Variance Decomposition of Fuzzy Random Variables

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Summary. The conditional variance of random variables plays an important role for well-known variance decomposition formulas. In this paper, the conditional variance for fuzzy random variables and some properties of it are considered. Moreover possible applications of the variance decomposition formula are presented.

Key words: Fuzzy random variable, conditional variance, variance decomposition

1 Introduction

Conditional expectation and conditional variance play an important role in probability theory. Let be X a random variable on the probability space $(\Omega, \mathfrak{F}, P)$ and $\mathfrak{A} \subseteq \mathfrak{F}$ a sub-σ-algebra of $\mathfrak{F}$. Then the conditional expectation $\mathbf{E}(X|\mathfrak{A})$, for example, is the best mean squared approximation (best prediction) of X by a more rough, i.e. only $\mathfrak{A}$-measurable function.

Conditioning is one of the principles of variance reduction, i.e. the "more rough" random variable $\mathbf{E}(X|\mathfrak{A})$ has a smaller variance than X,

$$\mathbf{Var}(\mathbf{E}(X|\mathfrak{A})) \leq \mathbf{Var}X.$$

The difference $\mathbf{Var}X - \mathbf{Var}(\mathbf{E}(X|\mathfrak{A}))$ can be expressed mainly by the conditional variance of X which is defined by

$$\mathbf{Var}(X|\mathfrak{A}) = \mathbf{E}((X - \mathbf{E}(X|\mathfrak{A}))^2|\mathfrak{A}) \tag{1}$$

and which leads to the well known formula of variance decomposition

$$\mathbf{Var}X = \mathbf{E}(\mathbf{Var}(X|\mathfrak{A})) + \mathbf{Var}(\mathbf{E}(X|\mathfrak{A})). \tag{2}$$

This formula plays an important role in applications (see section 4).

Very often we meet the situation where the random variables X has only fuzzy outcomes. E.g. if an insurance company is interested in the claim sum X of the next

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year, it would be wise to assume fuzzy claims. Or if we interested in the state X of health in a country the society of which is stratified e.g. wrt age or wrt to social groups then it seems to be a violation to restrict X on numbers. Linguistic expressions, however, lead more or less straightforward to a fuzzy valued X.

In section 2 we introduce necessary tools like frv's and their expectation and variance. In section 3 the conditional variance of a frv and some of its properties are investigated and in section 4 we discusses possible applications of the variance decomposition formula.

2 Preliminaries

A fuzzy subset $\widetilde{A}$ of $\mathbb{R}^n$ is characterized by its membership function $\mu_{\widetilde{A}} : \mathbb{R}^n \to [0,1]$ where $\mu_{\widetilde{A}}(x)$ is interpreted as the degree to which $x \in \mathbb{R}^n$ belongs to $\widetilde{A}$. The α-cuts of $\widetilde{A}$ for $0 < \alpha \leq 1$ are crisp sets and given by $\widetilde{A}_\alpha := \{x \in \mathbb{R}^n : \mu_{\widetilde{A}}(x) \geq \alpha\}$. Additionally, we call $\widetilde{A}_0 := \mathrm{cl}\{x \in \mathbb{R}^n : \mu_{\widetilde{A}}(x) > 0\}$, the support of $\widetilde{A}$.

Let $\mathscr{K}_c(\mathbb{R}^n)$ be the space of nonempty compact convex subsets of $\mathbb{R}^n$ and $\mathscr{F}_c(\mathbb{R}^n)$ the space of all fuzzy sets $\widetilde{A}$ of $\mathbb{R}^n$ with $\widetilde{A}_\alpha \in \mathscr{K}_c(\mathbb{R}^n)$ for all $\alpha \in (0,1]$. Using Zadeh's extension principle, addition between fuzzy sets from $\mathscr{F}_c(\mathbb{R}^n)$ and scalar multiplication (with $\lambda \in \mathbb{R}$) is defined as

$$\mu_{\widetilde{A}\oplus\widetilde{B}}(z) = \sup_{x+y=z} \min(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(y)) \quad ; \quad \mu_{\lambda\widetilde{A}}(x) = \mu_{\widetilde{A}}\left(\frac{x}{\lambda}\right), \lambda \neq 0.$$

Note that with Minkowski addition $\oplus$ between sets from $\mathscr{K}_c(\mathbb{R}^n)$ it holds

$$(\widetilde{A} \oplus \widetilde{B})_\alpha = \widetilde{A}_\alpha \oplus \widetilde{B}_\alpha \quad \text{and} \quad (\lambda\widetilde{A})_\alpha = \lambda\widetilde{A}_\alpha.$$

For $A \in \mathscr{K}_c(\mathbb{R}^n)$ the support function s_A is defined as

$$s_A(u) := \sup_{a\in A} a^T u \quad , \quad u \in \mathbb{S}^{n-1},$$

where $a^T u$ is the standard scalar product of a and u and $\mathbb{S}^{n-1} = \{t \in \mathbb{R}^n : ||t|| = 1\}$ the $(n-1)$-dimensional unit sphere in the Euclidean space $\mathbb{R}^n$. An natural extension of the support function of a fuzzy set $\widetilde{A} \in \mathscr{F}_c(\mathbb{R}^n)$ is:

$$s_{\widetilde{A}}(u,\alpha) = \begin{cases} s_{\widetilde{A}_\alpha}(u) & : \alpha > 0 \\ 0 & : \alpha = 0 \end{cases}, \quad u \in \mathbb{S}^{n-1}, \alpha \in [0,1].$$

Each fuzzy set $\widetilde{A} \in \mathscr{F}_c(\mathbb{R}^n)$ corresponds uniquely to its support function, i.e. different fuzzy subsets from $\mathscr{F}_c(\mathbb{R}^n)$ induce different support functions and for $\widetilde{A}, \widetilde{B} \in \mathscr{F}_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}^+$ it holds

$$s_{\widetilde{A}\oplus\widetilde{B}} = s_{\widetilde{A}} + s_{\widetilde{B}} \tag{3}$$
$$s_{\lambda\widetilde{A}} = \lambda s_{\widetilde{A}}. \tag{4}$$

So we can consider $\widetilde{A},\widetilde{B} \in \mathscr{F}_c(\mathbb{R}^n)$ in $L^2(\mathbb{S}^{n-1} \times [0,1])$ via its support function and we define

$$\delta_2(\widetilde{A},\widetilde{B}) := \left(n \int_0^1 \int_{\mathbb{S}^{n-1}} |s_{\widetilde{A}}(u,\alpha) - s_{\widetilde{B}}(u,\alpha)|^2 \nu(du)d\alpha \right)^{\frac{1}{2}},$$

$$\left\langle \widetilde{A},\widetilde{B} \right\rangle := \langle s_{\widetilde{A}}, s_{\widetilde{B}} \rangle = n \int_0^1 \int_{\mathbb{S}^{n-1}} s_{\widetilde{A}}(u,\alpha) s_{\widetilde{B}}(u,\alpha) \nu(du)d\alpha,$$

$$||\widetilde{A}||_2 := ||s_{\widetilde{A}}||_2 = \left(n \int_0^1 \int_{\mathbb{S}^{n-1}} s_{\widetilde{A}}(u,\alpha)^2 \nu(du)d\alpha \right)^{\frac{1}{2}}.$$

With $\delta_2(\widetilde{A},\widetilde{B}) = ||s_{\widetilde{A}} - s_{\widetilde{B}}||_2$, $\mathscr{F}_c(\mathbb{R}^n)$ can be embedded isometrically and isomorph as closed convex cone in $L^2(\mathbb{S}^{n-1} \times [0,1])$.

Now, a fuzzy random variable (frv) can be defined as a Borel measurable function

$$\widetilde{X} : \Omega \to \mathscr{F}_c(\mathbb{R}^n)$$

from $(\Omega, \mathfrak{F}, P)$ to $(\mathscr{F}_c(\mathbb{R}^n), \mathfrak{B}_2)$ where $\mathfrak{B}_2$ is the σ-algebra induced by δ_2.

Then all $\alpha-$cuts are compact convex random set (see Puri, Ralescu [11], too). There are further definitions of fuzzy random variables, which are equivalent under some constraints. For details see Krätschmer [6] [7]. The (Aumann-) expectation $\mathbf{E}\xi$ of a compact convex random set ξ is defined by the collection of all "pointwise" expectations $\mathbf{E}X$, the so called Bochner-integrals, with $X \in \xi$ almost surely and, i.e (see Aumann [1], too)

$$\mathbf{E}\xi = \{\mathbf{E}X : X : \Omega \to \mathbb{R}^n, X - \text{Bochner-integrable}, X(\omega) \in \xi(\omega) \text{ P-a.s.}\}.$$

Krätschmer shows in [8], that $\mathbf{E}\xi \in \mathscr{K}_c(\mathbb{R}^n)$ if and only if ξ is integrably bounded, i.e. if $\delta_2(\xi,\{0\})$ is integrable. A frv $\widetilde{X}$ is called integrably bounded if all α-cuts are integrably bounded. Then there exists a unique fuzzy set $\mathbf{E}\widetilde{X} \in \mathscr{F}_c(\mathbb{R}^n)$, called the Aumann expectation of $\widetilde{X}$, such that

$$(\mathbf{E}\widetilde{X})_\alpha = \mathbf{E}(\widetilde{X}_\alpha) \quad ; \quad 0 < \alpha \leq 1. \tag{5}$$

This expectation of a frv $\widetilde{X}$ was introduced by Puri/Ralescu [11]. Further we can define

$$\int_A \widetilde{X} dP := \mathbf{E}\left(\mathbf{I}_A \widetilde{X}\right),$$

where $\mathbf{I}_A$ denotes the indicator function of $A \in \mathfrak{F}$.

For an integrable bounded frv the measurable function

$$s_{\widetilde{X}(.)}(u,\alpha) : \quad \Omega \to \mathbb{R},\ \omega \mapsto s_{\widetilde{X}(\omega)}(u,\alpha)$$

is integrable and the support function of the expectation is equal, the expectation of the support function (Vitale [14]):

$$s_{\mathbf{E}\widetilde{X}}(u,\alpha) = \mathbf{E}s_{\widetilde{X}}(u,\alpha), \qquad u \in \mathbb{S}^{n-1}, \alpha \in (0,1]. \tag{6}$$

Following Körner [5] the variance of a frv $\widetilde{X}$ with $\mathbf{E}||\widetilde{X}||_2^2 < \infty$ is defined by

$$\begin{aligned}\mathbf{Var}\widetilde{X} &= \mathbf{E}\delta_2^2(\widetilde{X},\mathbf{E}\widetilde{X}) \\ &= \mathbf{E}\left\langle s_{\widetilde{X}} - s_{\mathbf{E}\widetilde{X}}, s_{\widetilde{X}} - s_{\mathbf{E}\widetilde{X}}\right\rangle \\ &= \mathbf{E}||\widetilde{X}||_2^2 - ||\mathbf{E}\widetilde{X}||_2^2.\end{aligned} \tag{7}$$

Using (5) and (6) this can be written as

$$\mathbf{Var}\widetilde{X} = n\int_0^1 \int_{\mathbb{S}^{n-1}} \mathbf{Var} s_{\widetilde{X}}(u,\alpha)\nu(du)d\alpha.$$

For more details on the expectation and variance of frv‘s see e.g. Näther [9].

3 Conditional Variance

In this section, we present the definition of the conditional variance of a frv and prove same properties of it. As a corollary, we obtain a variance decomposition formula analogously to (2). We start with the definition of the conditional expectation of a frv.

Assumption 1
Let $(\Omega,\mathfrak{F},P)$ be a probability space, $\mathfrak{A}$ a sub-σ-algebra of $\mathfrak{F}$ and $\widetilde{X}$ a frv with $\mathbf{E}(||\widetilde{X}||_2^2) < \infty$ (i.e. $\mathbf{Var}\widetilde{X} < \infty$).

Definition 1 (Conditional Expectation).
Under assumption 1, the *conditional expectation of a frv $\widetilde{X}$ with respect to* $\mathfrak{A}$ is the frv $\mathbf{E}(\widetilde{X}|\mathfrak{A})$ which:

(a) $\mathbf{E}(\widetilde{X}|\mathfrak{A})$ is $\mathfrak{A}$-measurable,

(b) $\displaystyle\int_A \mathbf{E}(\widetilde{X}|\mathfrak{A})dP = \int_A \widetilde{X}dP \quad \forall A \in \mathfrak{A}.$

Analogously to (5) it holds (see Puri and Ralescu [12])

$$(\mathbf{E}(\widetilde{X}|\mathfrak{A}))_\alpha = \mathbf{E}(\widetilde{X}_\alpha|\mathfrak{A}) \quad ; \quad 0 < \alpha \leq 1. \tag{8}$$

Moreover, similar to (6) it can be proven that

$$s_{\mathbf{E}(\widetilde{X}|\mathfrak{A})}(u,\alpha) = \mathbf{E}(s_{\widetilde{X}}(u,\alpha)|\mathfrak{A}), \quad u \in \mathbb{S}^{n-1}, \alpha \in (0,1], \tag{9}$$

see M. Stojakovic\Z. Stojakovic [13] and Wünsche\Näther [15] and Hiai\Umegaki [4]. With the equations (6) and (9) further it can be proven that

$$\mathbf{E}(\mathbf{E}(\widetilde{X}|\mathfrak{A})) = \mathbf{E}(\widetilde{X}). \tag{10}$$

Definition 2 (Conditional Variance).
Under assumption 1, the *conditional variance of* $\widetilde{X}$ *wrt* $\mathfrak{A}$ is the real random variable

$$\mathbf{Var}(\widetilde{X}|\mathfrak{A}) := \mathbf{E}(\delta_2^2(\widetilde{X}, \mathbf{E}(\widetilde{X}|\mathfrak{A}))|\mathfrak{A}) \tag{11}$$

In the following we present some properties of the conditional variance.

Assumption 2
Let X be a non-negative almost surely bounded random variable on $(\Omega, \mathfrak{F}, P)$ which is conditional independent (for conditional independence see, for instance Chow/Teicher [3])) of $\widetilde{X}$ wrt $\mathfrak{A} \subset \mathfrak{F}$.

Theorem 1. Under assumption 1 and assumption 2 it holds

$$\mathbf{Var}(X\widetilde{X}|\mathfrak{A}) = \mathbf{E}(X^2|\mathfrak{A})\mathbf{E}(||\widetilde{X}||_2^2|\mathfrak{A}) - \mathbf{E}(X|\mathfrak{A})^2||\mathbf{E}(\widetilde{X}|\mathfrak{A})||_2^2. \tag{12}$$

For the proofs of the theorem and the following corollary see
Näther\Wünsche[10].

As a direct conclusions of theorem 1 we obtain the following rules for the conditional variance. Note that for $\mathfrak{A} = \{\emptyset, \Omega\}$ the conditional variance is the variance of the frv. Take the assumptions of theorem 1 and let be $\widetilde{A} \in \mathscr{F}_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. Then it holds

$$\begin{aligned}
\mathbf{Var}(\widetilde{X}|\mathfrak{A}) &= \mathbf{E}(||\widetilde{X}||_2^2|\mathfrak{A}) - ||\mathbf{E}(\widetilde{X}|\mathfrak{A})||_2^2 && (13)\\
&= n\int_0^1 \int_{\mathbb{S}^{n-1}} \mathbf{Var}(s_{\widetilde{X}}(u,\alpha)|\mathfrak{A})\nu(du)d\alpha && (14)\\
\mathbf{Var}(\lambda\widetilde{X}|\mathfrak{A}) &= \lambda^2\mathbf{Var}(\widetilde{X}|\mathfrak{A}) && (15)\\
\mathbf{Var}(X\widetilde{A}|\mathfrak{A}) &= ||\widetilde{A}||_2^2\mathbf{Var}(X|\mathfrak{A}) && (16)\\
\mathbf{Var}\left(X\widetilde{X}|\mathfrak{A}\right) &= \mathbf{E}\left(X^2|\mathfrak{A}\right)\mathbf{Var}(\widetilde{X}|\mathfrak{A}) + \mathbf{Var}(X|\mathfrak{A})||\mathbf{E}(\widetilde{\mathbf{X}}|\mathfrak{A})||_2^2 && (17)\\
\mathbf{Var}\left(X\widetilde{X}|\mathfrak{A}\right) &= \mathbf{E}\left(X|\mathfrak{A}\right)^2\mathbf{Var}(\widetilde{X}|\mathfrak{A}) + \mathbf{Var}(X|\mathfrak{A})\mathbf{E}\left(||\widetilde{X}||_2^2|\mathfrak{A}\right) && (18)
\end{aligned}$$

Now, we easily can obtain an analogon of variance decomposition formula (2).

Corollary 1. Under assumption 1 it holds

$$\mathbf{Var}\widetilde{X} = \mathbf{E}\left(\mathbf{Var}\left(\widetilde{X}|\mathfrak{A}\right)\right) + \mathbf{Var}\left(\mathbf{E}\left(\widetilde{X}|\mathfrak{A}\right)\right). \tag{19}$$

4 Applications of the Variance Decomposition Formula

4.1 Wald's Identity

Consider, for example, an insurance company with a random claim number N per year and N individual claims $C_1,..,C_N$ which, for simplicity, are assumed to be iid.

like C. Obviously, the company is interested in the variance of the claim sum $S := \sum_{i=1}^{N} C_i$ which easily can be computed by use of variance decomposition formula (2) and which leads to the well known Wald's identity

$$\mathbf{Var}S = \mathbf{E}N\mathbf{Var}C + \mathbf{Var}N(\mathbf{E}C)^2. \tag{20}$$

Now, let us discuss Wald's formula for a random number N of iid. fuzzy claims $\widetilde{C}_i$; $i = 1,..,N$; distributed like the prototype claim $\widetilde{C}$. The claim sum

$$\widetilde{S} := \sum_{i=1}^{N} \widetilde{C}_i$$

is a frv, too. Applying (19) it holds

$$\mathbf{Var}\widetilde{S} = \mathbf{E}(\mathbf{Var}(\widetilde{S}|N)) + \mathbf{Var}(\mathbf{E}(\widetilde{S}|N)). \tag{21}$$

Obviously, we obtain

$$\begin{aligned} \mathbf{E}(\widetilde{S}|N) &= \mathbf{E}(\sum_{i=1}^{N} \widetilde{C}_i|N) \\ &= N\mathbf{E}\widetilde{C}. \end{aligned}$$

Using (16) (with $\mathfrak{A} = \{\emptyset, \Omega\}$), we have

$$\mathbf{Var}(\mathbf{E}(\widetilde{S}|N)) = ||\mathbf{E}\widetilde{C}||_2^2\mathbf{Var}N. \tag{22}$$

Since the $\widetilde{C}_i$ are iid the variance of the sum of the $\widetilde{C}_i$ is equal the sum of the variances i.e. it holds

$$\begin{aligned} \mathbf{Var}(\widetilde{S}|N) &= N\mathbf{Var}\widetilde{C} \\ \Longrightarrow \quad \mathbf{E}(\mathbf{Var}(\widetilde{S}|N)) &= \mathbf{E}N\mathbf{Var}\widetilde{C}. \end{aligned}$$

Hence, (21) can be written as

$$\mathbf{Var}\widetilde{S} = \mathbf{E}N\mathbf{Var}\widetilde{C} + \mathbf{Var}N||\mathbf{E}\widetilde{C}||_2^2 \tag{23}$$

which is the direct analogon of Wald's identity (20).

4.2 Stratified Sampling

Consider a random characteristic X with $\mathbf{E}X = \mu$ on a stratified sample space Ω with the strata (decomposition) $\Omega_1,..,\Omega_k$. Let μ_i and σ_i^2 be expectation and variance of X in stratum Ω_i and $p_i = P(\Omega_i)$; $i = 1,..,k$. Then, a consequence of (2) is

$$\mathbf{Var}X = \sum_{i=1}^{k} p_i \sigma_i^2 + \sum_{i=1}^{k} p_i (\mu_i - \mu)^2 \tag{24}$$

which is a well known formula in sampling theory (see e.g. Chaudhuri, Stenger [2]).

Consider a frv $\widetilde{X}$ on a probability space $(\Omega, \mathfrak{F}, P)$ which is stratified into strata $\Omega_i \in \mathfrak{F}$; $i = 1,..,k$; with $\bigcup_{i=1}^{k} \Omega_i = \Omega$, $\Omega_i \cup \Omega_j = \emptyset$ for $i \neq j$ and $P(\Omega_i) =: p_i$. Let $\mathfrak{A} = \sigma(\Omega_1,..,\Omega_k)$ be the $\sigma-$algebra generated by the strata Ω_i. Obviously, it is $\mathfrak{A} \subseteq \mathfrak{F}$. Following (7) and having in mind $\mathbf{E}(\mathbf{E}(\widetilde{X}|\mathfrak{A})) = \mathbf{E}\widetilde{X}$ we obtain

$$\begin{aligned}\mathbf{Var}(\mathbf{E}(\widetilde{X}|\mathfrak{A})) &= \mathbf{E}\delta_2^2(\mathbf{E}(\widetilde{X}|\mathfrak{A}), \mathbf{E}\widetilde{X}) \\ &= \sum_{i=1}^{k} p_i \delta_2^2(\mathbf{E}(\widetilde{X}|\Omega_i), \mathbf{E}\widetilde{X}).\end{aligned}$$

On the other hand it holds

$$\mathbf{E}(\mathbf{Var}(\widetilde{X}|\mathfrak{A})) = \sum_{i=1}^{k} p_i \mathbf{Var}(\widetilde{X}|\Omega_i).$$

Using the abbreviations $\widetilde{\mu} := \mathbf{E}\widetilde{X}$, $\widetilde{\mu}_i := \mathbf{E}(\widetilde{X}|\Omega_i)$, $\sigma_i^2 := \mathbf{Var}(\widetilde{X}|\Omega_i)$; $i = 1,..,k$; formula (19)

$$\mathbf{Var}\widetilde{X} = \mathbf{E}\left(\mathbf{Var}\left(\widetilde{X}|\mathfrak{A}\right)\right) + \mathbf{Var}\left(\mathbf{E}\left(\widetilde{X}|\mathfrak{A}\right)\right)$$

can be specified as

$$\mathbf{Var}\widetilde{X} = \sum_{i=1}^{k} p_i \sigma_i^2 + \sum_{i=1}^{k} p_i \delta_2^2(\widetilde{\mu}_i, \widetilde{\mu})$$

which is a direct generalization of formula (24).

For more details and proofs see Näther\Wünsche[10].

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