15

Computing Roadmaps and Connected Components of Algebraic Sets

In this chapter, we compute roadmaps and connected components of algebraic sets. Roadmaps provide a way to count connected components and to decide whether two points belong to the same connected component. Done in a parametric way the roadmap algorithm also gives a description of the semi-algebraically connected components of an algebraic set. The complexities of the algorithms given in this chapter are much better than the one provided by cylindrical decomposition in Chapter 11 (single exponential in the number of variables rather than doubly exponential).

We first define roadmaps. Let S be a semi-algebraic set. As usual, we denote by π the projection on the X_1-axis and set $S_x = \{y \in \mathrm{R}^{k-1} \mid (x, y) \in S\}$.

A **roadmap** for S is a semi-algebraic set M of dimension at most one contained in S which satisfies the following roadmap conditions:

- RM_1 For every semi-algebraically connected component D of S, $D \cap M$ is semi-algebraically connected.
- RM_2 For every $x \in \mathrm{R}$ and for every semi-algebraically connected component D' of S_x, $D' \cap M \neq \emptyset$.

The construction of roadmaps is based on the critical point method, using properties of pseudo-critical values provided in Section 15.1. In Section 15.2 we give an algorithm constructing a roadmap for $\mathrm{Zer}(Q, \mathrm{R}^k)$, for $Q \in \mathrm{R}[X_1, \ldots, X_k]$. As a consequence, we get an algorithm for computing the number of connected components (the zero-th Betti number) of an algebraic set, with single exponential complexity.

In Section 15.3 we obtain an algorithm giving a semi-algebraic description of the semi-algebraically connected components of an algebraic set. The idea behind the algorithm is simple: we perform parametrically the roadmap algorithm with a varying input point.

15.1 Pseudo-critical Values and Connectedness

We consider a semi-algebraic set S as the collection of its fibers S_x, $x \in \mathrm{R}$. In the smooth bounded case, critical values of π are the only places where the number of connected components in the fiber can change.

More precisely, we can generalize Proposition 7.6 to the case of a general real closed field.

Proposition 15.1. *Let* $\mathrm{Zer}(Q, \mathrm{R}^k)$ *be a non-singular bounded algebraic hypersurface,* $[a, b]$ *such that* π *has no critical value in* $[a, b]$*, and* $d \in [a, b]$*.*

a) The number of semi-algebraically connected components of $\mathrm{Zer}(Q, \mathrm{R}^k)_{[a,b]}$ *and* $\mathrm{Zer}(Q, \mathrm{R}^k)_d$ *are the same.*
b) Let S *be a semi-algebraically connected component of* $\mathrm{Zer}(Q, \mathrm{R}^k)_{[a,b]}$*. Then, for every* $d \in [a, b]$*,* S_d *is semi-algebraically connected.*

Proposition 15.1 immediately implies.

Proposition 15.2. *Let* $\mathrm{Zer}(Q, \mathrm{R}^k)$ *be a bounded non-singular algebraic hypersurface and* $[a, b]$ *such that* π *has no critical value in* $[a, b]$*. Let* S *be a semi-algebraically connected component of* $\mathrm{Zer}(Q, \mathrm{R}^k)_{[a,b]}$*. Then, for every* $d \in [a, b]$*,* S_d *is semi-algebraically connected.*

Proposition 15.3. *Let* $\mathrm{Zer}(Q, \mathrm{R}^k)$ *be a non-singular algebraic hypersurface and* S *a semi-algebraically connected component of* $\mathrm{Zer}(Q, \mathrm{R}^k)_{[a,b]}$*. If* $S_{[a,b)}$ *is not semi-algebraically connected then* b *is a critical value of* π *on* $\mathrm{Zer}(Q, \mathrm{R}^k)$*.*

Proof of Proposition 15.1: Over the reals (the case $\mathrm{R} = \mathbb{R}$), the two properties are true according to Proposition 7.6.

We now prove that Properties a and b hold for a general real closed field, using Theorem 5.46 (Semi-algebraic triviality) and the transfer principle (Theorem 2.80).

We first prove Property a.

Let $\{m_1, ..., m_N\}$ be a list of all monomials in the variables $x_1, ... x_k$ with degree at most the degree of Q. To an element $\mathrm{cof} = (c_1, ..., c_N)$ of R^N, we associate the polynomial

$$\mathrm{Pol}(\mathrm{cof}) = \sum_{i=1}^{N} c_i m_i.$$

Denoting by $\mathrm{cof}_i(Q)$ the coefficient of m_i in Q and by

$$\mathrm{cof}(Q) = (\mathrm{cof}_1(Q), ..., \mathrm{cof}_N(Q)),$$

we have $Q = \mathrm{Pol}(\mathrm{cof}(Q))$.

Consider the field $\mathbb{R}_{\mathrm{alg}}$ of real algebraic numbers and the subset $W \subset \mathbb{R}_{\mathrm{alg}}^{N+2+k}$ defined by

$$W = \{(\mathrm{cof}, a', b', x_1 ..., x_k) \mid a' \leq x_1 \leq b', \mathrm{Pol}(\mathrm{cof})(x_1, ..., x_k) = 0\}.$$

The set W can be viewed as the family of sets $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathbb{R}_{\mathrm{alg}}^{N+2+k})_{[a',b']}$, parametrized by $(\mathrm{cof}, a', b') \in \mathbb{R}_{\mathrm{alg}}^{N+2}$. We also consider the subset $W' \subset \mathbb{R}_{\mathrm{alg}}^{N+1+k}$ defined by

$$W' = \{(\mathrm{cof}, d', x_1 \ldots, x_k) \mid \mathrm{Pol}(\mathrm{cof})(d', \ldots, x_k) = 0\}.$$

The set W' can be viewed as the family of sets $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathbb{R}_{\mathrm{alg}}^{N+1+k})_{d'}$, parametrized by $(\mathrm{cof}, d') \in \mathbb{R}_{\mathrm{alg}}^{N+1}$. According to Theorem 5.46 (Hardt's triviality) applied to W (resp. W'), there is a finite partition $\mathcal{A}$ (resp. $\mathcal{B}$) of $\mathbb{R}_{\mathrm{alg}}^{N+2}$ (resp. $\mathbb{R}_{\mathrm{alg}}^{N+1}$) into semi-algebraic sets, and for every $A \in \mathcal{A}$ (resp. $B \in \mathcal{B}$) the sets $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathbb{R}_{alg}^{N+2+k})_{[a',b']}$ (resp. $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathbb{R}_{\mathrm{alg}}^{N+1+k})_{d'}$) are semi-algebraically homeomorphic as (cof, a', b') varies in A (resp. (cof, d') varies in B). Hence, they have the same number of bounded semi-algebraically connected components $\ell(A)$ (resp. $\ell(B)$).

Using the transfer principle (Theorem 2.80), for every real closed field R and every (cof, a', b') $\in \mathrm{Ext}(A, \mathrm{R})$ (resp. (cof, d') $\in \mathrm{Ext}(B, \mathrm{R})$), the set $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathrm{R}^{N+2+k})_{[a',b']}$ has $\ell(A)$ (resp. $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathrm{R}^{N+1+k})_{d'}$ has $\ell(B)$) bounded semi-algebraically connected components. Moreover, since the connected components of

$$W_A = \{(\mathrm{cof}, a', b', x_1, \ldots, x_k) \in W \mid (\mathrm{cof}, a', b') \in A\}$$

are semi-algebraic sets defined over $\mathbb{R}_{\mathrm{alg}}$, there exists, for every $A \in \mathcal{A}$, $\ell(A)$ quantifier free formulas

$$\Phi_1(A)(\mathrm{cof}, a', b', x_1, \ldots, x_k), \ldots, \Phi_{\ell(A)}(A)(\mathrm{cof}, a', b', x_1, \ldots, x_k),$$

such that for every real closed field R and for every (cof, a', b') $\in \mathrm{Ext}(A, \mathrm{R})$ the semi-algebraic sets

$$C_j = \{(x_1 \ldots, x_k) \in \mathrm{R}^k \mid \Phi_j(A)(\mathrm{cof}, a', b', x_1, \ldots, x_k)\}$$

for $1 \leq j \leq \ell(A)$ are the bounded semi-algebraically connected components of $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathrm{R}^{N+2+k})_{[a',b']}$.

Let A (resp. B) be the set of the partition $\mathcal{A}$ (resp. $\mathcal{B}$) such that $\mathrm{cof}(Q), a, b) \in \mathrm{Ext}(A, \mathrm{R})$ (resp. $(\mathrm{cof}(Q), d) \in \mathrm{Ext}(B, \mathrm{R})$), and let E be the semi-algebraic set of (cof, a', b', d') $\in (\mathbb{R}_{\mathrm{alg}})^{N+3}$ such that $(\mathrm{cof}, a', b') \in A$, $(\mathrm{cof}, d') \in B$, $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathbb{R}_{\mathrm{alg}}^{N+2+k})$ is a non-singular algebraic hypersurface, π has no critical value over $[a', b']$, and $a' < d' < b'$. Using the transfer principle (Theorem 2.80), the set E is non-empty since $\mathrm{Ext}(E, \mathrm{R})$ is non-empty, and hence $\mathrm{Ext}(E, \mathbb{R})$ is non-empty.

Given (cof, a', b', d') $\in \mathrm{Ext}(E, \mathbb{R})$, the number of bounded connected components of $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathbb{R}^{N+2+k})_{[a',b']}$ is equal to the number of bounded connected components of $\mathrm{Zer}(\mathrm{Pol}(\mathrm{cof}), \mathbb{R}^{N+2+k})_{d'}$, since Property 1 holds for the reals. It follows that $\ell(A) = \ell(B)$, so the number of bounded semi-algebraically connected components of $\mathrm{Zer}(Q, \mathrm{R}^k)_{[a,b]}$ is equal to the number of bounded semi-algebraically connected components of $\mathrm{Zer}(Q, \mathrm{R}^k)_d$.

To complete the proof of the proposition, it remains to prove Property b. According to the preceding paragraph, there exist j such that

$$S = \{(x_1 \ldots, x_k) \in \mathrm{R}^k \mid \Phi_j(A)(\mathrm{cof}(Q), a, b, x_1, \ldots, x_k)\}.$$

Since Property b is true over the reals, the formula expressing that for every $(\mathrm{cof}, a', b', d') \in \mathrm{Ext}(E, \mathbb{R})$ the set

$$\{(x_2, \ldots, x_k) \in \mathbb{R}^k \mid \Phi_j(A)(\mathrm{cof}(Q), a, b, d', \ldots, x_k)\}$$

is non-empty is true over the reals. Using the transfer principle (Theorem 2.80), this formula is thus true over any real closed field. Thus, S_d is non-empty. □

In the non-smooth case, we again consider X_1-pseudo-critical values introduced in Chapter 12. These pseudo critical-values will also be the only places where the number of connected components in the fiber can change. More precisely, generalizing Proposition 15.2 and Proposition 15.3, we prove the following two propositions, which play an important role for computing roadmaps.

Proposition 15.4. *Let* $\mathrm{Zer}(Q, \mathrm{R}^k)$ *be a bounded algebraic set and* S *a semi-algebraically connected component of* $\mathrm{Zer}(Q, \mathrm{R}^k)_{[a,b]}$*. If* $v \in (a, b)$ *and* $[a, b] \setminus \{v\}$ *contains no* X_1*-pseudo-critical value on* $\mathrm{Zer}(Q, \mathrm{R}^k)$*, then* S_v *is semi-algebraically connected.*

Proposition 15.5. *Let* $\mathrm{Zer}(Q, \mathrm{R}^k)$ *be a bounded algebraic set and let* S *be a semi-algebraically connected component of* $\mathrm{Zer}(Q, \mathrm{R}^k)_{[a,b]}$*. If* $S_{[a,b)}$ *is not semi-algebraically connected, then* b *is an* X_1*-pseudo-critical value of* $\mathrm{Zer}(Q, \mathrm{R}^k)$*.*

Before proving these two propositions, we need some preparation. Suppose that the polynomial $Q \in \mathrm{R}[X_1, \ldots, X_k]$, and $(d_1, \ldots, d_k)$ satisfy the following conditions:

- $Q(x) \geq 0$ for every $x \in \mathrm{R}^k$,
- $\mathrm{Zer}(Q, \mathrm{R}^k) \subset B(0, 1/c)$ for some $c \leq 1, c \in \mathrm{R}$,
- $d_1 \geq d_2 \cdots \geq d_k$,
- $\deg(Q) \leq d_1$, $\mathrm{tDeg}_{X_i}(Q) \leq d_i$, for $i = 2, \ldots, k$.

Let $\bar{d}_i$ be an even number $> d_i, i = 1, \ldots, k$, and $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_k)$.

Let $G_k(\bar{d}, c) = c^{\bar{d}_1} (X_1^{\bar{d}_1} + \cdots + X_k^{\bar{d}_k} + X_2^2 + \cdots + X_k^2) - (2k - 1)$, and note that $\forall\, x \in B(0, 1/c) \quad G_k(\bar{d}, c)(x) < 0$.

Using Notation 12.35, we consider

$$\begin{aligned} \mathrm{Def}(Q, \zeta) &= \zeta\, G_k(\bar{d}, c) + (1 - \zeta)\, Q, \\ \mathrm{Def}_+(Q, \zeta) &= \mathrm{Def}(Q, \zeta) + X_{k+1}^2. \end{aligned}$$

The algebraic set $\mathrm{Zer}(\mathrm{Def}_+(Q,\zeta),\mathrm{R}\langle\zeta\rangle^{k+1})$ has the following property which is not enjoyed by $\mathrm{Zer}(\mathrm{Def}(Q,\zeta),\mathrm{R}\langle\zeta\rangle^k)$.

Lemma 15.6. *Let* $\mathrm{Zer}(Q, \mathrm{R}^k) \subset B(0, 1/c)$ *be a bounded algebraic set. For every semi-algebraically connected component* D *of* $\mathrm{Zer}(Q,\mathrm{R}^k)_{[a,b]}$ *there exists a semi-algebraically connected component* D' *of* $\mathrm{Zer}(\mathrm{Def}_+(Q,\zeta),\mathrm{R}\langle\zeta\rangle^{k+1})_{[a,b]}$ *such that* $\lim_\zeta (D') = D \times \{0\}$.

Proof: Let $y = (y_1, \ldots, y_k)$ be a point of $\mathrm{Ext}(D, \mathrm{R}\langle\zeta\rangle)$. Since $y \in B(0, 1/c)$, we have $G_k(\bar{d}\ , c)(y) < 0$, hence $\mathrm{Def}(Q, \zeta)(y) < 0$. Thus, there exists a unique point $(y, f(y))$ in $\mathrm{Zer}(\mathrm{Def}_+(Q, \zeta), \mathrm{R}\langle\zeta\rangle^{k+1})$ for which $f(y) > 0$ and the mapping f is semi-algebraically continuous. Moreover for every z in D, $\mathrm{Def}(Q, \zeta)$ is infinitesimal, and hence $f(z) \in \mathrm{R}\langle\zeta\rangle$ is infinitesimal over R. So, $\lim_\zeta (z, f(z)) = (z, 0)$. Fix $x \in D$ and denote by D' the semi-algebraically connected component of $\mathrm{Zer}(\mathrm{Def}_+(Q,\zeta),\mathrm{R}\langle\zeta\rangle^{k+1})$ containing $(x, f(x))$. Since $\lim_\zeta (D')$ is connected (Proposition 12.43), contained in $\mathrm{Zer}(Q, \mathrm{R}^k)$, and contains x, it follows that $\lim_\zeta (D') \subset D$. Since f is semi-algebraic and continuous, and D is semi-algebraically path connected, for every z in D, the point $(z, f(z))$ belongs to the semi-algebraically connected component D' of $\mathrm{Zer}(\mathrm{Def}_+(Q, \zeta), \mathrm{R}\langle\zeta\rangle^{k+1})$ containing $(x, f(x))$. Since $\lim_\zeta (z, f(z)) = (z, 0)$, we have $\lim_\zeta (D') = D \times \{0\}$. □

Exercise 15.1. Prove that for

$$Q = ((X+1)^2 + Y^2 - 1)((X-1)^2 + Y^2 - 1)((X-2)^2 + Y^2 - 4)$$

the statement of Lemma 15.6 is false if $\mathrm{Def}_+(Q,\zeta)$ is replaced by $\mathrm{Def}(Q,\zeta)$.

We are now able to prove Proposition 15.4 and Proposition 15.5.

Proof of Proposition 15.4: By Lemma 15.6, there exists D', a semi-algebraically connected component of $\mathrm{Zer}(\mathrm{Def}_+(Q, \zeta), \mathrm{R}\langle\zeta\rangle^{k+1})_{[a,b]}$ such that $D \times \{0\} = \lim_\zeta (D')$. Since $[a, b] \setminus \{v\}$ contains no X_1-pseudo-critical value, there exists an infinitesimal β such that the X_1-critical values on $\mathrm{Zer}(\mathrm{Def}_+(Q,\zeta),\mathrm{R}\langle\zeta\rangle^{k+1})$ in the interval $[a, b]$, if they exist, lie in the interval $[v-\beta, v+\beta]$.

We claim that $D'_{[v-\beta,v+\beta]}$ is semi-algebraically connected.

Let x, y be any two points in $D'_{[v-\beta,v+\beta]}$. We show that there exists a semi-algebraic path connecting x to y lying within $D'_{[v-\beta,v+\beta]}$. Since, D' itself is semi-algebraically connected, there exists a semi-algebraic path, $\gamma\colon [0,1] \to D'$, with $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma(t) \in D'$, $0 \le t \le 1$. If $\gamma(t) \in D'_{[v-\beta,v+\beta]}$ for all $t \in [0,1]$, we are done. Otherwise, the semi-algebraic path γ is the union of a finite number of closed connected pieces γ_i lying either in $D'_{[a,v-\beta]}$, $D'_{[v+\beta,b]}$ or $D'_{[v-\beta,v+\beta]}$.

By Proposition 15.2 the connected components of $D'_{v-\beta}$ (resp. $D'_{v+\beta}$) are in 1-1 correspondence with the connected components of $D'_{[a,v-\beta]}$ (resp. $D'_{[v+\beta,b]}$) containing them. Thus, we can replace each of the γ_i lying in $D'_{[a,v-\beta]}$ (resp. $D'_{[v+\beta,b]}$) with endpoints in $D'_{v-\beta}$ (resp. $D'_{v+\beta}$) by another segment with the same endpoints but lying completely in $D'_{v-\beta}$ (resp. $D'_{v+\beta}$). We thus obtain a new semi-algebraic path γ' connecting x to y and lying inside $D'_{[v-\beta,v+\beta]}$.

It is clear that $\lim_\zeta (D'_{[v-\beta,v+\beta]})$ coincides with D_v. Since $D'_{[v-\beta,v+\beta]}$ is bounded, D_v is semi-algebraically connected by Proposition 12.43. □

Proof of Proposition 15.5: By Lemma 15.6, there exists D', a semi-algebraically connected component of $\mathrm{Zer}(\mathrm{Def}_+(Q,\zeta),\mathrm{R}\langle\zeta\rangle^{k+1})_{[a,b]}$ such that $D\times\{0\}=\lim_\zeta (D')$. According to Theorem 5.46 (Hardt's triviality), there exists $a'\in[a,\ b)$ such that for every $d\in[a',\ b)$, $D_{[a,d]}$ is not semi-algebraically connected. Hence, by Proposition 12.43, $D'_{[a,c]}$ is also not semi-algebraically connected for every $c\in\mathrm{R}\langle\zeta\rangle$ with $\lim_\zeta (c)=d$. Since D' is semi-algebraically connected, according to Proposition 15.3, there is an X_1-critical value c on $\mathrm{Zer}(\mathrm{Def}_+(Q,\zeta),\mathrm{R}\langle\zeta\rangle^{k+1})$, infinitesimally close to b. Hence b is an X_1-pseudo-critical value on $\mathrm{Zer}(Q,\mathrm{R}^k)$. □

15.2 Roadmap of an Algebraic Set

We describe the construction of a roadmap M for a bounded algebraic set $\mathrm{Zer}(Q,\ \mathrm{R}^k)$ which contains a finite set of points $\mathcal{N}$ of $\mathrm{Zer}(Q,\ \mathrm{R}^k)$. A precise description of how the construction can be performed algorithmically will follow.

We first construct X_2-pseudo-critical points on $\mathrm{Zer}(Q,\ \mathrm{R}^k)$ in a parametric way along the X_1-axis. This results in curve segments and their endpoints on $\mathrm{Zer}(Q,\mathrm{R}^k)$. The curve segments are continuous semi-algebraic curves parametrized by open intervals on the X_1-axis, and their endpoints are points of $\mathrm{Zer}(Q,\ \mathrm{R}^k)$ above the corresponding endpoints of the open intervals. Since these curves and their endpoints include, for every $x\in\mathrm{R}$, the X_2-pseudo-critical points of $\mathrm{Zer}(Q,\ \mathrm{R}^k)_x$, they meet every connected component of $\mathrm{Zer}(Q,\mathrm{R}^k)_x$. Thus the set of curve segments and their endpoints already satisfy RM_2. However, it is clear that this set might not be semi-algebraically connected in a semi-algebraically connected component, so RM_1 might not be satisfied (see Figure 15). We add additional curve segments to ensure that Mea is connected by recursing in certain distinguished hyperplanes defined by $X_1=z$ for distinguished values z.

The set of **distinguished values** is the union of the X_1-pseudo-critical values, the first coordinates of the input points $\mathcal{N}$ and the first coordinates of the endpoints of the curve segments. A **distinguished hyperplane** is an hyperplane defined by $X_1 = v$, where v is a distinguished value. The input points, the endpoints of the curve segments and the intersections of the curve segments with the distinguished hyperplanes define the set of **distinguished points** .

So we have constructed the distinguished values $v_1 < \cdots < v_\ell$ of X_1 among which are the X_1-pseudo-critical values. Above each interval (v_i, v_{i+1}), we have constructed a collection of curve segments $\mathcal{C}_i$ meeting every semi-algebraically connected component of $\mathrm{Zer}(Q, \mathrm{R}^k)_v$ for every $v \in (v_i, v_{i+1})$. Above each distinguished value v_i, we have constructed a set of distinguished points $\mathcal{N}_i$. Each curve segment in $\mathcal{C}_i$ has an endpoint in $\mathcal{N}_i$ and another in $\mathcal{N}_{i+1}$. Moreover, the union of the $\mathcal{N}_i$ contains $\mathcal{N}$.

We then repeat this construction in each distinguished hyperplane H_i defined by $X_1 = v_i$ with input $Q(v_i, X_2, \ldots, X_k)$ and the distinguished points in $\mathcal{N}_i$.

The process is iterated until for

$$I = (i_1, \ldots, i_{k-2}), 1 \le i_1 \le \ell, \ldots, 1 \le i_{k-2} \le \ell(i_1, \ldots, i_{k-3}),$$

we have distinguished values $v_{I,1} < \cdots < v_{I,\ell(I)}$ along the X_{k-1} axis with corresponding sets of curve segments and sets of distinguished points with the required incidences between them.

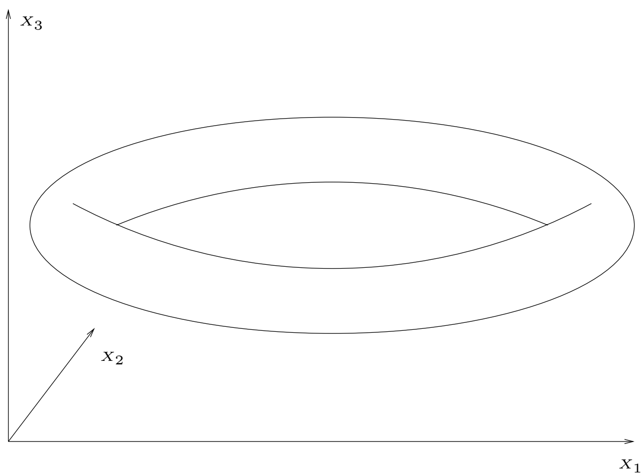

Fig. 15.1. A torus in R^3

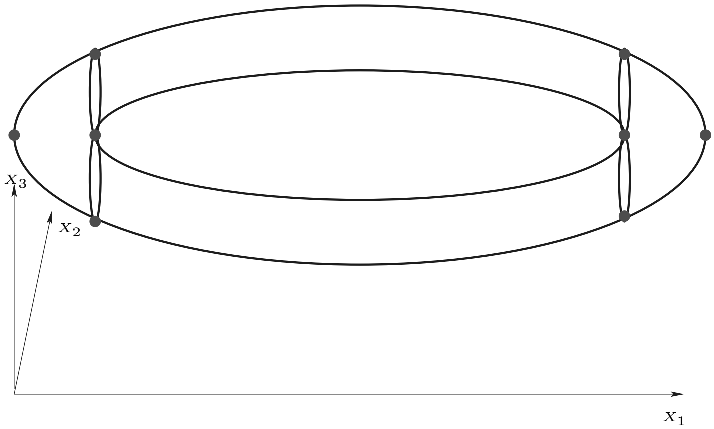

Fig. 15.2. The roadmap of the torus

Proposition 15.7. *The semi-algebraic set M obtained by this construction is a roadmap for* $\mathrm{Zer}(Q, \mathrm{R}^k)$.

The proof of Proposition 15.7 uses the following lemmas.

Lemma 15.8. *If $v \in (a, b)$ is a distinguished value such that $[a, b] \setminus \{v\}$ contains no distinguished value of π on* $\mathrm{Zer}(Q, \mathrm{R}^k)$ *and D is a semi-algebraically connected component of* $\mathrm{Zer}(Q, \mathrm{R}^k)_{[a,b]}$, *then $M \cap D$ is semi-algebraically connected.*

Proof: Since $[a, b] \setminus \{v\}$ contains no pseudo-critical value of the algebraic set $\mathrm{Zer}(Q, \mathrm{R}^k)$, we know, by Proposition 15.4, that D_v is semi-algebraically connected. Moreover, the points of $M \cap D$ are connected through curve segments to the points of $\mathcal{N}_v$. By induction hypothesis, the points of $\mathcal{N}_v$ are in the same semi-algebraically connected component of D_v, since D_v is semi-algebraically connected.

The construction makes a recursive call at every distinguished hyperplane and hence at H_v. The input to the recursive call is the algebraic set $\mathrm{Zer}(Q, \mathrm{R}^k)_v$ and the set of all distinguished points in H_v which includes the endpoints of the curves in $M \cap D \cap H_v$. Hence, by the induction hypothesis they are connected by the roadmap in the slice.

Therefore, $M \cap D$ is semi-algebraically connected. $\square$

Lemma 15.9. *If D is a semi-algebraically connected component of* $\mathrm{Zer}(Q, \mathrm{R}^k)$, *then $M \cap D$ is semi-algebraically connected.*

Proof: Let x, y be two points of $M \cap D$, and let γ be a semi-algebraic path in D from x to y such that $\gamma(0) = x$, $\gamma(1) = y$. We are going to construct another semi-algebraic path from x to y inside M. Let $\{v_1 < \cdots < v_\ell\}$ be the set of distinguished values and choose u_i such that

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_\ell < v_\ell < u_{\ell+1}.$$

There exist a finite number of points of γ, $x = x_0$, x_1, ..., $x_{N+1} = y$, with $\pi(x_i) = u_{n(i)}$, and semi-algebraic paths γ_i from x_i to x_{i+1} such that:

- $\gamma = \bigcup_{0 \leq i \leq N} \gamma_i$,
- $\gamma_i \subset D_{[u_{n(i)}, u_{n(i)+1}]}$ or $\gamma_i \subset D_{[u_{n(i)-1}, u_{n(i)}]}$.

Let D_i be the semi-algebraically connected component of $D_{[u_{n(i)}, u_{n(i)+1}]}$ (resp. $D_{[u_{n(i)-1}, u_{n(i)}]}$) containing γ_i. Since $D_{i-1} \cap D_i$ is a finite union of semi-algebraically connected components of $D_{\pi(x_i)}$, $M \cap D_{i-1} \cap D_i$ is not empty. Choose $y_0 = x, ..., y_i \in M \cap D_{i-1} \cap D_i, ..., y_{N+1} = y$. Then y_i and y_{i+1} are in the same semi-algebraically connected component of $M \cap D$ by Lemma 15.8. □

Proof of Proposition 15.7: We have already seen that M satisfies RM_2. We now prove that M satisfies RM_1.

The proof is by induction on the dimension of the ambient space. In the case of dimension one, the roadmap properties are obviously true for the set we have constructed. Now assume that the construction gives a roadmap for all dimensions less than k. That the construction gives a roadmap for dimension k follows from the following two lemmas. Lemma 15.8 and Lemma 15.9. □

We now describe precisely a way of performing algorithmically the preceding construction.

In our inductive construction of the roadmap, we are going to use the following specification describing points and curve segments:

A **real univariate triangular representation** $\mathcal{T}$, σ, u of level $i-1$ consists of:

- a triangular Thom encoding $\mathcal{T}, \sigma$ specifying $(z, t) \in \mathrm{R}^i$ with $z \in \mathrm{R}^{i-1}$
- a parametrized univariate representation

$$u(X_{<i}) = (\mathcal{T}_i(X_{<i}, T), g_0(X_{<i}, T), g_i(X_{<i}, T), ..., g_k(X_{<i}, T)),$$

  with parameters $X_{<i} = (X_1, ..., X_{i-1})$ (see Definition page 481).

The point associated to $\mathcal{T}, \sigma, u$ is

$$\left(z, \frac{g_i(z,t)}{g_0(z,t)}, ..., \frac{g_k(z,t)}{g_0(z,t)}\right).$$

A real univariate triangular representation $\mathcal{T}$, σ, u is **above** the triangular Thom encoding $\mathcal{T}', \sigma'$ if $\mathcal{T}' = \mathcal{T}_1, ..., \mathcal{T}_{i-1}, \sigma' = \sigma_1, ..., \sigma_{i-1}$.

It will be useful to compute the i-th projection of a point specified by a real univariate representation.

Algorithm 15.1. **[Projection]**

- **Structure:** a domain D contained in a field K.

- **Input:**
  a real univariate triangular representation $\mathcal{T}, \sigma, u$ of level $i-1$ with coefficients in D. We denote by z the root of $\mathcal{T}$ specified by σ and by x the point associated to $\mathcal{T}, \sigma, u$.
- **Output:** a Thom encoding $\mathrm{proj}_i(u), \mathrm{proj}_i(\tau)$ specifying the projection of the point associated to $\mathcal{T}, \sigma, u$ on the X_i axis.
- **Complexity:** $d^{O(i)}$, where d is a bound on the degree of the univariate representation and a bound on the degrees of the polynomials in $\mathcal{T}$.
- **Procedure:**
  - Compute the resultant $\mathrm{proj}_i(u)$ of $\mathcal{T}_i(X_{<i}, T),$ and
  $$X_i\, g_0(X_{<i}, T) - g_i(X_{<i}, T)$$
  with respect to $T,$ using Algorithm 8.21 (Signed subresultant).
  - Compute the Thom encoding of the root of $\mathrm{proj}_i(u)$ which is the i-th coordinate of x as follows: let d be the smallest even number not less than the degree of $\mathrm{proj}_i(u)$ with respect to X_i, and compute the sign of the derivatives of
  $$g_0(X_{<i}, T)^d \,\mathrm{proj}_i(u)\left(\frac{g_i(X_{<i}, T)}{g_0(X_{<i}, T)}\right)$$
  with respect to T at the root z of $\mathcal{T}$ specified by σ. This gives the Thom encoding $\mathrm{proj}_i(\tau)$ of the i-th coordinate of x. This is done using Algorithm 12.19 (Triangular Sign Determination).

Proof of correctness: Immediate. □

Complexity analysis: The complexity is $d^{O(ki)}$ using the complexity of Algorithm 12.19 (Triangular Sign Determination).

If $\mathrm{D} = \mathbb{Z},$ and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(i)}$. □

Let $\mathcal{V}_1, \tau_1, \mathcal{V}_2, \tau_2$ be two triangular Thom encodings above $\mathcal{T}, \sigma$. We denote by $z = (z_1, \ldots, z_{i-1}) \in \mathrm{R}^{i-1}$ the point specified by $\mathcal{T}, \sigma$ and by (z, a), (z, b) the points specified by $\mathcal{V}_1, \tau_1$ and $\mathcal{V}_2, \tau_2$ (see Definition page 496).

A **curve segment representation** u, ρ **above** $\mathcal{V}_1, \tau_1, \mathcal{V}_2, \tau_2$ is:

- a parametrized univariate representation with parameters $(X_{\le i})$
$$u = (f(X_{\le i}, T), g_0(X_{\le i}, T), g_{i+1}(X_{\le i}, T), \ldots, g_k(X_{\le i}, T)),$$
- a sign condition ρ on $\mathrm{Der}(f)$ such that for every $v \in (a, b)$ there exists a real root $t(v)$ of $f(z, v, T)$ with Thom encoding σ, ρ and $g_0(z, v, t(v)) \neq 0$.

The **curve segment associated to** u, ρ is the semi-algebraic function h which maps a point v of (a,b) to the point of R^k defined by

$$h(v)=\left(z,v,\frac{g_{i+1}(z,v,t(v))}{g_0(z,v,t(v))},\ldots,\frac{g_k(z,v,t(v))}{g_0(z,v,t(v))}\right).$$

It is a continuous injective semi-algebraic function.

The Curve Segments Algorithm will be the basic building block in our algorithm.

Algorithm 15.2. **[Curve Segments]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:**
  - a triangular Thom encoding $\mathcal{T}$, σ with coefficients in D specifying $z\in\mathrm{R}^{i-1}$,
  - a polynomial $Q\in\mathrm{D}[X_1\ldots,X_k]$, for which $\mathrm{Zer}(Q,\mathrm{R}^k)\subset B(0,1/c)$,
  - a finite set $\mathcal{N}$ of real univariate triangular representation above $\mathcal{T},\sigma$ with coefficients in D and associated points contained in $\mathrm{Zer}(Q,\mathrm{R}^k)$.
- **Output:**
  - an ordered list of triangular Thom encodings $\mathcal{V}_1,\tau_1,\ldots,\mathcal{V}_\ell,\tau_\ell$ above $\mathcal{T},\sigma$ specifying points $(z,v_1),\ldots,(z,v_\ell)$ with $v_1<\cdots<v_\ell$. The v_j are called distinguished values.
  - For every $j=1,\ldots,\ell$,
    - a finite set $\mathcal{D}_j$ of real univariate triangular representations representation above $\mathcal{V}_j,\tau_j$. The associated points are called distinguished points.
    - a finite set $\mathcal{C}_j$ of curve segment representations above $\mathcal{V}_j$, τ_j, $\mathcal{V}_{j+1}$, τ_{j+1}. The associated curve segments are called distinguished curves.
    - a list of pairs of elements of $\mathcal{C}_j$ and $\mathcal{D}_j$ (resp. $\mathcal{C}_{j+1}$ and $\mathcal{D}_j$) describing the adjacency relations between distinguished curves and distinguished points.

  The distinguished curves and points are contained in $\mathrm{Zer}(Q,\mathrm{R}^k)_z$. Among the distinguished values are the first coordinates of the points in $\mathcal{N}$ as well as the pseudo-critical values of $\mathrm{Zer}(Q,\mathrm{R}^k)_z$. The sets of distinguished values, distinguished curves, and distinguished points satisfy the following properties.
  - CS_1: For every $v\in\mathrm{R}$, the set of distinguished curve and distinguished points output intersect every semi-algebraically connected component of $\mathrm{Zer}(Q,\mathrm{R}^k)_{(z,v)}$.
  - CS_2: For each distinguished curve output over an interval with endpoint a given distinguished value, there exists a distinguished point over this distinguished value which belongs to the closure of the curve segment.

- **Complexity:** $d^{O(ik)}$, where d is a bound on the degree of Q and $O(d)^k$ is a bound on the degrees of the polynomials in $\mathcal{T}$, the degrees of the univariate representations in $\mathcal{N}$, and the number of these univariate representations.
- **Procedure:**
  - Step 1: Perform Algorithm 12.10 (Parametrized Multiplication Table) with input $\overline{\mathrm{Cr}}(Q^2,\zeta)$, (using Notation 12.46) and parameter $X_{\le i}$. Perform Algorithm 12.15 (Parametrized Limit of Bounded Points) and output $\mathcal{U}$.

    Consider for every $u=(f,\, g_0,\, g_{i+1},\, \ldots,\, g_k)\in\mathcal{U}$ the finite set $\mathcal{F}_u$ containing Q_u (Notation 13.8) and all the derivatives of f with respect to T, and compute

    $$\mathcal{D}_u = \mathrm{RElim}_T(f,\mathcal{F}_u) \subset \mathrm{D}[X_{\le i}],$$

    using Algorithm 11.19 (Restricted Elimination). Define $\mathcal{D}=\bigcup_{u\in\mathcal{U}}\mathcal{D}_u$.
  - Step 2: For every $\mathcal{T}', \tau, u \in \mathcal{N}$, compute $\mathrm{proj}_i(u), \mathrm{proj}_i(\tau)$ using Algorithm 15.1 (Projection), add to $\mathcal{D}$ the polynomial $\mathrm{proj}_i(u)$.
  - Step 3: Compute the Thom encodings of the zeroes of A, $A\in\mathcal{D}$ above $\mathcal{T}$, σ using Algorithms 12.20 (Triangular Thom Encoding), output their ordered list $A_1,\, \alpha_1,\, \ldots,\, A_\ell,\, \alpha_\ell$ and the corresponding ordered list $v_1<\cdots<v_\ell$ of distinguished values using Algorithm 12.21 (Triangular Comparison of Roots). Define $\mathcal{V}_i, \tau_i = \mathcal{T}, A_i, \sigma, \alpha_i$.
  - Step 4: For every $j=1,\, \ldots,\, \ell$ and every $(f,\, g_0,\, g_i,\, \ldots,\, g_k)$, $\tau\in\mathcal{N}$ such that $\mathrm{proj}_i(\tau)=\alpha_j$, append $(f,\, g_0,\, g_{i+1},\, \ldots,\, g_k)$, τ to $\mathcal{D}_j$, using Algorithm 12.19 (Triangular Sign Determination).
  - Step 5: For every $j=1,\ldots,\ell$ and every

    $$u=(f,g_0,g_{i+1},\ldots,g_k)\in\mathcal{U},$$

    compute the Thom encodings τ of the roots of f above $\mathcal{T}$, σ such that $\mathrm{proj}_i(\tau)=\alpha_j$, using Algorithm 12.20 (Triangular Thom Encoding). Append all pairs $(f,\, g_0,\, g_{i+1},\, \ldots,\, g_k)$, τ to $\mathcal{D}_j$ when the corresponding associated point belongs to $\mathrm{Zer}(Q,\mathrm{R}^k)_z$.
  - Step 6: For every $j=1,\ldots,\ell-1$ and every

    $$u=(f,g_0,g_{i+1},\ldots,g_k)\in\mathcal{U},$$

    compute the Thom encodings ρ of the roots of $f(z,v,T)$ over (v_j,v_{j+1}) using Algorithm 12.22 (Triangular Intermediate Points) and Algorithm 12.20 (Triangular Thom Encoding) and append pairs u,ρ to $\mathcal{C}_j$ when the corresponding associated curve is included in $\mathrm{Zer}(Q,\mathrm{R}^k)_z$.
  - Step 7: Determine adjacencies between curve segments and points. For every point of $\mathcal{D}_j$ specified by

    $$v'=(p,q_0,q_{i+1},\ldots,q_k),\tau', \text{with } \{p,q_0,q_{i+1},\ldots,q_k\}\subset\mathrm{D}[X_{\le i}][T]$$

and every curve segment representation of $\mathcal{C}_j$ specified by

$$v=(f, g_0, g_{i+1}, \ldots, g_k), \tau, \{f, g_0, g_{i+1}, \ldots, g_k\} \subset \mathrm{D}[X_{\leq i}][T],$$

decide whether the associated point t is adjacent to the associated curve segment as follows: compute the first ν such that $(\partial^\nu g_0/\partial X_i^\nu)(v_j, t)$ is not zero and decide whether for every $\ell = i+1, \ldots, k$

$$\frac{\partial^\nu g_\ell}{\partial X_i^\nu}(v_j, t) q_0(t) - \frac{\partial^\nu g_0}{\partial X_i^\nu}(v_j, t) q_\ell(t)$$

is zero. This is done using Algorithm 12.19 (Triangular Sign Determination) above $\mathcal{T}, \sigma$.

Repeat the same process for every element of $\mathcal{D}_{j+1}$ and every curve segment representation of $\mathcal{C}_j$.

Proof of correctness: It follows from Proposition 12.42, the correctness of Algorithm 12.10 (Parametrized Multiplication Table), Algorithm 12.15 (Parametrized Limit of Bounded Points), Algorithm 11.19 (Restricted Elimination), Algorithm 15.1, Algorithm 12.22 (Triangular Intermediate Points), Algorithm 12.20 (Triangular Thom Encoding), Algorithm 12.21 (Triangular Comparison of Roots) and Algorithm 12.19 (Triangular Sign Determination). □

Complexity analysis:

- Step 1: This step requires $d^{O(i(k-i))}$ arithmetic operations in D, using the complexity analysis of Algorithm 12.10 (Parametrized Multiplication Table), Algorithm 12.15 (Parametrized Limit of Bounded Points), Algorithm 11.19 (Restricted Elimination). There are $d^{O(k-i)}$ parametrized univariate representations computed in this step and each polynomial in these representations has degree $O(d)^{k-i}$.
- Step 2: This step requires $d^{O(ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 15.1 (Projection).
- Step 3: This step requires $d^{O(ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 12.20 (Triangular Thom Encoding).
- Step 4: This step requires $d^{O(ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 12.19 (Triangular Sign Determination)
- Step 5: This step requires $d^{O(ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 12.20 (Triangular Thom Encoding).
- Step 6: This step requires $d^{O(ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 12.22 (Triangular Intermediate Points), Algorithm 12.20 (Triangular Thom Encoding).
- Step 7: This step requires $d^{O(ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 12.19 (Triangular Sign Determination).

Thus, the complexity is $d^{O(ik)}$. The number of distinguished values is bounded by $d^{O(k)}$.

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(ik)}$. □

Given a polynomial Q and a set of real univariate representations $\mathcal{N}$, we denote by $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{N})$ a roadmap of $\mathrm{Zer}(Q, \mathrm{R}^k)$ which contains the points associated to $\mathcal{N}$.

We now describe a recursive roadmap algorithm for bounded algebraic sets.

Algorithm 15.3. **[Bounded Algebraic Roadmap]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:**
  - a triangular Thom encoding $\mathcal{T}$, σ with coefficients in D specifying $z \in \mathrm{R}^i$,
  - a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$, for which $\mathrm{Zer}(Q, \mathrm{R}^k) \subset B(0, 1/c)$,
  - a finite set $\mathcal{N}$ of real univariate representation u, τ above $\mathcal{T}$, σ with coefficients in D with associated points contained in $\mathrm{Zer}(Q, \mathrm{R}^k)_z$.
- **Output:** a roadmap $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k)_z, \mathcal{N})$ which contains the points associated to$\mathcal{N}$.
- **Complexity:** $d^{O(k^2)}$, where d is a bound on the degree of Q and $O(d)^k$ is a bound on the degrees of the polynomials in $\mathcal{T}$, the degrees of the univariate representations in $\mathcal{N}$, and the number of these univariate representations.
- **Procedure:**
  - Call Algorithm 15.2 (Curve Segments), output ℓ and, for every $j = 1, \ldots, \ell$, A_j, α_j, $\mathcal{D}_j$ and $\mathcal{C}_j$.
  - For every $j = 1, \ldots, \ell$, call Algorithm 15.3 (Bounded Algebraic Roadmap) recursively, with input $\mathcal{T}$, A_j, σ, α_j, specifying (z, v_j), Q and $\mathcal{D}_j$.

Proof of correctness: The correctness of the algorithm follows from Proposition 15.7 and the correctness of Algorithm 15.2 (Curve Segments). □

Complexity analysis: In the recursive calls to Algorithm 15.3 (Bounded Algebraic Roadmap), the number of triangular systems considered is at most $d^{O(k^2)}$ and the triangular systems involved have polynomials of degree $O(d)^k$. Thus the total number of arithmetic operations in D is bounded by $d^{O(k^2)}$ using the complexity analysis of Algorithm 15.2 (Curve Segments).

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^2)}$. □

Since $\mathrm{RM}(\mathrm{Zer}(Q,\,\mathrm{R}^k)_z,\,\mathcal{N})$ contains $\mathrm{RM}(\mathrm{Zer}(Q,\,\mathrm{R}^k)_z)$, it is possible to extract from $\mathrm{RM}(\mathrm{Zer}(Q,\mathrm{R}^k)_z,\,\{u,\tau\})$ a path connecting the point p associated to u,τ to $\mathrm{RM}(\mathrm{Zer}(Q,\mathrm{R}^k)_z)$.

Algorithm 15.4. **[Bounded Algebraic Connecting]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:**
  - a triangular Thom encoding $\mathcal{T}$, σ with coefficients in D specifying $z \in \mathrm{R}^i$,
  - a polynomial $Q \in \mathrm{D}[X_1,\ldots,X_k]$ for which $\mathrm{Zer}(Q,\mathrm{R}^k) \subset B(0,1/c)$,
  - a real univariate triangular representation $\mathcal{V}$, τ, u above $\mathcal{T}$, σ with coefficients in D, with associated point p contained in $\mathrm{Zer}(Q,\mathrm{R}^k)_z$.
- **Output:** a path $\gamma(p) \subset \mathrm{Zer}(Q,\mathrm{R}^k)_z$ connecting p to a distinguished point of $\mathrm{RM}(\mathrm{Zer}(Q,\mathrm{R}^k)_z)$.
- **Complexity:** $d^{O(k^2)}$, where d is a bound on the degree of Q and $O(d)^k$ is a bound on the degrees of the polynomials in $\mathcal{T}$ and the degree of the real univariate triangular representation $\mathcal{V},\tau,u$.
- **Procedure:** Call Algorithm 15.3 (Bounded Algebraic Roadmap) with input Q, $\mathcal{T}$, σ and $\{\mathcal{V},\,\tau,\,u\}$, and extract $\gamma(p)$ from $\mathrm{RM}(\mathrm{Zer}(Q,\,\mathrm{R}^k)$, $\{\mathcal{V},\tau,u\})$.

Proof of correctness: The correctness of the algorithm follows from the correctness of Algorithm 15.3 (Bounded Algebraic Roadmap). ☐

Complexity analysis: The total number of arithmetic operations in D is bounded by $d^{O(k^2)}$, using the complexity analysis of Algorithm 15.3 (Bounded Algebraic Roadmap).

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^2)}$. ☐

Remark 15.10. Note that the connecting path $\gamma(p)$ consists of two consecutive parts, $\gamma_0(p)$ and $\Gamma_1(p)$. The path $\gamma_0(p)$ is contained in $\mathrm{RM}(\mathrm{Zer}(Q,\mathrm{R}^k))$ and the path $\Gamma_1(p)$ is contained in $\mathrm{Zer}(Q,\,\mathrm{R}^k)_{p_1}$. The part $\gamma_0(p)$ consists of a sequence of sub-paths, $\gamma_{0,0},\ldots,\gamma_{0,m}$. Each $\gamma_{0,i}$ is a semi-algebraic path parametrized by one of the co-ordinates $X_1,\ldots,X_k$, over some interval $[a_{0,i},\,b_{0,i}]$ with $\gamma_{0,0}(a_{0,0}) = p$. The semi-algebraic maps, $\gamma_{0,0},\ldots,\gamma_{0,m}$ and the end-points of their intervals of definition $a_{0,0},b_{0,0},\ldots,a_{0,m},b_{0,m}$ are all independent of p (up to the discrete choice of the path $\gamma(p)$ in $\mathrm{RM}(\mathrm{Zer}(Q,\,\mathrm{R}^k),\,\{p\})$), except $b_{0,m}$ which depends on p_1.

Moreover, $\Gamma_1(p)$ can again be decomposed into two parts, $\gamma_1(p)$ and $\Gamma_2(p)$ with $\Gamma_2(p)$ contained in $\mathrm{Zer}(Q,\mathrm{R}^k)_{\bar{p}_2}$ and so on.

If $q = (q_1,\ldots,q_k) \in \mathrm{Zer}(Q,\mathrm{R}^k)$ is another point such that $p_1 \neq q_1$, then since $\mathrm{Zer}(Q,\mathrm{R}^k)_{p_1}$ and $\mathrm{Zer}(Q,\mathrm{R}^k)_{q_1}$ are disjoint, it is clear that

$$\mathrm{RM}(\mathrm{Zer}(Q,\mathrm{R}^k),\{p\}) \cap \mathrm{RM}(\mathrm{Zer}(Q,\mathrm{R}^k),\{q\}) = \mathrm{RM}(\mathrm{Zer}(Q,\mathrm{R}^k)).$$

Now consider a connecting path $\gamma(q)$ extracted from $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k), \{q\})$. The images of $\Gamma_1(p)$ and $\Gamma_1(q)$ are disjoint. If the image of $\gamma_0(q)$ (which is contained in $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k))$ follows the same sequence of curve segments as $\gamma_0(q)$ starting at p (that is, it consists of the same curves segments $\gamma_{0,0}, \ldots, \gamma_{0,m}$ as in $\gamma_0(p)$), then it is clear that the images of the paths $\gamma(p)$ and $\gamma(q)$ has the property that they are identical up to a point and they are disjoint after it. We call this the **divergence property**. $\square$

Next we show how to handle the case when the input algebraic set $\mathrm{Zer}(Q, \mathrm{R}^k)$ is not bounded.

Algorithm 15.5. **[Algebraic Roadmap]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:** a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$ together with a finite set $\mathcal{N}$ of real univariate representations with coefficients in D.
- **Output:** a roadmap $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{N})$ which contains $\mathcal{N}$.
- **Complexity:** $d^{O(k^2)}$, where d is a bound on the degree of Q and $O(d)^k$ is a bound on the degrees of the polynomials in $\mathcal{T}$, the degrees of the univariate representations in $\mathcal{N}$, and the number of these univariate representations.
- **Procedure:**
  - Introduce new variables X_{k+1} and ε and replace Q by the polynomial
  $$Q_\varepsilon = Q^2 + (\varepsilon^2 (X_1^2 + \cdots + X_{k+1}^2) - 1)^2.$$
  Replace $\mathcal{N} \subset \mathrm{R}^k$ by the set of real univariate representations specifying the elements of $\mathrm{Zer}(\varepsilon^2 (X_1^2 + \cdots + X_{k+1}^2) - 1, \mathrm{R}\langle\varepsilon\rangle^{k+1})$ above the points associated to $\mathcal{N}$ using Algorithm 12.11 (Univariate Representation).
  - Run Algorithm 15.3 (Bounded Algebraic Roadmap) without a triangular Thom encoding (i.e. with $i = 0$), Q_ε and $\mathcal{N}$ as input with structure $\mathrm{D}[\varepsilon]$. The algorithm outputs a roadmap of $\mathrm{RM}(\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1}), \mathcal{N})$ composed of points and curves whose description involves ε.
  - Denote by $\mathcal{L}$ the set of polynomials in $\mathrm{D}[\varepsilon]$ whose signs have been determined in the preceding computation and take $a = \min_{P \in \mathcal{L}} c'(P)$ (Definition 10.5). Replace ε by a in the polynomial Q_ε to get a polynomial Q_a. Replace ε by a in the output roadmap to obtain a roadmap $\mathrm{RM}(\mathrm{Zer}(Q_a, \mathrm{R}^{k+1}), \mathcal{N})$. When projected to R^k, this roadmap gives a roadmap for $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{N}) \cap B(0, 1/a)$.
  - In order to extend the roadmap outside the ball $B(0, 1/a)$ collect all the points $(y_1, \ldots, y_k, y_{k+1}) \in \mathrm{R}\langle\varepsilon\rangle^{k+1}$ in the roadmap $\mathrm{RM}(\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1}), \mathcal{N})$ which satisfies $\varepsilon(y_1^2 + \cdots + y_k^2) = 1$. Each such point is described by a real univariate representation involving ε. Add to the roadmap the curve segment obtained by first forgetting the last coordinate and then treating ε as a parameter which varies vary over $(0, a,]$ to get a roadmap $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{N})$.

Proof of correctness: The choice of a guarantees that the roadmap for Q_ε just computed specializes to a roadmap for Q_a when ε is replaced by a. The correctness follows from the correctness of Algorithm 15.3 (Bounded Algebraic Roadmap). □

Complexity analysis: According to the complexity analysis of Algorithm 15.3 (Bounded Algebraic Roadmap), the number of arithmetic operations in the ring $\mathrm{D}[\varepsilon]$ is $d^{O(k^2)}$. Moreover, the degrees of the polynomials in ε generated by the algorithm do not exceed $d^{O(k^2)}$, using the complexity analysis of Algorithm 12.10 (Parametrized Special Multiplication Table). The complexity is thus $d^{O(k^2)}$ in the ring D, taking into account the complexity analyses of Algorithm 8.4 (Addition of multivariate polynomials), Algorithm 8.5 (Multiplication of Multivariate Polynomials), and Algorithm 8.6 (Exact Division of Multivariate Polynomials).

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^2)}$. □

Algorithm 15.6. **[Algebraic Connecting]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:**
  - a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$,
  - a real univariate representation u, τ with coefficients in D, with associated point p contained in $\mathrm{Zer}(Q, \mathrm{R}^k)$.
- **Output:** a path $\gamma(p, \mathrm{Zer}(Q, \mathrm{R}^k)) \subset \mathrm{Zer}(Q, \mathrm{R}^k)$ connecting p to a distinguished point of $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k))$.
- **Complexity:** $d^{O(k^2)}$, where d is a bound on the degree of Q and $O(d)^k$ is a bound on the degrees of u.
- **Procedure:** Call Algorithm 15.5 (Algebraic Roadmap) with input Q and (u, τ) and extract γ from $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k), \{u, \tau\})$.

Proof of correctness: The correctness of the algorithm follows from the correctness of Algorithm 15.5 (Algebraic Roadmap). □

Complexity analysis: The total number of arithmetic operations in D is bounded by $d^{O(k^2)}$, using the complexity analysis of Algorithm 15.5 (Algebraic Roadmap).

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^2)}$. □

We can now summarize our results on the complexity of the computation of the roadmap for an algebraic set.

Theorem 15.11. *Let $Q \in \mathrm{R}[X_1, \ldots, X_k]$ be a polynomial whose total degree is at most d.*

a) There is an algorithm whose output is exactly one point in every semi-algebraically connected component of $\mathrm{Zer}(Q, \mathrm{R}^k)$. The complexity in the ring generated by the coefficients of Q is bounded by $d^{O(k^2)}$. In particular, this algorithm counts the number of semi-algebraically connected components of $\mathrm{Zer}(Q, \mathrm{R}^k)$ in time $d^{O(k^2)}$. If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^2)}$.

b) Let p and q in $\mathrm{Zer}(Q, \mathrm{R}^k)$ be two points which are represented by real k-univariate real representation u, σ v, τ of degree $O(d)^k$. There is an algorithm deciding whether p and q belong to the same connected component of $\mathrm{Zer}(Q, \mathrm{R}^k)$. The complexity in the ring generated by the coefficients of Q and the coefficients of the polynomials in u and v is bounded by $d^{O(k^2)}$. If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^2)}$.

Proof: For a), proceed as follows: first compute $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k))$, then describe its connected components using the adjacencies between curve segments and points, and finally take one point in each of these connected components.

For b), use Algorithm 15.6 (Algebraic Connecting) for p and q. The points p and q are connected to points p' and q' of the roadmap. Use the first item to decide whether they belong to the same connected component or not. □

15.3 Computing Connected Components of Algebraic Sets

This section is devoted to the proof of the following result.

Theorem 15.12. *If $\mathrm{Zer}(Q, \mathrm{R}^k)$ is an algebraic set defined as the zero set of a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$ of degree $\le d$, then there is an algorithm that outputs quantifier free formulas whose realizations are the semi-algebraically connected components of $\mathrm{Zer}(Q, \mathrm{R}^k)$. The complexity of the algorithm in the ring generated by the coefficients of Q is bounded by $d^{O(k^3)}$ and the degrees of the polynomials that appear in the output are bounded by $O(d)^{k^2}$. Moreover, if $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^3)}$.*

The proof is based on a parametrized version of the roadmap algorithm: we are going to find sign conditions on the parameters for which the description of the roadmap does not change.

For this purpose, we need parametrized versions of Algorithm 12.21 (Triangular Comparison of Roots) and Algorithm 12.22 (Triangular Intermediate Points). These algorithms will be based on Algorithm 14.6 (Parametrized Sign Determination).

Let $\mathcal{A} \subset \mathcal{B}$, ρ and $\bar{\rho}$ two sign conditions on $\mathcal{A}$ and $\mathcal{B}$. The sign condition $\bar{\rho}$ **refines** ρ if $\bar{\rho}(P) = \rho(P)$ for every $P \in \mathcal{A}$.

Notation 15.13. We denote by $\mathrm{SIGN}(\rho, \mathcal{B})$ the list of realizable sign conditions on $\mathcal{B}$ refining ρ. □

Algorithm 15.7. **[Parametrized Comparison of Roots]**

- **Structure**: an ordered integral domain D contained in a real closed field R.
- **Input**: a parametrized Thom encoding $\mathcal{A}$, ρ, $\mathcal{T}$, σ, of level $k-1$, with coefficients in D, two non-zero polynomials P and $Q \in \mathrm{D}[Y, X_1, \ldots, X_k]$.
- **Output**:
  - a finite set $\mathcal{B} \subset \mathrm{D}[Y]$ containing $\mathcal{A}$,
  - for every $\bar{\rho} \in \mathrm{SIGN}(\rho, \mathcal{B})$, a list of sign conditions on $\mathrm{Der}(\mathcal{T} \cup \{P\} \cup \{Q\})$ refining σ specifying for every $y \in \mathrm{Reali}(\rho)$ the ordered list of the triangular Thom encodings of the roots of P and Q above the point specified by σ.
- **Complexity:** $d^{O(k\ell)}$, where ℓ is the number of parameters and d is a bound on the degrees of the polynomials in $\mathcal{T}$, and the degree of P and Q.
- **Procedure:** Apply Algorithm 14.6 (Parametrized Sign Determination) to $\mathcal{T}, P$ and
$$\mathrm{Der}(\mathcal{T}) \cup \mathrm{Der}(P) \cup \mathrm{Der}(Q),$$
then to $\mathcal{T}, Q$ and
$$\mathrm{Der}(\mathcal{T}) \cup \mathrm{Der}(P) \cup \mathrm{Der}(Q)$$

Proof of correctness: Immediate. □

Complexity analysis: The complexity is $d^{O(k\ell)}$, using the complexity of Algorithm Algorithm 14.6 (Parametrized Sign Determination). The number of elements in $\mathcal{B}$ is $d^{O(k\ell)}$, and the degrees of the elements of $\mathcal{A}$ are bounded by $d^{O(k)}$.

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k\ell)}$. □

Algorithm 15.8. **[Parametrized Intermediate Points]**

- **Structure**: an ordered integral domain D contained in a real closed field R.

- **Input**: a parametrized Thom encoding $\mathcal{A}$, ρ, $\mathcal{T}$, σ of level $k-1$, with coefficients in D, two non-zero polynomials P and Q in $\mathrm{D}[Y, X_1, \ldots, X_k]$ of degree bounded by p.
- **Output**:
  - a finite set $\mathcal{B} \subset \mathrm{D}[Y]$ containing $\mathcal{A}$
  - for every $\bar{\rho} \in \mathrm{SIGN}(\rho, \mathcal{B})$, a list of sign conditions on $\mathrm{Der}(\mathcal{T} \cup \{(PQ)'\})$ specifying for every $y \in \mathrm{Reali}(\bar{\rho})$ the triangular Thom encodings of a set of points intersecting all the intervals between two consecutive roots of P and Q.
- **Complexity:** $d^{O(k\ell)}$, where ℓ is the number of parameters and d is a bound on the degrees of the polynomials in $\mathcal{T}$, and the degree of P and Q.
- **Procedure:** Apply Algorithm 14.7 (Parametrized Thom Encoding) with input $\mathcal{T}$, P, $\mathcal{T}$, Q and $\mathcal{T}$, $P'Q$. Sort them using Algorithm 15.7 (Parametrized Comparison of Roots).

Proof of correctness: Immediate. □

Complexity analysis: The complexity is $d^{O(k\ell)}$, using the complexity of Algorithm Algorithm 14.6 (Parametrized Sign Determination). The number of elements in $\mathcal{A}$ is $d^{O(k\ell)}$, and the degrees of the elements of $\mathcal{B}$ are bounded by $d^{O(k)}$.

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k\ell)}$. □

A **parametrized real univariate triangular representation** of level $i-1$ with parameters $Y = (Y_1, \ldots, Y_\ell)$ $\mathcal{T}, \sigma, u$ above $\mathcal{A}, \rho$ is

- a parametrized triangular Thom encoding $\mathcal{T}, \sigma$ of level i,
- a parametrized representation $u = (\mathcal{T}_i, g_0, g_i, \ldots, g_k) \subset \mathrm{D}[Y, X_{\le i}, T]$ such that for every $y \in \mathrm{Reali}(\rho)$ there is a root $(z(y), t(y))$ of $\mathcal{T}$ with triangular Thom encoding σ.

A parametrized real univariate triangular representation $\mathcal{T}, \sigma, u$ is **above** the parametrized triangular Thom encoding $\mathcal{A}, \rho, \mathcal{T}', \sigma'$ if $\mathcal{T}, \sigma$ is above $\mathcal{A}, \rho$ and if $\mathcal{T}' = \mathcal{T}_1, \ldots, \mathcal{T}_{i-1}$, and $\sigma' = \sigma_1, \ldots, \sigma_{i-1}$.

Algorithm 15.9. **[Parametrized Projection]**

- **Structure:** a domain D contained in a field K.
- **Input:** a parametrized real univariate representation $\mathcal{T}$, u, σ above a parametrized triangular Thom encoding $\mathcal{A}$, ρ, with coefficients in D. For every $y \in \mathrm{Reali}(\rho)$, we denote by $z(y)$ the root of $\mathcal{T}(y)$ specified by τ and by $x(y)$ the point associated to $u(y, z(y))$.
- **Output:**
  - a finite set $\mathcal{B} \subset \mathrm{D}[Y]$ containing $\mathcal{A}$,

  – for every $\bar{\rho} \in \mathrm{SIGN}(\rho, \mathcal{B})$ a Thom encoding $(\mathrm{proj}_i(u), \mathrm{proj}_i(\tau))$ specifying, for every $y \in \mathrm{Reali}(\bar{\rho})$, the projection of the point associated to $x(y)$ on the X_i axis.
- **Complexity:** $d^{O(ki\ell)}$, where ℓ is the number of parameters, d is a bound on the degrees of on the degree of the univariate representation and of the polynomials in $\mathcal{T}$.
- **Procedure:**
  – Compute the resultant $\mathrm{proj}_i(u)$ of $f(Y, X_{<i}, T)$, and

$$X_i\, g_0(Y, X_{<i}, T) - g_i(Y, X_{<i}, T)$$

  with respect to T, using Algorithm 8.21 (Signed subresultant).
  – Use Algorithm 14.6 (Parametrized Sign Determination) with $\mathcal{T}, f$ and the derivatives of

$$g_0(Y, X_{<i}, T)^d\, \mathrm{proj}_i(u)\left(\frac{g_i(Y, X_{<i}, T)}{g_0(Y, X_{<i}, T)}\right)$$

  with respect to T, where d is the smallest even number not less than the degree of $\mathrm{proj}_i(u)$ with respect to X_i. This gives a list of polynomials $\mathcal{B} \subset \mathrm{D}[Y]$ and for every $\bar{\rho} \in \mathrm{SIGN}(\rho, \mathcal{B})$ the Thom encoding $\mathrm{proj}_i(\tau)$ of the i-th coordinate of $x(y)$.

Prof of correctness: Immediate. □

Complexity analysis: The complexity is $d^{O(ki\ell)}$, using the complexity of Algorithm 14.6 (Parametrized Sign Determination).

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(ki\ell)}$. □

We now define parametrized curve segments.

Let $\mathcal{V}_1$, τ_1, $\mathcal{V}_2$, τ_2 be two parametrized triangular Thom encoding above $\mathcal{A}, \rho, \mathcal{T}, \sigma$. For every $y \in \mathrm{Reali}(\rho)$, we denote by $z(y) \in \mathrm{R}^{i-1}$ the point specified by $\mathcal{T}(y), \sigma$ and by $(z(y), a(y))$, $(z(y), b(y))$ the points specified by $\mathcal{V}_1(y)$, τ_1 and $\mathcal{V}_2(y)$, τ_2. A **parametrized curve segment representation** u, τ **above** $\mathcal{V}_1, \tau_1, \mathcal{V}_2, \tau_2$ is given by

– a parametrized univariate representation with parameters $(Y, X_{\le i})$,

$$u = (f(Y, X_{\le i}, T), g_0(Y, X_{\le i}, T), g_{i+1}(Y, X_{\le i}, T), \ldots, g_k(Y, X_{\le i}, T)),$$

– a sign condition τ on $\mathrm{Der}(f)$ such that for every $y \in \mathrm{Reali}(\rho)$ and for every $v \in (a(y), b(y))$ there exists a real root $t(v)$ of $f(z(y), v, T)$ with Thom encoding σ, ρ, τ and $g_0(z(y), v, t(v)) \neq 0$.

Our aim is first to describe a parametrized version of Algorithm 15.2 (Curve Segments).

Algorithm 15.10. **[Parametrized Curve Segments]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:**
  - a parametrized Thom encoding $\mathcal{A}$, ρ, $\mathcal{T}$, σ with parameters $Y=(Y_1,\ldots,Y_\ell)$ of level $i-1$, with coefficients in D. For every $y\in \mathrm{Reali}(\rho)$, $(y,z(y))$ denotes the point specified by σ.
  - a polynomial $Q\in \mathrm{D}[Y,X_1\ldots,X_k]$, for which $\mathrm{Zer}(Q,\mathrm{R}^k)\subset B(0,1/c)$
  - a finite set $\mathcal{N}$ of parametrized real univariate triangular representation above $\mathcal{A}$, ρ, $\mathcal{T}$, σ with, for every $y\in \mathrm{Reali}(\rho)$, associated points contained in $\mathrm{Zer}(Q,\mathrm{R}^k)$.
- **Output:**
  - a finite set $\mathcal{B}\subset \mathrm{D}[Y]$ containing $\mathcal{A}$,
  - for every $\bar{\rho}\in \mathrm{SIGN}(\rho,\mathcal{B})$,
    - an ordered list of parametrized Thom encodings

$$\mathcal{V}_{\bar{\rho},1},\tau_{\bar{\rho},1},\ldots,\mathcal{V}_{\bar{\rho},\ell(\bar{\rho})},\tau_{\bar{\rho},\ell(\bar{\rho})}$$

    above $\mathcal{B}$, $\bar{\rho}$, $\mathcal{T}$, σ
    - for every $i=1,\ldots,\ell(\bar{\rho})$,
      - a finite set $\mathcal{N}_{\bar{\rho},i}$ of parametrized real univariate triangular representations above

$$\mathcal{B},\bar{\rho},\mathcal{V}_{\bar{\rho},j},\tau_{\bar{\rho},j}$$

      - a finite set $\mathcal{C}_{\bar{\rho},j}$ of parametrized curve segments above

$$\mathcal{B},\bar{\rho},\mathcal{V}_{\bar{\rho},j},\tau_{\bar{\rho},j},\mathcal{V}_{\bar{\rho},j+1},\tau_{\bar{\rho},j+1}$$

      - a list of pairs of elements of $\mathcal{C}_{\bar{\rho},j}$ and $\mathcal{N}_{\bar{\rho},j}$ (resp. $\mathcal{C}_{\bar{\rho},j+1}$ and $\mathcal{N}_{\bar{\rho},j}$) describing the adjacency relation.

  For every $y\in \mathrm{Reali}(\bar{\rho})$, this defines a set of curves and points contained in $\mathrm{Zer}(Q,\mathrm{R}^k)_{y,z(y)}$. The specifications of these points and curves is fixed for every point $y\in \mathrm{Reali}(\bar{\rho})$. These points and curves satisfy the properties of the output of Algorithm 15.2 (Curve Segments).
- **Complexity:** $d^{O(ki\ell)}$, where ℓ is the number of parameters, d is a bound on the degree of Q, $O(d)^k$ is a bound on the degrees of on the degree of the parametrized univariate representations in $\mathcal{N}$ and of the polynomials in $\mathcal{T}$.
- **Procedure:**
  - Step 1: Perform Algorithm 12.10 (Parametrized Multiplication Table) with input $\overline{\mathrm{Cr}}(Q^2,\zeta,)$, using Notation 12.46, and parameter $Y,X_{\le i}$. Perform Algorithm 12.15 (Parametrized Limit of Bounded Points), and output a set $\mathcal{U}$ of parametrized univariate representations.

    Using Notation 13.8, consider for every $u=(f,g_0,g_{i+1},\ldots,g_k)\in\mathcal{U}$ the finite set $\mathcal{F}_u$ containing Q_u) and all the derivatives of f with respect to T, and compute $\mathcal{D}_u=\mathrm{RElim}_T(f,\mathcal{F}_u)\subset \mathrm{D}[Y,X_{\le i}]$ using Algorithm 11.19 (Restricted Elimination).
  - Define $\mathcal{D}=\bigcup_{u\in\mathcal{U}}\mathcal{D}_u$.

– Step 2: Use Algorithm 15.9 (Parametrized Projection) with input $\mathcal{N}$ and output a finite set $\mathcal{B}_2 \subset \mathrm{D}[Y]$ containing $\mathcal{A}$, such that for every $\bar{\rho} \in \mathrm{SIGN}(\rho, \mathcal{B}_2)$ and every $u \in \mathcal{N}$ the Thom encoding $\mathrm{proj}_i(u)$, $\mathrm{proj}_i(\tau)$ specifying the projection of the associated point on the X_i axis is fixed for every $y \in \mathrm{Reali}(\bar{\rho})$. Add to $\mathcal{D}$ the polynomials $\mathrm{proj}_i(u)$.
– Step 3: Apply Algorithms 14.7 (Parametrized Thom Encoding), 15.7 (Parametrized Comparison of Roots) to the set $\mathcal{D}$. Denote by $\mathcal{B}_3 \subset \mathrm{D}[Y]$ the family of polynomials output, and for every $\bar{\rho} \in \mathrm{SIGN}(\rho,\ \mathcal{B}_3)$, denote by

$$A_{\bar{\rho},1}\alpha_{\bar{\rho},1}, \ldots, A_{\bar{\rho},\ell(\bar{\rho})}, \alpha_{\bar{\rho},\ell(\bar{\rho})}$$

the list of Thom encodings output. For every $y \in \mathrm{Reali}(\bar{\rho})$, these are the Thom encodings of the corresponding distinguished values

$$v_1(y, z(y)) < \cdots < v_\ell(y, z(y)).$$

Define $\mathcal{V}_i, \tau_i = \mathcal{T}, A_i$ and $\tau_i = \sigma, \alpha_i$.
– Step 4: For every $\bar{\rho} \in \mathrm{SIGN}(\rho,\ \mathcal{B}_3)$, every $j = 1, \ldots, \ell(\bar{\rho})$ and every $u = (f, g_0, g_i, \ldots, g_k), \tau \in \mathcal{N}$, use Algorithm 14.7 (Parametrized Triangular Thom Encoding) and output $\mathcal{B}_4(\bar{\rho},\ j,\ u)$, containing $\mathcal{B}_3$. Append pairs $(f,\ g_0,\ g_{i+1},\ \ldots,\ g_k)$, τ to $\mathcal{N}_{\rho_1,j}$ for every $\rho_1 \in \mathrm{SIGN}(\bar{\rho}, \mathcal{B}_4(\bar{\rho}, j, u, \tau))$ such that for every $y \in \mathrm{Reali}(\rho_1)$ $\mathrm{proj}_i(\tau)$ is the Thom encoding of a point of $\mathrm{Zer}(Q, \mathrm{R}^k)_z(y)$ with projection having Thom encoding α_j. Define $\mathcal{B}_4(\bar{\rho}) = \cup \mathcal{B}_4(\bar{\rho}, j, u, \tau)$.
– Step 5: For every $\bar{\rho} \in \mathrm{SIGN}(\rho,\ \mathcal{B}_3)$, every $j = 1, \ldots, \ell(\bar{\rho})$ and every $u = (f, g_0, g_i \ldots, g_k) \in \mathcal{U}$, use Algorithm 15.8 (Parametrized Intermediate Points) and Algorithm 14.7 (Parametrized Triangular Thom Encoding) and output $\mathcal{B}_5(\bar{\rho},\ j,\ u)$, containing $\mathcal{B}_3$. Append pairs $(f, g_0, g_{i+1}, \ldots, g_k), \tau$ to $\mathcal{N}_{\rho_1,j}$ for every $\rho_1 \in \mathrm{SIGN}(\bar{\rho}, \mathcal{B}_5(\bar{\rho}, j, u))$ such that for every $y \in \mathrm{Reali}(\rho_1)$ $\mathrm{proj}_i(\tau)$ is the Thom encoding of a point of $\mathrm{Zer}(Q, \mathrm{R}^k)_z(y)$ with projection having Thom encoding α_j. Define $\mathcal{B}_5(\bar{\rho}) = \cup \mathcal{B}_5(\bar{\rho}, j, u)$.
– Step 6: For every $\bar{\rho} \in \mathrm{SIGN}(\rho,\ \mathcal{B}_3)$, every $j = 1, \ldots, \ell(\bar{\rho}) - 1$ and every $u = (f,\ g_0,\ g_i \ldots,\ g_k) \in \mathcal{U}$, use Algorithm 15.8 (Parametrized Intermediate Points) and Algorithm 14.7 (Parametrized Triangular Thom Encoding) and output a family $\mathcal{B}_6(\bar{\rho}, j, u)$ containing $\mathcal{B}_3$ such that for every sign condition ρ_1 on $\mathcal{B}_6$ and every $y \in \mathrm{Reali}(\rho_1)$ the Thom encodings τ of the roots of $f(y, z(y), v, T)$ over $(v_i(y), v_{i+1}(y))$ are fixed and the corresponding associated curves are contained in $\mathrm{Zer}(Q, \mathrm{R}^k)_z(y)$. Append all pairs $(f,\ g_0,\ g_{i+1} \ldots,\ g_k)$, τ to $\mathcal{C}_{\rho_3,i}$. Define $\mathcal{B}_6(\bar{\rho}) = \cup \mathcal{B}_6(\bar{\rho}, j, u)$.
– Step 7: Consider $\rho_1 \in \mathrm{SIGN}(\bar{\rho}, \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6)$. For every $j = 1, \ldots, \ell(\bar{\rho}_1)$ and every parametrized real univariate triangular representation of $\mathcal{N}_{\rho_1,j}$ specified by

$$v' = (p, q_0, q_2, \ldots, q_k), \tau', \{p, q_0, q_2, \ldots, q_k\} \subset \mathrm{D}[Y, X_{\le i}][T]$$

and every parametrized curve segment representation of $\mathcal{C}_{\rho_1,j}$ specified by

$$v=(f,g_0,g_2,\ldots,g_k),\tau,\{f,g_0,g_2,\ldots,g_k\}\subset \mathrm{D}[Y,X_{\le i}[T],$$

compute a family $\mathcal{B}_7(\rho_1, v', \tau', v, \tau)$ of polynomials containing $\mathcal{B}_4\cup\mathcal{B}_5\cup\mathcal{B}_6$ such that for every $\rho_2\in\mathrm{SIGN}(\rho_1,\mathcal{B}_7(\rho_1,v',\tau',v,\tau))$ and every $y\in\mathrm{Reali}(\rho_2)$ the algorithm deciding whether the corresponding point $t(y)$ is adjacent to the corresponding curve segment gives the same answer: compute the first ν such that $(\partial^\nu g_0/\partial X_i^\nu)(v_j,t)$ is not zero and decide whether for every $\ell=i+1,\ldots,k$

$$\frac{\partial^\nu g_\ell}{\partial X_i^\nu}(v_j,t)q_0(t)-\frac{\partial^\nu g_0}{\partial X_i^\nu}(v_j,t)q_\ell(t)$$

is zero, using Algorithm 14.6 (Parametrized Sign Determination).

Repeat the same process for every element of $\mathcal{N}_{\rho_1,i+1}$ and every curve segment of $\mathcal{C}_{\rho_1,i}$.

- Finally output $\mathcal{B}=\cup\mathcal{B}_7(\rho_1,v',\tau',v,\tau)$.

Proof of correctness: It follows from Proposition 12.42 and the correctness of Algorithm 12.10 (Parametrized Multiplication Table), Algorithm 12.15 (Parametrized Limit of Bounded Points), Algorithm 11.19 (Restricted Elimination), Algorithm 15.9 (Parametrized Projection), Algorithm 15.8 (Parametrized Intermediate Points), Algorithm 14.7 (Parametrized Thom Encoding), Algorithm 15.7 (Parametrized Comparison of Roots) and Algorithm 14.6 (Parametrized Sign Determination). □

Complexity analysis:

- Step 1: This step requires $d^{O((\ell+i)(k-i))}$ arithmetic operations in D, using the complexity analyses of Algorithm 12.10 (Parametrized Multiplication Table), Algorithm 12.15 (Parametrized Limit of Bounded Points), Algorithm 11.19 (Restricted Elimination). There are $d^{O(k-i)}$ parametrized univariate representations computed in this step and each polynomial in these representations has degree $O(d)^{k-i}$.
- Step 2: This step requires $d^{O((\ell+i)k)}$ arithmetic operations in D, using the complexity analysis of Algorithm 15.9 (Parametrized Projection).
- Step 3: This step requires $d^{O(\ell ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 14.7 (Parametrized Thom Encoding).
- Step 4: This step requires $d^{O(\ell ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 14.6 (Parametrized Sign Determination).
- Step 5: This step requires $d^{O(\ell ik)}$ arithmetic operations in D, using the complexity analysis of Algorithm 14.7 (Parametrized Thom Encoding).
- Step 6: This step requires $d^{O(\ell ik)}$ arithmetic operations in D, using the complexity analyses of Algorithm 15.8 (Parametrized Intermediate Points) and Algorithm 14.7 (Parametrized Thom Encoding).

– Step 7: This step requires $d^{O(\ell i k)}$ arithmetic operations, using the complexity analysis of Algorithm 14.6 (Parametrized Sign Determination).

Thus, the complexity is $d^{O(\ell i k)}$.

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(\ell i k)}$. □

Algorithm 15.11. **[Parametrized Bounded Algebraic Roadmap]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:**
  – a parametrized Thom encoding $\mathcal{A}$, ρ, $\mathcal{T}$, σ with parameters $Y = (Y_1, \ldots, Y_\ell)$ and variables $X_{\le i} = (X_1, \ldots, X_i)$, with coefficients in D. For every $y \in \mathrm{Reali}(\rho)$, $(y, z(y))$ denotes the point specified by σ,
  – a polynomial $Q \in \mathrm{D}[Y, X_1 \ldots, X_k]$, for which $\mathrm{Zer}(Q, \mathrm{R}^k) \subset B(0, 1/c)$,
  – a finite set $\mathcal{N}$ of parametrized real univariate triangular representations above $\mathcal{A}$, ρ, $\mathcal{T}$, σ with coefficients in D, with, for every $y \in \mathrm{Reali}(\rho)$, associated points contained in $\mathrm{Zer}(Q, \mathrm{R}^k)$.
- **Output:**
  – a subset $\mathcal{C}$ of $\mathrm{D}[Y]$ containing $\mathcal{A}$,
  – for every realizable sign condition τ on $\mathcal{C}$ refining ρ, a subset $\mathrm{RM}(\tau)$ such that, for every $y \in \mathrm{Reali}(\tau)$, $\mathrm{RM}(\tau)_y$ is a roadmap for $\mathrm{Zer}(Q, \mathrm{R}^k)_y$ that contains $\mathcal{N}_y$.
- **Complexity:** $d^{O(\ell k^2)}$, where ℓ is the number of parameters, $O(d)^k$ is a bound on the degrees of on the degree of the univariate representation and of the polynomials in $\mathcal{T}$.
- **Procedure:**
  – Call Algorithm 15.10 (Parametrized Curve Segments), output $\mathcal{B}$ and, for every realizable sign condition $\bar{\rho}$ on $\mathcal{B}$ refining ρ, $\ell(\rho)$. Output also, for every $j = 1, \ldots, \ell(\rho)$, $A_{\bar{\rho}, i}, \alpha_{\bar{\rho}, i}$, $\mathcal{N}_{\bar{\rho}, i}$ and $\mathcal{C}_{\bar{\rho}, i}$.
  – For every realizable sign condition $\bar{\rho}$ on $\mathcal{B}$ and for every i from 1 to $\ell(\bar{\rho})$, call Algorithm 15.11 (Parametrized Bounded Algebraic Roadmap) recursively, with input $\mathcal{B}, \bar{\rho}, \mathcal{T}, A_{\bar{\rho}, j}, \sigma, \alpha_{\bar{\rho}, j}$, Q and $\mathcal{N}_{\bar{\rho}, j}$.

Proof of correctness: The correctness of the algorithm follows from Proposition 15.7 and the correctness of Algorithm 15.2 (Curve Segments). □

Complexity analysis: In the recursive calls to Algorithm 15.11 (Parametrized Bounded Algebraic Roadmap), the number of triangular systems considered is at most $d^{O(k^2)}$ and the triangular systems involved have polynomials of degree $O(d)^k$.

Thus, the total number of arithmetic operations in D is bounded by $d^{O(\ell k^2)}$ using the complexity analysis of Algorithm 15.10 (Parametrized Curve Segments).

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(\ell k^2)}$. $\square$

We now want to obtain a parametrized connecting algorithm. We show how to obtain a covering of a given $\mathcal{P}$-closed semi-algebraic set contained in $\mathrm{Zer}(Q, \mathrm{R}^k)$ by a family of semi-algebraically contractible subsets. The construction is based on a parametrized version of the connecting algorithm: we compute a family of polynomials such that for each realizable sign condition σ on this family, the description of the connecting paths of different points in the realization, $\mathrm{Reali}(\sigma, \mathrm{Zer}(Q, \mathrm{R}^k))$, are uniform.

We first define parametrized paths. A parametrized path is a semi-algebraic set which is a union of semi-algebraic paths having the divergence property (see Remark 15.10).

More precisely,

Definition 15.14. A **parametrized path** γ is a continuous semi-algebraic mapping from $V \subset \mathrm{R}^{k+1} \rightarrow \mathrm{R}^k$, such that, denoting by $U = \pi_{1\ldots k}(V) \subset \mathrm{R}^k$, there exists a semi-algebraic continuous function $\ell: U \rightarrow [0, +\infty)$, and there exists a point a in R^k, such that

- $V = \{(x, t) \mid x \in U, 0 \leq t \leq \ell(x)\}$,
- $\forall\, x \in U,\ \gamma(x, 0) = a$,
- $\forall\, x \in U,\ \gamma(x, \ell(x)) = x$,
- $\forall\, x \in U, \forall\, y \in U, \forall\, s, 0 \leqslant s \leqslant \ell(x), \forall\, t 0 \leqslant t \leqslant \ell(y)$
  $(\gamma(x, s) = \gamma(y, t) \Rightarrow s = t)$,
- $\forall\, x \in U, \forall\, y \in U, \forall\, s \in [0, \min(\ell(x), \ell(y))]$
  $(\gamma(x, s) = \gamma(y, s) \Rightarrow \forall\, t \leq s\ \gamma(x, t) = \gamma(y, t))$. $\square$

Given a parametrized path, $\gamma: V \rightarrow \mathrm{R}^k$, we will refer to $U = \pi_{1\ldots k}(V)$ as its *base*. Also, any semi-algebraic subset $U' \subset U$ of the base of such a parametrized path, defines in a natural way the restriction of γ to the base U', which is another parametrized path, obtained by restricting γ to the set $V' \subset V$, defined by $V' = \{(x, t) \mid x \in U', 0 \leq t \leq \ell(x)\}$.

Proposition 15.15. *Let $\gamma: V \rightarrow R^k$ be a parametrized path such that $U = \pi_{1\ldots k}(V)$ is closed and bounded. Then, the image of γ is semi-algebraically contractible.*

Proof: Let $W = \mathrm{Im}(\gamma)$ and $M = \sup_{x \in U} \ell(x)$. We prove that the semi-algebraic mapping $\phi: W \times [0, M] \rightarrow W$ sending

$$\begin{array}{llll} (\gamma(x,t), s) & \text{to} & \gamma(x, s) & \text{if } t \geq s, \\ (\gamma(x,t), s) & \text{to} & \gamma(x, t) & \text{if } t < s \end{array}$$

is continuous. Note that the map ϕ is well-defined, since

$$\gamma(x,t)=\gamma(x',t')\Rightarrow t=t',$$

by condition (4).Since ϕ satisfies

$$\begin{aligned}\phi(\gamma(x,t),0) &= a,\\ \phi(\gamma(x,t),M) &= \gamma(x,t)\end{aligned}$$

this gives a semi-algebraic continuous contraction from W to $\{a\}$.

Let $w\in W, s\in[0,M]$. Let $\varepsilon>0$ be an infinitesimal, and let

$$(w',s')\in\mathrm{Ext}(W\times[0,M],\mathrm{R}\langle\varepsilon\rangle)$$

be such that $\lim_\varepsilon(w',s')=(w,s)$. In order to prove the continuity of ϕ at w it suffices to prove that

$$\lim_\varepsilon \mathrm{Ext}(\phi,\mathrm{R}\langle\varepsilon\rangle)(w',s')=\phi(w,s).$$

Let $w=\gamma(x,t)$ for some $x\in U, t\in[0,\ell(x)]$, and similarly let $w'=(x',t')$ for some $x'\in\mathrm{Ext}(U,\mathrm{R}\langle\varepsilon\rangle)$ and $t'\in[0,\mathrm{Ext}(\ell,\mathrm{R}\langle\varepsilon\rangle)(x')]$. Note that $\lim_\varepsilon(x')\in U$ since U is closed and bounded and $\lim_\varepsilon t'\in[0,\ell(\lim_\varepsilon x')]$.

Now,

$$\begin{aligned}\gamma(x,t) &= w\\ &= \lim_\varepsilon(w')\\ &= \lim_\varepsilon \mathrm{Ext}(\gamma,\mathrm{R}\langle\varepsilon\rangle)(x',t')\\ &= \gamma(\lim_\varepsilon x',\lim_\varepsilon t').\end{aligned}$$

Condition (4) now implies that $\lim_\varepsilon t'=t$.

Without loss of generality let $t'\geq t$. The other case is symmetric. We have the following two sub-cases.

- Case $s'>t'$: Since $s,t\in\mathrm{R}$ and $\lim_\varepsilon s'=s$ and $\lim_\varepsilon t'=t$, we must have that $s\geq t$. In this case $\mathrm{Ext}(\phi,\mathrm{R}\langle\varepsilon\rangle)(w',s')=\mathrm{Ext}(\gamma,\mathrm{R}\langle\varepsilon\rangle)(x',t')$. Then,

$$\begin{aligned}\lim_\varepsilon \mathrm{Ext}(\phi,\mathrm{R}\langle\varepsilon\rangle)(w',s') &= \lim_\varepsilon \mathrm{Ext}(\gamma,\mathrm{R}\langle\varepsilon\rangle)(x',t')\\ &= \lim_\varepsilon w'\\ &= w\\ &= \phi(w,s).\end{aligned}$$

- Case $s'\leq t'$: Again, since $s,t\in\mathrm{R}$ and $\lim_\varepsilon s'=s$ and $\lim_\varepsilon t'=t$, we must have that $s\leq t$.

  In this case we have,

$$\begin{aligned}\lim_\varepsilon \phi(w',s') &= \lim_\varepsilon \mathrm{Ext}(\gamma,\mathrm{R}\langle\varepsilon\rangle)(x',s')\\ &= \gamma(\lim_\varepsilon x',\lim_\varepsilon s')\\ &= \gamma(\lim_\varepsilon x',s).\end{aligned}$$

Now,

$$\begin{aligned}\gamma(\lim_\varepsilon x', t) &= \gamma(\lim_\varepsilon x', \lim_\varepsilon t')\\ &= \lim_\varepsilon \mathrm{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(x', t')\\ &= \lim_\varepsilon w'\\ &= w\\ &= \gamma(x, t).\end{aligned}$$

Thus, by condition (5) we have that $\gamma(\lim_\varepsilon x', s'') = \gamma(x, s'')$ for all $s'' \leq t$. Since, $s \leq t$, this implies,

$$\begin{aligned}\lim_\varepsilon \mathrm{Ext}(\phi, \mathrm{R}\langle\varepsilon\rangle)(w', s') &= \lim_\varepsilon \mathrm{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(w', s')\\ &= \gamma(\lim_\varepsilon x', \lim_\varepsilon s')\\ &= \gamma(x, s)\\ &= \phi(w, s).\end{aligned}$$

This proves the continuity of ϕ, using Proposition 3.5. □

Algorithm 15.12. **[Parametrized Bounded Algebraic Connecting]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:**
  - a parametrized Thom encoding $\mathcal{A}$, ρ, $\mathcal{T}$, σ with parameters $Y = (Y_1, \ldots, Y_\ell)$ and variables $X_{\leq i} = (X_1, \ldots, X_i)$, with coefficients in D. For every $y \in \mathrm{Reali}(\rho)$, $(y, z(y))$ denotes the point specified by σ,
  - a polynomial $Q \in \mathrm{D}[Y, X_1 \ldots, X_k]$, for which $\mathrm{Zer}(Q, \mathrm{R}^k) \subset B(0, 1/c)$
  - a parametrized real univariate triangular representation above $\mathcal{A}$, ρ, $\mathcal{T}$, σ with coefficients in D, with, for every $y \in \mathrm{Reali}(\rho)$, associated point $p(y)$ contained in $\mathrm{Zer}(Q, \mathrm{R}^k)$.
- **Output:**
  - a subset $\mathcal{C}$ of $\mathrm{D}[Y]$ containing $\mathcal{A}$,
  - for every realizable sign condition τ on $\mathcal{C}$ refining ρ, a parametrized path $\gamma(\tau)$ such that, for every $y \in \mathrm{Reali}(\tau)$, $\gamma(\tau)(y)$ is a path connecting $p(y)$ to a distinguished point of $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k))$.
- **Complexity:** $d^{O(\ell k^2)}$, where ℓ is the number of parameters, $O(d)^k$ is a bound on the degrees of on the degree of the univariate representation and of the polynomials in $\mathcal{T}$.
- **Procedure:** Call Algorithm 15.11 (Parametrized Bounded Algebraic Roadmap) and extract γ from $\mathrm{RM}(\tau)$.

Proof of correctness: The correctness of the algorithm follows from the correctness of Algorithm 15.11 (Parametrized Bounded Algebraic Roadmap).It is easy to see that γ is a parametrized path (see Definition 15.14), using the divergence property of the paths $\gamma(y, \cdot\,)$ (see Remark 15.10). □

Complexity analysis: The total number of arithmetic operations in D is bounded by $d^{O(\ell k^2)}$, using the complexity analysis of Algorithm 15.11 (Parametrized Bounded Algebraic Roadmap).

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(\ell k^2)}$. □

Algorithm 15.13. **[Connected Components of an Algebraic Set]**

- **Structure:** an ordered domain D contained in a real closed field R.
- **Input:** a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$.
- **Output:** a subset $\mathcal{A}$ of $\mathrm{D}[X_1, \ldots, X_k]$ and for every semi-algebraically connected component S of $\mathrm{Zer}(Q, \mathrm{R}^k)$ a finite subset $\Sigma \subset \mathrm{SIGN}(\mathcal{A})$ such that $S = \bigcup_{\sigma \in \Sigma} \mathrm{Reali}(\sigma, \mathrm{Zer}(Q, \mathrm{R}^k))$.
- **Complexity:** $d^{O(k^3)}$, where d is a bound on the degree of the polynomial Q.
- **Procedure:**
  - Take $Q_\varepsilon = Q^2 + (\varepsilon^2 (X_1^2 + \cdots + X_k^2 + X_{k+1}^2) - 1)^2$.
  - Call Algorithm 15.11 (Parametrized Bounded Algebraic Roadmap) without parametrized triangular Thom encoding, Q_ε, and

$$\mathcal{N} = \{(T - 1, 1, Y_1, \ldots, Y_k)\}.$$

    The output contains a family of polynomials $\mathcal{A}^\star \subset \mathrm{D}[\varepsilon][X]$ such that the realization of a non-empty sign condition ρ in $\mathcal{A}^\star$ is contained in a semi-algebraically connected component of $\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1})$.
  - Find a set $\mathcal{S}$ of sample points for every realizable sign condition on $\mathcal{A}^\star$ using Algorithm 13.1(Sampling). Compute $\mathrm{RM}(\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1})$ using Algorithm 15.3 (Bounded Algebraic Roadmap) and for every semi-algebraically connected component S' of $\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1})$, fix a point $y(S')$ of $S' \cap \mathrm{RM}(\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1})$. For every $x \in \mathcal{S}$ compute a roadmap $\mathrm{RM}(\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1}), x)$ of $\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1})$ containing x using Algorithm 15.3 (Bounded Algebraic Roadmap) and decide from $\mathrm{RM}(\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1}), x)$ whether x belongs to S'.
  - Output the description of S', i.e. the disjunction $\Phi(S')$ of realizable sign conditions on $\mathcal{A}^\star$ with a sample point belonging to S', for every semi-algebraically connected component S' of $\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1})$.
  - For every connected component S of $\mathrm{Zer}(Q, \mathrm{R}^k)$ there exists a connected component S' of $\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1})$, such that $\pi(S') \cap \mathrm{R}^k = S$, where π: $\mathrm{R}\langle\varepsilon\rangle^{k+1} \to \mathrm{R}\langle\varepsilon\rangle^k$ is the projection map forgetting the last coordinate.

    Consider the formula $\Phi(S')$ describing S' and, eliminating a quantifier, the formula Ψ describing $\pi(S')$. Then $\mathrm{Remo}_\varepsilon(\Psi(Y))$ (Notation 14.6) defines S.

Proof of correctness: All points satisfying the same sign condition on $\mathcal{A}$ can be connected by a semi-algebraic path in $\mathrm{Zer}(Q, \mathrm{R}^k)$ to some fixed curve segment of $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k))$ and hence must belong to the same connected component of $\mathrm{Zer}(Q, \mathrm{R}^k)$. Which realizable sign conditions on $\mathcal{A}$ belong to the same semi-algebraically connected component of $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k))$ follows from Step 2 and 3. We also use Proposition 14.7. □

Complexity analysis: The total number of arithmetic operations in D is bounded by $d^{O(k^3)}$, using the complexity analysis of Algorithm 15.11 (Parametrized Bounded Algebraic Roadmap). The degrees of the polynomials in $\mathcal{A}$ are bounded by $d^{O(k^2)}$.

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^3)}$. □

So we have proved Theorem 15.12.

15.4 Bibliographical Notes

The problem of deciding connectivity properties of algebraic sets considered here is a base case for deciding connectivity properties of semi-algebraic sets, studied in Chapter 16.

The notion of a roadmap for a semi-algebraic set was introduced by Canny in [36].

We discuss in more details the various contributions to the roadmap problem and the computation of connected components at the end of Chapter 16.

It is interesting to remark that the complexity of computing the number of connected components of an algebraic set given in this chapter is significantly worse than that of the algorithm for computing the Euler-Poincaré characteristic of an algebraic set given in Chapter 12. Thus, currently we are able to compute the Euler-Poincaré characteristic of real algebraic sets (which is the alternative sum of the Betti numbers) more efficiently than any of the individual Betti numbers.