
Applying Fuzzy Measures for Considering Interaction Effects in Fine Root Dispersal Models

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Summary. We present an example how fuzzy measures and discrete Choquet integrals can be used to model interactivities between trees within a stochastic fine root dispersal model.

1 Introduction

Fine roots are roots with diameter smaller than 2 mm which are responsible for the soil water reception of trees. Investigations of the spatial dispersion of the fine root biomass can help to improve the knowledge about effects trees impose on soil resources. One point within this research are dispersal models where interaction between trees says something about their rivalry, for example with regard to water resources. In this paper the multi-tree case with trees from two different species is discussed. In this multi-tree case the total biomass of fine roots consists of the contributions of the individual trees. Here, interactions can affect the root biomass. Therefore, we want to describe the total mean of the fine root biomass by a weighted sum of the individual biomass contributions where the weights depend on the interactivities. From fuzzy theory it is known that fuzzy integrals are flexible tools of aggregation considering interaction ([2]). Especially, so-called discrete Choquet integrals can be applied for the aggregation of interacting criteria which in our case are given by the individual tree biomasses.

2 Fuzzy Measures for Modelling Interactivities

We restrict ourselves to a finite universe of discourse, say $\mathcal{N} = \{1, \dots, N\}$. Denote $\mathcal{P}(\mathcal{N})$ the power set of \mathcal{N} . A fuzzy measure ν is a set function $\nu : \mathcal{P}(\mathcal{N}) \rightarrow [0, 1]$ with

$$\nu(\emptyset) = 0, \nu(\mathcal{N}) = 1, \nu(A) \leq \nu(B) \text{ for } A, B \in \mathcal{P}(\mathcal{N}) \text{ and } A \subseteq B, \quad (1)$$

see [1]. In general, a fuzzy measure ν is a non-additive set function. The ‘degree’ of nonadditivity expresses the ‘degree’ of interaction between two subsets A and B from

\mathcal{N} . A sub-additive v with $v(A \cup B) \leq v(A) + v(B)$ for $A \cap B = \emptyset$ models negative synergy or redundancy whereas a super-additive v with $v(A \cup B) \geq v(A) + v(B)$ for $A \cap B = \emptyset$ describes positive synergy.

Often the elements of \mathcal{N} are interpreted as criteria. Then sub-additivity for example says that the evaluation $v(A \cup B)$ of the ‘sum’ $A \cup B$ of criteria is less than the sum of the single evaluations $v(A) + v(B)$. Using the interpretation as criteria, the evaluation of a single criterion $A = \{i\}, i \in \mathcal{N}$, is of special interest. Let us introduce the so-called importance index of criterion i . Note that for i being unimportant it is not enough that $v(\{i\})$ is small. If it happens that for some $A \subset \mathcal{N}$ the value $v(A \cup \{i\})$ is much greater than $v(A)$, then i may be important although $v(\{i\})$ is small. Considering these effects, the importance index or Shapley value is defined by

$$\Phi_i(v) := \sum_{A \subset \mathcal{N} \setminus \{i\}} \frac{(N - |A| - 1)!|A|!}{N!} [v(A \cup \{i\}) - v(A)], \tag{2}$$

see [3]. Analogously to the concept of the importance index the interaction index between two criteria i and j is defined by

$$I_{i,j}(v) = \sum_{A \subset \mathcal{N} \setminus \{i,j\}} \frac{(N - |A| - 2)!|A|!}{(N - 1)!} \Delta_{i,j}(A, v) \tag{3}$$

$$\Delta_{i,j}(A, v) := v(A \cup \{i, j\}) - v(A \cup \{i\}) - v(A \cup \{j\}) + v(A).$$

If v reduces to a probability measure μ we always have $\Delta_{i,j}(A, \mu) = 0$, i.e. additive set functions cannot model interaction.

Consider a feature variable \mathbf{x} which takes values x_i for the criteria $i \in \mathcal{N}$. For global evaluation or for aggregation of the feature values on \mathcal{N} suitable tools seem to be certain means of x_i , more generally: certain integrals of \mathbf{x} over \mathcal{N} . Classical integrals are linear operators with respect to a given measure. A much more powerful tool for a suitable aggregation are Choquet integrals with respect to a given fuzzy measure v , see [2]. For the ordered feature values $x_{(1)} \leq \dots \leq x_{(N)}$ the discrete Choquet integral with respect to a fuzzy measure v is defined by

$$C_v(\mathbf{x}) := \sum_{i=1}^N w_{[i]} x_{(i)} \tag{4}$$

$$w_{[i]} := v(A(i)) - v(A(i + 1)), \quad i = 1, \dots, N; \quad A(i) := \{(i), (i + 1), \dots, (N)\}.$$

The set $A(i)$ collects the indices of the $N - i + 1$ largest feature values. Especially, it is $A(1) = \mathcal{N}, A(N + 1) = \emptyset$.

An ordered weighted average (OWA) of the feature values is given by

$$OWA(\mathbf{x}) = \sum_{i=1}^N w_i x_{(i)}, \quad \sum_{i=1}^N w_i = 1, w_i \geq 0. \tag{5}$$

Formally, $OWA(\mathbf{x}) = C_v(\mathbf{x})$ with respect to the special fuzzy measure

$$v(A) = \sum_{j=0}^{|A|-1} w_{N-j}$$

i.e. $w_{[j]}$ with respect to v from (4) coincides with w_i . For any fuzzy measure v with

$$v(T) = v(S) \text{ for all sets } T \text{ and } S \text{ with } |T| = |S| \tag{6}$$

$OWA(\mathbf{x})$ coincides with the corresponding Choquet integral. In this simple case, the interaction index (3) is the same for any pair (i, j) , given by

$$I_{i,j}(v) = \frac{w_1 - w_N}{N - 1}, i, j \in \mathcal{N}, i \neq j, \tag{7}$$

see [3].

One-parametric families of fuzzy measures where the parameter controls interaction in a transparent way are of special interest. Let us mention here the Yager family

$$v_q(A) = \left(\frac{|A|}{N}\right)^{\frac{1}{q}}, q > 0, \tag{8}$$

where $q > 1$ models negative synergy and $q \in (0, 1)$ positive synergy. The Yager family obviously satisfies (6) and leads to the weights

$$w_{[j]} = \left(\frac{N - i + 1}{N}\right)^{1/q} - \left(\frac{N - i}{N}\right)^{1/q}; i = 1, \dots, N; q \in (0, \infty). \tag{9}$$

This fuzzy measure contains all possible types of interaction though the corresponding OWA is not too far away from the arithmetic mean ($q = 1$) which is natural for many dispersal effects in forests.

3 A Stochastic Model for Root Dispersal and Estimation of the Model Parameters

To describe a real root dispersal situation, a number of soil cores (with diameter 2,65 cm and volume 440 cm³) is placed in the neighbourhood of the trees which collect a random number of root mass units (1 unit = 1 mg). At first let us consider a single tree and M soil cores, each of area a and fixed depth and with distance r_j from the tree, $j = 1, \dots, M$. For the random number n_j of root mass units in soil core j we use a special nonlinear regression model:

$$E(n_j) = amp(r_j, \vartheta) =: \rho(r_j; m, \vartheta), j = 1, \dots, M, \tag{10}$$

- m mean total mass of fine roots of the tree
- $p(r, \vartheta)$ probability density for the location of a single root mass unit at distance r from the tree
- ϑ unknown parameter.

The justification of (10) comes from theory of stochastic point processes which is suppressed here (see, e.g. [4]). An often used model for $p(r, \vartheta)$ assumes log-normality, i.e. with a normalizing constant c we have

$$p(r; \mu, \sigma^2) = \frac{c}{\sigma r} \exp\left(-\frac{(\ln r - \mu)^2}{2\sigma^2}\right), \sigma > 0, \mu \in R. \tag{11}$$

Much more interesting, especially with regard to interaction, is the multi-tree case. Firstly, we restrict ourselves to the case of N trees of the same species. Consider the model of N additive overlapping trees, i.e.

$$E(n_j) = \sum_{i=1}^N \rho(r_{ij}; m_i, \vartheta) =: \rho_j^N(m_1, \dots, m_N, \vartheta), \tag{12}$$

where r_{ij} is the distance of soil core j from the tree i and let m_i be the total root mass of tree i . To avoid too much parameters we use an empirical relation between m_i and dbh_i , the stem diameter at breast height of tree i :

$$m_i = m \left(\frac{dbh_i}{30}\right)^\beta. \tag{13}$$

This relation is often used in forest sciences (see e.g. [4]) and expresses m_i by the mass m of a standard tree of $dbh = 30$ cm and an allometry parameter β , i.e. the N parameters m_i reduce to two parameters m and β . Now, (12) can be written as

$$E(n_j) = \sum_{i=1}^N \rho(r_{ij}; m, \beta, \vartheta) = \sum_{i=1}^N \rho_{ij}^N(m, \beta, \vartheta) =: \rho_j^N(m, \beta, \vartheta). \tag{14}$$

Note that (14) is given by the unweighted sum of the root masses of the N trees. But from ecological point of view this is not realistic for all cases of root dispersion. In some cases it seems to be more realistic to prefer the ‘most intensive’ or ‘strong’ trees, e.g. the trees closest to soil core j and to put (more or less) the remaining ‘weak’ trees at a disadvantage. For example, the strong tree takes up the total soil volume at some location and forces the roots of weaker trees to use other soil regions. On the other hand, it is conceivable that a strong tree with a number of fine roots can afford to accept roots of other trees, maybe from the same species, at some locations. These remarks lead in a natural way to a discrete Choquet integral of the root intensities. Consider the ordered intensities $\rho_{(ij)}^N(m, \beta, \vartheta)$ of soil core j , i.e. $\rho_{(1j)}^N(m, \beta, \vartheta) \leq \dots \leq \rho_{(Nj)}^N(m, \beta, \vartheta)$ and aggregate them by (see (4))

$$E(n_j) = N \sum_{i=1}^N w_{[i]} \rho_{(ij)}^N(m, \beta, \vartheta) = NC_V(\rho_j^N(m, \beta, \vartheta)), \tag{15}$$

where the weights $w_{[j]}$ are defined in (4) and the vector $\rho_j^N(m, \beta, \vartheta)$ contains as elements $\rho(r_{ij}; m, \beta, \vartheta), 1 \leq i \leq N$.

Let us explain the fuzzy measure v in (15). Let $A \subset \mathcal{N}$ be a subset of trees. Then $v(A)$ stands for the overall root mass if the trees of A produce alone, without any contribution of trees from $\mathcal{N} \setminus A$.

Now, let us justify the use of the discrete Choquet integral. In (15) all trees contribute at least $\rho_{(1j)}^N(m, \beta, \vartheta)$ root mass units to the total root mass at j . This results in a total root mass at least equal to $N\rho_{(1j)}^N(m, \beta, \vartheta)v(\mathcal{N})$ with $v(\mathcal{N}) = 1$. Each tree in $\mathcal{N} \setminus A_1$ contributes at least $\rho_{(2j)}^N(m, \beta, \vartheta)$ additional root mass units where A_1 collects the tree with the smallest individual contribution. Therefore, the increment of total root mass is at least equal to

$$N \left(\rho_{(2j)}^N(m, \beta, \vartheta) - \rho_{(1j)}^N(m, \beta, \vartheta) \right) v(\mathcal{N} \setminus A_1).$$

And so on. Summing up all these increments of total root mass units results exactly in the expression of the Choquet integral (15), see (4). Note that the weight of the smallest contribution is equal to $v(\mathcal{N}) - v(\mathcal{N} \setminus A_1)$ given by $v(A(1)) - v(A(2))$ in (4).

For a symmetric fuzzy measure fulfilling (6) (15) reduces to

$$E(n_j) = \text{NOWA}(\rho_j^N(m, \beta, \vartheta)). \tag{16}$$

Obviously, if the chosen weights w_i of the OWA operator (see (4)) are increasing with i than - with regard to a given soil core - ‘strong’ trees suppress ‘weak’ trees, which expresses negative synergy. In the opposite case, if ‘weak’ trees contribute above the average, i.e. if the w_i ’s are decreasing in i , we have positive synergy, see (7).

Now, consider the more general case that root masses of trees from two species are given. Let N_1 be the number of trees from species 1 and N_2 the number of trees from species 2. We will propose a two-step approach for the total root mass consisting of the masses of the two species. At the first step, the mean of the root mass $n_j^{(l)}$ in a soil core j coming from species $l = 1, 2$, can be expressed following (15):

$$E(n_j^{(l)}) = N_l C_{v_l}(\rho_j^{N_l}(m_l, \beta_l, \vartheta_l)) =: \rho_j^{N_l}(m_l, \beta_l; \vartheta_l),$$

where v_l is the specific fuzzy measure of species l . m_l is the total root mass of a standard tree from species l , β_l is the corresponding allometry parameter for this species l . The distributional parameters for species l are given by ϑ_l . Obviously, v_l controls the type of interaction inside species l , the so-called intra-specific interaction of species l . But in ecological context, interaction between species - the so-called inter-specific interaction - is also of great interest. We will model such effects at the second step. Proceeding from $E(n_j^{(1)})$ and $E(n_j^{(2)})$ for a given soil core j we can describe the mean of the total root mass n_j at j by an additional discrete Choquet integral:

$$\begin{aligned}
 E(n_j) &= 2C_{v_{12}} \left(\left(\rho_j^{N_1}(m_1, \beta_1; \vartheta_1), \rho_j^{N_2}(m_2, \beta_2; \vartheta_2) \right) \right) \\
 &=: \rho_j^{N_1 N_2}(m_1, \beta_1, \vartheta_1, \gamma_1; m_2, \beta_2; \vartheta_2, \gamma_2; \gamma_{12})
 \end{aligned}
 \tag{17}$$

where the inter-specific interaction is controlled by the fuzzy measure v_{12} .

The unknown model parameters in (17) are

m_l mean total mass of a standard tree from species l with $dbh = 30$ cm

β_l allometry parameter for species l

ϑ_l vector of distribution parameters in $p(r)$ for species l

γ_l vector of the parameters in the fuzzy measure v_l for species l controlling intra-specific interactivities

γ_{12} vector of the parameters in the fuzzy measure v_{12} for the total mass controlling inter-specific interactivities.

Now, we have to estimate the parameters by the use of soil core results n_j , $j = 1, \dots, M$. The simplest way is a least squares approximation

$$\sum_{j=1}^M \left(n_j - \rho_j^{N_1 N_2}(m_1, \beta_1, \vartheta_1, \gamma_1; m_2, \beta_2; \vartheta_2, \gamma_2; \gamma_{12}) \right)^2 \rightarrow \min.$$

Denote the estimated parameters by $\widehat{m}_l, \widehat{\beta}_l, \widehat{\vartheta}_l, \widehat{\gamma}_l$ and $\widehat{\gamma}_{12}$. As usual, the goodness of model fit can be expressed by the (mean) sum of squared residuals

$$S_M^2 := \frac{1}{M} \sum_{j=1}^M \left(n_j - \rho_j^{N_1 N_2}(\widehat{m}_1, \widehat{\vartheta}_1, \widehat{\gamma}_1; \widehat{m}_2, \widehat{\beta}_2; \widehat{\vartheta}_2, \widehat{\gamma}_2; \widehat{\gamma}_{12}) \right)^2.
 \tag{18}$$

In our case it is also useful to regard the sum of mean squared residuals for species l , i.e.

$$S_{M,l}^2 := \frac{1}{M} \sum_{j=1}^{M_l} \left(n_j^{(l)} - w_{[l]} \rho_j^{N_1}(\widehat{m}_l, \widehat{\vartheta}_l, \widehat{\gamma}_l) \right)^2.
 \tag{19}$$

For further details and remarks see [5].

4 A Real-case Study

The study was carried out in a mixed spruce and beech stand consisting of 11 beech (species 1) and 17 spruce trees (species 2) in Germany (Saxony) near to Dresden. The study site is part of a greater nearly homogenous spruce stand. In 2003 soil cores were taken at 226 given sampling points with collections of the fine root biomass from the forest floor organic and mineral horizon.

At first, fine root biomass dispersion was modelled without considering interaction, i.e. inter- and intra-specific aggregations were carried out as additive sums of the contributions of the trees. The allometry parameters β_1 and β_2 were taken constant with value 2. Assuming a lognormal model, see (11), this leads to the following estimates

$$\begin{aligned} \widehat{m}_1 &= 3.32 \cdot 10^6, \widehat{\mu}_1 = 2.45, \widehat{\sigma}_1 = 1.11 \\ \widehat{m}_2 &= 1.64 \cdot 10^6, \widehat{\mu}_2 = 1.85, \widehat{\sigma}_2 = 1.2 \end{aligned} \tag{20}$$

with $S_M^2 = 2365$, $S_{M,1}^2 = 1809$ for species 1 (beech) and $S_{M,2}^2 = 1126$ for species 2 (spruce), see (18) and (19). The empirical standard deviation of the root mass of species 1 s_1 is equal to 73.04. For species 2 $s_2 = 64.88$ holds. We denote

$$r_i := \frac{\sqrt{S_{M,i}^2}}{s_i} \tag{21}$$

and obtain $r_1 = 0.58$ and $r_2 = 0.52$. From statistical point of view these values are not very good with respect to model fitting. But let us refer to some problems connected with fine root data. Fine roots are characterized by a high spatial and temporal variability depending for example on the changing availability of resources. Therefore, discussing investigations based on one spatio-temporal sample we have to accept some remaining variability of the residuals.

Now, interaction effects are considered applying an OWA operator with one-parametric fuzzy measures from the Yager family, see (8). The number of model parameters increases to 9. We obtain the following estimates:

$$\begin{aligned} \widehat{m}_1 &= 4.67 \cdot 10^6, \widehat{\mu}_1 = 2.73, \widehat{\sigma}_1 = 1.3, \widehat{q}_1 = 0.74 \\ \widehat{m}_2 &= 9.27 \cdot 10^5, \widehat{\mu}_2 = 1.71, \widehat{\sigma}_2 = 0.94, \widehat{q}_2 = 1.37; \widehat{q}_{12} = 1.38 \end{aligned} \tag{22}$$

with $S_M^2 = 1903$, $S_{M,1}^2 = 1328$ and $S_{M,2}^2 = 951$. This leads to $r_1 = 0.54$ and $r_2 = 0.47$, see (21). Considering interactions results in visible improvement of the mean squared error of both species and the total root mass. To sum it up it can be said that the two species suppress each other, whereas fine root dispersal of the beeches is characterized by positive synergy. Considering that the study site comprises more spruces than beeches the supposition that positive synergy between the beeches enables their survival against the superiority of spruces is quite logical in ecological sense. In contrast to this the spruces are able to develop without intra-specific support or even with intra-specific suppression.

If the inter-specific interaction is modelled by an OWA operator, it is not possible to decide which species the other suppresses. This can be seen regarding the interaction index from (3). We obtain $\Phi_{spruce} = \Phi_{beech} = 0,5$. In ecological sense it is not satisfactory that the effect of suppression is equally distributed over the two species. Therefore, if we want to model that spruces suppress beeches we need a discrete Choquet integral as introduced in (17). Fortunately, it is easy to define a non-symmetric fuzzy measure for a set of two criteria. For negative synergy a sub-additive fuzzy measure is necessary. By

$$v(\{spruce, beech\}) = 1, v(\{beech\}) = w_1, v(\{spruce\}) = w_2, v(\emptyset) = 0 \tag{23}$$

with $w_1, w_2 \leq 1$ and $w_1 + w_2 > 1$ such a fuzzy measure is given. In this case

$$\Phi_{spruce} = \frac{1}{2} + \frac{1}{2}(w_1 - w_2) \text{ and } \Phi_{beech} = \frac{1}{2} + \frac{1}{2}(w_2 - w_1)$$

with $I_{12} = 1 - w_1 - w_2 < 0$ holds.

Now, the discrete Choquet integral (17) can be evaluated. Model fitting leads to

$$\begin{aligned}\hat{m}_1 &= 1.04 \cdot 10^7, \hat{\mu}_1 = 2.33, \hat{\sigma}_1 = 1.15, \hat{q}_1 = 0.73 \\ \hat{m}_2 &= 1.40 \cdot 10^6, \hat{\mu}_2 = 2.18, \hat{\sigma}_2 = 1.21, \hat{q}_2 = 1.82 \\ \hat{w}_1 &= 0.42, \hat{w}_2 = 0.8\end{aligned}\quad (24)$$

with $S^2 = 2050, S_{M,1}^2 = 1345, S_{M,2}^2 = 997, r_1 = 0.54, r_2 = 0.49$ and

$$I_{12}(v) = -0.22, \Phi_{beech} = 0.31, \Phi_{spruce} = 0.69.$$

Obviously, negative synergy is given for inter-specific interactions. Further, the spruces are more important than the beeches with respect to fine root biomass dispersal, i.e. the spruces suppress the beeches. Analogously to the OWA case (22) the beeches support each other whereas interaction within the spruces is shaped by negative synergy.

5 Conclusions

The paper presents only some first results and shows that modelling of interaction effects by fuzzy measures leads to ecologically meaningful results. In a future project, we will analyze further ecologically interesting parameters, e.g. humus thickness and quality or the shape of the tree-tops, and we expect much more clear interaction effects.

Acknowledgement

This research was supported by grants from the Deutsche Forschungsgemeinschaft (DFG). The authors are grateful to S. Wagner for useful discussions and to A. Zeibig, who collected the root data used in Section 4.

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