Option Pricing Theory in Financial Engineering from the Viewpoint of Fuzzy Logic

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Summary. A mathematical model for European/American options with uncertainty is presented. The uncertainty is represented by both randomness and fuzziness. The randomness and fuzziness are evaluated respectively by probabilistic expectation and fuzzy expectation defined by a possibility measure from the viewpoint of decision-maker's subjective judgment. Prices of European call/put options with uncertainty are presented, and their valuation and properties are discussed under a reasonable assumption. The hedging strategies are also considered for marketability of the European options in portfolio selection. Further, the American options model with uncertainty is discussed by a numerical approach and is compared with the analytical case of the infinite terminal time. The buyer's/seller's rational range of the optimal expected price in each option is presented and the meaning and properties of the optimal expected prices are discussed.

Keywords: American option; European option; Black-Scholes formula; hedging strategy; stopping time; Sugeno integral; Choquet integral; fuzzy goal; fuzzy stochastic process.

1 Introduction and Notations

Option pricing theory in financial market has been developing together with financial engineering based on the famous Black-Scholes model. When we sell or buy stocks in financial market, there sometimes exists a difference between the actual prices and the theoretical value which derived from Black-Scholes method. Actually we cannot utilize timely some of fundamental data regarding the market, and therefore there exists uncertainty which we cannot represent by only probability theory because the concept of probability is constructed on mathematical representation whether something occurs or not in the future. When the market is unstable and changing rapidly, the losses/errors often become bigger between the decision maker's expected price and the actual price. Introducing fuzzy logic to the log-normal stochastic processes designed for the financial market, we present a model with uncertainty of both randomness and

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fuzziness in output, which is a reasonable and natural extension of the original log-normal stochastic processes in Black-Scholes model. To valuate American options, we need to deal with an optimal stopping problem in log-normal stochastic processes (Elliott and Kopp [4], Karatzas and Shreve [5], Ross [10] and so on). We introduce a *fuzzy stochastic process* by fuzzy random variables to define prices in American options, and we evaluate the randomness and fuzziness by probabilistic expectation and fuzzy expectation defined by a possibility measure from the viewpoint of Yoshida [13]. We discuss the following themes on the basis of the results in Yoshida [14, 15, 17].

- American put option in a stochastic and fuzzy environment
  - The case with an expiration date
  - The perpetual option case without expiration dates
- European call/put options in a stochastic and fuzzy environment
  - Option pricing formula
  - Hedging strategies

In the next section, we introduce a fuzzy stochastic process by fuzzy random variables to define prices for American put option with uncertainty. We call the prices *fuzzy prices*. The randomness and fuzziness in the fuzzy stochastic process are evaluated by both probabilistic expectation and fuzzy expectation defined by a possibility measure, taking account of decision-maker's subjective judgment (Yoshida [13]). In Section 2, we deal with two models in American options with uncertainty, the case with an expiration date and the perpetual option case without expiration dates, and it is shown that the optimal fuzzy price is a solution of an optimality equation under a reasonable assumption. In Sections 3 and 4, we consider the optimal expected price in the American put option and we discuss seller's permissible range of expected prices, and we also give an optimal exercise time for the American put option. In Section 5, we give prices in European call/put options with uncertainty and we discuss their valuation and properties under a reasonable assumption. Finally, we give an explicit formula for the fuzzy prices in European options. We consider a rational expected price of the European options and buyer's/seller's permissible range of expected prices. The meaning and properties of rational expected prices are discussed in a numerical example. In the last section, we consider hedging strategies for marketability of the European options.

In the remainder of this section, we describe notations regarding bond price processes and stock price processes. We consider American put option in a finance model where there is no arbitrage opportunities ([4, 5]). Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field and P is a non-atomic probability measure. $\mathbb{R}$ denotes the set of all real numbers. For a stock, let μ be the appreciation rate and let σ be the volatility ($\mu \in \mathbb{R}$, $\sigma > 0$). Let $\{B_t\}_{t \geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{M}, P)$. $\{\mathcal{M}_t\}_{t \geq 0}$ denotes a family of nondecreasing right-continuous complete sub-σ-fields of $\mathcal{M}$ such that $\mathcal{M}_t$ is generated by $B_s (0 \leq s \leq t)$. We consider two assets, a bond price and a stock price, where the bond price process $\{R_t\}_{t \geq 0}$ is riskless and the stock price process $\{S_t\}_{t\geq 0}$ is risky. Let r ($r \geq 0$) be a instantaneous interest rate, i.e. a interest factor, on the bond, and the bond price process $\{R_t\}_{t\geq 0}$ is given by $R_t = e^{rt}$ ($t \geq 0$). Let the stock price process $\{S_t\}_{t\geq 0}$ satisfy the following log-normal stochastic differential equation in Black-Scholes model: S_0 is a positive constant, and

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t, \tag{1}$$

$t \geq 0$. It is known ([4]) that there exists an equivalent probability measure Q. Under Q, $W_t := B_t - ((r-\mu)/\sigma)t$ is a standard Brownian motion and it holds that $\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t$. By Ito's formula, we have $S_t = S_0 \exp((r - \sigma^2/2)t + \sigma W_t)$ ($t \geq 0$). The present stock price is determined by the information regarding the market until the previous time, and the present stock price S_t contains a certain uncertainty since we cannot utilize some of fundamental data actually at the current time t. The uncertainty comes from imprecision of information in the present market and is different from randomness, which is based on whether something occurs or not in the future. In the next section, we introduce fuzzy random variables to represent the uncertainty using fuzzy set theory.

2 Fuzzy Stochastic Processes

Fuzzy random variables, which take values in fuzzy numbers, have been studied by Puri and Ralescu [9] and many authors. It is known that the fuzzy random variable is one of the successful hybrid notions of randomness and fuzziness. First we introduce fuzzy numbers. Let $\mathcal{I}$ be the set of all non-empty bounded closed intervals. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support (Zadeh [19], Klir and Yuan [6]). We identify a fuzzy number with its corresponding membership function. $\mathcal{R}$ denotes the set of all fuzzy numbers. The α-cut of a fuzzy number $\tilde{a} (\in \mathcal{R})$ is given by $\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\}$ ($\alpha \in (0,1]$) and $\tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\}$, where cl denotes the closure of an interval. We write the closed intervals as $\tilde{a}_\alpha := [\tilde{a}^-_\alpha, \tilde{a}^+_\alpha]$ for $\alpha \in [0,1]$. We also use a metric δ_∞ on $\mathcal{R}$ defined by $\delta_\infty(\tilde{a}, \tilde{b}) = \sup_{\alpha\in[0,1]} \delta(\tilde{a}_\alpha, \tilde{b}_\alpha)$ for fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, where δ is the Hausdorff metric on $\mathcal{I}$. Hence we introduce a partial order $\succeq$, so called the fuzzy max order, on fuzzy numbers $\mathcal{R}$([6]): Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers. Then $\tilde{a} \succeq \tilde{b}$ means that $\tilde{a}^-_\alpha \geq \tilde{b}^-_\alpha$ and $\tilde{a}^+_\alpha \geq \tilde{b}^+_\alpha$ for all $\alpha \in [0,1]$. Then $(\mathcal{R}, \succeq)$ becomes a lattice ([13]). For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, we define the maximum $\tilde{a} \vee \tilde{b}$ with respect to the fuzzy max order $\succeq$ by the fuzzy number whose α-cuts are $(\tilde{a} \vee \tilde{b})_\alpha = [\max\{\tilde{a}^-_\alpha, \tilde{b}^-_\alpha\}, \max\{\tilde{a}^+_\alpha, \tilde{b}^+_\alpha\}]$ ($\alpha \in [0,1]$). An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows: For $\tilde{a}, \tilde{b} \in \mathcal{R}$ and $\lambda \geq 0$, the addition and subtraction $\tilde{a} \pm \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\lambda\tilde{a}$ of λ and $\tilde{a}$ are fuzzy numbers given by $(\tilde{a} + \tilde{b})_\alpha := [\tilde{a}^-_\alpha + \tilde{b}^-_\alpha, \tilde{a}^+_\alpha + \tilde{b}^+_\alpha]$, $(\tilde{a} - \tilde{b})_\alpha := [\tilde{a}^-_\alpha - \tilde{b}^+_\alpha, \tilde{a}^+_\alpha - \tilde{b}^-_\alpha]$ and $(\lambda\tilde{a})_\alpha := [\lambda\tilde{a}^-_\alpha, \lambda\tilde{a}^+_\alpha]$ for $\alpha \in [0,1]$.

A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a fuzzy random variable if $\{(\omega, x) \in \Omega \times \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} \in \mathcal{M} \times \mathcal{B}$ for all $\alpha \in [0,1]$, where $\mathcal{B}$ is the Borel σ-field of $\mathbb{R}$. We can find some equivalent conditions in general cases ([9]), however we adopt a simple characterization in the following lemma.

Lemma 1 (Wang and Zhang [12, Theorems 2.1 and 2.2]). *For a map $\tilde{X}$: $\Omega \mapsto \mathcal{R}$, the following (i) and (ii) are equivalent:*

(i) *$\tilde{X}$ is a fuzzy random variable.*
(ii) *The maps $\omega \mapsto \tilde{X}^-_\alpha(\omega)$ and $\omega \mapsto \tilde{X}^+_\alpha(\omega)$ are measurable for all $\alpha \in [0,1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}^-_\alpha(\omega), \tilde{X}^+_\alpha(\omega)] := \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$.*

From Lemma 1, we obtain the following lemma regarding fuzzy random variables $\tilde{X}$ and their α-cuts $\tilde{X}_\alpha(\omega) = [\tilde{X}^-_\alpha(\omega), \tilde{X}^+_\alpha(\omega)]$.

Lemma 2.

(i) *Let $\tilde{X}$ be a fuzzy random variable. The α-cuts $\tilde{X}_\alpha(\omega) = [\tilde{X}^-_\alpha(\omega), \tilde{X}^+_\alpha(\omega)]$, $\omega \in \Omega$, have the following properties (a) — (c):*
 (a) *$\tilde{X}_\alpha(\omega) \subset \tilde{X}_{\alpha'}(\omega)$ for $\omega \in \Omega$, $0 \leq \alpha' < \alpha \leq 1$.*
 (b) *$\lim_{\alpha' \uparrow \alpha} \tilde{X}_{\alpha'}(\omega) = \tilde{X}_\alpha(\omega)$ for $\omega \in \Omega$, $\alpha > 0$.*
 (c) *The maps $\omega \mapsto \tilde{X}^-_\alpha(\omega)$ and $\omega \mapsto \tilde{X}^+_\alpha(\omega)$ are measurable for $\alpha \in [0,1]$.*
(ii) *Conversely, suppose that a family of interval-valued maps $X_\alpha = [X^-_\alpha, X^+_\alpha]$: $\Omega \mapsto \mathcal{I}$ ($\alpha \in [0,1]$) satisfies the above conditions (a) – (c). Then, a membership function*

$$\tilde{X}(\omega)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{X_\alpha(\omega)}(x)\}, \quad \omega \in \Omega,\ x \in \mathbb{R},$$

gives a fuzzy random variable and $\tilde{X}_\alpha(\omega) = X_\alpha(\omega)$ for $\omega \in \Omega$ and $\alpha \in [0,1]$, where $1_{\{\cdot\}}$ denotes the characteristic function of an interval.

Next we need to introduce expectations of fuzzy random variables in order to describe fuzzy-valued European option models in the next section. A fuzzy random variable $\tilde{X}$ is called integrably bounded if both $\omega \mapsto \tilde{X}^-_\alpha(\omega)$ and $\omega \mapsto \tilde{X}^+_\alpha(\omega)$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{X}$ be an integrally bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}, \quad x \in \mathbb{R},$$

where $E(\tilde{X})_\alpha := [\int_\Omega \tilde{X}^-_\alpha(\omega)\, \mathrm{d}P(\omega), \int_\Omega \tilde{X}^+_\alpha(\omega)\, \mathrm{d}P(\omega)]$ ($\alpha \in [0,1]$).

Now, we consider a continuous-time fuzzy stochastic process by fuzzy random variables. Let $\{\tilde{X}_t\}_{t \geq 0}$ be a family of integrally bounded fuzzy random variables. We assume that the map $t \mapsto \tilde{X}_t(\omega) (\in \mathcal{R})$ is continuous on $[0, \infty)$ for almost all $\omega \in \Omega$. $\{\mathcal{M}_t\}_{t \geq 0}$ is a family of nondecreasing sub-σ-fields of $\mathcal{M}$

which is right continuous, and fuzzy random variables $\tilde{X}_t$ are $\mathcal{M}_t$-adapted, i.e. random variables $\tilde{X}^-_{r,\alpha}$ and $\tilde{X}^+_{r,\alpha}$ $(0 \le r \le t; \alpha \in [0,1])$ are $\mathcal{M}_t$-measurable. Then $(\tilde{X}_t, \mathcal{M}_t)_{t\ge 0}$ is called a fuzzy stochastic process.

We introduce a valuation method of fuzzy prices, taking into account of decision maker's subjective judgment. Give a fuzzy goal by a fuzzy set $\varphi : [0,\infty) \mapsto [0,1]$ which is a continuous and increasing function with $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = 1$. Then we note that the α-cut is $\varphi_\alpha = [\varphi^-_\alpha, \infty)$ for $\alpha \in (0,1)$. For an exercise time T and call/put options with fuzzy values $\tilde{X}_T = \tilde{C}_T$ or $\tilde{X}_T = \tilde{P}_T$, which will be given as fuzzy prices in the next section, we define a fuzzy expectation of the fuzzy numbers $E(\tilde{X}_T)$ by

$$\tilde{E}(E(\tilde{X}_T)) := ⨍_{[0,\infty)} E(\tilde{X}_T)(x)\, \mathrm{d}\tilde{m}(x) = \sup_{x\in[0,\infty)} \min\{E(\tilde{X}_T)(x), \varphi(x)\}, \quad (2)$$

where $\tilde{m}$ is the possibility measure generated by the density φ and $⨍ \mathrm{d}\tilde{m}$ denotes Sugeno integral ([11]). The fuzzy number $E(\tilde{X}_T)$ means a fuzzy price, and the fuzzy expectation (2) implies the degree of buyer's/seller's satisfaction regarding fuzzy prices $E(\tilde{X}_T)$. Then the fuzzy goal $\varphi(x)$ means a kind of utility function for expected prices x in (2), and it represents a buyer's/seller's subjective judgment from the idea of Bellman and Zadeh [1]. Hence, a real number $x^*(\in [0,\infty))$ is called a rational expected price if it attains the supremum of the fuzzy expectation (2), i.e.

$$\tilde{E}(\tilde{V}) = \sup_{x\in[0,\infty)} \min\{\tilde{V}(x), \varphi(x)\} = \min\{\tilde{V}(x^*), \varphi(x^*)\}, \quad (3)$$

where

$$\tilde{V} := E(\tilde{X}_T)$$

is a fuzzy price of options.

We also consider about an estimation of imprecision regarding fuzzy numbers. One of the methods to evaluate the imprecision regarding a fuzzy number $\tilde{a}$ is given by Choquet integral ([3, 7]):

$$(C)\int \tilde{a}(\cdot)\, \mathrm{d}\tilde{Q}(\cdot) = \int_0^1 \tilde{Q}\{x \in \mathbb{R} | \tilde{a}(x) \ge \alpha\}\, \mathrm{d}\alpha,$$

where $\tilde{Q}$ is a fuzzy measure on $\mathbb{R}$. We take $\tilde{Q}$ Lebesgue measure, then

$$(C)\int \tilde{a}(\cdot)\, \mathrm{d}\tilde{Q}(\cdot) = \int_0^1 (\tilde{a}^+_\alpha - \tilde{a}^-_\alpha)\, \mathrm{d}\alpha.$$

Therefore, the estimation of fuzziness regarding a fuzzy random variable $\tilde{X}_t$ follows

$$(C)\int \tilde{X}_t(\omega)(\cdot)\, \mathrm{d}\tilde{Q}(\cdot) = \int_0^1 (\tilde{X}^+_{t,\alpha}(\omega) - \tilde{X}^-_{t,\alpha}(\omega))\, \mathrm{d}\alpha \quad (4)$$

for $t \ge 0$, $\omega \in \Omega$.

3 American Put Option in Uncertain Environment

In this section, we introduce American put option with fuzzy prices and we discuss its properties. Let $\{a_t\}_{t\geq 0}$ be an $\mathcal{M}_t$-adapted stochastic process such that the map $t \mapsto a_t(\omega)$ is continuous on $[0,\infty)$ and $0 < a_t(\omega) \leq S_t(\omega)$ for almost all $\omega \in \Omega$. We give a fuzzy stochastic process of the stock prices $\{\tilde{S}_t\}_{t\geq 0}$ by the following fuzzy random variables:

$$\tilde{S}_t(\omega)(x) := L((x - S_t(\omega))/a_t(\omega))$$

for $t \geq 0$, $\omega \in \Omega$ and $x \in \mathbb{R}$, where $L(x) := \max\{1 - |x|, 0\}$ $(x \in \mathbb{R})$ is the triangle-type shape function(Fig. 1) and $\{S_t\}_{t\geq 0}$ is defined by (1). Then, from (4), the fuzziness regarding the fuzzy random variables $\tilde{S}_t$ is estimated by Choquet integral

$$(C)\int \tilde{S}_t(\omega)(\cdot)\,\mathrm{d}\tilde{Q}(\cdot) = \int_0^1 (\tilde{S}^+_{t,\alpha}(\omega) - \tilde{S}^-_{t,\alpha}(\omega))\,\mathrm{d}\alpha = a_t(\omega). \tag{5}$$

Therefore $a_t(\omega)$ means the amount of fuzziness regarding the stock price $\tilde{S}_t(\omega)$ and is the spread of the triangular fuzzy number in Fig. 1.

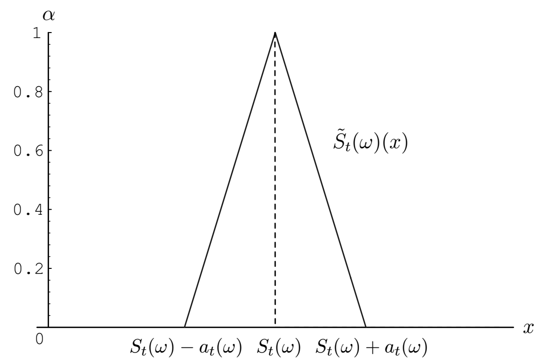

Fig. 1. Fuzzy random variable $\tilde{S}_t(\omega)(x)$.

Let K $(K > 0)$ be a strike price. We define a fuzzy price process by the following fuzzy stochastic process $\{\tilde{P}_t\}_{t\geq 0}$:

$$\tilde{P}_t(\omega) := e^{-rt}(1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}}$$

for $t \geq 0$, $\omega \in \Omega$, where $\vee$ is the maximum by the fuzzy max order, and $1_{\{K\}}$ and $1_{\{0\}}$ denote the crisp numbers K and zero respectively. By using stopping times τ, we consider a problem to maximize fuzzy price process of American put option. Fix an initial stock price y $(y = S_0 > 0)$. Put the optimal fuzzy price of American put option by

$$\tilde{V} = \bigvee_{\tau\text{: stopping times with values in } \mathbb{T}} E(\tilde{P}_\tau), \tag{6}$$

where $E(\cdot)$ denotes the expectation with respect to the equivalent martingale measure Q, and $\vee$ means the supremum induced from the fuzzy max order.

We consider a valuation method of fuzzy prices, taking into account of decision maker's subjective judgment. From (2), for a stopping time τ, we define a fuzzy expectation of the fuzzy numbers $E(\tilde{P}_\tau)$ by

$$\tilde{E}(E(\tilde{P}_\tau)) = \sup_{x\in[0,\infty)} \min\{E(\tilde{P}_\tau)(x), \varphi(x)\}, \tag{7}$$

where φ is the seller's fuzzy goal. In this section, we discuss the following optimal stopping problem regarding American put option with fuzziness.

Problem P. Find a stopping time τ^* with values in $\mathbb{T}$ such that

$$\tilde{E}(E(\tilde{P}_{\tau^*})) = \tilde{E}(\tilde{V}),$$

where $\tilde{V}$ is given by (6).

Then, τ^* is called an *optimal exercise time* and a real number $x^*(\in[0,\infty))$ is called an *optimal expected price* under the fuzzy expectation generated by possibility measures if it attains the supremum of the fuzzy expectation (7), i.e.

$$\tilde{E}(\tilde{V}) = \min\{\tilde{V}(x^*), \varphi(x^*)\}. \tag{8}$$

The fuzzy random variables $\tilde{Z}_t$ correspond to Snell's envelope in probability theory. Hence, by using dynamic programming approach, we obtain the following optimality characterization for the fuzzy stochastic process regarding the optimal fuzzy price $\tilde{V}$ by fuzzy random variables $\tilde{Z}_t$. Now we introduce a reasonable assumption for computation.

Assumption S. The stochastic process $\{a_t\}_{t\geq 0}$ is represented by

$$a_t(\omega) := cS_t(\omega),$$

$t \geq 0, \omega \in \Omega$, where c is a constant satisfying $0 < c < 1$.

Since (1) can be written as

$$\mathrm{d}\log S_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \tag{9}$$

$t \geq 0$, one of the most difficulties is estimation of the volatility σ of a stock in actual cases ([10, Sect.7.5.1]). Therefore, Assumption S is reasonable since $a_t(\omega)$ corresponds to the size of fuzziness from (5) and so it is reasonable that $a_t(\omega)$ should depend on the fuzziness of the volatility σ and the stock price $S_t(\omega)$ of the term $\sigma S_t(\omega)$ in (1). In this model, we represent by c the fuzziness

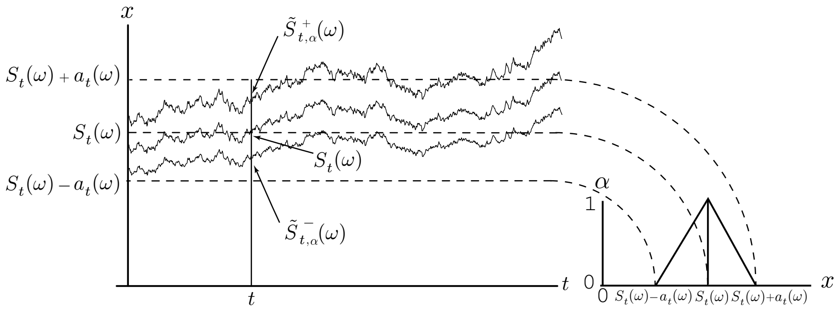

Fig. 2. The stochastic process $\{a_t\}_{t\geq 0}$.

of the volatility σ, and we call c a *fuzzy factor of the stock process*. From now on, we suppose that Assumption S holds. By putting $b^{\pm}(\alpha) := 1 \pm (1-\alpha)c$ ($\alpha \in [0,1]$), from Assumption S we have

$$\tilde{S}^{\pm}_{t,\alpha}(\omega) = b^{\pm}(\alpha) S_t(\omega),$$

$t \geq 0, \omega \in \Omega, \alpha \in [0,1]$, where $\tilde{S}^{\pm}_{t,\alpha}(\omega)$ is the α-cut of $\tilde{S}^{\pm}_t(\omega)$.

Define fuzzy price processes by

$$\tilde{V}^{\pm}_{\alpha}(y,t) := \sup_{\tau \geq t} E(e^{-r(\tau - t)} \max\{K - \tilde{S}^{\mp}_{\tau,\alpha}, 0\} \mid S_t = y)$$

for $y > 0$ and $t \in \mathbb{T}$. Here we consider the following two cases (I) and (II).

Case (I) (American put option with an expiration date T, $\mathbb{T} = [0,T]$). Define an operator

$$\mathcal{L} := \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2} + ry\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$$

on $[0,\infty) \times [0,T)$. Then we obtain the following optimality conditions by dynamic programming([15]).

Theorem 1 (Free boundary problem). *The fuzzy price* $V(y,t) = \tilde{V}^{\pm}_{\alpha}(y,t)$ *satisfies the following equations:*

$$\begin{aligned} \mathcal{L}(e^{-rt}V(y,t)) &\leq 0, \\ \mathcal{L}(e^{-rt}V(y,t)) &= 0 \quad \text{on } D, \\ V(y,t) &\geq \max\{K - b^{\mp}(\alpha)y, 0\}, \\ V(y,T) &= \max\{K - b^{\mp}(\alpha)y, 0\}, \end{aligned}$$

where $D := \{(y,t) \in [0,\infty) \times [0,T) \mid \tilde{V}^{\pm}_{\alpha}(y,t) > \max\{K - b^{\mp}(\alpha)y, 0\}\}$. *The corresponding optimal exercise time is*

$$\tau_{\alpha}(\omega) = \inf\left\{t \in \mathbb{T} \mid \tilde{V}^{\pm}_{\alpha}(S_t(\omega),t) = \max\{K - b^{\mp}(\alpha)S_t(\omega), 0\}\right\}.$$

Case (II) (A perpetual American put option, $\mathbb{T} = [0, \infty)$). The both ends of the α-cuts are

$$\tilde{V}_\alpha^\pm(y, 0) := \sup_{\tau \geq 0} E(e^{-r\tau} \max\{K - \tilde{S}_{\tau,\alpha}^\mp, 0\} 1_{\{\tau < \infty\}} \mid S_0 = y)$$

for $y > 0$. Then we obtain the following results ([15]).

Theorem 2. *The fuzzy price $\tilde{V}_\alpha^\pm(y, 0)$ is represented by*

$$\tilde{V}_\alpha^\pm(y, 0) = \begin{cases} K - b^\mp(\alpha)y & \text{if } y \leq s^\pm(\alpha) \\ (K - b^\mp(\alpha)s^\pm(\alpha))(y/s^\pm(\alpha))^{-\gamma} & \text{if } y > s^\pm(\alpha), \end{cases}$$

where $s^\pm(\alpha) := 2rK/(b^\mp(\alpha)(2r + \sigma^2))$ and $\gamma := 2r/\sigma^2$. The optimal exercise time is

$$\tau_\alpha(\omega) = \inf \left\{ t \geq 0 \mid (r - \frac{\sigma^2}{2})t + \sigma W_t(\omega) = \log \left(\frac{s^\pm(\alpha)}{y} \right) \right\}.$$

4 The Optimal Expected Price and the Optimal Exercise Times

Fix an initial stock price $y(> 0)$. In this section, we discuss the optimal expected price of American put option $\tilde{V} := \tilde{V}(y, 0)$, which is introduced in the previous section, and we give an optimal exercise time for Problem P. Define a grade α^* by

$$\alpha^* := \sup\{\alpha \in [0, 1] | \ \varphi_\alpha^- \leq \tilde{V}_\alpha^+\}, \tag{10}$$

where $\varphi_\alpha = [\varphi_\alpha^-, \infty)$ for $\alpha \in (0, 1)$, and the supremum of the empty set is understood to be 0. The following theorem, which is obtained by a modification of the proofs in [13, Theorems 3.1 and 3.2], implies that α^* is the grade of the fuzzy expectation of American put option price $\tilde{V}$ (see (8)).

Theorem 3. *Under the fuzzy expectation generated by possibility measures (8), the following (i) – (iii) hold.*

(i) *The grade of the fuzzy expectation of American put option price $\tilde{V}$ is given by*

$$\alpha^* = \tilde{E}(\tilde{V}) = \sup_{\tau:\text{ stopping times with values in } \mathbb{T}} \tilde{E}(E(\tilde{P}_\tau)).$$

(ii) *Further, the optimal expected price of American put option is given by*

$$x^* = \varphi_{\alpha^*}^-. \tag{11}$$

(iii)*Define a stopping time*

$$\tau^*(\omega) := \tau_{\alpha^*}(\omega) = \inf\{t \in \mathbb{T} \mid S_t(\omega) \le s^+(\alpha^*)\}, \tag{12}$$

where the infimum of the empty set is understood to be $\sup \mathbb{T}$. *If* τ^* *is finite, then* τ^* *is an optimal stopping time for Problem P, and it is the optimal exercise time.*

In Theorem 3, we need the assumption *the finiteness of* τ^*, only when $\mathbb{T} = [0, \infty)$. Since the fuzzy expectation (8) is defined by possibility measures, (11) gives an upper bound on optimal expected prices of American put option. Similarly to (10) we can define another grade, which gives a lower bound on optimal expected prices of American put option as follows:

$$x_* = \varphi^-_{\alpha_*}, \tag{13}$$

where α_* is defined by

$$\alpha_* := \sup\{\alpha \in [0,1] | \ \varphi^-_\alpha \le \tilde{V}^-_\alpha\}.$$

Then, its corresponding stopping time is given by

$$\tau_*(\omega) := \tau_{\alpha_*}(\omega) = \inf\{t \in \mathbb{T} \mid S_t(\omega) \le s^-(\alpha_*)\}.$$

Hence, from (11) and (13), we can easily check the interval $[x_*, x^*]$ is written as

$$[x_*, x^*] = \{x \in \mathbb{R} \mid \tilde{V}(x) \ge \varphi(x)\},$$

which is the range of prices x such that the reliability degree of the optimal expected price, $\tilde{V}(x)$, is greater than the degree of seller's satisfaction, $\varphi(x)$. Therefore, $[x_*, x^*]$ means *seller's permissible range of expected prices* under his fuzzy goal φ.

Example 1. Consider a perpetual American put option ($\mathbb{T} = [0, \infty)$). Put a fuzzy goal

$$\varphi(x) = \begin{cases} 1 - e^{-0.2x}, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

Then, $\varphi^-_\alpha = -0.2^{-1} \log(1 - \alpha)$ for $\alpha \in (0, 1)$. Put a volatility $\sigma = 0.25$, an interest factor $r = 0.05$, a fuzzy factor $c = 0.1$, an initial stock price $y = 20$ and a strike price $K = 25$. We can easily calculate that the optimal grades are $\alpha_* \approx 0.700468$ and $\alpha^* \approx 0.73281$. From (11) and (13), the *permissible range of expected prices* under seller's fuzzy goal φ is (see Fig. 3) $[x_*, x^*] \approx [6.02767, 6.59897]$. The corresponding optimal exercise times are

$$\tau_*(\omega) = \inf\{t \ge 0 \mid W_t(\omega) - 0.45t = \log(s^-(\alpha_*)/20)\},$$
$$\tau^*(\omega) = \inf\{t \ge 0 \mid W_t(\omega) - 0.45t = \log(s^+(\alpha^*)/20)\}$$

with $s^-(\alpha_*) \approx 14.9372$ and $s^+(\alpha^*) \approx 15.807$.

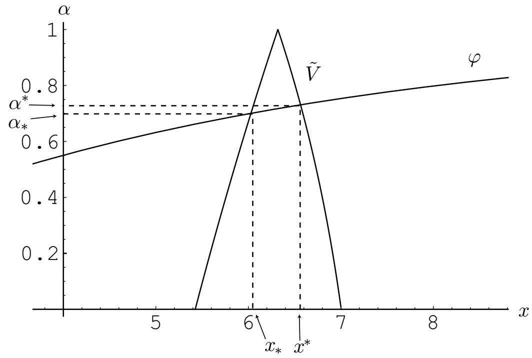

Fig. 3. Optimal fuzzy price $\tilde{V}(x)$ and fuzzy goal $\varphi(x)$.

5 European Options in Uncertain Environment

In this section, we introduce European option with fuzzy prices and we discuss their properties. We define fuzzy stochastic processes of European call/put options by $\{\tilde{C}_t\}_{t\geq 0}$ and $\{\tilde{P}_t\}_{t\geq 0}$:

$$\tilde{C}_t(\omega) := e^{-rt}(\tilde{S}_t(\omega) - 1_{\{K\}}) \vee 1_{\{0\}},$$
$$\tilde{P}_t(\omega) := e^{-rt}(1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}},$$

$t \geq 0,\ \omega \in \Omega$. We evaluate these fuzzy stochastic processes by the expectations introduced in the pervious sections. Then, the fuzzy price processes of European call/put options are given as follows:

$$\tilde{V}^{\tilde{C}}(y,t) := e^{rt} E(\tilde{C}_T \mid S_t = y)$$
$$\tilde{V}^{\tilde{P}}(y,t) := e^{rt} E(\tilde{P}_T \mid S_t = y)$$

for an initial stock price y $(y > 0)$ and $t \in [0,T]$, where $E(\cdot)$ denotes expectation with respect to the equivalent martingale measure Q. Their α-cuts are

$$\tilde{V}_\alpha^{\tilde{C},\pm}(y,t) = E(e^{-r(T-t)} \max\{\tilde{S}_{T,\alpha}^{\pm} - K, 0\} \mid S_t = y);$$
$$\tilde{V}_\alpha^{\tilde{P},\pm}(y,t) = E(e^{-r(T-t)} \max\{K - \tilde{S}_{T,\alpha}^{\mp}, 0\} \mid S_t = y).$$

Then we obtain the following formulae to calculate fuzzy price in European options ([14]).

Theorem 4 (Black-Scholes formula for fuzzy prices). *Suppose that Assumption S holds. Let $\alpha \in [0,1]$. Let an initial stock price $y(:= S_0 > 0)$.*

(i) *The rational fuzzy price of European call option is given by*

$$\tilde{V}_\alpha^{\tilde{C},\pm}(y,0) = b^{\pm}(\alpha) y \Phi(z_1) - K e^{-rT} \Phi(z_2), \tag{14}$$

where $\Phi(z) = (1/\sqrt{2\pi})\int_{-\infty}^{z} e^{-w^2/2}dw$ $(z \in \mathbb{R})$ *is the standard normal distribution function, and* z_1 *and* z_2 *are given by*

$$z_1 = \frac{\log b^{\pm}(\alpha) + \log(y/K) + T(r + \sigma^2/2)}{\sigma\sqrt{T}};$$
$$z_2 = \frac{\log b^{\pm}(\alpha) + \log(y/K) + T(r - \sigma^2/2)}{\sigma\sqrt{T}}.$$

(ii) *The rational fuzzy price of European put option is given by the following call-put parity:*

$$\tilde{V}_\alpha^{\tilde{P},\pm}(y,0) = \tilde{V}_\alpha^{\tilde{C},\mp}(y,0) - b^{\mp}(\alpha)y + Ke^{-rT}. \tag{15}$$

Fix an initial stock price $y(> 0)$. Next, we discuss the expected price, which is introduced in the previous section, of European call/put options $\tilde{V} = \tilde{V}^{\tilde{C}}(y,0)$ or $\tilde{V} = \tilde{V}^{\tilde{P}}(y,0)$. Define a grade $\alpha^{\tilde{C},+}$ by

$$\alpha^{\tilde{C},+} := \sup\{\alpha \in [0,1] |\ \varphi_\alpha^- \le \tilde{V}_\alpha^{\tilde{C},+}(y,0)\},$$

where $\varphi_\alpha = [\varphi_\alpha^-, \infty)$ for $\alpha \in (0,1)$, and the supremum of the empty set is understood to be 0. From the continuity of φ and $\tilde{V}^{\tilde{C}}$, we can easily check that the grade $\alpha^{\tilde{C},+}$ satisfies

$$\varphi^-_{\alpha^{\tilde{C},+}} = \tilde{V}^{\tilde{C},+}_{\alpha^{\tilde{C},+}}(y,0). \tag{16}$$

The following theorem, which is obtained by a modification of the proofs in [13, Theorems 3.1 and 3.2], implies that $\alpha^{\tilde{C},+}$ is the grade of the fuzzy expectation of European call option $\tilde{V}^{\tilde{C}}(y,0)$.

Theorem 5. *Under the fuzzy expectation generated by possibility measures (2), the following (i) and (ii) hold.*

(i) *The grade of the fuzzy expectation of European call option price* $\tilde{V}^{\tilde{C}}$ *is given by*

$$\alpha^{\tilde{C},+} = \tilde{E}(\tilde{V}^{\tilde{C}}(y,0)) = \tilde{E}(E(\tilde{C}_T)).$$

(ii) *Further, the rational expected price of European call option is given by*

$$x^{\tilde{C},+} = \varphi^-_{\alpha^{\tilde{C},+}}. \tag{17}$$

Since the fuzzy expectation (3) is defined by possibility measures, (17) gives an upper bound on rational expected prices of European call option. Similarly to (16) we can define another grade, which gives a lower bound on rational expected prices of European call option as follows:

$$x^{\tilde{C},-} = \varphi^-_{\alpha^{\tilde{C},-}}, \tag{18}$$

where $\alpha^{\tilde{C},-}$ is defined by

$$\alpha^{\tilde{C},-} := \sup\{\alpha \in [0,1] |\ \varphi_\alpha^- \leq \tilde{V}_\alpha^{\tilde{C},-}(y,0)\}.$$

Hence, from (17) and (18), we can easily check the interval $[x^{\tilde{C},-}, x^{\tilde{C},+}]$ is written as

$$[x^{\tilde{C},-}, x^{\tilde{C},+}] = \{x \in \mathbb{R} \mid \tilde{V}^{\tilde{C}}(y,0)(x) \geq \varphi(x)\},$$

which is the range of prices x such that the reliability degree of the optimal expected price, $\tilde{V}^{\tilde{C}}(y,0)(x)$, is greater than the degree of buyer's satisfaction, $\varphi(x)$. Therefore, $[x^{\tilde{C},-}, x^{\tilde{C},+}]$ means *buyer's permissible range of expected prices* under his fuzzy goal φ. Regarding European put option, similarly we obtain *seller's permissible range of rational expected prices* by $[x^{\tilde{P},-}, x^{\tilde{P},+}]$, where $x^{\tilde{P},-} := \varphi^-_{\alpha^{\tilde{P},-}}$ and $x^{\tilde{P},+} := \varphi^+_{\alpha^{\tilde{P},+}}$, and the grades $\alpha^{\tilde{P},-}$ and $\alpha^{\tilde{P},+}$ are given by $\varphi^-_{\alpha^{\tilde{P},-}} = \tilde{V}^{\tilde{P},-}_{\alpha^{\tilde{P},-}}(y,0)$ and $\varphi^-_{\alpha^{\tilde{P},+}} = \tilde{V}^{\tilde{P},+}_{\alpha^{\tilde{P},+}}(y,0)$.

Example 2. Consider a fuzzy goal

$$\varphi(x) = \begin{cases} 1 - e^{-2x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Then $\varphi_\alpha^- = -2^{-1}\log(1-\alpha)$ for $\alpha \in (0,1)$. Put an exercise time $T = 1$, a volatility $\sigma = 0.25$, an interest factor $r = 0.05$, a fuzzy factor $c = 0.05$, an initial stock price $y = 20$ and a strike price $K = 25$. From Theorem 5, we can easily calculate that the grades of the fuzzy expectation of the fuzzy price are $\alpha^{\tilde{C},-} \approx 0.767815$ and $\alpha^{\tilde{C},+} \approx 0.845076$. These grades means the degree of buyer's satisfaction in pricing. From (17) and (18), the corresponding permissible range of rational expected prices in European call option under his fuzzy goal φ is $[x^{\tilde{C},-}, x^{\tilde{C},+}] \approx [0.730111, 0.84642]$. Consider another fuzzy goal

$$\varphi(x) = \begin{cases} 1 - e^{-0.5x}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

and take the other parameters in the same as the above. Similarly, in European put option, we can easily calculate the grades, the degree of seller's satisfaction, and the corresponding permissible range of rational expected prices is as follows: $[x^{\tilde{P},-}, x^{\tilde{P},+}] \approx [4.4975, 4.64449]$. Buyer/seller should take into account of *the permissible range of rational expected prices* under their fuzzy goal φ.

6 Hedging Strategies

Finally, we deal with hedging strategies in European call option. Fix any $\alpha \in [0,1]$. A hedging strategy is an $\mathcal{M}_t$-predictable process $\{(\pi_t^{0,\pm}, \pi_t^{1,\pm})\}_{t \geq 0}$

with values in $\mathbb{R} \times \mathbb{R}$, where $\pi_t^{0,\pm}$ means the amount of the bond and $\pi_t^{1,\pm}$ means the amount of the stock at time t, and it satisfies

$$V_{t,\alpha}^{\pm} = \pi_t^{0,\pm} R_t + \pi_t^{1,\pm} \tilde{S}_{t,\alpha}^{\pm}, \quad t \geq 0, \tag{19}$$

where $V_{t,\alpha}^{\pm} := e^{rt} E(\tilde{C}_{T,\alpha}^{\pm} | \mathcal{M}_t)$ is called a wealth process. Then, we obtain the following results ([14]).

Theorem 6. *The minimal hedging strategy* $\{(\pi_t^{0,\pm}, \pi_t^{1,\pm})\}_{t \in [0,T]}$ *for the fuzzy price of European call option is given by*

$$\pi_t^{0,\pm} = \Phi(z_t^{0,\pm}) \quad \text{and} \quad \pi_t^{1,\pm} = -\, e^{-rT} K \Phi(z_t^{1,\pm})$$

for $t < T$, *where*

$$z_t^{0,\pm} = \log b^{\pm}(\alpha) + \frac{\log(S_t/K) + (T-t)(r + \sigma^2/2)}{\sigma\sqrt{T-t}},$$
$$z_t^{1,\pm} = \log b^{\pm}(\alpha) + \frac{\log(S_t/K) + (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}}.$$

The corresponding wealth process is (19).

7 Concluding Remarks

In this paper, the uncertainty is represented by both randomness and fuzziness. The fuzziness is evaluated by fuzzy expectation defined by a possibility measure from the viewpoint of decision-maker's subjective judgment. We can find other estimation methods instead of fuzzy expectation. For example, mean values by evaluation measures in Yoshida [18] are applicable to this model in the evaluation of fuzzy numbers with the decision maker's subjective judgment.

This paper takes theoretical approach to the option pricing theory under uncertainty. However, in a fuzzy environment, the other approaches from the viewpoint of the option pricing for actual stocks are discussed by Carlsson et al. [2] and Zmeškal [20].

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