11. Theory of Stochastic Optimal Economic Growth

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11.1 Introduction

Stochastic optimal growth involves the study of optimal intertemporal allocation of capital and consumption in an economy where production is subject to random disturbances. The theory traces its roots to the seminal work on deterministic optimal growth by Ramsey [108], Cass [21] and Koopmans [56]. Its influence has been enhanced by research that shows how the convex stochastic growth model can be decentralized to represent the behavior of consumers and firms in a dynamic competitive equilibrium of a productive economy ([104], [117], [15]). This makes the stochastic optimal growth model useful both as a normative exercise and in the development of positive theories of how the economy works. As a consequence, the theory has emerged as one of the central paradigms of dynamic economics. It is based on a simple, yet powerful model that encompasses fundamental questions that are basic to any theory of dynamic economic behavior: What are the characteristics and determinants of optimal policies? What are the economic incentives that govern the optimal intertemporal allocation of resources? What is the transient and long run behavior of variables in the model? Under different assumptions the model admits a rich set of answers to these questions.

Historically, the main focal point of the theory has been issues of aggregate economic growth. At the same time its primary variable, capital, has a flexible interpretation that allows the model and its extensions to represent a wide variety of economic problems ranging from the study of business cycles ([60], [64]) and asset pricing ([14], [15]) to the allocation of renewable natural resources ([78], [83], [84]). Equally important, the model provides a strong theoretical foundation for applied analysis of these problems. The model can be solved

numerically and has proved a testing ground for many numerical techniques used today in the analysis of dynamic economic problems.

This chapter provides an overview of key results in the *theory* of discounted stochastic optimal growth in discrete time.[1] The paper begins with an analysis of the classical stochastic growth model of Brock and Mirman [18] for a one-sector economy with a convex technology and utility that depends only on consumption.[2] We then consider extensions of the theory to problems with irreversible investment, increasing returns or a non-convex technology, experimentation and learning, and problems where utility depends on more than consumption alone. We develop the competitive price characterization of optimal policies that can be used to establish the equivalence between optimal and competitive outcomes; our focus, however, is on optimal solutions and their properties. The large literature on dynamic competitive equilibria is, therefore, left to the reader to explore. Likewise, we do not survey the many applications of the stochastic growth model. Instead, we focus on how the theory can be extended in different directions that have proved useful in application. Finally, we provide a glimpse of practical methods for solving the model, but the literature on numerical methods is too large for us to review here.

11.2 The Classical Framework

11.2.1 The One Sector Classical Model: Basic Properties

The stochastic growth model has three essential elements: an exogenous stochastic environment corresponding to random productivity disturbances, the production possibilities that determine the set of feasible allocations for consumption, investment and output, and an instantaneous welfare or utility function that represents the preferences of the agent or economic decision-maker. Productivity shocks at dates $t = 1, 2, ...,$ are denoted by $\{r_t\}$, a sequence of i.i.d. real-valued random variables, with common distribution ν on $B(\Phi)$, the Borel σ-field of $\Phi \subset \Re$. In particular, Φ is the support of ν and is assumed to be compact. Associated with this stochastic environment is a measure space $(\Omega, \mathcal{F}, \mu)$, where Ω is the set of all real sequences, $\mathcal{F}$ is the σ-field generated by cylinder sets of the form $\prod_{t=0}^{\infty} A_t$, where A_t belongs to $B(\Phi)$ for all t, and μ is the product distribution induced by ν. The statements: for a.e. ω and μ-a.s. mean "except for a subset of Ω of μ-measure zero". The random variable r_t is simply the tth coordinate function on Ω. In the economy, output of a homogeneous consumption/capital good is produced via a production function that is homogeneous of degree one in capital and labor. This allows the economy to be

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[1] There is a large literature on stochastic growth in continuous time that builds on Merton's [79] early work (see also, [16]).

[2] Previous surveys of stochastic growth such as [82] and [6] focus primarily on this case.

represented in per capita terms where c_t, k_t and y_t denote per capita consumption, capital and output at time t. Given a capital stock at time $t-1$ and the productivity disturbance at the beginning of period t, $y_t = f(k_{t-1}, r_t)$, where $f : \Re_+ \times \Phi \to \Re_+$ is the production function. The *feasible set* for consumption and investment is: $\Gamma(y_t) = \{(c_t, k_t) | 0 \leq c_t, 0 \leq k_t, \text{ and } c_t + k_t \leq y_t\}$.

Each period the economic agent receives utility $u(c_t)$, where $u : \Re_+ \to \Re$. The discount factor for future utility is δ, where $0 \leq \delta < 1$. At the beginning of period t the agent observes y_t and chooses c_t and k_t. The productivity disturbance, r_{t+1}, occurs and a new output, y_{t+1}, is produced. The objective of the agent is to maximize the expected discounted sum of utility over time subject to the feasibility constraints on consumption and capital, and the transition equation that maps capital to output in the following period. Given an initial output, y_0, the objective is to:

$$Max \quad E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right]$$

subject to: $0 \leq c_t, 0 \leq k_t, c_t + k_t \leq y_t, y_{t+1} = f(k_t, r_{t+1}), t \geq 0. \quad (11.1)$

This problem can be formulated as a stochastic dynamic programming problem ([11], [127] and [66]). At date t, the *partial history* is $h_t = \{y_0, c_0, k_0, y_1, ..., c_{t-1}, k_{t-1}, y_t\}$. A *policy*, π, is a sequence $\{\pi_0, \pi_1, ...\}$, where π_t is a conditional probability on $B(\Re_+)$, given h_t, such that $\pi_t(\Gamma(y_t) \mid h_t) = 1$. Let F be the set of all measurable functions ϕ such that $\phi(y) \in \Gamma(y)$ for all $y \in \Re_+$. A policy is *Markovian* if $\pi_t \in F$ for all t. A Markov policy is *stationary* if there exists a Borel measurable function, $\hat{\pi}(y)$, such that $\pi_t(y) = \hat{\pi}(y)$ for all t. A policy, π, and an initial state, y, induce a *feasible program*, $(\mathbf{y},\mathbf{c},\mathbf{k})=(y_t, c_t, k_t)_{t=0}^{\infty}$, a stochastic process for output, consumption and capital such that $(c_t, k_t) \in \Gamma(y_t)$ and $y_{t+1} = f(k_t, r_{t+1})$ a.s. for all t. Associated with each policy is an expected discounted sum of utility $V_\pi(y) = E\sum_{t=0}^{\infty} \delta^t u(c_t)$, where $(\mathbf{y},\mathbf{c},\mathbf{k})$ is the feasible program generated by π and f in the obvious manner. A policy, π^*, is optimal if $V_{\pi^*}(y) \geq V_\pi(y)$ for all π and y, and the associated program is called an *optimal program*. The value function $V(y)$ is defined on $\Re_+$ by $V(y) = \sup\{V_\pi(y) \mid \pi \text{ is a policy}\}$. It follows that π^* is an optimal policy if, and only if, $V_{\pi^*}(y) = V(y)$ for all $y \geq 0$.

Throughout the paper, derivatives are denoted using subscripts, so that u_c represents marginal utility of consumption and so on. The production technology and preferences are assumed to satisfy the following assumptions:

A.1. $f(0, r) = 0$, $f(k, r) > 0$ for all $r \in \Phi$ and all $k > 0$.

A.2. f is continuous on $\Re_+ \times \Phi$ and for each $r \in \Phi$, $f(\cdot, r)$ is continuously differentiable on $\Re_{++}$.

A.3. $f_k(k, r) > 0$ and $\inf_{r \in \Phi} f_k(0, r) > 1$.

A.4. $f(., r)$ is strictly concave on $\Re_+$ for all $r \in \Phi$.

A.5. There exists a $\bar{k} > 0$ such that $f(k, r) < k$ for all $k > \bar{k}$ and all $r \in \Phi$.

A.6. u is continuous on $\Re_+$ and continuously differentiable on $\Re_{++}$.

A.7. $u_c(c) > 0$ on $\Re_{++}$.

A.8. u is strictly concave on $\Re_+$.

Under these assumptions the dynamic optimization problem is well defined, the value is finite from any initial state and it satisfies the functional equation:

$$V(y) = \max_{c \in \Gamma(y)} [u(c) + \delta \int V(f(y - c, r))d\nu(r)]. \qquad (11.2)$$

Further, there exist stationary optimal policy functions for consumption, $C(y) = \arg\max_{c \in \Gamma(y)}[u(c) + \delta \int V(f(y - c, r))d\nu(r)]$, and capital, $K(y) = y - C(y)$.[3]

To characterize economic behavior in the model it is important to understand the basic properties of the optimal value and policy functions. Further, such knowledge is necessary to examine how departures from the classical model affect economic outcomes. In the classical model, the feasible set correspondence $\Gamma(y)$ is expanding and has a convex graph,[4]. Using the assumption that the production and utility functions are strictly increasing and strictly concave and the fact that the functional equation (11.2) maps the set of continuous, increasing and strictly concave functions into itself this implies (e.g., [126]):

Lemma 11.2.1. *Under A.1-A.8, $V(y)$ is continuous, strictly increasing and strictly concave.*

The value function is a measure of lifetime economic welfare and to a first order approximation is proportional to traditional measures of GDP. The economic implication of Lemma 11.2.1 is that small increases in output have small effects on welfare, and that welfare increases at a diminishing rate as output increases.

Strict concavity of u, f and V implies that the maximization problem on the right hand side of (11.2), has a unique solution for every $y \geq 0$. Using the Maximum Theorem, one can then show that the optimal policy functions $C(y)$ and $K(y)$ are continuous. Monotonicity properties of $C(y)$ and $K(y)$ are determined by the complementarity[5] between k and y, and c and y, respectively.

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[3] Note that existence and all other results in this section continue to hold for logarithmic or CES utility functions that are unbounded below, though $V(0) = u(0) = -\infty$ (e.g., [118]).

[4] The feasible set is expanding if $y \leq y'$ implies $\Gamma(y) \subseteq \Gamma(y')$ and has a convex graph if $\{(c, k, y) \mid (c, k) \in \Gamma(y)\}$ is a convex set.

[5] Formally this is represented by the concept of supermodularity. Let $y \wedge y' = min[y, y']$ and $y \vee y' = max[y, y']$. A function $F(k, y)$ is supermodular in (k, y) if $F(k \wedge k', y \wedge y') + F(k \vee k', y \vee y') \geq F(k, y) + F(k', y')$. For C^2 functions this is equivalent to $F_{ky} \geq 0$ so that an increase in one argument raises the marginal value or marginal productivity of the other.

Lemma 11.2.2. *Under A.1-A.8, $C(y)$ and $K(y)$ are single-valued, continuous and increasing functions.*[6]

Proof. The fact that $C(y)$ and $K(y)$ are single-valued and continuous follows from the maximum theorem and the strict concavity of u and f. Let $k \in K(y)$ and $k' \in K(y')$ for $y < y'$. Suppose that $k' < k$. Then $k' \in \Gamma(y)$ and $k \in \Gamma(y')$. Further, $0 < u(y - k) + \delta EV(f(k, r)) - [u(y - k') + \delta EV(f(k', r))] < u(y' - k) + \delta EV(f(k, r)) - [u(y' - k') + \delta EV(f(k', r))] < 0$, where the first and last inequalities follow from the principle of optimality and the middle inequality follow from the fact that A.8 implies u is strictly supermodular in (y, k). Hence, it must be that $k' > k$. Next suppose that $c' \leq c$. Then, $0 \leq u(c) + \delta EV(f(y - c, r)) - [u(c) + \delta EV(f(y - c', r))] + u(c') + \delta EV(f(y' - c', r)) - [u(c') + \delta EV(f(y' - c))] < 0$, where the first inequality follows from the principle of optimality and the last inequality is due to the strict concavity of f and V. Hence, $c' > c$.

When the stochastic growth model is representative of aggregate economic behavior, it is natural that consumption and investment should always be in the interior of the feasible set. In disaggregate or microeconomic settings, this may not always be true. Since the interiority of optimal policies facilitates the use of differentiable optimization methods it is common to impose an assumption that guarantees interiority.

A.9. $\lim_{c\downarrow 0} u_c = \infty$.

Lemma 11.2.3. *Under A.1-A.9, $C(y) > 0$ and $K(y) > 0$.*

The condition $\lim_{c\downarrow 0} u_c = \infty$ is known as the Inada [44] condition at zero. In the classical model, the intuition for its use is as follows. To invest y yields finite discounted expected marginal value of investment but an infinite marginal utility from consumption. Hence, one can do better by reallocating some output from investment to consumption. Analogous arguments can be used to rule out investment of zero.

When optimal policies are interior, the value function in the classical model is differentiable.

Lemma 11.2.4. *(Mirman-Zilcha [81], Lemma 1). Under A.1-A.9, $V(y)$ is differentiable for all $y > 0$ and $V_y(y) = U_c(C(y))$.*

Proof. As a concave function, V has left and right-hand derivatives, $V_-(y) \geq V_+(y)$. Let k and c be optimal from y. As $c > 0, k$ is feasible from $y + \epsilon$ and $y - \epsilon$ for sufficiently small $\epsilon > 0$. By optimality, $V(y + \epsilon) - V(y) \geq u(c + \epsilon) +$

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[6] Lemma 11.2.2 was first established by Brock and Mirman [18]. The monotonicity of $K(y)$ does not depend on the concavity of u or f and can be generalized to the case where $K(y)$ is a correspondence using the methods of Topkis [129] (see also,[121]).

$\delta EV(f(k,r)) - [u(c) + \delta EV(f(k,r))]$, which implies $V_+(y) \geq u_c(c)$. By a similar argument $V_-(y) \leq u_c(c)$.[7]

When V is differentiable, output has a unique shadow price given by $V_y(y)$. This shadow price is useful in examining the intertemporal tradeoff between consumption and investment, and in showing that the optimization problem can be decentralized.

Proposition 11.2.1. *Let $(\boldsymbol{c},\boldsymbol{k})$ be an optimal program induced by $C(y), K(y)$. Under A.1-A.9, necessary and sufficient conditions for $C(y), K(y)$ to be optimal are:*

$$u_c(c_t) = \delta \int u_c(c_{t+1}(r)))f_k(k_t, r)d\nu(r). \tag{11.3}$$

$$\lim_{t \to \infty} \delta^t E[u_c(c_t)k_t] = 0. \tag{11.4}$$

Proof. The necessary condition (11.3) is typically proved in one of two ways. The first method is a variational approach that assumes period t output and the period t+1 capital stock are optimal. It then examines how a change in period t consumption affects discounted expected utility across the two periods. The second method proceeds as follows. If V is differentiable (Lemma 11.2.4) then maximizing the right hand side of equation (11.2) implies: $u_c(c_t) = \delta \int V_y(f(k_t, r)f_k(k_t, r)dv(r)$. Further, $V_y(y_t) = u_c(c_t)$ by the envelope theorem. Combining these yields (11.3). As commonly used, this approach requires both interior solutions and a differentiable value function; but a more general statement using inequalities is possible in other cases.

A proof of (11.4) is given in [87].[8]

Equation (11.3) is known as the stochastic Ramsey-Euler equation. It is a dynamic optimality condition that equates the marginal utility from consumption to the discounted expected marginal value of investment. The latter can be decomposed into the marginal productivity of investment times the marginal utility from consuming the additional output next period.

Equation (11.4) is the transversality condition. It implies that marginal utility is bounded in expectation.[9] It is also important to note that there may

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[7] An alternative approach in [12] assumes that the disturbance distribution has a C^n density. This smooths out possible points of discontinuity in the derivative of V. The approach has the advantage that it can be used to obtain higher order differentiability of both V and the optimal policy function, the latter via the implicit function theorem. Santos and Vigo-Aguiar[116] contains sufficient conditions for the value and policy functions to be C^2 and C^1, respectively. They use their results to place analytical bounds on the approximation error of a numerical solution.

[8] Kamihigashi [54] establishes the necessity of the transversality condition for optimality in a class of multisector stochastic growth models with single consumption good and bounded or constant relative risk aversion utility.

[9] Mirman and Zilcha [86] show that marginal utilities themselves may be unbounded.

be many non-optimal programs that satisfy the Ramsey-Euler equation. The transversality condition selects an optimal program from among those satisfying (11.3).

One of the most important results in the stochastic growth literature relates to the validity of the fundamental theorems of welfare economics in infinite horizon, stochastic economies. The two basic issues are the existence of prices that support an optimal program and the optimality of a dynamic, competitive equilibrium. In their seminal work Malinvaud [72] and Koopmans [55] make clear that the fundamental welfare theorems do not extend to infinite horizon settings without some additional conditions. The importance of these issues is apparent in [18], [81], [86] and [87] even though prices are often implicit in the necessary and/or sufficient conditions for optimality. Zilcha ([135], [136], [137]) examines the fundamental welfare theorems in a setting in which competitive prices are explicit throughout.

A feasible program $(\mathbf{y},\mathbf{c},\mathbf{k})$ is *competitive* if there exists a sequence $\mathbf{p} = (p_t)_{t=0}^{\infty}$ of discounted prices such that $p_t > 0$ a.s. for all t and:

$$\delta^t u(c_t) - p_t c_t \geq \delta^t u(c) - p_t c \ a.s., \forall \ c \geq 0. \tag{11.5}$$

$$Ep_{t+1}f(k_t, r_{t+1}) - p_t k_t \geq Ep_{t+1}f(k, r_{t+1}) - p_t k \ a.s., \forall k \geq 0 \tag{11.6}$$

Proposition 11.2.2. *A feasible program is optimal if and only if it is competitive and satisfies:*

$$\lim_{t \to \infty} Ep_t k_t = 0. \tag{11.7}$$

Proof. See [135]. ∎

As in Proposition 11.2.1, the existence of competitive prices alone is not sufficient to guarantee optimality. For that, the transversality condition (11.7) is also required.

The supporting price p_t is the discounted shadow price of the consumption-capital good. Equation (11.5) requires that consumption maximize utility less expenditure for almost every realized path and every time period. Equation (11.6) captures intertemporal (expected) profit maximization. When a competitive program is interior it implies $p_t = Ep_{t+1}f_k(k_t, r_{t+1})$. A primary difference between the deterministic and stochastic models is that in the former prices reflect temporal values, while in the latter prices also reflect values across different random states of nature. As a consequence, prices and the marginal willingness to substitute consumption are an important determinant of economic behavior even in the long run.

11.2.2 Stochastic Steady States and Convergence Properties in the One Sector Classical Model

A central concern of optimal growth theory is the study of the long run dynamics of an economy. The deterministic literature focusses on the existence and stability of non-trivial (strictly positive) optimal steady states and on turnpike properties of optimal capital accumulation paths. An optimal steady state or stationary program is a limit point of an optimal program. If optimal paths from all initial states converge to a steady state then this unique optimal steady state is globally stable and the long run behavior of the economy is independent of initial conditions.

When the evolution of capital stocks is stochastic, an optimal program of capital stocks is a sequence of random variables. The optimal policy, the production function, and the random shock map the probability distribution of current capital stocks to the probability distribution of the next period's capital stock. A stochastic steady state is a fixed point of this mapping or a distribution of capital that is invariant under the optimal policy. The stochastic analogue of a globally stable steady state is a unique invariant distribution to which the stochastic process of capital stocks converges from every initial state. In such a steady state the capital stock is not constant over time. Instead, it exhibits endogenous fluctuations in response to random productivity disturbances.[10]

Turnpike theorems study the conditions under which differences in initial conditions have negligible effects on the process of economic growth over long time horizons.[11] In the deterministic case, this involves analyzing when optimal paths from different initial states approach each other asymptotically. The stochastic analogue is convergence to zero in probability (or sometimes, almost surely) of an appropriately defined distance between the optimal capital stocks in each period.

In the classical one sector stochastic optimal growth model the unique optimal stationary policy generates a Markov process of capital stocks k_t. Recall that the optimal investment function $K(y)$ is a continuous and strictly increasing function on $\Re_+$. Define $H(k, r) \equiv K(f(k, r))$ to be the realized capital stock for the next period under the optimal policy. Then, H is continuous in (k, r) and increasing in k. Let S denote the interval $[0, \overline{k}]$. Given $y_0 \in S$, $k_0 = K(y_0) \in S$, the evolution of optimal capital stocks over time is given by:

$$k_t = H(k_{t-1}, r_t) \qquad (11.8)$$

Recall that ν is the common probability distribution of the i.i.d random shocks r_t, with support Φ, a compact subset of $\Re$. Let ν^t be the joint distribution of $r^t \equiv (r_1, ...r_t)$ on the product space Φ^t and define $k^n(k_0, r^n) \equiv H(H(....(H(k_0, r_1), r_2)..., r_n)$. In other words, $k^n(k_0, r^n)$ is the nth-period capital stock k_n, given k_0 and realization $r^n = (r_1, ...r_n)$ of random shocks in the

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[10] The literature also examines stronger concepts of an optimal steady state [133].
[11] See, [77], [75], [76].

first n periods. For any probability measure μ defined on S (and the Borel σ-field generated by S), define the probability measure $\nu^n \mu$ on S by the relation

$$\nu^n \mu(B) = \int_S \nu^n(\{r^n \in \Phi^n \mid k^n(k_0, r^n) \in B\})\mu(dk_0)$$

where B is any Borel-subset of S. Thus, $\nu^t \mu$ gives us the probability distribution of k_t, when k_0 is distributed according to the probability measure μ. Let S' be a closed interval in S. Then, S' is said to be ν-invariant if $\nu\{r \in \Phi \mid H(k,r) \in S'$ for all $k \in S'\} = 1$. A probability measure μ on S is said to be an invariant probability measure on S' if the support of μ is a subset of S' and for any Borel set B in S,

$$\nu\mu(B) = \mu(B) \tag{11.9}$$

In other words, if k_0 is distributed according to an invariant probability μ, then the distribution of optimal capital stocks in every subsequent period follows the same distribution. The distribution function corresponding to an invariant probability measure is an invariant distribution.

There is a large body of work in the mathematical theory of Markov processes and random dynamical systems that provides sufficient conditions for the existence and stability of invariant distributions for a given stochastic process[12]. Let

$$H_m(k) = \min_{r \in \Phi} H(k,r) \text{ and } H_M(k) = \max_{r \in \Phi} H(k,r)$$

denote the lower and upper envelopes, respectively, of the transition function $H(k,r)$ defining the Markov process (11.8). Note that the continuity of H and the fact that Φ is compact imply that $H_m(k)$ and $H_M(k)$ are well defined and continuous. Further, since H is increasing in k, $H_m(k)$ and $H_M(k)$ are increasing functions.

In addition to the assumptions made in the previous section, the standard proof of existence and global stability of the invariant distribution requires that the production function $f(k,r)$ and the optimal transition function $H(k,r)$ satisfy two additional conditions:

A.10. There does not exist any $k > 0$ and $\widetilde{y} \in S$ such that $\nu\{r \mid f(k,r) = \widetilde{y}\} = 1$.

A.11. There exists an $\epsilon > 0$ such that $H_m(k) > k$ for all $k \in (0, \epsilon)$.

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[12] See, among many others, [35], [41], [42], [2], [8], [9]. Models of descriptive stochastic growth (such as the stochastic Solow model) where the consumption and investment rules are exogenously specified have also applied these conditions ([81], [13] and [109]).

A.10 requires that every investment level is associated with some non-trivial uncertainty over output. A.11 is a restriction on the optimal policy. It implies that even if the lowest possible output is realized every period, the optimal program from every initial stock is bounded away from zero. In a deterministic model, the optimal policy satisfies this condition as long as marginal productivity at zero is large enough. This is no longer true in the stochastic model. Mirman and Zilcha [86] develop an example where the production function has infinite slope at zero, yet the optimal program from any initial stock comes arbitrarily close to zero with probability one.[13] One can impose restrictions on the production function and distribution of random shocks to ensure that A.11 is satisfied. For example, if there is a strictly positive probability mass on the "worst" production function in the sense that there exists some $\varrho > 0$ such that $\nu\{r \mid f(k,r) = \min_{r \in \Phi} f(k,r)\} > \varrho, \forall k > 0$, then infinite marginal productivity at zero is sufficient for A.11. For conditions that are applicable to atomless distributions, see [93].

Define the maximal fixed point of H_m by $k_m = \max\{k > 0 \mid H_m(k) = k\}$ and the minimal fixed point of H_M by $k_M = \min\{k > 0 \mid H_M(k) = k\}$. Assumption A.11 implies that $k_m, k_M > 0$.

Lemma 11.2.5. $k_m < k_M$.

Proof. Since $H(k,r)$ is continuous in r, there exists $r_m, r_M \in \Phi$ such that $k_m = H_m(k_m) = H(k_m, r_m)$ and $k_M = H_M(k_M) = H(k_M, r_M)$. Further, $f(k_m, r_m) \leq f(k_m, r)$ for all $r \in \Phi$. From the stochastic Ramsey-Euler equation:

$$u'(C(f(k_m, r_m))) = \delta \int_\Phi u'(C(f(H(k_m, r_m), r)))f'(H(k_m, r_m), r)\nu(dr)$$

$$= \delta \int_\Phi u'(C(f(k_m, r)))f'(k_m, r)\nu(dr).$$

Since u is strictly concave and C is increasing $u'(C(f(k_m, r))) \geq u'(C(f(k_m, r_m)))$ for all $r \in \Phi$. Hence, the inequality above yields $1 \leq \delta \int_\Phi f'(k_m, r)\nu(dr)$. Similarly, one can show that $1 \geq \delta \int_\Phi f'(k_M, r)\nu(dr)$ so that $\int_\Phi f'(k_M, r)\nu(dr) \leq \int_\Phi f'(k_m, r)\nu(dr)$. The fact that $k_m \leq k_M$ follows from the strict concavity of f. Finally, if $k_m = k_M$ then $f(k,r)$ is constant in r which violates A.10.

Lemma 11.2.5 implies that the highest fixed point of H_m lies below the smallest positive fixed point H_M. We now state the main result regarding the existence and global stability of the optimal stochastic steady state. For the stochastic process of optimal capital stocks k_t defined by (11.8), let $F_t(k)$ be the distribution function of k_t i.e., $F_t(k) = \nu^t\{r^t \in \Phi^t \mid k_t \leq k\}$.

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[13] Mitra and Roy [93] develop general conditions under which $Prob\{\liminf_{t \to \infty} k_t = 0\}$ is 0 and 1.

Proposition 11.2.3. *Assume A.1 - A.11. There exists a unique non-zero in-variant distribution $F(k)$ on S and its support is the interval $[k_m, k_M]$. For any initial capital stock $k_0 > 0$, as $t \to \infty, F_t(k)$ converges uniformly in k (on S) to $F(k)$.*

Proof. (Sketch).Instead of giving a full proof, we sketch the main arguments for the simple case of multiplicative shock, $f(k, r) = rf(k)$, which assumes just two possible values a and b, $0 < a < b < \infty$. Then, $H_m(k) = K(af(k))$ and $H_M(k) = K(bf(k))$. The proof consists of the following key arguments. First, for the Markov process (11.8), the set of states $(0, k_m)$ and (k_M, ∞) are transient. With probability one, capital stocks move out of these sets in finite time, never to return. Second, once the process enters the set $[k_m, k_M]$ it remains there with probability one. Further, $[k_m, k_M]$ is the smallest $\nu - invariant$ set. Let $y_m = \min\{k : H_m(k) = k\}$ and $y_M = \max\{k : H_M(k) = k\}$. Then, $0 < y_m \le k_m < k_M \le y_M$. From any stock $k \in (0, y_m)$ the optimal capital stocks increase almost surely and reach the set $[y_m, k_M]$ in finite time with probability one. Similarly, from any stock $k \in [y_M, \infty)$ the optimal capital stocks decrease almost surely and reach the set $[k_m, y_M]$ in finite time with probability one. Further, for $k \in [y_m, k_m)$ one can show that the probability that the optimal path from such a stock does not enter $[k_m, k_M]$ in finite time is zero. To move the capital stock y_m to the interval $[k_m, k_M]$ only takes a sufficiently long, but finite run $r_t = b$, such that the realized transition occurs along the function $H_M(k)$. Any such run must occur $\omega-$almost surely as shocks are independent. In fact, no strict subset of $[k_m, k_M]$ is invariant. The next step is to show that a well-known "splitting" condition due to Dubins and Freedman [35] (or some variation/extension) holds on the interval $[k_m, k_M]$. For any $n = 1, 2...,$ the probability ν^n is said to split on a $\nu-$invariant subset S' of S if there exists $z \in S'$ and $\eta > 0$ such that:

$$\nu^n\{r^n \in \Phi^n \mid k^n(k, r^n) \le z \text{ for all } k \in S'\} > \eta$$
$$\nu^n\{r^n \in \Phi^n \mid k^n(k, r^n) \ge z \text{ for all } k \in S'\} > \eta.$$

To verify that the splitting condition holds fix any $z \in (k_m, k_M)$. There exists some $N \ge 1$, such that: (i) if $r_t = a, t = 1, ...N$, then $k^N(k_M, r^N) \le z$, and (ii) if $r_t = b, t = 1, ...N$, then $k^N(k_m, r^N) \ge z$. For $0 < \eta < \min\{(\nu(a))^N, (\nu(b))^N\}$, $n = N$, it is easy to see that the splitting condition is satisfied on $S' = [k_m, K_m]$. Dubins and Freedman [35] then show that this implies there exists a unique invariant distribution F on S' and that $F_t(k)$ converges uniformly in k to $F(k)$.[14] Finally, since the set $S - S'$ is transient

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[14] Recent extensions of the result that are applicable to situations where $H(k, r)$ is monotonic but not necessarily continuous and situations where the capital process is multidimensional can be found in [8],[9].

and S' is the smallest $\nu-$invariant set on S, it must be that F is the unique invariant distribution on S and $F_t(k)$ converges uniformly in k to $F(k)$ on S.[15]

The basic results on the existence and global stability of an invariant distribution for the classical one sector stochastic model were originally developed in the pioneering work by Brock and Mirman [18] and subsequently refined by Mirman and Zilcha [81]. Majumdar, Mitra and Nyarko [69] were the first to explicitly use the Dubins-Freedman splitting condition. Versions of this problem have also been analyzed by [126] and [42]. [19] [25] contain similar results for the undiscounted model ($\delta = 1$) where optimality is based on the "overtaking criterion".[16] Donaldson and Mehra [34] extend these results to the case of correlated shocks that enter the production function multiplicatively and follow a stationary process.

When shocks are unbounded, Stachurski [124] shows that there is always a unique globally stable steady state for the special case of a multiplicative shock where r has a density function that is strictly positive everywhere on $\Re_{++}$. With an interior optimal policy, the structure imposed on the random shock ensures that the system moves with positive probability from any positive stock to any interval on $\Re_{++}$ in *one* step.

¿From an empirical point of view one may be interested in the asymptotic statistical properties of the stochastic processes for capital and consumption. For example, if the law of large numbers holds so that sample averages from time series converge to the mean of the limiting steady state distribution, then one can test a model by comparing the sample average over a sufficiently long period with the theoretical prediction. Alternatively, one can forecast the mean of the long run distribution by using the sample average. The central limit theorem or asymptotic normality of the partial sums can be used for inference of likelihood of values in a parameter space. Many of the conditions that guarantee global stability of an invariant distribution also ensure that both the law of large numbers and the central limit theorem hold. In addition, they imply a minimum bound on the rate of convergence.[17]

An important implication of global stability is that the long run behavior of the economy is independent of the initial state. This is also brought out in

[15] A more traditional approach in theory of Markov processes is to directly verify that the process is irreducible on $[k_m, k_M]$, that intervals disjoint from $[k_m, k_M]$ are transient and an equicontinuity condition on the sequence of probability measures for the capital stock (defined through the stochastic kernel of the Markov process). See, [78]. Another approach is to show that the iterated random functions satisfy a Lipschitz condition and are "contracting on the average" (see, [33]). In a framework with multiplicative shocks that are not necessarily i.i.d. and have a positive valued density on $\mathbb{R}_+$, Nishimura and Stachurski [96] use a new approach by defining Foster-Lyapunov functions to characterize stability.

[16] For convergence in a stochastic open economy, see for example, [28].

[17] See, [9]. [125] contains similar results for the case of multiplicative unbounded shocks.

turnpike results that directly examine the conditions under which differences in initial conditions have negligible effects on the process of economic growth over long time horizons. Majumdar and Zilcha [71] establish a "late" turnpike theorem in a model that is far more general than the classical model of Section 2. Their model allows for unbounded expansion of capital and consumption, time varying utility and random shocks that may follow a non-stationary stochastic process. Under a condition that requires the elasticity of marginal product to be bounded away from zero (implying a lower bound on the degree of concavity of the production function), they show that the number of periods for which the relative distance between the optimal capital stocks (from any two initial stocks) exceeds any positive threshold is bounded almost surely, where the bound depends on how far apart the initial states are. In other words, optimal paths from different initial states eventually approach each other with probability one. Note that this result is quite independent of whether there is a globally stable invariant distribution. The condition on the elasticity of marginal product ensures a that a uniform "value-loss" argument (originally due to Radner [105]) holds.[18] Joshi [49] provides similar turnpike results in a one-sector model with recursive preferences and time varying technology.

Apart from convergence, the other important question in economic growth relates to characterization of the properties of the limiting steady state; in particular, the relationship between the preferences and technology underlying the economy and the nature of the invariant distribution to which it converges. In the one sector convex deterministic model of optimal growth, there is fairly rich characterization of the steady state. For example, with a strictly concave production function f, the unique steady state or modified golden rule is a capital stock $\widehat{k}$ that is the unique maximizer of $[\delta f(k) - k]$, where the latter can be interpreted as the net gain from investment. For the no-discounting case, the steady state is the well-known golden rule capital stock that maximizes the level of sustainable consumption $[f(k) - k]$. There are also other decentralized or support price-based characterizations of the optimal steady state. In general, in the deterministic one sector model, it is possible to look at the steady state as a solution to an independent *static* optimization problem that has desirable economic properties. In the multisector deterministic model, it is a solution to a static optimization and a fixed point problem.

Surprisingly, there is very little by way of general qualitative characterization of the limit invariant distribution in the stochastic growth literature. One of the reasons behind this is the fact that, unlike the deterministic model, the

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[18] The value loss argument uses support prices of optimal paths to look at the accumulation of shortfalls in values (shadow profits and losses) of input-output combinations along one optimal path relative to another at the other's support prices. Loosely speaking, for two optimal paths that do not approach each other asymptotically, if the value loss is uniformly bounded away from zero over all states and time periods, then the accumulated loss is infinite and that contradicts optimality. [51] contains a turnpike result without requiring uniformity of value-loss across time and states.

steady state is not determined solely by the production function and the discount factor. Both the utility function (and its curvature) and the distribution of the random shocks play important roles. Specific examples show that for the same technology, discount factor and distribution of random shocks, the steady state distribution can change dramatically with variations in the utility function [26].

Further, even for very standard utility and production functions, the limiting distribution can be very sensitive to parameter values when the shock does not have a continuous distribution.[19] For the case with logarithmic utility, Cobb-Douglas production, and a binary multiplicative shock, Mirman and Zilcha [81] show that the invariant distribution can be degenerate for some parameter values and uniform for others. Montrucchio and Privileggi [94] show that the invariant distribution can also be a Cantor function. Mitra, Montrucchio and Privileggi [92] expand on this example to establish precise bounds on the parameters under which Cantor and more general singular invariant distributions can arise as well as bounds under which the distribution is absolutely continuous. Recently, Mitra and Privileggi [91] extend the example to the class of all iso-elastic utility functions and establish sufficient conditions for a Cantor type invariant distribution.

11.2.3 Stochastic Steady States and Convergence Properties in the Multisector Classical Model

In the literature on deterministic models of optimal economic growth, the multisector case has been extensively studied. In particular, the literature has focused on two key issues - the existence of an optimal steady state and turnpike results or the convergence properties of optimal paths.[20] In comparison, the stochastic multisector literature is relatively thin and there is only a small literature on the existence and stability of steady states in the stochastic, multisector case.

In the deterministic literature, it is well recognized that with discounting, the existence of a globally stable optimal steady state and other turnpike results may not hold in the multisector case (even though it always holds under very mild restrictions in the one-sector model).[21] With significant discounting, optimal paths in the multisector model may not be convergent. They may exhibit cyclical and even chaotic dynamics.

A general stochastic multisector optimal growth model with i.i.d. shocks has been analyzed by Brock and Majumdar [17]. The model is a natural extension of the classical one-sector model to the case of m capital goods. For each vector of current capital stocks and realization of the random shock there is a correspondence that defines the set of attainable utilities and capital stocks for

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[19] For the case of multiplicative shock with continuous density, Danthine and Donaldson [27] show that the limiting invariant distribution has a continuous density function.

[20] For an excellent review of the basic results see McKenzie [76].

[21] See, [128].

the next period, which in turn can be used to define the set of feasible *programs* from any given initial vector of capital stocks. The objective is to maximize the discounted expected sum of utilities, or for the undiscounted case, a stochastic version of the overtaking criterion. The paper imposes four conditions:

(i) there is a compact set $S' \subset \Re_+^m/\{0\}$ such that for any initial vector of capital stocks lying in S', there exists an *optimal* program such that the stochastic process for capital lies almost surely in S'.

(ii) there exist continuous stationary optimal investment and consumption policies.

(iii) an optimal program is "competitive" relative to a non-trivial price process in a similar sense as in the previous section and satisfies a transversality condition that the expected values of the capital stocks (at the competitive prices) go to zero, for the case of discounting, and are bounded, in the undiscounted case.

(iv) the Hamiltonian system corresponding to the optimal process has "suitable curvature" so that a stochastic value-loss condition is satisfied.

Under conditions (i) - (iv), Brock and Majumdar show that the distance between the probability distributions of tth−period optimal capital vectors from two distinct initial capital vectors in S' converges to zero as $t \to \infty$. Further, the difference between the two optimal paths converges to zero in probability. Thus, conditions (i) - (iv) are sufficient to ensure that the optimal paths from alternative capital stocks come close to one another asymptotically and that the long run behavior of optimal paths does not depend on initial conditions. The existence of a unique and globally stable invariant distribution for the stochastic process of optimal capital stocks can also be established under these conditions. Unlike the conditions for global stability of an invariant distribution and other turnpike results in the one-sector stochastic growth model, (i) - (iv) are fairly strong restriction imposed directly on the optimal policy rather than the primitives of the model. Conditions (i) - (iii) are readily satisfied in the one sector stochastic growth model. In the multisector case there are plausible conditions on preferences and technology for (i) and (ii) to hold. For example, Majumdar and Radner [70] consider a stochastic nonlinear activity analysis model in which neoclassical conditions on the technology and preferences are sufficient for (i) and (ii).[22] Condition (iii) is motivated by the equivalence between optimal programs and competitive programs that satisfy a transversality condition (see, [135], [136], [137]). Condition (iv) is a stochastic extension of conditions for asymptotic stability in the deterministic multisector model due to Cass and Shell [22] and Rockafellar [110] that are, in turn, based on the well known "value loss" argument alluded to in the previous subsection (see also, McKenzie [75]). In particular, condition (iv) requires that the Cass-Shell-Rockafeller version of the value-loss restriction holds uniformly for all states of the environment.

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[22] See also the discussion in [88].

Chang [24] shows that a weaker version of condition (iv) based on *expected value loss* is actually sufficient and further, that the difference between any two optimal paths converges not only in probability, but almost surely. It is worth noting that in a multisector model condition (iv) involves a strong restriction on the extent of discounting in the model, and unlike the one-sector case, it does not follow directly from a restriction on the curvature of the production function.

Föllmer and Majumdar [39] follow a somewhat different approach using the theory of martingales to show that even if one does not impose a condition such as (iv), a weaker result is possible. That is, for any two optimal paths, the number of periods for which the value loss exceeds any given positive threshold is finite with probability one. Under uniformity of value loss and a specific distance metric, optimal paths approach each other asymptotically almost surely.

For the case of no discounting with the "overtaking" criterion of optimality, global stability of the stochastic steady state and other turnpike results can be established under much less restrictive conditions (see, among others, [46], [25], [136]).[23,24]

11.3 Extensions of the Classical Framework

11.3.1 Sustained (Long Run) Growth

The past two decades have seen a renewed interest in the economics of long run growth where unbounded expansion of output, capital and consumption is possible. In the deterministic convex one-sector model, sustained growth is optimal if the marginal productivity of capital at infinity exceeds the discount rate [47]. Much of the literature on stochastic optimal growth theory focusses primarily on models where the technology exhibits bounded growth that rules out indefinite expansion of consumption and capital and sustained long run growth. An exception is the class of models on optimal intertemporal household savings under uncertainty. A portion of this literature considers a linear production function with a multiplicative shock, $f(k, r) = rk$, so that optimal paths may diverge to infinity (see, Phelps [103], Levhari and Srinivasan [63] and subsequent contributions). A closely related literature on the permanent income hypothesis has examined optimal savings where the wealth next period

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[23] Dutta [37] provides sufficient conditions under which as $\delta \to 1$, the optimal policies (and value functions) in the discounted stochastic model converge to the optimal policy using two alternative optimality criteria - the undiscounted overtaking criterion and the long run average reward criterion.

[24] In a stochastic multisector model with a double infinity of time periods and discount factors close to 1, Yano [133] establishes the existence and continuity in the discount factor of a stronger concept of an optimal stationary program (where a stationary program is one where the vector of current capital stocks associated with any infinite realized sequence of past history is time invariant with probability one). See also, [74].

is composed of a deterministic return on current savings (interest income) plus an additive income shock (non-interest income).[25]

De Hek and Roy [31] examine the possibility of sustained long run growth in optimal consumption and capital stocks in a one sector model with i.i.d. shocks and a concave production function that is not necessarily linear. Consider the model in Section 2 without assumption A.5. In particular, suppose that $f(k,r) = rf(k)$ and let $\theta = \lim_{k\to\infty} \frac{f(k)}{k}$. They show that under the following two conditions, optimal capital and consumption diverge to infinity with probability one from every positive initial stock:

(i) $E[\ln(\theta r)] > 0$

(ii) $\inf_{y>0} \delta E[\frac{u_c(rf(sy))rf_k(sy)}{u_c((1-s)y)}] > 1$, where $s = \exp[-E[\ln(\theta r)]]$.

Note that these conditions involve the utility function and its curvature. The possibility of long run growth depend on more that a simple comparison of the discount rate and average marginal product at infinity. Once again, this reflects the general fact that in a stochastic growth model, the utility function and distribution of shocks play important roles in determining the nature of long run behavior of the economy. To illustrate this further we consider a specific example of iso-elastic utility and linear production function for which we can derive the optimal policy explicitly and thus provide an almost exact characterization of the condition for sustained long run growth.

Example 11.3.1. $u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \sigma > 0, \sigma \neq 1, f(k,r) = rk$. One can show that the optimal policy function $K(y)$ is linear and given by $K(y) = [\delta E(r^{1-\sigma})]^{\frac{1}{\sigma}} y$ so that

$$k_{t+1} = \alpha r_{t+1} k_t \text{ where } \alpha = [\delta E(r^{1-\sigma})]^{\frac{1}{\sigma}} \qquad (11.10)$$

which implies $\ln k_{t+1} = \ln k_0 + (t+1)[\frac{1}{t+1}\sum_{j=0}^{t} \ln \alpha r_{j+1}]$. Using the law of large numbers, it is easy to show that an "almost" exact condition for $\ln k_{t+1}$ to diverge to infinity with probability one is that $E[\ln(\alpha r)] > 0$ which can be rewritten as $\sigma E(\ln r) + \ln \delta + \ln E(r^{1-\sigma}) > 0$. This indicates that the risk aversion/intertemporal elasticity of substitution parameter of utility, σ, plays an important role in determining whether sustained growth occurs.

11.3.2 Stochastic Growth with Irreversible Investment

In the classical framework analyzed in the previous section, investment is either reversible or the existing capital stock depreciates completely at the end of a period. In reality, it is costly to transform capital into consumption and there are limits to how fast the aggregate capital stock depreciates. The stochastic growth model with irreversible investment was first examined by Sargent [117]. In his setting output can either be consumed or invested, but once invested,

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[25] See [132] and [119] for the undiscounted case, [120] and [122] for the discounted case and also [23].

capital cannot be converted for consumption. Individual agents transact in a competitive market for existing capital. This allows individual investment decisions to be reversed while maintaining the irreversibility of investment in the aggregate. As Sargent shows, irreversibility in the aggregate provides the necessary friction for Tobin's q, the relative price of used to new capital, to diverge from unity. This enables aggregate investment to be positively correlated with q. However, this same friction implies that agents' investment decisions are necessarily a function of their expectations about the future which cannot be summarized by q. The implication is that q-theory of investment functions are of little use for econometric policy evaluation.

The analysis in Sargent is based on the properties of the value function. Olson [100] develops an alternative approach that characterizes optimal policies using stochastic Kuhn-Tucker conditions. Let $f(k, r) = F(k, r) + (1-d)k$ where k is the depreciation rate of capital. If λ_t is the Lagrange multiplier on the period t irreversibility constraint, $k_{t+1} \geq (1-d)k_t$, the Ramsey-Euler equation can be written as:

$$u_c(c_t) - \lambda_t = \delta E \left[u_c(c_{t+1}(r)) f_k(k_t, r) - (1-d)\lambda_{t+1}(r) \right]. \qquad (11.11)$$

Solving for λ_t and substituting forward this can be expressed as:

$$u_c(c_t) = \delta \sum_{i=1}^{T} (\delta(1-d))^{i-1} E \left[u_c(c_{t+i}) F_k(k_{t+i-1}, r_{t+i}) \right] + \delta^T (1-d)^T E[u_c(c_{t+T})].$$

$$(11.12)$$

This derivation uses the fact that eventual depreciation of the entire capital stock is not optimal so there is a uniform upper bound, T, on the number of time periods for which the irreversibility constraint binds. Sargent's point that agent's decisions are a function of expectations about the future is clearly evident from (11.12). Evaluating (11.12) at the minimal and maximal optimal transition functions for capital it can be shown that the support of the limiting distribution under irreversible investment is a subset of the support when investment is reversible.

11.3.3 Stochastic Growth with Experimentation and Learning

The stochastic growth model has been extended to environments where there is learning about productivity or the capital stock itself. This requires expanding the state space to represent the agent's beliefs. The transition equation for beliefs follows Bayes' rule. In this setting, the possibility of learning affects the optimization problem in two important ways. First, even if information signals are exogenous so that learning is passive and not affected by the current action, the mere prospect of learning may alter current period decisions. Second, when the current action affects how much learning occurs, there is an incentive to

experiment to obtain better information. Friexas [40] was the first to examine this problem. Assume output is produced by a technology $f(k, \theta, r)$, where θ is an unknown parameter. The distribution of r is known. Given an initial value for y and current beliefs about θ, the agent chooses consumption and investment. Output in the following period is observed and provides information that can be used to update beliefs about θ. Friexas examines how learning and experimentation affect the initial consumption/investment decision. The learning effect depends on whether learning increases or decreases the marginal value of investment. Friexas then uses Blackwell's [10] theorem to assert that if larger investment yields more information then the experimentation effect leads to an increase in investment. Subsequently, it was shown in [5] and [29] that this need not always be true. The reason is that investment affects both state variables in the value function so that Blackwell's theorem does not apply. While higher investment may be more informative, the value of information at higher levels of output may be lower. When the second effect dominates, an expected utility maximizer may prefer to invest less even if it is more informative. These tradeoffs have made it difficult to obtain a general set of verifiable conditions to characterize how information affects consumption and investment in the infinite horizon model. Precise results are limited largely to problems where there are only two relevant decision periods.

Nyarko and Olson [99] examine experimentation and learning in a stochastic growth model where there is imperfect information about the capital stock itself. Consumption is observable, but output and investment are not. Beliefs about the state are summarized by a probability distribution over y. After choosing consumption, an information signal is observed that can be used to update beliefs about y. The mapping from beliefs in period t to beliefs in period $t+1$ is determined jointly by consumption, the information signal and the stochastic production function. Here there is learning about a moving target, in contrast to the case above where the unknown parameter is fixed. Nyarko and Olson show that if $u(0) = -\infty$[26] then the optimal policy is to assume the worst and optimize against that. That is, the initial state is assumed to be the lower bound of the support of the agent's beliefs about output and the transition equation is $\inf_r f(k, r)$. When information alters the lower bound of the support of the agent's beliefs there is an endogenous capital discovery process. When it does not, the problem with learning has an equivalent, deterministic representation. In that case, output and investment are more volatile than consumption and there is excess saving compared to the case where the capital stock is observable. In cases where $u(0) > -\infty$, the solution either corresponds to that above, or the capital stock becomes zero with strictly positive probability.

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[26] This assumption holds for the class of all constant relative risk averse utility functions with coefficient at least one.

11.4 Non-classical Models of Optimal Stochastic Growth

The models of optimal economic growth under uncertainty reviewed in the previous section are based on the classical assumptions of convex technology and utility that depends only on consumption. This section reviews some extensions of the theory that allow for non-classical features such as non-convexities and state-dependent utility. These non-classical features imply that even in a one-sector model, continuity and monotonicity properties of optimal policies need not hold and optimal paths need not converge to a unique stochastic steady state. The long run behavior of the economy may depend critically on the initial state.

11.4.1 Stochastic Growth with Non-convex Technology

Non-convexities enter the production technology of an economy through numerous sources, such as fixed costs, threshold effects, increasing returns to scale, economies of scope, and depensation in the reproduction of natural resources. In applications of optimal growth models to areas such as environmental management there is also the need to study the implications of a non-convex technology. A separate chapter of this handbook focuses on optimal growth in non-convex economies. In this subsection, we concentrate on explaining how a non-concave production function (non-convex technology) alters the basic results of the classical stochastic growth literature reviewed in the previous subsections.

Majumdar, Mitra and Nyarko [69] were the first to comprehensively analyze the problem of optimal stochastic growth in a one sector model where the production function, $f(k, r)$, is not necessarily concave, though it exhibits bounded growth.[27] In this framework, the set of feasible programs is not necessarily convex and therefore, the value function for the dynamic optimization problem is not necessarily concave even though the utility function satisfies classical concavity restrictions. This non-convexity means that the maximization problem on the right hand side of the functional equation may have multiple solutions so that instead of a unique optimal policy function, the solution is characterized by a measurable selection from an upper semi-continuous optimal policy correspondence. Further, there need not exist any continuous selection and every policy function may exhibit jump discontinuities on a set that is at most countable. Also, non-convexity in the economy implies that the optimal path is not necessarily decentralizable - in particular, support prices may not exist.

As the value function is not necessarily concave, the expected future marginal value of capital may be increasing in current investment.[28] This, in turn,

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[27] Some notable contributions to deterministic optimal growth with a non-convex technology include [68],[32].

[28] The term "marginal" is used loosely here as the value function is not necesarily differentiable no matter how smooth the utility and production functions are.

implies that optimal consumption may actually decline with an increase in output.[29] Indeed, in the deterministic model it has been shown that there may not exist an optimal consumption function that is globally monotonic. The optimal investment policy correspondence is, however, an ascending correspondence. Further, if the utility function is strictly concave, then it can be shown that every measurable selection from this correspondence is non-decreasing and an optimal investment policy function $K(y)$ is always non-decreasing in output.[30]

A central question is whether there exists a globally stable invariant distribution. In the deterministic literature with non-concave production functions, it has been shown that there may be a multiplicity of steady states and the limit of the optimal path of capital stocks may depend on the initial state. For example, with an S-shaped production function, it is quite possible that optimal paths from small stocks converge to zero (extinction), while for initial stocks above a critical level,[31] optimal paths converge to a strictly positive optimal steady state. This initial state dependence can be expected to be true in the stochastic model too.

Consider the model of Section 2 without assumption $A.4$. For any measurable selection from the optimal policy correspondence, the transition function $H(k, r)$ for the Markov process of optimal capital stocks (11.8) is non-decreasing in k, but not necessarily continuous.[32] Recall that k_m, k_M are the largest positive fixed point of the lowest transition function $H_m(k)$ and the smallest positive fixed point of the highest transition function $H_M(k)$, respectively. A critical step in the proof of global stability in Proposition 11.2.3 is Lemma 11.2.5 that showed $k_m < k_M$. Indeed, if $A.10$ and $A.11$ hold and $k_m < k_M$, there exists a globally stable invariant distribution even if assumption A.4 does not hold. However, in the non-convex model it is quite possible that $k_m > k_M$ so that Lemma 11.2.5 does not hold. To see what happens in that case, suppose that

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[29] Unlike both the classical stochastic model and the deterministic model with non-concave production function, it is difficult to guarantee that optimal consumption is strictly positive in the stochastic model with non-concave production, even if Inada conditions are imposed on the utility and production functions. An interior optimal policy is ensured in [69] by assuming that $u(0) = -\infty$, which is a very strong restriction on the class of admissible utility functions. More recently, [95] establishes interiority by assuming the Inada condition on utility, sufficiently high marginal productivity at zero, and that the random shock is multiplicative and has a density function so that the maximand on the right hand side of the functional equation of dynamic programming is smooth.

[30] If the utility function is concave but not strictly concave, then there may be an optimal investment function that is non-monotonic though, even in that case, there is at least one optimal investment function that is non-decreasing.

[31] This critical level is referred to as a safe standard of conservation in the literature on renewable resource economics.

[32] An innovative approach to the non-convex model can be found in Amir [1]. It takes advantage of the averaging associated with the random disturbances to derive conditions for the monotonicity of optimal policies and higher order differentiability of the value function. As in [12], differentiability of the optimal policy functions follows from the implicit function theorem.

$A.10$ and $A.11$ hold and optimal policy is interior ($0 < K(y) < y$ for all $y > 0$).
As in the sketch of the proof of Proposition 11.2.3, confine attention to the
case where $f(k, r) = rf(k)$ where r assumes one of two possible values a, b.
As before, let $y_m > 0$ be the smallest positive fixed point of the lowest tran-
sition function $H_m(k)$. Then, for all $k \in (0, y_m), H_M(k) > H_m(k) > k$ and
$H_M(y_m) > H_m(y_m) = y_m$ so that $y_m < k_M$. Similarly, it is easy to show that
$k_m < y_M$, where y_M is the largest fixed point for the highest transition function
$H_M(k)$. Thus, $k_m > k_M$ implies $y_m < k_M < k_m < y_M$. It is easy to check that
the two disjoint intervals $[y_m, k_M]$ and $[k_m, y_M]$ are both $\nu-$invariant; from any
initial state in either interval, the optimal capital process remains in that inter-
val almost surely. For $k_0 \in (0, k_M]$, all optimal paths eventually enter and stay
in the interval $[y_m, k_M]$ while for $k_0 \in [k_m, \infty)$, all optimal paths eventually
enter and stay in the interval $[k_m, y_M]$. There is no globally stable invariant
distribution. Using arguments based on the splitting condition referred to ear-
lier, Majumdar, Mitra and Nyarko [69] show that if $k_m > k_M$, then for all
$k_0 \in (0, k_M]$, the distribution of capital stocks converges to the same invariant
distribution whose support is $[y_m, k_M]$, while for all $k_0 \in [k_m, \infty)$, the distribu-
tion of capital stocks converges to another invariant distribution whose support
is $[k_m, y_M]$. For any fixed initial stock in the intermediate range (k_M, k_m), the
optimal path may enter either of the two invariant sets and remain there, de-
pending on the realization of random shocks. This last possibility illustrates an
aspect of path dependence that has no parallel in the deterministic literature.

In general, non-convexities in production may lead to multiple invariant
distributions. However, if production is "sufficiently stochastic", then there
exists a globally stable invariant distribution despite the non-convexity [69].
Here, the precise condition that ensures global stability is:

$A.12$. There exists some $\vartheta > 0$ in S such that $\nu(\{r \in \Phi \mid f(k, r) \leq \vartheta$ for
each $k \in S\}) > 0$ and $\nu(\{r \in \Phi \mid f(k, r) \geq \vartheta$ for each $k \in S\}) > 0$.

Observe that assumption $A.12$ a condition on the production function, not
the transition function for the optimal capital process. It captures the idea
that the random output that results from any given investment is sufficiently
spread out, i.e., the technology exhibits sufficient variability. Under this condi-
tion, if we let $z = K(\vartheta)$,then one can easily verify that the splitting condition
described in the proof of Proposition 11.2.3 is immediately satisfied. This en-
sures global stability. Thus, the possibility of multiple stochastic steady states
depends on the stochasticity of the model. This is another instance where the
stochastic growth model (with sufficient uncertainty) is qualitatively different
from the deterministic analogue. We summarize the above discussion in the
next proposition:

Proposition 11.4.1. *Assume $A.1 - A.7, A.10, A.11$ and that optimal policy is
interior. Then, (i) if $k_m < k_M$ or if $A.12$ holds, there is a unique invariant
distribution on S whose support is $[k_m, k_M]$ and from every $k_0 > 0$,the optimal*

capital stocks converge in distribution (uniformly) to this invariant distribution; (ii) if $k_m \geq k_M$, then for all $k_0 \in (0, k_M]$, the distribution of optimal capital stocks converges to an invariant distribution whose support is $[y_m, k_M]$, while for all $k_0 \in [k_m, \infty)$, the distribution of capital stocks converges to another invariant distribution whose support is $[k_m, y_M]$.

As in Section 3.1, A.11 implicitly imposes restrictions on the technology. For example, in [69] it is obtained from the model primitives by assuming (in addition to a condition for interiority of optimal policy) that the random shock has finite support and that the marginal productivity at zero is infinite. The latter is a rather serious restriction on the class of admissible non-concave production functions. It rules out the S-shaped production function that is a widely used canonical form to capture increasing returns to scale and other threshold effects.[33]

Nishimura, Rudnicki and Stachurski [95] analyze a non-convex model with multiplicative i.i.d. random shock that has a density function that is strictly positive on $\Re_{++}$. Under restrictions on the expectation of the random shock, they show that the Markov process of optimal capital stocks either converges to zero from every initial state or there is a globally stable non-zero steady state (and identify conditions for these events). To place their results in context, their assumption on the density function automatically satisfies the "very stochastic" assumption in [69] discussed above. Their result does not require Inada conditions on the production function and, in fact, allows the marginal product at zero to be less than one with positive probability. In a similar framework, Nishimura and Stachurski [96] use the Euler equation to analyze stability of the stochastic optimal capital process; in particular, they use the marginal utilities as Foster-Lyapunov functions in order to obtain stability.

The literature on non-convex stochastic growth also develops turnpike conditions under which optimal paths approach each other asymptotically. In a model with non-convex and non-stationary technology Joshi [50] uses the monotonicity properties of the optimal policy and a supermartingale process generated by the stochastic Ramsey-Euler equation to show that, under a strong "value loss" condition that is uniform with respect to time and state, the asymptotic distance between optimal paths from two distinct initial states converges to zero with probability one. However, as in the case of turnpike theorems in the stochastic multisector convex models, the uniform value loss condition is not very transparent in terms of its implications for the model primitives.

One of the interesting questions in stochastic growth models with non-convexity is the possibility of extinction where optimal paths converge to zero. This is particularly important in applications of the optimal growth model to problems of renewable resource management where utility reflects the net

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[33] Mitra and Roy [93] provide weaker conditions that ensure A.11 even when the marginal productivity at zero is finite and the distribution of the random shock is absolutely continuous.

benefit from harvesting and the production function reflects natural biological growth. Assuming a bounded growth production function and i.i.d. shocks that have compact support, Mitra and Roy [93] show that there are only three possibilities: (i) optimal paths from all initial states get arbitrarily close to zero infinitely often with probability one (this includes extinction in finite time), (ii) optimal paths from all initial states are bounded away from zero with probability one, and (iii) there exists a critical capital stock or safe standard above which all optimal paths are bounded away from zero with probability one. They develop sufficient conditions on the preferences and technology that lead to each of these outcomes. In contrast to the deterministic literature, these conditions involve not just the discount factor and marginal productivity, but also marginal utility - one compares the discount rate to expected "welfare-modified" return on investment (marginal productivity) as in the condition in Proposition 11.4.2. Another result on optimal extinction is due to Kamihigashi [53] who shows that if the marginal productivity at zero is finite, then sufficient variability in the random shock implies that *all* feasible programs (including therefore, the optimal program) converge to zero almost surely.

11.4.2 Stochastic Growth with Stock-Dependent Utility

For some important capital theoretic allocation problems welfare depends on both consumption and the beginning of period output, as represented by $u(c, y)$.[34] Utility is assumed to be nondecreasing in y, jointly concave in (c, y) and A.7 is no longer imposed.[35] In the deterministic case stock-dependent utility has two important consequences. The first consequence arises if investment and output are substitutes in utility in the sense that $u(y - k, y)$ is submodular in k and y. In that case, an interior optimal investment policy may be decreasing in output. At the same time, there may be intervals of the state space on which corner solutions are optimal and the optimal transition function coincides with the production function. Combining these two possibilities opens the door for the optimal transition function to be like a tent map, or even more complex. When this happens an optimal program may exhibit nonlinear dynamics including cycles or chaos [3]. The second important consequence is that multiple optimal steady states are possible, even if the utility function is supermodular in k and y and the optimal investment policy is monotone (Kurz [59]). In such cases, the asymptotic behavior of an optimal program depends on the initial state.

The first analysis of stochastic models with stock-dependent utility can be traced to the literature on renewable resource allocation. In that literature, the production function represents biological growth of the renewable

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[34] Such models include the allocation of natural capital or renewable resources and the effects of wealth on consumption-savings behavior.

[35] In renewable resource allocation problems welfare declines if consumption exceeds the quantity that equates demand and supply.

resource and the random shock represents the effect of environmental distur-
bances on resource growth. The state variable is the resource stock (output) at
the beginning of the period. Stock-dependent utility arises when the harvest
costs depend on the resource stock or when the resource stock has amenity or
other social value. Early papers ([45], [107], [123]) focused on the case where
$u_{cc}(c, y) + u_{cy}(c, y) = 0$. In this case the direct and indirect utility effects of an
increase in output offset exactly and investment and output are neither strict
complements nor strict substitutes in utility. As a result, the optimal policy is a
constant investment policy, which in the presence of fixed costs becomes an (s,S)
inventory rule. Mendelssohn and Sobel [78] prove monotonicity of the optimal
investment policy under the supermodularity condition $u_{cc}(c, y) + u_{cy}(c, y) \leq 0$.
Nyarko and Olson [97] show that the optimal consumption policy is nondecreas-
ing when $u_{cy}(c, y) \geq 0$ and u and f are concave. They also use the Dubins and
Freedman splitting condition to characterize the convergence of optimal pro-
grams to a limiting distribution. Without additional restrictions the invariant
distribution may not be unique and the long run behavior of an optimal pro-
gram may depend on initial conditions. Subsequently, Nyarko and Olson [98]
show that additional sufficient conditions for the existence of a unique invariant
distribution are: (i) $u_c(c, y) = 0$ implies $u_y > 0$ for sufficiently large y, and (ii)
for all $y > 0, c \in \Gamma(y)$ and $\alpha > 1$, if $u_c(c, y) > 0$ and $u_c(\alpha c, \alpha y) > 0$ then
$\frac{u_y(c,y)}{u_c(c,y)} \geq \frac{u_y(\alpha c, \alpha y)}{u_c(\alpha c, \alpha y)}$. The last assumption is a complementarity condition that
implies that the slope of indifference curves for u decrease as output and con-
sumption increase along a ray through the origin in (c, y) space. Nyarko and
Olson provide examples to show that multiple invariant distributions can be
optimal when either (i) or (ii) are violated. The existence of a unique invariant
distribution is also ensured when there is sufficient divergence between produc-
tion in the best and worst states. [69] and [98] show that there is more than
one way to define sufficient variation in production. The underlying intuition
is the same. A model with multiple limiting distributions can be transformed
into one with a unique invariant distribution by the mixing that results from
increasing the variance in production. On the other hand, if the variability in
production is small enough and if $u(y - k, y)$ is submodular in (k, y), then an
optimal program may oscillate between cyclic sets [3].

The economic possibilities associated with the stochastic growth model ex-
pand considerably when a non-convex production technology is combined with
stock-dependent utility. To date this combination has primarily been used to
examine the conditions under which capital stocks remain strictly bounded
away from zero, issues related to conservation and extinction. In the determin-
istic model with both non-convex production and stock-dependent utility it
is possible for there to be disjoint intervals in the state space from which an
optimal program converges to zero. That is, an optimal program starting from
intermediate states may remain bounded away from zero, while optimal pro-
grams starting from lower or higher states converge to zero [101]. The addition
of random productivity disturbances leads to the somewhat surprising possi-

bility that a first-order improvement in the distribution of disturbances can reduce the set of initial states from which optimal output and capital stocks have a positive lower bound.

One useful technique to analyze some questions in the non-convex model is to examine behavior under the convex-hull of the technology. If capital stocks under an optimal program always remain in an interval where the convex-hull coincides with the non-convex technology then the two optimization problems coincide on that interval. This can be used, for example, to provide conditions for the existence of a safe standard of conservation.

Proposition 11.4.2. *Assume $u(y - k, y)$ is supermodular in (k, y), u is increasing in c, and f is concave in k for all r. Let $\underline{f}(k) = \inf_r f(k, r)$. If*

$$\inf_{z \in [0,k]} \delta E\Big[\frac{u_c(f(k,r) - z, f(k,r)) + u_y(f(k,r) - z, f(k,r))}{u_c(\underline{f}(k) - z, \underline{f}(k))}\Big] > 1 \qquad (11.13)$$

then $\liminf y_t \geq k$ for all $y_0 \geq k$.

A general version of this result in the model with non-convex technology and stock-dependent utility can be found in Olson and Roy [102], along with other results dealing with conservation or extinction. The conclusions depend on the joint properties of the technology, utility, and the distribution of disturbances. As can be seen above, $\underline{f}(k)$ or productivity under the worst disturbance is an important determinant of conservation or extinction.

11.5 Comparative Dynamics

An important question in stochastic growth theory is the sensitivity of optimal decision rules and paths with respect to preference and technology parameters that describe the underlying economy. In a one sector model, continuity of optimal investment and consumption decisions with respect to various parameters of the model generally holds under far weaker assumptions than those described in Section 2.[36]

The theory of monotone comparative statics using supermodular functions and complementarity developed in Topkis [129] has been extended to stochastic dynamic models (see, for example, [121], [42]). One can apply results from this literature to derive the comparative dynamics of the optimal policy function with respect to various preference and technology parameters by looking at the maximization problem on the right hand side of the functional equation of dynamic programming [43]. Most of these results have been derived in a one sector framework.

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[36] See for example, [38]. Conditions for parametric continuity of stationary distributions of Markov processes are discussed, among others, by [126] and [61]. These properties are important for numerical simulations.

Danthine and Donaldson [26] show that an increase in the discount factor increases optimal investment and shifts the distribution of optimal capital stocks to the right and hence, the invariant distribution to which the stocks converge.[37] Moreover, they show that an increase in the curvature of the utility function (loosely speaking, an increase in risk aversion), leads to higher consumption (i.e., lower investment) at low levels of output, and lower consumption (i.e., higher investment) at high levels of output; further, the range of the limiting distribution expands as risk aversion increases.[38]

Another important issue in comparative dynamics is the effect of a change in the degree of riskiness or volatility of the random shocks. This relates to a central concern in macroeconomics about the relationship between riskiness of productive assets and the optimal intertemporal precautionary saving decisions of individuals as well as more aggregative analysis of the relationship between growth and economic fluctuations (see for example, [48]). Unfortunately, there is no general characterization of the effect of a second order stochastic change in the distribution of shocks on the optimal policy.[39]

In the specific case of optimal savings under uncertainty [40] discussed in Example 11.3.1, one can characterize the comparative dynamics of riskiness fairly tightly. ¿From (11.10), we have

$$K(y) = \alpha y, \ E[\frac{k_{t+1}}{k_t}] = \alpha E(r_{t+1}), \ \text{where} \ \alpha = [\delta E(r^{1-\sigma})]^{\frac{1}{\sigma}}$$

so that the propensity to invest/save and the expected growth rate of capital are both proportional to α and the latter is increasing (decreasing) in riskiness of the random shock if $\sigma > (<)1$ because $r^{1-\sigma}$ is a convex (concave) function of r in that case. Thus, depending on the curvature of the utility function, an increase riskiness may increase or decrease optimal investment and cause a first order increase or decrease in the distribution of optimal capital stocks. Roughly speaking, if utility is more concave than the logarithmic function, an increase in riskiness of the random shock increases the optimal savings rate and the expected rate of growth. The reverse holds if utility is less concave than the logarithmic function. In the case of log utility, the optimal policy depends only on the average realization of the random shock and not on its higher moments.[41]

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[37] Dutta [36] shows that lengthening the time horizon for a fixed discount factor and increasing the discount factor for a fixed time horizon are, in a precise sense, equivalent.

[38] In the case of logarithmic utility, Cobb-Douglas production with multiplicative shock, an increase in the discount factor increases the variance of capital stock and output. Danthine and Donaldson [27] provide sufficient conditions for this to occur. They also characterize conditions under which an increase in the curvature of the utility function has a similar effect.

[39] An exception is the model of optimal dynamic consumption with deterministic linear interest and additive labor income shock. See, for example, [80].

[40] See, among others, [103], [63], [111], [62].

[41] For an extension of this kind of result to a model of endogenous growth see [30].

A less ambitious question relates to a comparison of the moments of the limiting distribution to the steady state in the deterministic model.

Example 11.5.1. $u(c) = \ln c$ and $f(k, r) = rk^{\beta}, 0 < \beta < 1$. For this example Mirman and Zilcha [81] show that the optimal investment policy is given by $K(y) = \beta\delta y$. Danthine and Donaldson [27] use this to analyze the properties of the optimal program for capital:

$$k_t = (\beta\delta)^{1+\beta+\cdots+\beta^{t-1}} k_0^{\beta^t} r_0^{\beta^t} r_1^{\beta^{t-1}} \ldots r_{t-1}^{\beta} r_t.$$

This implies:

$$Ek_t = (\beta\delta)^{\sum\limits_{s=0}^{t-1} \beta^s} k_0^{\beta^t} \prod_{s=0}^{t-1} E(r_{t-s-1}^{\beta s}).$$

Assume $E(r_t) = 1$ and a non-degenerate distribution for r_t. Then taking limits as $t \to \infty$, the first moment of the limiting invariant distribution of capital satisfies: $Ek = (\beta\delta)^{\frac{1}{1-\beta}} L$, where, $L = \lim_{t\to\infty} \prod\limits_{s=0}^{t-1} E(r_{t-s-1}^{\beta s})$. Jensen's inequality implies $L < 1$. Expected consumption and output in the limiting invariant distribution are given by: $Ec = (1 - \beta\delta)(\beta\delta)^{\frac{\beta}{1-\beta}} L$, $Ey = (\beta\delta)^{\frac{\beta}{1-\beta}} L$. In the deterministic version of the model (where $r_t = 1$ almost surely), steady state capital, consumption and output are given by: $k = (\beta\delta)^{\frac{1}{1-\beta}}$, $c = (1 - \beta\delta)(\beta\delta)^{\frac{\beta}{1-\beta}}$, $y = (\beta\delta)^{\frac{\beta}{1-\beta}}$. This shows that in the stochastic model, the steady state distribution has smaller average capital stock, output and consumption than in the certainty equivalent version of the model.

In Example 11.5.1, uncertainty only affects the evolution of an optimal program and not the optimal policy function itself. This simplifies the task of characterizing the effect of uncertainty on the limiting distribution. In general, uncertainty will also affect the optimal policy function. As we have seen earlier, in the case of iso-elastic utility and linear production, uncertainty may increase or reduce optimal investment depending on the nature of the utility function. This makes it difficult to compare the moments of the limiting distribution of capital for the stochastic model with its certainty equivalent.

11.6 Solving the Stochastic Growth Model

The stochastic growth model is inherently nonlinear. There is no known general, closed form solution. Instead, analysis of the model with general functional forms aims to qualitatively characterize optimal policies and the resulting implications for economic behavior. There are two main approaches to achieving more specific solutions, all of which require assumptions regarding functional forms for production and utility. The only cases with known closed form analytical solutions are those discussed in Examples 11.3.1 and 11.5.1.

Approximation and numerical methods are the alternative when an analytical solution is not available.[42] By far the most common approximation technique is to linearize the Euler equations around the steady state of the model, an idea pioneered by Magill [65] in continuous time and Kydland and Prescott [60] in discrete time. This approach was subsequently extended by [57], [58], [20] and many others. In a model with Cobb-Douglas technology and CES/CRRA utility, [131] develops central limit and large deviation principles that characterize the manner in which capital trajectories in the stochastic model converge to those in the deterministic case as the standard deviation of the random shock goes to zero. In practice, most approximation methods are not entirely analytical and the approximate solution is analyzed using simulations where the underlying parameters are calibrated to data. Solutions are accurate in the neighborhood of a stochastic steady state with support on a small interior interval. Approximation methods are less useful in situations where the disturbance term has support on a large interval, where the solution is not interior, where second order effects are important, and in the study of transition dynamics.

Numerical dynamic programming can be used to solve parametric specifications of the stochastic growth model. The two most common approaches involve iteration of discrete or parametric approximations to the value or policy functions. Recent surveys of numerical methods can be found in [112], [60], [74], and [113]. Once the model is solved, the policy functions can be used to compute moments for the limiting distributions of the economic variables of interest. The main advantage of numerical dynamic programming is that attention need not be restricted to a neighborhood of the steady state. This allows one to investigate almost any question of interest within the context of a given parametric specification, including a study of global dynamics. The primary disadvantage has to do with robustness to model specification, calibration and choice of numerical method.[43] In general, different numerical procedures can yield substantially different results so care must be exercised in their implementation.

11.7 Conclusion

The literature on optimal stochastic growth theory is over three decades old now and there are many important ways in which the theory has contributed

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[42] An early survey and comparison of different methods can be found in [130].

[43] For algorithms generated by a contraction mapping of modulus δ, the approximation error is bounded by $\| V_n - V_{n+1} \| < \epsilon/(1-\beta)$, where ϵ is the tolerance level under the given metric and V_n is the nth iterate of the algorithm. Santos [114] shows how the Euler equation residuals can be used to bound the approximation error for other types of algorithms. Santos and Peralta-Alva [115] examine when the simulated moments from a numerical solution converge to their exact values as the approximation errors converge to zero.

to our understanding of capital accumulation, growth and more generally, optimal intertemporal resource allocation. In this section, we summarize some of the contributions of the stochastic growth literature and we point out where the introduction of uncertainty has done little to alter the conclusions of the deterministic model.

First, stochastic growth theory has provided a different explanation of economic volatility. In contrast to the deterministic case, an optimal program in the stochastic model is a sequence of random variables generated jointly by optimal decisions and random productivity disturbances. Realized capital paths fluctuate even when the optimal policy is time stationary and well-behaved. This way of looking at economic volatility has been successfully utilized by the business cycle literature as a way to capture various stylized facts about economic fluctuations.

Second, in the stochastic growth model the utility function plays a prominent role in determining the long run behavior of the economy. Even in a one-sector model and for the same production technology, discount factor and distribution of random shock, the limiting steady state distribution typically differs with the specification of the utility function. The role of the utility function is also seen in conditions for long run growth and avoidance of extinction. This role is absent in deterministic models.

It is possible to examine how optimal paths and the limiting distribution are affected by changes in the riskiness of productive assets, risk aversion, or the willingness to substitute consumption across time. Unfortunately, not much general analytical characterization is available outside a few examples in the log-linear family. These examples nonetheless serve to illustrate how the qualitative nature of comparative dynamics can depend on the parameters of the utility function.

Third, key qualitative features of optimal policies such as continuity and monotonicity are not significantly altered by the presence of uncertainty in the production technology.

Fourth, extending results on the existence and global stability of an optimal steady state to the stochastic model requires verifying that the transition law for the optimal process satisfies certain conditions, which have been discussed in previous sections. This has necessitated strong technical assumptions that have no counterpart in the deterministic literature. In the multisector case, this difficulty has been more pronounced and the conditions for global stability of a stochastic steady state are only specified in terms of the transition law for the optimal process, making it difficult to evaluate their economic implications.

Fifth, in non-classical one-sector models that generate multiple invariant distributions that act as local attractors, it has been shown that if the volatility of technological disturbances is increased sufficiently, one can establish global convergence of optimal processes to a unique stochastic steady state. Loosely speaking, higher stochasticity in the production technology makes it more likely that realized optimal paths exhibit a high degree of economic fluctuations over time, but it also increases the likelihood that the distribution of optimal capital

stocks converges globally to a unique invariant distribution independent of the initial state. In other words, greater production uncertainty may be associated with higher economic volatility and at the same time, may ensure long run "convergence" in probability distribution of economies that differ in their initial states. This is a fundamental insight into the process of growth and fluctuations in an economy.

Finally, the stochastic growth literature has followed the deterministic literature very closely in establishing a set of turnpike results that show how optimal paths approach each other almost surely in the long run.

As for the important theoretical questions that remain unanswered, our survey indicates that a general characterization of the stochastic steady state or invariant distribution, is lacking. Steps toward such a characterization would improve our understanding of the forces that determine long run economic behavior in a convergent stochastic economy. We not only need to understand how complex the limiting distribution can be, but also have some idea of the relationship between the fundamentals of the model and the properties of the limiting distribution. That is, what do technology and preferences imply about the nature of the limiting distribution? Much work remains to be done there.

Other important open questions in the one sector model are: a *complete* characterization of conditions under which optimal paths converge to zero almost surely, to a non-trivial invariant distribution and diverge to infinity almost surely (the existing literature only provides strong sufficient conditions for each of these events); relaxing the conditions for convergence and stability in the non-convex model; and the question of asymptotic convergence in versions of the model with non-monotone optimal investment policy (such as the stock-dependent model). Developing more transparent conditions for convergence and stability in the multisector stochastic model and conditions for sustained long run growth in such models are also problems that remain open to the current generation of growth theorists.

Finally, the methodology of stochastic optimal growth is increasingly applied to other problems of dynamic resource allocation ranging from models of financial markets and macroeconomic fluctuations to the management of natural and environmental assets. These applications often require extensions and modifications to the basic framework in order to suit the stylized facts that characterize these problems. This, in turn, poses new questions for the growth theorist. The development of new applications and extensions of existing ones may well continue to be the most fruitful source of new ideas related to the stochastic growth model.

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