7

Quantitative Semi-algebraic Geometry

In this chapter, we study various quantitative bounds on the number of connected components and Betti numbers of algebraic and semi-algebraic sets. The key method for this study is the critical point method, i.e. the consideration of the critical points of a well chosen projection. The critical point method also plays a key role for improving the complexity of algorithms in the last chapters of the book.

In Section 7.1, we explain a few basic results of Morse theory and use them to study the topology of a non-singular algebraic hypersurface in terms of the number of critical points of a well chosen projection. Bounding the number of these critical points by Bézout's theorem provides a bound on the sum of the Betti numbers of a non-singular bounded algebraic hypersurface in Section 7.2. Then we prove a similar bound on the sum of the Betti numbers of a general algebraic set.

In Section 7.3, we prove a bound on the sum of the i-th Betti numbers over all realizable sign conditions of a finite set of polynomials. In particular, the bound on the zero-th Betti numbers gives us a bound on the number of realizable sign conditions of a finite set of polynomials. We also explain why these bounds are reasonably tight.

In Section 7.4, we prove bounds on Betti numbers of closed semi-algebraic sets. In Section 7.5 we prove that any semi-algebraic set is semi-algebraically homotopic to a closed one and prove bounds on Betti numbers of general semi-algebraic sets.

7.1 Morse Theory

We first define the kind of hypersurfaces we are going to consider.

A **non-singular algebraic hypersurface** is the zero set $\mathrm{Zer}(Q, \mathrm{R}^k)$ of a polynomial $Q \in \mathrm{R}[X_1, \ldots, X_k]$ such that the **gradient** of Q, i.e. the vector

$$\mathrm{Grad}(Q)(p) = \left(\frac{\partial Q}{\partial X_1}(p), \ldots, \frac{\partial Q}{\partial X_k}(p) \right) \text{is never 0 for } p \in \mathrm{Zer}(Q, \mathrm{R}^k).$$

Exercise 7.1. Prove that a non-singular algebraic hypersurface is an $\mathcal{S}^\infty$ submanifold of dimension $k-1$. (Hint. Use the Semi-algebraic implicit function theorem (Theorem 3.25).)

Exercise 7.2. Let $\mathrm{Zer}(Q,\mathrm{R}^k)$ be a non-singular algebraic hypersurface. Prove that the gradient vector of Q at a point $p\in\mathrm{Zer}(Q,\mathrm{R}^k)$ is orthogonal to the tangent space $T_p(\mathrm{Zer}(Q,\mathrm{R}^k))$ to $\mathrm{Zer}(Q,\mathrm{R}^k)$ at p.

We denote by π the projection from R^k to the first coordinate sending $(x_1,\ldots,x_k)$ to x_1.

Notation 7.1. [Fiber] For $S\subset\mathrm{R}^k$, $X\subset\mathrm{R}$, let S_X denote $S\cap\pi^{-1}(X)$. We also use the abbreviations S_x, $S_{<x}$, and $S_{\le x}$ for $S_{\{x\}}$, $S_{(-\infty,x)}$, and $S_{(-\infty,x]}$. □

Let $\mathrm{Zer}(Q, \mathrm{R}^k)$ be a non-singular algebraic hypersurface and $p\in\mathrm{Zer}(Q,\mathrm{R}^k)$. Then, the derivative $\mathrm{d}\pi(p)$ of π on $\mathrm{Zer}(Q, \mathrm{R}^k)$ is a linear map from $T_p(\mathrm{Zer}(Q,\mathrm{R}^k))$ to R. Clearly, p is a critical point of π on $\mathrm{Zer}(Q,\mathrm{R}^k)$ if and only if

$$\frac{\partial Q}{\partial X_i}(p)=0, 2\le i\le k$$

(see Definition 5.55). In other words, p is a critical point of π on $\mathrm{Zer}(Q,\mathrm{R}^k)$ if and only if the gradient of Q is parallel to the X_1-axis, i.e. $T_p(\mathrm{Zer}(Q,\mathrm{R}^k))$ is orthogonal to the X_1 direction. A critical value of π on $\mathrm{Zer}(Q,\mathrm{R}^k)$ is the projection to the X_1-axis of a critical point of π on $\mathrm{Zer}(Q,\mathrm{R}^k)$.

Lemma 7.2. *Let $\mathrm{Zer}(Q,\mathrm{R}^k)$ be a bounded non-singular algebraic hypersurface. The set of values that are not critical for π is non-empty and open.*

Proof: The set of values that are not critical for π is clearly open, from the definition of a critical value. It is also non-empty by Theorem 5.56 (Sard's theorem) since the set of critical values is a finite subset of R. □

Also, as an immediate consequence of the Semi-algebraic implicit function theorem (Theorem 3.25), we have:

Proposition 7.3. *Let $\mathrm{Zer}(Q,\mathrm{R}^k)$ be a bounded non-singular algebraic hypersurface. If x is not a critical value of π on $\mathrm{Zer}(Q, \mathrm{R}^k)$ and p is a point of $\mathrm{Zer}(Q,\mathrm{R}^k)_x$, then for ϵ small enough $\mathrm{Zer}(Q,\mathrm{R}^k)\cap B(p,\epsilon)_{<x}$ is non-empty and semi-algebraically connected.*

We also have the following proposition.

Proposition 7.4. *Let $\mathrm{Zer}(Q,\mathrm{R}^k)$ be a bounded non-singular algebraic hypersurface. The set of critical points of π on $\mathrm{Zer}(Q, \mathrm{R}^k)$ meets every semi-algebraically connected component of $\mathrm{Zer}(Q,\mathrm{R}^k)$.*

Proof: Let C be a semi-algebraically connected component of $\mathrm{Zer}(Q, \mathrm{R}^k)$. Since C is semi-algebraic, closed, and bounded, its image by π is semi-algebraic, closed, and bounded, using Theorem 3.20. Thus $\pi(C)$ is a finite number of points and closed intervals and has a smallest element v. Using Proposition 7.3, it is clear that any $x \in C$ such that $\pi(x) = v$ is critical. □

We will now state and prove the first basic ingredient of Morse theory. In the remainder of the section, we assume $\mathrm{R} = \mathbb{R}$. We suppose that $\mathrm{Zer}(Q, \mathbb{R}^k)$ is a bounded algebraic non-singular hypersurface and denote by π the projection map sending $(x_1, \ldots, x_k)$ to x_1.

Consider the sets $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq x}$ as x varies from $-\infty$ to ∞. Thinking of X_1 as the horizontal axis, the set $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq x}$ is the part of $\mathrm{Zer}(Q, \mathbb{R}^k)$ to the left of the hyperplane defined by $X_1 = x$, and we study the changes in the homotopy type of this set as we sweep the hyperplane in the rightward direction. Theorem 7.5 states that there is no change in the homotopy type as x varies strictly between two critical values of π.

Theorem 7.5. [Morse lemma A] *Let $[a, b]$ be an interval containing no critical value of π. Then $\mathrm{Zer}(Q, \mathbb{R}^k)_{[a,b]}$ and $\mathrm{Zer}(Q, \mathbb{R}^k)_a \times [a, b]$ are homeomorphic, and $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq a}$ is homotopy equivalent to $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq b}$.*

Theorem 7.5 immediately implies:

Proposition 7.6. *Let $\mathrm{Zer}(Q, \mathbb{R}^k)$ be a non-singular bounded algebraic hypersurface, $[a, b]$ such that π has no critical value in $[a, b]$, and $d \in [a, b]$.*

- *The sets $\mathrm{Zer}(Q, \mathbb{R}^k)_{[a,b]}$ and $\mathrm{Zer}(Q, \mathbb{R}^k)_d$ have the same number of semi-algebraically connected components.*
- *Let S be a semi-algebraically connected component of $\mathrm{Zer}(Q, \mathbb{R}^k)_{[a,b]}$. Then, for every $d \in [a, b]$, S_d is semi-algebraically connected.*

The proof of Theorem 7.5 is based on local existence and uniqueness of solutions to systems of differential equations. Let U be an open subset of $\mathbb{R}^k$. A **vector field** Γ on U is a C^∞ map from an open set U of $\mathbb{R}^k$ to $\mathbb{R}^k$. To a vector field is associated a system of differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}T} = \Gamma_i(x_1, \ldots, x_k), 1 \leq i \leq k.$$

A **flow line** of the vector field Γ is a C^∞ map $\gamma: I \to \mathbb{R}^k$ defined on some interval I and satisfying

$$\frac{\mathrm{d}\gamma}{\mathrm{d}T}(t) = \Gamma(\gamma(t)), t \in I.$$

Theorem 7.7. *Let Γ be a vector field on an open subset V of $\mathbb{R}^k$ such that for every $x \in V$, $\Gamma(x) \neq 0$. For every $y \in V$, there exists a neighborhood U of y and $\epsilon > 0$, such that for every $x \in U$, there exists a unique flow line $\gamma_x: (-\epsilon, \epsilon) \to \mathbb{R}^k$ of Γ satisfying the initial condition $\gamma_x(0) = x$.*

Proof: Since Γ is C^∞, there exists a bounded neighborhood W of y and $L>0$ such that $|\Gamma(x_1)-\Gamma(x_2)|<L|x_1-x_2|$ for all $x_1,x_2\in W$. Let $A=\sup_{x\in W}|\Gamma(x)|$. Also, let $\epsilon>0$ be a small enough number such that the set

$$W'=\{x\in W \mid B_k(x,\epsilon A)\subset W\}$$

contains an open set U containing y.

Let $x\in U$. If $\gamma_x\colon [-\epsilon,\ \epsilon]\to\mathbb{R}^k$, with $\gamma_x(0)=x$, is a solution, then $\gamma_x([-\epsilon,\epsilon])\subset W'$. This is because $|\Gamma(x')|\le A$ for every $x'\in W$, and hence applying the Mean Value Theorem, $|x-\gamma_x(t)|\le|t|A$ for all $t\in[-\epsilon,\epsilon]$. Now, since $x\in U$, it follows that $\gamma_x([-\epsilon,\epsilon])\subset W'$.

We construct the solution $\gamma_x\colon[-\epsilon,\ \epsilon]\to W$ as follows. Let $\gamma_{x,0}(t)=x$ for all t and

$$\gamma_{x,n+1}(t)=x+\int_0^t\Gamma(\gamma_{x,n}(t))dt.$$

Note that $\gamma_{x,n}([-\epsilon,\epsilon])\subset W'$ for every $n\ge 0$. Now,

$$\begin{aligned}|\gamma_{x,n+1}(t)-\gamma_{x,n}(t)| &= \left|\int_0^t(\Gamma(\gamma_{x,n}(t))-\Gamma(\gamma_{x,n-1}(t)))dt\right|\\ &\le \left|\int_0^t|\Gamma(\gamma_{x,n}(t))-\Gamma(\gamma_{x,n-1}(t))|dt\right|\\ &\le \epsilon L|\gamma_{x,n}(t)-\gamma_{x,n-1}(t)|\end{aligned}$$

Choosing ϵ such that $\epsilon<1/L$, we see that for every fixed $t\in[-\epsilon,\epsilon]$, the sequence $\gamma_{x,n}(t)$ is a Cauchy sequence and converges to a limit $\gamma_x(t)$.

Moreover, it is easy to verify that $\gamma_x(t)$ satisfies the equation,

$$\gamma_x(t)=x+\int_0^t\Gamma(\gamma_x(t))dt.$$

Differentiating both sides, we see that $\gamma_x(t)$ is a flow line of the given vector field Γ, and clearly $\gamma_x(0)=x$.

The proof of uniqueness is left as an exercise. □

Given a C^∞ hypersurface $M\subset\mathbb{R}^k$, a C^∞ **vector field on** M, Γ, is a C^∞ map that associates to each $x\in M$ a tangent vector $\Gamma(x)\in T_x(M)$.

An important example of a vector field on a hypersurface is the gradient vector field. Let $\mathrm{Zer}(Q,\mathbb{R}^k)$ be a non-singular algebraic hypersurface and (a',b') such that π has no critical point on $\mathrm{Zer}(Q,\mathbb{R}^k)_{(a',b')}$. The **gradient vector field of** π on $\mathrm{Zer}(Q,\ \mathbb{R}^k)_{(a',b')}$ is the C^∞ vector field on $\mathrm{Zer}(Q,\mathbb{R}^k)_{(a',b')}$ that to every $p\in\mathrm{Zer}(Q,\ \mathbb{R}^k)_{(a',b')}$ associates $\Gamma(p)$ characterized by the following properties

– it belongs to $T_p(\mathrm{Zer}(Q,\mathbb{R}^k))$,

– it belongs to the plane generated by the gradient $\mathrm{Grad}(Q)(p)$, and the unit vector of the X_1-axis,
– its projection on the X_1-axis is the negative of the unit vector.

The flow lines of the gradient vector field correspond to curves on the hypersurface along which the X_1 coordinate decreases maximally. A straightforward computation shows that, for $p \in \mathrm{Zer}(Q, \mathbb{R}^k)$,

$$\Gamma(p) = -\frac{G(p)}{\sum_{2 \le i \le k} \left(\frac{\partial Q}{\partial X_i}(p)\right)^2},$$

where

$$G(p) = \left(\sum_{2 \le i \le k} \left(\frac{\partial Q}{\partial X_i}(p)\right)^2, -\frac{\partial Q}{\partial X_1}\frac{\partial Q}{\partial X_2}(p), \ldots, -\frac{\partial Q}{\partial X_1}\frac{\partial Q}{\partial X_k}(p)\right).$$

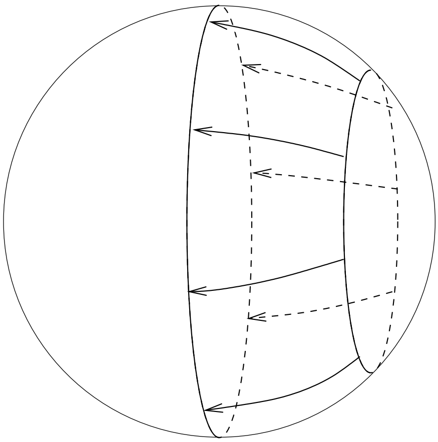

Fig. 7.1. Flow of the gradient vector field on the 2-sphere

Proof of Theorem 7.5: By Lemma 7.2 we can chose $a' < a$, $b' > b$ such that π has no critical point on $\mathrm{Zer}(Q, \mathbb{R}^k)_{(a', b')}$. Consider the gradient vector field of π on $\mathrm{Zer}(Q, \mathbb{R}^k)_{(a', b')}$.

By Corollary 5.51, the set $\mathrm{Zer}(Q, \mathbb{R}^k)_{(a', b')}$ can be covered by a finite number of open sets such that for each open set U' in the cover, there is an open U of $\mathbb{R}^{k-1}$ and a diffeomorphism $\Phi: U \to U'$.

Using the linear maps $\mathrm{d}\Phi_x^{-1}: T_x(M) \to T_{\Phi^{-1}(x)}\mathbb{R}^{k-1}$, we associate to the gradient vector field of π on $U' \subset \mathrm{Zer}(Q, \mathbb{R}^k)_{(a', b')}$ a C^∞ vector field on U.

By Theorem 7.7, for each point $x \in \mathrm{Zer}(Q, \mathbb{R}^k)_{[a,b]}$, there exists a neighborhood W of $\Phi^{-1}(x)$ and an $\epsilon > 0$ such that the induced vector field in $\mathbb{R}^{k-1}$ has a solution $\gamma_x(t)$ for $t \in (-\epsilon, \epsilon)$ and such that $\gamma_x(0) = \Phi^{-1}(x)$. We consider its image $\Phi(\gamma_x)$ on $\mathrm{Zer}(Q, \mathbb{R}^k)_{(a',b')}$. Thus, for each point $x \in \mathrm{Zer}(Q, \mathbb{R}^k)_{[a,b]}$, we have a neighborhood U_x, a number $\epsilon_x > 0$, and a curve

$$\Phi \circ \gamma_x: (-\epsilon_x, \epsilon_x) \rightarrow \mathrm{Zer}(Q, \mathbb{R}^k)_{(a',b')},$$

such that $\Phi \circ \gamma_x(0) = x$, and $\mathrm{d}\Phi \circ \gamma_x \mathrm{d}t = \Gamma(\Phi \circ \gamma_x(t))$ for all $t \in (-\epsilon_x, \epsilon_x)$.

Since $\mathrm{Zer}(Q, \mathbb{R}^k)_{[a,b]}$ is compact, we can cover it using a finite number of the neighborhoods U_x and let $\epsilon_0 > 0$ be the least among the corresponding ϵ_x's. For $t \in [0, b-a]$, we define a one-parameter family of smooth maps

$$\alpha_t: \mathrm{Zer}(Q, \mathbb{R}^k)_b \rightarrow \mathrm{Zer}(Q, \mathbb{R}^k)_{\leq b}$$

as follows:

Let $x \in \mathrm{Zer}(Q, \mathbb{R}^k)_b$. If $|t| \leq \epsilon_0/2$, we let $\alpha_t(x) = \gamma_x(t)$. If $|t| > \epsilon_0/2$, we write $t = n\,\epsilon_0/2 + \delta$, where n is an integer and $|\delta| < \epsilon_0/2$. We let

$$\alpha_t(x) = \overbrace{\alpha_{\epsilon_0/2} \circ \cdots \circ \alpha_{\epsilon_0/2}}^{n \text{ times}} \circ \alpha_\delta(x).$$

Observe the following.

- For every $x \in \mathrm{Zer}(Q, \mathbb{R}^k)_b$, $\alpha_0(x) = x$.
- By construction, $\dfrac{\mathrm{d}\alpha_t(x)}{\mathrm{d}t} = \Gamma(\alpha_t(x))$. Since the projection on the X_1 axis of $\Gamma(\alpha_t(x)) = (-1, 0, \ldots, 0)$, it follows that $\pi(\alpha_t(x)) = b - t$.
- $\alpha_t(\mathrm{Zer}(Q, \mathbb{R}^k)_b) = \mathrm{Zer}(Q, \mathbb{R}^k)_{b-t}$.
- It follows from the uniqueness of the flowlines through every point of the gradient vector field on $\mathrm{Zer}(Q, \mathbb{R}^k)_{[a,b]}$ (Theorem 7.7) that each α_t defined above is injective.

We now claim that the map $f: \mathrm{Zer}(Q, \mathbb{R}^k)_{[a,b]} \rightarrow \mathrm{Zer}(Q, \mathbb{R}^k)_a \times [a,b]$ defined by

$$f(x) = (\alpha_{b-a}(\alpha_{b-\pi(x)}^{-1}(x)), \pi(x))$$

is a homeomorphism. This is an immediate consequence of the properties of α_t listed above.

Next, consider the map $F(x, t)$: $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq b} \times [0, 1] \rightarrow \mathrm{Zer}(Q, \mathbb{R}^k)_{\leq b}$ defined as follows:

$$\begin{aligned} F(x,s) &= x, && \text{if } \pi(x) \leq b - s\,(b-a) \\ &= \alpha_{s(b-a)}(\alpha_{b-\pi(x)}^{-1}(x)), && \text{otherwise.} \end{aligned}$$

Clearly, F is a deformation retraction from $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq b}$ to $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq a}$, so that $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq b}$ is homotopy equivalent to $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq a}$.

This completes the proof. □

Theorem 7.5 states that there is no change in homotopy type on intervals containing no critical values. The remainder of the section is devoted to studying the changes in homotopy type that occur at the critical values. In this case, we will not be able to use the gradient vector field of π to get a flow as the gradient becomes zero at a critical point. We will, however, show how to modify the gradient vector field in a neighborhood of a critical point so as to get a new vector field that agrees with the gradient vector field outside a small neighborhood. The flow corresponding to this new vector field will give us a homotopy equivalence between $\mathrm{Zer}(Q,\mathbb{R}^k)_{\leq c+\epsilon}$ and $\mathrm{Zer}(Q,\mathbb{R}^k)_{\leq c-\epsilon}\cup B$, where c is a critical value of π, $\epsilon>0$ is sufficiently small, and B a topological ball attached to $\mathrm{Zer}(Q,\mathbb{R}^k)_{\leq c-\epsilon}$ by its boundary. The key notion necessary to work this idea out is that of a Morse function.

Definition 7.8. [Morse function] Let $\mathrm{Zer}(Q,\ \mathbb{R}^k)$ be a bounded non-singular algebraic hypersurface and π the projection on the X_1-axis sending $x=(x_1,\ldots,x_k)\in\mathbb{R}^k$ to $x_1\in\mathbb{R}$. Let $p\in\mathrm{Zer}(Q,\ \mathbb{R}^k)$ be a critical point of π. The tangent space $T_p(\mathrm{Zer}(Q,\ \mathbb{R}^k))$ is the $(k-1)$-dimensional space spanned by the $X_2,\ \ldots,\ X_k$ coordinates with origin p. By virtue of the Implicit Function Theorem (Theorem 3.25), we can choose $(X_2,\ \ldots,\ X_k)$ to be a local system of coordinates in a sufficiently small neighborhood of p. More precisely, we have an open neighborhood $U\subset\mathbb{R}^{k-1}$ of $p'=(p_2,\ldots,p_k)$ and a mapping $\phi\colon U\to\mathbb{R}$, such that, with $x'=(x_2,\ldots,x_k)$, and

$$\Phi(x')=(\phi(x'),x')\in\mathrm{Zer}(Q,\mathbb{R}^k), \tag{7.1}$$

the mapping Φ is a diffeomorphism from U to $\Phi(U)$.

The critical point p is **non-degenerate** if the $(k-1)\times(k-1)$ Hessian matrix

$$\mathrm{Hes}_\pi(p')=\left[\frac{\partial^2\phi}{\partial X_i\partial X_j}(p')\right],\ 2\leq i,j\leq k, \tag{7.2}$$

is invertible. Note that $\mathrm{Hes}_\pi(p')$ is a real symmetric matrix and hence all its eigenvalues are real (Theorem 4.42). Moreover, if p is a non-degenerate critical point, then all eigenvalues are non-zero. The number of positive eigenvalues of $\mathrm{Hes}_\pi(p')$ is the **index** of the critical point p.

The function π is a **Morse function** if all its critical points are non-degenerate and there is at most one critical point of π above each $x\in\mathrm{R}$. □

We next show that to require π to be a Morse function is not a big loss of generality, since an orthogonal change of coordinates can make the projection map π a Morse function on $\mathrm{Zer}(Q,\mathbb{R}^k)$.

Proposition 7.9. *Up to an orthogonal change of coordinates, the projection π to the X_1-axis is a Morse function.*

The proof of Proposition 7.9 requires some preliminary work.

We start by proving:

Proposition 7.10. *Let d be the degree of Q. Suppose that the projection π on the X_1-axis has only non-degenerate critical points. The number of critical points of π is finite and bounded by $d(d-1)^{k-1}$.*

Proof: The critical points of π can be characterized as the real solutions of the system of k polynomial equations in k variables

$$Q = \frac{\partial Q}{\partial X_2} = 0 , \ldots, \frac{\partial Q}{\partial X_k} = 0 .$$

We claim that every real solution p of this system is non-singular, i.e. the Jacobian matrix

$$\begin{bmatrix} \frac{\partial Q}{\partial X_1}(p) & \frac{\partial^2 Q}{\partial X_2 \partial X_1}(p) & \cdots & \frac{\partial^2 Q}{\partial X_k \partial X_1}(p) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial Q}{\partial X_k}(p) & \frac{\partial^2 Q}{\partial X_2 \partial X_k}(p) & \cdots & \frac{\partial^2 Q}{\partial X_k \partial X_k}(p) \end{bmatrix}$$

is non-singular. Differentiating the identity (7.3) and evaluating at p, we obtain for $2 \leq i, j \leq k$, with $p' = (p_2, \ldots, p_k)$,

$$\frac{\partial^2 Q}{\partial X_j \partial X_i}(p) = -\frac{\partial Q}{\partial X_1}(p) \frac{\partial^2 \phi}{\partial X_j \partial X_i}(p') .$$

Since $\frac{\partial Q}{\partial X_1}(p) \neq 0$ and $\frac{\partial Q}{\partial X_i}(p) = 0$, for $2 \leq i \leq k$, the claim follows.By Theorem 4.106 (Bézout's theorem), the number of critical points of π is less than or equal to the product

$$\deg(Q) \deg\left(\frac{\partial Q}{\partial X_2}\right) \cdots \deg\left(\frac{\partial Q}{\partial X_k}\right) = d\,(d-1)^{k-1} . \qquad \square$$

We interpret geometrically the notion of non-degenerate critical point.

Proposition 7.11. *Let $p \in \mathrm{Zer}(Q, \mathbb{R}^k)$ be a critical point of π. Let g: $\mathrm{Zer}(Q, \mathbb{R}^k) \to S^{k-1}(0,1)$ be the Gauss map defined by*

$$g(x) = \frac{\mathrm{Grad}(Q(x))}{\|\mathrm{Grad}(Q(x))\|}.$$

The Gauss map is an $\mathcal{S}^\infty$-diffeomorphism in a neighborhood of p if and only if p is a non-degenerate critical point.

Proof: Since p is a critical point of π, $g(p) = (\pm 1, 0, \ldots, 0)$. Using Notation 7.8, for $x' \in U$, $x = \Phi(x') = (\phi(x'), x')$, and applying the chain rule,

$$\frac{\partial Q}{\partial X_i}(x) + \frac{\partial Q}{\partial X_1}(x) \frac{\partial \phi}{\partial X_i}(x') = 0, \ 2 \leq i \leq k. \qquad (7.3)$$

Thus

$$g(x) = \pm \frac{1}{\sqrt{1 + \sum_{i=2}^{k} \left(\frac{\partial \phi}{\partial X_i}(x') \right)^2}} \left(-1, \frac{\partial \phi}{\partial X_2}(x'), \ldots, \frac{\partial \phi}{\partial X_k}(x') \right).$$

Taking the partial derivative with respect to X_i of the j-th coordinate g_j of g, for $2 \le i, j \le k$, and evaluating at p, we obtain

$$\frac{\partial g_j}{\partial X_i}(p) = \pm \frac{\partial^2 \phi}{\partial X_j \, \partial X_i}(p'),\ 2 \le i, j \le k.$$

The matrix $[\partial g_i / \partial X_i(p)], 2 \le i, j \le k,$ is invertible if and only if p is a non-degenerate critical point of ϕ by (7.2). □

Proposition 7.12. *Up to an orthogonal change of coordinates, the projection π to the X_1-axis has only non-degenerate critical points.*

Proof: Consider again the Gauss map $g\colon \mathrm{Zer}(Q, \mathbb{R}^k) \to S^{k-1}(0,1)$, defined by

$$g(x) = \frac{\mathrm{Grad}(Q(x))}{\|\mathrm{Grad}(Q(x))\|}.$$

According to Sard's theorem (Theorem 5.56) the dimension of the set of critical values of g is at most $k-2$. We prove now that there are two antipodal points of $S^{k-1}(0,\,1)$ such that neither is a critical value of g. Assume the contrary and argue by contradiction. Since the dimension of the set of critical values is at most $k-2$, there exists a non-empty open set U of regular values in $S^{k-1}(0,1)$. The set of points that are antipodes to points in U is non-empty, open in $S^{k-1}(0,1)$ and all critical, contradicting the fact that the critical set has dimension at most $k-2$.

After rotating the coordinate system, we may assume that $(1,\,0,\,\ldots,\,0)$ and $(-1, 0, \ldots, 0)$ are not critical values of g. The claim follows from Proposition 7.11. □

It remains to prove that it is possible to ensure, changing the coordinates if necessary, that there is at most one critical point of π above each $x \in \mathrm{R}$.

Suppose that the projection π on the X_1-axis has only non-degenerate critical points. These critical points are finite in number according to Proposition 7.10. We can suppose without loss of generality that all the critical points have distinct X_2 coordinates, making if necessary an orthogonal change of coordinates in the variables $X_2, \ldots, X_k$ only.

Lemma 7.13. *Let δ be a new variable and consider the field $\mathbb{R}\langle\delta\rangle$ of algebraic Puiseux series in δ. The set S of points $\overline{p} = (\overline{p_1}, \ldots, \overline{p_k}) \in \mathrm{Zer}(Q, \mathbb{R}\langle\delta\rangle^k)$ with gradient vector $Grad(Q)(\overline{p})$ proportional to $(1, \delta, 0, \ldots, 0)$ is finite. Its number of elements is equal to the number of critical points of π. Moreover there is a point $\overline{p}$ of S infinitesimally close to every critical point p of π and the signature of the Hessian at p and $\overline{p}$ coincide.*

Proof: Note that, modulo the orthogonal change of variable

$$X_1' = X_1 + \delta X_2, X_2' = X_2 - \delta X_1, X_i' = X_i, i \geq 3,$$

a point $\overline{p}$ such that $\mathrm{Grad}(Q)(\overline{p})$ is proportional to $(1, \delta, 0, \ldots, 0)$ is a critical point of the projection π' on the X_1'-axis, and the corresponding critical value of π' is $\overline{p_1} + \delta\,\overline{p_2}$.

Since $\mathrm{Zer}(Q, \mathbb{R}^k)$ is bounded, a point $\overline{p} \in \mathrm{Zer}(Q, \mathbb{R}\langle\delta\rangle^k)$ always has an image by $\lim_\delta$. If $\overline{p}$ is such that $\mathrm{Grad}(Q)(\overline{p})$ is proportional to $(1, \delta, 0, \ldots, 0)$, then $\mathrm{Grad}(Q)(\lim_\delta (\overline{p}))$ is proportional to $(1, 0, \ldots, 0, 0)$, and thus $p = \lim_\delta (p)$ is a critical point of π. Suppose without loss of generality that $\mathrm{Grad}(Q)(p) = (1, 0, \ldots, 0, 0)$. Since p is a non-degenerate critical point of π, Proposition 7.11 implies that there is a semi-algebraic neighborhood U of $p' = (p_2, \ldots, p_k)$ such that $g \circ \Phi$ is a diffeomorphism from U to a semi-algebraic neighborhood of $(1, 0, \ldots, 0, 0) \in S^{k-1}(0, 1)$. Denoting by g' the inverse of the restriction of g to $\Phi(U)$ and considering

$$\mathrm{Ext}(g', \mathbb{R}\langle\delta\rangle) \colon \mathrm{Ext}(g(\Phi(U)), \mathbb{R}\langle\delta\rangle) \rightarrow \mathrm{Ext}(\Phi(U), \mathbb{R}\langle\delta\rangle),$$

there is a unique $\overline{p} \in \mathrm{Ext}(\Phi(U), \mathbb{R}\langle\delta\rangle)$ such that $\mathrm{Grad}(Q)(\overline{p})$ is proportional to $(1, \delta, 0, \ldots, 0)$. Moreover, denoting by J the Jacobian of $\mathrm{Ext}(g', \mathbb{R}\langle\delta\rangle)$, the value $J(1, 0, 0, \ldots, 0) = t$ is a non-zero real number. Thus the signature of the Hessian at p and $\overline{p}$ coincide. □

Proof of Proposition 7.9: Since J is the Jacobian of $\mathrm{Ext}(g', \mathbb{R}\langle\delta\rangle)$, the value $J(1, 0, 0, \ldots, 0) = t$ is a non-zero real number, $\lim_\delta (J(y)) = t$ for every $y \in \mathrm{Ext}(S^{k-1}(0, 1), \mathbb{R}\langle\delta\rangle)$ infinitesimally close to $(1, 0, 0, \ldots, 0)$. Using the mean value theorem (Corollary 2.23)

$$o(|\overline{p} - p|) = o\left(\left|\frac{1}{\sqrt{1+\delta^2}}(1, \delta, 0, \ldots, 0) - (1, 0, 0, \ldots, 0)\right|\right) = 1.$$

Thus $o(\overline{p_i} - p_i) \geq 1, i \geq 1$.

Let $b_{i,j} = \dfrac{\partial^2 \phi}{\partial X_i \partial X_j}(p),\ 2 \leq i \leq k, 2 \leq j \leq k$. Taylor's formula at p for ϕ gives

$$\overline{p_1} = p_1 + \sum_{2 \leq i \leq k, 2 \leq j \leq k} b_{i,j}\,(\overline{p_i} - p_i)\,(\overline{p_j} - p_j) + c,$$

with $o(c) \geq 2$. Thus $o(\overline{p_1} - p_1) \geq 2$.

It follows that the critical value of π' at $\overline{p}$ is $\overline{p_1} + \delta\,\overline{p_2} = p_1 + \delta\, p_2 + w$, with $o(w) > 1$.

Thus, all the critical values of π' on $\mathrm{Zer}(Q, \mathbb{R}\langle\delta\rangle^k$ are distinct since all values of p_2 are. Using Proposition 3.17, we can replace δ by $d \in \mathbb{R}$, and we have proved that there exists an orthogonal change of variable such that π is a Morse function. □

We are now ready to state the second basic ingredient of Morse theory, which is describing precisely the change in the homotopy type that occurs in $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq x}$ as x crosses a critical value when π is a Morse function.

Theorem 7.14. [Morse lemma B] *Let* $\mathrm{Zer}(Q, \mathbb{R}^k)$ *be a non-singular bounded algebraic hypersurface such that the projection* π *to the* X_1*-axis is a Morse function. Let* p *be a non-degenerate critical point of* π *of index* λ *and such that* $\pi(p) = c$.

Then, for all sufficiently small $\epsilon > 0$, *the set* $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq c+\epsilon}$ *has the homotopy type of the union of* $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq c-\epsilon}$ *with a ball of dimension* $k - 1 - \lambda$, *attached along its boundary.*

We first prove a lemma that will allow us to restrict to the case where $x = 0$ and where Q is a quadratic polynomial of a very simple form.

Let $\mathrm{Zer}(Q, \mathbb{R}^k), U, \phi, \Phi$ be as above (see page 243).

Lemma 7.15. *Let* $\mathrm{Zer}(Q, \mathbb{R}^k)$ *be a non-singular bounded algebraic hypersurface such that the projection* π *to the* X_1*-axis is a Morse function. Let* $p \in \mathrm{Zer}(Q, \mathbb{R}^k)$ *be a non-degenerate critical point of the map* π *with index* λ. *Then there exists an open neighborhood* V *of the origin in* $\mathbb{R}^{k-1}$ *and a diffeomorphism* Ψ *from* U *to* V *such that, denoting by* Y_i *the* i*-th coordinate of* $\Psi(X_2, \ldots, X_k)$,

$$\phi(Y_2, \ldots, Y_k) = \sum_{2 \leq i \leq \lambda+1} Y_i^2 - \sum_{\lambda+2 \leq i \leq k} Y_i^2.$$

Proof: We assume without loss of generality that p is the origin. Also, by Theorem 4.42, we assume that the matrix

$$\mathrm{Hes}(0) = \left[\frac{\partial^2 \phi}{\partial X_i \partial X_j}(0)\right], \ 2 \leq i, j \leq k,$$

is diagonal with its first λ entries $+1$ and the remaining -1.

Let us prove that there exists a C^∞ map M from U to the space of symmetric $(k-1) \times (k-1)$ matrices, $X \mapsto M(X) = (m_{ij}(X))$, such that

$$\phi(X_2, \ldots, X_k) = \sum_{2 \leq i, j \leq k} m_{ij}(X)\, X_i\, X_j.$$

Using the fundamental theorem of calculus twice, we obtain

$$\begin{aligned} \phi(X_2, \ldots, X_k) &= \sum_{2 \leq j \leq k} X_j \int_0^1 \frac{\partial \phi}{\partial X_j}(t\, X_2, \ldots, t\, X_k) \mathrm{d}t \\ &= \sum_{2 \leq i \leq k} \sum_{2 \leq j \leq k} X_i\, X_j \int_0^1 \int_0^1 \frac{\partial^2 \phi}{\partial X_i \partial X_j}(s\, t\, X_2, \ldots, s\, t\, X_k) \mathrm{d}t\, \mathrm{d}s. \end{aligned}$$

Take

$$m_{ij}(X_2, \ldots, X_k) = \int_0^1 \int_0^1 \frac{\partial^2 \phi}{\partial X_i \partial X_j}(s\, t\, X_2, \ldots, s\, t\, X_k) \mathrm{d}t\, \mathrm{d}s.$$

Note that the matrix $M(X_2, \ldots, X_k)$ obtained above clearly satisfies $M(0) = H(0)$, and $M(x_2, \ldots, x_k)$ is close to $H(x_2, \ldots, x_k)$ for $(x_2, \ldots, x_k)$ in a sufficiently small neighborhood of the origin.

Using Theorem 4.42 again, there exists a C^∞ map N from a sufficiently small neighborhood V of 0 in R^{k-1} to the space of $(k-1)\times(k-1)$ real invertible matrices such that

$$\forall x \in V, N(x)^t M(x) N(x) = H(0).$$

Let $Y = N(X)^{-1} X$. Since $N(X)$ is invertible, the map sending X to Y maps V diffeomorphically into its image. Also,

$$\begin{aligned} X^t M(X) X &= Y^t N(X)^t M(X) N(X) Y \\ &= Y^t H(0) Y \\ &= \sum_{2 \le i \le \lambda+1} Y_i^2 - \sum_{\lambda+2 \le i \le k} Y_i^2. \end{aligned}$$

□

Using Lemma 7.15, we observe that in a small enough neighborhood of a critical point, a hypersurface behaves like one defined by a quadratic equation. So it suffices to analyze the change in the homotopy type of $\mathrm{Zer}(Q, \mathbb{R}^k)_{\le x}$ as x crosses 0 and the hypersurface defined by a quadratic polynomial of a very simple form. The change in the homotopy type consists in "attaching a handle along its boundary", which is the process we describe now.

A j**-ball** is an embedding of $\overline{B_j}(0,1)$, the closed j-dimensional ball with radius 1, in $\mathrm{Zer}(Q,\mathbb{R}^k)$. It is a homeomorphic image of $\overline{B_j}(0,1)$ in $\mathrm{Zer}(Q,\mathbb{R}^k)$.

Let

$$P = X_1 - \sum_{2 \le i \le \lambda+1} X_i^2 + \sum_{\lambda+2 \le i \le k} X_i^2,$$

and π the projection onto the X_1 axis restricted to $\mathrm{Zer}(P, \mathbb{R}^k)$.

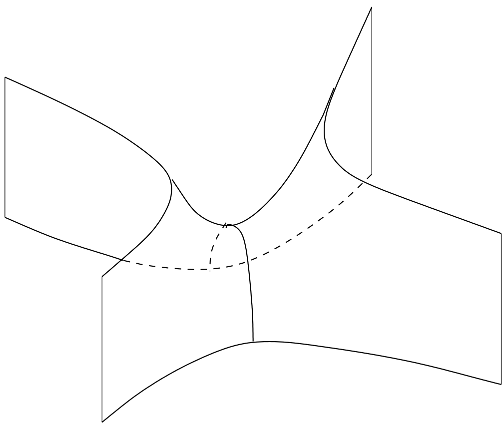

Fig. 7.2. The surface $\mathrm{Zer}(X_1 - X_2^2 + X_3^2, \mathbb{R}^3)$ near the origin

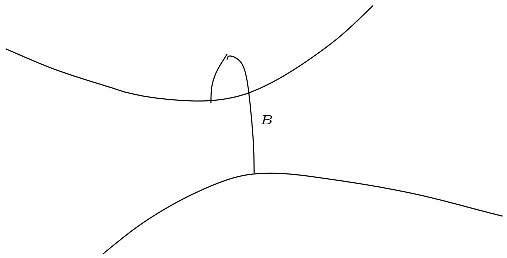

Fig. 7.3. The retract of $\mathrm{Zer}(X_1 - X_2^2 + X_3^2, \mathbb{R}^3)$ near the origin

Let B be the set defined by

$$X_2 = \cdots = X_{\lambda+1} = 0, X_1 = -\sum_{\lambda+2 \le i \le k} X_i^2, -\epsilon \le X_1 \le 0.$$

Note that B is a $(k-\lambda-1)$-ball and $B \cap \mathrm{Zer}(P, \mathbb{R}^k)_{\le -\epsilon}$ is the set defined by

$$X_2 = \cdots = X_{\lambda+1} = 0, X_1 = -\epsilon, \sum_{\lambda+2 \le i \le k} X_i^2 = \epsilon,$$

which is also the boundary of B.

Lemma 7.16. *For all sufficiently small $\epsilon > 0$, and $r > 2\sqrt{\epsilon}$, there exists a vector field Γ' on $\mathrm{Zer}(P, \mathbb{R}^k)_{[-\epsilon,\epsilon]} \setminus B$, having the following properties:*

1. *Outside the ball $B_k(r)$, $2\epsilon\Gamma'$ equals the gradient vector field, Γ, of π on $\mathrm{Zer}(P, \mathbb{R}^k)_{[-\epsilon,\epsilon]}$.*
2. *Associated to Γ' there is an one parameter continuous family of smooth maps α_t: $\mathrm{Zer}(P, \mathbb{R}^k)_\epsilon \to \mathrm{Zer}(P, \mathbb{R}^k)_{[-\epsilon,\epsilon]}$, $t \in [0, 1)$, such that for $x \in \mathrm{Zer}(P, \mathbb{R}^k)_\epsilon, t \in [0,1)$,*
   a) *Each α_t is injective,*
   b) $\dfrac{\mathrm{d}\alpha_t(x)}{\mathrm{d}t} = \Gamma'(\alpha_t(x))$,
   c) $\alpha_0(x) = x$,
   d) $\lim_{t\to 1} \alpha_t(x) \in \mathrm{Zer}(P, \mathbb{R}^k)_{-\epsilon} \cup B$,
   e) *for every $y \in \mathrm{Zer}(P, \mathbb{R}^k)_{[-\epsilon,\epsilon]} \setminus B$ there exists a unique $z \in \mathrm{Zer}(P, \mathbb{R}^k)_\epsilon$ and $t \in [0,1)$ such that $\alpha_t(z) = y$.*

Proof of Lemma 7.16: In the following, we consider $\mathbb{R}^{k-1}$ as a product of the coordinate subspaces spanned by $X_2, \ldots, X_{\lambda+1}$ and $X_{\lambda+2}, \ldots, X_k$, respectively, and denote by Y(resp. Z) the vector of variables $(X_2, \ldots, X_{\lambda+1})$ (resp. $(X_{\lambda+2}, \ldots, X_k)$). We denote by $\phi: \mathbb{R}^k \to \mathbb{R}^{k-1}$ the projection map onto the hyperplane $X_1 = 0$. Let $S = \phi(\overline{B}_k(r))$.

We depict the flow lines of the flow we are going to construct (projected onto the hyperplane defined by $X_1 = 0$) in the case when $k = 3$ and $\lambda = 1$ in Figure 7.4.

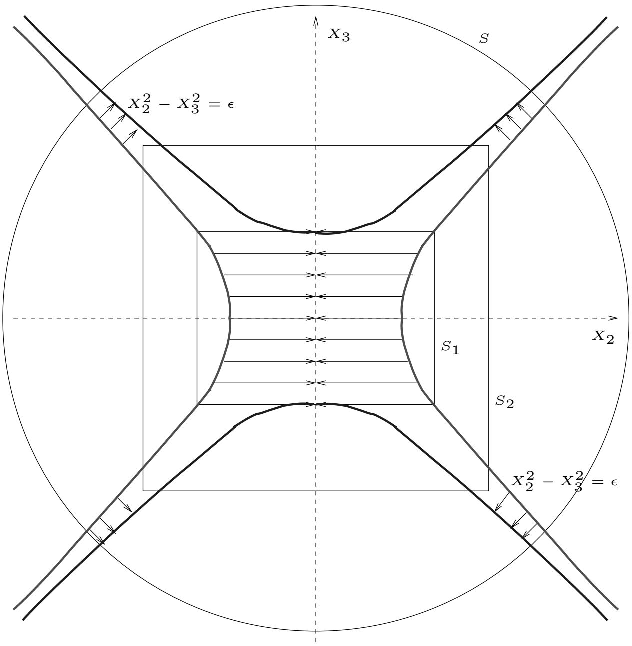

Fig. 7.4. S_1 and S_2

Consider the following two subsets of S.

$$S_1 = \overline{B}_\lambda(\sqrt{2\,\epsilon}) \times \overline{B}_{k-1-\lambda}(\sqrt{\epsilon})$$

and

$$S_2 = \overline{B}_\lambda(2\,\sqrt{\epsilon}) \times \overline{B}_{k-1-\lambda}(2\,\sqrt{\epsilon}).$$

In $\mathrm{Zer}(P,\ \mathbb{R}^k)_{[-\epsilon,\epsilon]} \cap \phi^{-1}(S_1)$, consider the flow lines whose projection onto the hyperplane $X_1 = 0$ are straight segments joining the points $(y_2, \ldots, y_k) \in \phi(\mathrm{Zer}(P, \mathbb{R}^k)_\epsilon)$ to $(0, \ldots 0, y_{\lambda+2}, \ldots, y_k)$.

These correspond to the vector field on $\mathrm{Zer}(P,\ \mathbb{R}^k)_{[-\epsilon,\epsilon]} \cap \phi^{-1}(S_1) \setminus B$ defined by

$$\Gamma_1 = \left(-\frac{1}{|Z|^2 + \epsilon}, \frac{-Y}{2|Y|^2(|Z|^2 + \epsilon)}, 0 \right).$$

Let $p = (\epsilon, y, z) \in \mathrm{Zer}(P, \mathbb{R}^k)_\epsilon \cap \phi^{-1}(S_1)$ and q the point in $\mathrm{Zer}(P, \mathbb{R}^k)$ having the same Z coordinates but having $Y = 0$. Then, $\pi(q) = |z|^2 + \epsilon$. Thus, the decreases uniformly from ϵ to $-|z|^2$ along the flow lines of the vector field Γ_1. For a point $p = (x_1, y, z) \in \mathrm{Zer}(Q, \mathbb{R}^k)_{[-\epsilon,\epsilon]} \cap \phi^{-1}(S_1) \setminus B$, we denote by $g(p)$ the limiting point on the flow line through p of the vector field Γ_1 as it approaches $Y = 0$.

In $\mathrm{Zer}(P,\mathbb{R}^k)_{[-\epsilon,\epsilon]}\cap\phi^{-1}(S\setminus S_2)$, consider the flow lines of the vector field

$$\Gamma_2=\left(-\frac{1}{2\,\epsilon},-\frac{Y}{4\,\epsilon(|Y|^2+|Z|^2)},\frac{Z}{4\,\epsilon(|Y|^2+|Z|^2)}\right).$$

Notice that Γ_2 is $\frac{1}{2\epsilon}$ times the gradient vector field on

$$\mathrm{Zer}(P,\mathbb{R}^k)_{[-\epsilon,\epsilon]}\cap\phi^{-1}(S\setminus S_2).$$

For a point $p=(x_1,y,z)\in\mathrm{Zer}(P,\mathbb{R}^k)_{[-\epsilon,\epsilon]}\cap\phi^{-1}(S\setminus S_2)$, we denote by $g(p)$ the point on the flow line through p of the vector field Γ_2 such that $\pi(g(p))=-\epsilon$.

We patch these vector fields together in

$$\mathrm{Zer}(P,\mathbb{R}^k)_{[-\epsilon,\epsilon]}\cap\phi^{-1}(S_2\setminus S_1)$$

using a C^∞ function that is 0 in S_1 and 1 outside S_2. Such a function $\mu:\mathbb{R}^{k-1}\to\mathbb{R}$ can be constructed as follows. Define

$$\lambda(x)=\begin{cases}0 & \text{if } x\leq 0,\\ 1-2^{-\frac{1}{4x^2}} & \text{if } 0<x\leq\frac{1}{2},\\ 2^{-\frac{1}{4(1-x)^2}} & \text{if } \frac{1}{2}<x\leq 1,\\ 1 & \text{if } x\geq 1.\end{cases}$$

Take

$$\mu(y,z)=\lambda\left(\frac{|y|-\sqrt{2\,\epsilon}}{\sqrt{2\epsilon}(\sqrt{2}-1)}\right)\lambda\left(\frac{|z|-\sqrt{\epsilon}}{\sqrt{\epsilon}}\right).$$

Then, on $\mathrm{Zer}(P,\mathbb{R}^k)_{[-\epsilon,\epsilon]}\cap\phi^{-1}(S_2\setminus S_1)$ we consider the vector field

$$\Gamma'(p)=\mu(\phi(p))\Gamma_2(p)+(1-\mu(\phi(p)))\Gamma_1(p).$$

Notice that it agrees with the vector fields defined on

$$\mathrm{Zer}(P,\mathbb{R}^k)_{[-\epsilon,\epsilon]}\cap\phi^{-1}(S\setminus S_2),\ \mathrm{Zer}(P,\mathbb{R}^k)_{[-\epsilon,\epsilon]}\cap\phi^{-1}(S_1).$$

For a point $p=(x_1,\ y,\ z)\in\mathrm{Zer}(Q,\ \mathbb{R}^k)_{[-\epsilon,\epsilon]}\cap\phi^{-1}(S_2\setminus S_1)$, we denote by $g(p)$ the point on the flow line through p of the vector field Γ_2 such that $\pi(g(p))=-\epsilon$.

Denote the flow through a point $p\in\mathrm{Zer}(P,\mathbb{R}^k)_\epsilon\cap\phi^{-1}(S)$ of the vector field Γ' by $\gamma_p:[0,1]\to\mathrm{Zer}(P,\mathbb{R}^k)_{[-\epsilon,\epsilon]}$, with $\gamma_p(0)=p$.

For $x\in\mathrm{Zer}(P,\mathbb{R}^k)_\epsilon$ and $t\in[0,1]$, define $\alpha_t(x)=\gamma_x(t)$. By construction of the vector field Γ, α_t has the required properties. □

Before proving Theorem 7.14 it is instructive to consider an example.

Example 7.17. Consider a smooth torus in R^3 (see Figure 7.5). There are four critical points p_1, p_2, p_3 and p_4 with critical values v_1, v_2, v_3 and v_4 and indices $2, 1, 1$ and 0 respectively, for the projection map to the X_1 coordinate.

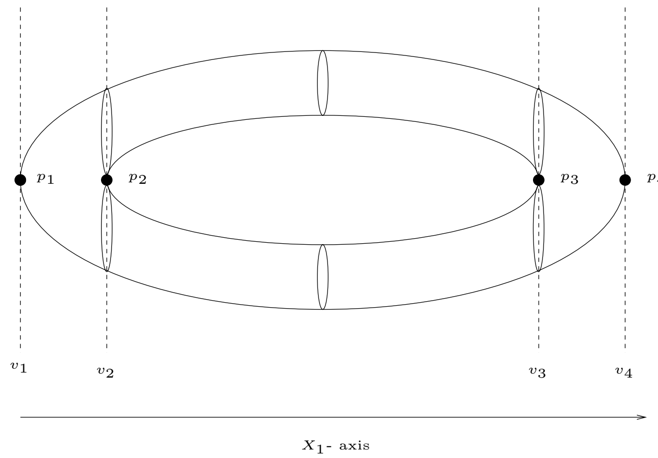

Fig. 7.5. Changes in the homotopy type of the smooth torus in $\mathbb{R}^3$ at the critical values

The changes in homotopy type at the corresponding critical values are described as follows: At the critical value v_1 we add a 0-dimensional ball. At the critical values v_2 and v_3 we add 1-dimensional balls and finally at v_4 we add a 2-dimensional ball. □

Proof of Theorem 7.14: We construct a vector field Γ' on $\mathrm{Zer}(Q, \mathbb{R}^k)_{[c-\epsilon,c+\epsilon]}$ that agrees with the gradient vector field Γ everywhere except in a small neighborhood of the critical point p. At the critical point p, we use Lemma 7.15 to reduce to the quadratic case and then use Lemma 7.16 to construct a vector field in a neighborhood of the critical point that agrees with Γ outside the neighborhood. We now use this vector field, as in the proof of Theorem 7.5, to obtain the required homotopy equivalence. □

We also need to analyze the topological changes that occur to sets bounded by non-singular algebraic hypersurfaces.

We are also going to prove the following versions of Theorem 7.5 (Morse Lemma A) and Theorem 7.14 (Morse Lemma B).

Proposition 7.18. *Let S be a bounded set defined by $Q \geq 0$, bounded by the non-singular algebraic hypersurface* $\mathrm{Zer}(Q, \mathbb{R}^k)$*. Let $[a, b]$ be an interval containing no critical value of π on* $\mathrm{Zer}(Q, \mathbb{R}^k)$*. Then $S_{[a,b]}$ is homeomorphic to $S_a \times [a, b]$ and $S_{\leq a}$ is homotopy equivalent to $S_{\leq b}$.*

Proposition 7.19. *Let S be a bounded set defined by $Q \geq 0$, bounded by the non-singular algebraic hypersurface* $\mathrm{Zer}(Q, \mathbb{R}^k)$. *Suppose that the projection π to the X_1-axis is a Morse function. Let p be the non-degenerate critical point of π on ∂W of index λ such that $\pi(p) = c$. For all sufficiently small $\epsilon > 0$, the set $S_{\leq c+\epsilon}$ has*

- *the homotopy type of $S_{\leq c-\epsilon}$ if* $(\partial Q/\partial X_1)(p) < 0$,
- *the homotopy type of the union of $S_{\leq c-\epsilon}$ with a ball of dimension $k-1-\lambda$ attached along its boundary, if* $(\partial Q/\partial X_1)(p) > 0$.

Example 7.20. Consider the set in $\mathbb{R}^3$ bounded by the smooth torus. Suppose that this set is defined by the single inequality $Q \geq 0$. In other words, Q is positive in the interior of the torus and negative outside. Referring back to Figure 7.5, we see that at the critical points p_2 and p_4, $(\partial Q/\partial X_1)(p) < 0$ and hence according to Proposition 7.19 there is no change in the homotopy type at the two corresponding critical values v_2 and v_4. However, $(\partial Q/\partial X_1)(p) > 0$ at p_1 and p_3 and hence we add a 0-dimensional and an 1-dimensional balls at the two critical values v_1 and v_3 respectively. □

Proof of Proposition 7.18: Suppose that S, defined by $Q \geq 0$, is bounded by the non-singular algebraic hypersurface $\mathrm{Zer}(Q, \mathbb{R}^k)$. We introduce a new variable, X_{k+1}, and consider the polynomial $Q_+ = Q - X_{k+1}^2$ and the corresponding algebraic set $\mathrm{Zer}(Q_+, \mathbb{R}^{k+1})$. Let $\phi\colon \mathbb{R}^{k+1} \rightarrow \mathbb{R}^k$ be the projection map to the first k coordinates.

Topologically, $\mathrm{Zer}(Q_+, \mathbb{R}^{k+1})$ consists of two copies of S glued along $\mathrm{Zer}(Q, \mathbb{R}^k)$. Moreover, denoting by π' the projection from $\mathbb{R}^{k+1}$ to $\mathbb{R}$ forgetting the last k coordinates, $\mathrm{Zer}(Q_+, \mathbb{R}^{k+1})$ is non-singular and the critical points of π' on $\mathrm{Zer}(Q_-, \mathbb{R}^{k+1})$ are the critical points of π on $\mathrm{Zer}(Q, \mathbb{R}^k)$ (considering $\mathrm{Zer}(Q, \mathbb{R}^k)$ as a subset of the hyperplane defined by the equation $X_{k+1} = 0$). We denote by Γ_+ the gradient vector field on $\mathrm{Zer}(Q_+, \mathbb{R}^{k+1})$.

Since Q_+ is a polynomial in $X_1, \ldots, X_k$ and X_{k+1}^2, the gradient vector field Γ_+ on $\mathrm{Zer}(Q_+, \mathbb{R}^{k+1})$ is symmetric with respect to the reflection changing X_{k+1} to $-X_{k+1}$. Hence, we can project Γ_+ and its associated flowlines down to the hyperplane defined by $X_{k+1} = 0$ and get a vector field as well as its flowlines in S.

Now, the proof is exactly the same as the proof of Theorem 7.5 above, using the vector field Γ_+ instead of Γ, and projecting the associated vector field down to $\mathbb{R}^k$, noting that the critical values of the projection map onto the first coordinate restricted to $\mathrm{Zer}(Q_+, \mathbb{R}^{k+1})$ are the same as those of $\mathrm{Zer}(Q, \mathbb{R}^k)$. □

For the proof of Proposition 7.19, we first study the quadratic case.

Let π the projection onto the X_1 axis and

$$P = X_1 - \sum_{2 \leq i \leq \lambda+1} X_i^2 + \sum_{\lambda+2 \leq i \leq k} X_i^2.$$

Let B_+ be the set defined by

$$X_2 = \cdots = X_{\lambda+1} = 0, X_1 = -\sum_{\lambda+2 \le i \le k} X_i^2, -\epsilon \le X_1 \le 0,$$

and let B_- be the set defined by

$$X_2 = \cdots = X_{\lambda+1} = 0, X_1 \le -\sum_{\lambda+2 \le i \le k} X_i^2, -\epsilon \le X_1 \le 0.$$

Note that, B_+ is a $(k-\lambda-1)$-ball and $B_- \cap \mathrm{Zer}(P, \mathbb{R}^k)_{\le -\epsilon}$ is the set defined by

$$X_2 = \cdots = X_{\lambda+1} = 0, X_1 = -\epsilon, \sum_{\lambda+2 \le i \le k} X_i^2 \le \epsilon,$$

which is also the boundary of B_+.

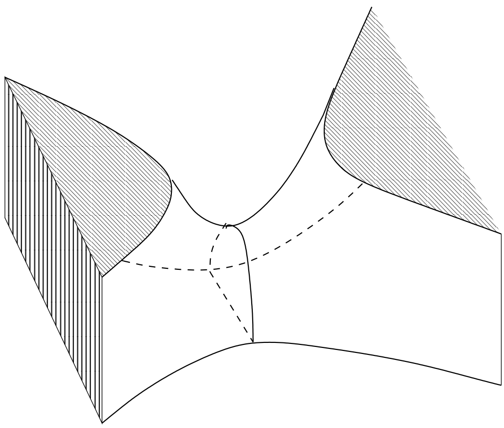

Fig. 7.6. Set defined by $X_1 - X_2^2 + X_3^2 \le 0$ near the origin

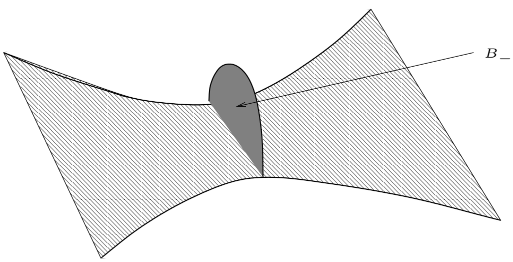

Fig. 7.7. Retract of the set $X_1 - X_2^2 + X_3^2 \le 0$

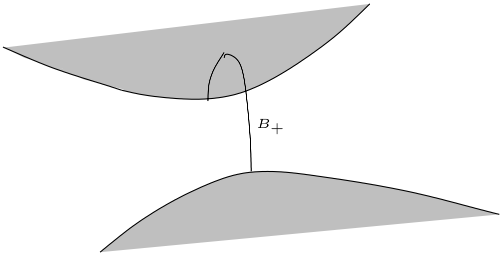

Fig. 7.8. Retract of the set $X_1 - X_2^2 + X_3^2 \geq 0$

Lemma 7.21. *Let* $P_+ = P - X_{k+1}^2, P_- == P + X_{k+1}^2$.

1. *Let* S' *be the set defined by* $P \geq 0$. *Then, for all sufficiently small* $\epsilon > 0$ *and* $r > 2\sqrt{(}\epsilon)$, *there exists a vector field* Γ'_+ *on* $S'_{[-\epsilon,\epsilon]} \setminus B_+$, *having the following properties:*
   a) *Outside the ball* $B_k(r)$, $2\epsilon\Gamma'_+$ *equals the projection on* $\mathbb{R}^k$ *of the gradient vector field,* Γ_+, *of* π *on* $\mathrm{Zer}(P_+, \mathbb{R}^{k+1})_{[-\epsilon,\epsilon]}$.
   b) *Associated to* Γ'_+, *there is a one parameter family of smooth maps* $\alpha_t^+ : S'_\epsilon \to S'_{[-\epsilon,\epsilon]}, t \in [0,1)$, *such that for* $x \in S'_\epsilon, t \in [0,1)$,
      i. *Each* α_t^+ *is injective,*
      ii.
      $$\frac{\mathrm{d}\alpha_t^+(x)}{\mathrm{d}t} = \Gamma'_+(\alpha_t^+(x)),$$
      iii. $\alpha_0^+(x) = x$,
      iv. $\lim_{t \to 1} \alpha_t^+(x) \in S'_{-\epsilon} \cup B_+$ *and,*
      v. *for every* $y \in S_{[-\epsilon,\epsilon]} \setminus B_+$ *there exists a unique* $z \in S_\epsilon$ *and* $t \in [0,1)$ *such that* $\alpha_t(z) = y$.
2. *Similarly, let* T' *be the set defined by* $P \leq 0$. *Then, for all sufficiently small* $\epsilon > 0$ *and* $r > 2\sqrt{(}\epsilon)$, *there exists a vector field* Γ'_- *on* $T'_{[-\epsilon,\epsilon]} \setminus B_+$ *having the following properties:*
   a) *Outside the ball* $B_k(r)$, $2\epsilon\Gamma'_-$ *the projection on* $\mathbb{R}^k$ *of the gradient vector field,* Γ_-, *of* π *on* $\mathrm{Zer}(P_-, \mathbb{R}^{k+1})_{[-\epsilon,\epsilon]}$.
   b) *Associated to* Γ'_-, *there is a one parameter continuous family of smooth maps* $\alpha_t^- : T_\epsilon \to T_{[-\epsilon,\epsilon]}, t \in [0,1)$, *such that for* $x \in T_\epsilon, t \in [0,1)$
      i. *Each* α_t^- *is injective,*
      ii.
      $$\frac{\mathrm{d}\alpha_t^-(x)}{\mathrm{d}t} = \Gamma'_-(\alpha_t^-(x)),$$
      iii. $\alpha_0^-(x) = x$,
      iv. $\lim_{t \to 1} \alpha_t^-(x) \in T'_{-\epsilon} \cup B_-$ *and,*
      v. *for every* $y \in T'_{[-\epsilon,\epsilon]} \setminus B_-$, *there exists a unique* $z \in T'_\epsilon$ *and* $t \in [0,1)$ *such that* $\alpha_t(z) = y$.

Proof: Since P_+ (resp. P_-) is a polynomial in X_1, ..., X_k and X_{k+1}^2, the gradient vector field Γ_+ (resp. Γ_-) on $\mathrm{Zer}(P_+, \mathbb{R}^{k+1})$ (resp. $\mathrm{Zer}(P_-, \mathbb{R}^{k+1})$) is symmetric with respect to the reflection changing X_{k+1} to $-X_{k+1}$. Hence, we can project Γ_+ (resp. Γ_-) and its associated flowlines down to the hyperplane defined by $X_{k+1} = 0$ and get a vector field $\Gamma_+^\star$ (resp. $\Gamma_-^\star$) as well as its flowlines in S' (resp. T').

1. Apply Lemma 7.16 to $\mathrm{Zer}(P_+, \mathbb{R}^k)$ to obtain a vector field Γ'_+ on
$$\mathrm{Zer}(P_+, \mathbb{R}^{k+1})_{[-\epsilon,\epsilon]} \setminus B_+$$
coinciding with $\Gamma_+^\star$. Figure 7.8 illustrates the situation in the case $k = 3$ and $\lambda = 1$.
2. Apply Lemma 7.16 to $\mathrm{Zer}(Q_-, \mathbb{R}^k)$ to obtain a vector field Γ'_- on
$$\mathrm{Zer}(Q_-, \mathbb{R}^{k+1})_{[-\epsilon,\epsilon]} \setminus \phi^{-1}(B_-)$$
coinciding with $\Gamma_-^\star$. Figures 7.8 and 7.8 illustrate the situation in the case $k = 3$ and $\lambda = 1$. □

We are now in a position to prove Proposition 7.19.

Proof of Proposition 7.19: First, use Lemma 7.15 to reduce to the quadratic case, and then use Lemma 7.21, noting that the sign of $\partial Q / \partial X_1\}(p)$ determines which case we are in. □

7.2 Sum of the Betti Numbers of Real Algebraic Sets

For a closed semi-algebraic set S, let $\mathrm{b}(S)$ denote the sum of the Betti numbers of the simplicial homology groups of S. It follows from the definitions of Chapter 6 that $\mathrm{b}(S)$ is finite (see page 198).

According to Theorem 5.47, there are a finite number of algebraic subsets of R^k defined by polynomials of degree at most d, say $V_1, \ldots, V_p,$ such that any algebraic subset V of R^k so defined is semi-algebraically homeomorphic to one of the V_i. It follows immediately that any algebraic subset of R^k defined by polynomials of degree at most d is such that $\mathrm{b}(V) \le \max\{\mathrm{b}(V_1), \ldots, \mathrm{b}(V_p)\}$. Let $\mathrm{b}(k, d)$ be the smallest integer which bounds the sum of the Betti numbers of any algebraic set defined by polynomials of degree d in R^k. The goal of this section is to bound the Betti numbers of a bounded non-singular algebraic hypersurface in terms of the number of critical values of a function defined on it and to obtain explicit bounds for $\mathrm{b}(k, d)$.

Remark 7.22. Note that $\mathrm{b}(k,\, d) \ge d^k$ since the solutions to the system of equations,
$$(X_1 - 1)\,(X_1 - 2) \cdots (X_1 - d) = \cdots = (X_k - 1)\,(X_k - 2) \cdots (X_k - d) = 0$$

consist of d^k isolated points and the only non-zero Betti number of this set is $\mathrm{b}_0 = d^k$. (Recall that for a closed and bounded semi-algebraic set S, $\mathrm{b}_0(S)$ is the number of semi-algebraically connected components of S by Proposition 6.34.) □

We are going to prove the following theorem.

Theorem 7.23. [Oleinik-Petrovski/Thom/Milnor bound]

$$\mathrm{b}(k,d) \le d\,(2\,d-1)^{k-1}.$$

The method for proving Theorem 7.23 will be to use Theorems 7.5 and 7.14, which give enough information about the homotopy type of $\mathrm{Zer}(Q,\,\mathbb{R}^k)$ to enable us to bound $\mathrm{b}(\mathrm{Zer}(Q,\,\mathbb{R}^k))$ in terms of the number of critical points of π.

A first consequence of Theorems 7.5 and 7.14 is the following result.

Theorem 7.24. *Let $\mathrm{Zer}(Q,\mathbb{R}^k)$ be a non-singular bounded algebraic hypersurface such that the projection π on the X_1-axis is a Morse function. For $0 \le i \le k-1$, let c_i be the number of critical points of π restricted to $\mathrm{Zer}(Q,\mathbb{R}^k)$, of index i. Then,*

$$\begin{aligned}\mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k)) &\le \sum_{i=0}^{k-1} c_i,\\ \chi(\mathrm{Zer}(Q,\mathbb{R}^k)) &= \sum_{i=0}^{k-1} (-1)^{k-1-i}\, c_i.\end{aligned}$$

In particular, $\mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k))$ is bounded by the number of critical points of π restricted to $\mathrm{Zer}(Q,\mathbb{R}^k)$.

Proof: Let $v_1 < v_2 < \cdots < v_\ell$ be the critical values of π on $\mathrm{Zer}(Q,\mathbb{R}^k)$ and p_i the corresponding critical points, such that $\pi(p_i) = v_i$. Let λ_i be the index of the critical point p_i. We first prove that $\mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_i}) \le i$.

First note that $\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_1}$ is $\{p_1\}$ and hence

$$\mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_1}) = \mathrm{b}_0(\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_1}) = 1.$$

By Theorem 7.5, the set $\mathrm{Zer}(Q,\,\mathbb{R}^k)_{\le v_{i+1}-\epsilon}$ is homotopy equivalent to the set $\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_i+\epsilon}$ for any small enough $\epsilon > 0$, and thus

$$\mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_{i+1}-\epsilon}) = \mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_i+\epsilon}).$$

By Theorem 7.14, the homotopy type of $\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_i+\epsilon}$ is that of the union of $\mathrm{Zer}(Q,\mathbb{R}^k)_{\le v_i-\epsilon}$ with a topological ball. Recall from Proposition 6.44 that if S_1, S_2 are two closed semi-algebraic sets with non-empty intersection, then

$$\mathrm{b}_i(S_1 \cup S_2) \le \mathrm{b}_i(S_1) + \mathrm{b}_i(S_2) + \mathrm{b}_{i-1}(S_1 \cap S_2), 0 \le i \le k-1.$$

Recall also from Proposition 6.34 that for a closed and bounded semi-algebraic set S, $\mathrm{b}_0(S)$ equals the number of connected components of S. Since, $S_1 \cap S_2 \neq \emptyset$, for $i = 0$ we have the stronger inequality,

$$\mathrm{b}_0(S_1 \cup S_2) \leq \mathrm{b}_0(S_1) + \mathrm{b}_0(S_2) - 1.$$

By Proposition 6.37, for $\lambda > 1$ we have that

$$\begin{aligned}
\mathrm{b}_0(B_\lambda) &= \mathrm{b}_0(S^{\lambda-1}) \\
&= \mathrm{b}_{\lambda-1}(S^{\lambda-1}) \\
&= 1, \\
\mathrm{b}_i(B_\lambda) &= 0, i > 0, \\
\mathrm{b}_i(S^{\lambda-1}) &= 0, 0 < i < \lambda - 1.
\end{aligned}$$

It follows that, for $\lambda > 1$, attaching a λ-ball can increase b_λ by at most one, and none of the other Betti numbers can increase.

For $\lambda = 1$, $\mathrm{b}_{\lambda-1}(S^{\lambda-1}) = \mathrm{b}_0(S^0) = 2$. It is an exercise to show that in this case, b_1 can increase by at most one and no other Betti numbers can increase. (Hint. The number of cycles in a graph can increase by at most one on addition of an edge.)

It thus follows that

$$\mathrm{b}(\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_i + \epsilon}) \leq \mathrm{b}(\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_i - \epsilon}) + 1.$$

This proves the first part of the lemma.

We next prove that for $1 < i \leq \ell$ and small enough $\epsilon > 0$,

$$\chi(\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_i + \epsilon}) = \chi(\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_{i-1} + \epsilon}) + (-1)^{k-1-\lambda_i}.$$

By Theorem 7.5, the set $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_i - \epsilon}$ is homotopy equivalent to the set $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_{i-1} + \epsilon}$ for any small enough $\epsilon > 0$, and thus

$$\chi(\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_i - \epsilon}) = \chi(\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_{i-1} + \epsilon}).$$

By Theorem 7.14, the homotopy type of $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_i + \epsilon}$ is that of the union of $\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_i - \epsilon}$ with a topological ball of dimension $k - 1 - \lambda_i$. Recall from Corollary 6.36 (Equation 6.36) that if S_1, S_2 are two closed and bounded semi-algebraic sets with non-empty intersection, then

$$\chi(S_1 \cup S_2) = \chi(S_1) + \chi(S_2) - \chi(S_1 \cap S_2).$$

Hence,

$$\begin{aligned}
\chi(\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_i + \epsilon}) &= \chi(\mathrm{Zer}(Q, \mathbb{R}^k)_{\leq v_{i-1} + \epsilon}) \\
&= \chi(\overline{B}_{k-1-\lambda_i}) \\
&- \chi(S^{k-2-\lambda_i}).
\end{aligned}$$

Now, it follows from Proposition 6.37 and the definition of Euler-Poincaré characteristic, that $\chi(\overline{B}_{k-1-\lambda_i}) = 1$ and $\chi(S^{k-2-\lambda_i}) = 1 + (-1)^{k-2-\lambda_i}$.

Substituting in the equation above we obtain that

$$\chi(\mathrm{Zer}(Q,\mathbb{R}^k)_{\leq v_i+\epsilon}) = \chi(\mathrm{Zer}(Q,\mathbb{R}^k)_{\leq v_{i-1}+\epsilon}) + (-1)^{k-1-\lambda_i}.$$

The second part of the theorem is now an easy consequence. □

We shall need the slightly more general result.

Proposition 7.25. *Let* $\mathrm{Zer}(Q,\mathbb{R}^k)$ *be a non-singular bounded algebraic hypersurface such that the projection* π *on the* X_1*-axis has non-degenerate critical points on* $\mathrm{Zer}(Q,\mathbb{R}^k)$*. For* $0 \leq i \leq k-1$*, let* c_i *be the number of critical points of* π *restricted to* $\mathrm{Zer}(Q,\mathbb{R}^k)$*, of index* i*. Then,*

$$\begin{aligned}\mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k)) &\leq \sum_{i=0}^{k-1} c_i,\\ \chi(\mathrm{Zer}(Q,\mathbb{R}^k)) &= \sum_{i=0}^{k-1} (-1)^{k-1-i} c_i.\end{aligned}$$

In particular, $\mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k))$ *is bounded by the number of critical points of* π *restricted to* $\mathrm{Zer}(Q,\mathbb{R}^k)$*.*

Proof: Use Lemma7.13 and Theorem 7.24. □

Using Theorem 7.24, we can estimate the sum of the Betti numbers in the bounded case.

Proposition 7.26. *Let* $\mathrm{Zer}(Q,\mathbb{R}^k)$ *be a bounded non-singular algebraic hypersurface with* Q *a polynomial of degree* d*. Then*

$$\mathrm{b}(\mathrm{Zer}(Q,\mathbb{R}^k)) \leq d\,(d-1)^{k-1}.$$

Proof: Using Proposition 7.9, we can suppose that π is a Morse function. Applying Theorem 7.24 to the function $\pi\colon \mathrm{Zer}(Q,\mathbb{R}^k) \to \mathbb{R}$, it follows that the sum of the Betti numbers of $\mathrm{Zer}(Q,\mathbb{R}^k)$ is less than or equal to the number of critical points of π. Now apply Proposition 7.10. □

In order to obtain Theorem 7.23, we will need the following Proposition.

Proposition 7.27. *Let* S *be a bounded set defined by* $Q \geq 0$*, bounded by the non-singular algebraic hypersurface* $\mathrm{Zer}(Q,\mathbb{R}^k)$*. Let the projection map* π *be a Morse function on* $\mathrm{Zer}(Q,\mathbb{R}^k)$*. Then, the sum of the Betti numbers of* S *is bounded by half the number of critical points of* π *on* $\mathrm{Zer}(Q,\mathbb{R}^k)$*.*

Proof: We use the notation of the proof of Proposition 7.18. Let $v_1 < v_2 < \cdots < v_\ell$ be the critical values of π on $\mathrm{Zer}(Q,\mathbb{R}^k)$ and $p_1, \ldots, p_\ell$ the corresponding critical points, such that $\pi(p_i) = v_i$. We denote by J the subset of $\{1, \ldots, \ell\}$ such that the direction of $\mathrm{Grad}(Q)(p)$ belongs to S (see Proposition 7.18).

We are going to prove that

$$\mathrm{b}(S_{\leq v_i}) \leq \#(j \in J, j \leq i).$$

First note that $S_{\leq v_1}$ is $\{p_1\}$ and hence $\mathrm{b}(S_{\leq v_1}) = 1$. By Proposition 7.18 $S_{\leq v_{i+1}-\epsilon}$ is homotopic to $S_{\leq v_i+\epsilon}$ for any small enough $\epsilon > 0$, and thus

$$\mathrm{b}(S_{\leq v_{i+1}-\epsilon}) = \mathrm{b}(S_{\leq v_i+\epsilon}).$$

By Theorem 7.14, the homotopy type of $S_{\leq v_i+\epsilon}$ is that of $S_{\leq v_i-\epsilon}$ if $i \notin J$ and that of the union of $S_{\leq v_i-\epsilon}$ with a topological ball if $i \in J$.

It follows that

$$\begin{cases} \mathrm{b}(S_{\leq v_i+\epsilon}) = \mathrm{b}(S_{\leq v_i-\epsilon}) & \text{if } i \notin J \\ \mathrm{b}(S_{\leq v_i+\epsilon}) \leq \mathrm{b}(S_{\leq v_i-\epsilon}) + 1 & \text{if } i \in J. \end{cases}$$

By switching the direction of the X_1 axis if necessary, we can always ensure that $\#(J)$ is at most half of the critical points. □

Proposition 7.28. *If* $\mathrm{R} = \mathbb{R}$,

$$\mathrm{b}(k, d) \leq d\,(2\,d - 1)^{k-1}.$$

Proof: Let $V = \mathrm{Zer}(\{P_1, \ldots, P_\ell\}, \mathbb{R}^k)$ with the the degrees of the P_i's bounded by d. By remark on page 226, it suffices to estimate the sum of the Betti numbers of $V \cap \overline{B}_k(0, r)$. Let

$$F(X) = \frac{P_1^2 + \cdots + P_\ell^2}{r^2 - \|X\|^2}.$$

By Sard's theorem (Theorem 5.56), the set of critical values of F is finite. Hence, there is a positive $a \in \mathbb{R}$ so that no $b \in (0, a)$ is a critical value of F and thus the set $W_b = \{x \in \mathbb{R}^k \mid P(x, b) = 0\}$, where

$$P(X, b) = P_1^2 + \cdots + P_\ell^2 + b\,(\|X\|^2 - r^2))$$

is a non-singular hypersurface in $\mathbb{R}^k$. To see this observe that, for $x \in \mathbb{R}^k$

$$P(x, b) = \partial P / \partial X_1(x, b) = \cdots = \partial P / \partial X_k(x, b) = 0$$

implies that $F(x) = b$ and $\partial F / \partial X_1(x) = \cdots = \partial F / \partial X_k(x) = 0$ implying that b is a critical value of F which is a contradiction.

Moreover, W_b is the boundary of the closed and bounded set

$$K_b = \{x \in \mathbb{R}^k \mid P(x, b) \leq 0\}.$$

By Proposition 7.26, the sum of the Betti numbers of W_b is less than or equal to $2\,d\,(2\,d - 1)^{k-1}$.

Also, using Proposition 7.27, the sum of the Betti numbers of K_b is at most half that of W_b.

We now claim that $V \cap \overline{B}_k(0,r)$ is homotopy equivalent to K_b for all small enough $b > 0$. We replace b in the definition of the set K_b by a new variable T, and consider the set $K \subset \mathbb{R}^{k+1}$ defined by $\{(x,t) \in \mathbb{R}^{k+1} | P(x,t) \leq 0\}$. Let π_X (resp. π_T) denote the projection map onto the X (resp. T) coordinates.

Clearly, $V \cap \overline{B}_k(0, r) \subset K_b$. By Theorem 5.46 (Semi-algebraic triviality), for all small enough $b > 0$, there exists a semi-algebraic homeomorphism,

$$\phi: K_b \times (0,b] \rightarrow K \cap \pi_T^{-1}((0,b]),$$

such that $\pi_T(\phi(x,\ s)) = s$ and ϕ is a semi-algebraic homeomorphism from $V \cap B_k(0,r) \times (0,b]$ to itself.

Let G: $K_b \times [0,\ b] \rightarrow K_b$ be the map defined by $G(x,\ s) = \pi_X(\phi(x,\ s))$ for $s > 0$ and $G(x,\ 0) = \lim_{s \rightarrow 0+} \pi_X(\phi(x,\ s))$. Let g: $K_b \rightarrow V \cap \overline{B}_k(0,\ r)$ be the map $G(x,0)$ and i: $V \cap \overline{B}_k(0,\ r) \rightarrow K_b$ the inclusion map. Using the homotopy G, we see that $i \circ g \sim \mathrm{Id}_{K_b}$, and $g \circ i \sim \mathrm{Id}_{V \cap B_k(0,r)}$, which shows that $V \cap B_k(0,r)$ is homotopy equivalent to K_b as claimed.

Hence,

$$\mathrm{b}(V \cap \overline{B}_k(0,r)) = \mathrm{b}(K_b) \leq 1/2\, \mathrm{b}(W_b) \leq d\,(2\,d-1)^{k-1}.$$ □

Proof of Theorem 7.23: It only remains to prove that Proposition 7.28 is valid for any real closed field R. We first work over the field of real algebraic numbers $\mathbb{R}_{\mathrm{alg}}$. We identify a system of ℓ polynomials $(P_1, \ldots, P_\ell)$ in k variables of degree less than or equal to d with the point of $\mathbb{R}_{\mathrm{alg}}^N$, $N = \ell\binom{k+d-1}{d}$, whose coordinates are the coefficients of $P_1, \ldots, P_\ell$. Let

$$Z = \{(P_1, \ldots, P_\ell, x) \in \mathbb{R}_{\mathrm{alg}}^N \times \mathbb{R}_{\mathrm{alg}}^k \mid P_1(x) = \cdots = P_\ell(x) = 0\},$$

and let Π: $Z \rightarrow \mathbb{R}_{\mathrm{alg}}^N$ be the canonical projection. By Theorem 5.46 (Semi-algebraic Triviality), there exists a finite partition of $\mathbb{R}_{\mathrm{alg}}^N$ into semi-algebraic sets $A_1, \ldots, A_m$, semi-algebraic sets $F_1, \ldots, F_m$ contained in $\mathbb{R}_{\mathrm{alg}}^k$, and semi-algebraic homeomorphisms θ_i: $\Pi^{-1}(A_i) \rightarrow A_i \times F_i$, for $i = 1, \ldots, m$, such that the composition of θ_i with the projection $A_i \times F_i \rightarrow A_i$ is $\Pi|_{\Pi^{-1}(A_i)}$. The F_i are algebraic subsets of $\mathbb{R}_{\mathrm{alg}}^k$ defined by ℓ equations of degree less than or equal to d. The sum of the Betti numbers of $\mathrm{Ext}(F_i, \mathbb{R})$ is less than or equal to $d\,(2\,d-1)^{k-1}$. So, by invariance of the homology groups under extension of real closed field (Section 6.2), the same bound holds for the sum of the Betti numbers of F_i. Now, let $V \subset \mathrm{R}^k$ be defined by k equations $P_1 = \cdots = P_\ell = 0$ of degree less than or equal to d with coefficients in R. We have

$$\mathrm{Ext}(\Pi^{-1}, \mathrm{R})(P_1, \ldots, P_\ell) = \{(P_1, \ldots, P_\ell)\} \times V.$$

The point $(P_1, \ldots, P_\ell) \in \mathrm{R}^N$ belongs to some $\mathrm{Ext}(A_i, \mathrm{R})$, and the semi-algebraic homeomorphism $\mathrm{Ext}(\theta_i, \mathrm{R})$ induces a semi-algebraic homeomorphism from V onto $\mathrm{Ext}(F_i,\ \mathrm{R})$. Again, the sum of the Betti numbers of $\mathrm{Ext}(F_i,\ \mathrm{R})$ is less than or equal to $d\,(2\,d-1)^{k-1}$, and the same bound holds for the sum of the Betti numbers of V. □

7.3 Bounding the Betti Numbers of Realizations of Sign Conditions

Throughout this section, let $\mathcal{Q}$ and $\mathcal{P} \neq \emptyset$ be finite subsets of $\mathrm{R}[X_1, \ldots, X_k]$, let $Z = \mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$, and let k' be the dimension of $Z = \mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$.

Notation 7.29. [Realizable sign conditions] We denote by

$$\mathrm{SIGN}(\mathcal{P}) \subset \{0, 1, -1\}^{\mathcal{P}}$$

the set of all realizable sign conditions for $\mathcal{P}$ over R^k, and by

$$\mathrm{SIGN}(\mathcal{P}, \mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{P}}$$

the set of all realizable sign conditions for $\mathcal{P}$ over $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$. □

For $\sigma \in \mathrm{SIGN}(\mathcal{P}, \mathcal{Q})$, let $\mathrm{b}_i(\sigma)$ denote the i-th Betti number of

$$\mathrm{Reali}(\sigma, Z) = \{x \in \mathrm{R}^k \mid \bigwedge_{Q \in \mathcal{Q}} Q(x) = 0, \bigwedge_{P \in \mathcal{P}} \mathrm{sign}(P(x)) = \sigma(P)\}.$$

Let $\mathrm{b}_i(\mathcal{Q}, \mathcal{P}) = \sum_\sigma \mathrm{b}_i(\sigma)$. Note that $\mathrm{b}_0(\mathcal{Q}, \mathcal{P})$ is the number of semi-algebraically connected components of basic semi-algebraic sets defined by $\mathcal{P}$ over $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$.

We denote by $\deg(\mathcal{Q})$ the maximum of the degrees of the polynomials in $\mathcal{Q}$ and write $\mathrm{b}_i(d, k, k', s)$ for the maximum of $\mathrm{b}_i(\mathcal{Q}, \mathcal{P})$ over all $\mathcal{Q}, \mathcal{P}$, where $\mathcal{Q}$ and $\mathcal{P}$ are finite subsets of $\mathrm{R}[X_1, \ldots, X_k]$, $\deg(\mathcal{Q}, \mathcal{P}) \leqslant d$ whose elements have degree at most d, $\#(\mathcal{P}) = s$ (i.e. $\mathcal{P}$ has s elements), and the algebraic set $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$ has dimension k'.

Theorem 7.30.

$$\mathrm{b}_i(d, k, k', s) \leq \sum_{1 \leq j \leq k' - i} \binom{s}{j} 4^j d (2d - 1)^{k-1}.$$

So we get, in particular a bound on the total number of semi-algebraically connected components of realizable sign conditions.

Proposition 7.31.

$$\mathrm{b}_0(d, k, k', s) \leq \sum_{1 \leq j \leq k'} \binom{s}{j} 4^j d (2d - 1)^{k-1}.$$

Remark 7.32. When $d = 1$, i.e. when all equations are linear, it is easy to find directly a bound on the number of non-empty sign conditions. The number of non-empty sign conditions $f(k', s)$ defined by s linear equations on a flat of dimension k' satisfies the recurrence relation

$$f(k', s + 1) \leq f(k', s) + 2 f(k' - 1, s),$$

since a flat L of dimension $k'-1$ meets at most $f(k'-1,s)$ non-empty sign condition defined by s polynomials on a flat of dimension k', and each such non-empty sign condition is divided in at most three pieces by L.

In Figure 7.9 we depict the situation with four lines in R^2 defined by four linear polynomials. The number of realizable sign conditions in this case is easily seen to be 33.

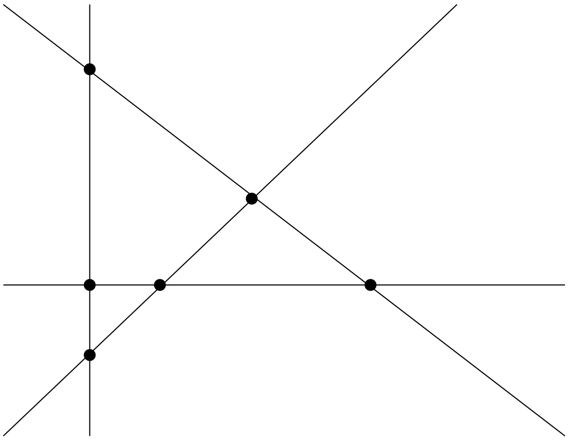

Fig. 7.9. Four lines in R^2

Moreover, when the linear equations are in general position,

$$f(k', s+1) = f(k', s) + 2\, f(k'-1, s). \tag{7.4}$$

Since $f(k', 0) = 1$, the solution to Equation (7.4) is given by

$$f(k', s) = \sum_{i=0}^{k'} \sum_{j=0}^{k'-i} \binom{s}{i} \binom{s-i}{j}. \tag{7.5}$$

Since all the realizations are convex and hence contractible, this bound on the number of non-empty sign conditions is also a bound on

$$\mathrm{b}_0(1, k, k', s) = \mathrm{b}(1, k', k', s)$$

We note that

$$f(k', s) \le \sum_{1 \le j \le k'} \binom{s}{j} 4^j,$$

the right hand side being the bound appearing in Proposition 7.31 with $d=1$. □

The following proposition, Proposition 7.33, plays a key role in the proofs of these theorems. Part (a) of the proposition bounds the Betti numbers of a union of s semi-algebraic sets in R^k in terms of the Betti numbers of the intersections of the sets taken at most k at a time.

Part (b) of the proposition is a dual version of Part (a) with unions being replaced by intersections and vice-versa, with an additional complication arising from the fact that the empty intersection, corresponding to the base case of the induction, is an arbitrary real algebraic variety of dimension k', and is generally not acyclic.

Let $S_1, \ldots, S_s \subset \mathrm{R}^k$, $s \geq 1$, be closed semi-algebraic sets contained in a closed semi-algebraic set T of dimension k'. For $1 \leq t \leq s$, let $S_{\leq t} = \bigcap_{1 \leq j \leq t} S_j$, and $S^{\leq t} = \bigcup_{1 \leq j \leq t} S_j$. Also, for $J \subset \{1, \ldots, s\}$, $J \neq \emptyset$, let $S_J = \bigcap_{j \in J} S_j$, and $S^J = \bigcup_{j \in J} S_j$. Finally, let $S^{\emptyset} = T$.

Proposition 7.33.

a) For $0 \leq i \leq k'$,

$$\mathrm{b}_i(S^{\leq s}) \leq \sum_{j=1}^{i+1} \sum_{\substack{J \subset \{1, \ldots, s\} \\ \#(J) = j}} \mathrm{b}_{i-j+1}(S_J). \tag{7.6}$$

b) For $0 \leq i \leq k'$,

$$\mathrm{b}_i(S_{\leq s}) \leq \sum_{j=1}^{k'-i} \sum_{\substack{J \subset \{1, \ldots, s\} \\ \#(J) = j}} \mathrm{b}_{i+j-1}(S^J) + \binom{s}{k'-i} \mathrm{b}_{k'}(S^{\emptyset}). \tag{7.7}$$

Proof : a)We prove the claim by induction on s. The statement is clearly true for $s = 1$, since $\mathrm{b}_i(S_1)$ appears on the right hand side for $j = 1$ and $J = \{1\}$.

Using Proposition 6.44 (6.44), we have that

$$\mathrm{b}_i(S^{\leq s}) \leq \mathrm{b}_i(S^{\leq s-1}) + \mathrm{b}_i(S_s) + \mathrm{b}_{i-1}(S^{\leq s-1} \cap S_s). \tag{7.8}$$

Applying the induction hypothesis to the set $S^{\leq s-1}$, we deduce that

$$\mathrm{b}_i(S^{\leq s-1}) \leq \sum_{j=1}^{i+1} \sum_{\substack{J \subset \{1, \ldots, s-1\} \\ \#(J) = j}} \mathrm{b}_{i-j+1}(S_J). \tag{7.9}$$

Next, we apply the induction hypothesis to the set

$$S^{\leq s-1} \cap S_s = \bigcup_{1 \leq j \leq s-1} (S_j \cap S_s)$$

to get that

$$\mathrm{b}_{i-1}(S^{\leq s-1} \cap S_s) \leq \sum_{j=1}^{i} \sum_{\substack{J \subset \{1, \ldots, s-1\} \\ \#(J) = j}} \mathrm{b}_{i-j}(S_{J \cup \{s\}}). \tag{7.10}$$

Adding the inequalities (7.9) and (7.10), we get

$$\mathrm{b}_i(S^{\leq s-1}) + \mathrm{b}_i(S_s) + \mathrm{b}_{i-1}(S^{\leq s-1} \cap S_s) \leq \sum_{j=1}^{i+1} \sum_{\substack{J \subset \{1, \ldots, s\} \\ \#(J) = j}} \mathrm{b}_{i-j+1}(S_J).$$

We conclude using (7.8).

b) We first prove the claim when $s=1$. If $0\le i\le k'-1$, the claim is

$$\mathrm{b}_i(S_1)\le \mathrm{b}_{k'}(S^{\emptyset})+\big(\mathrm{b}_i(S_1)+\mathrm{b}_{k'}(S^{\emptyset})\big),$$

which is clear. If $i=k'$, the claim is $\mathrm{b}_{k'}(S_1)\le \mathrm{b}_{k'}(S^{\emptyset})$. If the dimension of S_1 is k', consider the closure V of the complement of S_1 in T. The intersection W of V with S_1, which is the boundary of S_1, has dimension strictly smaller than k' by Theorem 5.42 thus $\mathrm{b}_{k'}(W)=0$. Using Proposition 6.44

$$\mathrm{b}_{k'}(S_1)+\mathrm{b}_{k'}(V)\le \mathrm{b}_{k'}(S^{\emptyset})+\mathrm{b}_{k'}(W),$$

and the claim follows. On the other hand, if the dimension of S_1 is strictly smaller than k', $\mathrm{b}_{k'}(S_1)$=0.

The claim is now proved by induction on s. Assume that the induction hypothesis (7.7) holds for $s-1$ and for all $0\le i\le k'$.

From Proposition 6.44 (6.44), we have

$$\mathrm{b}_i(S_{\le s})\le \mathrm{b}_i(S_{\le s-1})+\mathrm{b}_i(S_s)+\mathrm{b}_{i+1}(S_{\le s-1}\cup S_s). \tag{7.11}$$

Applying the induction hypothesis to the set $S_{\le s-1}$, we deduce that

$$\begin{aligned}\mathrm{b}_i(S_{\le s-1}) \;&\le\; \sum_{j=1}^{k'-i}\sum_{\substack{J\subset\{1,\ldots,s-1\}\\ \#(J)=j}}\mathrm{b}_{i+j-1}(S^J)\\ &\quad+\binom{s-1}{k'-i}\mathrm{b}_{k'}(S^{\emptyset}).\end{aligned}$$

Next, applying the induction hypothesis to the set,

$$S_{\le s-1}\cup S_s=\bigcap_{1\le j\le s-1}(S_j\cup S_s),$$

we get that

$$\begin{aligned}\mathrm{b}_{i+1}(S_{\le s-1}\cup S_s) \;&\le\; \sum_{j=1}^{k'-i-1}\sum_{\substack{J\subset\{1,\ldots,s-1\}\\ \#(J)=j}}\mathrm{b}_{i+j}(S^{J\cup\{s\}})\\ &\quad+\binom{s-1}{k'-i-1}\mathrm{b}_{k'}(S^{\emptyset}).\end{aligned} \tag{7.12}$$

Adding the inequalities (7.11) and (7.11), we get

$$\mathrm{b}_i(S_{\le s})\le \sum_{j=1}^{k'-i}\sum_{\substack{J\subset\{1,\ldots,s\}\\ \#(J)=j}}\mathrm{b}_{i+j-1}(S^J)+\binom{s}{k'-i}\mathrm{b}_{k'}(S^{\emptyset}).$$

We conclude using (7.11). □

Let $\mathcal{P}=\{P_1, \ldots, P_s\}$, and let δ be a new variable. We will consider the field $\mathrm{R}\langle\delta\rangle$ of algebraic Puiseux series in δ, in which δ is an infinitesimal.

Let $S_i = \mathrm{Reali}\big(P_i^2(P_i^2 - \delta^2) \geq 0, \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)\big)$, $1 \leq i \leq s$, and let S be the intersection of the S_i with the closed ball in $\mathrm{R}\langle\delta\rangle^k$ defined by

$$\delta^2\left(\sum_{1 \leq i \leq k} X_i^2\right) \leq 1.$$

In order to estimate $\mathrm{b}_i(S)$, we prove that $\mathrm{b}_i(\mathcal{P}, \mathcal{Q})$ and $\mathrm{b}_i(S)$ are equal and we estimate $\mathrm{b}_i(S)$.

Proposition 7.34.

$$\mathrm{b}_i(\mathcal{P}, \mathcal{Q}) = \mathrm{b}_i(S).$$

Proof: Consider a sign condition σ on $\mathcal{P}$ such that, without loss of generality,

$$\begin{aligned} \sigma(P_i) &= 0 \quad \text{if} \, i \in I \\ \sigma(P_j) &= 1 \quad \text{if} \, j \in J \\ \sigma(P_\ell) &= -1 \quad \text{if} \, \ell \in \{1, \ldots, s\} \setminus (I \cup J), \end{aligned}$$

and denote by $\overline{\mathrm{Reali}}(\sigma)$ the subset of $\mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)$ defined by

$$\delta^2\left(\sum_{1 \leq i \leq k} X_i^2\right) \leq 1, P_i = 0, i \in I,$$

$$P_j \geq \delta, j \in J, P_\ell \leq -\delta, \ell \in \{1, \ldots, s\} \setminus (I \cup J).$$

Note that S is the disjoint union of the $\overline{\mathrm{Reali}}(\sigma)$ for all realizable sign conditions σ.

Moreover, by definition of the homology groups of sign conditions (Notation 6.46) $\mathrm{b}_i(\sigma) = \mathrm{b}_i(\overline{\mathrm{Reali}}(\sigma))$, so that

$$\mathrm{b}_i(\mathcal{P}, \mathcal{Q}) = \sum_\sigma \mathrm{b}_i(\sigma) = \mathrm{b}_i(S).$$ □

Proposition 7.35.

$$\mathrm{b}_i(S) \leq \sum_{j=1}^{k'-i} \binom{s}{j} 4^j d\,(2d - 1)^{k-1}.$$

Before estimating $\mathrm{b}_i(S)$, we estimate the Betti numbers of the following sets.

Let $j \geq 1$,

$$V_j = \mathrm{Reali}\left(\bigvee_{1 \leq i \leq j} P_i^2(P_i^2 - \delta^2) = 0, \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)\right),$$

and

$$W_j = \mathrm{Reali}\left(\bigvee_{1 \leq i \leq j} P_i^2(P_i^2 - \delta^2) \geq 0, \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)\right).$$

Note that W_j is the union of $S_1, \ldots, S_j$.

Lemma 7.36.

$$\mathrm{b}_i(V_j) \leq (4^j - 1)\, d\, (2\, d - 1)^{k-1}.$$

Proof: Each of the sets

$$\mathrm{Reali}\big(P_i^2(P_i^2 - \delta^2)) = 0, \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)\big)$$

is the disjoint union of three algebraic sets, namely

$$\mathrm{Reali}(P_i = 0, \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)),$$

$$\mathrm{Reali}(P_i = \delta, \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)),$$

$$\mathrm{Reali}(P_i = -\delta, \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)).$$

Moreover, each Betti number of their union is bounded by the sum of the Betti numbers of all possible non-empty sets that can be obtained by taking, for $1 \leq \ell \leq j$, ℓ-ary intersections of these algebraic sets, using part (a) of Proposition 7.33. The number of possible ℓ-ary intersection is $\binom{j}{\ell}$. Each such intersection is a disjoint union of 3^ℓ algebraic sets. The sum of the Betti numbers of each of these algebraic sets is bounded by $d\,(2\,d - 1)^{k-1}$ by using Theorem 7.23.

Thus,

$$\mathrm{b}_i(V_j) \leq \sum_{\ell=1}^{j} \binom{j}{\ell} 3^\ell\, d\, (2\, d - 1)^{k-1} = (4^j - 1)\, d\, (2\, d - 1)^{k-1}. \qquad \square$$

Lemma 7.37.

$$\mathrm{b}_i(W_j) \leq (4^j - 1)\, d\, (2\, d - 1)^{k-1} + \mathrm{b}_i(Z).$$

Proof: Let $Q_i = P_i^2(P_i^2 - \delta^2)$ and

$$F = \mathrm{Reali}\left(\bigwedge_{1 \leq i \leq j} (Q_i \leq 0) \vee \bigvee_{1 \leq i \leq j} (Q_i = 0), \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle)\right).$$

Apply inequality (6.44), noting that

$$W_j \cup F = \mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle),\ W_j \cap F = W_{j,0}.$$

Since $\mathrm{b}_i(Z) = \mathrm{b}_i(\mathrm{Ext}(Z, \mathrm{R}\langle\delta\rangle))$, we get that

$$\mathrm{b}_i(W_j) \leq \mathrm{b}_i(W_j \cap F) + \mathrm{b}_i(W_j \cup F) = \mathrm{b}_i(V_j) + \mathrm{b}_i(Z).$$

We conclude using Lemma 7.36. $\square$

Proof of Proposition 7.35: Using part b) of Proposition 7.33 and Lemma 7.37, we get

$$\begin{aligned}\mathrm{b}_i(S) &\leq \sum_{j=1}^{k'-i} \binom{s}{j}((4^j-1)\,d\,(2\,d-1)^{k-1}+\mathrm{b}_i(Z))\\ &+ \binom{s}{k'-i}\mathrm{b}_{k'}(Z).\end{aligned}$$

By Theorem 7.23, for all $i<k'$,

$$\mathrm{b}_i(Z)+\mathrm{b}_{k'}(Z)\leq d\,(2\,d-1)^{k-1}.$$

Thus, we have

$$\mathrm{b}_i(S)\leq \sum_{j=1}^{k'-i}\binom{s}{j}4^j\,d\,(2\,d-1)^{k-1}.$$ □

Theorem 7.30 follows clearly from Proposition 7.34 and Proposition 7.35.

7.4 Sum of the Betti Numbers of Closed Semi-algebraic Sets

Let $\mathcal{P}$ and $\mathcal{Q}$ be finite subsets of $\mathrm{R}[X_1,\ldots,X_k]$.

A $(\mathcal{Q},\mathcal{P})$-**closed formula** is a formula constructed as follows:

- For each $P\in\mathcal{P}$,

$$\bigwedge_{Q\in\mathcal{Q}} Q=0\wedge P=0,\ \bigwedge_{Q\in\mathcal{Q}} Q=0\wedge P\geq 0,\ \bigwedge_{Q\in\mathcal{Q}} Q=0\wedge P\leq 0,$$

- If Φ_1 and Φ_2 are $(\mathcal{Q},\mathcal{P})$-closed formulas, $\Phi_1\wedge\Phi_2$ and $\Phi_1\vee\Phi_2$ are $(\mathcal{Q},\mathcal{P})$-closed formulas.

Clearly, $\mathrm{Reali}(\Phi)$, the realization of a $(\mathcal{Q},\mathcal{P})$-closed formula Φ, is a closed semi-algebraic set. We denote by $\mathrm{b}(\Phi)$ the sum of its Betti numbers.

We write $\overline{\mathrm{b}}(d,k,k',s)$ for the maximum of $\mathrm{b}(\Phi)$, where Φ is a $(\mathcal{Q},\mathcal{P})$-closed formula, $\mathcal{Q}$ and $\mathcal{P}$ are finite subsets of $\mathrm{R}[X_1,\ldots,X_k]$ $\deg(\mathcal{Q},\mathcal{P})\leqslant d$, $\#(\mathcal{P})=s$, and the algebraic set $\mathrm{Zer}(\mathcal{Q},\mathrm{R}^k)$ has dimension k'.

Our aim in this section is to prove the following result.

Theorem 7.38.

$$\overline{\mathrm{b}}(d,k,k',s)\leq\sum_{i=0}^{k'}\sum_{j=1}^{k'-i}\binom{s}{j}6^j\,d\,(2\,d-1)^{k-1}.$$

For the proof of Theorem 7.38, we are going to introduce several infinitesimal quantities. Given a list of polynomials $\mathcal{P}=\{P_1,\ldots,P_s\}$ with coefficients in R, we introduce s new variables $\delta_1,\cdots,\delta_s$ and inductively define

$$\mathrm{R}\langle\delta_1,\ldots,\delta_{i+1}\rangle=\mathrm{R}\langle\delta_1,\ldots,\delta_i\rangle\langle\delta_{i+1}\rangle.$$

Note that δ_{i+1} is infinitesimal with respect to δ_i, which is denoted by

$$\delta_1 \gg \ldots \gg \delta_s.$$

We define $\mathcal{P}_{>i} = \{P_{i+1}, \ldots, P_s\}$ and

$$\begin{aligned}\Sigma_i &= \{P_i = 0, P_i = \delta_i, P_i = -\delta_i, P_i \geq 2\,\delta_i, P_i \leq -2\,\delta_i\},\\ \Sigma_{\leq i} &= \{\Psi \mid \Psi = \bigwedge_{j=1,\ldots,i} \Psi_i, \Psi_i \in \Sigma_i\}.\end{aligned}$$

If Φ is a $(\mathcal{Q}, \mathcal{P})$-closed formula, we denote by $\mathrm{Reali}_i(\Phi)$ the extension of $\mathrm{Reali}(\Phi)$ to $\mathrm{R}\langle\delta_1, \ldots, \delta_i\rangle^k$. For $\Psi \in \Sigma_{\leq i}$, we denote by $\mathrm{Reali}_i(\Phi \wedge \Psi)$ the intersection of the realization of Ψ with $\mathrm{Reali}_i(\Phi)$ and by $\mathrm{b}(\Phi \wedge \Psi)$ the sum of the Betti numbers of $\mathrm{Reali}_i(\Phi \wedge \Psi)$.

Proposition 7.39. *For every $(\mathcal{Q}, \mathcal{P})$-closed formula Φ,*

$$\mathrm{b}(\Phi) \leq \sum_{\substack{\Psi \in \Sigma_{\leq s}\\ \mathrm{Reali}_s(\Psi) \subset \mathrm{Reali}_s(\Phi)}} \mathrm{b}(\Psi).$$

The main ingredient of the proof of the proposition is the following lemma.

Lemma 7.40. *For every $(\mathcal{Q}, \mathcal{P})$-closed formula Φ and every $\Psi \in \Sigma_{\leq i}$,*

$$\mathrm{b}(\Phi \wedge \Psi) \leq \sum_{\psi \in \Sigma_{i+1}} \mathrm{b}(\Phi \wedge \Psi \wedge \psi).$$

Proof: Consider the formulas

$$\begin{aligned}\Phi_1 &= \Phi \wedge \Psi \wedge (P_{i+1}^2 - \delta_{i+1}^2) \geq 0,\\ \Phi_2 &= \Phi \wedge \Psi \wedge (0 \leq P_{i+1}^2 \leq \delta_{i+1}^2).\end{aligned}$$

Clearly, $\mathrm{Reali}_{i+1}(\Phi \wedge \Psi) = \mathrm{Reali}_{i+1}(\Phi_1 \vee \Phi_2)$. Using Proposition 6.44, we have that,

$$\mathrm{b}(\Phi \wedge \Psi) \leq \mathrm{b}(\Phi_1) + \mathrm{b}(\Phi_2) + \mathrm{b}(\Phi_1 \wedge \Phi_2).$$

Now, since $\mathrm{Reali}_{i+1}(\Phi_1 \wedge \Phi_2)$ is the disjoint union of

$$\mathrm{Reali}_{i+1}(\Phi \wedge \Psi \wedge (P_{i+1} = \delta_{i+1})),\ \mathrm{Reali}_{i+1}(\Phi \wedge \Psi \wedge (P_{i+1} = -\delta_{i+1})),$$

$$\mathrm{b}(\Phi_1 \wedge \Phi_2) = \mathrm{b}(\Phi \wedge \Psi \wedge (P_{i+1} = \delta_{i+1})) + \mathrm{b}(\Phi \wedge \Psi \wedge (P_{i+1} = -\delta_{i+1})).$$

Moreover,

$$\begin{aligned}\mathrm{b}(\Phi_1) &= \mathrm{b}(\Phi \wedge \Psi \wedge (P_{i+1}\\ &\quad \geq 2\,\delta_{i+1})) + \mathrm{b}(\Phi \wedge \Psi \wedge (P_{i+1} \leq -2\,\delta_{i+1})),\\ \mathrm{b}(\Phi_2) &= \mathrm{b}(\Phi \wedge \Psi \wedge (P_{i+1} = 0)).\end{aligned}$$

Indeed, by Theorem 5.46 (Hardt's triviality), denoting

$$F_t = \{x \in \mathrm{Reali}_i(\Phi \wedge \Psi) \mid P_{i+1}(x) = t\},$$

there exists $t_0 \in \mathrm{R}\langle\delta_1, \ldots, \delta_i\rangle$ such that

$$F_{[-t_0,0)\cup(0,t_0]} = \{x \in \mathrm{Reali}_i(\Phi \wedge \Psi) \mid t_0^2 \geq P_{i+1}(x) > 0\}$$

and

$$([-t_0, 0) \times F_{-t_0}) \cup ((0, t_0] \times F_{t_0})$$

are homeomorphic. This implies clearly that

$$F_{[\delta_{i+1},t_0]} = \{x \in \mathrm{Reali}_{i+1}(\Phi \wedge \Psi) \mid t_0 \geq P_{i+1}(x) \geq \delta_{i+1}\}$$

and

$$F_{[2\delta_{i+1},t_0]} = \{x \in \mathrm{Reali}_{i+1}(\Phi \wedge \Psi) \mid t_0 \geq P_{i+1}(x) \geq 2\,\delta_{i+1}\}$$

are homeomorphic, and moreover the homeomorphism can be chosen such that it is the identity on the fibers F_{-t_0} and F_{t_0}.

Hence,

$$\mathrm{b}(\Phi_1) = \mathrm{b}(\Phi \wedge \Psi \wedge (P_{i+1} \geq 2\,\delta_{i+1})) + \mathrm{b}(\Phi \wedge \Psi \wedge (P_{i+1} \leq -2\delta_{i+1})).$$

Note that $F_0 = \mathrm{Reali}_{i+1}(\Phi \wedge \Psi \wedge (P_{i+1} = 0))$ and $F_{[-\delta_{i+1},\delta_{i+1}]} = \mathrm{Reali}_{i+1}(\Phi_2)$.

Thus, it remains to prove that $\mathrm{b}(F_{[-\delta_{i+1},\delta_{i+1}]}) = \mathrm{b}(F_0)$. By Theorem 5.46 (Hardt's triviality), for every $0 < u < 1$, there is a fiber preserving semi-algebraic homeomorphism

$$\phi_u\colon F_{[-\delta_{i+1},-u\delta_{i+1}]} \to [-\delta_{i+1}, -u\delta_{i+1}] \times F_{-u\delta_{i+1}}$$

and a semi-algebraic homeomorphism

$$\psi_u\colon F_{[u\delta_{i+1},\delta_{i+1}]} \to [u\,\delta_{i+1}, \delta_{i+1}] \times F_{u\delta_{i+1}}.$$

We define a continuous semi-algebraic homotopy g from the identity of $F_{[-\delta_{i+1},\delta_{i+1}]}$ to $\lim_{\delta_{i+1}}$ (from $F_{[-\delta_{i+1},\delta_{i+1}]}$ to F_0) as follows:

- $g(0, -)$ is $\lim_{\delta_{i+1}}$,
- for $0 < u \leq 1$, $g(u, -)$ is the identity on $F_{[-u\delta_{i+1},u\delta_{i+1}]}$ and sends $F_{[-\delta_{i+1},-u\delta_{i+1}]}$ (resp. $F_{[u\delta_{i+1},\delta_{i+1}]}$) to $F_{-u\delta_{i+1}}$ (resp. $F_{u\delta_{i+1}}$) by ϕ_u (resp. ψ_u) followed by the projection to $F_{u\delta_{i+1}}$ (resp. $F_{-u\delta_{i+1}}$).

Thus,

$$\mathrm{b}(F_{[-\delta_{i+1},\delta_{i+1}]}) = \mathrm{b}(F_0).$$

Finally,

$$\mathrm{b}(\Phi \wedge \Psi) \leq \sum_{\psi \in \Sigma_{i+1}} \mathrm{b}(\Phi \wedge \Psi \wedge \psi). \qquad \square$$

Proof of Proposition 7.39: Starting from the formula Φ, apply Lemma 7.40 with Ψ the empty formula. Now, repeatedly apply Lemma 7.40 to the terms appearing on the right-hand side of the inequality obtained, noting that for any $\Psi \in \Sigma_{\leq s}$,

- either $\mathrm{Reali}_s(\Phi \wedge \Psi) = \mathrm{Reali}_s(\Psi)$ and $\mathrm{Reali}_s(\Psi) \subset \mathrm{Reali}_s(\Phi)$,
- or $\mathrm{Reali}_s(\Phi \wedge \Psi) = \emptyset$. □

Using an argument analogous to that used in the proof of Theorem 7.30, we prove the following proposition.

Proposition 7.41. *For* $0 \le i \le k'$,

$$\sum_{\Psi \in \Sigma_{\le s}} \mathrm{b}_i(\Psi) \le \sum_{j=1}^{k'-i} \binom{s}{j} 6^j\, d\, (2\, d - 1)^{k-1}.$$

We first prove the following Lemma 7.42 and Lemma 7.43.

Let $\mathcal{P} = \{P_1, \ldots, P_j\} \subset R[X_1, \ldots, X_k]$, and let $Q_i = P_i^2 (P_i^2 - \delta_i^2)^2 (P_i^2 - 4\, \delta_i^2)$. Let $j \ge 1$,

$$\begin{aligned} V_j' &= \mathrm{Reali}\left(\bigvee_{1 \le i \le j} Q_i = 0, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle) \right), \\ W_j' &= \mathrm{Reali}\left(\bigvee_{1 \le i \le j} Q_i \ge 0, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle) \right). \end{aligned}$$

Lemma 7.42.

$$\mathrm{b}_i(V_j') \le (6^j - 1)\, d\, (2\, d - 1)^{k-1}.$$

Proof: The set $\mathrm{Reali}((P_i^2 (P_i^2 - \delta_i^2)^2 (P_i^2 - 4\, \delta_i^2) = 0), Z)$ is the disjoint union of

$$\begin{gathered} \mathrm{Reali}(P_i = 0, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle)), \\ \mathrm{Reali}(P_i = \delta_i, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle)), \\ \mathrm{Reali}(P_i = -\delta_i, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle)), \\ \mathrm{Reali}(P_i = 2\, \delta_i, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle)), \\ \mathrm{Reali}(P_i = -2\, \delta_i, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle)). \end{gathered}$$

Moreover, the i-th Betti number of their union V_j' is bounded by the sum of the Betti numbers of all possible non-empty sets that can be obtained by taking intersections of these sets using part (a) of Proposition 7.33.

The number of possible ℓ-ary intersection is $\binom{j}{\ell}$. Each such intersection is a disjoint union of 5^ℓ algebraic sets. The i-th Betti number of each of these algebraic sets is bounded by $d\, (2\, d - 1)^{k-1}$ by Theorem 7.23.

Thus,

$$\mathrm{b}_i(V_j') \le \sum_{\ell=1}^{j} \binom{j}{\ell} 5^\ell\, d\, (2\, d - 1)^{k-1} = (6^j - 1)\, d\, (2\, d - 1)^{k-1}.$$ □

Lemma 7.43.

$$\mathrm{b}_i(W_j') \leq (6^j - 1)\, d\, (2\, d - 1)^{k-1} + \mathrm{b}_i(Z).$$

Proof: Let

$$F = \mathrm{Reali}\left(\bigwedge_{1 \leq i \leq j} Q_i \leq 0 \vee \bigvee_{1 \leq i \leq j} Q_i = 0, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_i \rangle) \right).$$

Now,

$$W_j' \cup F = Z, W_j' \cap F = V_j'.$$

Using inequality (6.44) we get that

$$\mathrm{b}_i(W_j') \leq \mathrm{b}_i(W_j' \cap F) + \mathrm{b}_i(W_j' \cup F) = \mathrm{b}_i(V_j') + \mathrm{b}_i(Z)$$

since $\mathrm{b}_i(Z) = \mathrm{b}_i(\mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_i \rangle))$. We conclude using Lemma 7.42. □

Now, let

$$S_i = \mathrm{Reali}\big(P_i^2 (P_i^2 - \delta_i^2)^2 (P_i^2 - 4\delta_i^2) \geq 0, \mathrm{Ext}(Z, \mathrm{R}\langle \delta_1, \ldots, \delta_s \rangle)\big),\ 1 \leq i \leq s,$$

and let S be the intersection of the S_i with the closed ball in $\mathrm{R}\langle \delta_1, \ldots, \delta_s, \delta \rangle^k$ defined by $\delta^2 \left(\sum_{1 \leq i \leq k} X_i^2 \right) \leq 1$. Then, it is clear that

$$\sum_{\Psi \in \Sigma_{\leq s}} \mathrm{b}_i(\Psi) = \mathrm{b}_i(S).$$

Proof of Proposition 7.41: Since, for all $i < k'$,

$$\mathrm{b}_i(Z) + \mathrm{b}_{k'}(Z) \leq d\, (2\, d - 1)^{k-1}$$

by Theorem 7.23 we get that,

$$\sum_{\Psi \in \Sigma_{\leq s}} \mathrm{b}_i(\Psi) = \mathrm{b}_i(S) \leq \sum_{j=1}^{k'-i} \binom{s}{j} (6^j - 1) d\, (2\, d - 1)^{k-1} + \binom{s}{k' - i} \mathrm{b}_{k'}(Z)$$

using part (b) of Proposition 7.33 and Lemma 7.43.

Thus, we have that

$$\sum_{\Psi \in \Sigma_{\leq s}} \mathrm{b}_i(\Psi) \leq \sum_{j=1}^{k'-i} \binom{s}{j} 6^j\, d\, (2\, d - 1)^{k-1}.$$ □

Proof of Theorem 7.38: Theorem 7.38 now follows from Proposition 7.39 and Proposition 7.41. □

7.5 Sum of the Betti Numbers of Semi-algebraic Sets

We first describe a construction for replacing any given semi-algebraic subset of a bounded semi-algebraic set by a closed bounded semi-algebraic subset and prove that the new set has the same homotopy type as the original one. Moreover, the polynomials defining the bounded closed semi-algebraic subset are closely related (by infinitesimal perturbations) to the polynomials defining the original subset. In particular, their degrees do not increase, while the number of polynomials used in the definition of the new set is at most twice the square of the number used in the definition of the original set. This construction will be useful later in Chapter 16.

Definition 7.44. Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a finite set of polynomials with t elements, and let S be a bounded $\mathcal{P}$-closed set. We denote by $\mathrm{SIGN}(S)$ the set of realizable sign conditions of $\mathcal{P}$ whose realizations are contained in S.

Recall that, for $\sigma \in \mathrm{SIGN}(\mathcal{P})$ we define the level of σ as $\#\{P \in \mathcal{P} | \sigma(P) = 0\}$. Let, $\varepsilon_{2t} \gg \varepsilon_{2t-1} \gg \cdots \gg \varepsilon_2 \gg \varepsilon_1 > 0$ be infinitesimals, and denote by R_i the field $\mathrm{R}\langle\varepsilon_{2t}\rangle \cdots \langle\varepsilon_i\rangle$. For $i > 2t$, $\mathrm{R}_i = \mathrm{R}$ and for $i \leqslant 0, \mathrm{R}_i = \mathrm{R}_1$.

We now describe the construction. For each level m, $0 \leq m \leq t$, we denote by $\mathrm{SIGN}_m(S)$ the subset of $\mathrm{SIGN}(S)$ of elements of level m.

Given $\sigma \in \mathrm{SIGN}_m(\mathcal{P}, S)$, let $\mathrm{Reali}(\sigma_+^c)$ be the intersection of $\mathrm{Ext}(S, \mathrm{R}_{2m})$ with the closed semi-algebraic set defined by the conjunction of the inequalities,

$$\begin{array}{ll} -\varepsilon_{2m} \leq P \leq \varepsilon_{2m} & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 0, \\ P \geq 0 & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 1, \\ P \leq 0 & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = -1. \end{array}$$

and let $\mathrm{Reali}(\sigma_+^o)$ be the intersection of $\mathrm{Ext}(S, \mathrm{R}_{2m-1})$ with the open semi-algebraic set defined by the conjunction of the inequalities,

$$\begin{array}{ll} -\varepsilon_{2m-1} < P < \varepsilon_{2m-1} & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 0, \\ P > 0 & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 1, \\ P < 0 & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = -1. \end{array}$$

Notice that, denoting $\mathrm{Reali}(\sigma)_i = \mathrm{Ext}(\mathrm{Reali}(\sigma), \mathrm{R}_i)$,

$$\begin{array}{rcl} \mathrm{Reali}(\sigma)_{2m} & \subset & \mathrm{Reali}(\sigma_+^c), \\ \mathrm{Reali}(\sigma)_{2m-1} & \subset & \mathrm{Reali}(\sigma_+^o). \end{array}$$

Let $X \subset S$ be a $\mathcal{P}$-semi-algebraic set such that

$$X = \bigcup_{\sigma \in \Sigma} \mathrm{Reali}(\sigma)$$

with $\Sigma \subset \mathrm{SIGN}(S)$. We denote $\Sigma_m = \Sigma \cap \mathrm{SIGN}_m(S)$ and define a sequence of sets, $X^m \subset \mathrm{R}'^k$, $0 \leq m \leq t$ inductively by

– $X^0 = \mathrm{Ext}(X, \mathrm{R}_1)$.

– For $0 \le m \le t$,

$$X^{m+1} = \left(X^m \cup \bigcup_{\sigma \in \Sigma_m} \mathrm{Reali}(\sigma_+^c)_1 \right) \setminus \bigcup_{\sigma \in \mathrm{SIGN}_m(S) \setminus \Sigma_m} \mathrm{Reali}(\sigma_+^o)_1 ,$$

with $\mathrm{Reali}(\sigma_+^c)_i = \mathrm{Ext}(\mathrm{Reali}(\sigma_+^c), \mathrm{R}_i)$, $\mathrm{Reali}(\sigma_+^o)_i = \mathrm{Ext}(\mathrm{Reali}(\sigma_+^o), \mathrm{R}_i)$.

We denote by X' the set X^{t+1}. □

Theorem 7.45. *The sets* $\mathrm{Ext}(X, R_1)$ *and* X' *are semi-algebraically homotopy equivalent. In particular,*

$$\mathrm{H}_*(X) \cong \mathrm{H}_*(X').$$

For the purpose of the proof we introduce several new families of sets defined inductively.

For each p, $0 \le p \le t+1$ we define sets, $Y_p \subset \mathrm{R}_{2p}^k$, $Z_p \subset \mathrm{R}_{2p-1}^k$ as follows.

– We define

$$\begin{aligned} Y_p^p &= \mathrm{Ext}(X, \mathrm{R}_{2p}) \cup \bigcup_{\sigma \in \Sigma_p} \mathrm{Reali}(\sigma_+^c)_{2p} \\ Z_p^p &= \mathrm{Ext}(Y_p^p, \mathrm{R}_{2p-1}) \setminus \bigcup_{\sigma \in \mathrm{SIGN}_p(S) \setminus \Sigma_p} \mathrm{Reali}(\sigma_+^o)_{2p-1}. \end{aligned}$$

– For $p \le m \le t$, we define

$$\begin{aligned} Y_p^{m+1} &= \left(Y_p^m \cup \bigcup_{\sigma \in \Sigma_m} \mathrm{Reali}(\sigma_+^c)_{2p} \right) \setminus \bigcup_{\sigma \in \mathrm{SIGN}_m(S) \setminus \Sigma_m} \mathrm{Reali}(\sigma_+^o)_{2p} \\ Z_p^{m+1} &= \left(Z_p^m \cup \bigcup_{\sigma \in \Sigma_m} \mathrm{Reali}(\sigma_+^c)_{2p-1} \right) \setminus \bigcup_{\sigma \in \mathrm{SIGN}_m(S) \setminus \Sigma_m} \mathrm{Reali}(\sigma_+^o)_{2p-1}. \end{aligned}$$

We denote by $Y_p \subset \mathrm{R}_{2p}^k$ the set Y_p^{t+1} and by $Z_p \subset \mathrm{R}_{2p-1}^k$ the set Z_p^{t+1}.

Note that

– $X = Y_{t+1} = Z_{t+1}$,
– $Z_0 = X'$.

Notice also that for each $p, 0 \le p \le t$,

– $\mathrm{Ext}(Z_{p+1}^{p+1}, \mathrm{R}_{2p}) \subset Y_p^p$,
– $Z_p^p \subset \mathrm{Ext}(Y_p^p, \mathrm{R}_{2p-1})$

The following inclusions follow directly from the definitions of Y_p and Z_p.

Lemma 7.46. *For each* $p, 0 \le p \le t$,

– $\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p}) \subset Y_p$,
– $Z_p \subset \mathrm{Ext}(Y_p, \mathrm{R}_{2p-1})$.

We now prove that in both the inclusions in Lemma 7.46 above, the pairs of sets are in fact semi-algebraically homotopy equivalent. These suffice to prove Theorem 7.45.

Lemma 7.47. *For $1 \leq p \leq t$, Y_p is semi-algebraically homotopy equivalent to* $\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$.

Proof: Let $Y_p(u) \subset \mathrm{R}_{2p+1}^k$ denote the set obtained by replacing the infinitesimal ε_{2p} in the definition of Y_p by u, and for $u_0 > 0$, we will denote by

$$Y_p((0, u_0]) = \{(x, u) | x \in Y_p(u), u \in (0, u_0]\} \subset \mathrm{R}_{2p+1}^{k+1}.$$

By Hardt's triviality theorem there exist $u_0 \in \mathrm{R}_{2p+1}$, $u_0 > 0$ and a homeomorphism,

$$\psi: Y_p(u_0) \times (0, u_0] \rightarrow Y_p((0, u_0]),$$

such that

- $\pi_{k+1}(\phi(x, u)) = u$,
- $\psi(x, u_0) = (x, u_0)$ for $x \in Y_p(u_0)$,
- for all $u \in (0, u_0]$, and for every sign condition σ on

$$\cup_{P \in \mathcal{P}} \{P, P \pm \varepsilon_{2t}, \ldots, P \pm \varepsilon_{2p+1}\},$$

  $\psi(\,\cdot\,, u)$ defines a homeomorphism of $\mathrm{Reali}(\sigma, Y_p(u_0))$ to $\mathrm{Reali}(\sigma, Y_p(u))$.

Now, we specialize u_0 to ε_{2p} and denote the map corresponding to ψ by ϕ. For $\sigma \in \Sigma_p$, we define, $\mathrm{Reali}(\sigma_{++}^o)$ to be the set defined by

$$\begin{array}{ll} -2\,\varepsilon_{2p} < P < 2\,\varepsilon_{2p} & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 0, \\ P > -\,\varepsilon_{2p} & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 1, \\ P < \varepsilon_{2p} & \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = -1. \end{array}$$

Let $\lambda: Y_p \rightarrow \mathrm{R}_{2p}$ be a semi-algebraic continuous function such that,

$$\begin{array}{ll} \lambda(x) = 1 & \text{on } Y_p \cap \cup_{\sigma \in \Sigma_p} \mathrm{Reali}(\sigma_+^c), \\ \lambda(x) = 0 & \text{on } Y_p \setminus \cup_{\sigma \in \Sigma_p} \mathrm{Reali}(\sigma_{++}^o), \\ 0 < \lambda(x) < 1 & \text{else.} \end{array}$$

We now construct a semi-algebraic homotopy,

$$h: Y_p \times [0, \varepsilon_{2p}] \rightarrow Y_p,$$

by defining,

$$\begin{array}{lcll} h(x, t) & = & \pi_{1\ldots k} \circ \phi(x, \lambda(x)t + (1 - \lambda(x))\varepsilon_{2p}) & \text{for } 0 < t \leq \varepsilon_{2p} \\ h(x, 0) & & \lim_{t \rightarrow 0+} h(x, t), & \text{else.} \end{array}$$

Note that the last limit exists since S is closed and bounded. We now show that, $h(x, 0) \in \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$ for all $x \in Y_p$.

Let $x \in Y_p$ and $y = h(x, 0)$.

There are two cases to consider.

- $\lambda(x) < 1$. In this case, $x \in \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$ and by property (3) of ϕ and the fact that $\lambda(x) < 1$, $y \in \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$.

- $\lambda(x) \geqslant 1$. Let σ_y be the sign condition of $\mathcal{P}$ at y and suppose that $y \notin \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$. There are two cases to consider.
  - $\sigma_y \in \Sigma$. In this case, $y \in X$ and hence there must exist
$$\tau \in \mathrm{SIGN}_m(S) \setminus \Sigma_m,$$
    with $m > p$ such that $y \in \mathrm{Reali}(\tau_+^o)$.
  - $\sigma_y \notin \Sigma$. In this case, taking $\tau = \sigma_y$, level$(\tau) > p$ and $y \in \mathrm{Reali}(\tau_+^o)$. It follows from the definition of y, and property (3) of ϕ, that for any $m > p$, and every $\rho \in \mathrm{SIGN}_m(S)$,
    - $y \in \mathrm{Reali}(\rho_+^o)$ implies that $x \in \mathrm{Reali}(\rho_+^o)$,
    - $x \in \mathrm{Reali}(\rho_+^c)$ implies that $y \in \mathrm{Reali}(\rho_+^c)$.

    Thus, $x \notin Y_p$ which is a contradiction.

  It follows that,
  - $h(\cdot, \varepsilon_{2p}): Y_p \rightarrow Y_p$ is the identity map,
  - $h(Y_p, 0) = \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$,
  - $h(\cdot, t)$ restricted to $\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$ gives a semi-algebraic homotopy between
$$h(\cdot, \varepsilon_{2p})|_{\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})} = \mathrm{id}_{\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})}$$
    and
$$h(\cdot, 0)|_{\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})}.$$

  Thus, Y_p is semi-algebraically homotopy equivalent to $\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$.

□

Lemma 7.48. *For each* p, $0 \leq p \leq t$, Z_p *is semi-algebraically homotopy equivalent to* $\mathrm{Ext}(Y_p, \mathrm{R}_{2p-1})$.

Proof: For the purpose of the proof we define the following new sets for $u \in \mathrm{R}_{2p}$.

- Let $Z_p'(u) \subset \mathrm{R}_{2p}^k$ be the set obtained by replacing in the definition of Z_p, ε_{2j} by $\varepsilon_{2j} - u$ and ε_{2j-1} by $\varepsilon_{2j-1} + u$ for all $j > p$, and ε_{2p} by $\varepsilon_{2p} - u$, and ε_{2p-1} by u. For $u_0 > 0$ we will denote
$$Z_p'((0, u_0]) = \{(x, u) \mid x \in Z_p'(u), u \in (0, u_0]\}.$$
  $Z_p'((0, u_0])$ the set $\{(x, u) \mid x \in Z_p'(u), u \in (0, u_0]\}$.
- Let $Y_p'(u) \subset \mathrm{R}_{2p}^k$ be the set obtained by replacing in the definition of Y_p, ε_{2j} by $\varepsilon_{2j} - u$ and ε_{2j-1} by $\varepsilon_{2j-1} + u$ for all $j > p$ and ε_{2p} by by $\varepsilon_{2p} - u$.
- For $\sigma \in \mathrm{Sign}_m(S)$, with $m \geq p$, let $\mathrm{Reali}(\sigma_+^c)(u) \subset \mathrm{R}_{2p}^k$ denote the set obtained by replacing ε_{2m} by $\varepsilon_{2m} - u$ in the definition of $\mathrm{Reali}(\sigma_+^c)$.
- For $\sigma \in \mathrm{Sign}_m(S)$, with $m > p$, let $\mathrm{Reali}(\sigma_+^o)(u) \subset \mathrm{R}_{2p}^k$ denote the set obtained by replacing ε_{2m-1} by $\varepsilon_{2m-1} + u$ in the definition of $\mathrm{Reali}(\sigma_+^o)$.
- Finally, for $\sigma \in \mathrm{Sign}_p(S)$ let $\mathrm{Reali}(\sigma_+^o)(u) \subset \mathrm{R}_{2p-1}^k$ denote the set obtained by replacing in the definition of $\mathrm{Reali}(\sigma_o^c)$, ε_{2p-1} by u.

Notice that by definition, for any $u, v \in \mathrm{R}_{2p}$ with $0 < u \leq v$, $Z'_p(u) \subset Y'_p(u)$, $Z'_p(v) \subset Z'_p(u)$, $Y'_p(v) \subset Y'_p(u)$, and

$$\bigcup_{0<s\leq u} Y'_p(s) = \bigcup_{0<s\leq u} Z'_p(s).$$

We denote by Z'_p (respectively, Y'_p) the set $Z'_p(\varepsilon_{2p-1})$ (respectively, $Y'_p(\varepsilon_{2p-1})$). It is easy to see that Y'_p is semi-algebraically homotopy equivalent to $\mathrm{Ext}(Y_p, \mathrm{R}_{2p-1})$, and Z'_p is semi-algebraically homotopy equivalent to Z_p. We now prove that, Y'_p is semi-algebraically homotopy equivalent to Z'_p, which suffices to prove the lemma.

Let $\mu: Y'_p \to \mathrm{R}_{2p-1}$ be the semi-algebraic map defined by

$$\mu(x) = \sup_{u \in (0, \varepsilon_{2p-1}]} \{u \mid x \in Z'_p(u)\}.$$

We prove separately (Lemma 7.49 below) that μ is continuous. Note that the definition of the set $Z'_p(u)$ (as well as the set $Y'_p(u)$) is more complicated than the more natural one consisting of just replacing ε_{2p-1} in the definition of Z_p by u, is due to the fact that with the latter definition the map μ defined below is not necessarily continuous.

We now construct a continuous semi-algebraic map,

$$h: Y'_p \times [0, \varepsilon_{2p-1}] \to Y'_p$$

as follows.

By Hardt's triviality theorem there exist $u_0 \in \mathrm{R}_{2p}$, with $u_0 > 0$ and a semi-algebraic homeomorphism,

$$\psi: Z'_p(u_0) \times (0, u_0] \to Z'_p((0, u_0]),$$

such that

- $\pi_{k+1}(\psi(x, u)) = u$,
- $\psi(x, u_0) = (x, u_0)$ for $x \in Z'_p(u_0)$,
- for all $u \in (0, u_0]$ and for every sign condition σ of the family,

  $$\bigcup_{P \in \mathcal{P}} \{P, P \pm \varepsilon_{2t}, \ldots, P \pm \varepsilon_{2p+1}\},$$

  the map $\psi(\,\cdot\,, u)$ restricts to a homeomorphism of $\mathrm{Reali}(\sigma, Z'_p(u_0))$ to $\mathrm{Reali}(\sigma, Z'_p(u))$.

We now specialize u_0 to ε_{2p-1} and denote by ϕ the corresponding map,

$$\phi: Z'_p \times (0, \varepsilon_{2p-1}] \to Z'_p((0, \varepsilon_{2p-1}]).$$

Note, that for every u, $0 < u \leq \varepsilon_{2p-1}$, ϕ gives a homeomorphism,

$$\phi_u: Z'_p(u) \to Z'_p.$$

Hence, for every pair, u, u', $0 < u \leq u' \leq \varepsilon_{2p-1}$, we have a homeomorphism, $\theta_{u,u'}: Z'_p(u) \to \mathbb{Z}'_p(u')$ obtained by composing ϕ_u with $\phi_{u'}^{-1}$.

For $0 \leq u' < u \leq \varepsilon_{2p-1}$, we let $\theta_{u,u'}$ be the identity map. It is clear that $\theta_{u,u'}$ varies continuously with u and u'.

For $x \in Y'_p, t \in [0, \varepsilon_{2p-1}]$ we now define,

$$h(x,t) = \theta_{\mu(x),t}(x).$$

It is easy to verify from the definition of h and the properties of ϕ listed above that, h is continuous and satisfies the following.

- $h(\cdot, 0): Y'_p \to Y'_p$ is the identity map,
- $h(Y'_p, \varepsilon_{2p-1}) = Z'_p$,
- $h(\cdot, t)$ restricts to a homeomorphism $Z'_p \times t \to Z'_p$ for every $t \in [0, \varepsilon_{2p-1}]$.

This proves the required homotopy equivalence. □

We now prove that the function μ used in the proof above is continuous.

Lemma 7.49. *The semi-algebraic map* $\mu: Y'_p \to \mathrm{R}_{2p-1}$ *defined by*

$$\mu(x) = \sup_{u \in (0, \varepsilon_{2p-1}]} \{u \mid x \in Z'_p(u)\}$$

is continuous.

Proof : Let $0 < \delta \ll \varepsilon_{2p-1}$ be a new infinitesimal. In order to prove the continuity of μ (which is a semi-algebraic function defined over R_{2p-1}), it suffices, by Proposition 3.5 to show that

$$\lim_\delta \mathrm{Ext}(\mu, \mathrm{R}_{2p-1}\langle\delta\rangle)(x') = \lim_\delta \mathrm{Ext}(\mu, \mathrm{R}_{2p-1}\langle\delta\rangle)(x)$$

for every pair of points $x, x' \in \mathrm{Ext}(Y'_p, \mathrm{R}_{2p-1}\langle\delta\rangle)$ such that $\lim_\delta x = \lim_\delta x'$.

Consider such a pair of points $x, x' \in \mathrm{Ext}(Y'_p, \mathrm{R}_{2p-1}\langle\delta\rangle)$. Let $u \in (0, \varepsilon_{2p-1}]$ be such that $x \in Z'_p(u)$. We show below that this implies $x' \in Z'_p(u')$ for some u' satisfying $\lim_\delta u' = \lim_\delta u$.

Let m be the largest integer such that there exists $\sigma \in \Sigma_m$ with $x \in \mathrm{Reali}(\sigma_+^c)(u)$. Since $x \in Z'_p(u)$ such an m must exist.

We have two cases:

- $m > p$: Let $\sigma \in \Sigma_m$ with $x \in \mathrm{Reali}(\sigma_+^c)(u)$. Then, by the maximality of m, we have that for each $P \in \mathcal{P}$, $\sigma(P) \neq 0$ implies that $\lim_\delta P(x) \neq 0$. As a result, we have that $x' \in \mathrm{Reali}(\sigma_+^c)(u')$ for all

$$u' < u - \max_{P \in \mathcal{P}, \sigma(P) = 0} |P(x) - P(x')|,$$

  and hence we can choose u' such that $x' \in \mathrm{Reali}(\sigma_+^c)(u')$ and $\lim_\delta u' = \lim_\delta u$.
- $m \leq p$: If $x' \notin Z'_p(u)$ then since $x' \in Y'_p \subset Y'_p(u)$,

$$x' \in \cup_{\sigma \in \mathrm{SIGN}_p(\mathcal{P}, S) \setminus \Sigma_p} \mathrm{Reali}(\sigma_+^o)(u).$$

Let $\sigma \in \mathrm{SIGN}_p(S) \setminus \Sigma_p$ be such that $x' \in \mathrm{Reali}(\sigma_+^o)(u)$. We prove by contradiction that $\lim_\delta \max_{P \in \mathcal{P}, \sigma(P)=0} |P(x')| = u$.

Assume that

$$\lim_\delta \max_{P \in \mathcal{P}, \sigma(P)=0} |P(x')| \neq u.$$

Since, $x \notin \mathrm{Reali}(\sigma_+^o)(u)$ by assumption, and $\lim_\delta x' = \lim_\delta x$, there must exist $P \in \mathcal{P}$, $\sigma(P) \neq 0$, and $\lim_\delta P(x) = 0$. Letting τ denote the sign condition defined by $\tau(P) = 0$ if $\lim_\delta P(x) = 0$ and $\tau(P) = \sigma(P)$ else, we have that $\mathrm{level}(\tau) > p$ and x belongs to both $\mathrm{Reali}(\tau_+^o)(u)$ as well as $\mathrm{Reali}(\tau_+^c)(u)$.

Now there are two cases to consider depending on whether τ is in Σ or not. If $\tau \in \Sigma$, then the fact that $x \in \mathrm{Reali}(\tau_+^c)(u)$ contradicts the choice of m, since $m \leq p$ and $\mathrm{level}(\tau) > p$. If $\tau \notin \Sigma$ then x gets removed at the level of τ in the construction of $Z_p'(u)$, and hence $x \in \mathrm{Reali}(\rho_+^c)(u)$ for some $\rho \in \Sigma$ with $\mathrm{level}(\rho) > \mathrm{level}(\tau) > p$. This again contradicts the choice of m. Thus, $\lim_\delta \max_{P \in \mathcal{P}, \sigma(P)=0} |P(x')| = u$ and since $x' \notin \cup_{\sigma \in \mathrm{SIGN}_p(\mathcal{C}, S) \setminus \Sigma_p} \mathrm{Reali}(\sigma_+^o)(u')$ for all $u' < \max_{P \in \mathcal{P}, \sigma(P)=0} |P(x')|$, we can choose u' such that $\lim_\delta u' = \lim_\delta u$, and $x' \notin \cup_{\sigma \in \mathrm{SIGN}_p(\mathcal{P}, S) \setminus \Sigma_p} \mathrm{Reali}(\sigma_+^o)(u')$.

In both cases we have that $x' \in Z_p'(u')$ for some u' satisfying $\lim_\delta u' = \lim_\delta u$, showing that $\lim_\delta \mu(x') \geq \lim_\delta \mu(x)$. The reverse inequality follows by exchanging the roles of x and x' in the previous argument. Hence,

$$\lim_\delta \mu(x') = \lim_\delta \mu(x),$$

proving the continuity of μ. □

Proof of Theorem 7.45: The theorem follows immediately from Lemmas 7.47 and 7.48. □

We now define the **Betti numbers** of a general $\mathcal{P}$-semi-algebraic set and bound them. Given a $\mathcal{P}$-semi-algebraic set $Y \subset \mathrm{R}^k$, we replace it by

$$X = \mathrm{Ext}(Y, \mathrm{R}\langle\varepsilon\rangle) \cap \overline{B}_k(0, 1/\varepsilon).$$

Taking $S = \overline{B}_k(0, 1/\varepsilon)$, we know by Theorem 7.45 that there is a closed and bounded semi-algebraic set $X' \subset \mathrm{R}\langle\varepsilon\rangle_1^k$ such that $\mathrm{Ext}(X, \mathrm{R}\langle\varepsilon\rangle_1)$ and X' are semi-algebraically homotopy equivalent.We define the Betti numbers $\mathrm{b}_i(Y) := \mathrm{b}_i(X')$. Note that this definition is clearly homotopy invariant since Y and X' has are semi-algebraically homotopy equivalent. We denote by $\mathrm{b}(Y) = \mathrm{b}(X')$ the sum of the Betti numbers of Y.

Theorem 7.50. *Let Y be a $\mathcal{P}$-semi-algebraic set where $\mathcal{P}$ is a family of at most s polynomials of degree d in k variables. Then*

$$\mathrm{b}(Y) \leq \sum_{i=0}^{k} \sum_{j=1}^{k-i} \binom{2s^2+1}{j} 6^j d (2d-1)^{k-1}.$$

Proof: Take $S = \overline{B}_k(0, 1/\varepsilon)$ and $X = \mathrm{Ext}(Y, \mathrm{R}\langle\varepsilon\rangle) \cap \overline{B}_k(0, 1/\varepsilon)$. Defining X' according to Definition 7.44, apply Theorem 7.38 to X', noting that the number of polynomials defining X' is $2\,s^2{+}1$, and their degrees are bounded by d. □

7.6 Bibliographical Notes

The inequalities relating the number of critical points of a Morse function (Theorem 7.24) with the Betti numbers of a compact manifold was proved in its full generality by Morse [121], building on prior work by Birkhoff ([25], page 42). Their generalization "with boundary" (Propositions 7.18 and 7.18) can be found in [81, 82]. Using these inequalities, Thom [157], Milnor[118], Oleinik and Petrovsky [124, 125] proved the bound on the sum of the Betti numbers of algebraic sets presented here. Subsequently, using these bounds Warren [165] proved a bound of $(4e\,s\,d/k)^k$ on the number of connected components of the realizations of strict sign conditions of a family of polynomials of s polynomials in k variables of degree at most d and Alon [3] derived a bound of $(8e\,s\,d/k)^k$ on the number of all realizable sign conditions (not connected components). All these bounds are in terms of the product $s\,d$.

The first result in which the dependence on s (the combinatorial part) was studied separately from the dependence on d (algebraic) appeared in [14], where a bound of $\binom{O(s)}{k'}\,O(d)^k$ was proved on the number of connected components of all realizable sign conditions of a family of polynomials restricted to variety of dimension k'. The generalization of the Thom-Milnor bound on the sum of the Betti numbers of basic semi-algebraic sets to more general closed semi-algebraic sets restricted to a variety was first done in [11] and later improved in [17]. The first result bounding the individual Betti numbers of a basic semi-algebraic set appears in [10] and that bounding the individual Betti numbers over all realizable sign conditions in [17]. The construction for replacing any given semi-algebraic subset of a bounded semi-algebraic set by a closed bounded semi-algebraic subset in Section 7.5, as well as the bound on the sum of the Betti numbers of a general semi-algebraic set, appears in [62].