Fig. 2 wherein circle nodes are fixed nodes and square nodes are adaptive nodes whose parameters are changed during training process. ANFIS architecture is composed of the following layers:

Layer 1: All the nodes in this layer are adaptive. It contains membership functions of input variables. Each node i in this layer is presented by a function as follows.

$$O_i^1 = \mu_{A_i}(x) \tag{1}$$

$$O_i^1 = \mu_{B_i}(x) \tag{2}$$

where A_i is the linguistic label, and x is input to node i, O_i^1 is the membership of A_i which is usually defined by a bell-shape function with maximum and minimum values equal to 1 and 0 as follows:

$$\mu_{A_i}(x) = \exp\left[-\left(\frac{x - c_i}{a_i}\right)^2\right] \qquad i = 1, 2 \tag{3}$$

where a_i is the sigma and c_i is the center of the above Gaussian membership function.

Layer 2: The fixed nodes in this layer are T-norm operators like AND operator. The output of each node in this layer represents the firing strength of the associated rule as follows:

$$w_i = \mu_{A_i}(x) \times \mu_{B_i}(x), \qquad i = 1, 2 \tag{4}$$

Layer 3: Nodes in this layer are fixed nodes and the normalized ratio of the ith rule's firing strength to the sum of all rules' firing strength is calculated in this layer as:

$$\overline{w} = \frac{w_i}{w_1 + w_2} \qquad i = 1, 2 \tag{5}$$

Layer 4: All nodes in this layer are adaptive. The output of each node (rule) is simply the product of normalized firing strength and a first order polynomial.

$$O_i^4 = \overline{w}_i f_i = \overline{w_i}\,(p_i x + q_i y + r_i) \tag{6}$$

where p_i, q_i, r_i are parameters of the consequent part of rule i.

Layer 5: This layer has only one node labeled $\sum$ to indicate that it performs as the simple sum over all outputs coming from layer 4.

A hybrid learning algorithm is used in ANFIS wherein the parameters of membership functions of input variables in antecedent part of fuzzy rules are optimized using a steepest descent algorithm while the linear parameters of the output variable in consequent part are optimized using least square method. If all the parameters defined are fixed, the final output of network is as follows:

$$\begin{aligned} f &= \frac{w_1}{w_1 + w_2} f_1 + \frac{w_2}{w_1 + w_2} f_2 = \overline{w}_1 f_1 + \overline{w_2} f_2 \\ &= \overline{w_1}(p_1 x + q_1 y + r_1) + \overline{w}_2 (p_2 x + q_2 + r_2) \\ &= (\overline{w}_1 x)\, p_1 + (\overline{w}_1 y)\, q_1 + (\overline{w}_1)\, r_1 + (\overline{w}_2 x)\, p_2 + (\overline{w}_2 y)\, q_2 + (\overline{w}_2)\, r \end{aligned} \tag{7}$$

As there is a linear combination of adaptive parameters, one can divide the set of all parameters (S) in two separate sets as $S = S_1 \oplus S_2$ where S is the set of total parameters, S_1 is the set of nonlinear antecedent parameters, S_2 is the set of linear consequent parameters and $\oplus$ is the summation operator. Least square method is used in forward path for optimizing consequent parameters through minimizing the error function as $\|AX - B\|^2$ where X is an unknown vector whose elements are parameters in S_2, A and B are, respectively, the coefficient matrix and the right hand side vector whose elements are determined using input–output data available as training data set. In the second step antecedent parameters are optimized using the steepest descent method.

3 Subtractive Clustering

As it is crucial in ANFIS to have the number of fuzzy rules as minimum as possible, a subtractive clustering method (Chiu et al., 1994) is used whose parameters would control the number of fuzzy rules. The proper cluster parameters may be obtained by experience or by using a trial and error procedure.

In subtractive clustering, the potential of each data point to be a cluster center is estimated and the points with high potential values are selected as candidates for cluster centers in an iterative manor. The points whose distance from the cluster center is less than a prespecified value (cluster radius) are then subtracted and the procedure continues until all the data are examined. Taking a collection of K data points $\mathbf{x}_k, k = 1, 2, \ldots, K$ each of which is specified by a D-dimensional vector, the potential of x_k to be a cluster center may be estimated as below:

$$P_k = \sum_{j=1}^{K} \exp\left(-4\sqrt{\sum_{i=1}^{D}\left(\frac{x_k^i - x_j^i}{r_i}\right)^2}\right) \tag{8}$$

where P_k is the potential of kth data point and r_i is the cluster radius associated with ith dimension of the point. Therefore, a data point would have a high potential value if it has more neighboring points. After determining the potential value of each data point, the point with the highest potential is selected as the first cluster center. Assume that $\mathbf{x}_{C1}$ is that point and P_1^* is its potential value. Then, the potential value of each data point $\boldsymbol{x}_k$ is reduced by using the following equation:

$$P_k' = P_k - P_1^* \exp\left(-4\sqrt{\sum_{i=1}^{D}\left(\frac{x_k^i - x_{C1}^i}{\eta\, r_i}\right)^2}\right) \tag{9}$$

where P_k' is the reduced potential value of the kth data point and η is a parameter called as squash factor, which is multiplied by radius values to determine the neighboring clusters within which the existence of other cluster centers are discouraged. New cluster centers are determined based on a potential value depending upon an acceptance ratio $\overline{\varepsilon}$, rejection ratio $\underline{\varepsilon}$, and the relative distance criterion. A data point with a potential greater than the acceptance threshold is directly accepted as a cluster center. The acceptance level of data points with potential values between the upper and lower thresholds depends on the relative distance equation as follows:

$$d_{\min} + \frac{P_k^*}{P_1^*} \geq 1 \tag{10}$$

where $d_{\min}$ is the nearest distance between the candidate cluster center and all cluster centers previously found as follows:

$$d_{k,c} = \left(\sqrt{\sum_{i=1}^{D}\left(\frac{x_k^i - x_{C1}^i}{r_i}\right)^2}\right) \tag{11}$$

where $d_{k,c}$ is the distance of kth data point from c_i th cluster previously found. In ANFIS each cluster center would represent a fuzzy if-then rule as

the cluster center i is assumed to be the mean value (c_i) of the following Gaussian membership function for the ith fuzzy input variable of the fuzzy rule base:

$$\mu_{A_i}(x) = \exp\left[-\left(\frac{x - c_i}{a_i}\right)^2\right] \tag{12}$$

The sigma parameter of Gaussian membership function (a_i) is calculated as follows:

$$a_i = r_i\left(\frac{\max(x_i) - \min(x_i)}{\sqrt{8}}\right) \tag{13}$$

Therefore the cluster centers and squash factors may be viewed as parameters which the number of fuzzy rules in an initial FIS depend on, before the rule base parameters of that initial FIS is tuned by ANNs in ANFIS.

4 Hybrid GA-ANFIS Model

It was mentioned that clustering parameters on which ANFIS parameters depend may be determined by a trial and error procedure and therefore they would not necessarily be optimal. The hybrid GA-ANFIS model proposed in this study is a model wherein the clustering parameters are optimized using a GA model within which an AFIS model is called for fitness function evaluation of any candidate solution generated by GA. Therefore GA acts as a stochastic search-based model within which the hybrid learning algorithm used in ANFIS performs as a local search optimizer. The parameters of the FIS designed for mapping input values to desired outputs are optimized by GA-ANFIS model proposed to get the total prediction error of the final model minimized. The objective function of the GA optimizer would therefore be the minimization of root mean square error (RMSE) of prediction made by an ANFIS model whose number of rules has been generated by GA. The abstract form of the optimization problem being considered in Hybrid GA-ANFIS model may be written as follows:

$$\min\left(\sqrt{\frac{\sum_{k=1}^{N}(O^k - P^k)^2}{N}} + P_1\right)$$

$$if(2 \leq NumRule \leq MaxNumRule)$$

$$P_1 = 0$$

$$Else$$

$$P_1 = \text{large}\, number$$

$P^k = f(radius\ of\ clustring\ for\ any\ input,\ squashfactor,\ Governing\ Eqs.\ of\ ANFIS)$ where *MaxNumRule* is the maximum number of fuzzy rules. P_1 is a penalty function penalizing the objective function where the number of fuzzy

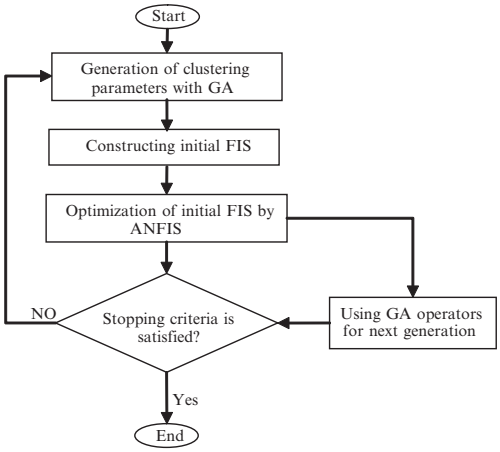

Fig. 3. Flow diagram of the hybrid GA-ANFIS model

rules become greater than *MaxNumRule*. However the number of fuzzy rules *NumRule* resulting to the best objective function is a variable which depend on clustering parameters that are varied by GA. Note that P^k is the ANFIS output for the kth member of the training data set. Figure 3 shows the flow diagram of the hybrid GA-ANFIS model presented.

5 Application

The hybrid GA-ANFIS model explained has been applied in prediction of wave parameters in Lake Michigan. The wave data is gathered by National Data Buoy Center (NDBC) at station 45007. The station is located at $42°40'30''$ N, $87°01'30''$ W in a depth of 176.4 m with a maximum measured peak spectral period of 7.3 s. The data available belongs to a period from March to January, 2001 and from January to December 2004, in 1-hr intervals. Wind-induced waves generally depend on wind speed, fetch length and the wind duration. In duration-limited condition, fetch length itself depends on wind duration and wind speed. Therefore the wave significant parameters would be a function of wind duration and wind speed. The data set was separated to 2 sets including duration-limited and fetch-limited data sets. 550 duration-limited data points of total 4,554 hourly data points were selected of which 350 data points (2001 data set) were used as training data and the remaining (2004 data set) as testing data. 300 data points of 350 training data were directly used in training procedure and 50 data points as checking data to ensure over-fitting would not be occurred. Separate hybrid GA-ANFIS models were developed for predicting significant wave height (GA-ANFIS-H) and peak spectral period (GA-ANFIS-P), respectively. The GA and ANFIS parameters used are given in Table 1.

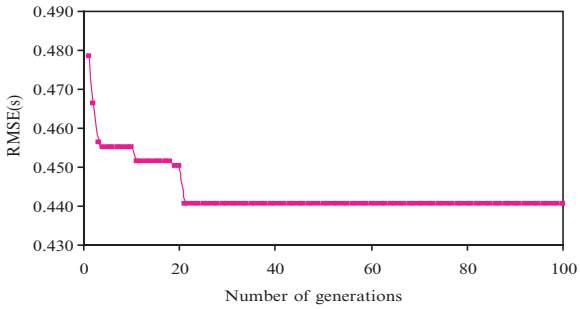

Fig. 4. Variation of fitness function in GA-ANFIS-H model over generations

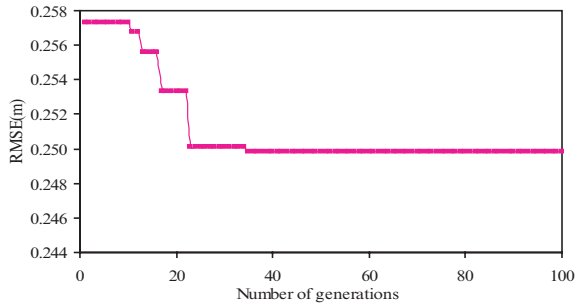

Fig. 5. Variation of fitness function in GA-ANFIS-P model over generations

Table 1. Suppositions of GA and ANFIS in wave prediction models

Model	GA-ANFIS-H	GA-ANFIS-P
Maximum number of generations in GA	100	100
Population size	20	20
Maximum number of epochs in ANFIS	300	300
Mutation probability in GA	0.27	0.17
Crossover probability in GA	0.64	0.72
Number of elitism	2	2
MaxNumRule	10	10

Figures 4 and 5, respectively, show the best fitness (objective) function of the GA-ANFIS-H and GA-ANFIS-P models over evolving generations. The optimal clustering parameters in GA-ANFIS-H were obtained as $[r_{tr}, r_{U_{10}}, r_{H_s}, H_{s_{squash\ factor}}] = [0.43, 0.09, 0.378, 2.15]$ with an optimal number of fuzzy rules equal to 5. The initial and final optimal membership shapes of the premise variables in GA-ANFIS-H associated with that optimal clustering parameters are also illustrated in Fig. 6. For the GA-ANFIS-P, the optimal clustering parameters were obtained as $[r_{tr}, r_{U_{10}}, r_{H_s}, T_{p_{squash\ factor}}] = [0.47, 0.48, 0.49, 0.53]$ with an optimal number of fuzzy rules equal to 4. It

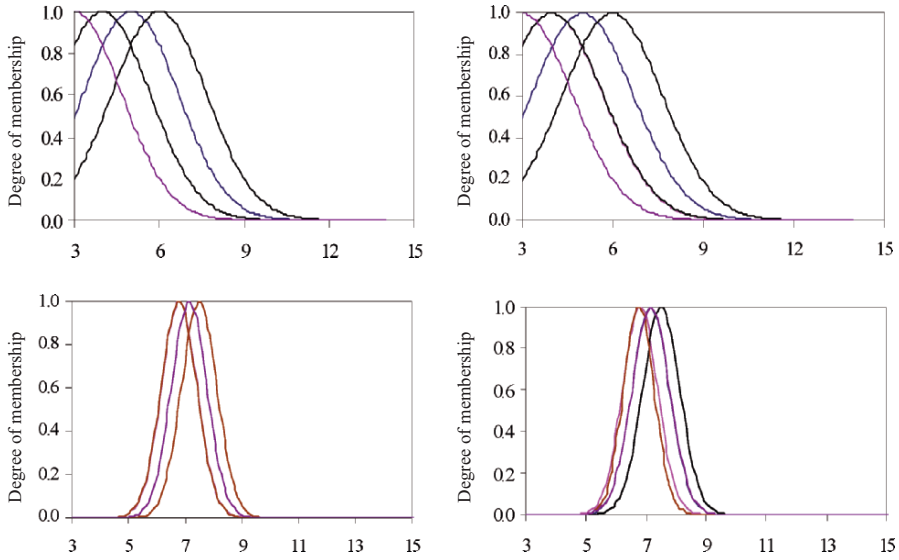

Fig. 6. Initial and final membership functions of input premise variables in GA-ANFIS-H model

is seen that the optimal radius of wind speed in GA-ANFIS-H model is less than that one in GA-ANFIS-P model indicating that wind speed has more effect on wave height rather than on peak spectral period.

6 Summary and Conclusions

In this paper a hybrid GA-ANFIS model was presented in which GA optimizes the structure and number of fuzzy if-then rules in a FIS by finding the best values for parameters of a subtractive clustering method, while ANFIS is used to optimize the initial FIS constructed based on the clustering parameter values generated by GA. The model was applied in predicting wave significant parameters, i.e., significant wave height and peak spectral period in Lake Michigan. Results show the satisfactory performance of the GA-ANFIS model proposed in reducing the prediction error of the FIS to be used in prediction of wave parameters.

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Estimation of Degree of Polymerisation and Residual Age of Transformers Based on Furfural Levels in Insulating Oil Through Generalized Regression Neural Networks

K.S. Madhavan, T.S.R. Murthy, and R. Sethuraman

Summary. Furfural Analysis and Degree of Polymerisation measurements give a measure of the degradation of paper insulation in transformers. These in turn relate to the ageing of transformers. Independent analysis of each of these chemical parameters gives an idea of the residual age of a transformer. But there have been no specific standards established to determine the ageing in transformers. In this paper, there is an attempt to estimate/predict the Degree of Polymerisation and the residual age of a transformer using Artificial Neural Networks given the Furfural component in oil.

Key words: Degree of polymerisation, Furfural analysis, Generalized regression neural networks, Probability density function, Residual age.

1 Introduction

Furfurals are major degradation products of cellulose insulation paper and are found in insulation oils of operating transformers. Furfural analysis is an indirect method to estimate the integrity of cellulose insulation compared to the direct measurement of Degree of Polymerisation of insulating paper. The tensile strength of the paper decreases corresponding to an increase in the concentrations of the Furfural in the oil. 5-Hydroxymethyl-2-Furfuraldehyde and 2-Furfuraldehyde are present in the oil at significantly greater concentrations than any other Furfural components. Furfural levels range from 0.1 ppm to 10 ppm depending on the age and condition of the transformer insulation. The residual life of the transformer can be predicted by estimating Furfural content in the oil or by the Degree of Polymerisation of cellulose paper taken from lead insulation. The life assessment can be made faster by estimating furfural from oil which can be collected from the transformer in running condition. The collection of cellulose paper involves cumbersome procedure of shutdown of the transformer and removal of paper from lead insulation after opening the transformer. Hence life assessment by furfural estimation is more popular and rapid method as compared to DP estimation of paper.

2 Transformer Insulation Measurement and Residual Life Assessment

The solid insulation used in transformers is a sheet of material made from cellulose. The solid insulation could be paper, pressboard or transformer-board. Cellulose is a linear polymer composed of individual anhydrous glucose units linked at the first and fourth carbon atoms through a glucosidic bond. The good mechanical properties of cellulose and its derivatives are due to their polymeric and fibrous nature. The number of monomer units in the polymers indicates Degree of Polymerisation (DP) [1].

Degree of Polymerisation has been used as a diagnostic tool to determine the condition of transformer. New Kraft paper has an average length of 1,000–1,500. After long period of service at high temperatures, its Degree of Polymerisation falls to 200–250.

Presently, Degree of Polymerisation (DP) estimation is not possible without collection of cellulose samples wherein paper is removed from lead insulation of the transformer windings after opening the transformer. Measurement of Furfural content is done on the oil sample that is collected from a transformer, the residual life of the transformer is also calculated.

A large database is available on DP, Furfural content and the corresponding age of transformers. Using this database, a software package has been developed such that the DP of paper insulation as well as the residual age of the transformer can be predicted more accurately, by applying ANN technique. The present aim is that in future the DP and the age of the transformer can be predicted without paper sample. It is easier to collect the oil sample from the transformer offline.

3 Choice of Artificial Neural Networks

Artificial Neural Networks have been applied to study/classify the patterns established in Degree of Polymerisation and Residual Age of transformer calculated as the Furfural levels vary in the transformer oil. Artificial Neural Networks have proven to be very versatile techniques for prediction on sparse data sets or when data input is vague and indeterminate in specific cases.

The Degree of Polymerisation and Residual Age of transformers are continuous functions of Furfural levels of transformers. Therefore continuous function approximation of multiple outputs through Generalized Regression Neural Networks is one solution for predicting output patterns.

GRNN based on radial units, giving estimates of continuous variables rather than discrete decisions, overcoming the disadvantage of slow training inherent in backpropagation thereby lending itself well to real-time application, is the appropriate choice for our present analysis. Least square method has been used to minimize the error in prediction [2].

4 Application of Generalized Regression Neural Networks

The basic structure of this network consists of four layers, a layer of input units, (radial basis layer) a layer of pattern units, a layer of summation units and a layer of output units. Each output unit corresponds to a continuous variable. It has a second hidden layer, a summation layer, unlike the usual radial basis network

Two main functions are required for GRNN neurons and its output prediction. Those functions are (1) to calculate the difference between all pairs of input pattern vectors and (2) to estimate the probability density function (pdf) of the input variables. Equation 1 namely the difference between input vectors is the simple Euclidean distance (D_i) between the data values in attribute space. Multiplying the measured value of the output with the appropriate probability function of the Euclidean distance (D_i) of any input variable X from other input variables occurring in the attribute space and averaging yields the estimated value of the predicted output [3].

$$D_i^2 = (X - X_i)^2 \quad \ldots \tag{1}$$

$$E[y|X] = \frac{\int_{-\infty}^{\infty} y f(X,y)dy}{\int_{-\infty}^{\infty} f(X,y)dy} \quad \ldots \tag{2}$$

where E[y|X] is regression of y on X, f(X,y) is the probability density function

Estimation of pdf i.e. f(X,y) is based on Parzen's theory. As per the theory each point is assigned a sphere of influence (sigma, known as the Parzen window or smoothing factor), similar to variance, which is centered over the point. The most common sphere of influence given to each data point is the Gaussian curve.

Many spheres of influence will be formed for various points. The appropriate sphere of influence is defined as the one that produces the smallest mean square error between the actual and predicted output values. Determination of this appropriate sphere of influence i.e. σ (smoothing factor) is where learning takes place in GRNN. In accordance with the above theory, Equation 2 takes the form of Equation 3 where f(X_iy) is given by exp $(-D_i^2/2\sigma^2)$. O_i is called the output of pattern unit i.

$$\hat{Y}(X) = \frac{\sum_{i=1}^{n} Y_i \exp(-D_i^2)/2\sigma^2}{\sum_{i=1}^{n} \exp(-D_i^2/2\sigma^2)} \quad \ldots \tag{3}$$

where $O_i = Y_i \exp(-D_i^2/2\sigma^2)$ or $\exp(-D_i^2/2\sigma^2)$

Yet another estimator of the output activation O_i as per the concept of Parzen is $\exp(-C_i/\sigma)$ instead of $\exp(-D_i^2/2\sigma^2)$ Where $C_i = |X - X_I|$, the city block distance. The predicted probability value Y (based on the least square error between the predicted value and the actual value) is finally calculated in the output layer by dividing the value in the numerator A unit by the value in the denominator unit B.

For the determination of the optimum smoothing factor σ, initial weights of any value are assigned to each variable. Then the weights are varied to get the lowest mean square error (mse) between the estimated output and actual output The architecture of the GRNN is shown in Fig. 1 [4].

The Furfural Concentration as input with corresponding Degree of Polymerisation and Residual Age of transformer as outputs are fed into the Neural Network. A set of 31 patterns, 28 training patterns and 3 test patterns are trained with the neural network. The error curves are shown in Fig. 2. The data is trained through 23 iterations with a smoothing factor of 0.00254. The variation of smoothing factor with mean square error is shown in Fig. 3. The correlation coefficient obtained on the final result for both the outputs is 0.997. The statistical analysis is shown in Tables 1 and 2.

The results of Statistical Analysis after training by GRNN are shown in Table 1 and 2.

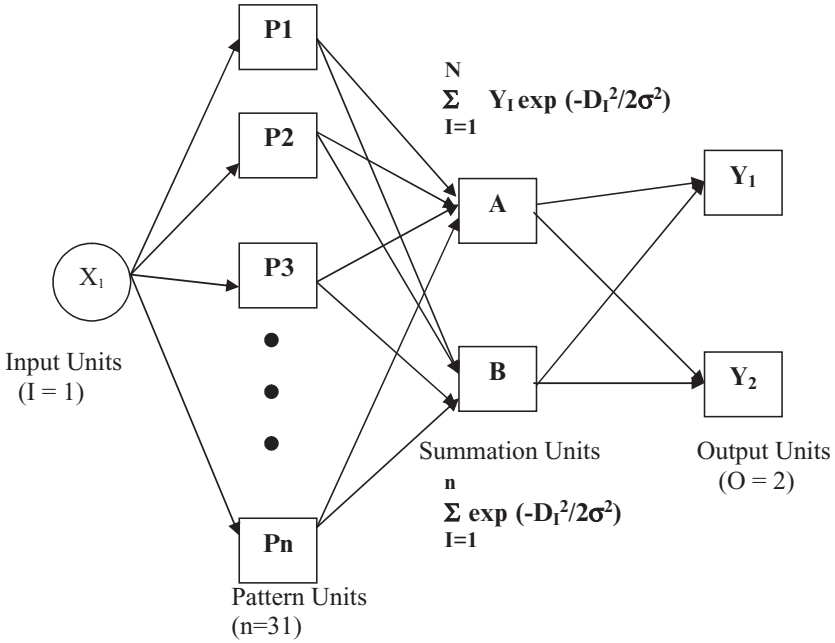

Fig. 1. The general regression neural network for estimating continuous variables

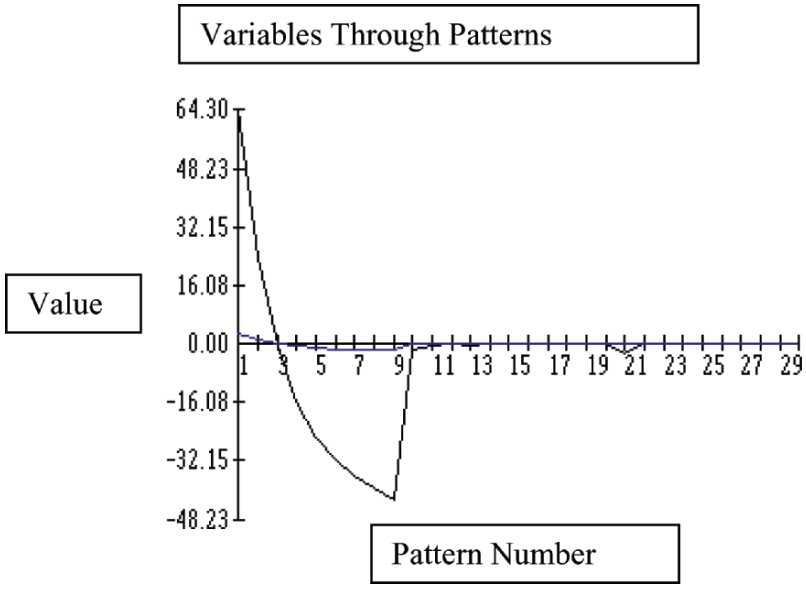

Fig. 2. Variation through patterns of pattern number vs error

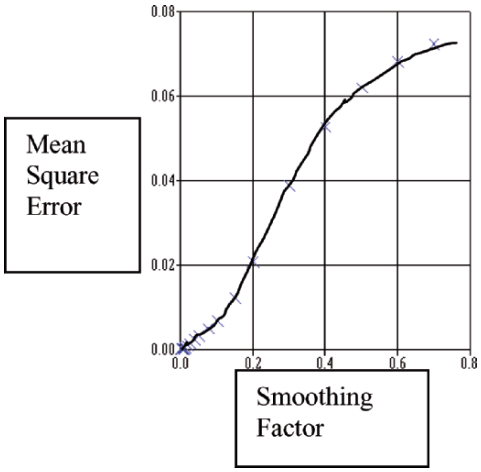

Fig. 3. Variation of smoothing factor vs mean square error

5 Conclusion

This is a novel method for predicting Degree of Polymerisation (DP) and Residual Age (RA) of transformers based on Furfural levels given. The method would be of great help in Residual Life Assessment studies of transformer. Henceforth, the DP and the life of the transformer can be predicted by simply

Table 1. Training parameters

Training Parameters	Value
Training Patterns	28
Test Patterns	3
Smoothing Test Events	3
Current Best Smoothing Factor	0.00254
Smoothing Test Epochs	23

Table 2. Error parameters

Error Parameters	C1	C2
Mean Square Error	365.8	0.618
Mean Absolute Error	9.231	0.379
Min. Absolute Error	0	0
Max. Absolute Error	64.3	2.639
Correlation Coefficient	0.997	0.997

C1 = DP Coefficient; C2 = RA Coefficient

measuring Furfural content and feeding into an ANN. Transformers need not be changed for paper samples which is most desirable. Also the testing time is reduced.

Acknowledgement

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Fuzzy Shortest Paths in Fuzzy Graphs

Amir Baniamerian and Mohammad Bagher Menhaj

Summary. This paper presents a new method for finding the shortest path in fuzzy graphs. This is indeed a "*modified Dijkstra*" in which there is no need to compare all paths between any two nodes. A new approach has been developed in this paper for fuzzy numbers comparison if needed. To show how good the method is, a computer simulation is done. The simulation results are very promising.

Key words: Fuzzy graphs, Shortest path, Modified dijkstra.

1 Introduction

Graph theory has various applications in different fields of science, like system analysis, research operation, economics, etc. In graph theory, weighted directed graphs are of special importance. Edge weights can represent money, time, etc.

There are so many cases that a unique number cannot be assigned to a given edge weight because of cost variations. One example is a computer network in which link costs are changed as traffic changes in the network [1]. One of the existing solutions is to use fuzzy graphs which have fuzzy numbers as edge weights.

One of the most important problems in graph theory is to find the shortest path available between two nodes. In crisp graphs, the optimum algorithm is *Dijkstra's algorithm* which has an order equal to $(n \log n)$ [1]. *Dijkstra's algorithm* needs only one execution in order to determine the shortest path between every specific source node to all other nodes.

In fuzzy graph algorithms, it is crucial to compare all available paths between every two nodes. On the other hand, in [3, 4] the algorithm must be executed for each node from the source node to acquire the shortest path.

With the above condition, it is obvious that the number of available paths between two nodes increases dramatically as the number of edges or nodes increases. In the suggested approach, numbers of fuzzy comparisons are limited to the number of edges. Note that this algorithm is applicable on directed acyclic fuzzy graphs. This algorithm, in fact, is a modified Dijkstra's algorithm for fuzzy graphs.

The rest of the paper is organized as follows:
At the end of this section, the notation used in this paper is introduced. In Sect. 2, the necessary mathematical preliminaries for other sections are explained. In Sect. 3, a new approach to compare two fuzzy numbers is developed and two basic theorems are derived. Section 4 presents the algorithm in a step by step format. In Sect. 5, a computer simulation is carried out and finally the future work are explained in Sect. 6.

Notation:

$\tilde{A}$	: A triangular fuzzy number that determined by (a_1, a_2, a_3).
e_i	: An edge of graph.
Sh^{a-b}	: Fuzzy shortest path set from node "a" to node "b" (goal set).
$Sh^{a-b}(e)$	: Membership degree of the edge "e" to the shortest path set.
L_i	: A path in graph that is defined by a series of edges (i.e. $(e_{m_1}, e_{m_2}, ..., e_{m_k})$).
$\mu_{L_i \approx L_j}$	: Reflects how much L_i is near to L_j.
E	: The edge set of a graph.
V	: The node set of a graph.

2 Preliminaries

In this section some useful definitions are introduced.

Definition 1. ***Triangular Fuzzy number***

A fuzzy number is a normal, convex fuzzy set whose support[1] is bounded. In this study triangular fuzzy numbers are used. See Fig. 1.

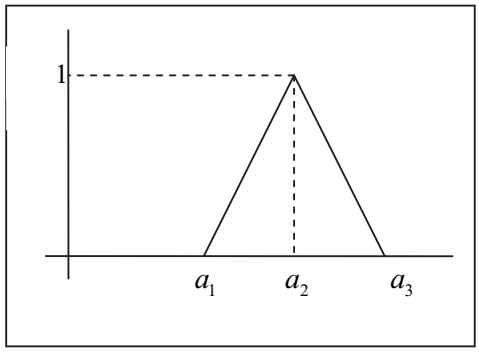

Fig. 1. A triangular fuzzy number

[1] Support of $\tilde{A}$ is $\{x \,|A(x) > 0\}$.

Definition 2. ***Fuzzy graph***

In [4] five types of fuzzy graphs are introduced. In this study, fuzzy graphs with the following properties are considered:

1. The graph is directed and acyclic.
2. Nodes and edges are both crisp.
3. Edge length is a triangular fuzzy number.

Definition 3. ***Fuzzy shortest path***

In crisp graphs, shortest path between two nodes is a specific path which is composed of some certain edges. It can be defined as:

$$\left\{ (e_i, \mu) \middle| \mu = \begin{cases} 1, & e_i \in Shortest\,path \\ 0, & e_i \notin Shortest\ path \end{cases} \right\} \tag{1}$$

where e_i is an edge.

The fuzzy shortest path set can then be guessed from (1). Membership degree of the shortest path set is assigned to each edge. In other words we can write:

$$\left\{ (e_i, Sh^{s-d}(e_i)) \,\middle|\, Sh^{s-d}(e_i) \in [0,1] \right\} \tag{2}$$

It is apparent that separate shortest paths must be calculated for a specific source node to other nodes. The goal of the proposed algorithm is to find them in one run. Membership values for each edge are determined as:

$$Sh^{a-b}(e_i) = \bigvee_{P \in \Pi} \Big(\bigwedge_{Q \in \Pi} \mu_{L_Q \approx L_P} \Big) \tag{3}$$

where Π is set of all paths between a and b, P is each path between a and b that contains e_i, and $\wedge$, $\vee$ are infimum and suprimum operations, respectively.

From (3), it is obvious that all paths between a and b containing e_i must be taken into account.

3 The Proposed Method

In this section, a new method to compare two fuzzy numbers is introduced. Furthermore, two theorems which are the bases of the algorithm are investigated.

3.1 Comparing Two Fuzzy Numbers

To compare two fuzzy numbers, an index for fuzzy numbers should be first evaluated[2]:

[2] This index was introduced completely in [2].

$$A_I = \int_0^1 D(0, [A_\alpha^L, A_\alpha^U])d\alpha \tag{4}$$

where $[A_\alpha^L, A_\alpha^U]$ is $\alpha - cut$ of $\tilde{A}$ fuzzy number and

$$D(0, [A_\alpha^L, A_\alpha^U]) = 1/2(A_\alpha^L + A_\alpha^U).$$

Now, the following equation can be used to compare two fuzzy numbers.

$$\begin{array}{l} g : \Re^+ \longrightarrow [0,1] \\ \mu_{\tilde{A}\approx\tilde{B}} = g(B_I - A_I) \end{array} \tag{5}$$

In the above equation $g(\cdot)$ is a monotonically decreasing function in which A_I and B_I are calculated from (4).

3.2 Theorems

Theorem 1[3]:

Assume that, the binary operator $\hat{*}$ is a t-norm with the following two properties:

1. $a\hat{*}a < a$
2. $\hat{*}$ is continuous for all of its arguments

Then, a monotonically decreasing function $f(\cdot)$ can be found which satisfies:

$$a\hat{*}b = f^{-1}(f(a) + f(b)) \quad ; a, b \in [0,1] \tag{6}$$

$f(\cdot)$ is called "t-norm generator function".

Theorem 2.

Assume that the fuzzy numbers $\tilde{A}$,$\tilde{B}$,$\tilde{C}$ are given. We can write

$$\mu_{\tilde{A}\approx\tilde{B}} = \mu_{(\tilde{A}+\tilde{C})\approx(\tilde{B}+\tilde{C})} \tag{7}$$

where $\mu_{\tilde{X}\approx\tilde{Y}}$ is defined by (5).

Proof.

For a triangular fuzzy number $\tilde{X} = (x_1, x_2, x_3)$, the index defined in (4) becomes

$$X_I = 1/2(2x_2 + x_1 + x_3) \tag{8}$$

Meanwhile, for any tow triangular fuzzy numbers $\tilde{X}, \tilde{Y}$, we can have

$$(\tilde{X} + \tilde{Y}) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \tag{9}$$

[3] This theorem was explained in [5].

The above equation is obtained as follows

$$\tilde{Z} = (\tilde{X} + \tilde{Y})$$

$$Z(z) = \underset{z=x+y}{\vee}(X(x) \wedge Y(y))$$

Use (8) and (9) to write

$$(\tilde{X} + \tilde{Y})_I = \tilde{X}_I + \tilde{Y}_I$$

where $(\tilde{X} + \tilde{Y})_I$ is index of $(\tilde{X} + \tilde{Y})$ defined by (4). Use (5) to complete the proof as:

$$\begin{aligned} \mu_{\left(\widetilde{A}+\widetilde{C}\right) \approx \left(\widetilde{B}+\widetilde{C}\right)} &= g((B_I + C_I) - (A_I + C_I)) \\ &= g(B_I - A_I) = \mu_{\widetilde{A} \approx \widetilde{B}} \end{aligned}$$

This theorem is used to compare two paths that share some common edges.

Theorem 3.

Consider three triangular fuzzy numbers $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and a t-norm, $\hat{*}$ with the properties given in theorem 1. We further assume $A_I \leq B_I \leq C_I$ and $g(\cdot) = f^{-1}(\cdot)$. Note that function $g(\cdot)$ is defined in (5). Then, it can be shown that

$$\mu_{\tilde{A} \approx \tilde{B}} \hat{*} \mu_{\tilde{B} \approx \tilde{C}} = \mu_{\tilde{A} \approx \tilde{C}} \tag{10}$$

Proof.

It is clear that

$$\mu_{\tilde{X} \approx \tilde{Y}} = f^{-1}(Y_I - X_I) \tag{11}$$

In the light of (6), we can write

$$\begin{aligned} \mu_{\tilde{A} \approx \tilde{B}} * \mu_{\tilde{B} \approx \tilde{C}} &= f^{-1}(f(\mu_{\tilde{A} \approx \tilde{B}}) + f(\mu_{\tilde{B} \approx \tilde{C}})) \\ &= f^{-1}(f(f^{-1}(B_I - A_I)) + f(f^{-1}(C_I - B_I))) \end{aligned}$$

$$\begin{aligned} \mu_{\tilde{A} \approx \tilde{B}} * \mu_{\tilde{B} \approx \tilde{C}} &= f^{-1}(B_I - A_I + C_I - B_I) \\ &= f^{-1}(C_I - A_I) = \mu_{\tilde{A} \approx \tilde{C}} \end{aligned}$$

Now we are ready to develop the proposed algorithm using the above theorems. Next subsection is devoted to do so.

Consider Fig. 2 and calculate the fuzzy shortest path set from node a to node d as illustrated below.

Let us assume

$$L_1 = (5), L_2 = (1, 2, 4), L_3 = (1, 3, 4), L_4 = (1, 2), L_5 = (1, 3)$$

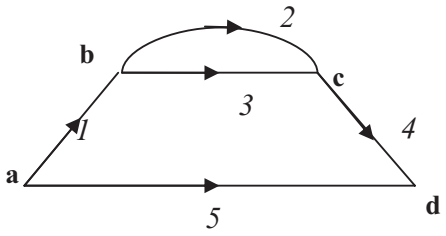

Fig. 2. A simple fuzzy graph

Denote

L_1 as the shortest path from a to d

L_4 as the shortest path from a to c

It is clear that the shortest path from a to d, containing edge (4), is L_2. Hence,

$$Sh^{a-d}(4) = \mu_{L_1 \approx L_2}$$

Now, we want to calculate $Sh^{a-d}(3)$. From (3) and Fig. 2, it is obvious that

$$Sh^{a-d}(3) = \mu_{L_1 \approx L_2}.$$

Assuming that Sh^{a-c} was completely calculated[4] and

$$Sh^{a-c}(3) = \mu_{L_4 \approx L_5}$$

Theorem 2 implies

$$\mu_{L_2 \approx L_3} = \mu_{L_4 \approx L_5}$$

Use theorem 3 and the above fact to obtain

$$\begin{aligned} \mu_{L_1 \approx L_3} &= \mu_{L_2 \approx L_3} \hat{*} \mu_{L_1 \approx L_2} \\ &= \mu_{L_4 \approx L_5} \hat{*} \mu_{L_1 \approx L_2} \\ \Rightarrow \mu_{L_1 \approx L_3} &= Sh^{a-c}(3) \hat{*} Sh^{a-d}(4) \\ \Rightarrow Sh^{a-d}(3) &= Sh^{a-c}(3) \hat{*} Sh^{a-d}(4) \end{aligned}$$

The last result shows that no fuzzy comparison is needed to compute $Sh^{a-d}(3)$.

Now the steps of the proposed algorithm are summarized in the next section.

4 The Proposed Algorithm

1. Calculate all edge indexes[5] and $\{Sh^{s-u}(e_i) = 0 \quad \forall u \in V\ , \forall e_i \in E\}$
2. Use "*Dijkstra algorithm*" with the following actions.

[4] In the algorithm, an edge, e_i, is selected when the shortest path set from the source node to the e_i head node was calculated completely.

[5] The indexes are obtained by (11).

1. If the selected node "v" is picked up for the first time (this means that the selected path is the shortest path), then set

$$Sh^{s-v}(e_0) = \mu' = 1$$

where e_0 is the last edge in the selected path.
2. For a repeatedly selected node "v" find

$$Sh^{a-v}(e_0) = \mu'$$

where e_0 is the last edge in the selected path and μ' is calculated by comparing the shortest path from "a" to "v" and a path which is obtained from e_0, and the shortest path from "a" to the head of e_0("v'").
3. For every edge, do the following

$$Sh^{a-v}(e) = \max\left\{(Sh^{a-v'}(e)\hat{*}\mu'), Sh^{a-v}(e)\right\}$$

where $\hat{*}$ is a *t-norm* with the specifications given in theorem 3.

5 Simulation

Consider the following graph.

For this example, we have applied the proposed algorithm in *C environment* and Table 1 summarizes the results. In this simulation, dot product is used as t-norm and $g(x) = e^{-x}$ is considered in (5).

Table 1 also summarizes the results for the fuzzy shortest path set from node "1" to node "8". Total number of comparisons for this graph became 10 as expected.

Table 1. The fuzzy shortest path set for node "8" in graph shown in Fig. 3 (The lengths are triangular fuzzy number).

arc	Length	Degree
(1,2)	(19,25,29)	1
(2,3)	(15,20,25)	1
(2,4)	(58,63,68)	0.86
(3,4)	(38,41,46)	1
(4,5)	(12,15,18)	0.68
(3,6)	(54,57,62)	0.22
(4,6)	(8,9,10)	1
(5,7)	(70,75,80)	0.68
(6,7)	(65,75,85)	1
(7,8)	(20,25,30)	1

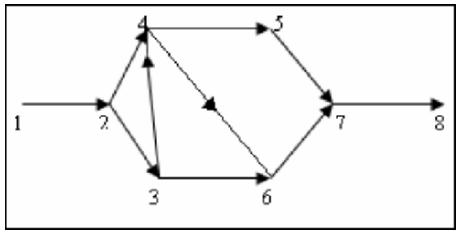

Fig. 3. A simple fuzzy graph for simulation

6 Conclusion

In this paper, a new approach was presented to find fuzzy shortest paths in fuzzy graphs. In the algorithm, a method for fuzzy comparison was derived. This algorithm reduces the total number of comparisons to the maximum number of the edges. This algorithm with a few modifications would be able to exploit s-norm for finding the longest path. This is under current investigation and will be reported later.

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Improving Vegas Algorithm Using PID and Fuzzy PID Controllers

Aria Shahingohar, Mohammad Bagher Menhaj, Mehdi Karrari, and Mohammad Hossein Yaghamae

Summary. Since in the TCP protocol the source has no idea which congestion window size will result in the best performance, it tunes the window size according to parameters such as loss, duplicate acknowledge and etc. The Vegas algorithm is one of the most intelligent flow control algorithms that despite other protocols, tunes the window size by estimation of actual throughput. In this paper, we focus on the fact that controlling the traffic flow is indeed a control problem in nature. We developed a PID and Fuzzy PID based Vegas algorithm. To show the performance of our proposed methods, we performed simulations on NS-2 simulator. The simulation results highlight better performance of our methods.

1 Introduction

TCP Tahoe [1] and TCP Reno [2] are two common type of TCP algorithms. TCP Vegas [3] enhanced the Congestion Avoidance and Fast Retransmission algorithms of TCP Reno [2]. In resent years several attempts have been made to develop Vegas algorithm [4–7]. In this paper, we first briefly introduce congestion control mechanism. Then, we explain Vegas algorithm and show that congestion avoidance phase is a simple feedback control problem that is easily solved by a PID controller. Afterwards, we adaptively tune the parameters of the PID controller using fuzzy logic. The simulation results easily approve the outperformance of the proposed methods over the ordinary Vegas.

2 Congestion Control

TCP congestion control, prevents a source from exceeding network capacity by allowing it to adapt its transmission rate to avoid congestion in routers.

When a TCP connection is first established, the source TCP does not transmit a full receiver's advertised window of segments. The source TCP initiates slow-start by transmitting one segment and waiting for its ACK. When

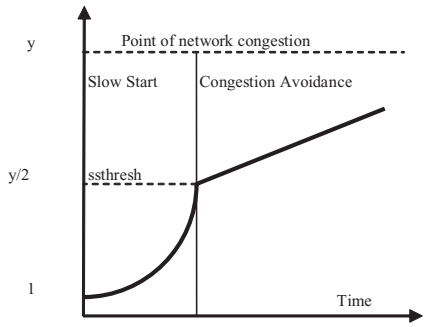

Fig. 1. Slow start and Congestion Avoidance in TCP

the ACK is received, the source increases congestion window size (CWND) from one to two, and two segments are sent. When these two segments are acknowledged, the source increases its CWND from two to four and four segments are sent. The exponential growth of CWND continues until either its value exceeds the destination's advertised window or packets are dropped due to congestion.

When the source TCP discovers that a packet has been dropped by the network, it sets the variable ssthresh (slow-start threshold) equal tone-half of the current value of CWND. The source reduces its transmission rate by returning to slow-start mode, but this time it exponentially increases its transmission rate until CWND is equal to the value of ssthresh. At this point, the sender increases CWND linearly (by at most one segment per RTT), allowing it to slowly increase its transmission rate as it begins to approach the previous CWND value that caused packets to be dropped. Figure 1 shows an example of TCP behavior in slow-start and congestion avoidance.

3 TCP Vegas

In the TCP Vegas, in the Congestion Avoidance phase, the congestion window size is not increased linearly. Instead, the actual throughput of network is compared with its expected throughput. If these values are close or if the actual throughput is greater than the actual value, the window size is increased but if the actual throughput is very smaller than expected throughput, then the window size is decreased. The main difference between Vegas with other algorithms is using throughput rather than loss of packets for detection of severity of congestion.

The actual and expected throughputs of network are computed according to (1) and (2), respectively. In (2), BaseRTT is the smallest measured RTT(Round Time Trip).

$$\text{Actual Throughput} = \frac{CWND}{RTT} \tag{1}$$

$$\text{Expected Throughput} = \frac{CWND}{BaseRTT} \tag{2}$$

Vegas then define Delta according to the following equation:

$$\text{Delta} = ExpectedThroughput - ActualThroughput \tag{3}$$

Vegas uses Delta to set the window size. For this purpose two thresholds α and β ($\alpha < \beta$) are defined. If $Delta > \beta$, Vegas decreases the window size at each RTT and if $Delta < \alpha$ Vegas increases the window size at each RTT.

To adjust the window size in the Vegas algorithm, Delta is compared with two constant thresholds. This is the main weakness of Vegas algorithm since in fact it is an On–Off control. We show that better results could be achieved if a proper controller is used to adjust the window size.

4 Developing TCP Vegas using PID Controller

The block diagram of the PID controller that we have used to adjust the CWND is shown if Fig. 2. In this figure K_p, K_d, and K_i are proportional, differential and integral gains respectively. D_0 is the set point of Delta. A small value for D_0 will utilize maximum network resources while preventing from congestion. The same idea exists in ordinary Vegas, since the thresholds α and β are experimentally selected positive values.

We have also selected the PID gains based on experience. The simulation results show that PID controller has better performance than ordinary Vegas.

5 Developing TCP Vegas using Fuzzy PID Controller

The PID Vegas controller surpasses ordinary Vegas. However, it is possible to enhance the PID controller's performance by online tuning of PID gains. We

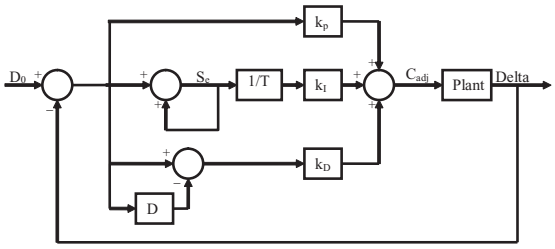

Fig. 2. PID Vegas algorithm

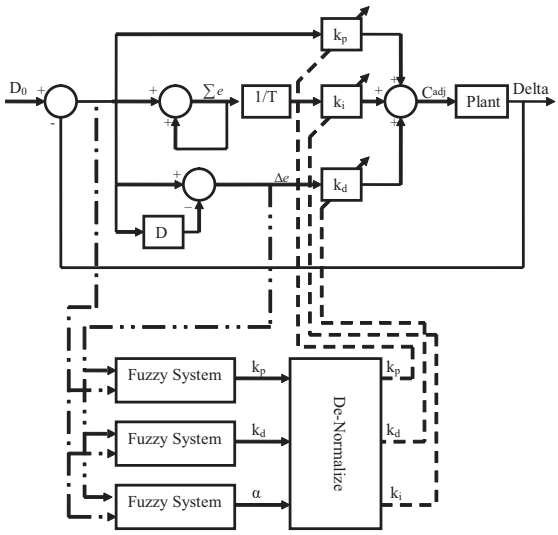

Fig. 3. Fuzzy PID Vegas algorithm

will follow the approach proposed by Zhao, Tomizuka and Isakaf for Fuzzy gain scheduling of our PID controller [8]. The block diagram of proposed method is shown in Fig. 3. We use three groups of rules each consisting 49 rules to tune the PID gains. We combine the 49 rules in each set using product inference engine, singleton fuzzifier, and center average defuzzifier.

For tuning of the parameters we shall first define the intervals $[K_{pmin}, K_{pmax}] \subset \Re$ and $[K_{dmin}, K_{dmax}] \subset \Re$ so that proportional gain$K_p \in [K_{pmin}, K_{pmax}]$ and differentia gain $K_d \in [K_{dmin}, K_{dmax}]$. For convenience K_p and K_d are normalized to the ranges between zero and one according to the following transformations:

$$K'_p = \frac{K_p - K_{pmin}}{K_{pmax} - K_{pmin}} \tag{4}$$

$$K'_d = \frac{K_d - K_{dmin}}{K_{dmax} - K_{dmin}} \tag{5}$$

Assume that the integral time constant is determined with reference to the derivative time constant by:

$$T_i = \alpha \times T_d \tag{6}$$

From which we can obtain:

$$K_i = \frac{K_p}{\alpha \times T_d} = \frac{K_p^2}{\alpha \times K_d} \tag{7}$$

Hence the parameters to be tuned by the fuzzy system are K'_p, K'_d, and α. Assume that the inputs to the fuzzy system are e(n) and Δe(n)=e(n)-e(n-1). So our fuzzy systems consist of three two-input one-output fuzzy systems, as shown in Fig. 3. We now derive the fuzzy If-Then rules that constitute these fuzzy systems. Let the fuzzy If-Then rules be of the following form:

IF e(n) is A^l and Δe(n) is B^l, THEN K'_p is C^l, K'_d is D^l, α is E^l

Where A^l, B^l, C^l, D^l, and E^l are fuzzy sets and suppose that the domains of interest of e(n) and Δe(n) are$[e_M^-, e_M^+]$ and $[e_{Md}^-, e_{Md}^+]$, respectively. We define seven fuzzy sets as shown in Fig. 4 to cover them. Thus, a complete fuzzy rule base consists of 49 rules. For simplicity, assume that C^l and D^l are either the fuzzy big or the fuzzy set small whose membership functions are shown in Fig. 5. Finally, assume that E^l can be the four fuzzy sets shown in Fig. 6.

According to [8], we have derived the fuzzy rules such that the error of the step response of a sample system becomes minimum. The fuzzy rules for K'_p, K'_d, and α are shown in Tables 1–3, respectively. We combine the 49 rules in each set using product inference engine, singleton fuzzifier, and center average defuzzifire; that is, parameters K'_p, K'_d, and α are tuned online according to equations (8)–(10).

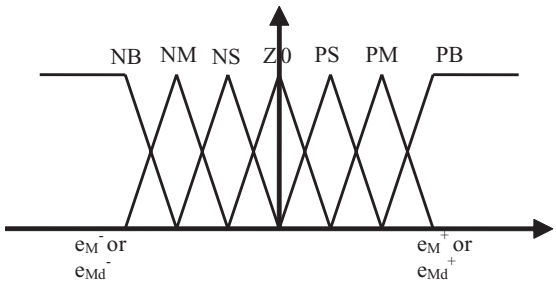

Fig. 4. Membership functions of e(n) and Δe(n)

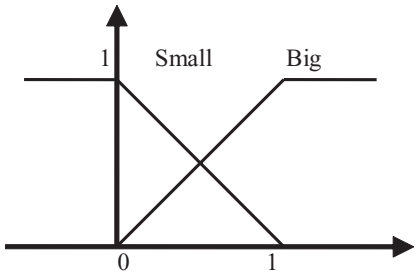

Fig. 5. Membership functions of K'_p, K'_d, and α

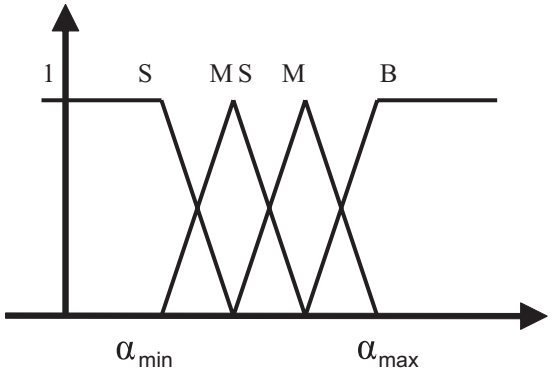

Fig. 6. Membership functions of α

Table 1. Fuzzy tuning rules for K'_p

		e(n)						
		NB	NM	NS	Z0	PS	PM	PB
Δe(n)	NB	B	B	B	B	B	B	B
	NM	S	B	B	B	B	B	S
	NS	S	S	B	B	B	S	S
	Z0	S	S	S	B	S	S	S
	PS	S	S	B	B	B	S	S
	PM	S	B	B	B	B	B	S
	PB	B	B	B	B	B	B	B

$$K'_p(n+1) = \frac{\sum_{l=1}^{49} y_p^{-l} \mu_{A^l}(e(n)) \mu_{B^l}(\Delta e(n))}{\sum_{l=1}^{49} \mu_{A^l}(e(n)) \mu_{B^l}(\Delta e(n))} \tag{8}$$

$$K'_d(n+1) = \frac{\sum_{l=1}^{49} y_d^{-l} \mu_{A^l}(e(n)) \mu_{B^l}(\Delta e(n))}{\sum_{l=1}^{49} \mu_{A^l}(e(n)) \mu_{B^l}(\Delta e(n))} \tag{9}$$

$$\alpha(n+1) = \frac{\sum_{l=1}^{49} y_\alpha^{-l} \mu_{A^l}(e(n)) \mu_{B^l}(\Delta e(n))}{\sum_{l=1}^{49} \mu_{A^l}(e(n)) \mu_{B^l}(\Delta e(n))} \tag{10}$$

6 Performance Evaluation

To evaluate the performance of the proposed controllers, we use the NS-2 simulator [9]. We have used four measures as the bases of comparison. These measures are overall throughput of TCP sources, channel utilization, loss probability, and number of duplicate acknowledges. We shall express these concepts first.

Table 2. Fuzzy tuning rules for K'_d

		e(n)						
		NB	NM	NS	Z0	PS	PM	PB
Δe(n)	NB	S	S	S	S	S	S	S
	NM	B	B	S	S	S	B	B
	NS	B	B	B	S	B	B	B
	Z0	B	B	B	B	B	B	B
	PS	B	B	B	S	B	B	B
	PM	B	B	S	S	S	B	B
	PB	S	S	S	S	S	S	S

Table 3. Fuzzy tuning rules for α

		e(n)						
		NB	NM	NS	Z0	PS	PM	PB
Δe(n)	NB	S	S	S	S	S	S	S
	NM	MS	MS	S	S	S	MS	MS
	NS	M	MS	MS	S	MS	MS	M
	Z0	B	M	MS	MS	MS	M	B
	PS	M	MS	MS	S	MS	MS	M
	PM	MS	MS	S	S	S	MS	MS
	PB	S	S	S	S	S	S	S

– *Overal Throughput of TCP sources*: This measure shows the amount of acknowledged packets. This quantity is in fact the average of instant throughput that is computed according to the (1). The overall throughput of TCP source is computed according to following equation:

$$\text{Overal Thoughput(Time)} = \frac{ACKNum \times PacketSize \times 8}{Time \times 1024 \times 1024} \tag{11}$$

– *Channel Utilization*: Channel utilization shows how much capacity of the link is used. This value is the ratio of the departed packets to the dropped packets and it is computed according to the following equation. The greater this value is close to 1 the more the link is utilized.

$$\text{Utilization(Time)} = \frac{DepartmentPackets \times PacketSize \times 8}{Time \times LinkSpeed} \tag{12}$$

– *Loss Probability*: This value shows the extent of dropped packets and it is computed according to this equation:

$$\text{Loss Probablity(Time)} = \frac{DropedPackets}{DepartmentPackets + DropedPackets} \tag{13}$$

– *Number of duplicate acknowledges*: The receiver creates duplicate acknowledge when it receives a packet out of order or in case of missing a single

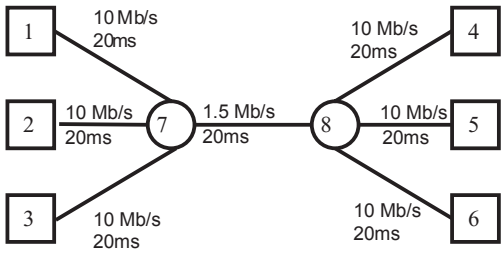

Fig. 7. Topology of simulation

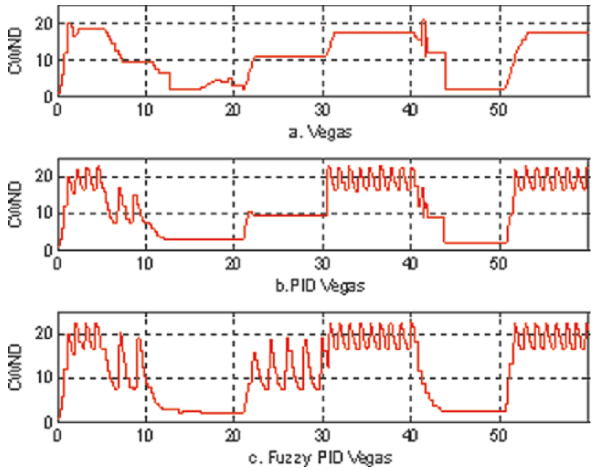

Fig. 8. CWND in (**a**) Ordinary Vegas (**b**) PID Vegas (**c**) Fuzzy PID Vegas

packet in the stream of packets. So if a TCP controls the CWND more efficiently, the number of duplicate acknowledges decreases.

The topology of the simulation is shown in Fig. 7. Bandwidth and delay of all links are shown. All links are duplex and their buffer management is Tail Drop. A TCP connection is set between the nodes 1 and 4 and the connection between the nodes 2 to 5 and 3 to 6 is UDP. An FTP application is connected to the node 1 and 2 CBR applications with rate .5 Mb/s are connected to the nodes 2 and 3. The node 1 starts sending data at the beginning of the simulation to the end. The nodes 2 and 3 starts at 5s. The node 2 stops at 15s and the node 3 stops at 20s. In this regard, a bottleneck is created between the nodes 7 and 8 in the duration 5s to 15s and we can examine the performance of protocols.

Figure 8 shows the CWND of TCP node for ordinary Vegs, PID Vegas, and Fuzzy PID Vegas. Figure 9–11 and Table 4 show the overall throughput,

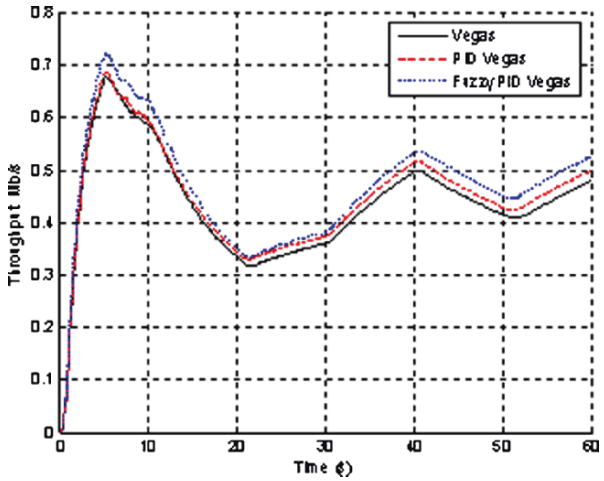

Fig. 9. Comparison of overall throughput of Ordinary Vegas, .PID Vegas, and Fuzzy PID Vegas

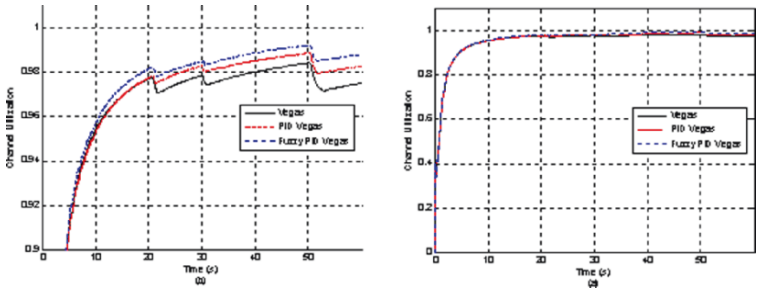

Fig. 10. Comparison of channel utilization of Ordinary Vegas, .PID Vegas and Fuzzy PID Vegas (fig b is magnification of fig a)

utilization and loss probability and number of double acknowledges respectively for three algorithms. As it is shown in Fig. 9, the throughput of the PID Vegas has been greater than that of the ordinary Vegas and the throughput of the Fuzzy PID Vegas has been greater than that of the PID Vegas. In Fig. 10 we have compared the channel utilization of three algorithms. This figure shows that the PID Vegas and the Fuzzy PID Vegas has utilized the link more efficiently. Figure 11 shows the trends of loss probability for the three algorithms. The proposed algorithms has been more success full to avoid losses. Table 4 shows the number of duplicate acknowledges for three algorithms. It is seen that the performance of the PID Vegas has been better than that of the ordinary Vegas and the Fuzzy tuning has improved the performance of the PID Vegas.

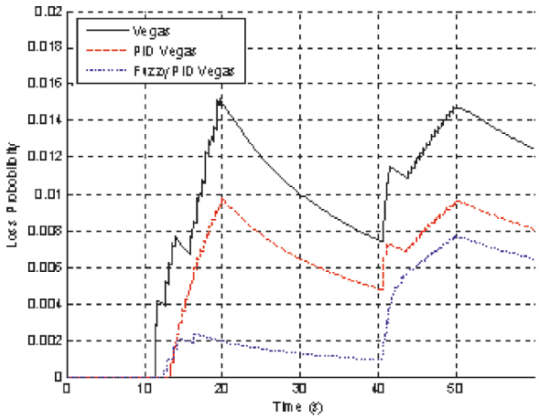

Fig. 11. Comparison of loss probability of Ordinary Vegas, .PID Vegas and Fuzzy PID Vegas

Table 4. Number of duplicate acknowledges

TCP	Number of Duplicate Acknowledges
Ordinary Vegas	1525
PID Vegas	613
Fuzzy PID Vegas	583

7 Conclusion

We reviewed the TCP Vegas algorithm and showed that there is a feedback control loop in its congestion avoidance phase which uses a simple On/Off controller to adjust the CWND. Then, we introduced the PID Vegas algorithm in which, a PID controller is used for tuning of CWND. The PID controller enables the algorithm to control the sending rate from the source such that more capacity of the network is utilized while it avoids congestion. Simulation results show that performance of the PID Vegas algorithm is better than ordinary Vegas. The reason is that PID Vegas is more flexible than ordinary Vegas and it is sensible to deviations in Delta. This enables the TCP to continually examine the network and increase the CWND if possible and at the same time if there is any problem in the network it rapidly decreases the CWND to avoid congestion.

Finally, we developed a Fuzzy methode to tune the parameters of the PID controller. Simulation results approve better performance of TCP as expected. Since the Fuzzy tuning of parameters is done no matter what the plant, it ensures that the algorithm behaves properly in unexpected network conditions.

8 Acknowledgment

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A Fuzzy-Based Automation Level Analysis in Irrigation Equipment

Mohsen Davoudi, Mohammad Bagher Menhaj, and Mehdi Davoudi

Summary. In underdeveloped countries because of old irrigation/agricultural equipment, there are some problems with automation. In some cases, automation is not efficient. In this paper, we introduce an idea to make automation plausible for an old equipment environment using fuzzy logic based analysis. The proposed fuzzy analyzer uses data collected from farm, climate, energy, etc to determine which equipment has enough reasons to be automated. The analyzer helps us decide where and when to switch from the mechanization level to the automation level.

Key words: Fuzzy control, Irrigation, Automation level.

1 Introduction

Agriculture has, throughout History, played a major role in human societies endeavors to be self-sufficient in food. However, water shortage has seriously impeded the attainment of such an objective. This is why, for Mankind, agricultural land irrigation has increasingly become a challenge and water resource control is a priority [1]. Around the middle of twentieth century a period started where design and operation of the irrigation systems were increasingly determined by scientific methods and techniques. This scientific approach was based on production objectives: optimization of yields and efficiencies. Soil, water, and plant relationship became the focus [2].

As a result of extensive research into this relationship, water requirements for crops at various growth stages under different climatic and soil conditions could be determined with a great accuracy [3].

Today, irrigation systems performance has been increased using mechanized equipment. A mechanized system needs automation to be more efficient because of correct and efficient management. In this context, new approaches are needed for more insight into ways of achieving greater efficiency at decision-making stages involved in irrigation and water resource management, in order to optimize the available irrigation systems and to help decision making for agriculture management.

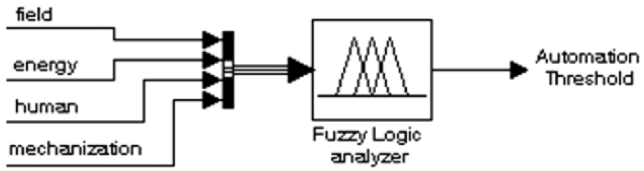

Fig. 1. A typical irrigation fuzzy analyzer

The rest of the paper is organized as follows. Section 2 introduces a procedure to calculate both automation level and automation threshold. In Sect. 3, we discuss the so called field parameter followed by Section 4 that presents the energy parameter. Section 5 describes the mechanization parameter. Section 6 is devoted to explain the fuzzy analyzer. Section 7 present the simulation results and finally, Section 8 concludes the paper.

2 Automation Level and Automation Threshold

Automation level is maximum capability of performing tasks automatically by machines. Automation level determination is the first step to determine the efficiency of the automation. Automation level is not dependent on environmental conditions, and it only depends on structure and specification of the equipment [4]. The second step is the calculation of the Automation Threshold (AT) for a given the farm or field where the equipment is installed.

Parameters such as geographical attributes, cost of energy resources (electricity and water), human resource accessibility, and existing level of mechanization in equipment are all effective in calculation of AT. Figure 1 shows a typical irrigation fuzzy analyzer [5].

The value of AT cannot be calculated precisely, because it depends on inputs such as management, economic, and technical problems that cannot be clearly quantified. Fuzzy analyzer employs specific rules commonly used by farmers, combines environment data to calculate AT value [6]. At last by comparing Automation Levels calculated in the first step and the AT value calculated in the second step, we can select the equipment that has technical and economical plausibility for performing automation plan in a field. The analysis of farmers' perceptions is based on the data generated using a farmer survey questionnaire [7].

The aims of each controller in the Irrigation system is flexibility, accuracy, simplicity, and reliability [8].

3 Automation Level Determination

In this paper, we consider a number of irrigation equipment. This method can be also generalized to other agricultural/irrigation equipment. In Table 1 irrigation equipment are categorized into different types. To determine Automation levels, we assign normalized scores to each item of equipment given in Table 2.

Table 1. Irrigation equipment

Local Irrigation (Drip)	Global Irrigation (Rain)
Drip	Center pivot
Bubbler	Linear
Spray	Wheel move
Tape	Classic
Subsurface	Traveling Gun

Table 2. Items of equipment

Item	Score
Accessible electricity existence in the equipment	0 - 10
Control level of the electricity box (if exists)	0 - 10
Cost of automation implementation	0 - 10
Data transmission capability (wired /Wireless/none)	0 - 10
Equipment's Distribution Uniformity (DU)	0 - 5
Irrigation frequency by this equipment	0 - 10
Alarm based installed sensors and auto stop	0 - 5
Water supply (Fixed pipe/Canal/Hydrant)	0 - 10
Movement structure (Fix/Pivot/Linear move)	0 - 10
Movement driver (Fix/Electromotor/Diesel motor/manual)	0 - 10
Cost of mechanized system	0 - 10

Each item in Table 2 corresponding to each equipment has a specific score; therefore, the total score for equipment can be computed. The total value quantifies the Automation level. Typically, for the Center pivot, Hose Fed Linear, Canal Fed Linear, Wheel Move, Drip irrigation and Turf, the values of Automation level, respectively, are: 100, 48, 70, 33, 77, and 87. No matter what the environmental conditions are, the automation level only depends on the specifications of the equipment. The interactions between Automation level and farm conditions are formulated by fuzzy inference systems developed in the paper and will be used in the fuzzy analyzer. Now, we briefly introduce the parameters important in determination of the AT.

4 Energy Resources

Electricity power and water resources are two important factors which play key role in automation of the agricultural equipment. Rising from mechanization level to automation level needs applying control methods in water resources. Variable Frequency Drive (VFD) on irrigation pumping plant is an effective approach for supplying water in irrigation equipment with minimum power consumption [8]. In order to stabilize water pressure stabilization into the irrigation pipes, which is one of the automation duties, the relationship

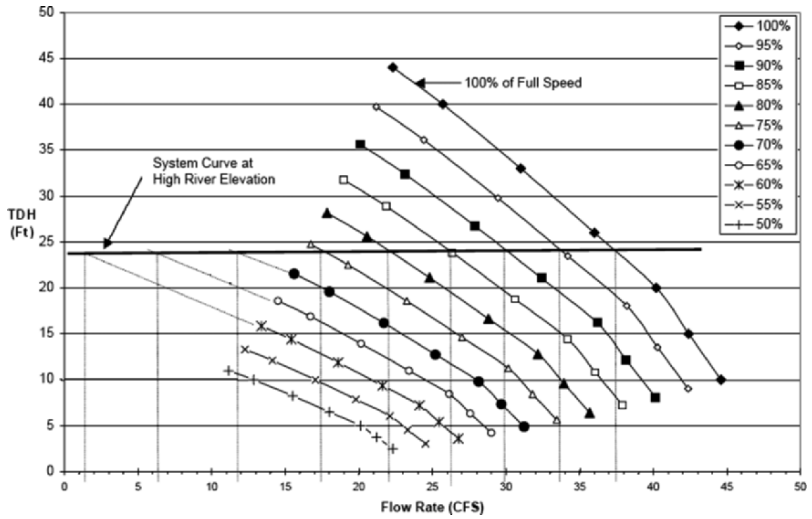

Fig. 2. Variable Frequency Drive on pumping plant

between "change in pump speed" and "change in Q" must be known. In practice, there is neither a constant nor a precisely known relationship because (I) the water level in the pump changes considerably during the year, and (II) pump characteristics can change with time. Figure 2 illustrates the pump and system curves.

This paper defines five fuzzy term sets for the energy cost factor: "Expensive, Expensive-medium, Medium, Cheep-medium, Cheep". These fuzzy sets cover the nonlinearity and changes in irrigation pump plants and indirectly effects on the Automation Threshold which is the basis for selection of equipment to be automated.

Statistics shows that implementation of the mechanized irrigation system reduces the water consumption to $5400 m^3/hectare$. For example, in nonmechanized systems it may be $11000 m^3/hectare$. Automation improves the management of water and farm and increases the efficiency of the mechanized system.

5 Field

Climatic factor in the agricultural field or farm where irrigation equipment are installed is the second parameter that is studied in this paper. This parameter is another input of the fuzzy analyzer. Table 3 illustrates how the major climatic factors affect crop water needs.

The AT value directly depends on the Climatic factors. Typically fuzzy sets for field parameter are: "Dry, Dry-medium, Medium, Wet-medium, Wet". Figure 3 shows map of a typical area which can be divided into these area sets. Generally the map of areas is available and fuzzy sets can be extracted simply on map [9]. Here, we describe how humidity affects from the irrigation

Table 3. Effects of major climatic factors on crop water needs

Climatic Factor	Crop water need (High)	(Low)
Temperature	hot	cool
Humidity	low (dry)	high (humid)
Wind speed	windy	little wind
Sunshine	sunny (no clouds)	cloudy (no sun)

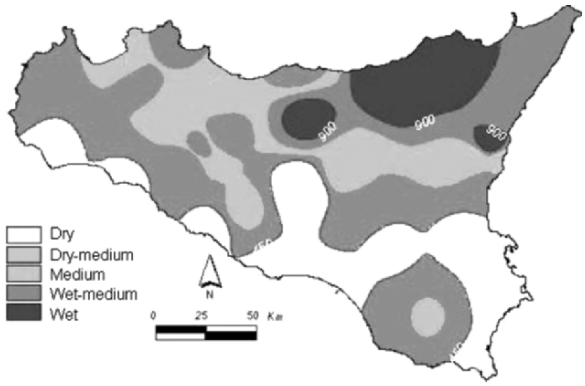

Fig. 3. Map of a typical area

management and automation. The basic equation for Evapotranspiration (ET) based scheduling is:

$$ET_{crop} = Kc \times ETo \tag{1}$$

ET_o the crop evapotranspiration, usually measured in inches/day, ETo is the evapotranspiration of a reference crop (usually either alfalfa or grass) Kc is a crop coefficient, and is a dimensionless number. Kc tells how much water your crop uses in comparison with the reference crop [10]. Much research has allowed us to estimate ETo based on four weather variables which are shown in Table 3.

The weather may vary from day to day and can vary with location. But for a given set of weather conditions, ETo can be used for all crops grown on that day in that location. Kc will vary with crop stage of growth (time of the season), so we can use Kc at all locations where that crop is grown. More precisely Kc also depends on irrigation management, which may not be the same at all locations. However we often assume the "typical" or "normal" irrigation practice is being used, so Kc is regarded as being transferable from one location to another. Kc is subject to two adjustments that irrigation management can affect. One has to do with a wet soil surface after an irrigation, and the other has to do with possible soil moisture stress in the last days before an irrigation [11]. The basic equation for adjusting Kc is:

$$Kc = Kcb \times K_{stress} + K_{evap} \tag{2}$$

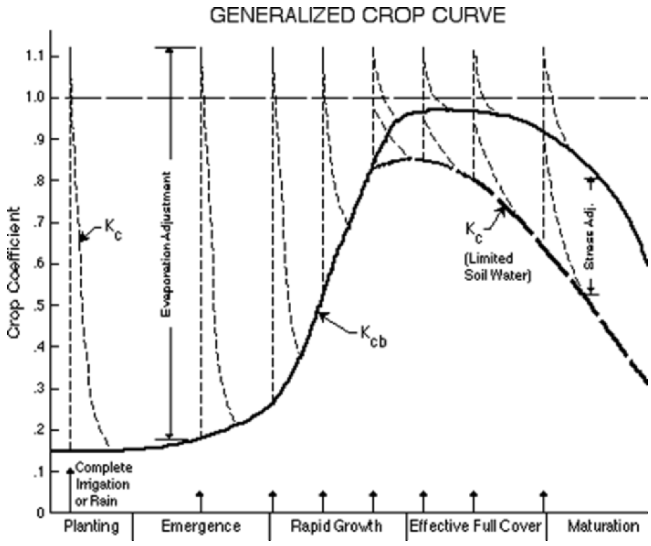

Fig. 4. Crop coefficient variations

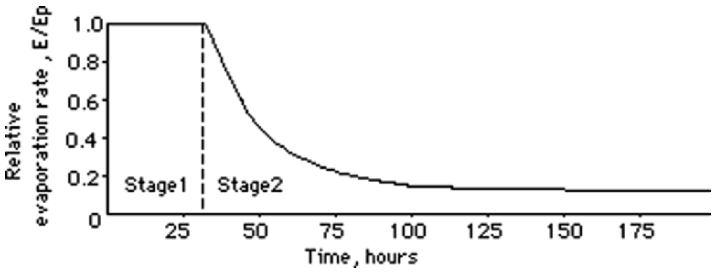

Fig. 5. Stages of the evaporation

Kcb is called the "basal crop coefficient." Kcb is defined at the crop coefficient that will be observed when the soil surface is dry (no soil surface evaporation), and there is adequate moisture in the root zone to prevent any decrease in the relative ET (Fig. 4). K_{stress} is an adjustment factor that makes Kc decrease when the soil is so dry that relative ET is decreased. K_{evap} accounts for the extra water use due to evaporation from a wet soil surface. Evaporation from a wet soil surface is at its maximum rate for about 1 day after the soil surface is wetted, and then the evaporation amount decreases in stage 2 evaporation, as shown in Fig. 5.

As stage 2 evaporation decreases to essentially zero, $K_{evap} = 0$. Note that when the soil surface is dry and there is enough moisture in the root zone so that the relative ET is at 100 percent, $K_{sress} = 1.0$. Under these conditions, thus:

$$Kc = Kcb \times 1 + 0 \Rightarrow Kc = Kcb \tag{3}$$

"When to irrigate?" is a TIME question. Our basic expression for TIME questions is this:

$$Time = Amount/Rate \tag{4}$$

so, for irrigation time,

$$IrrigationFrequency = \frac{SMD_{max}, (in)}{ET_{crop}, (in/day)} \tag{5}$$

SMD = Soil Moisture Depletion.

Above equations show dependence between climatic factors and irrigation frequency which is done by means of irrigation equipment under management [1,12]. There is no clear relationships that can be analytically formulated between climatic factor and other parameters like field, etc.

6 Mechanization

Irrigation System Effectiveness, Water Use Index (WUI) and Agricultural Water Productivity (AWP) are the main objective of the irrigation management. Irrigation System Effectiveness depends on the mechanization type. Mechanization type affects the irrigation depth and water infiltration into soiled.

$$WUI = \frac{Crop(AnyUnit)}{Water(AnyUnit)}, DU = \frac{MDWA}{ADWA} \tag{6}$$

MDWA= Minimum Depth of Water Accumulated in an Element.

ADWA=Average Minimum Depth of Water Accumulated in all Elements Statistics shows that in a mechanized irrigation system $WUI = 1.7Kg/m^3$ but in a nonmechanized system $WUI = 0.79Kg/m^3$ in an underdeveloped country. DU is the ratio of the "minimum" to the "average" amounts of water received by plants in a field or system [13].

The water destination diagrams shown in Fig. 6 display the fate of the water that has infiltrated into the ground, however not all the water pumped is absorbed by ground. There is runoff, evaporation, and spray loss according to irrigation type. It is customary to show the amount of these pre-infiltration losses above the Depth = 0 line, as shown in Fig. 6.

The Pre-Infiltration Loss ($PILoss$) is usually indicated as a percent of the gross amount applied, or in inches if known directly. The mechanization system determines how much water will be absorbed by plant reflected by the value of DU. Runoff is included in pre-infiltration losses but is handled separately because it may be recovered for re-use [14].

7 Fuzzy Analyzer

A very wide range of applications of fuzzy logic based methodology has been found in the literature especially for industrial control systems which are very

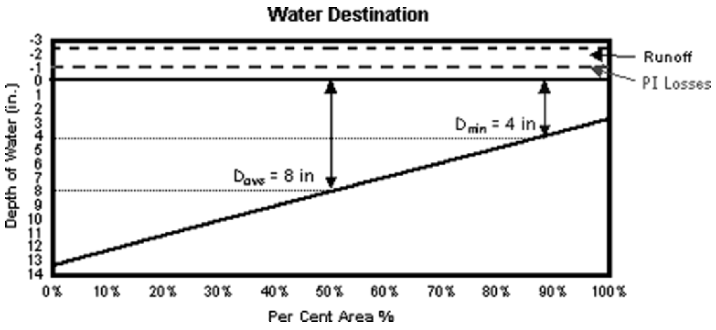

Fig. 6. Water destination diagram

complex, uncertain and cannot be modeled precisely even under various assumptions and approximations. Here, we introduce another application of fuzzy rule base system to irrigation.

The fuzzy analyzer designed in the paper consists of some rules which are derived from relationships between soiled/water/crop, perception of the farmer using questionnaire and managers considerations in irrigation systems. After a deep look at the information gathered from different questionary sheets and resources, we came up with a set of 42 fuzzy if-then rules. Some samples of these rules are given below.

R^1: IF(F is dry) and (E is expensive) and (H is cheep) and (M is medium), then (AT is High)
R^2: IF(F is wet) and (E is cheep) and (H is medium) and (M is high), then (AT is Low)
R^3: IF(F is wet) and (E is cheep) and (H is medium) and (M is medium), then (AT is Medium)

Where Dry, Dry-medium, Medium, Wet-medium, and Wet are linguistic terms of antecedent fuzzy sets for the Field parameter (F). Energy (E), human (H), and mechanization (M) have their own fuzzy sets else. Fuzzy set of AT is: Low Low, Low, Medium, High, and High High. We use a general form to describe these fuzzy rules:

R^i: IF(F is x1) and (E is x2) and (H is x3), and (M is x4), then (AT is y), i = 1..42

Where x1, x2, x2, x4 are triangle-shaped fuzzy number and y is fuzzy singleton.

Let first X and Y be the input and output space, and F, E, H, M be arbitrary fuzzy sets in X. Then a fuzzy set, $[F, E, H, M] \circ R^i$ in Y, can be determined by each R^i. We use the sup-min compositional rule of inference:

$$m^i_{AT^i} = \mu_{F^i}(x1).\mu_{E^i}(x2).\mu_{H^i}(x3).\mu_{M^i}(x4), i = 1..42 \tag{7}$$

By using the center of area (centroid method) defuzzifier, we can obtain a crisp output AT:

$$AT = \frac{\sum m^i_{AT^i} \cdot \bar{y}^i}{\sum m^i_{AT^i}}, i = 1..42 \tag{8}$$

where $\bar{y}^i$ is center of the AT^i area.

$3-D$ surfaces of Fig. 7 show relationship between rules which considered above. In order to demonstrate these relationships, each surface consists of two parameters and rules that create the surface [6]. Three typical surface are shown in Fig. 7.

8 Simulation Results

In the simulation, we considered six farms with different conditions shown in Fig. 8. The first farm is a dry farm with mechanized equipment whose cost of energy and human resources are very high. The fuzzy analyzer suggests us to go to a higher level of automation because the value of AT obtained is low. The AT values of other farms are shown in Fig. 9 and 10.

Based on these results the equipment for automation should be selected as: Center Pivot, Canal fed linear, Turf, and Drip irrigation. The analyzer also indicates that the above equipment have enough management/economic/technical justification for the automation.

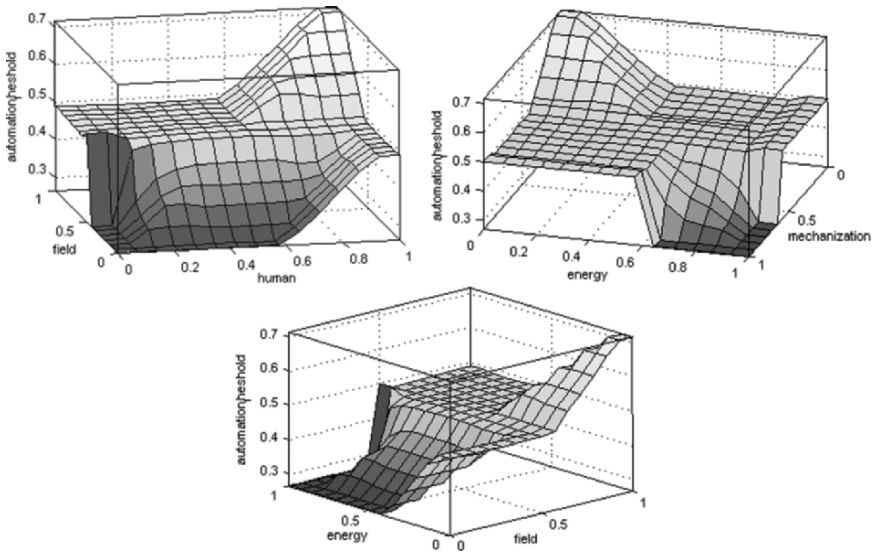

Fig. 7. $3-D$ surfaces of the fuzzy rules

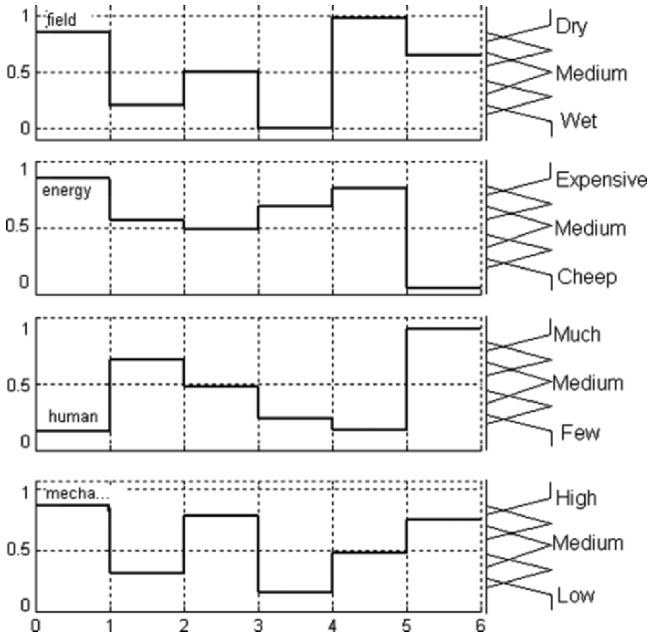

Fig. 8. Curve of the supposed farms with different conditions

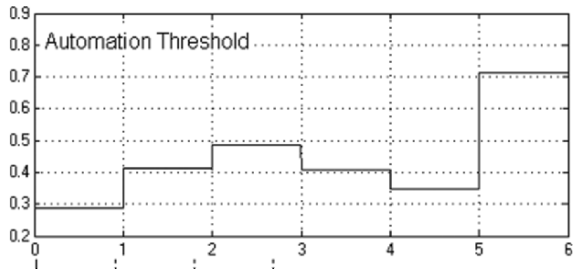

Fig. 9. Values of AT for the given farms

9 Conclusion

This paper conclusively reached to the point that any satisfactory operation of an automated irrigation system, particularly in a water-poor environment, is complex and requires realistic planning and rigorous implementation. This has been achieved by developing proper fuzzy based decision making process. We developed a software in MATLAB environment for the purpose of simulation. Some typical farms have been considered and implemented in our toolbox. The simulation results approved the aforementioned claim.

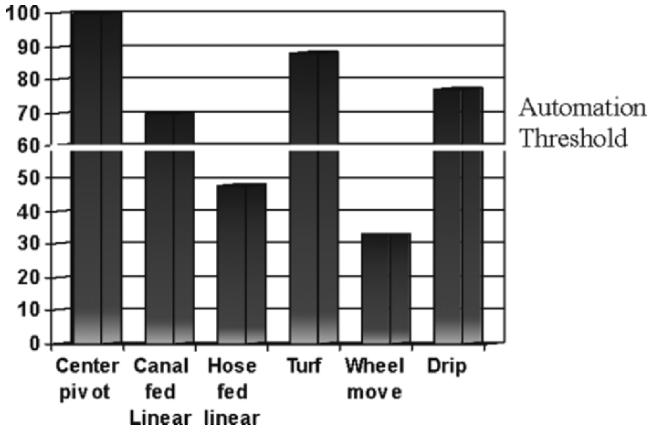

Fig. 10. Comparison of the AT with automation levels

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Motorized Skateboard Stabilization Using Fuzzy Controller

Mohsen Davoudi, Mohammad Bagher Menhaj, and Mehdi Davoudi

Summary. In this paper, a fuzzy tuner is designed for a motorized skateboard to tune KI and KP coefficients of the PI controller to stabilize the skateboard rider. The fuzzy if-then rules are derived from physical reactions of body against external forces. The PI controller tracks the set point chosen by riders through a handle and applies a proper force to the skateboard to keep dynamic equilibrium of the person stable during the travel. In this paper, through different simulations, it has been shown that the proposed controller make the system track the set point as quickly as possible while having a remarkably bigger traveling distance without any sort of instability problem.

Key words: Skateboard, Fuzzy controller, Stability, Motor scooter.

1 Introduction

Motor scooter is a two-wheeled motor vehicle similar to a motorcycle or two-wheeled children's vehicle resembling a skateboard with a handlebar. Motorized skateboard is a four-wheeled vehicle. The percentage of people do Skate boarding is 5 percent [1]. Staff from the US Consumer Product Safety Commission (CPSC) recently conducted a special study to track injuries associated with powered scooters, a recreational product growing in popularity. From July 2003 through June 2004, an estimated 10,015 powered scooter-related emergency room-treated injuries were reported through CPSC's National Electronic Injury Surveillance System (NEISS) [1].

Usually control of a motorized skateboard is done by a handle. However for many people it is very difficult to learn how to control the skateboard and keep balance [2]. The concept which is outlined in this paper is controlling a Motorized skateboard with a person standing on to keep dynamic equilibrium easily and increasing velocity and traveled distance as well. Suppose that the skateboard has an electric motor, battery pack, a handle containing Start/Stop button which is run on a standard road. The control of Skateboard consist of two main parts (1) determination of the physical parameters of the person standing on the skateboard, (2) changing the parameters of the PI controller

using data gathered in Section 1. Physical parameters of a person needed in this analysis are weight, flexibility of the joints, length of every organ of the body such as shank, leg, chest, head, etc. [3]. To reach these objectives, we design a system to estimate amount of changes in the PI controller using the data driven from sensors on the skateboard and a fuzzy tuner.

In the first step, a pulse wave is sent to the driver of the electric motor to run the motor in a limited time, e.g., 5 s. This action leads to a pulse like force to both the skateboard and the person making a movement. The person reacts to this force and starts to oscillate. Every one has its own specific reaction [4]. It depends on age, flexibility of joints, length, weight, etc. [5]. Then, the system records the velocity signal during the time. Practically the velocity signal can be obtained through a shaft encoder sensor installed in a wheel of the skateboard. Finally, by processing the velocity signal, getting information needed for the fuzzy tuner and applying some rules in the Fuzzy Inference System (FIS), the fuzzy tuner changes the coefficients KP and KI in the PI controller. The rules are derived from practical skateboarding situations, physics of motion and perception of the people reaction on the skateboard, see figure 1.

2 Skateboard Model

The modeled skateboard has 50 cm length and 5 kg weight. A general model is used for skateboard based on Euler Springs. Euler Springs model stores no static energy in the skateboard and dynamic energy causes vibration in motion as shown in figure 2. The resonant frequency is:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}} \tag{1}$$

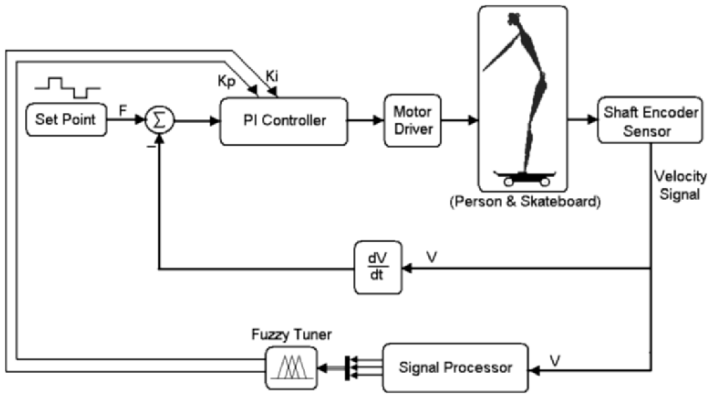

Fig. 1. The block diagram of the skateboard fuzzy controller

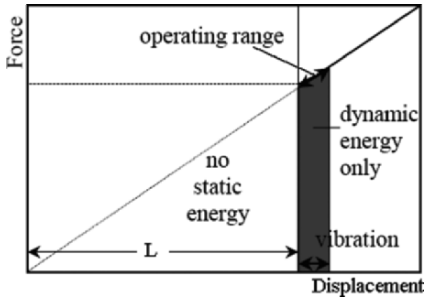

Fig. 2. Dynamic energy and vibration in motion

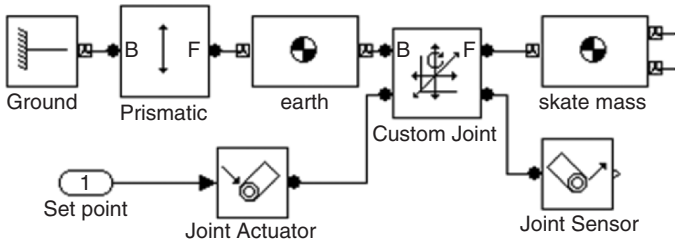

Fig. 3. Skate model implemented in MATLAB/Sim mechanics

where k is the coefficient of the springs connected to the skate (related to the body model), m is the weight of the skateboard ($m = 5\,\text{kg}$), g is the Gravity and L is the displacement. Because of puddles on the road the dynamic energy is changed during the time. The resonant frequency usually is above 10 Hz ($\omega \geq 10\,\text{Hz}$). This frequency is seen in the velocity signal but does not give us any useful information about the body parameters.

Signal processor contains a filter to eliminate the above frequencies. Figure 3 shows the skate model implemented in MATLAB/Sim mechanics. This model contains an actuator (motor driver) and a sensor (shaft encoder).

3 Body Model

The postural balance system is one of the most fundamental functions for human voluntary motion. This system has been analyzed and modeled by many researchers in the past. In the field of biomechanics, many researchers work on the human balance control. Some of them investigated features of balance control of the real human [3, 6].

They actually applied perturbations to the real human, and measured the force, the velocity, or several physical parameters. Others investigated the motion of balance recovery by stepping [6, 7]. In these researches, the motion of the real human is analyzed. For postural balance, Horak et al. [8] found that

human use three strategies for keeping the balance, the ankle, hip, and arm strategy. The ankle strategy is a strategy to use mainly the ankle joint to restore the position of the center of gravity back to the equilibrium state. This strategy is chosen when the foot surface is long enough relative to the foot length, so that the subject can fully use his/her toes to push back the body. When the foot surface is short relative to the foot length, the subject uses the hip and trunk joint to keep the balance, which is called the hip strategy.

Kuo et al. [9] theoretically analyzed such strategies using the musculoskeletal model. When a person is about to lose his/her balance, and is under a condition that he/she cannot step out one leg, the arms are recursively rotated to work as the final servo to move the center of mass back over the feet. This strategy is effective in returning the center of mass over the feet, because the angular momentum of the trunk of the body is canceled out by the angular momentum generated by the rotation of the arms [10]. In this paper, we propose a new human body model which is composed of five body segments, the shank, thigh, trunk, head, and arm.

The human-like body motion is obtained by rigid body, spring, and damper which are used for every joint in our body model. The arm strategy appears without any prior feed-forward input when large perturbation force is applied to the body. The motion of recovery closely resembles those by real human. Figure 4 shows the model of a sample joint in MATLAB/Sim mechanics. Other joints are similar to this joint. In order to model the above three strategies, feedback force is applied to the ankle, hip, and arm joints.

4 Signal processor

The velocity signal is only reachable signal that is derived from shaft encoder sensor practically. Velocity signal is given to two Band Pass Filters (BPF). Therefore, the vibration frequency is divided into two frequency bands (a) 2–4 Hz, (b) 4–10 Hz.

The first band indicates the low frequency vibration of the body and the second indicates high frequency vibration. Relationships between signal parameters in these two bands help us find out the person's body parameters and lead to model them into IF-THEN fuzzy rules. For example, we found

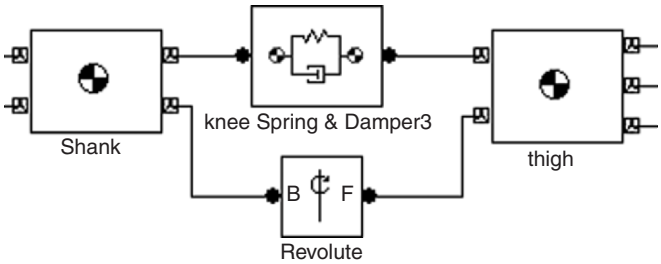

Fig. 4. model of a sample joint in MATLAB/Sim mechanics

out that the signal of a short strong person on the skate in band (b) has a higher value of energy in comparison with band (a) because of rapid reactions to changes. In this case, the skate can be run with a higher speed by means of increasing the KI coefficient. For a tall weak inflexible person that has slow reactions, energy of band (a) is higher than energy of band (b). So a minimum value of KI and KP is needed to control the motion equilibrium. We first calculate the energy of the velocity signal in these bands:

$$S_1 = (\int f_1^2 dt)^{1/2} \tag{2}$$

$$S_2 = (\int f_2^2 dt)^{1/2} \tag{3}$$

where f_1 is the output of the 2–4 Hz BPF and $S1$ is energy of the signal in this band. f_2 is output of the 4–10 Hz BPF and S2 is energy of the signal in this band. Then, we count zero crossings in f_1 and f_2 signals within the experiment time to obtain the zc_1 value and the zc_2 value, respectively. The following parameters

$$A = \frac{zc_1}{zc_2} \tag{4}$$

$$B = \frac{S_1}{S_2} \tag{5}$$

$$C = \frac{S_1}{zc_1} + \frac{S_2}{zc_2} \tag{6}$$

are used to tune KP and KI by fuzzy tuner based on the aforementioned rules [10]. The next step is to design the fuzzy tuner which is strongly dependent upon the physical reactions of the rider. This information helps us determine fuzzy sets boundaries on every joint's position and organs' weights. The Fuzzy tuner which is outlined in the next section has fixed rules.

5 Fuzzy Tuner

In this section, a fuzzy tuner is introduced to tune the PI controller. This tuner is described by the Following set of IF-THEN rules [11,12].
R^1: IF (A is SMALL) and (B is SMALL) and (C is HIGH), then (KI is H), (KP is H)
R^2: IF (A is SMALL) and (B is SMALL) and (C is SMALL), then (KI is M), (KP is M)
R^3: IF (A is SMALL) and (B is HIGH) and (C is HIGH), then (KI is H), (KP is LL)
R^4: IF (A is HIGH) and (B is HIGH) and (C is SMALL), then (KI is L), (KP is LL)
R^5: IF (A is HIGH) and (B is SMALL) and (C is MEDIUM), then (KI is HH), (KP is H)

In the above SMALL, MEDIUM, HIGH, and LL, L, M, H, HH are linguistic terms of antecedent fuzzy sets, and in the then parts LL, L, M, H, HH stand for very low, low, medium, high, very high, respectively. We use a general form to describe these fuzzy rules [13]:

R^i: IF (A is x1) and (B is x2) and (C is x3), then (KI is y1), (KP is y2), i = 1,...,16

where $x1, x2, x3$ are triangle-shaped fuzzy numbers and $y1, y2$ are fuzzy singletons. An arbitrary fuzzy set A is depicted in Figure 5. Figure 6 represents the term sets of the output linguistic variable KI.

Let X and Y be the input and output space, and A, B, C be arbitrary fuzzy sets in X. Then, a fuzzy set, $[A, B, C] \circ R^i$ in Y, can be determined by each R^i. We use the sup-min compositional rule of inference [13, 14]:

$$m^i_{KI^i} = \mu_{A^i}(x1).\mu_{B^i}(x2).\mu_{C^i}(x3) \tag{7}$$

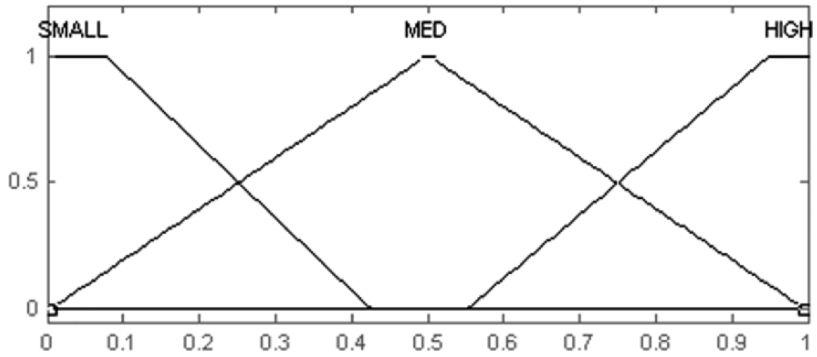

Fig. 5. An arbitrary fuzzy set A

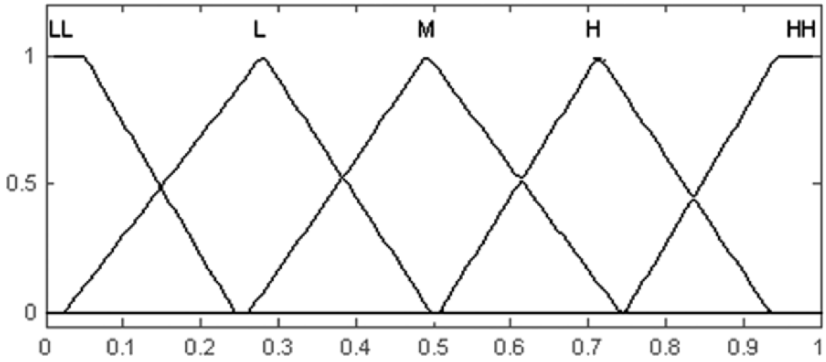

Fig. 6. Term sets of output linguistic variable

$$m^i_{KP^i} = \mu_{A^i}(x1).\mu_{B^i}(x2).\mu_{C^i}(x3) \tag{8}$$

By using the center of area defuzzifier, we can obtain a crisp outputs KI, KP:

$$KI = \frac{\sum m^i_{KI^i} \cdot \bar{y}^i}{\sum m^i_{KI^i}}, i = 1, ..., 16 \tag{9}$$

where $\bar{y}^i$ is center of the KI^i area.

$$KP = \frac{\sum m^i_{KP^i} \cdot \bar{y}^i}{\sum m^i_{KP^i}}, i = 1, ..., 16 \tag{10}$$

where $\bar{y}^i$ is center of the KP^i area.

Figure 7 shows a 3-D surface plot of the above rules.

6 Simulation Results

In this section we study four types of person's body shown in Table 1 [15,16].

The objective here is the PI controller makes the skateboard track the desired trajectory indicated by set point signal in block diagram shown in figure 1.

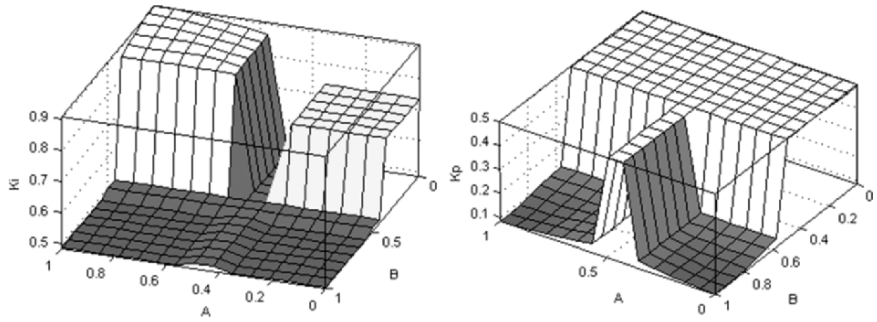

Fig. 7. surfaces of the fuzzy rules

Table 1. Four type of person's body

Number	Tallness	Weight	Strength
1	180 cm	74 kg	Average
2	160 cm	74 kg	Strong
3	190 cm	66 kg	Weak
4	140 cm	74 kg	Strong

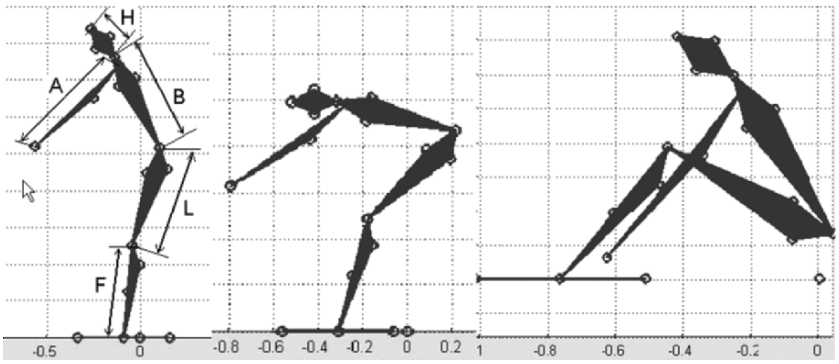

Fig. 8. A person on the skateboard in three states

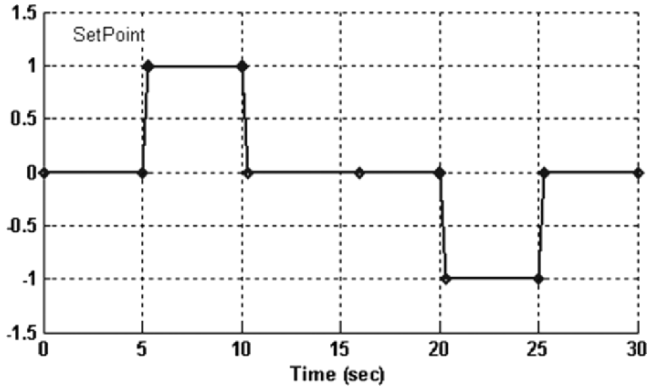

Fig. 9. The Set point signal

As the set point signal increases, the risk of instability increases though it is dependent on the rider physical parameters [17, 18]. Figure 8a shows a person in equilibrium state, (b) is a person in the threshold equilibrium state, and (c) is a person in tumble state. In this experiment a positive pulse applied in t = 5 s with a 5 sec duration ($altitude = 1$) and a negative pulse applied in t = 20 s with 5 s duration ($altitude = 1$) (Figure 9)

The experiment contains a case in which an over exciting force leading to an immediate instability condition is applied. Figure 10 shows the open loop simulation results for the first person whose parameters are given in Table 1. Here we assumed that the traveled distance in 30 s is 8 m. Figure 11 shows a closed loop simulation results for $KP = 1$, $KI = 0.2$. In this case the traveled distance becomes 25 m.

Figure 12 shows the simulation results (velocity signal) when the parameters Kp and Ki which are adjusted by the proposed fuzzy tuner. In this

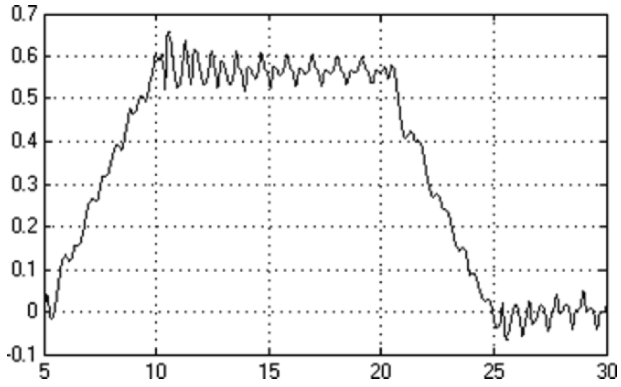

Fig. 10. Velocity signal (without using any feedback controller)

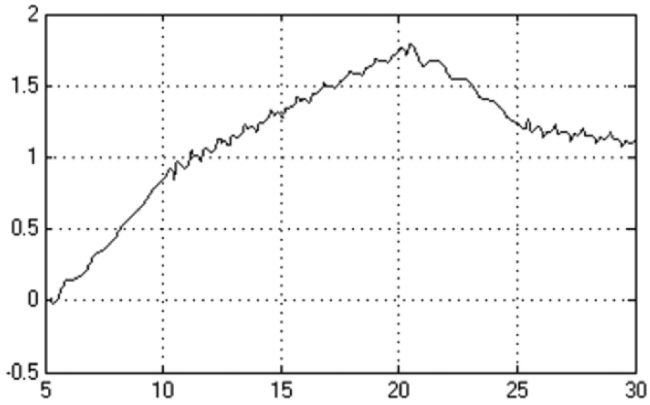

Fig. 11. Velocity signal (PI controller with constant parameters)

case the traveled distance becomes 51 meters in 30 s. Figure 13a shows the trajectory of controller parameter KI and figure 13b shows the trajectory of controller parameter KP.

7 Conclusion

In this paper, a Fuzzy tuner has been developed for a motorized skateboard. The tuner adjusts the parameters of a PI controller online. The rule base designed for the tuner came from reactions of riders against external forces so that to keep dynamic equilibrium state. As the set point signal increases, the risk of instability increases. Thus, the PI controller tracks the desired set

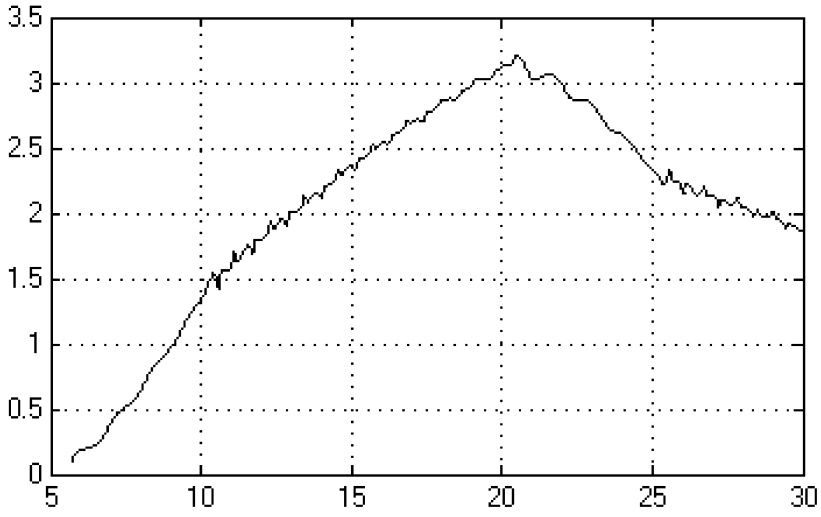

Fig. 12. Velocity signal (with fuzzy tuner)

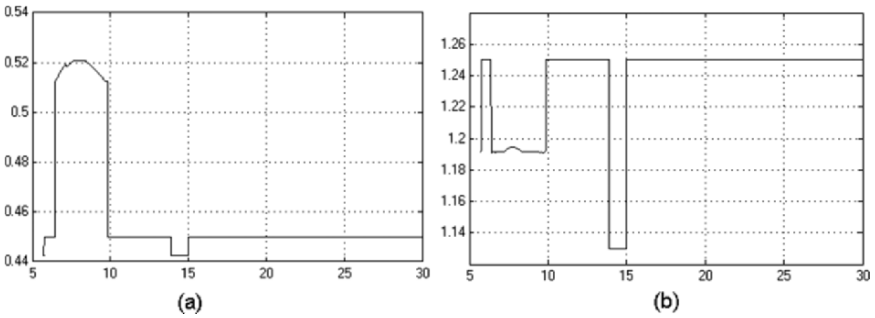

Fig. 13. KI (**a**) and KP (**b**) coefficients

point signal by applying proper forces to the skateboard to keep dynamic equilibrium state of the rider during traveling.

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