Applications of Fuzzy Capital Budgeting Techniques

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Summary. In an uncertain economic decision environment, an expert's knowledge about dicounting cash flows consists of a lot of vagueness instead of randomness. Cash amounts and interest rates are usually estimated by using educated guesses based on expected values or other statistical techniques to obtain them. Fuzzy numbers can capture the difficulties in estimating these parameters. In this chapter, the formulas for the analyses of fuzzy present value, fuzzy equivalent uniform annual value, fuzzy future value, fuzzy benefit-cost ratio, and fuzzy payback period are developed and some numeric examples are given. Then the examined cash flows are expanded to geometric and trigonometric cash flows and using these cash flows fuzzy present value, fuzzy future value, and fuzzy annual value formulas are developed for both discrete compounding and continuous compounding. Finally, a fuzzy versus stochastic investment analysis is examined by using the probability of a fuzzy event.

Key words: fuzzy number, capital budgeting, cash flow, ranking method

1 Introduction

The purpose of this chapter is to develop the fuzzy capital budgeting techniques. The analyses of fuzzy future value, fuzzy present value, fuzzy rate of return, fuzzy benefit/cost ratio, fuzzy payback period, fuzzy equivalent uniform annual value are examined for the case of discrete compounding.

To deal with vagueness of human thought, Zadeh [1] first introduced the fuzzy set theory, which was based on the rationality of uncertainty due to imprecision or vagueness. A major contribution of fuzzy set theory is its capability of representing vague knowledge. The theory also allows mathematical operators and programming to apply to the fuzzy domain.

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A fuzzy number is a normal and convex fuzzy set with membership function $\mu_A(x)$ which both satisfies normality: $\mu_A(x)$=1, for at least one $x \in R$ and convexity: $\mu_A(x') \geq \mu_A(x_1) \Lambda \mu_A(x_2)$, where $\mu_A(x) \in [0,1]$ and $\forall x' \in [x_1, x_2]$. '$\Lambda$' stands for the minimization operator.

Quite often in finance future cash amounts and interest rates are estimated. One usually employs educated guesses, based on expected values or other statistical techniques, to obtain future cash flows and interest rates. Statements like *approximately between \$ 12,000 and \$ 16,000* or *approximately between 10% and 15%* must be translated into an exact amount, such as *\$ 14,000* or *12.5%* respectively. Appropriate fuzzy numbers can be used to capture the vagueness of those statements.

A tilde will be placed above a symbol if the symbol represents a fuzzy set. Therefore, $\tilde{P}, \tilde{F}, \tilde{G}, \tilde{A}, \tilde{i}, \tilde{r}$ are all fuzzy sets. The membership functions for these fuzzy sets will be denoted by $\mu(x|\tilde{P}), \mu(x|\tilde{F}), \mu(x|\tilde{G})$, etc. A fuzzy number is a special fuzzy subset of the real numbers. A triangular fuzzy number (TFN) is shown in Figure 1. The membership function of a TFN $(\tilde{M})$ defined by

$$\mu(x|\tilde{M}) = (m_1, f_1(y|\tilde{M}) / m_2, m_2 / f_2(y|\tilde{M}), m_3) \tag{1}$$

where $m_1 \prec m_2 \prec m_3$, $f_1(y|\tilde{M})$ is a continuous monotone increasing function of y for $0 \leq y \leq 1$ with $f_1(0|\tilde{M}) = m_1$ and $f_1(1|\tilde{M}) = m_2$ and $f_2(y|\tilde{M})$ is a continuous monotone decreasing function of y for $0 \leq y \leq 1$ with $f_2(1|\tilde{M}) = m_2$ and $f_2(0|\tilde{M}) = m_3$. $\mu(x|\tilde{M})$ is denoted simply as $(m_1/m_2, m_2/m_3)$.

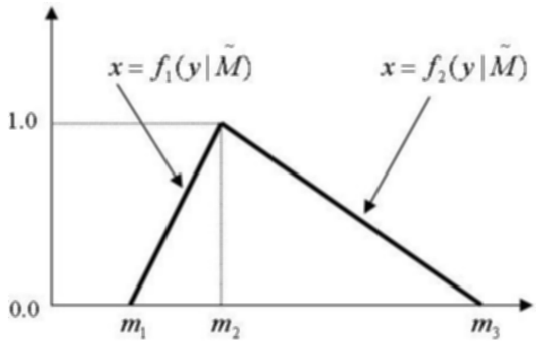

Fig. 1. A Triangular Fuzzy Number, $\tilde{M}$

A flat fuzzy number (FFN) is shown in Figure 2. The membership function of a FFN, $\tilde{V}$ is defined by

$$\mu(x|\tilde{V}) = (m_1, f_1(y|\tilde{V})/m_2, m_3/f_2(y|\tilde{V}), m_4) \tag{2}$$

where $m_1 \prec m_2 \prec m_3 \prec m_4$, $f_1(y|\tilde{V})$ is a continuous monotone increasing function of y for $0 \le y \le 1$ with $f_1(0|\tilde{V}) = m_1$ and $f_1(1|\tilde{V}) = m_2$ and $f_2(y|\tilde{V})$ is a continuous monotone decreasing function of y for $0 \le y \le 1$ with $f_2(1|\tilde{V}) = m_3$ and $f_2(0|\tilde{V}) = m_4$. $\mu(y|\tilde{V})$ is denoted simply as $(m_1/m_2, m_3/m_4)$.

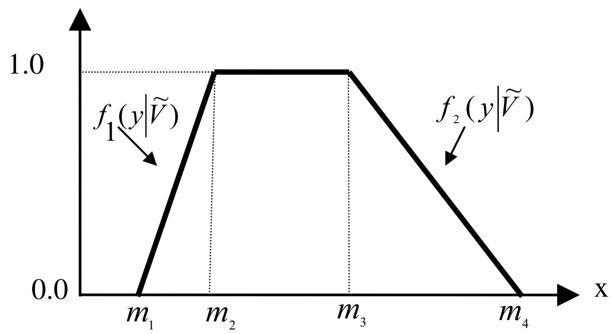

Fig. 2. A Flat Fuzzy Number, $\tilde{V}$

The fuzzy sets $\tilde{P}, \tilde{F}, \tilde{G}, \tilde{A}, \tilde{i}, \tilde{r}$ are usually fuzzy numbers but n will be discrete positive fuzzy subset of the real numbers [2]. The membership function $\mu(x|\tilde{n})$ is defined by a collection of positive integers n_i, $1 \le i \le K$, where

$$\mu(x|\tilde{n}) = \begin{cases} \mu(n|\tilde{n}) = \lambda, & 0 \le \lambda \le 1 \\ 0, & otherwise \end{cases} \tag{3}$$

2 Fuzzy Present Value (PV) Method

The present-value method of alternative evaluation is very popular because future expenditures or receipts are transformed into equivalent dollars now. That is, all of the future cash flows associated with an alternative are converted into present dollars. If the alternatives have different lives, the alternatives must be compared over the same number of years.

Chiu and Park [3] propose a present value formulation of a fuzzy cash flow. The result of the present value is also a fuzzy number with nonlinear membership function. The present value can be approximated by a TFN. Chiu and Park [3]'s formulation is

$$P\tilde{V} = \left[\sum_{t=0}^{n}\left(\frac{\max(P_t^{l(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{r(y)})} + \frac{\min(P_t^{l(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{l(y)})}\right),\right.$$
$$\left.\sum_{t=0}^{n}\left(\frac{\max(P_t^{r(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{l(y)})} + \frac{\min(P_t^{r(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{r(y)})}\right)\right] \tag{4}$$

where $P_t^{l(y)}$: the left representation of the cash at time *t*, $P_t^{r(y)}$: the right representation of the cash at time *t*, $r_t^{l(y)}$: the left representation of the interest rate at time *t*, $r_t^{r(y)}$: the right representation of the interest rate at time *t*.

Buckley's [2] membership function for $\tilde{P}_n$,

$$\mu(x|\tilde{P}_n) = (p_{n1}, f_{n1}(y|\tilde{P}_n)/p_{n2}, p_{n2}/f_{n2}(y|\tilde{P}_n), p_{n3}) \tag{5}$$

is determined by

$$f_{ni}(y|\tilde{P}_n) = f_i(y|\tilde{F})(1+f_k(y|\tilde{r})^{-n} \tag{6}$$

for i = 1,2 where $k=i$ for negative $\tilde{F}$ and $k=3-i$ for positive $\tilde{F}$.

Ward [4] gives the fuzzy present value function as

$$P\tilde{V} = (1+r)^{-n}(a,b,c,d) \tag{7}$$

where *(a, b, c, d)* is a flat fuzzy filter function (4F) number.

Example 1. A $ (-14,000, -12,000, -10,000) investment will return annual benefits $(2,650, 2,775, 2,900) for six years with no salvage value at the end of six years. Compute the fuzzy present worth of the cash flow using an interest of (7.12%, 10.25%, 13.42%) per year.

$$f_{6,1}(y|\widetilde{P}) = \sum_{j=0}^{6} f_{j,1}(y|\widetilde{F}_j)[1 + f_{k(j)}(y|\widetilde{r}_f)]^{-j} \tag{8}$$

$$f_{6,2}(y|\widetilde{P}) = \sum_{j=0}^{6} f_{j,2}(y|\widetilde{F}_j)[1 + f_{k(j)}(y|\widetilde{r}_f)]^{-j} \tag{8}$$

i=1, 2 where $k(j)=i$ for negative $\widetilde{F}_j$ and $k(j) = 3 - i$ for positive $\widetilde{F}_j$.

For y = 0, $f_{6,1}(y|\widetilde{P}) = \$ - 3{,}525.57$

For y = 1, $f_{6,1}(y|\widetilde{P}) = f_{6,2}(y|\widetilde{P}) = \$ - 24.47$

For y = 0, $f_{6,2}(y|\widetilde{P}) = \$ + 3{,}786.34$

The possibility of NPV = 0 for this triangular fuzzy number can be calculated using a linear enterpolation:

$x = -3{,}810.81y + 3{,}786.34$

For x = 0, $Poss(NPV = 0) = 0.9936$.

3 Fuzzy Capitalized Value Method

A specialized type of cash flow series is a perpetuity, a uniform series of cash flows which continues indefinetely. An infinite cash flow series may be appropriate for such very long-term investment projects as bridges, highways, forest harvesting, or the establishment of endowment funds where the estimated life is 50 years or more.

In the nonfuzzy case, if a present value P is deposited into a fund at interest rate r per period so that a payment of size A may be withdrawn each and every period forever, then the following relation holds between P, A, and r:

$$P = \frac{A}{r} \tag{10}$$

In the fuzy case, lets assume all the parameters as triangular fuzzy numbers: $\tilde{P}=(p_1,p_2,p_3)$ or $\tilde{P}=((p_2-p_1)y+p_1),(p_2-p_3)y+p_3)$ and $\tilde{A}=(a_1,a_2,a_3)$ or $\tilde{A}=((a_2-a_1)y+a_1),(a_2-a_3)y+a_3)$ and $\tilde{r}=(r_1,r_2,r_3)$ or $\tilde{r}=((r_2-r_1)y+r_1,(r_2-r_3)y+r_3)$, where y is the membership degree of a certain point of A and r axis. If $\tilde{A}$ and $\tilde{r}$ are both positive,

$$\tilde{P}=\tilde{A}\div\tilde{r}=(a_1/r_3,a_2/r_2,a_3/r_1) \tag{11}$$

or

$$\tilde{P}=(((a_2-a_1)y+a_1)/((r_2-r_3)y+r_3),((a_2-a_3)y+a_3)/((r_2-r_1)y+r_1)) \tag{12}$$

If $\tilde{A}$ is negative and $\tilde{r}$ is positive,

$$\tilde{P}=\tilde{A}\div\tilde{r}=(a_1/r_1,a_2/r_2,a_3/r_3) \tag{13}$$

or

$$\tilde{P}=(((a_2-a_1)y+a_1)/((r_2-r_1)y+r_1),((a_2-a_3)y+a_3)/((r_2-r_3)y+r_3)) \tag{14}$$

Now, let $\tilde{A}$ be an expense every *n*th period forever, with the first expense occurring at *n*. For example, an expense of (\$5,000,\$7,000,\$9,000) every third year forever, with the first expense occurring at t=3. In this case, the fuzzy efffective rate $\tilde{e}$ may be used as in the following:

$$f_i(y|\tilde{e})=(1+(1/m)f_i(y|\tilde{r}'))^m-1 \tag{15}$$

where $i=1,2$; $f_1(y|\tilde{e})$: a continuous monotone increasing function of y; $f_2(y|\tilde{e})$: a continuous monotone decreasing function of y; m: the number of compoundings per period; $\tilde{r}'$: the fuzzy nominal interest rate per period. The membership function of $\tilde{e}$ may be given as

$$\mu(x|\tilde{e})=(e_1,f_1(y|\tilde{e})/e_2,e_2/f_2(y|\tilde{e}),e_3) \tag{16}$$

If $\tilde{A}$ and $f_i(y|\tilde{e})$ are both positive,

$$\tilde{P} = \tilde{A} \oslash \tilde{e} = [((a_2 - a_1)y + a_1) / f_2(y|\tilde{e}), ((a_2 - a_3)y + a_3) / f_1(y|\tilde{e})] \tag{17}$$

If $\tilde{A}$ is negative and $f_i(y|\tilde{e})$ is positive,

$$\tilde{P} = \tilde{A} \oslash \tilde{e} = [((a_2 - a_1)y + a_1) / f_1(y|\tilde{e}), ((a_2 - a_3)y + a_3 / f_2(y|\tilde{e})] \tag{18}$$

$(a_2 - a_1)y + a_1$ and $(a_2 - a_3)y + a_3$ can be symbolized as $f_1(y|\tilde{a})$ and $f_2(y|\tilde{a})$ respectively.

Example 2: Project ABC consists of the following requirements. Find the capitalized worth of the project if $\tilde{r} = (\%12, \%15, \%18)$ annually.

1. A ($40,000, $50,000, $60,000) first cost $(F\tilde{C})$ at *t*=0,
2. A ($4,000, $5,000, $6,000) expense $(\tilde{A}_1)$ every year,
3. A ($20,000, $25,000, $30,000) expense $(\tilde{A}_2)$ every third year forever, with the first expense occurring at *t*=3.

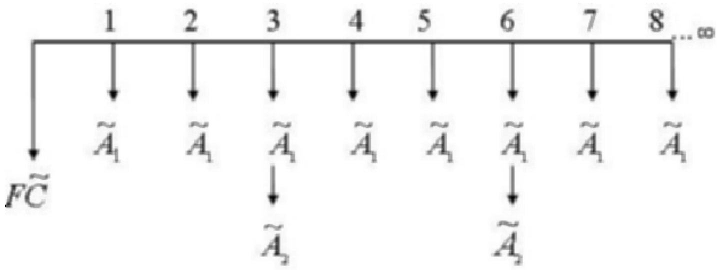

Fig. 3. The Cash Flow Diagram for Example 2

$$\tilde{P} = F\tilde{C} \oplus (\tilde{A}_1 \oslash \tilde{r}) + (\tilde{A}_2 \oslash \tilde{e}) \tag{19}$$

$\tilde{P} = ((10{,}000y + 40{,}000), (-10{,}000y + 60{,}000)) +$
$((1{,}000y + 4{,}000) / (-0.03y + 0.18), (-1{,}000y + 6{,}000) / (0.03y + 0.12)) +$
$((5{,}000y + 20{,}000) / ((1.18 - 0.03y)^3 - 1),$
$(-5{,}000y + 30{,}000) / ((1.12 + 0.03y)^3 - 1))$

For *y*=0, $f_{\infty,1}(y|\tilde{P}) = \$93{,}326.42$

For y=1, $f_{\infty,1}(y|\widetilde{P}) = f_{\infty,2}(y|\widetilde{P}) = \$131{,}317.97$

For y=0, $f_{\infty,2}(y|\widetilde{P}) = \$184{,}074.07$

4 Fuzzy Future Value Method

The future value (*FV*) of an investment alternative can be determined using the relationship

$$FV(r) = \sum_{t=0}^{n} P_t (1+i)^{n-t} \tag{20}$$

where *FV(r)* is defined as the future value of the investment using a minimum attractive rate of return *(MARR)* of *r%*. The future value method is equivalent to the present value method and the annual value method.

Chiu and Park's [3] formulation for the fuzzy future value has the same logic of fuzzy present value formulation:

$$\{\sum_{t=0}^{n-1}[\max(P_t^{l(y)},0)\prod_{t'=t+1}^{n}(1+r_{t'}^{l(y)}) + \min(P_t^{l(y)},0)\prod_{t'=t+1}^{n}(1+r_{t'}^{r(y)})] + P_n^{l(y)}, \tag{21}$$

$$\sum_{t=0}^{n-1}[\max(P_t^{r(y)},0)\prod_{t'=t+1}^{n}(1+r_{t'}^{r(y)}) + \min(P_t^{r(y)},0)\prod_{t'=t+1}^{n}(1+r_{t'}^{l(y)})] + P_n^{r(y)}\}$$

Buckley's [2] membership function $\mu(x|\widetilde{F})$ is determined by

$$f_i(y|\widetilde{F}_n) = f_i(y|\widetilde{P})(1 + f_i(y|\widetilde{r}))^n \tag{22}$$

For the uniform cash flow series, $\mu(x|\widetilde{F})$ is determined by

$$f_{ni}(y|\widetilde{F}) = f_i(y|\widetilde{A})\beta(n, f_i(y|\widetilde{r})) \tag{23}$$

where i=1,2 and $\beta(n,r) = (((1+r)^n - 1)/r)$ and $\widetilde{A} \succ 0$ and $\widetilde{r} \succ 0$.

5 Fuzzy Benefit/Cost Ratio Method

The benefit/cost ratio (*BCR)* is often used to assess the value of a municipal project in relation to its cost; it is defined as

$$BCR = \frac{B - D}{C} \tag{24}$$

where B represents the equivalent value of the benefits associated with the project, D represents the equivalent value of the disbenefits, and C represents the project's net cost. A BCR greater than 1.0 indicates that the project evaluated is economically advantageous. In BCR analyses, costs are not preceded by a minus sign.

When only one alternative must be selected from two or more mutually exclusive (stand-alone) alternatives, a multiple alternative evaluation is required. In this case, it is necessary to conduct an analysis on the incremental benefits and costs. Suppose that there are two mutually exclusive alterantives. In this case, for the incremental BCR analysis ignoring disbenefits the following ratios must be used:

$$\Delta B_{2-1} / \Delta C_{2-1} = \Delta PV\, B_{2-1} / \Delta PVC_{2-1} \tag{25}$$

where PVB: present value of benefits, PVC: present value of costs. If $\Delta B_{2-1} / \Delta C_{2-1} \geq 1.0$, the alternative 2 is preferred.

In the case of fuzziness, first, it will be assumed that the largest possible value of Alternative 1 for the cash in year t is less than the least possible value of Alternative 2 for the cash in year t. The fuzzy incremental BCR is

$$\Delta\tilde{B} / \Delta\tilde{C} = \left(\frac{\sum_{t=0}^{n} (B_{2t}^{l(y)} - B_{1t}^{r(y)})(1 + r^{r(y)})^{-t}}{\sum_{t=0}^{n} (C_{2t}^{r(y)} - C_{1t}^{l(y)})(1 + r^{l(y)})^{-t}}, \frac{\sum_{t=0}^{n} (B_{2t}^{r(y)} - B_{1t}^{l(y)})(1 + r^{l(y)})^{-t}}{\sum_{t=0}^{n} (C_{2t}^{l(y)} - C_{1t}^{r(y)})(1 + r^{r(y)})^{-t}} \right) \tag{26}$$

If $\Delta \tilde{B} / \Delta\tilde{C}$ is equal or greater than (1, 1, 1), Alternative 2 is preferred.

In the case of a regular annuity, the fuzzy $\tilde{B} / \tilde{C}$ ratio of a single investment alternative is

$$\tilde{B} / \tilde{C} = \left(\frac{A^{l(y)} \gamma(n, r^{r(y)})}{C^{r(y)}}, \frac{A^{r(y)} \gamma(n, r^{l(y)})}{C^{l(y)}} \right) \tag{27}$$

where $\tilde{C}$ is the first cost and $\tilde{A}$ is the net annual benefit, and $\gamma(n, r) = ((1 + r)^n - 1) / (1 + r)^n r)$.

The $\Delta\tilde{B} / \Delta\tilde{C}$ ratio in the case of a regular annuity is

$$\Delta\tilde{B} / \Delta\tilde{C} = \left(\frac{(A_2^{l(y)} - A_1^{r(y)}) \gamma(n, r^{r(y)})}{C_2^{r(y)} - C_1^{l(y)}}, \frac{(A_2^{r(y)} - A_1^{l(y)}) \gamma(n, r^{l(y)})}{C_2^{l(y)} - C_1^{r(y)}} \right) \tag{28}$$

6 Fuzzy Equivalent Uniform Annual Value (EUAV) Method

The *EUAV* means that all incomes and disbursements (irregular and uniform) must be converted into an equivalent uniform annual amount, which is the same each period. The major advantage of this method over all the other methods is that it does not require making the comparison over the least common multiple of years when the alternatives have different lives [5]. The general equation for this method is

$$EUAV = A = NPV\gamma^{-1}(n,r) = NPV[\frac{(1+r)^n r}{(1+r)^n - 1}] \quad (29)$$

where *NPV* is en the fuzzy $\widetilde{EUAV}$ ($\tilde{A}_n$) will be found. The membership function the net present value. In the case of fuzziness, $\widetilde{NPV}$ will be calculated and th $\mu(x|\tilde{A}_n)$ for $\tilde{A}_n$ is determined by

$$f_{ni}(y|\tilde{A}_n) = f_i(y|\widetilde{NPV})\gamma^{-1}(n, f_i(y|\tilde{r})) \quad (30)$$

and *TFN(y)* for fuzzy *EUAV* is

$$\tilde{A}_n(y) = (\frac{NPV^{l(y)}}{\gamma(n, r^{l(y)})}, \frac{NPV^{r(y)}}{\gamma(n, r^{r(y)})}) \quad (31)$$

Example 3. Assume that $\widetilde{NPV} = (-\$3{,}525.57, -\$24.47, +\$3{,}786.34)$ and $\tilde{r} = (3\%, 5\%, 7\%)$. Calculate the fuzzy *EUAV*.

$$f_{6,1}(y|\tilde{A}_6) = (3{,}501.1y - 3{,}525.57)[\frac{(1.03+0.02y)^6(0.02y+0.03)}{(1.03+0.02y)^6 - 1}]$$

$$f_{6,2}(y|\tilde{A}_6) = (-3,,810.81y + 3{,}786.34)[\frac{(1.07-0.02y)^6(0.07-0.02y)}{(1.07-0.02y)^6 - 1}]$$

For y=0, $f_{6,1}(y|\tilde{A}_6) = -\$650.96$

For y=1, $f_{6,1}(y|\tilde{A}_6) = f_{6,2}(y|\tilde{A}_6) = -\$4.82$

For y=0, $f_{6,2}(y|\tilde{A}_6) = +\$795.13$

7 Fuzzy Payback Period (FPP) Method

The payback period method involves the determination of the length of time required to recover the initial cost of investment based on a zero interest rate ignoring the time value of money or a certain interest rate recognizing the time value of money. Let C_{j0} denote the initial cost of investment alternative *j*, and R_{jt} denote the net revenue received from investment *j* during period *t*. Assuming no other negative net cash flows occur, the smallest value of m_j ignoring the time value of money such that

$$\sum_{t=1}^{m_j} R_{jt} \geq C_{j0} \tag{32}$$

or the smallest value of m_j recognizing the time value of money such that

$$\sum_{t=1}^{m_j} R_{jt}(1+r)^{-t} \geq C_{j0} \tag{33}$$

defines the payback period for the investment *j*. The investment alternative having the smallest payback period is the preferred alternative. In the case of fuzziness, the smallest value of m_j ignoring the time value of money such that

$$(\sum_{t=1}^{m_j} r_{1jt}, \sum_{t=1}^{m_j} r_{2jt}, \sum_{t=1}^{m_j} r_{3jt}) \geq (C_{1j0}, C_{2jo}, C_{3j0}) \tag{34}$$

and the smallest value of m_j recognizing the time value of money such that

$$\begin{aligned}&(\sum_{t=1}^{m_j} (R_{jt}^{l(y)}/(1+r^{r(y)})^t), \sum_{t=1}^{m_j} (R_{jt}^{r(y)}/(1+r^{l(y)})^t) \geq \\ &((C_{2j0} - C_{1j0})y + C_{1j0}, (C_{2j0} - C_{3j0})y + C_{3j0})\end{aligned} \tag{35}$$

defines the payback period for investment *j*, where r_{kjt} : the *k*th parameter of a triangular fuzzy R_{jt}; C_{kj0} : the *k*th parameter of a triangular fuzzy C_{j0}; $R_{jt}^{l(y)}$: the left representation of a triangular fuzzy R_{jt}; $R_{jt}^{r(y)}$: the right representation of a triangular fuzzy R_{jt}. If it is assumed that the discount rate changes from one period to another, $(1+r^{r(y)})^t$ and $(1+r^{l(y)})^t$ will be $\prod_{t'=1}^{t}(1+r_{t'}^{r(y)})$ and $\prod_{t'=1}^{t}(1+r_{t'}^{l(y)})$ respectively.

It is now necessary to use a ranking method to rank the triangular fuzzy numbers such as Chiu and Park's [3], Chang's [6] method , Dubois and Prade's [7] method, Jain's [8] method, Kaufmann and Gupta's [9] method, Yager's [10] method. These methods may give different ranking results and most methods are tedious in graphic manipulation requiring complex mathematical calculation. In the following, two of the methods which does not require graphical representations are given. Chiu and Park's (1994) weighted method for ranking TFNs with parameters (*a, b, c*) is formulated as

$$\left((a+b+c)/3\right)+wb \tag{36}$$

where *w* is a value determined by the nature and the magnitude of the most promising value. The preference of projects is determined by the magnitude of this sum.

Kaufmann and Gupta (1988) suggest three criteria for ranking TFNs with parameters (a,b,c). The dominance sequence is determined according to priority of:

1. Comparing the ordinary number (a+2b+c)/4
2. Comparing the mode, (the corresponding most promise value), b, of each TFN.
3. Comparing the range, c-a, of each TFN.

The preference of projects is determined by the amount of their ordinary numbers. The project with the larger ordinary number is preferred. If the ordinary numbers are equal, the project with the larger corresponding most promising value is preferred. If projects have the same ordinary number and most promising value, the project with the larger range is preferred.

Example 4. Assume that there are two alternative machines that are under consideration to replace an aging production machine. The associated cash flows are given in the following table. Determine the best alternative by using the payback period method recognizing the time value of money. The fuzzy interest rate is (12%, %15, %18) annually.

Table 1. Fuzzy Cash Flow (x$1,000)

End of year	0	1	2	3	4
Alt. A	(-7, -5, -3)	(2, 3, 4)	(4, 4.5, 5)	(1, 1.5, 2)	(3.5, 4, 4.5)
Alt. B	(-12, -10, -8)	(3, 4, 5)	(4.5, 5, 5.5)	(3, 3.5, 4)	(4, 4.5, 5)

If Chiu and Park 's [3] method (*CP*) is used for ranking *TFN*s, it is obtained (w=0.3):

For Alternative *A*,

$$CP_0 = (\frac{C_{1jt} + C_{2jt} + C_{3jt}}{3}) + wC_{2jt} = -6{,}500$$

$$TFN_1 = (\frac{1{,}000y + 2{,}000}{1.18 - 0.03y}, \frac{-1{,}000y + 4{,}000}{1.12 + 0.03y}) = (1695, 2608.7, 3571.4)$$

$$CP_1 = 3{,}407.6$$

$$TFN_2 = (\frac{500y + 4{,}000}{(1.18 - 0.03y)^2}, \frac{-500y + 5{,}000}{(1.12 + 0.03y)^2}) = (2872.7, 3402..6, 3986)$$

$$CP_2 = 4{,}441.2$$

$$\sum_{i=1}^{2} CP_i \succ CP_0 \rightarrow PP_A = 1.656 years .$$

For Alternative *B*,

$$CP_0 = -13{,}000$$

$$TFN_1 = (2542.4, 3478.3, 4464.3)$$

$$CP_1 = 4538.5$$

$$TFN_2 = (3231.8, 3780.7, 4384.6)$$

$$CP_2 = 4{,}993.2$$

$$TFN_3 = (1825.9, 2301.3, 2847.1)$$

$$CP_3 = 3{,}015.2$$

$$TFN_4 = (2063.2, 2572.9, 3177.6)$$

$$CP_4 = 3{,}376.4$$

$$\sum_{i=1}^{4} CP_i \succ CP_0 \rightarrow PP_B = 3.3 years.$$

Alternative *A* is the preferred one.

8 Fuzzy Internal Rate of Return (IRR) Method

The *IRR* method is referred to in the economic analysis literature as the discounted cash flow rate of return, internal rate of return, and the true rate of reurn. The internal rate of return on an investment is defined as the rate of interest earned on the unrecovered balance of an investment. Letting r^* denote the rate of return, the equation for obtaining r^* i

$$\sum_{t=1}^{n} P_t (1+r^*)^{-t} - FC = 0 \tag{37}$$

where P_t is the net cash flow at the end of period *t*.

Assume the cash flow $\widetilde{F} = \widetilde{F}_0, \widetilde{F}_1, ..., \widetilde{F}_N$ is fuzzy. $\widetilde{F}_n$ is a negative fuzzy number and the other $\widetilde{F}_i$ may be positive or negative fuzzy numbers. The fuzzy $IRR(\widetilde{F}, n)$ is a fuzzy interest rate $\widetilde{r}$ that makes the present value of all future cash amounts equal to the initial cash outlay. Therefore, the fuzzy number $\widetilde{r}$ satisfies

$$\sum_{i=1}^{n} PV_{k(i)}(\widetilde{F}_i, i) = -\widetilde{F}_0 \tag{38}$$

where $\sum$ is fuzzy addition, *k(i)*=1 if $\widetilde{F}_i$ is negative and *k(i)*=2 if $\widetilde{F}_i$ is postive.

Buckley [2] shows that such simple fuzzy cash flows may not have a fuzzy *IRR* and concludes that the *IRR* technique does not extend to fuzzy cash flows. Ward [4] considers Eq. (37) and explains that such a procedure can not be applied for the fuzzy case because the right hand side of Eq. (37) is fuzzy, 0 is crisp, and an equality is impossible.

9 An Expansion to Geometric and Trigonometric Cash Flows

When the value of a given cash flow differs from the value of the previous cash flow by a constant percentage, *j%*, then the series is referred to as a *geometric series*. If the value of a given cash flow differs from the value of the previous cash flow by a sinusoidal wave or a cosinusoidal wave, then the series is referred to as a *trigonometric series*

9.1 Geometric Series - Fuzzy Cash Flows in Discrete Compounding

The present value of a crisp geometric series is given by

$$P = \sum_{n=1}^{N} F_1 (1+g)^{n-1} (1+i)^{-n} = \frac{F_1}{1+g} \sum_{n=1}^{N} (\frac{1+g}{1+i})^n \tag{39}$$

where F_1 is the first cash at the end of the first year. When this sum is made, the following present value equation is obtained:

$$P = \begin{cases} F_1\left[\dfrac{1-(1+g)^N(1+i)^{-N}}{i-g}\right], i \neq g \\ \dfrac{NF}{1+i}, i = g \end{cases} \tag{40}$$

and the future value

$$F = \begin{cases} F_1\left[\dfrac{(1+i)^N-(1+g)^N}{i-g}\right], i \neq g \\ NF_1(1+i)^{N-1}, i = g \end{cases} \tag{41}$$

In the case of fuzziness, the parameters used in Eq.(4) will be assumed to be fuzzy numbers, except project life. Let $\gamma(i,g,N) = [\frac{1-(1+g)^N(1+i)^{-N}}{i-g}], i \neq g$. As it is in Figure 1 and Figure 2, when k=1, the left side representation will be depicted and when k=2, the right side representation will be depicted. In this case, for $i \neq g$

$$f_{Nk}(y|\widetilde{P}_N) = f_k(y|\widetilde{F}_1)\gamma(f_{3-k}(y|\widetilde{i}), f_{3-k}(y|\widetilde{g}), N) \tag{42}$$

In Eq. (42), the least possible value is calculated for k = 1 and y = 0; the largest possible value is calculated for k = 2 and y = 0; the most promising value is calculated for k = 1 or k = 2 and y = 1.

To calculate the future value of a fuzzy geometric cash flow, let $\zeta(i,g,N) = [\frac{(1+i)^N-(1+g)^N}{i-g}], i \neq g$. Then the fuzzy future value is

$$f_{Nk}(y|\widetilde{F}_N) = f_k(y|\widetilde{F}_1)\zeta(f_k(y|\widetilde{i}), f_k(y|\widetilde{g}), N) \tag{43}$$

In Eq. (43), the least possible value is calculated for k = 1 and y = 0; the largest possible value is calculated for k = 2 and y = 0; the most promising value is calculated for k = 1 or k = 2 and y = 1. This is also valid for the formulas developed at the rest of the paper.

The fuzzy uniform equivalent annual value can be calculated by using Eq. (44):

$$f_{Nk}(y|\widetilde{A}) = f_k(y|\widetilde{P}_N)\vartheta(f_k(y|\widetilde{i}), N) \tag{44}$$

where $\vartheta(i,N) = [\frac{(1+i)^N i}{(1+i)^N-1}]$ and $f(y|\widetilde{P}_N)$ is the fuzzy present value of the fuzzy geometric cash flows.

9.2 Geometric Series - Fuzzy Cash Flows in Continuous Compounding

In the case of crisp sets, the present and future values of discrete payments are given by Eq. (45) and Eq. (46) respectively:

$$P = \begin{cases} F_1\left[\dfrac{1-e^{(g-r)N}}{e^r - e^g}\right], r \neq g \\ \dfrac{NF_1}{e^r}, g = e^r - 1 \end{cases} \tag{45}$$

$$F = \begin{cases} F_1\left[\dfrac{e^{rN} - e^{gN}}{e^r - e^g}\right], r \neq g \\ NF_1 e^{r(N-1)}, g = e^r - 1 \end{cases} \tag{46}$$

and the present and future values of continuous payments are given by Eq. (47) and Eq. (48) respectively:

$$P = \begin{cases} F_1\left[\dfrac{1-e^{(g-r)N}}{r-g}\right], r \neq g \\ \dfrac{NF_1}{1+r}, r = g \end{cases} \tag{47}$$

$$F = \begin{cases} F_1\left[\dfrac{e^{rN} - e^{gN}}{r-g}\right], r \neq g \\ \dfrac{NF_1 e^{rN}}{1+r}, r = g \end{cases} \tag{48}$$

The fuzzy present and future values of the fuzzy geometric discrete cash flows in continuous compounding can be given as in Eq. (49) and Eq. (50) respectively:

$$f_{Nk}(y|\tilde{P}_N) = f_k(y|\tilde{F}_1)\beta(f_{3-k}(y|\tilde{r}), f_{3-k}(y|\tilde{g}), N) \tag{49}$$

$$f_{Nk}(y|\tilde{F}) = f_k(y|\tilde{F}_1)\tau(f_k(y|\tilde{r}), f_k(y|\tilde{g}), N) \tag{50}$$

where $\beta(r,g,N) = F_1[\frac{1-e^{(g-r)N}}{e^r - e^g}], r \neq g$ for present value and $\tau(r,g,N) = F_1[\frac{e^{rN} - e^{gN}}{e^r - e^g}], r \neq g$ for future value.

The fuzzy present and future values of the fuzzy geometric continuous cash flows in continuous compounding can be given as in Eq. (51) and Eq. (52) respectively:

$$f_{Nk}(y|\widetilde{P}_N) = f_k(y|\widetilde{F}_1)\eta(f_{3-k}(y|\widetilde{r}), f_{3-k}(y|\widetilde{g}), N) \quad (51)$$

$$f_{Nk}(y|\widetilde{F}_N) = f_k(y|\widetilde{F}_1)\upsilon(f_k(y|\widetilde{r}), f_k(y|\widetilde{g}), N) \quad (52)$$

where $\eta(r,g,N) = F_1[\frac{1-e^{(g-r)N}}{r-g}]$, $\upsilon(r,g,N) = F_1[\frac{e^{rN} - e^{gN}}{r-g}], r \neq g$

9.3 Trigonometric Series - Fuzzy Continuous Cash Flows

In Figure 4, the function of the semi-sinusodial wave cash flows is depicted. This function, *h(t),* is given by Eq. (53) in the crisp case:

$$h(t) = \begin{cases} D\sin \pi t, \; 0 \leq t \leq 1 \\ 0, \; otherwise \end{cases} \quad (53)$$

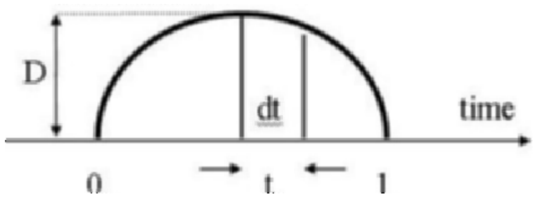

Fig. 4. Semi-Sinusoidal Wave Cash Flow Function

The future value of a semi-sinusoidal cash flow for T=1 and g is defined by Eq. (54) :

$$V(g,1) = D\int_0^1 e^{r(1-t)} \sin \pi.t.tdt = D[\frac{\pi(2+g)}{r^2+\pi^2}] \quad (54)$$

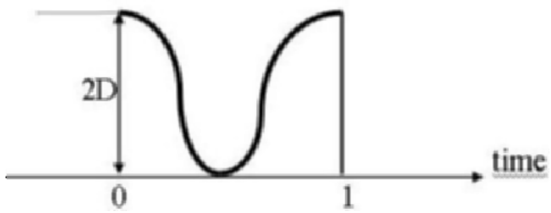

Fig. 5. Cosinusoidal Wave Cash Flow Function

Figure 5 shows the function of a cosinusoidal wave cash flow. This function, *h(t),* is given by Eq. (55):

$$h(t) = \begin{cases} D(Cos2\pi t + 1), \ 0 \le t \le 1 \\ 0, \ otherwise \end{cases} \tag{55}$$

The future value of a cosinusoidal cash flow for T=1 and g is defined as

$$V(g,1) = D\int_0^1 e^{r(t-1)}(\cos 2\pi .t + 1)dt = D[\frac{gr}{r^2 + 4\pi^2} + \frac{g}{r}] \tag{56}$$

Let the parameters in Eq. (54), r and g, be fuzzy numbers. The future value of the semi-sinusoidal cash flows as in Figure 6 is given by

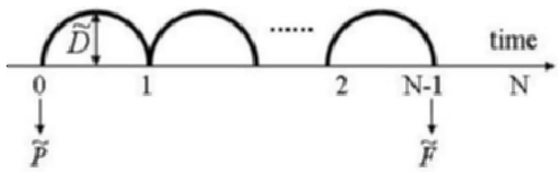

Fig. 6. Fuzzy Sinusoidal Cash Flow Diagram

$$f_{Nk}(y|\widetilde{F}_N) = f_k(y|\widetilde{D})\phi(f_{3-k}(y|\widetilde{r}), f_k(y|\widetilde{g}))\varphi(f_k(y|\widetilde{r}), N) \tag{57}$$

where $\phi(r,g) = \pi(2+g)/(r^2+\pi^2)$, $\varphi(r,N) = (e^{rN}-1)/(e^r-1)$.

The present value of the semi-sinusoidal cash flows is given by Eq. (58):

$$f_{Nk}(y|\widetilde{P}_N) = f_k(y|\widetilde{D})\phi(f_{3-k}(y|\widetilde{r}), f_k(y|\widetilde{g}))\psi(f_{3-k}(y|\widetilde{r}), N) \tag{58}$$

where $\psi(r,N) = (e^{rN}-1)/((e^r-1)e^{rN})$.

The present and future values of the fuzzy cosinusoidal cash flows can be given by Eq. (59) and Eq. (60) respectively:

$$f_{Nk}(y|\tilde{P}_N) = f_k(y|\tilde{D})\xi(f_{3-k}(y|\tilde{r}), f_k(y|\tilde{g}))\psi(f_{3-k}(y|\tilde{r}), N) \quad (59)$$

where $\xi(r,g) = [\frac{gr}{r^2 + 4\pi^2} + \frac{g}{r}]$ and the fuzzy future value is

$$f_{Nk}(y|\tilde{F}_N) = f_k(y|\tilde{D})\xi(f_{3-k}(y|\tilde{r}), f_k(y|\tilde{g}))\varphi(f_k(y|\tilde{r}), N) \quad (60)$$

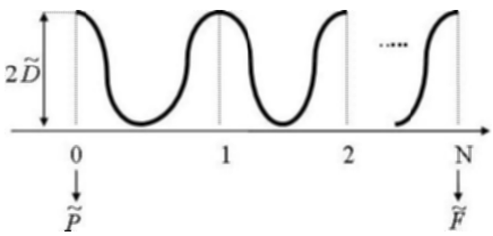

Fig. 7. Fuzzy Cosinusoidal Wave Cash Flow Diagram

Numerical Example-1

The continuous profit function of a firm producing ice cream during a year is similar to semi-sinusoidal wave cash flows whose g is around 4%. The maximum level in $ of the ice-cream sales is between the end of June and the beginning of July. The profit amount obtained on this point is around $120,000. The firm manager uses a minimum attractive rate of return of around 10%, compounded continuously and he wants to know the present worth of the 10-year profit and the possibility of having a present worth of $1,500,000.

'Around $120,000' can be represented by a TFN, ($100,000; $120,000;$130,000). 'Around 10%' can be represented by a TFN, (9%;10%;12%). 'Around 4%' can be represented by a TFN, (3%;4%;6%).

$f_1(y|\tilde{r}) = 0.09 + 0.01y \qquad f_1(y|\tilde{D}) = 100{,}000 + 20{,}000y$

$f_2(y|\tilde{r}) = 0.12 - 0.02y \qquad f_2(y|\tilde{D}) = 130{,}000 - 10{,}000y$

$$f_{10,1}(y|\tilde{P}_{10}) = f_1(y|\tilde{D})\Phi(f_2(y|\tilde{r}), f_1(y|\tilde{g}))\Psi(f_2(y|\tilde{r}), 10)$$

$$f_{10,2}(y|\tilde{P}_{10}) = f_2(y|\tilde{D})\Phi(f_1(y|\tilde{r}), f_2(y|\tilde{g}))\Psi(f_1(y|\tilde{r}), 10)$$

$f_1(y|\tilde{g}) = 0.03 + 0.01y \qquad f_2(y|\tilde{g}) = 0.06 - 0.02y$

$$f_{10,1}\left(y\middle|\tilde{P}_{10}\right)=\left(100{,}000+20{,}000y\right)\times$$
$$\left[\frac{\pi\left(2.03+0.01y\right)}{\left(0.12-0.02y\right)^2+\pi^2}\right]\left[\frac{e^{(0.12-0.02y)10}-1}{e^{(0.12-0.02y)}-1}\right]\frac{1}{e^{(0.12-0.02y)10}}$$

$$f_{10,2}\left(y\middle|\tilde{P}_{10}\right)=\left(130{,}000-10{,}000y\right)\times$$
$$\left[\frac{\pi\left(2.06-0.02y\right)}{\left(0.09+0.01y\right)^2+\pi^2}\right]\left[\frac{e^{(0.09+0.01y)10}-1}{e^{(0.09+0.01y)}-1}\right]\frac{1}{e^{(0.09+0.01y)10}}$$

For y = 0, the smallest possible value is $f_{10,1}\left(y\middle|\tilde{P}_{10}\right)=\$353{,}647.1$

For y = 1, the most possible value is $f_{10,1}\left(y\middle|\tilde{P}_{10}\right)=f_{10,2}\left(y\middle|\tilde{P}_{10}\right)=\$467{,}870.9$

For y = 0, the largest possible value is $f_{10,2}\left(y\middle|\tilde{P}_{10}\right)=\$536{,}712.8$

It seems to be impossible to have a present worth of $1,500,000.

Numerical Example-2

The continuous cash flows of a firm is similar to cosinusoidal cash flows. The maximum level of the cash flows during a year is around $780,000. The fuzzy nominal cost of capital is around 8% per year. The fuzzy geometric growth rate of the cash flows is around 4% per year. Let us compute the future worth of a 10 year working period.

Let us define

$\tilde{D}=\left(\$300{,}000;\$390{,}000;\$420{,}000\right)$

$f_2\left(y\middle|\tilde{D}\right)=420{,}000-30{,}000y$

$f_1\left(y\middle|\tilde{D}\right)=350{,}000+40{,}000y$

$\tilde{r}=\left(6\%,8\%,10\%\right)$

$f_1\left(y\middle|\tilde{r}\right)=0.06+0.02y$

$f_2\left(y\middle|\tilde{r}\right)=0.10-0.02y$

$\tilde{g}=\left(3\%,4\%,5\%\right)$

$f_1\left(y\middle|\tilde{g}\right)=0.03+0.01y$

$f_2\left(y\middle|\tilde{g}\right)=0.05-0.01y$

$$f_{10,1}\left(y\middle|\tilde{F}_{10}\right)=(350{,}000+40{,}000y)\times$$
$$\left[\frac{(0.03+0.01y)(0.10-0.02y)}{(0.19-0.02y)^2+4\pi^2}+\frac{0.03+0.01y}{0.10-0.02y}\right]\left[\frac{e^{(0.06+0.02y)10}-1}{e^{0.06+0.02y}-1}\right]$$
$$f_{10,2}\left(y\middle|\tilde{F}_{10}\right)=(420{,}000-30{,}000y)\times$$
$$\left[\frac{(0.05-0.01y)(0.06+0.02y)}{(0.06+0.02y)^2+4\pi^2}+\frac{0.05-0.01y}{0.06+0.02y}\right]\left[\frac{e^{(0.10-0.02y)10}-1}{e^{0.10-0.02y}-1}\right]$$

For $y = 0$, the smallest possible value is $f_{10,1}\left(y\middle|\tilde{F}_{10}\right)=\$1{,}396{,}331.5$

For $y = 1$, the most possible value is $f_{10,1}\left(y\middle|\tilde{F}_{10}\right)=f_{10,2}\left(y\middle|\tilde{F}_{10}\right)=\$2{,}869{,}823.5$

For $y = 0$, the largest possible value is $f_{10,2}\left(y\middle|\tilde{F}_{10}\right)=\$5{,}718{,}818.9$

10 Investment Analysis under Fuzziness Using Possibilities of Probabilities

A typical investment may involve several factors such as the initial cost, expected life of the investment, the market share, the operating cost and so on. Values for factors are projected at the time the investment project is first proposed and are subject to deviations from their expected values. Such variations in the outcomes of future events, often termed risk, have been of primary concern to most decision makers in evaluating investment alternatives.

The term risk analysis has different interpretations among various agencies and units. However, there is a growing acceptance that risk analysis involves the development of the probability distribution for the measure of effectiveness. Furthermore, the risk associated with an investment alternative is either given as the probability of an unfavorable value for the measure of effectiveness or measured by the variance of the measure of effectiveness.

Probability of a Fuzzy Event

The formula for calculating the probability of a fuzzy event A is a generalization of the probability theory:

$$P(A)=\begin{cases}\int \mu_A(x)P_X(x)dx, & \text{if X is continuous}\\ \sum_i \mu_A(x_i)P_X(x_i), & \text{if X is discrete}\end{cases} \tag{61}$$

where P_x denotes the probability distribution function of X.

Fuzzy verses Stochastic Investment Analyses

Typical parameters for which conditions of risk can reasonably be expected to exist include the initial investment, yearly operating and maintenance expenses, salvage values, the life of an investment, the planning horizon, and the minimum attractive rate of return. The parameters can be statistically independent, correlated with time and/or correlated with each other.

In order to determine analytically the probability distribution for the measure of effectiveness, a number of simplifying assumptions are normally made. The simplest situation is one involving a known number of random and statistically independent cash flows. As an example, suppose the random variable A_j denotes the net cash flow occurring at the end of period j, $j=0,1,..,N$. Hence, the present worth (PW) is given by

$$PW=\sum_{j=0}^{N} A_j\left(1+i\right)^{-j} \tag{62}$$

Since the expected value, $E\left[.\right]$, of a sum of random variables equals the sum of the expected values of the random variables, then the expected present worth is given by:

$$E\left[PW\right]=\sum_{j=0}^{N} E\left[A_j\right]\left(1+i\right)^{-j} \tag{63}$$

Furthermore, since the A_j's are statistically independent, then the variance, V(.), of present worth is given by:

$$V\left(PW\right)=\sum_{j=0}^{N} V\left(A_j\right)\left(1+i\right)^{-2j} \tag{64}$$

The central limit theorem, from probability theory, establishes that the sum of independently distributed random variables tends to be normally distributed as the number of terms in the summation increases. Hence, as N increases, PW tends to be normally distributed with a mean value of E[PW] and a variance of V(PW).

An illustrative example:

A new cost reduction proposal is expected to have annual expenses of $20,000 with a standard deviation of $3,000, and it will likely save $24,000 per year with a standard deviation of $4,000. The proposed operation will be in effect for 3 years, and a rate of return of 20 percent before taxes is required. Determine the probability that implementation of the proposal will actually result in an overall loss and the probability that the PW of the net savings will exceed $10,000.

The expected value of the present worth of savings and cost is

$$E[PW] = (\$24{,}000 - \$20{,}000)(P/A, 20\%, 3) = \$8{,}426$$

The variance is calculated from the relation

$$\sigma^2_{\text{savings}-\text{costs}} = \sigma^2_{\text{savings}} + \sigma^2_{\text{costs}} \tag{65}$$

to obtain

$$\begin{aligned} \mathrm{Var}[PW] &= (\$3{,}000)^2 (P/F,20,2) + \\ &(\$3{,}000)^2 (P/F,20,4) + (\$3{,}000)^2 (P/F,20,6) \\ &+ (\$4{,}000)^2 (P/F,20,2) + (\$4{,}000)^2 (P/F,20,2) \\ &+ (\$4{,}000)^2 (P/F,20,2) = 37{,}790{,}000 \end{aligned}$$

from which

$$\sigma_{PW} = \sqrt{\mathrm{Var}[PW]} = \$6{,}147$$

Assuming that the PW is normally distributed, we find that

$$P(\text{loss}) = P\left(Z < \frac{0 - \$8{,}426}{\$6{,}147}\right)$$

$$= P(Z < 1.37) = 0.0853$$

and

$$P(PW > \$10{,}000) = P\left(Z > \frac{\$10{,}000 - \$8{,}426}{\$6{,}147}\right)$$

$$= P(Z > 0.256) = 0.40$$

Under fuzziness, the fuzzy expected net present value in a triangular fuzzy number form is calculated as in the following way. The expected fuzzy annual expenses are around $20,000 with a fuzzy standard deviation of around $3,000, and it will possibly save around $24,000 per year with a fuzzy standard deviation of around $4,000. The proposed operation will be in effect for 3 years, and a rate of return of around 20 percent before taxes is required. Determine the possibility that implementation of the proposal will actually result in an overall loss and the possibility that the PW of the net savings will exceed around $10,000. The fuzzy annual expenses are

$E_e\left[\tilde{X}\right]=(\$19,000;\$20,000;\$24,000)$

$\sigma_e\left(\tilde{X}\right)=(\$2,500;\$3,000;\$3,500)$

The fuzzy annual savings are

$E_s\left[\tilde{X}\right]=(\$23,000;\$24,000;\$25,000)$

$\sigma_s\left(\tilde{X}\right)=(\$3,500;\$4,000;\$4,500)$

The required fuzzy rate of return is

$\tilde{i}_{annual}=(18\%,20\%,22\%)$.

The fuzzy variance of the cash flows is calculated using

$$\tilde{\sigma}_{s-e}^2=\tilde{\sigma}_s^2+\tilde{\sigma}_e^2 \tag{66}$$

and it is equal to $\tilde{\sigma}_{s-e}^2=(18,500,000;25,000,000;32,500,000)$ with the left side representation $f_l\left[y|\tilde{\sigma}_{s-e}\right]=\sqrt{(18,500,000+6,500,000y)}$ and the right side representation $f_r\left[y|\tilde{\sigma}_{s-e}\right]=\sqrt{(32,500,000-7,500,000y)}$ where y shows the degree of membership.

The fuzzy present worth is calculated by using the Formula

$$P\widetilde{W}=\left[E_s\left[\widetilde{X}\right]-E_e\left[\widetilde{X}\right]\right]\left(P/A,\tilde{i}_{annual},n\right) \tag{67}$$

The left side representation of the difference between savings and expenses is $f_l\left(y|\tilde{X}_{s-e}\right)=(-1,000+5,000y)$ and the right side representation is $f_r\left(y|\tilde{X}_{s-e}\right)=(6,000-2,000y)$ where y shows the degree of membership. Similarly, the left side representation of the fuzzy interest rate is $f_l\left(y|\tilde{i}_{annual}\right)=(0.18+0.02y)$ and the right side representation is $f_r\left(y|\tilde{i}_{annual}\right)=(022-0.02y)$. Using these representations we find the left side representation of the $P\widetilde{W}$ as

$$f_l\left(y|P\widetilde{W}\right)=(-1,000+5,000y)\times\left[\frac{(1.22-0.02y)^3-1}{(1.22-0.02y)^3(0.22-0.02y)}\right]$$

and the right side representation as

$$f_r\left(y|P\widetilde{W}\right)=(6,000-2,000y)\times\left[\frac{(1.18+0.02y)^3-1}{(1.18+0.02y)^3(0.18+0.02y)}\right]$$

When these two functions are combined on x-y axes, we obtain the graph of fuzzy PW.

Now, to calculate the possibility of loss, we can use the following equalities

$$P_1(loss) = P_1\left(\tilde{Z} \prec \frac{\tilde{0} - f_r\left(y \middle| P\tilde{W}\right)}{f_r\left(y \middle| \tilde{\sigma}_{s-e}\right)} \right) \tag{68}$$

$$P_r(loss) = P_r\left(\tilde{Z} \prec \frac{\tilde{0} - f_l\left(y \middle| P\tilde{W}\right)}{f_l\left(y \middle| \tilde{\sigma}_{s-e}\right)} \right) \tag{69}$$

The graph of these two functions is illustrated in Figure 8.

To calculate the possibility that the PW of the net savings will exceed around $10,000, the following equations can be used.

$$P_1\left(P\tilde{W} > \$10{,}000\right) = P_1\left(\tilde{Z} \prec \frac{10{,}\tilde{0}00 - f_r\left(y \middle| P\tilde{W}\right)}{f_r\left(y \middle| \tilde{\sigma}_{s-e}\right)} \right)$$

and

$$P_r\left(P\tilde{W} > \$10{,}000\right) = P_r\left(\tilde{Z} \prec \frac{10{,}\tilde{0}00 - f_l\left(y \middle| P\tilde{W}\right)}{f_l\left(y \middle| \tilde{\sigma}_{s-e}\right)} \right)$$

where $\$10{,}00\tilde{0}$ is accepted as $(\$9{,}000; \$10{,}000; \$11{,}000)$ with the left side representation $(\$9{,}000 + \$1{,}000y)$ and the right side representation $(\$11{,}000 - \$1{,}000y)$. The graph of these two functions is illustrated in Figure 8.

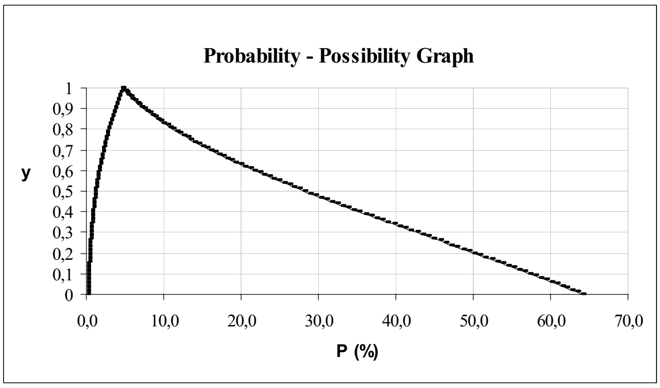

Fig. 8. The possibilities of probabilities P(Loss)

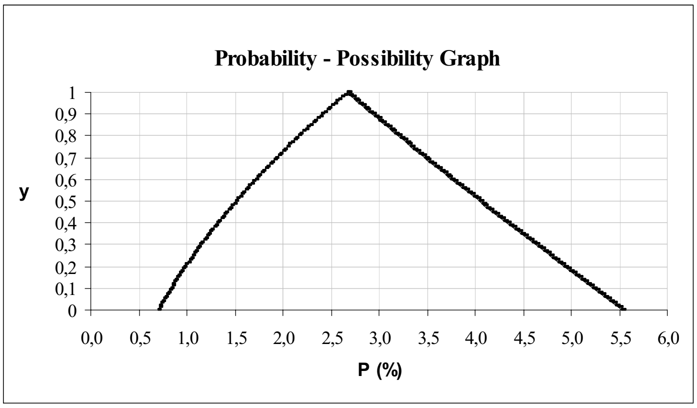

Fig. 9. The possibilities of probabilities P(Profit>10,000)

11 Conclusion

In this chapter, capital budgeting techniques in the case of fuzziness and discrete compounding have been studied. The cash flow profile of some investments projects may be geometric or trigonometric. For these kind of projects, the fuzzy present, future, and annual value formulas have been also developed under discrete and continuous compounding in this chapter. Fuzzy set theory is a powerful tool in the area of management when sufficient objective data has not been obtained. Appropriate fuzzy numbers can capture the vagueness of knowledge. The other financial subjects such as replacement analysis, income tax considerations, continuous compounding in the case of fuzziness can be also applied [11], [12]. Comparing projects with unequal lives has not been considered in this paper. This will also be a new area for further study.

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