Fuzzy Capital Budgeting: Investment Project Evaluation and Optimization

Pavel Sevastjanov[1], Ludmila Dimova[1], and Dmitry Sevastianov[2]

[1] Institute of Comp.& Information Sci., Technical University of Czestochowa,Dabrowskiego 73, 42-200 Czestochowa, Poland
sevast@icis.pcz.czest.pl

[2] Reuters Research, Inc.
3 Times Square, 17 floor, location 17-W15
New York, NY 10036, USA
Tel.: +1-646-223-8182; fax: +1-646-223-4123.
dmitry.sevastyanov@reuters.com
The statements made and the views expressed are solely those of the author and should not be attributed to his employer.

Summary. Capital budgeting is based on the analysis of some financial parameters of considered investment projects. It is clear that estimation of investment efficiency, as well as any forecasting, is rather an uncertain problem. In a case of stock investment one can to some extent predict future profits using stock history and statistical methods, but only in a short time horizon. In the capital investment one usually deals with a business-plan which takes a long time — as a rule, some years — for its realization. In such cases, a description of uncertainty within a framework of traditional probability methods usually is impossible due to the absence of objective information about probabilities of future events. This is a reason for the growing for the last two decades interest in applications of interval and fuzzy methods in budgeting. In this paper a technique for fuzzy-interval evaluation of financial parameters is presented. The results of technique application in a form of fuzzy-interval and weighted non-fuzzy values for main financial parameters NPV and IRR as well as the quantitative estimation of risk of an investment are presented.Another problem is that one usually must consider a set of different local criteria based on financial parameters of investments. As its possible solution, a numerical method for optimization of future cash-flows based on the generalized project's quality criterion in a form of compromise between local criteria of profit maximisation and financial risk minimisation is proposed.

Keywords: Capital budgeting; investment project optimisation, fuzzy-interval evaluation; risk minimisation, profit maximisation.

P. Sevastjanov et al.: *Fuzzy Capital Budgeting: Investment Project Evaluation and Optimization*, StudFuzz **201**, 205–228 (2006)
www.springerlink.com © Springer-Verlag Berlin Heidelberg 2006

1 Introduction

Consider common non-fuzzy approaches to a capital budgeting problem. There are a lot of financial parameters proposed in literature [1, 2, 3, 4] for budgeting.The main are: Net Present Value (NPV), Internal Rate of Return (IRR), Payback Period (PB), Profitability Index (PI). These parameters are usually used for a project quality estimation, but in practice they have different importance. It is earnestly shown in [5] that the most important parameters are NPV and IRR.

Therefore, further consideration will be based only on the analysis of the NPV and IRR. Good review of other useful financial parameters can be found in [6]. Net Present Value is usually calculated as follows:

$$NPV = \sum_{t=t_n}^{T} \frac{P_t}{(1+d)^t} - \sum_{t=0}^{t_c} \frac{KV_t}{(1+d)^t}, \tag{1}$$

where d - discount rate, t_n - first year of production, t_c - last year of investments, KV_t - capital investment in year t, P_t - income in year t, T - duration of an investment project in years. Usually, the discount rate is taken equal to an average bank interest rate in a country of investment or other value corresponding to a profit rate of alternate capital investments. An economic nature of the Internal Rate of Return (IRR) can be explained as follows. As an alternative to analyzed project, the deposit under some bank interest distributed in time the same way as analyzed investments is considered. All earned profits are also deposited with the same interest rate. If the discount rate is equal IRR, an investment in the project will give the same total income as in a case of the deposit. Thus, both alternatives are economically equivalent. If the actual bank discount rate is less then IRR, the investment into the project is more preferable. Therefore IRR is a threshold discount rate dividing effective and ineffective investment projects. The value of IRR is a solution of a non-linear equation with respect to d:

$$\sum_{t=t_n}^{T} \frac{P_t}{(1+d)^t} - \sum_{t=0}^{t_c} \frac{KV_t}{(1+d)^t} = 0. \tag{2}$$

An estimation of IRR is frequently used as a first step of the financial analysis. Only projects with IRR not below of some accepted threshold value, e.g., 15–20%, can be chosen for further consideration.

There are two conjoint discussable points in the budgeting realm. The first is the multiple roots of Eq. (2), i.e., so called multiple IRR problem. The second is the negative NPV problem. The problem of multiple roots of Eq. (2) rises when the negative cash flows take place after starting investment. In practice, an appearance of some negative cash flow after initial investment is usually treated as a local "force majeur" or even a total project's failure. That is why, on the stage of planning, investors try to avoid situations when

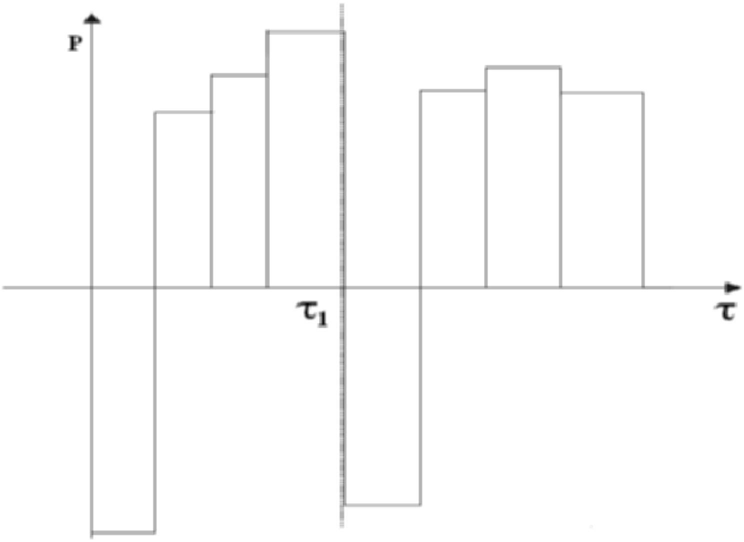


Fig. 1. Two stage investment project.

such negative cash flows are possible, except the cases when they are dealing with long-term projects consist of some phases. Let us see to the Fig. 1. This is a typical two-phase project: after initial investment the project brings considerable profits and at the time τ_1 a part of accumulated earnings and, perhaps, an additional banking credit are invested once again. Factually, an investor buys new production equipment and buildings (in fact creating the new enterprise) and from his/her point of view a quite new project is started. It is easy to see that investor's creditors which are interested in repayment of a credit always analyze phases $\tau < \tau_1$ and $\tau > \tau_1$ separately. It worth noting that what we describe is only an investment planning routine, not some theoretical considerations we can find in finansial books. On the other hand, a separate assessment of different projects' phases reflects economic sense of capital investment better. Indeed, if we consider a two phase project as a whole, we often get the IRRs performed by two roots so different that it is impossible to make any decision. For example, we can obtain IRR_1 =4% and IRR_2 =120%. It is clear that average IRR= (4+120)/2=62% seems as rather fantastic estimation, whereas when considering the two phases of project separately we usually get quite acceptable values, e.g., for the first phase IRR_1 =20% and for the second phase IRR_2 =25%. So we can say that the problem of "multiple IRR values" exists only in some finansial textbooks, not in the practice of capital investment. Therefore, only the case when Eq. (2) has a single root will be analyzed in the current paper. Similarly, the negative NPV

problem seems as a rather artificial one. Obviously, any investment project with negative NPV should be rejected at the planning stage. On the other hand, all possible undesirable events leading to the financial losses or even to the failure of the projects should be taken into account too. In the framework of probabilistic approach, e.g., when using the Monte-Carlo method, there may be local results of calculations with negative NPV and the problem of their interpretation in terms of risk management or in other context arises. The different situation we meet when future cash flows are presented by fuzzy numbers. It is clear the full body of uncertainty is involved in such a description. So if the decision maker find some negative part in predicted cash flow he/she consider such a case as a source of risk and try to improve the project to avoid this risk. As the result in a fuzzy budgeting the negative cash flows and especially NPV, seem rather as the exotics. Nevertheless, the probabilistic approach to interval and fuzzy value comparison we describe in Section 2, makes it possible to deal with such situation as well, i.e., to compare NPV comprising negative part with some real or fuzzy number representing acceptable risk associated with future NPV .

The focus of current paper is that nowadays traditional approach to the evaluation of NPV, IRR and other financial parameters is subjected to quite deserved criticism, since the future incomes P_t, capital investments KV_t and rates d are rather uncertain parameters. Uncertainties which one meets in capital budgeting differ from those in a case of share prices forecasting and cannot be adequately described in terms of the probability theory. In a capital investment one usually deals with a business-plan that takes a long time — as a rule, some years — for its realization. In such cases, the description of uncertainty within a framework of traditional probability methods usually is impossible due to the absence of objective information about probabilities of future events. Thus, what really is available in such cases are some expert estimates. In real-world situations, investors or experts involved are able to predict confidently only intervals of possible values P_t, KV_t and d and sometimes the most expected values inside these intervals. Therefore, during last two decades the growing interest in applications of interval arithmetic [7] and fuzzy sets theory methods [8] in budgeting was observing.

After pioneer works of T.L.Ward [9] and J.U. Buckley [10], some other authors contributed to the development of the fuzzy capital budgeting theory [11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. It is safe to say that almost all problems of the fuzzy NPV estimation are solved now, but an interesting and important problem of project risk assessment using fuzzy NPV gets higher priority.

An unsolved problem is a fuzzy estimation of the IRR. Ward [9] considers Eq. (2) and states that such an expression cannot be applied to fuzzy case because the left side of Eq. (2) is fuzzy, 0 is crisp and an equality is impossible. Hence, the Eq. (2) is senseless from fuzzy viewpoint.

In [23], a method for the fuzzy IRR estimation is proposed where α-cut representation of fuzzy numbers [26] is used. The method is based on an assumption (see [23, p. 380]) that a set of equations for IRR determination on each α-level may be presented as (in our notation)

$$(CF_0^\alpha)_1 + \sum_{i=1}^{n} \frac{(CF_i^\alpha)_1}{(1+IRR_1^\alpha)^i} = 0, \quad (CF_0^\alpha)_2 + \sum_{i=1}^{n} \frac{(CF_i^\alpha)_2}{(1+IRR_2^\alpha)^i} = 0, \qquad (3)$$

where $CF_i^\alpha = [(CF_i^\alpha)_1, (CF_i^\alpha)_2]$, $i = 0$ to n, are crisp interval representations of fuzzy cash flows on α-levels. Of course, from Eqs. (3) all crisp intervals $IRR^\alpha = [IRR_1^\alpha, IRR_2^\alpha]$ expressing the fuzzy valued IRR may be obtained. Regrettable, there is a little mistake in (3). Taking into account the conventional interval arithmetic rules, the right crisp interval representation of Eq. (2) on α-levels must be written as

$$(CF_0^\alpha)_1 + \sum_{i=1}^{n} \frac{(CF_i^\alpha)_1}{(1+IRR_2^\alpha)^i} = 0, \quad (CF_0^\alpha)_2 + \sum_{i=1}^{n} \frac{(CF_i^\alpha)_2}{(1+IRR_1^\alpha)^i} = 0. \qquad (4)$$

There is no way to get intervals IRR^α from (4), but the crisp ones may be obtained (see Section 3, below). Another problem not presented in literature is an optimization of cash flows. The rest of the paper is set out as follows. In Section 2, a method for a fuzzy estimation of NPV is presented and possible approaches to the risk estimation are considered. In Section 3, a method for crisp solving of Eq. (2) for a case of fuzzy cash flows is described. As an outcome of the method a set of useful crisp parameters is proposed and analyzed. In Section 4, a numerical method of an optimization of cash flows as a compromise between local criteria of a profit maximisation and financial risk minimisation is proposed.

2 Fuzzy NPV and Risk Assessment

The technique is based on the fuzzy extension principle [8]. According to it, the values of uncertain parameters P_t, KV_t and d are substituted for corresponding fuzzy intervals. In practice it means that an expert sets lower — P_{t1} (pessimistic value) and upper — P_{t4} (optimistic value) boundaries of the intervals and internal intervals of the most expected values [P_{t2}, P_{t3}] for analyzed parameters (see Fig. 2). The function $\mu(P_t)$ is usually interpreted as a membership function, i.e., a degree to which values of a parameter belong to an interval (in this case [P_{t1}, P_{t4}]). A membership function changes continuously from 0 (an area out of the interval) up to maximum value 1 in an area of the most possible values. It is obvious that a membership function is a generalization of a characteristic function of usual set, which equals 1 for all values inside a set and 0 in all other cases.

The linear character of the function is not obligatory, but such a mode is most used and it allows to represent the fuzzy intervals in a convenient form of a quadruple $P_t = \{P_{t1}, P_{t2}, P_{t3}, P_{t4}\}$. Then all necessary calculations are

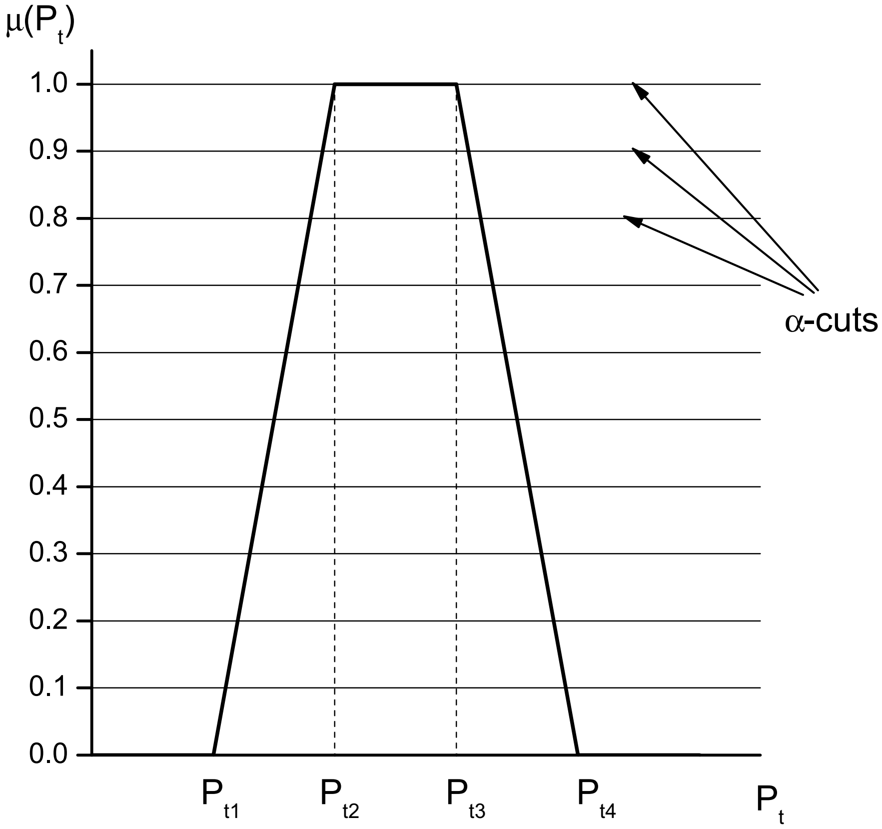


Fig. 2. Fuzzy interval of the uncertain parameter P_t and its membership function $\mu(P_t)$

carried out using special fuzzy-interval arithmetic rules. Consider some basic principles of the fuzzy arithmetic [26]. In general, for an arbitrary form of a membership function the technique of fuzzy-interval calculations is based on representation of initial fuzzy intervals in a form of so-called α-cuts (Fig. 2) which are, in fact, crisp intervals associated with corresponding degrees of the membership. All further calculations are made with those α-cuts according to well known crisp interval-arithmetic rules and resulting fuzzy intervals are obtained as disjunction of corresponding final α-cuts.

Thus, if A is a fuzzy number then $A = \bigcup_{\alpha} \alpha A_\alpha$, where A_α is a crisp interval $\{x : \mu_A(x) \geq \alpha\}$, αA_α is a fuzzy interval $\{(x, \alpha) : x \in A_\alpha\}$. So if A, B, Z are fuzzy numbers (intervals) and @ is an operation from $\{+, -, *, /\}$ then

$$Z = A@B = \bigcup_{\alpha} (A@B)_\alpha = \bigcup_{\alpha} A_\alpha @ B_\alpha. \tag{5}$$

Since in a case of α-cut representation the fuzzy arithmetic is based on crisp interval arithmetic rules, basic definitions of applied interval analysis also must be presented. There are several definitions of interval arithmetic (see [7, 27]), but in practical applications so-called "naive" form proved to be the best one. According to it, if $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are crisp intervals, then

$$Z = A@B = \{z = x@y, \forall x \in A, \forall y \in B\}. \tag{6}$$

As a direct outcome of the basic definition (6) following expressions were obtained:

$$\begin{aligned} A + B &= [a_1 + b_1, b_2 + b_2], \\ A - B &= [a_1 - b_2, a_2 - b_1], \\ A * B &= [\min(a_1b_1, a_2b_2, a_1b_2, a_2b_1), \max(a_1b_1, a_2b_2, a_1b_2, a_2b_1)], \\ A/B &= [a_1, a_2] * [1/b_2, 1/b_1] \end{aligned}$$

Of course, there are many internal problems within applied interval analysis, for example, a division by zero-containing interval, but in general, it can be considered as a good mathematical tool for modelling under conditions of uncertainty.

To illustrate, consider an investment project, in which building phase proceeds two years with investments KV_0 and KV_1 accordingly. Profits are expected only after the end of the building phase and will be obtained during two years (P_2 and P_3). It is suggested that the fuzzy interval for the discount d remains stable during the time of project realisation. The sample trapezoidal initial fuzzy intervals are presented in Table 1.

Table 1. Parameters of sample project

KV_0	$\{2, 2.8, 3.5, 4\}$	P_0	$\{0, 0, 0, 0\}$
KV_1	$\{0, 0.88, 1.50, 2\}$	P_1	$\{0, 0, 0, 0\}$
KV_2	$\{0, 0, 0, 0\}$	P_2	$\{6.5, 7.5, 8.0, 8.5\}$
KV_3	$\{0, 0, 0, 0\}$	P_3	$\{5.5, 6.5, 7.0, 7.5\}$

It was assumed that $d = \{0.08, 0.13, 0.22, 0.35\}$. Resulting fuzzy interval NPV calculated using fuzzy extension of Eq. (1) is presented in Fig. 3.

Obtained fuzzy interval allows to estimate the boundaries of possible values of predicted NPV, the interval of the most expected values, and also — that is very important — to evaluate a degree of financial risk of investment. There may be different ways to define the measure of financial risk in the framework of fuzzy sets based methodology. Therefore we consider here only the three, in our opinion, most interesting and scientifically grounded approaches.

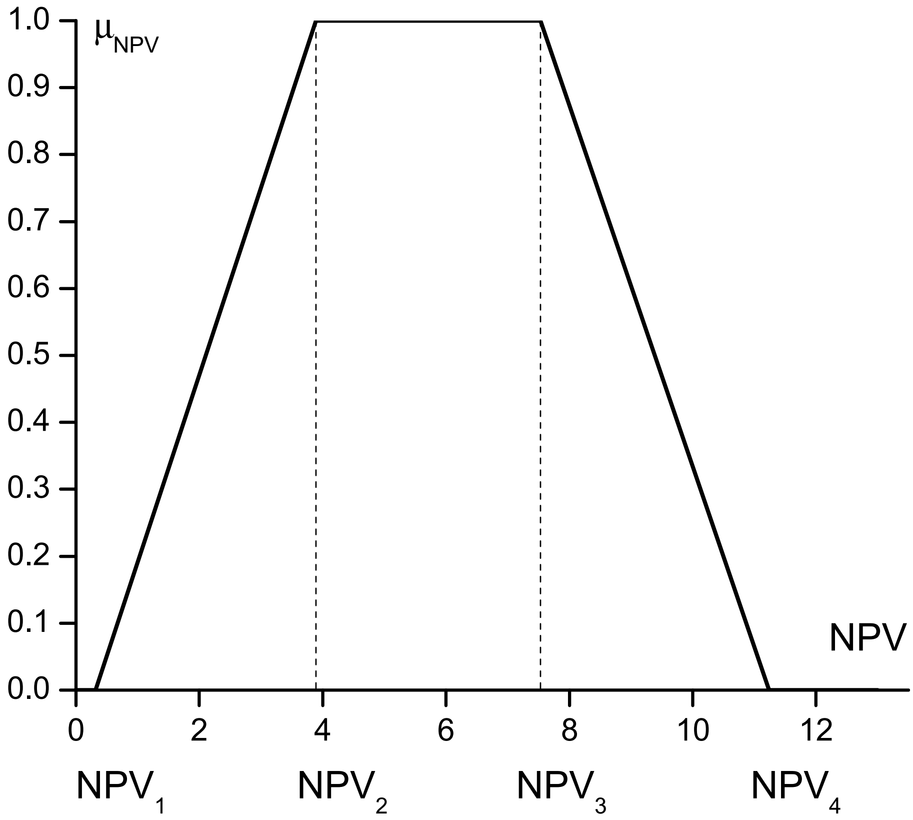


Fig. 3. Resulting fuzzy interval NPV

1. To estimate the financial risk, the following inherent property of fuzzy sets was taken into account. Let A be some fuzzy subset of X, being described by a membership function $\mu(A)$. Then complementary fuzzy subset $\bar{A}$ has a membership function $\mu(\bar{A}) = 1 - \mu(A)$. The principal difference between fuzzy subset and usual precise one is that an intersection of fuzzy A and $\bar{A}$ is not empty, that is $A \cap \bar{A} = B$, where B is also not an empty fuzzy subset. It is clear that the closer A to $\bar{A}$, the more power of set B and more A differs from ordinary sets.

   Using this circumstance R. Yager [30] proposed a set of grades of non-fuzziness of fuzzy subsets

$$D_p(A, \bar{A}) = \frac{1}{n} \left| \sum_{i=1}^{n} |\mu_A(x_i) - \mu_{\bar{A}}(x_i)|^p \right|^{\frac{1}{p}}, p = 1, 2, \ldots, \infty. \tag{7}$$

   Hence, the grade of fuzziness may be defined as

$$dd_p(A, \bar{A}) = 1 - D_p(A, \bar{A}). \tag{8}$$

The definition (8) is in compliance with obvious requests to a grade of fuzziness. If A is a fuzzy subset on X, $\mu(A)$ is its membership function and dd is a corresponding grade of fuzziness, then following properties should be observed:

a) $dd(A) = 0$, if A is a crisp subset.
b) $dd(A)$ has a maximum value if $\mu(A) = 1/2$ for $x \in X$.
c) $dd(A_1) > dd(A)$ if $\mu(x) < \mu(y)(x \in A_1, y \in A)$.

It is proved that introduced measure is similar to the Shannon entropy measure [30].

In the most useful case ($p = 1$), expression (8) is transformed to

$$dd = 1 - \frac{1}{n}\sum_{i=1}^{n} |2\mu_A(x_i) - 1| \,. \tag{9}$$

It is clear (see Eq. (9)) that the grade of fuzziness is rising from 0 when $\mu(A) = 1$ (crisp subset) up to 1 when $\mu(A) = 1/2$ (maximum degree of fuzziness).

With respect to considering problem the grade of nonfuzziness of a fuzzy interval NPV can linguistically be interpreted as a risk or uncertainty of obtaining the Net Present Value in interval $[NPV_1, NPV_4]$. Really, the more precise, (more "rectangular") interval obtained, the more a degree of uncertainty and risk. At first glance, this assertion seems to be paradoxical. However, any precise (crisp) interval contains no additional information about relative preference of values placed inside it. Therefore, it contains less useful information than any fuzzy interval being constructed on its basis. In the later case an additional information reducing uncertainty is derived from a membership function of considered fuzzy interval.

2. The second approach is based on the α-cut representation of fuzzy value and the measure of its fuzziness. Let A be fuzzy value and A_r be rectangular fuzzy value defined on the support of A and represented by characteristic function $\eta_A(x) = 1, x \in A; \eta_A(x) = 0, x \notin A$. Obviously, such rectangular value is not a fuzzy value at all, but it is asymptotic limit (object) we obtain when fuzziness of A tends to zero. Hence, it seems quite natural to define a measure of fuzziness of A as its distinction from A_r. To do this we define primarily the measure of non fuzziness as

$$MNF(A) = \int_0^1 f(\alpha)((A_{\alpha 2} - A_{\alpha 1})/(A_{02} - A_{01}))d\alpha,$$

where $f(\alpha)$ is some function of α , e.g, $f(\alpha) = 1$ or $f(\alpha) = \alpha$. Of course, last expression makes sense only for the fuzzy or interval values, i.e., only for non zero width of support $A_{02} - A_{01}$. It is easy to see that if $A \rightarrow A_r$ then $MNF(A) \rightarrow 1$. Obviously, the measure of fuzziness can be defined as $MF(A) = 1 - MNF(A)$.

We can say that rectangular value A_r defined on the support of A is a more uncertain object than A . Really, only what we know about A_r is that all $x \in A$ belong to A_r with equal degrees, whereas the membership function, $0 \leq \mu(x) \leq 1$, characterizing the fuzzy value A, brings more information to the description and as a consequence, represents a more certain object. Therefore, we can treat the measure of non fuzziness, MNF, as the uncertainty measure. Hence, if some decision is made concerning fuzzy NPV, the uncertainty and, consequently, the risk of such decision can be calculated as $MNF(MPV)$.

3. The authors of [29] proposed approach that can be treated as fuzzy analogue of the sound VAR method [28]. According to this approach the risk associated with fuzzy NPV can presented as

$$Risk = Prob(NPV < G),$$

where G is the fuzzy, interval or real valued effectiveness constrain [29], in other words, G in the low bound on acceptable values of NPV. It is clear the focus of this approach is the method for interval and fuzzy value comparison. In [29], such method based on the geometrical reasoning has been proposed which leads to the resulting formulas nearly the same as earlier were obtained in [31]with a help of probabilistic approach to fuzzy value comparison. In [33] [34], we have presented an overview of existing methods for fuzzy value comparison based on probabilistic approach. It is shown in [33] [34] that analyzed methods have a common drawback- the lack of separate equality relations- leading to the absurd results in the asymptotical cases and some others inconsistencies. The same can be said about of non- probabilistic method proposed in [29]. To solve the problem, in [33] [34] a new method based on the probabilistic approach has been elaborated which generates the complete set of probabilistic interval and fuzzy value relations involving separated equality and inequality relations, comparisons of real numbers with interval or fuzzy values. Let us recall briefly the basics of this approach. There are only two nontrivial situation of intervals setting: the overlapping and inclusion cases (see Fig. 4) are deserved to be considered.

Let $\mathrm{A} = [\mathrm{a}_1, \mathrm{a}_2]$ and $\mathrm{B} = [\mathrm{b}_1, \mathrm{b}_2]$ be independent intervals and $a \in [a_1, a_2]$, $b \in [b_1, b_2]$ be random values distributed on these intervals. As we are dealing with crisp (nonfuzzy) intervals, the natural assumption is that the random values a and b are distributed uniformly. There are some subintervals, which play an important role in our analysis. For example see Fig. 4), falling of random variables $a \in [a_1, a_2]$, $b \in [b_1, b_2]$ in the subintervals $[a_1, b_1]$, $[b_1, a_2]$, $[a_2, b_2]$ may be treated as a set of independent random events. Let us define the events $H_k : a \in A_i, b \in B_j$, for $k = 1$ to n, where A_i and B_j are subintervals formed by the boundaries of compared intervals A and B such that $A = \bigcup\limits_i A_i$, $B = \bigcup\limits_j B_j$. It is easy to see that events H_k form the complete group of events, which describes

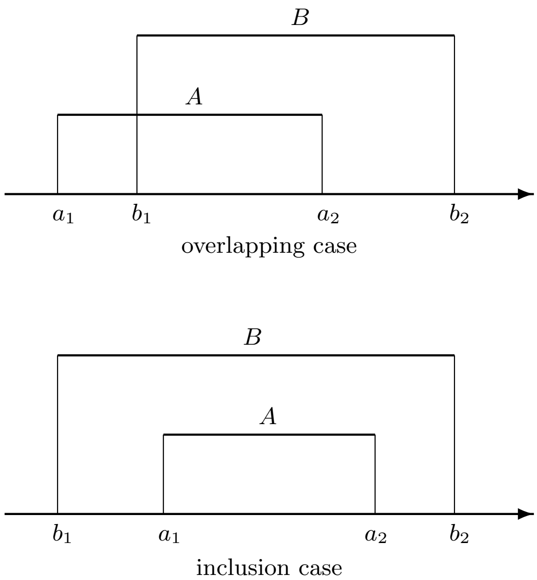


Fig. 4. Examples of interval relations

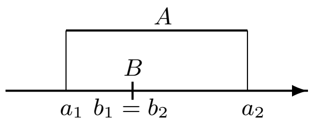
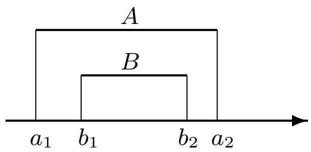
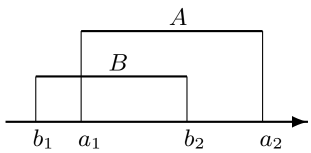
all the cases of falling random values a and b in the various subintervals A_i and B_j, respectively.

Let $P(H_k)$ be the probability of event H_k, and $P(B > A/H_k)$ be the conditional probability of $B > A$ given H_k. Hence, the composite probability may be expressed as follows:

$$P(B > A) = \sum_{k=1}^{n} P(H_k)P(B > A/H_k)$$

As we are dealing with uniform distributions of the random values a and b in the given subintervals, the probabilities $P(H_k)$ can be easily obtained by simple geometric reasoning. These basic assumptions make it possible to infer the complete set of probabilistic interval relations involving separated equality and inequality relations and comparisons of real numbers and intervals. The complete set of expressions for interval relations is shown in Table 2, obvious cases (without overlapping and inclusion) are omitted. In Table 2, only half of cases that may be realized when considering interval overlapping and including are presented since other three cases, e.q., $b_2 > a_2$ for overlapping and so on, can be easily obtained by changing letter a through b and otherwise in the expressions for the probabilities.

Table 2. The probabilistic interval relations

$P(B > A)$	$P(B < A)$	$P(B = A)$
1. $b_1 > a_1 \wedge b_1 < a_2 \wedge b_1 = b_2$		
		
$\dfrac{b_1 - a_1}{a_2 - a_1}$	$\dfrac{a_2 - b_1}{a_2 - a_1}$	0
2. $b_1 \geq a_1 \wedge b_2 \leq a_2$		
		
$\dfrac{b_1 - a_1}{a_2 - a_1}$	$\dfrac{a_2 - b_2}{a_2 - a_1}$	$\dfrac{b_2 - b_1}{a_2 - a_1}$
3. $a_1 \geq b_1 \wedge a_2 \geq b_2 \wedge a_1 \leq b_2$		
		
0	$1 - \dfrac{(b_2 - a_1)^2}{(a_2 - a_1)(b_2 - b_1)}$	$\dfrac{(b_2 - a_1)^2}{(a_2 - a_1)(b_2 - b_1)}$

It easy to see that in all cases $P(A < B) + P(A = B) + P(A > B) = 1$. Of course, we can state that $B > A$ if $P(B > A) > max(P(A > B), P(A = B))$, $B = A$ if $P(A = B) > max(P(A > B), P(A < B))$ and $B < A$ if $P(A > B) > max(P(A < B), P(A = B))$. We think that treating an interval equality as an identity ($A = B$ only if $a_1 = b_1$,$a_2 = b_2$) will not bring good solutions to some practical problems, e.g., when dealing with interval extension of optimization task under equality type restrictions. Obviously, there may be a lot of real-world situations when, from common sense, the intervals $A = [0, 1000.1]$ and $B = [0, 1000.2]$ will be considered as somewhat equal ones. In our approach, equality is not equivalent to identity, since $P(A = B) \leq 1$.

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers on X with corresponding membership functions $\mu_A(x), \mu_B(x) : X \to [0,1]$. We can represent $\tilde{A}$ and $\tilde{B}$ by the sets of α-levels: $\tilde{A} = \bigcup\limits_{\alpha} A_\alpha$, $\tilde{B} = \bigcup\limits_{\alpha} B_\alpha$, where $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}, B_\alpha = \{x \in X : \mu_B(x) \geq \alpha\}$ are crisp intervals. Then all fuzzy number relations $\tilde{A} rel \tilde{B}$, $rel = \{<, =, >\}$, may be presented by sets of α-cut relations

$$\tilde{A} rel \tilde{B} = \bigcup_{\alpha} A_\alpha \, rel \, B_\alpha.$$

Since A_α and B_α are crisp intervals, the probability $P_\alpha(B_\alpha > A_\alpha)$ for each pair A_α and B_α can be calculated in the way described above. The set of the probabilities $P_\alpha(\alpha \in (0,1])$ may be treated as the support of the fuzzy subset

$$P(\tilde{B} > \tilde{A}) = \{\alpha / P_\alpha(B_\alpha > A_\alpha)\},$$

where the values of α may be considered as grades of membership to fuzzy interval $P(\tilde{B} > \tilde{A})$. In this way, the fuzzy subset $P(\tilde{B} = \tilde{A})$ may also be easily created.

Obtained results are simple enough and reflect in some sense the nature of fuzzy arithmetic. The resulting "fuzzy probabilities" can be used directly. For instance, let $\tilde{A}, \tilde{B}, \tilde{C}$ be fuzzy intervals and $P(\tilde{A} > \tilde{B})$, $P(\tilde{A} > \tilde{C})$ be fuzzy intervals expressing the probabilities $A > \tilde{B}$ and $\tilde{A} > \tilde{C}$, respectively. Hence the probability $P(P(\tilde{A} > \tilde{B}) > P(\tilde{A} > \tilde{C}))$ has a sense of probability's comparison and is expressed in the form of fuzzy interval as well. Such fuzzy calculations may be useful at the intermediate stages of analysis, since they preserve the fuzzy information available. Indeed, it can be shown that in any case $P(\tilde{B} > \tilde{A}) + P(\tilde{B} = \tilde{A}) + P(\tilde{B} < \tilde{A}) =$"near 1", where "near 1" is a symmetrical relative to 1 fuzzy number. It is worth noting here that the main properties of probability are remained in the introduced operations, but in a fuzzy sense. However, a detailed discussion of these questions is out of the scope of this paper.

Nevertheless, in practice, the real-valued number indices are needed for fuzzy interval ordering. For this purpose, some characteristic numbers of fuzzy sets could be used. But it seems more natural to use the defuzzification, which for a discrete set of α-cuts takes the form:

$$\overline{P}(\tilde{B} > \tilde{A}) = \sum_{\alpha} \alpha P_\alpha(B_\alpha > A_\alpha) / \sum_{\alpha} \alpha.$$

Last expression indicates that the contribution of α- level to the overall probability estimation is rising along with the rise in its number. Some typical cases of fuzzy interval comparison are represented in the Fig. 5.

It is easy to see that the resulting quantitative estimations are in a good accordance with our intuition. Obviously, the other approaches to the risk assessment in budgeting can be proposed and can be relevant in the specific

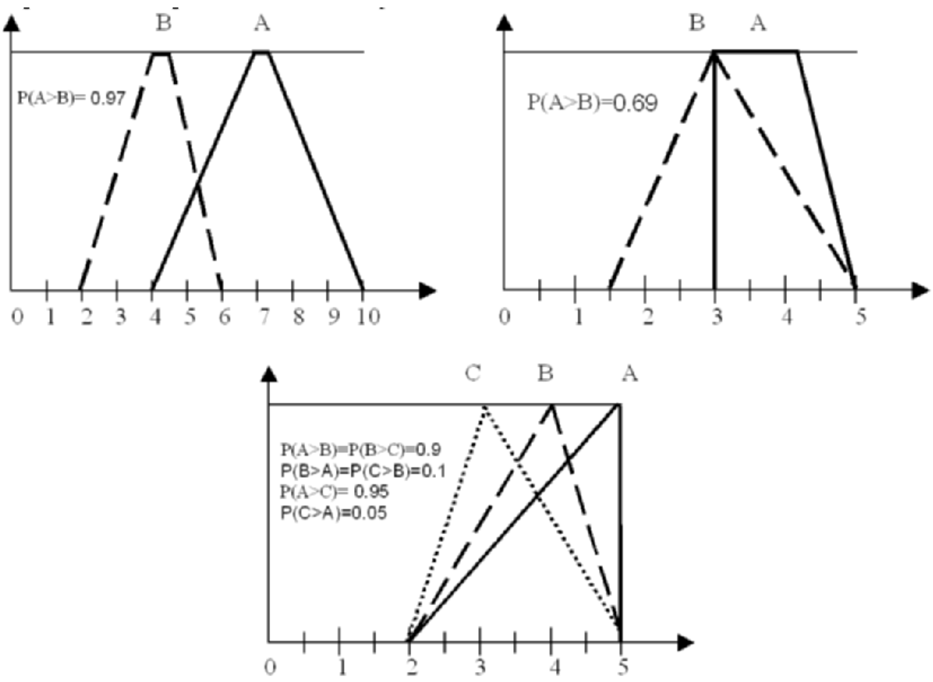


Fig. 5. The typical cases of fuzzy interval ordering.

situations. It is clear they should lead to the different results of investment projects estimation or optimization tasks as they reflect the different decision maker's attitudes to the risk and its importance to the concrete problem considered. Therefore, we think for methodical purposes it is quite enough to consider only one of above approaches. So in further analysis the first model of risk, based on the Exp. (9) will be used. It is important that all considered approaches based on an evaluation of fuzzy NPV inevitably generate two criteria for the estimation of future profits: the fuzzy interval NPV and the degree of its uncertainty (degree of risk).

Therefore, a problem of evaluation of investment efficiency on a base of NPV becomes two-criteria and requires special approach and an appropriate technique. Recently, authors proposed such a technique [32], [14] based on the fuzzy set theory; however, its detailed consideration is out of scope of this paper.

3 The Set of Crisp *IRR* Estimations Based on Fuzzy Cash Flows

In general, the problem of the Internal Rate of Return (IRR) evaluation looks as a fuzzy interval solution of the Eq. (2) with respect to d.

It is proved that a solution of equations with fuzzy parameters (in this case, P_t , KV_t and d) is possible using representation of fuzzy parameters in a form of sets of corresponding α-cuts. For the evaluating IRR, a system of non-linear crisp-interval equations can be obtained:

$$\sum_{t=t_n}^{T} \frac{[P_t]_\alpha}{(1+[d]_\alpha)^t} - \sum_{t=0}^{t_c} \frac{[KV_t]_\alpha}{(1+[d]_\alpha)^t} = [0,0], \tag{10}$$

where $[Pt]_\alpha, [KV_t]_\alpha$ and $[d]_\alpha$ are crisp intervals on corresponding α-cuts.

Of course, it can be claimed that naive assumption, that the degenerated zero interval $[0,0]$ should be placed in the right side of Eq. (10), does not ensure obtaining of adequate outcomes since a non-degenerated interval expression is in the left side of Eq. (10), but this situation needs more thorough consideration.

As the simplest example consider a two-year project when all investments are finished in the first year and all revenues are obtained in the second year. Then each of the equations for α-cuts (10) should be divided on two:

$$\frac{P_{11}}{1+d_2} - KV_{02} = 0, \frac{P_{12}}{1+d_1} - KV_{01} = 0. \tag{11}$$

The formal solution Eq. (11) with respect to d_1 and d_2 is trivial:

$$d_1 = \frac{P_{12}}{KV_{01}} - 1; \quad d_2 = \frac{P_{11}}{KV_{02}} - 1,$$

however it is senseless, as the right boundary of the interval $[d_1, d_2]$ always appears to be less than the left one. This absurd, on a first glance, result is easy to explain from common methodological positions. Really, the rules of the interval mathematics are constructed in such a manner that any arithmetical operation with intervals results in an interval as well. These rules fully coinside with well known common viewpoint stating that any arithmetical operation with uncertainties must increase total uncertainty and the entropy of a system. Therefore, placing the degenerated zero interval in right sides of (10) and (11) is equivalent to the request of reducing uncertainty of the left sides down to zero, which is possible only in case of inverse character of the interval $[d_1, d_2]$, which is in turn can be interpreted as a request to introduce negative entropy into the system.

Thus, the presence of the degenerated zero interval in right sides of interval equations is incorrect. More acceptable approach to solving this problem has been constructed with a help of following reasons. When analysing expressions (11) it is easy to see that for any value d_1 the minimal width of the interval NPV is reached when $d_2 = d_1$. This is in accordance with a common viewpoint: the minimum uncertainty of an outcome (NPV) is reached when uncertainty of all system parameters is minimal. It is clear (see Fig. 6) that the most reasonable decision of “zero” problem is a request for a middle of

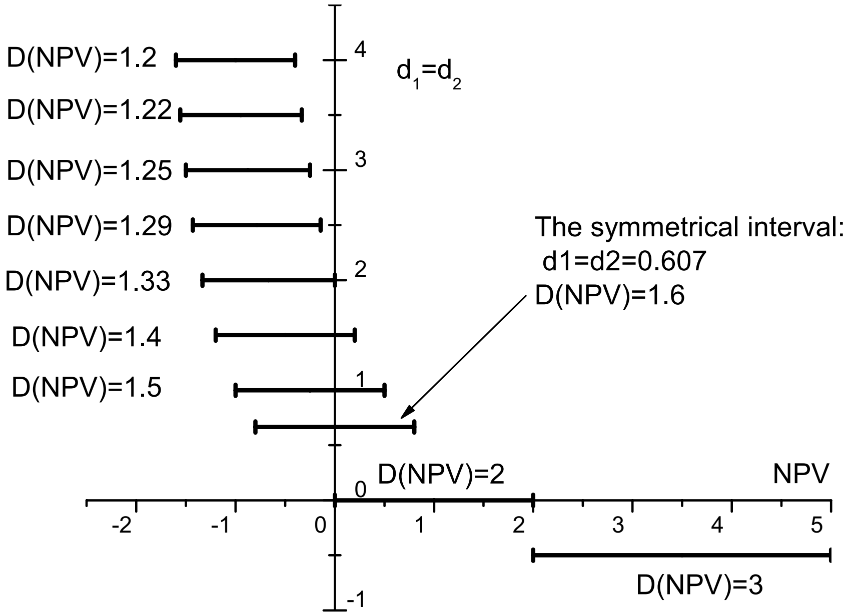


Fig. 6. Interval NPV for different real valued discounts, d, for the case when the investments in the first year is $KV_0 = [1, 2]$, the income in the second year is $P_1 = [2, 3]$, $D(NPV)$ is a width of the interval NPV.

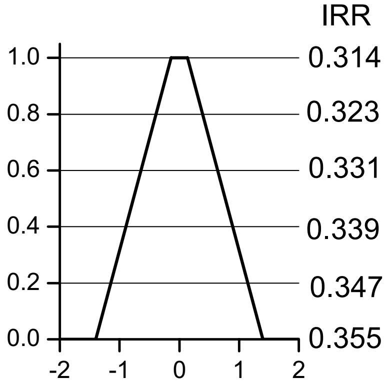
the interval NPV to be placed on a zero point (request of symmetry of the interval against zero). An obvious, on a first glance, intention to minimise the length of interval NPV results in deriving positive or negative intervals of minimum width, but not containing zero point, that does not correspond to a natural definition of zero containing interval. Besides, it can be easily proved that only the request of symmetry of zero containing interval ensures an asymptotically valid outcome when contracting boundaries of all considered intervals to their centres. Thus, the problem is reduced to a search of exact (non-interval) values d that will provide a symmetry of zero resulting intervals NPV on each α-cut in the equations (10), i.e. would guarantee fulfilment of the request $(NPV_1 + NPV_2) = 0$, for each $a = 0, 0.1, 0.2 \ldots, 1$.

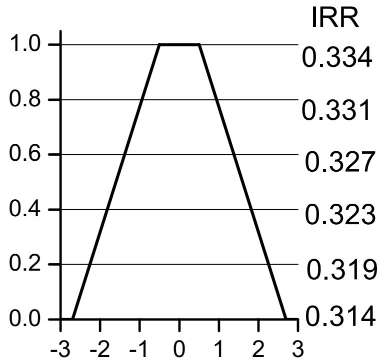
Obviously, the problem is solved using numerical methods. To illustrate previous theoretical considerations, compare two investment pro-jects of 4 years duration. Fuzzy cash flows $K_t = P_t - KV_t$ are defined with a help of the four-reference point form described above (see Table 3). It is worth noting that data of the first project are more certain.

The results of calculations for two investment projects with different fuzzy cash flows are also presented in Table 3. It is seen that values of IRR_α obtained for each α-cut can increase or decrease with growth of α. As a result the set of possible crisp values of IRR is obtained for each project. Thus, a problem of

Table 3. The results of IRR_α calculation

Project 1		Project 2	
Year	Cash flow	Year	Cash flow
1	$\{-6.95, -6,95, -7,05, -8.00\}$	1	$\{-6.00, -6.95, -7.50, -8.00\}$
2	$\{4.95, 4.95, 5.05, 6.00\}$	2	$\{4.00, 4.95, 5.50, 6.00\}$
3	$\{3.95, 3.95, 4.05, 5.00\}$	3	$\{3.00, 3.95, 4.50, 5.00\}$
4	$\{1.95, 1.95, 2.05, 3.00\}$	4	$\{1.00, 1.95, 2.50, 3.00\}$





the results interpretation rises. To solve this problem it is proposed to reduce the sets of IRR_α obtained on each α-cut to a small set of parameters which can be easily interpreted. The first elementary parameter — average value IRR_{m} — is certainly convenient, however it does not take into account that with growth of α the reliability of an outcome increases as well, i.e., IRR_α, obtained on higher α-cuts are more expected than those obtained on lower α-cuts according to the α-cut definition. On the other hand, the width of the crisp interval $[NPV_1, NPV_2]_\alpha$ corresponding to the IRR_α can be considered as, in some sense, a measure of uncertainty for the obtained crisp value IRR_α, since such width quantitatively characterises the difference of the left side of Eq. (10) from the degenerated zero interval $[0, 0]$. This allows to introduce two weighted estimations of IRR on a set IRR_α: least expected (least reliable) $IRR_{\min}$ and most expected (most reliable) $IRR_{\max}$:

$$IRR_{\min} = \frac{\sum\limits_{i=0}^{n-1} IRR_i \left(NPV_{2i} - NPV_{1i}\right)}{\sum\limits_{i=0}^{n-1} \left(NPV_{2i} - NPV_{1i}\right)}, \tag{12}$$

$$IRR_{\max} = \frac{\sum\limits_{i=0}^{n-1} IRR_i \alpha_i}{\sum\limits_{i=0}^{n-1} \alpha_i}, \tag{13}$$

where n is a number of α-cuts.

In a decision making practice it is worth to use all three proposed parameters IRR_{m}, $IRR_{\min}$, $IRR_{\max}$ when choosing the best project. An interpretation of length of $[NPV_1, NPV_2]_\alpha$ as an indexes of uncertainty of IRR_α allows to propose a quantitative, expressed in monetary units essessment of financial risk of a project (the degree of uncertainty of the values IRR_{m}, $IRR_{\min}$, $IRR_{\max}$ derived from uncertainty of initial data):

$$R_{\mathrm{r}} = \frac{\sum\limits_{i=0}^{n-1} (NPV_{2i} - NPV_{1i})}{n}, \tag{14}$$

Parameter R_{r} can play a key role in project efficiency estimation. THe values of introduced derivative paramters for the considered sample projects are presented in Table 4.

Table 4. The derivative (based on IRR)parameter of sample projects.

Project#	$IRR_{\min}$	$IRR_{\max}$	IRR_{m}	R_{r}
1	0.34	0.327	0.335	1.56
2	0.322	0.329	0.325	3.52

It is seen, the projects have rather the close values of IRR_{m}, $IRR_{\min}$, $IRR_{\max}$. At the same time, the risk R_{r} for the second project is considerably higher than risk of the first one. Hence, the first project is the best one. In addition, some other useful paramters have been proposed: IRR_{mr} — most reliable value of IRR_α — derived from the minimum interval $[NPV_1, NPV_2]_{mr}$ among all $[NPV_1, NPV_2]_\alpha$ and IRR_{lr} — the least reliable value of IRR_α — derived from the maximum interval $[NPV_1, NPV_2]_{\mathrm{lr}}$ among all $[NPV_1, NPV_2]_\alpha$. It is clear, that $[NPV_1, NPV_2]_{\mathrm{mr}}$ and $[NPV_1, NPV_2]_{\mathrm{lr}}$ are the risk estimations for the considering IRR_{mr} and IRR_{lr}. It should be noted (see Table 3) that the difference between values IRR_{mr} for the projects is rather small, but the difference in risk estimations is considerable.

4 A Method for a Numerical Solution of the Project Optimization Problem

Proposed here approach to the optimization problem solving is based on consideration of all initial fuzzy intervals P_t and KV_t as the restrictions on controlled input data, as well as on assumption that d_t is a random parameter describing external, in relation to a considered project, uncertainty. The fact that some preferences for an interval of possible values of d may be expressed by certain membership function $\mu_d(d)$, is also taken into account. Thus, while describing the discount factor one deals with uncertainties of both random and fuzzy nature. The problem is solved in two steps. At first, according to the fuzzy extension principle all parameters P_t , KV_t and d_t in Eq. (1) are substituted for corresponding fuzzy-intervals. As a result the fuzzy-interval NPV is obtained. On the next step, obtained fuzzy-interval NPV is considered as a restriction on a profit when building a local criterion for NPV maximisation. For a mathematical description of local criteria, so-called desirability functions are used. In essence, they can be described as a special interpretation of usual membership functions. Briefly, the desirability function rises from 0 (in area of unacceptable values of its argument) up to 1 (in area of the most preferable values). Thus, a construction of desirability function for NPV is rather obvious: the desirability function $\mu_{NPV}(NPV)$ can be considered only on the interval of possible values restricted by the interval $[NPV_1, NPV_4]$. Hence,the more value of the NPV, the more degree of desirability (see Fig. 7).

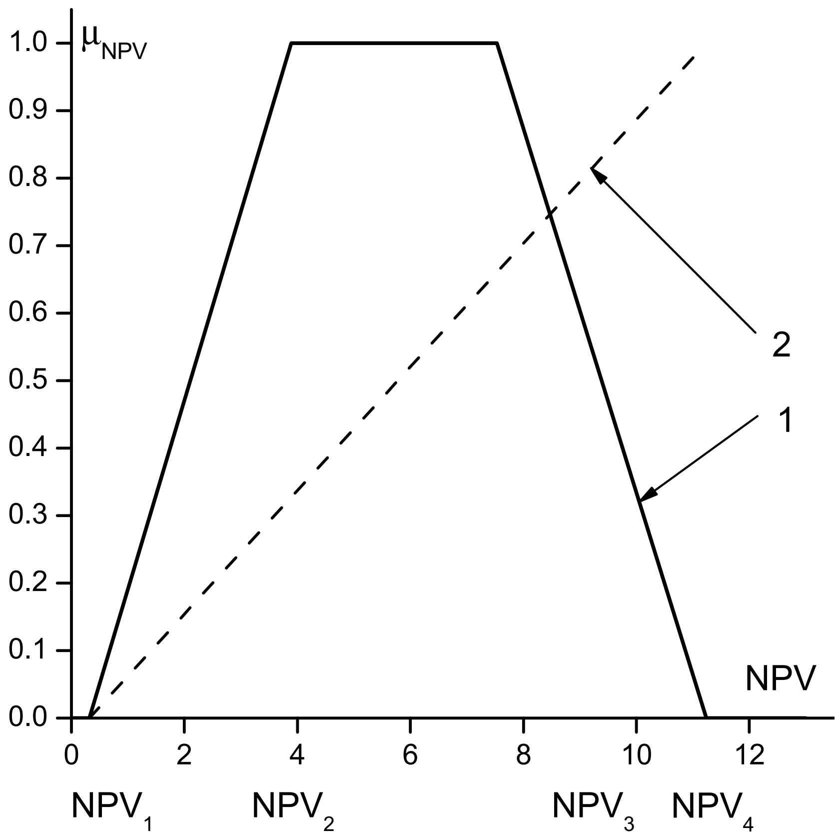


Fig. 7. Connection between the restriction and the local criterion: 1 - the intitial fuzzy interval of NPV (fuzzy restriction); 2 - the desirability function $\mu_{NPV}(NPV)$.

The initial fuzzy intervals P_t and KV_t are also considered as desirability functions $\mu_{P_1}, \mu_{P_2}, \ldots, \mu_{KV_1}, \mu_{KV_2}, \ldots$ describing restrictions on controlled input variables. It is clear that initial intervals were already constructed in such a way that when they are interpreted as desirability functions the more preferable values in intervals of P_t and KV_t appear to be those more realisable (possible). Since these desirability functions are connected with a possibility of realisation of corresponding values of variables P_t and KV_t, they implicitly describe financial risk of the project.

As the result, the general criterion based on the set of all desirability functions has been defined as

$$D\left(P_t, KV_t, d_t\right) = \mu_{NPV}^{\alpha_1}\left(NPV\left(P_t, KV_t, d_t\right)\right) \wedge \left(\mu_{P_1} \wedge \mu_{P_2} \wedge \ldots \wedge \mu_{KV_1}, \mu_{KV_2} \wedge \ldots\right)^{\alpha_2}, \quad (15)$$

where α_1 and α_2 are ranks characterising the relative importance of local criteria of profit maximisation and risk minimisation, $\wedge$ is min operator, $\mu_{NPV}\left(NPV\left(P_t, KV_t, d_t\right)\right)$ is a desirability function of NPV.

Many different forms of the general criterion are in use. As emphasised in [35],the choice of particular aggregating operator, usually called t-norm, is rather an application dependent problem. However, the choice of min operator in Eq. (15) is the most straightforward approach, when a compensation of small values of some criteria by the great values of others is not permited [32], [14]. The problem is reduced to a search for crisp values of $PP_1, PP_2, \ldots, KKV_1, KKV_2, \ldots$ on corresponding fuzzy intervals $P_1, P_2, \ldots, KV_1, KV_2, \ldots$, maximising the general criterion (15).

The problem is complicated by the fact that the discount d is a random parameter, distributed in a specific interval. The solution was carried out as follows.

Firstly, from interval of possible values a fixed value of discount d_i is selected randomly. Further, with a help of the Nollaw-Furst random method an optimum solution is obtained as the best compromise between uncertainty of basic data and intention to maximise profit, i.e., the optimisation problem reduces to maximisation of the general criterion (15). Obtained optimal values PP_t^d and KKV_t^d present the local optimum solution for given discount value. Above procedure is repeated with random discount values until the statistically representative sample of optimum solutions for various d_i is obtained. Final optimum values PP_t^0 , KKV_t^0 are calculated by weighting with degrees of possibility of d_i, which are defined by initial fuzzy interval d with a membership function $\mu_d(d_i)$:

$$PP_t^0 = \frac{\sum\limits_{i=1}^{m} PP_t^d(d_i)\mu_d(d_i)}{\sum\limits_{i=1}^{m} \mu_d(d_i)}, \quad (16)$$

where m is a number of random discount values used for the solution of a problem. Similarly, all KKV_t^0 can be calculated.

It is also possible to take into account the values of the general criterion in optimum points:

$$PP_t^0 = \frac{\sum_{i=1}^{m} \left(PP_t^d(d_i) \left(\mu_d^{\beta_1}(d_i) \wedge D^{\beta_2}(d_i) \right) \right)}{\sum_{i=1}^{m} \left(\mu_d^{\beta_1}(d_i) \wedge D^{\beta_2}(d_i) \right)}, \tag{17}$$

where β_1, β_2 are corresponding weights.

The similar expression can be constructed for KKV_t^0. It is worth noting that last expression gives an ability to take into account, apart from reliability of the values d_i, the degree of compatibility (in other words, the degree of consensus) for each of selected values of discount.

Obtained optimal PP_t^0 and KK_t^0 may be used for a final project's quality estimation. The results of calculation for the first example from the previous Section (Table 3, project 1) are presented in Table 5.

Table 5. The results of optimization

	Expression (16)		Expression (17)	
Years	PP_t^0	KK_t^0	PP_t^0	KK_t^0
0	0.00	2.49	0.00	2.50
1	0.00	0.83	0.00	0.79
2	8.05	0.00	8.04	0.00
3	7.12	0.00	7.09	0.00

Further, with substituting PP_t^0, KK_t^0 and fuzzy interval d in the expression (1) an optimal fuzzy value of NPV was obtained.

For considered example we get the following results:
$NPV_{16} = \{4.057293, 6.110165, 8.073906, 9.454419\}$ using (16)
and
$NPV_{17} = \{4.065489, 6.109793, 8.064094, 9.436519\}$ using (17).

It is clear that there is no great deference between the results obtained using expressions (16) and (17) in this case.

In Fig. 8, the fuzzy NPV_{16} obtained with PP_t^0 and KK_t^0 is compared with the initial one, obtained with the initial fuzzy values P_t and KV_t, without optimisation. It is obvious that in the optimal case the mean value of fuzzy interval NPV is greater.

Using optimal PP_t^0 and KK_t^0 and applying the method described in Section 2, the degree of project risk may be also estimated. This risk can be considered as financial risk of a project as a whole.

For the needs of common accounting practice it is possible to calculate an average weighted value of NPV using following expression:

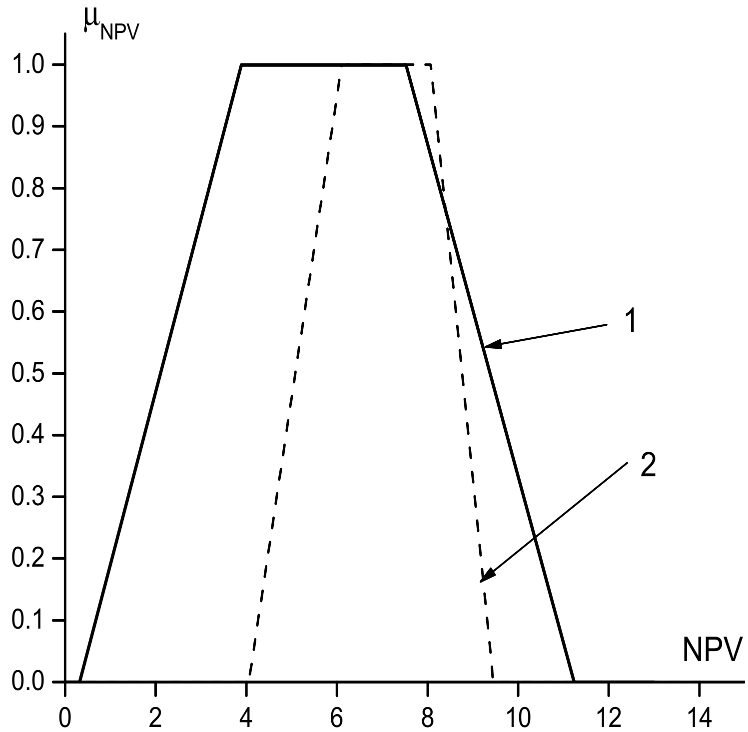


Fig. 8. Comparison of the initial and optimal fuzzy intervals NPV: 1 - initial NPV; 2 - optimal NPV.

$$NPV = \frac{\sum_{i=1}^{m} NPV_i * \mu_{NPV_i}}{\sum_{i=1}^{m} \mu_{NPV_i}}, \tag{18}$$

For the considered example $NPV_{16} = 6.8931$ and $NPV_{17} = 6.8942$ were obtained.

5 Conclusion

The problems of calculation of NPV and IRR and investment project risk assessment i a fuzzy setting are considered. It is shown that the straightforward way of project risk assessment is to consider this risk as a degree of fuzziness of the fuzzy Net Present Value, NPV. Nevertheless, other method for risk astimation on the probability approach to interval and fuzzy value comparison can be relevant in the fuzzy budgeting as well. It is shown that although it is impossible to obtain fuzzy Internal Rate of Return, IRR, the crisp IRR may be obtained as a solution of a fuzzy equation and a set of new useful derivative parameters characterising uncertainty of the problem may be obtained as an additional result.The problem of the multiobjective optimization of a project in a mixed fuzzy and random enviroment is formulated in a form of compromise between local criteria of profit maximisation and risk minimisation. Numerical method for the problem solving is described and tested.

References

1. Belletante B, Arnaud H (1989) Choisir ses investissements. Paris Chotard et Associés Éditeurs
2. Brigham E F (1992) Fundamentals of Financial Management. The Dryden Press, New York
3. Chansa-ngavej Ch, Mount-Campbell CA (1991) Decision criteria in capital budgeting under uncertainties: implications for future research. Int J Prod Economics 23:25–35
4. Liang P, Song F (1994) Computer-aided risk evaluation system for capital investment. Omega 22(4):391–400
5. Bogle HF, Jehenck GK (1985) Investment Analysis: US Oil and Gas Producers Score High in University Survey. In: Proc of Hydrocarbon Economics and Evaluation Symposium. Dallas 234-241
6. Babusiaux D, Pierru A (2001) Capital budgeting, project valuation and financing mix: Methodological proposals. Europian Journal of Operational Research 135:326–337
7. Moore RE (1966) Interval analysis. Englewood Cliffs, Prentice-Hall N J
8. Zadeh LA (1965) Fuzzy sets. Inf.Control 8:338–353
9. Ward TL (1985) Discounted fuzzy cash flow analysis. In: 1985 Fall Industrial Engineering Conference Proceedings 476–481
10. Buckley JJ (1987) The fuzzy mathematics of finance. Fuzzy Sets and Systems 21:257–273
11. Chen S (1995) An empirical examination of capital budgeting techniques: impact of investment types and firm characteristics. Eng Economist 40(2):145–170
12. Chiu CY, Park CS (1994) Fuzzy cash flow analysis using present worth criterion. Eng Economist 39(2):113–138
13. Choobineh F, Behrens A (1992) Use of intervals and possibility distributions in economic analysis. J Oper Res Soc 43(9):907–918
14. Dimova L, Sevastianov P, Sevastianov D (2005) MCDM in a fuzzy setting: investment projects assessment application. International Journal of Production Economics (in press)
15. Li Calzi M (1990) Towards a general setting for the fuzzy mathematics of finance. Fuzzy Sets and Systems 35:265–280
16. Perrone G (1994) Fuzzy multiple criteria decision model for the evaluation of AMS. Comput Integrated Manufacturing Systems 7(4):228–239
17. Chiu CY, Park CS (1994) Fuzzy cash flow analysis using present worth criterion. Eng Economist 39(2):113–138

18. Kahraman C, Tolga E, Ulukan Z (2000) Justification of manufacturing technologies using fuzzy benefit/cost ratio analysis. Int J Product Econom 66(1):45–52
19. Kahraman C, Ulukan Z (1997) Continuous compounding in capital budgeting using fuzzy concept. In: Proc of the 6th IEEE International Conference on Fuzzy Systems 1451–1455
20. Kahraman C, Ulukan Z (1997) Fuzzy cash flows under inflation. In: Proc of the Seventh International Fuzzy Systems Association World Congress (IFSA 97) 4:104–108
21. Sevastianov P, Sevastianov D (1997) Risk and capital budgeting parameters evaluation from the fuzzy sets theory position. Reliable software 1:10–19
22. Dimova L, Sevastianov D, Sevastianov P (2000) Application of fuzzy sets theory, methods for the evaluation of investment efficiency parameters. Fuzzy economic review 5(1):34–48
23. Kuchta D (2000) Fuzzy capital budgeting. Fuzzy Sets and Systems 111:367–385
24. Kahraman C (2001) Fuzzy versus probabilistic benefit/cost ratio analisis for public work projects. Int J Appl Math Comp Sci 11(3):705–718
25. Kahraman C, Ruan D, Tolga E (2002) Capital budgeting techniques using discounted fuzzy versus probabilistic cash flows. Information Sciences 142:57–76
26. Kaufmann A, Gupta M (1985) Introduction to fuzzy-arithmetic theory and applications. Van Nostrand Reinhold, New York
27. Jaulin L, Kieffir M, Didrit O, Walter E (2001) Applied Interval Analysis. Springer-Verlag, London
28. Longerstaey J, Spenser M (1996) RiskMetric-Technical document. RiskMetric Group, J.P. Morgan, New York
29. Nedosekin A, Kokosh A (2004) Investment risk estimation for arbitrary fuzzy factors of investment project. In: Proc. of Int. Conf. on Fuzzy Sets and Soft Computing in Economics and Finance. St. Petersburg 423–437
30. Yager RA (1979) On the measure of fuzziness and negation. Part 1. Membership in the Unit. Interval Int J Gen Syst 5:221–229
31. Yager RR, Detyniecki M, Bouchon-Meunier B (2001) A context-dependent method for ordering fuzzy numbers using probabilities, Information Sciences 138:237–255
32. Sevastianov P, Dimova L, Zhestkova E (2000) Methodology of the multicriteria quality estimation and its software realizing. In: Proc of the Fourth International Conference on New Information Technologies NITe' 1:50–54
33. Sevastianov P, Rog P (2002) A probabilistic approach to fuzzy and interval ordering, Task Quarterly. Special Issue Artificial and Computational Intelligence 7:147–156
34. Sewastianow P, Rog P (2005) Two-objective method for crisp and fuzzy interval comparison in optimization. Computers and Operation Research (in press)
35. Zimmermann HJ, Zysno P (1980) Latest connectives in human decision making. Fuzzy Sets and Systems 4:37–51