Interval Random Variables and Their Application in Queueing Systems with Long–Tailed Service Times

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1 Introduction

The theory of interval random variables has been introduced by the authors in [12]. Up to now, it has been used in a few applications connected with Internet servers admission control and queueing systems.

The theory may be considered to be remotely related to the p–bound (i.e. "probability bound") concept (see e.g. [2], [4]), using bounds on the CDF of a random variable. For example, [13] and [15] consider some links between these theories.

Likewise, the approach based on the Evidence Theory (e.g. [7]) exhibits several similarities.

Actually, the interval random variables theory may also be considered to be the antitype of all approaches using set–valued probabilities – it operates on events having certain, non–interval probabilities. Nevertheless, values of random variables, assigned to these events are uncertain.

Earlier papers, e.g. [12], concentrated on an application connected with the estimation of uncertain parameters. Suitable notions and propositions, including the analog of Kolmogorov's theorem were introduced.

This paper is devoted to another problem, for which the developed theory is useful – numerical computation of the Laplace transform of random variable's PDF, which is useful e.g. in queueing theory with long–tailed distributions of service (or interarrival) times, e.g. [5], [18].

2 Basic Notions of Probability Theory

One of the most fundamental notions is the random variable.

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Definition 1. *Let the probability space (Ω, S, P) be given, where Ω is the set of elementary events, S – the σ-field of events and $P\colon S \to [0,1]$ – the (σ-additive) probability measure.*

A mapping $X\colon \Omega \to \mathbb{R}$ is called a (real) random variable*, if the inverse images of Borel subsets of $\mathbb{R}$ are events (i.e. the elements of S).*

This condition is necessary to define a probability distribution of the variable.

By the *probability distribution* we mean a function $P_X(X_0) = P\big(\{\omega\colon X(\omega) \in X_0\}\big)$ where X_0 is a Borel subset of $\mathbb{R}$.

Several important (and well–known) notions of the probability theory (like the expected value of a random variable) are beyond the scope of this paper. What we have to mention is the Laplace transform of the PDF.

For continuous random variables, with the PDF $f(x)$, the Laplace transform of the PDF may be defined as follows (see e.g. [5], [18]):

$$\widetilde{f}(s) = \int_0^\infty e^{-sx} f(x)\, dx\,. \tag{1}$$

Equation (1) may be transformed to the form of the integral w.r.t. (with respect to) the measure $P(\cdot)$. Then, in the case when the random variable is non–negative, it takes the form:

$$\widetilde{f}(s) = \int_\Omega e^{-sX(\omega)}\, dP(\omega)\,. \tag{2}$$

3 Basics of Interval Computations

To define the notion of interval random variable, we need to have some basic notions of intervals and their arithmetic. We follow a wide literature, like the article [8] or books [6], [9], [17], to name just a few.

We define the (closed) interval $[\underline{x},\overline{x}]$ as a set $\{x \in \mathbb{R} \mid \underline{x} \le x \le \overline{x}\}$. We denote all intervals by brackets; open ones will be denoted as $]\underline{x},\overline{x}[$ and partially open as: $[\underline{x},\overline{x}[$, $]\underline{x},\overline{x}]$. (We prefer this notation than using the parenthesis that are used also to denote sequences, vectors, etc.)

We also use boldface lowercase letters to denote interval variables, e.g. $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$. Following [10], $\mathbb{IR}$ denotes the set of all real intervals and $\mathbb{IC}_{\text{rect}}$ – the set of "rectangular" complex intervals (i.e. pairs of intervals for real and imaginary part).

We design arithmetic operations on intervals so that the following condition was fulfilled: if $\odot \in \{+,-,\cdot,/\}$, $a \in \boldsymbol{a}$, $b \in \boldsymbol{b}$, then $a \odot b \in \boldsymbol{a} \odot \boldsymbol{b}$. We omit the actual formulae for arithmetic operations; they can be found in a wide literature e.g. ([6], [8], [9], [17]).

Now, we define a notion to set links between real and interval functions.

Definition 2. *A function $\mathsf{f}\colon \mathbb{IR} \to \mathbb{IR}$ is an* inclusion function *of $f\colon \mathbb{R} \to \mathbb{R}$, if for all intervals $\boldsymbol{x}$ within the domain of f the following condition is satisfied:*

$$\{f(x) \mid x \in \boldsymbol{x}\} \subseteq \mathsf{f}(\boldsymbol{x})\,. \tag{3}$$

The definition is analogous for functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

Remark 1. There is a confusion in interval community; some researchers use terms "interval enclosure" or "interval extension" instead of "interval inclusion function". We also use the term "interval enclosure" below, but in a slightly different case: a function assigning intervals to non–interval arguments (see Definition 4).

4 The Notion of an Interval Random Variable

Now, we define a "random variable" that maps the events not to real numbers, but rather to intervals of real numbers.

Definition 3. *Let the probability space (Ω, S, P) be given (as in Definition 1). Let us define a partition of Ω into sets $A(x)$ of the form:*

$$A_X(x) = \{\omega \in \Omega \mid X(\omega) = x\} \ , \ \textit{where } x \in \mathbb{I}_X \ . \tag{4}$$

Any mapping $X\colon \Omega \to \mathbb{I}_X \subseteq {}^*\mathbb{IR}$, satisfying the condition that for each $x \in \mathbb{I}_X$ the set $A_X(x)$ is an event, is called an interval random variable.

According to [10], ${}^*\mathbb{IR}$ denotes the set of intervals, the endpoints of which may be not only finite real numbers, but also $-\infty$ or $+\infty$.

The definition of an interval random variable differs from the definition of a real random variable not only in the set of values. We omit here the condition about the reverse images of the Borel sets, replacing it by a simpler one. Why ? To formulate a relevant condition it would be necessary to define a reverse image first. And this notion is not explicitly defined.

There are several possible definitions of a reverse image of an interval valued function (see e.g. [11] and Section 4.1). In this paper we consider (and so we do in the earlier works, e.g. [12], [13], [15]) only those random variables that have a finite set $\mathbb{I}_X$ of intervals as its possible values. This assumption allows us to define the probability function only for interval arguments from the set $\mathbb{I}_X$:

$$P_X(x) = P\Big(A_X(x)\Big) \ ,$$

for any interval $x \in \mathbb{I}_X$.

Papers [12], [13], [15] consider also several other notions (e.g. the expected value of an interval random variable) that may be important in many applications, but are of no importance for the considered problem.

Now, we define some notions that will allow to associate interval variables with real variables, namely an *interval enclosure* and an *interval discretization* of a real random variable.

Definition 4. *Suppose, we have a real random variable X. The interval random variable X that fulfills the condition:*

$$X(\omega) \in X(\omega) \qquad \forall\, \omega \in \Omega \ , \tag{5}$$

will be called an interval enclosure *of the random variable X.*

Definition 5. *Suppose, we have a real random variable X. Let the values of X be contained in the interval $[a,b]$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$.*

Let us divide the interval $[a,b]$ into n subintervals. We denote their endpoints by x_i. We obtain the sequence $(x_i)_{i=0}^n$, where:

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \ .$$

The interval random variable X will be called an interval discretization *of the random variable X, if the following conditions are fulfilled:*

- *X is an interval enclosure of X,*
- *the set of values of X is equal to $\mathbb{I}_X = \Big\{ [x_{i-1}, x_i] \quad i = 1, \ldots, n \Big\}$.*

Remark 2. In recent papers (e.g. [12], [13], [15]) a less restrictive condition was used in the definition of an interval discretization. The interval discretization was supposed to take the value $x_i = [x_{i-1}, x_i]$ with probability:

$$p_i = \int_{x_{i-1}}^{x_i} f(x)\,dx = F(x_i) - F(x_{i-1}) \ ,$$

for each $i = 1, \ldots, n$.

Defining the interval discretization by the sets of elementary events instead of the values of probability measure seems more appropriate.

Property 1. If X is an interval discretization of X, then:

$$P\big(\{X \in x\}\big) = P\Big(A_X(x)\Big) \ \forall x \in \mathbb{I}_X \ .$$

This property does not hold for interval enclosures that are not interval discretizations.

Remark 3. Precisely, Property 1 is fulfilled for interval discretizations of continuous random variables. Nevertheless, it can be generalized for the case of discrete random variables, relatively simply.

Namely, we have to consider only disjoint intervals, which means we cannot use closed ones only. If the probability that a random variable X takes a single value x_1 is nonzero, then computing probabilities: $P\Big(\big\{X \in [x_0, x_1]\big\}\Big)$ and $P\Big(\big\{X \in [x_1, x_2]\big\}\Big)$, we add $P\Big(\{X = x_1\}\Big)$ to both these quantities. We have to use either intervals $[x_0, x_1[$ and $[x_1, x_2]$ or $[x_0, x_1]$ and $]x_1, x_2]$.

It is well known that the distribution does not determine the random variable uniquely. Different random variables may have exactly the same distribution, but associate different values with different elementary events.

Obviously, the same holds for interval–valued random variables. We shall introduce now a notion to represent the distribution of an interval random variable.

Definition 6. *Consider a finite subset of $\mathbb{IR}$, $\mathbb{I}_X = \{x_1, \ldots, x_n\}$.*

A generalized histogram *is a mapping $P\colon \mathbb{I}_X \to \mathbb{R}_+ \cup \{0\}$, such that:*

$$\sum\nolimits_{i=1}^{n} P(x_i) = 1 \ .$$

Remark 4. What is the difference between a generalized histogram and an ordinary one ? Only such that the intervals x_i, $i = 1,\ldots,n$ do not have to be pairwise disjoint.

Obviously, each interval random variable defines a generalized histogram of the form:

$$P(x) = P\Big(A_X(x)\Big) \quad x \in \mathbb{I}_X \ .$$

Remark 5. In many cases we are more interested in the distribution of a random variable than in the assignment of values to specific elementary events. Hence, researchers sometimes do not distinguish between the random variable and its distribution when it is not important. Also, we shall use notions "interval random variable" and "generalized histogram" alternately, when elementary events are not explicitly considered.

4.1 Interval Random Variables or Random Sets ?

Yet one more important question has to be answered: what is the relation between the theory of interval random variables and the more general theory of *set–valued random variables*, also known as *random sets* (e.g. [11], [16]).

A random set is a measurable mapping from the space Ω of elementary events to some family of sets. Measurable means that all sets $\{\omega\colon X(\omega) \cap x \neq \emptyset\}$ are events.

Though the interval random variables' theory was developed independently from the theory of random sets ([12]), it is obvious that interval random variables are a particular case of set–valued random variables.

Nevertheless, they are an important specific case, because they are computationally far more tractable and their theory is simpler. It is especially worth noting, that for interval–valued random variables it is reasonably simple to consider unbounded random variables (which we actually do in this paper), while papers on set–valued random variables usually assume, they are bounded and compact (see e.g. [16].

5 TAM – Transform Approximation Method

The $\mathscr{L}$–transform is well–defined and finite for the PDF of each random variable. Unfortunately, for some probability distribution functions the Laplace transform does not have an analytic form. According to e.g. [18], this is the case for all power–tailed distributions (e.g. Pareto distribution) and most other long–tailed ones (including lognormal and Weibull distributions).

The Laplace transforms, useful e.g. in $M/GI/1$ and $GI/M/1$ queueing systems analysis (see e.g. [1]), have to be approximated somehow. Below, we present a popular method to approximate such transforms.

5.1 Transform Approximation Method

TAM (Transform Approximation Method) is described e.g. in [5], [18].

Let us consider a random variable X with the PDF $f(x)$ and CDF $F(x)$.

The essence of TAM is very simple: we discretize the domain of the random variable X (at least the set $[0, +\infty[$), obtaining n points: $x_1 < x_2 < \ldots < x_n$.

Let us denote the CDF's values in these points in the following way:

$$y_i = F(x_i) \qquad i = 1, \ldots, n \, .$$

We associate some probability masses with these points:

$$\begin{aligned} p_1 &= \frac{y_1 + y_2}{2} \, , \\ p_n &= 1 - \frac{y_{n-1} + y_n}{2} \, , \\ p_i &= \frac{y_{i+1} - y_{i-1}}{2} \quad i = 2, \ldots, n-1 \, . \end{aligned} \tag{6}$$

Then we can approximate the $\mathscr{L}$–transform $\widetilde{f}(s)$ of the PDF of X by a finite sum:

$$\breve{f}(s) = \sum_{i=1}^{n} p_i \cdot e^{-s \cdot x_i} \, . \tag{7}$$

The above description does not specify how to choose points x_i (or y_i). There are a few approaches to do it (see below), but how to do it optimally remains an open problem.

Possible parameterizations.

The method was first developed in 1998 by Gross and Harris. The formula $\breve{f}(s) = \frac{1}{n} \cdot \sum_{i=1}^{n} e^{-s \cdot x_i}$, where $x_i = F^{-1}\left(\frac{i}{n+1}\right)$ was used there. Such an approach is called *uniform–TAM*, or shortly UTAM.

Currently, more widely used is the GTAM (*geometric–TAM*), which sets: $y_i = 1 - q^i$ (for some q such that $0 < q < 1$) and $x_i = F^{-1}(y_i)$.

6 Interval Transform Approximation Method

To introduce the interval analog of TAM, let us consider a real–valued random variable X with the PDF $f_X(x)$ and CDF $F_X(x)$. Consider an interval discretization $\boldsymbol{X}$ of X.

We can formulate the interval inclusion function for the Laplace transform of the PDF of X. It is the function $\breve{\mathsf{f}}_{\boldsymbol{X}} : \mathbb{IC}_{\text{rect}} \to \mathbb{IC}_{\text{rect}}$ of the form:

$$\breve{\mathsf{f}}_{\boldsymbol{X}}(s) = \sum_{i=1}^{n} p_i \cdot e^{-s \cdot \boldsymbol{x}_i} \, ,$$

where $\boldsymbol{x}_i = [x_{i-1}, x_i]$ and $s = \left[\underline{s}, \overline{s}\right]$ is an interval complex variable, i.e. $\underline{s}$ and $\overline{s}$ are complex numbers.

We want to use this approximation for Laplace transforms of the PDFs of services time in queueing systems. It should be especially useful when the distribution of service time is long–tailed. In such a case (and actually each case when the time is unbounded) one of the intervals $[x_i, x_{i+1}]$ will be of the form: $[x_{n-1}, +\infty]$.

So, the interval extension of the $\mathscr{L}$–transform will be finite only for values of the argument s, satisfying $\operatorname{Re}\underline{s} > 0$.

Now, let us prove that $\check{\mathsf{f}}_X(s)$ is indeed an inclusion function of $\widetilde{f}_X(s)$.

Theorem 1. *Let an interval random variable X be interval enclosure of a real random variable X.*

Then, for each complex s such that $s \in \boldsymbol{s}$, the following condition is fulfilled:

$$\widetilde{f}_X(s) \in \check{\mathsf{f}}_{\boldsymbol{X}}(\boldsymbol{s}) \ .$$

The theorem is holds specifically for $\boldsymbol{s} = [s, s]$.

Proof.

According to (2), we obtain:

$$\widetilde{f}_X(s) = \int_{\Omega} e^{-s \cdot X(\omega)} \, dP(\omega) \ .$$

Using the partition of Ω into sets $A(x_i)$, defined by equation (4), we can reformulate the above integral into the form of the following sum:

$$\widetilde{f}_X(s) = \sum_{i=1}^{n} \int_{A(x_i)} e^{-s \cdot X(\omega)} \, dP(\omega) \ . \tag{8}$$

From the definition of an interval enclosure we have that $X(\omega) \in \boldsymbol{X}(\omega)$. The rules of interval computations (e.g. [6], [8], [9], [17]) imply that for each $x \in \boldsymbol{x}_i$ and $s \in \boldsymbol{s}$ we have: $e^{-s \cdot x} \in e^{-\boldsymbol{s} \cdot \boldsymbol{x}_i}$.

Hence, we obtain:

$$\Big(\int\limits_{A(x_i)} e^{-s \cdot X(\omega)} \, dP(\omega) \Big) \in \Big(e^{-\boldsymbol{s} \cdot \boldsymbol{x}_i} \int\limits_{A(x_i)} dP(\omega) \Big) \ .$$

The right side simply reduces to the form:

$$e^{-\boldsymbol{s} \cdot \boldsymbol{x}_i} \cdot p_i \ .$$

So:

$$\Big(\sum_{i=1}^{n} \int_{A(x_i)} e^{-sX(\omega)} \, dP(\omega) \Big) \in \Big(\sum_{i=1}^{n} e^{-\boldsymbol{s} \cdot \boldsymbol{x}_i} \cdot p_i \Big) \ . \tag{9}$$

Then, from (8) and (9) we obtain:

$$\widetilde{f}_X(s) \in \sum_{i=1}^{n} e^{-\boldsymbol{s} \cdot \boldsymbol{x}_i} \cdot p_i = \check{\mathsf{f}}_{\boldsymbol{X}}(\boldsymbol{s}) \ .$$

QED

The Essence of the Method

Let us now refer to TAM, described in Section 5.

Having defined the notion of interval discretization and the interval inclusion of $\mathscr{L}$–transform of a random variable, it was simple to develop an interval analog of TAM. We may call it "Interval TAM" or ITAM for short.

This approach is similar to classical TAM, except for using interval discretization instead of a traditional one and computing the interval inclusion of the $\mathscr{L}$–transform basing on this interval discretization.

The advantages of such approach in comparison with the traditional TAM are obvious:

- we use correct probabilities associated with the intervals, not probability masses quite arbitrarily associated with chosen points, as in (6),
- we can naturally bound the discretization error and truncation error,
- as in other interval methods, we can bound the numerical error (see e.g. [6], [8], [9], [17]).

7 Laplace Transform for Queueing Systems

In case of $M/GI/1$ systems the $\mathscr{L}$–transform of the sojourn time is given by the so–called Pollaczek–Khinchin formula (see e.g. [1]):

$$\widetilde{w}(s) = \frac{(1-\rho)\cdot\widetilde{b}(s)\cdot s}{s+\lambda\cdot\left(\widetilde{b}(s)-1\right)}\,, \tag{10}$$

where $\widetilde{b}(s)$ is the $\mathscr{L}$–transform of PDF of the service time B, λ is the arrival rate and $\rho = \lambda\cdot\mathbb{E}B$.

Assume the service time to be Pareto–distributed; this is a typical power–tailed distribution, commonly used to model various levels of computer network traffic. The most commonly encountered form uses two parameters: the shaping parameter $\alpha > 0$ and the location parameter $\beta > 0$. A Pareto–distributed variable X has the CDF $F_X(x) = 1 - \left(\frac{\beta}{x}\right)^{\alpha}$ (for $x \geq \beta$; otherwise $F_X(x) = 0$) and PDF $f_X(x) = \frac{\beta^{\alpha}}{x^{\alpha+1}}$ (also for $x \geq \beta$).

As it was mentioned before, PDF of a Pareto–distributed random variable posses an $\mathscr{L}$–transform (as PDFs of all random variables do), but that transform does not have a closed analytical form. Hence, some approximation of $\widetilde{b}(s)$ has to be used, in particular in formula (10), to get an approximation $\breve{w}(s)$ of $\widetilde{w}(s)$.

So, we can now approximate the Laplace transform of the sojourn time. Where can we use such an approximation ? Obviously, we can invert it numerically, to obtain the distribution of the sojourn time. But the next subsection describes a different application.

7.1 Optimization

Consider the following problem: we want to find the arrival and service rate of a queueing system, to optimize some performance measure for users waiting for the completion of their tasks. It can be set as the following optimization problem:

$$\max_{\lambda,\mu} \Big(Q = V(\lambda) - \lambda \cdot G(\lambda,\mu) - C(\mu) \Big)$$

s.t.

$$0 \leq \lambda \leq \Lambda \ ,$$

$$\lambda - \mu \leq -\varepsilon \ .$$

The meaning of the above notions is as follows:

- $V(\lambda)$ – increasing, concave and strictly differentiable is the aggregated utility of the users,
- $G(\lambda,\mu)$ – is the delay cost of the user,
- $C(\mu)$ – is the capacity cost (usually a linear structure is assumed $C(\mu) = c \cdot \mu$,
- ε is a small positive number used to avoid a strict inequality $\lambda < \mu$.

What about the delay cost G ? In [3] a few measures are proposed: linear cost, polynomial cost, etc. However in the case of a Pareto–distributed service time (with $\alpha < 2$) most of them are useless: they are infinite regardless the values of parameters λ and μ (proof given in [15], Subsection 4.1.4). The only useful measure of the delay cost is the exponential one (see [3]), expressed as:

$$G = \frac{v}{k} \cdot \big(1 - \widetilde{w}(k)\big) \ ,$$

where $v > 0$ and $k > 0$ are some real–valued parameters, estimation of which is beyond our interest (interval random variables might be useful there too, though; [12].

So, $\mathscr{L}$–transform of the sojourn time is explicitly used here to measure performance of the queueing system. More details may be found in [15].

Numerical Experiments

The lack of space makes the authors to present only a limited number of experiments. They are presented in Tables 1, 2 and 3.

Table 1 presents the results for Erlang distribution. Obviously, this is not a long–tailed distribution and TAM does not have to be used here. We present it, however, to show the failure of traditional real–valued TAM, which provides incorrect values there. Intervals computed by ITAM are somewhat wide, but correct.

Table 2 presents the results for Pareto distribution. We do not know the actual values of the $\mathscr{L}$–transform, so we can only use some kind of TAM.

Finally, Table 3 shows the performance of an interval optimization algorithm, setting the parameters of a queueing system with the Pareto service time (i.e. solving the problem from Subsection 7.1).

Table 1. Approximate values of the Laplace transform of the Erlang–distributed random variable's PDF ($r = 2$, $\mu = 100.000000$); TAM with 100 points

s	$\mathscr{L}$–transform	real–valued TAM	ITAM
(5.000000, -3.000000)	(0.951203, 0.027846)	(0.789720, 0.089621)	([0.579440, 1.000000],[4.705679E-009, 0.295521])
(5.000000, -2.000000)	(0.951543, 0.018568)	(0.797220, 0.060250)	([0.594440, 1.000000],[3.163486E-009, 0.198670])
(5.000000, -1.000000)	(0.951746, 0.009286)	(0.801750, 0.030276)	([0.603500, 1.000000],[1.589685E-009, 0.099834])
(5.000000, 0.000000)	(0.951814, 0.000000)	(0.803265, 0.000000)	([0.606530, 1.000000],[-0.000000, 0.000000])
(5.000000, 1.000000)	(0.951746, -0.009286)	(0.801750, -0.030276)	([0.603500, 1.000000],[-0.099834,-1.589685E-009])
(5.000000, 2.000000)	(0.951543, -0.018568)	(0.797220, -0.060250)	([0.594440, 1.000000],[-0.198670,-3.163486E-009])
(5.000000, 3.000000)	(0.951203, -0.027846)	(0.789720, -0.089621)	([0.579440, 1.000000],[-0.295521,-4.705679E-009])

Table 2. Approximate values of the Laplace transform of the Pareto–distributed random variable's PDF ($\alpha = 1.1$, $\beta = 1.0$; TAM with 1000 discretization points

s	real–valued TAM	ITAM
(0.100000, 0.000000)	(0.733017, 0.000000)	([0.732661, 0.757252],[0.000000, 0.000000])
(0.200000, 0.000000)	(0.593593, 0.000000)	([0.593003, 0.602140],[0.000000, 0.000000])
(0.500000, 0.000000)	(0.343648, 0.000000)	([0.342789, 0.344802],[0.000000, 0.000000])
(1.000000, 0.000000)	(0.157934, 0.000000)	([0.157144, 0.158725],[0.000000, 0.000000])
(2.000000, 0.000000)	(0.040318, 0.000000)	([0.039914, 0.040722],[0.000000, 0.000000])
(5.000000, 0.000000)	(0.001082, 0.000000)	([0.001054, 0.001109],[0.000000, 0.000000])
(10.000000, 0.000000)	(4.183039E-006, 0.000000)	([3.974061E-006,4.392017E-006],[0.000000, 0.000000])

Table 3. Results of the interval branch–and–bound for the single $M/P/1$ queue, capacity cost $c = 1.0$ and exponential delay cost with different values of v and k; ITAM with 100 discretization points

v	k	execution time	function evaluations	number of boxes that can contain a solution
10	10	0.66 sec.	73	7
10	2	25.94 sec.	3491	121
5	2	2.78 sec.	374	68
0.1	0.4	1.7 sec.	290	16

8 Conclusions

The proposed ITAM is an efficient way to approximate the Laplace transform of PDFs of random variables. Its computation may be a bit more costly than in the case of traditional, real–valued TAM, but it is significantly more precise and safe. It seems to be another useful application of the presented interval random variables theory.

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