Measure-Free Martingales with Application to Classical Martingales

S.F. Cullender, W.-C. Kuo, C.C.A. Labuschagne and B.A. Watson

School of Mathematics, University of the Witwatersrand, Private Bag 3, WITS 2050, South Africa
scullender@maths.wits.ac.za, wpkuo@global.co.za,
cola@maths.wits.ac.za, bwatson@maths.wits.ac.za

Summary. The aim of this work is to give a summary of some of the known properties of sets of measure-free martingales in vector lattices and Banach spaces. In particular, we consider the relationship between such sets of martingales and the ranges of the underlying filtration of conditional expectation operators.

1 Introduction

There are examples in the literature where certain aspects of martingale theory are considered in a suitable framework which avoids the use of an underlying measure space (cf. [2, 3, 4, 5, 6, 7, 8, 9, 15, 16, 17, 18]). The aim of this work is to give a summary of some of the known properties of sets of measure-free martingales. In particular, we consider the relationship between such sets of martingales and the ranges of the underlying filtration of conditional expectation operators.

We assume that the reader is familiar with the terminology and notation of vector lattices (i.e. Riesz spaces) and Banach lattices, as can be found in [12, 14, 19].

Some general notation and terminology on martingales are in order at this stage, so as to avoid unnecessary repetition later.

Let E be a vector space. A sequence (T_i) of linear projections defined on E for which $T_iT_m = T_mT_i = T_i$ for each $m \geq i$ is called a *filtration* of linear projections on E. If $\mathscr{R}(T_i)$ denotes the range of T_i, then a filtration of linear projections (T_i) is a commuting family of linear projections with increasing ranges, i.e. $\mathscr{R}(T_i) \uparrow_i$. A sequence $(f_i, T_i)_{i\in\mathbb{N}}$, where (T_i) a filtration of linear projections on E and $f_i \in \mathscr{R}(T_i)$ for each $i \in \mathbb{N}$, is called a *martingale* if $f_i = T_i f_m$, for each $m \geq i$.

Let E be a vector space and (T_i) a filtration of linear projections on E and $M(E,T_i) := \big\{(f_i,T_i) : (f_i,T_i) \text{ is a martingale on } E\big\}$. Then $M(E,T_i)$ is a vector space if we define addition and scalar multiplication by

$$(f_i,T_i) + (g_i,T_i) = (f_i + g_i, T_i) \text{ and } \lambda(f_i,T_i) = (\lambda f_i, T_i) \text{ for each } \lambda \in \mathbb{R}.$$

If E is an ordered vector space and (T_i) a filtration of positive (i.e. $T_ix \geq 0$ for all $x \geq 0$) linear projections on E, define $\leq$ on $M(E,T_i)$ by $(f_i,T_i) \leq (g_i,T_i) \iff f_i \geq$

S.F. Cullender et al.: *Measure-Free Martingales with Application to Classical Martingales*, Advances in Soft Computing **6**, 121–128 (2006)
www.springerlink.com

g_i for all $i \in \mathbb{N}$, and let $M_+(E,T_i) := \big\{(f_i,T_i) : f_i \geq 0 \text{ for all } i \in \mathbb{N}\big\}$. Then $M(E,T_i)$ is an ordered vector space with positive cone $M_+(E,T_i)$.

If E is a vector lattice, then $e \in E_+$ is called a *weak order unit* for E if $x \in E_+$ implies that $x \wedge ne \uparrow x$. If (Ω,Σ,μ) is a probability space, then $\mathbf{1}$, defined by $\mathbf{1}(s) = 1$ for all $s \in \Omega$, is a weak order unit for $L^p(\mu)$ for all $1 \leq p < \infty$.

2 Martingales in Vector Lattices

In the setting of vector lattices with weak order units, the following definition is taken from [5], where a motivation is also given:

Definition 1. *Let E be a vector lattice with weak order unit e. A positive order continuous projection $T: E \to E$ for which $T(w)$ is a weak order unit in E for each weak order unit $w \in E_+$ and $\mathscr{R}(T)$ is a Dedekind complete Riesz subspace of E, is called a* conditional expectation *on E.*

A proof is given in [7] that the statement "$T(w)$ is a weak order unit in E for each weak order unit $w \in E_+$" in the preceding definition is equivalent to the statement "$T(e) = e$".

Let E be a vector lattice and (T_i) a filtration of positive linear projections on E. Let

$$M_{oc}(E,T_i) := \big\{(f_i,T_i) \in M(E,T_i) : (f_i) \text{ is order convergent in } E\big\},$$

$$M_{ob}(E,T_i) := \big\{(f_i,T_i) \in M(E,T_i) : (f_i) \text{ is an order bounded in } E\big\}.$$

It is easy to show that the above defined sets of martingales are ordered vector subspaces of $M(E,T_i)$. Moreover, since order convergent sequences are order bounded, $M_{oc}(E,T_i) \subseteq M_{ob}(E,T_i)$. However, [8, Corollary 5.2] shows that equality holds in the setting of vector lattices with weak order units:

Theorem 1. *Let E be a Dedekind complete vector lattice with weak order unit e, and let (T_i) be a filtration of conditional expectations on E. Then $M_{oc}(E,T_i) = M_{ob}(E,T_i)$.*

If E is a vector lattice and $T: E \to E$ is a positive linear map, then T is said to be *strictly positive* if $\big\{x \in E : T(|x|) = 0\big\} = \{0\}$.

There is a connection between $M_{oc}(E,T_i)$ and $\overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)}$, where the latter denotes the order closure of $\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)$:

Theorem 2. [8, Theorem 5.8] *Let E be a Dedekind complete vector lattice with weak order unit e and let (T_i) be a filtration of conditional expectations on E with T_1 strictly positive. Then $M_{oc}(E,T_i)$ is a Dedekind complete vector lattice and $L: M_{oc}(E,T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)}$, defined by $L((f_i,T_i)) = \lim_i f_i$ (order), is an order continuous surjective Riesz isomorphism.*

Let E be a vector lattice and (T_i) a filtration of positive linear projections on E. Let $M_r(E,T_i)$ denote the set of all *regular martingales* on E; i.e., those martingales (f_i,T_i) on E for which there exist $(g_i,T_i),(h_i,T_i) \in M_+(E,T_i)$ such that $f_i = g_i - h_i$.

It is readily verified that $M_r(E,T_i)$ is an ordered vector subspace of $M(E,T_i)$ and $M_r(E,T_i) = M_+(E,T_i) - M_+(E,T_i)$.

The simple proof given in [11] for the following result, is based on the ideas in [8] and the main idea in the proof of [18, Theorem 7]:

Theorem 3. *If E is a Dedekind* [σ*-Dedekind*] *complete vector lattice and* (T_n) *a filtration of order* [σ*-order*] *continuous positive linear projections on E, then* $M_r(E,T_i)$ *is a Dedekind* [σ*-Dedekind*] *complete vector lattice.*

3 Martingales in Banach Spaces and Banach Lattices

Let (Ω,Σ,μ) denote a probability space. Then, for $1 \le p < \infty$ and X a Banach space, let $L^p(\mu,X)$ denote the space of (classes of a.e. equal) Bochner p-integrable functions $f\colon \Omega \to X$ and denote the Bochner norm on $L^p(\mu,X)$ by Δ_p, i.e. $\Delta_p(f) = \left(\int_\Omega \|f\|_X^p d\mu\right)^{1/p}$.

If one wants to apply a measure-free approach to martingales on $L^p(\mu,X)$-spaces, a measure-free approach to Banach spaces has to be considered. In [2, 3], such an approach is followed:

Let X be a Banach space and (T_i) a filtration of contractive linear projections on X. Define $\|\cdot\|$ on $M(X,T_i)$ by $\|(f_i,T_i)\| = \sup_i \|f_i\|$ and let $\mathscr{M}(X,T_i)$ denote the space of *norm bounded martingales* on X; i.e., $\mathscr{M}(X,T_i) = \{(f_i,T_i) \in M(X,T_i) : \|(f_i,T_i)\| < \infty\}$. Then $\mathscr{M}(X,T_i)$ is a Banach space with respect to $\|\cdot\|$.

Let $\mathscr{M}_{nc}(X,T_i)$ denote the space of *norm convergent martingales* on X; i.e., $\mathscr{M}_{nc}(X,T_i) = \{(f_i,T_i) \in \mathscr{M}(X,T_i) : (f_i) \text{ is norm convergent in } X\}$.

To describe $\mathscr{M}_{nc}(X,T_i)$, the following results are used in [2]:

Proposition 1. *Let X be a Banach space and let* (T_i) *be a filtration of contractive linear projections on X. Then* $f \in \overline{\bigcup_{i=1}^\infty \mathscr{R}(T_i)}$*, the latter denoting the norm closure of* $\bigcup_{i=1}^\infty \mathscr{R}(T_i)$*, if and only if* $\|T_i f - f\| \to 0$.

Corollary 1. *Let X be a Banach space and let* (f_i,T_i) *be a martingale in X, where* (T_i) *is a filtration of contractive linear projections on X. Then* (f_i,T_i) *converges to f if and only if* $f \in \overline{\bigcup_{i=1}^\infty \mathscr{R}(T_i)}$ *and* $f_i = T_i f$ *for all* $i \in \mathbb{N}$.

An application in [2] of Proposition 1 and Corollary 1 yields:

Proposition 2. *Let X be a Banach space and* (T_i) *a filtration of contractive linear projections on X. Then* $L\colon \mathscr{M}_{nc}(X,T_i) \to \overline{\bigcup_{i=1}^\infty \mathscr{R}(T_i)}$*, defined by* $L((f_i,T_i)) = \lim_i f_i$ (*norm*)*, is a surjective isometry.*

Another application in [3] of Proposition 1 and Corollary 1 provides a proof, via martingale techniques, for the following well known result:

Proposition 3. *Let X be a Banach space and (x_i) a basic sequence in X. Then (x_i) is an unconditional basic sequence if and only if the closure of the span of (x_i), denoted $[x_i]$, can be renormed so that it is an order continuous Banach lattice with positive cone*

$$C_+^{(x_i)} := \left\{ \sum_{i=1}^{\infty} \alpha_i x_i \in [x_i] : \alpha_i \geq 0 \textit{ for each } i \in \mathbb{N} \right\}.$$

Motivated by [18], Proposition 2 is specialized in [2] to Banach lattices to obtain:

Proposition 4. *Let E be a Banach lattice and (T_i) a filtration of positive contractive linear projections on E for which $\overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)}$ is a closed Riesz subspace of E. If $L\colon \mathscr{M}_{nc}(E,T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)}$ is defined by $L((f_i,T_i)) = \lim_i f_i$, then $\mathscr{M}_{nc}(E,T_i)$ is a Banach lattice and $L\colon \mathscr{M}_{nc}(E,T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)}$ is a surjective Riesz isometry.*

By Corollary 1 we have

$$\mathscr{M}_{nc}(E,T_i) = \big\{ (f_i,T_i) \in \mathscr{M}_{nb}(E,T_i) : \exists f \in E \text{ such that } f_i = T_i f \to f \big\}.$$

Corollary 2. *Let E be a Banach lattice and (T_i) a filtration of positive contractive linear projections on E for which $\overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)}$ is a closed Riesz subspace of E. Then $\mathscr{M}_{nc}(E,T_i)$ is a Banach lattice in which the following formulas hold:*

$$\begin{aligned}
\big(\lim_{m\to\infty} T_n f_m, T_n\big)^+ &= \big(\lim_{m\to\infty} T_n f_m^+, T_n\big);\\
\big(\lim_{m\to\infty} T_n f_m, T_n\big)^- &= \big(\lim_{m\to\infty} T_n f_m^-, T_n\big);\\
\big(\lim_{m\to\infty} T_n f_m, T_n\big) \vee \big(\lim_{m\to\infty} T_n g_m, T_n\big) &= \big(\lim_{m\to\infty} T_n (f_m \vee g_m), T_n\big);\\
\big(\lim_{m\to\infty} T_n f_m, T_n\big) \wedge \big(\lim_{m\to\infty} T_n g_m, T_n\big) &= \big(\lim_{m\to\infty} T_n (f_m \wedge g_m), T_n\big);\\
\Big|\big(\lim_{m\to\infty} T_n f_m, T_n\big)\Big| &= \big(\lim_{m\to\infty} T_n |f_m|, T_n\big).
\end{aligned} \tag{1}$$

Proof. By Proposition 4, we have that $\mathscr{M}_{nc}(E,T_i)$ is a Banach lattice.

The formulas are easy to prove. Since L is a bijective Riesz homomorphism, it follows from $L(T_n|f|,T_n) = |f| = |L(T_n f,T_n)|$ that $(T_n|f|,T_n) = L^{-1}(L(T_n|f|,T_n)) = L^{-1}(|L(T_n f,T_n)|) = |(L^{-1}(L(T_n f,T_n))| = |(T_n f,T_n)|$. The other formulas follow in a similar manner. □

Let (T_i) be a filtration of positive contractive linear projections on a Banach lattice E. As in [18], we now consider the space

$$\mathscr{M}_r(E,T_i) = \big\{ (f_i,T_i) \in \mathscr{M}(E,T_i) : \exists (g_i,T_i) \in M_+(E,T_i), f_i \leq g_i \; \forall\, i \in \mathbb{N} \big\},$$

the elements of which are called *regular norm bounded* martingales.

Troitski proves in [18] that the formulas in (1) also hold in $\mathscr{M}_r(E,T_i)$ and in $\mathscr{M}(E,T_i)$. He uses less stringent assumptions on (T_i) than in Corollary 2, but he makes additional assumptions on E:

Theorem 4. ([18, Theorems 7 and 13]) *Let E be a Banach lattice and (T_i) a filtration of positive contractive linear projections on a Banach lattice E.*

(a) If E is an order continuous Banach lattice, then $\mathscr{M}_r(E,T_i)$ is a Dedekind complete Banach lattice with lattice operations given by (1) *and martingale norm given by $\|(f_n,T_n)\| = \sup_n \|f_n\|$.*
(b) If E is a KB-space, then $\mathscr{M}(E,T_i)$ is a Banach lattice with lattice operations given by (1) *and martingale norm $\|(f_n,T_n)\| = \sup_n \|f_n\|$.*

It follows easily that if $\overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)}$ is a closed Riesz subspace of E, then

$$\mathscr{M}_{oc}(E,T_i) \subseteq \mathscr{M}_r(E,T_i) \subseteq \mathscr{M}(E,T_i). \tag{2}$$

One can say more about the inclusions in (2) under additional assumptions on E (see [18, Proposition 16]):

Corollary 3. *Let E be a Banach lattice with order continuous norm and (T_i) a filtration of positive contractive linear projections on a Banach lattice E for which $\overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)}$ is a closed Riesz subspace of E.*

(a) If E is an order continuous Banach lattice, then $\mathscr{M}_{nc}(E,T_i)$ is an ideal in $\mathscr{M}_r(E,T_i)$.
(b) If E is a KB-space, then $\mathscr{M}_r(E,T_i) = \mathscr{M}(E,T_i)$ and $\mathscr{M}_{nc}(E,T_i)$ is a projection band in $\mathscr{M}(E,T_i)$.

4 Martingales in $L^p(\mu,X)$

Chaney and Schaefer extended the Bochner norm to the tensor product of a Banach lattice and a Banach space (see [1] and [14]). If E is a Banach lattice and Y is a Banach space, then the l-norm of $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes Y$ is given by $\|u\|_l = \inf\{\, \| \sum_{i=1}^{n} \|y_i\| |x_i| \,\| : u = \sum_{i=1}^{n} x_i \otimes y_i\}$.

Furthermore, if $E = L^p(\mu)$ where (Ω,Σ,μ) is a σ-finite measure space, then we have that $E\widetilde{\otimes}_l Y$ is isometric to $L^p(\mu,Y)$.

Let E and F be Banach lattices. We denote the *projective cone* of $E\otimes F$ by $E_+ \otimes F_+ := \{\sum_{i=1}^{n} x_i \otimes y_i : (x_i,y_i) \in E_+ \times F_+\}$. It was shown by Chaney and Schaefer that $E\widetilde{\otimes}_l F$ is a Banach lattice with positive cone the l-closure of $E_+ \otimes F_+$.

Let E_1 and E_2 be Banach lattices and Y_1 and Y_2 Banach spaces. If $S\colon E_1 \to E_2$ is a positive linear operator and $T\colon Y_1 \to Y_2$ a bounded linear operator, then $\big\|(S\otimes T)u\big\|_l \le \|S\|\,\|T\|\,\|u\|_l$ for all $u \in E_1 \otimes Y_1$ (see [10]).

The following is proved in [2]:

Theorem 5. *Let E be a Banach lattice and Y be a Banach space* [*lattice*]. *If (S_i) is a filtration of positive contractive linear projections on E with each $\mathscr{R}(S_i)$ a closed Riesz subspace of E, and (T_i) is a filtration of* [*positive*] *contractive linear projections on Y* [*and each $\mathscr{R}(T_i)$ is a closed Riesz subspace of Y*]*, then $(S_i \otimes_l T_i)$ is a filtration of* [*positive*] *contractive linear projections on $E\widetilde{\otimes}_l Y$ with each $S(E_i)\widetilde{\otimes}_l T(Y_i)$ a closed* [*Riesz*] *subspace of $E\widetilde{\otimes}_l Y$.*

To consider tensor product versions of some of the martingale results stated earlier, we need the following result noted by Popa, [13]:

Theorem 6. *Let E and F be Banach lattices.*

(a) If E and F are order continuous Banach lattices, then $E\tilde{\otimes}_l F$, is an order continuous Banach lattice.
(b) If E and F are KB-spaces, then $E\tilde{\otimes}_l F$ is a KB-space.

The following is an l-tensor product version of Corollary 3.

Theorem 7. *Let E and F be Banach lattices and let (S_i) and (T_i) be filtrations of positive contractive linear projections on E and F respectively with each $\mathscr{R}(S_i)$ and each $\mathscr{R}(T_i)$ a closed Riesz subspace of E and F respectively.*

(a) If E and F are order continuous Banach lattices, then $\mathscr{M}_r(E\tilde{\otimes}_l F, T_i \otimes_l S_i)$ is a Banach lattice and $\mathscr{M}_{nc}(E\tilde{\otimes}_l F, T_i \otimes_l S_i)$ is an ideal in $\mathscr{M}_r(E\tilde{\otimes}_l F, T_i \otimes_l S_i)$.
(b) If E and F are KB-spaces, then $\mathscr{M}(E\tilde{\otimes}_l F, T_i \otimes_l S_i)$ is a Banach lattice and $\mathscr{M}_{nc}(E\tilde{\otimes}_l F, T_i \otimes_l S_i)$ is a projection band in $\mathscr{M}(E\tilde{\otimes}_l F, T_i \otimes_l S_i)$.

Proof (a) Since E and F are order continuous Banach lattices, $E\tilde{\otimes}_l F$ is an order continuous Banach lattice, by Popa's result. By Proposition 5, we get that $(S_i \otimes_l T_i)$ is a filtration of positive contractive linear projections on $E\tilde{\otimes}_l F$ with $\overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i \otimes S_i)}$ a closed Riesz subspace of $E\tilde{\otimes}_l F$. But then $\mathscr{M}_{nc}(E\tilde{\otimes}_\alpha F, T_i \otimes S_i)$ is an ideal in the Banach lattice $\mathscr{M}_r(E\tilde{\otimes}_l F, T_i \otimes_l S_i)$, by Corollary 3 (a).

(b) Since E and F are KB-spaces, $E\tilde{\otimes}_l F$ is a KB-space, by Popa's result. Similar reasoning as in (a), but by using Corollary 3 (b), shows that $\mathscr{M}_{nc}(E\tilde{\otimes}_l F, T_i \otimes S_i)$ is a projection band in the Banach lattice $\mathscr{M}(E\tilde{\otimes}_l F, T_i \otimes S_i) = \mathscr{M}_r(E\tilde{\otimes}_l F, T_i \otimes S_i)$. □

In [2], we show that, if (S_i) is a filtration of positive contractive linear projections on the Banach lattice E such that each $\mathscr{R}(S_i)$ is a closed Riesz subspace of E and (T_i) is a filtration of contractive linear projections on the Banach space Y, then $\overline{\bigcup_{i=1}^{\infty} \mathscr{R}(S_i)} \tilde{\otimes}_l \overline{\bigcup_{i=1}^{\infty} \mathscr{R}(T_i)} = \overline{\bigcup_{i=1}^{\infty} \mathscr{R}(S_i \otimes_l T_i)}$.

In [10], it is shown that, if E is a Banach lattice and Y a Banach space, then $u \in E\tilde{\otimes}_l Y$ if and only if $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, where $\left\| \sum_{i=1}^{\infty} |x_i| \right\|_E < \infty$ and $\lim_{i\to\infty} \|y_i\|_Y = 0$.

As a consequence, the following result is derived in [2].

Theorem 8. *Let (S_i) be a filtration of positive contractive linear projections on the Banach lattice E such that each $\mathscr{R}(S_i)$ is a closed Riesz subspace of E and (T_i) a filtration of contractive linear projections on the Banach space Y. Then, in order for $M = (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ to be a convergent martingale in $E\tilde{\otimes}_l Y$, it is necessary and sufficient that, for each $i \in \mathbb{N}$, there exist convergent martingales $\left(x_i^{(n)}, S_n\right)_{n=1}^{\infty}$ and $\left(y_i^{(n)}, T_n\right)_{n=1}^{\infty}$ in E and Y respectively such that, for each $n \in \mathbb{N}$, we have*

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)},$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\| < \infty \textit{ and } \lim_{i\to\infty} \left\| \lim_{n\to\infty} y_i^{(n)} \right\| = 0.$$

As a simple consequence of Theorem 8, the following representation result is noted in [2]:

Theorem 9. *Let* (Ω, Σ, μ) *denote a probability space,* $(\Sigma_n)_{n=1}^{\infty}$ *a filtration, X a Banach space and* $1 \leq p < \infty$*. Then, in order for* $(f_n, \Sigma_n)_{n=1}^{\infty}$ *to be a convergent martingale in* $L^p(\mu, X)$*, it is necessary and sufficient that, for each* $i \in \mathbb{N}$*, there exist a convergent martingale* $\left(x_i^{(n)}, \Sigma_n\right)_{n=1}^{\infty}$ *in* $L^p(\mu)$ *and* $y_i \in X$ *such that, for each* $n \in \mathbb{N}$*, we have*

$$f_n(s) = \sum_{i=1}^{\infty} x_i^{(n)}(s) y_i \textit{ for all } s \in \Omega,$$

where $\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\|_{L^p(\mu)} < \infty$ *and* $\lim_{i\to\infty} \|y_i\| = 0$.

Acknowledgement

The third and fourth named authors were funded by the John Knopfmacher Centre for Applicable Analysis and Number Theory. The third named author also gratefully acknowledges funding by the Anderson Capelli Fund and thanks Professor Shoumei Li and her students – Li Guan and Xiang Li – for their warm hospitality during his stay at the Beijing University of Technology in March/April 2006.

References

[1] Chaney, J. (1972). Banach lattices of compact maps. *Math. Z.* **129**, 1-19.
[2] Cullender, S.F. and Labuschagne, C.C.A. (2005). A description of norm-convergent martingales on vector valued L^p-spaces. *J. Math. Anal. Appl.*, (to appear).
[3] Cullender, S.F. and Labuschagne, C.C.A. (2006). Unconditional martingale difference sequences in a Banach space. *J. Math. Anal. Appl.*, (to appear).
[4] DeMarr, R. (1966). A martingale convergence theorem in vector lattices. *Canad. J. Math.* **18**, 424-432.
[5] Kuo, W.-C., Labuschagne, C.C.A. and Watson, B.A. (2004). Discrete-time stochastic processes on Riesz spaces. *Indag. Math. (N.S.)* **15**, 435-451.
[6] Kuo, W.-C., Labuschagne, C.C.A. and Watson, B.A. (2004). Riesz space and fuzzy upcrossing theorems. In: Soft methodology and random information systems. *Advances in Soft Computing Springer*, Berlin, pp. 101-108.
[7] Kuo, W.-C., Labuschagne, C.C.A. and Watson, B.A. (2005). Conditional expectations on Riesz spaces. *J. Math. Anal. Appl.* **303**, 509-521.

[8] Kuo, W.-C., Labuschagne, C.C.A. and Watson, B.A. (2005). Convergence of Riesz space martingales. *Indag. Math. (N.S.)*, (to appear).

[9] Kuo, W.-C., Labuschagne, C.C.A. and Watson, B.A. (2006). Ergodic theory and the strong law of large numbers on Riesz spaces. *J. Math. Anal. Appl.*, (to appear).

[10] Labuschagne, C.C.A. (2004). Characterizing the one-sided tensor norms Δ_p and ${}^t\Delta_p$. *Quaest. Math.* **27**, 339-363.

[11] Labuschagne, C.C.A., Pinchuck, A.L. and van Alten, C.J. (2006). R*r*adström embeddings of vector lattice cones. (Preprint).

[12] Meyer-Nieberg, P. (1991). *Banach lattices*. Springer Verlag.

[13] Popa, N. (1979). Die Permanenzeigenschaften der Tensorprodukte von Banachverbänden. In: *Romanian-Finnish Seminar on Complex Analysis (Proc. Bucharest 1976)*, LNM Berlin-Heidelberg-New York, **743**, 627-647.

[14] Schaefer, H.H. (1974). *Banach Lattices and Positive Operators*. Springer Verlag.

[15] Stoica, G. (1990). Martingales in vector lattices. *Bull. Math. Soc. Sci. Math. Roumanie. (N.S.)* **34(82)**, 357-362.

[16] Stoica, G. (1991). Martingales in vector lattices II. *Bull. Math. Soc. Sci. Math. Roumanie. (N.S.)* **35(83)**, 155-157.

[17] Stoica, G. (1994). The structure of stochastic processes in normed vector lattices. *Stud. Cerc. Mat.* **46**, 477-486.

[18] Troitsky, V. (2005). Martingales in Banach lattices. *Positivity* **9**, 437-456.

[19] Zaanen, A.C. (1997). *Introduction to Operator Theory in Riesz Space*. Springer Verlag.