Rose-Anne Dana
Cuong Le Van · Tapan Mitra
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Editors

Handbook on Optimal Growth

1 Discrete Time

Springer

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(Editors)

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With 5 Figures

Springer

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Preface

The problem of efficient or optimal allocation of resources is a fundamental concern of economic analysis. The theory of optimal economic growth can be viewed as an aspect of this central theme, which emphasizes in general the issues arising in the allocation of resources over an infinite time horizon, and in particular the consumption-investment decision process in models in which there is no natural "terminal date". This broad scope of "optimal growth theory" is one which has evolved over time, as economists have discovered new interpretations of its central results, as well as new applications of its basic methods. The purpose of this handbook is to provide surveys of some significant results of the theory of optimal growth, as well as the techniques of dynamic optimization theory on which they are based. Armed with the results and methods of this theory, a researcher should be in an advantageous position to apply these versatile methods of analysis to new issues in the area of dynamic economics, as well as to contribute to the further development of the mathematical techniques of optimization over time. The survey included in this volume all have as a common starting point the seminal contribution of Frank Ramsey (1928). This paper was concerned with the problem of how much a nation should save, which in modern terminology would be called the social planner's problem of optimal savings. The impact of the contribution has, however, been well beyond the scope of this specific problem. It has influenced the modern theories of planning, economic development, international economics, public finance, macroeconomics, monetary theory, economics of finance and natural resource economics. We elaborate a bit on one aspect of optimal growth theory which has ensured the ever-growing importance of Ramsey's paper. As indicated above, the paper was explicitly concerned with an omniscient central planner's optimal savings problem. This issue continued to be the concern of optimal growth theorists in the 1950s and especially in the 1960s when there was really a boom in research in this area. However, difficulties with the planning process were voiced theoretically, and realized in the actual experience of many socialist centrally planned economies. Thus, studying the social planner's problem appeared to lead to less insights about the performance of actual economies than had been previously supposed. One implication of this was that there was a need to carefully re-examine the fundamental theorems of classical welfare economics, which provide conditions for a competitive equilibrium to be an optimal allocation and conversely. Specifically, it was important to identify the circumstances under which such an equivalence theorem failed to be valid. Central to this investigation is the concept of an equilibrium over time, based exclusively on the microeconomic foundations of rational decision making agents, and entirely without the assistance of any central planner. This concept, and its applications in various sub-disciplines of economics, has been the object of systematic study over the last thirty years. A basic ingredient in developing this concept is the recognition of the fact that in making current investment decisions, which yield returns in the future, agents are necessarily "forward looking" in the sense that their actions will be influenced by their beliefs regarding these future returns. These investment decisions could be regarding a variety of variables such as physical capital accumulation, the amount of education one acquires through schooling, the rate of extraction of mineral deposits or the development of environmental resources for industrial purposes. The beliefs of agents regarding future returns on their investments can, of course, turn out to be incorrect. But, it is plausible to proceed with the notion that beliefs that are at odds with the actual development of events cannot persist; any collection of agents actions that we wish to call an "equilibrium" must validate (or at least not contradict) the beliefs on which they are based. Thus, one focuses on a notion of equilibrium over time, in which (i) given their beliefs, agents choose optimal actions according to their preferences, subject to the constraints they face; (ii) markets clear at each date; and (iii) the beliefs of the agents turn out to be correct. When this notion of equilibrium is studied in the context of the infinitely lived agent model, where agents are modeled as dynasties, having no natural termination date to their "lifetimes", the techniques that Ramsey used to analyse the social planner's optimal savings problem become directly relevant. In fact, his paper provides precisely the tools to examine the equilibrium dynamics of infinite-horizon economies, as well as the benchmark of the social planners's optimum as an ideal to which such an equilibrium can be compared, whenever the two solutions differ. The starting point of the analysis of optimal growth theory is, of course, the existence of an optimum; that is, the existence of a solution to the social planner's or representative agent's infinite horizon optimization problem.

Dynamic programming with undiscounted return is reviewed in Chapter 1. The topic goes back to Ramsey who felt that from a planner's point of view, discounting future utilities was ethically indefensible. Two versions of the overtaking criterion are considered. Convergence of a class of fesible paths (the good programmes) to the stationary optimal path assumed to be unique and existence of optimal policies are first established. Optimal policies are then described in terms of a Bellman's equation and of Euler's equation.

Chapter 2 is devoted to dynamic programming with discounted return. The two main methods of analysis of solutions are recalled under convex and non convex hypotheses. The first uses Bellman's equation: the value function is shown to verify a Bellman's equation and the optimal path from any initial condition to be a dynamical system, generated by the optimal policy associated with Bellman's equation. The second is based on Euler's equation. Numerous examples are given.

Chapter 3 provides an exposition of duality theory in infinite horizon dynamic optimization models. The basic results on price characterization of optimality, when future utilities are discounted, are discussed in the framework of a general reduced-form intertemporal allocation model. The theory is applied (i) to the existence of a stationary optimal stock, (ii) to a study of optimal behavior in a model in which utility is derived from consumption alone, and (iii) to analyze a rule, proposed by Weitzman, which relates the net national product of an economy to its dynamic social welfare. The role of the transversality condition in these characterization results is investigated, and the possibility of replacing it by a condition, verifiable in finite time, is discussed.

The rationazibility literature is surveyed in Chapter 4. Boldrin and Montrucchio (1986) took a significant step in showing that any twice continuously differentiable function could be obtained as policy function of an appropriate dynamic optimization model. Since policy functions are known to be continuous, the question arose whether their result could be extended to the class of all continuous functions. This is known as " the rationazibility problem". While Neuman (1988) demonstrated that this was not possible in general, Mitra and Sorger (1998) showed that Boldrin and Montrucchio's result could be extended to the class of Lipschitz continuous functions.

The concept of a stationary optimal stock is central to optimal growth theory. Chapter 5 surveys the main results on existence and uniqueness of stationary optimal stocks. The existence issue is analyzed via the concept of a discounted golden-rule stock, and following the primal approach used in Khan and Mitra (1986). Two quite distinct approaches are illustrated in the discussion of uniqueness. The first uses the methods of duality theory and emphasizes the role of non-joint production and normality in consumption behavior. The second uses the method of differential topology and establishes a link between uniqueness in the discounted case, and the (known) uniqueness when future utilities are not discounted.

Optimal growth models have not only been used to study capital accumulation and long-run growth but also to demonstrate that cyclical or chaotic equilibria can emerge even in the absence of any market imperfections. Chapter 6 surveys that part of this literature which deals with the standard neoclassical optimal growth model with a single state variable. It starts by emphasizing the different roles played by submodularity and concavity of the reduced form utility function. The former provides an incentive for the decision maker to switch between low and high capital stocks whereas the latter creates a desire for consumption smoothing. Which of the two incentives dominates is mostly determined by the time-preference of the decision maker. The chapter therefore discusses in detail how the existence and structure of optimal cycles depends on the discount factor. The chapter then identifies three sources of optimal chaos: submodularity, strong time-preference, and some form of inertia. Inertia, which is necessary to prevent the model from generating periodic cycles, can either be created by limited production possibilities at small capital stocks or by partial (rather than full) depreciation of capital. It is finally shown that strong time-preference facilitates but it is not strictly necessary for the existence of optimal chaos.

Optimal growth models with non-concave production functions are considered in Chapter 7. A one sector closed economy is studied as a benchmark. The production function displays increasing returns for small outputs and decreasing returns for large outputs. Existence and properties of optimal paths are discussed both in the no discounting and in the discounting cases. Optimal paths are compared with those of the Ramsey model. Extensions are given to multi-sectors models and to open economies.

Chapter 8 uses isotone recursive methods, first introduced in operations research by Veinott and Topkis, to analyze economies with homogeneous agents. These methods have provided a unified catalog of results on existence, characterization and computation of Markov Equilibrium Decision Processes (MEDPs) in infinite horizon economies where the second welfare theorem fails. Examples include models with production nonconvexities, taxes, valued fiat money, monopolistic competition, behavioral heterogeneity and incomplete markets. These methods emphasizes the role of order and provide monotone comparison theorems on the space of economies and foundations for a theory of numerical solutions for MEDPs and stationary Markov equilibrium.

Chapter 9 is devoted to recursive utilities. Given a preference order on the space of sequences of intertemporal consumptions, a utility representation of the preference is a recursive utility function if it depends only on present consumptions and future utility. Such utilities give rise to an aggregator. Conversely, given an aggregator which satisfies appropriated properties, one can construct a recursive utility function. Existence of optimal paths, of equilibria in infinite horizon with recursive preferences, dynamics of optimal paths are reviewed. Optimal paths are characterized in the one-sector model and a turnpike theorem is derived.

Chapter 10 studies indeterminacy of equilibrium in discrete-time models of economic growth. The main concern of the chapter is to explore the conditions under which representative-agent models of capital accumulation with market distortions may hold infinite number of converging trajectories around the steady state equilibrium. Conditions for the occurrence of indeterminacy are examined in one-sector and two-sector models of exogenous growth and in the prototype model of endogenous growth. While the discussion mainly focuses on local indeterminacy conditions, global indeterminacy is also considered.

Chapter 11 provides an overview of key results in the theory of discounted stochastic optimal growth in discrete time. It begins with an analysis of the classical stochastic growth model of Brock and Mirman (1972) for a one-sector economy with convex technology and utility of consumption. The theory is then extended to problems with irreversible investment, increasing returns, non-convex technologies and more general utilities. Equivalence between optimal and competitive outcomes is established by using the competitive price characterization of optimal policies.

Chapter 12 deals with deterministic and stochastic versions of the Von-Neumann-Gale model. Von-Neumann's original concern was to determine a balanced path growing at a maximal rate for a linear and stationary deterministic technology and prices supporting that path. Such a pair was called a Von-Neumann's equilibrium. The chapter first provides non linear and non stationary deterministic generalizations of the model, a topic initiated by Gale. Stochastic generalizations of a von Neumann equilibrium and of efficient paths are then considered. Existence, uniqueness and turnpike results are proven. Lastly the chapter outlines new applications of the von Neumann-Gale model to finance related to asset pricing and hedging in securities markets.

Chapter 13 surveys one interpretation of Ramsey's multi-agent model, solves for the steady state distribution and examines the models' dynamics within well-specified theories of intertemporal equilibrium. The resulting analysis shows that there are fundamental differences between the dynamics of the representative agent model and one with heterogenous households. Not only do the long-run distribution of income and wealth differ from the representative agent outcome, but so does the dynamics. Indeed, the convergence of the economy to the long-run steady state from arbitrary initial conditions characteristic of Ramsey's optimal accumulation – representative agent equilibrium model only holds for **some** specifications of preferences and technology in the multi-agent setup. Complicated dynamics at the aggregate level can arise even with very unequal income and wealth distributions evolving over time.

Chapter 14 provides an extensive account of research in economics based on dynamic games. It is organized along methodological as well as field-related criteria. For methodologically defined frameworks, a somewhat detailed coverage of the general set-ups and their main results is provided, including for the open-loop concept, the linear-quadratic model, myopic equilibrium, general existence theory, games of perfect information, and stochastic games with a continuum of players. As to area-by-area coverage, it is mostly in the form of an overview, and includes applications in capital theory/resource economics, industrial organization, and experimental economics.

January 2006

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1. Optimal Growth Without Discounting

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1.1 Introduction

As it is well known, the standard approach to infinite time horizon optimal growth problem is to discount future consumptions utilities by some factor and to maximize the resulting infinite series. Another approach pionnered by Ramsey [12] and reworked by Samuelson and Solow [13], Koopmans [9] and von Weizsäcker [14] uses a discount factor equal to one. The criterion is then sensitive to all increases in consumptions and treats generations equally. As was shown by Koopmans [27] and Diamond [5], no continuous preference ordering can be, at the same time, sensitive and treat generations equally but they do exist partial orderings which satisfy these axioms. The overtaking criterion is such an example. Gale [7] showed that, although it was not possible to compare all programs under that criterion, a partition of programs could be made into good or bad programs and that one could restrict himself to good programs. The concept of "optimal program" was first discussed. Various concepts of optimality were considered by Gale [7] and Brock [2] and "optimal programs" were shown to exist under various sets of assumptions about the technology and preferences. Then, on one hand, in order to relate the "undiscounted" case to the "discounted" case, Dana and Le Van [3],[4] introduced value functions for the overtaking criterion and showed that, under further hypotheses, an optimal program could be described as in the discounted case, by an optimal policy. On the other hand, non stationary versions of the overtaking criterion were used by Mckenzie [10] and more particularly by Michel[11] who characterized optimality by transversality conditions. We emphasize the fact that the literature on the overtaking criterion makes extensive use of price theory and turnpike results.

The chapter is organized as follows: In section 1.2, we set the model and show existence of a stationary optimal program and prices supporting it. In section 1.3, we define good programmes and show that a partition of programs can be made into bad or good programs. Assuming strict concavity of the utility fonction at the stationary optimal program (respectively uniqueness of a stationary optimal program), we then show convergence (respectively average convergence) of good programs to the stationary optimal program. In section 1.4, we reconsider the two concepts of optimality ("optimality" and "weak optimality") introduced by Gale [7] and Brock [2] and show existence of optimal solutions for both concepts. We further characterize optimal solutions in terms of an Euler equation. In section 1.5, we introduce as in the discounted case, a value function and a Bellman's equation and on further assumptions on the technology and the criterion, show that an optimal program is unique and can be described by an optimal policy.

1.2 The Model

We consider an intertemporal economy where the instantaneous utility of the representative consumer depends on k_t, the capital stock on hand at date t and on k_{t+1}, the capital stock for date $t+1$. Given k_t, the set of feasible capital stocks for the next period $t+1$ is $\Gamma(k_t)$. We assume that at any period t, the feasible capital stock on hand belongs to X, a subset of $\mathbb{R}^n_+$. More explicitly, we make the following assumptions:

H1: X is a compact, convex set of $\mathbb{R}^n_+$ with non-empty interior and X contains 0.
H2: Γ is a continuous correspondence from X into X with non-empty convex images. Its graph, graph $\Gamma = \{(x, y) \in X \times X : y \in \Gamma(x)\}$, is convex.
H3: (Free disposal) If $y \in \Gamma(x), x' \geq x$ and $y' \leq y$, then $y' \in \Gamma(x')$.
H4: (Existence of expansible capital stocks) There exist $(x, y) \in$ graph Γ, with $y >> x$, i.e. $y_i > x_i$, for all $i = 1, ..., n$.
H5: The instantaneous utility function F : graph $\Gamma \to \mathbb{R}$ is concave, continuous, increasing in first variable and decreasing in the second variable.

Remark 1.2.1. Assumption **H3** implies that $0 \in \Gamma(0)$. Assumptions **H3** and **H4** imply that the interior of graph Γ, denoted by int (graph Γ) is non-empty.

Definition 1.2.1. *A sequence* $\mathbf{x}$ *is feasible from* $x_0 \in X$ *if* $x_{t+1} \in \Gamma(x_t)$ *for all* $t \geq 0$. *A programme from* x_0 *is a feasible sequence from* x_0. *We denote by* $\Pi(x_0)$ *the set of feasible sequences from* x_0. *The set of programmes is denoted by* Π, *i.e.* $\Pi = \cup_{x \in X} \Pi(x)$.

We next define optimal stationary programmes and prove the existence of an optimal stationary programme and supporting prices.

Definition 1.2.2. *An optimal stationary programme is a solution* $\bar{x}$ *to the problem*

$$\max_{x \in X} \{F(x, x) : (x, x) \in \text{ graph } \Gamma\}$$

Let $\bar{F}$ be the value function of the above problem. It follows from **H4** that

$$\bar{F} = \max\{F(x,y) : (x,y) \in \text{ graph } \Gamma, y \geq x\}$$

Proposition 1.2.1. *Assume* **H1-H5**. *Then:*
(i) there exists an optimal stationary programme $\bar{x}$ *,*
(ii) there exists $p \in \mathbb{R}^n_+$ *such that:*

$$F(x,y) + p \cdot y - p \cdot x \leq F(\bar{x},\bar{x}), \text{ for all } (x,y) \in \text{graph } \Gamma$$

Proof. (i) From **H4** and **H5**, the set $\{(x,y) \in \text{ graph } \Gamma \ \mid y = x\}$ is non-empty and compact. As F is continuous on graph Γ, a maximal pair $(\bar{x},\bar{x})$ exists proving the existence of an optimal stationary programme.
(ii) Let $F_0(x,y) = F(x,y)$ if $(x,y) \in$ graph Γ, and $F_0(x,y) = -\infty$ if $(x,y) \notin$graph Γ. We have

$$\max\{F_0(x,y) : y \geq x\} = \max\{F(x,y) : (x,y) \in \text{ graph } \Gamma, y \geq x\}$$

Assertion (ii) follows from the Kuhn-Tucker condition (see Corollary 7.2.1 in Florenzano, Le Van and Gourdel [6]).

For further use, let δ : graph $\Gamma \to \mathbb{R}$ be defined by

$$\delta(x,y) = F(\bar{x},\bar{x}) - F(x,y) + p \cdot (x - y), \text{ for all } (x,y) \in \text{graph } \Gamma \tag{1.1}$$

From Proposition 1.2.1, $\delta(x,y) \geq 0,$ for all $(x,y) \in$ graph Γ.

1.3 Good Programmes

1.3.1 Good Programmes

The next lemma shows that the utility of a programme is always bounded above.

Lemma 1.3.1. *Assume* **H1-H5**. *For any programme* $\mathbf{x}$*, there exists* $M > 0$ *such that*

$$\sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x},\bar{x})] \leq M, \text{ for all } T$$

Proof. From Proposition 1.2.1,

$$F(x_t, x_{t+1}) - F(\bar{x},\bar{x}) \leq p \cdot x_{t+1} - p \cdot x_t$$

Since X is compact

$$\sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x},\bar{x})] \leq p \cdot x_{t+1} - p.x_0 \leq p \cdot x_{t+1} \leq M$$

proving the desired assertion.

Lemma 1.3.2. *Assume* **H1-H5**. *For any programme* $\mathbf{x}$, $\sum_{t=0}^{+\infty} \delta(x_t, x_{t+1})$ *exists in* $\mathbb{R} \cup \{+\infty\}$.

Proof. By Proposition 1.2.1, $\delta(x_t, x_{t+1}) \geq 0$. The assertion follows.

For further use, we have

$$\sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] = -\sum_{t=0}^{T} \delta(x_t, x_{t+1}) + p \cdot (x_0 - x_{T+1}) \tag{1.2}$$

Lemma 1.3.3. *Assume* **H1-H5**. *For any programme* $\mathbf{x}$, *then either*
(i) $\liminf_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] > -\infty$ *or*
(ii) $\lim_{T \to +\infty} \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] = -\infty$.

Proof. From Lemma 1.3.2, $\sum_{t=0}^{+\infty} \delta(x_t, x_{t+1})$ exists in $\mathbb{R} \cup \{+\infty\}$. Hence

– If $\sum_{t=0}^{+\infty} \delta(x_t, x_{t+1}) < \infty$, then, from (1.2) and since X is compact,

$$\sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] \geq [-\sum_{t=0}^{\infty} \delta(x_t, x_{t+1}) - \sup_{x \in X} p \cdot x] = A,$$

for all T, hence assertion (i).

– If $\sum_{t=0}^{+\infty} \delta(x_t, x_{t+1})$ is infinite

$$\limsup_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] \leq -\sum_{t=0}^{+\infty} \delta(x_t, x_{t+1}) + p \cdot x_0 = -\infty,$$

hence assertion (ii).

From lemma 1.3.3, we may now introduce the definition of a good programme.

Definition 1.3.1. *Assume* **H1-H5**. *A programme* $\mathbf{x}$ *is* ***good*** *if it fulfills one of the following equivalent condition:*

1) $\sum_{t=0}^{+\infty} \delta(x_t, x_{t+1}) < \infty$,

2) There exists $A \in \mathbb{R}$ *such that* $\sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] \geq A$, *for all* T.

3) $\liminf_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] > -\infty$

4) For any programme $\mathbf{x}'$, $\liminf_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(x'_t, x'_{t+1}] > -\infty$

Proof. Let us show that the four assertions are equivalent.
1) is equivalent to 2) follows from lemma 1.3.3 and (1.2).
2) is equivalent to 3) follows from the fact that $\liminf_T u_T$ is the smallest cluster point of the sequence (u_T).
2) implies 4): Assume 2). Let $\mathbf{x}$ be **good** and let $\mathbf{x}'$ be any programme. From 2) and lemma 5, we have that for all T,

$$\sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(x'_t, x'_{t+1}] = \sum_{t=0}^{T} [F(x_t, x_{t+1}) - \bar{F}] - \sum_{t=0}^{T} [F(x'_t, x'_{t+1}) - \bar{F}] \geq B,$$

with $B = A - M$, hence assertion 4.
4) implies 3) by taking for $\mathbf{x}'$, the optimal stationary sequence $\bar{\mathbf{x}}$.

1.3.2 Convergence Properties of Good Programmes

We next assume:

H6: The stationary point $(\bar{x}, \bar{x})$ is unique.

The following stronger assumption insures the uniqueness of the stationary point:

H6G: The utility function F is strictly concave at $(\bar{x}, \bar{x})$, i.e. for every $(x, y) \neq (\bar{x}, \bar{x})$ and $\lambda \in]0, 1[$,

$$F(\lambda x + (1 - \lambda)\bar{x}, \lambda y + (1 - \lambda)\bar{x}) > \lambda F(x, y) + (1 - \lambda)F(\bar{x}, \bar{x})$$

Lemma 1.3.4. *Assume* **H1**-**H2**. *Then* **H6G** *implies* **H6**

Proof. Assume **H6G** and that there exists another stationary point $\bar{x'}$. Let $\lambda \in]0, 1[$. Since graph Γ is convex, $(\lambda\bar{x} + (1 - \lambda)\bar{x'}, \lambda\bar{x} + (1 - \lambda)\bar{x'}) \in$graph Γ. Since F is strictly concave at $(\bar{x}, \bar{x})$, we have:

$$F(\lambda\bar{x} + (1 - \lambda)\bar{x'}, \lambda\bar{x} + (1 - \lambda)\bar{x'}) > \lambda F(\bar{x}, \bar{x}) + (1 - \lambda)F(\bar{x'}, \bar{x'}) = \bar{F}$$

hence a contradiction.

We know prove a turnpike or average turnpike property for good programmes.

Proposition 1.3.1.
1) Assume **H1- H6G**. *If* $\mathbf{x}$ *is good, then* $x_n \to \bar{x}$.
2) Assume **H1- H6**. *If* $\mathbf{x}$ *is good, then* $\bar{x}_n = \frac{1}{n}\sum_{t=0}^{n} x_t \to \bar{x}$.

Proof. To prove the first assertion, if $\mathbf{x}$ is good, then from definition 1.3.1, assertion 1, $\sum_{t=0}^{+\infty} \delta(x_t, x_{t+1}) < \infty$, hence $\delta(x_t, x_{t+1}) \to 0$ when $t \to +\infty$. Let ϕ :graph $\Gamma \to \mathbb{R}$ be defined by:

$$\phi(x,y) = F(x,y) - p\cdot x + p\cdot y = -\delta(x,y) + F(\bar{x},\bar{x}) \tag{1.3}$$

Since $\delta(x,y) \geq 0,$ for all $(x,y) \in$ graph Γ,

$$\phi(x,y) \leq F(\bar{x},\bar{x}) = \phi(\bar{x},\bar{x}), \text{ for all } (x,y) \in \text{graph } \Gamma$$

Since ϕ is strictly concave at $(\bar{x},\bar{x})$, $(\bar{x},\bar{x})$ is a unique maximizer of ϕ. Since $\delta(x_t,x_{t+1}) \to 0$, $\phi(x_t,x_{t+1}) \to F(\bar{x},\bar{x}) = \phi(\bar{x},\bar{x})$. Let (x,x') be a cluster point of the sequence $\{(x_t,x_{t+1})\}$. Since ϕ is continuous, $\phi(x,x') = \phi(\bar{x},\bar{x})$. The maximizer being unique, $(x,x') = (\bar{x},\bar{x})$. The sequence $\{(x_t,x_{t+1})\}$ having $(\bar{x},\bar{x})$ as a unique cluster point, it converges to $(\bar{x},\bar{x})$. Equivalently, $x_t \to \bar{x}$ which proves the first assertion.
To prove the second assertion, let us first remark that, as graph Γ is convex, $(\bar{x}_T, \bar{x}_{T+1}) \in$ graph Γ, for all T. As F is concave and $\mathbf{x}$ is good, it follows from definition 1.3.1 (ii) that, for some G, we have, for all T,

$$F(\bar{x}_T,\bar{x}_{T+1}) - F(\bar{x},\bar{x}) \geq \frac{1}{T}\sum_{t=0}^{T}[F(x_t,x_{t+1}) - F(\overline{x},\overline{x})] \geq \frac{G}{T}$$

Let (x,x') be a cluster point of the sequence $(\bar{x}_T,\bar{x}_{T+1})$. Then $(x,x') \in$graph Γ and $F(x,x') - F(\bar{x},\bar{x}) \geq 0$, hence $F(x,x') = F(\bar{x},\bar{x})$. The function F having a unique maximizer on graph Γ, $(x,x') = (\bar{x},\bar{x})$ and the sequence $(\bar{x}_T,\bar{x}_{T+1})$ converges to $(\bar{x},\bar{x})$ proving the second assertion.

Corollary 1.3.1. *Assume* **H1- H6** *and let* $\mathbf{x}$ *and* $\mathbf{x}'$ *be two good programmes. Then*

$$\liminf_T (px_T - px_T^{'}) \leq 0$$

Proof. From Cesaro Lemma, for a sequence $\{a_t\}$ such that $\lim_T \frac{1}{T}\sum_{t=0}^{T} a_t$ exists, we have:

$$\liminf_t a_t \leq \lim_T \frac{1}{T}\sum_{t=0}^{T} a_t \leq \limsup_t a_t$$

From proposition 10, $\frac{1}{T}\sum_{t=0}^{T} x_t \to \bar{x}$ and $\frac{1}{T}\sum_{t=0}^{T} x'_t \to \bar{x}$, hence

$$\liminf_t (p \cdot x_t - p \cdot x_t^{'}) \leq \lim_T \frac{1}{T} p \cdot \sum_0^T (x_t - x_t^{'}) = 0$$

proving the desire result.

Corollary 1.3.2. *Assume* **H1- H6G***, then* $\lim_{T \to +\infty} \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})]$ *exists in* $\mathbb{R} \cup \{-\infty\}$.

A programme $\mathbf{x} \in \Pi$ *is good iff* $\lim_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})]$ *exists in* $\mathbb{R}$.

A programme $\mathbf{x} \in \Pi$ *is not good iff* $\lim_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] = -\infty$.

Proof. From (1.2), definition 1.3.1 (i) and proposition 1.3.1, $\mathbf{x} \in \Pi$ is *good* iff

$$\sum_{t=0}^{\infty} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] = -\sum_{t=0}^{\infty} \delta(x_t, x_{t+1}) + \bar{p} \cdot x_0 - \bar{p} \cdot \bar{x}$$

When $\mathbf{x}$ is not good, from definition 1.3.1 and lemma 1.3.3,

$$\sum_{t=0}^{+\infty} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] = -\infty.$$

1.3.3 Existence of Good Programmes from x_0

We now give sufficient conditions for non emptiness of the set of good programmes from x_0.

Proposition 1.3.2. *Assume* **H1-H5** *and that there exists a neighborhood* $\mathcal{U}$ *of* $(\bar{x}, \bar{x})$ *in graph* Γ*, and* $\sigma > 0$ *such that*

$$|F(x,y) - F(\bar{x}, \bar{x})| \leq \sigma[\|x - \bar{x}\| + \|y - \bar{x}\|].$$

Then $\Pi_G(x_0)$ *is non-empty for any* x_0 *such that* $\Gamma(x_0)$ *contains a strictly positive* y_0 *(i.e., all the components in* $\mathbb{R}^n$ *of* y_0 *are strictly positive.)*

Proof. By **H4**, there exists (x_p, y_p) in graph Γ such that $x_p << y_p$. Since $0 \in \Gamma(0)$, we have $\lambda y_p \in \Gamma(\lambda x_p)$, for $\lambda \in]0,1[$. If λ is sufficiently small, we have $\lambda y_p << y_0$ and $\lambda y_p >> \lambda x_p$. So, we can assume $y_p << y_0$ and $y_p >> x_p$.
Let $\lambda \in]0,1[$. For any integer n, we remark that

$$(1 - \lambda^{n+1})\bar{x} + \lambda^{n+1} x_p \leq (1 - \lambda^n)\bar{x} + \lambda^n y_p \text{ iff } (1 - \lambda)\bar{x} + \lambda x_p \leq y_p.$$

Choose λ sufficiently close to 1 so that $(1 - \lambda)\bar{x} + \lambda x_p \leq y_p << y_0$.

Define, for $t \geq 1$, $x_t = (1-\lambda^t)\bar{x} + \lambda^t x_p$. By the choice of λ and by **H3**, we have $x_1 \in \Gamma(x_0)$. For $t \geq 1$,

$$x_{t+1} \leq (1-\lambda^t)\bar{x} + \lambda^t y_p \in \Gamma((1-\lambda^t)\bar{x} + \lambda^t x_p) = \Gamma(x_t).$$

By **H3**, $x_{t+1} \in \Gamma(x_t)$.
Since $x_t \to \bar{x}$, there exists T_0 such that for any $t \geq T_0$, $(x_t, x_{t+1}) \in \mathcal{U}$ and hence

$$|F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})| \leq 2\sigma\lambda^t \|\bar{x} - x_p\|.$$

Thus,

$$\sum_{t=T_0}^{+\infty} |F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})| \leq 2\sigma \frac{\lambda^{T_0}}{1-\lambda} \|\bar{x} - x_p\|,$$

and $\mathbf{x}$ is good from x_0.

Let $\Pi_G(x_0)$ denote the set of good programmes starting from $x_0 \in X$.

1.4 Optimal and Weakly Optimal Programmes

1.4.1 Definition and First Properties

We will now define an optimal programme for optimal growth models without discounting. The first definition we next give is due to Gale [7], the second to Brock [2].

1.4.2 Definition and Characterisation

Definition 1.4.1. *Assume* **H1-H5**. *A programme* $\mathbf{x}^* \in \Pi(x_0)$ *is optimal if, for any programme* $\mathbf{x} \in \Pi(x_0)$, *we have*

$$\limsup_{T \to +\infty} \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)] \leq 0.$$

A programme $\mathbf{x}^* \in \Pi(x_0)$ *is weakly optimal if, for any programme* $\mathbf{x} \in \Pi(x_0)$, *we have*

$$\liminf_{T} \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)] \leq 0$$

Clearly an optimal programme is weakly optimal.

Proposition 1.4.1. *Assume that* $\Pi_G(x_0) \neq \emptyset$. *Then any weakly optimal programme is good. Hence any optimal programme is good.*

Proof. Let $\mathbf{x}^* \in \Pi(x_0)$ be weakly optimal. Since X is compact, we have, for some $m > 0$,

$$\sum_{t=0}^{T}[F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)] =$$
$$-\sum_{t=0}^{T}\delta(x_t, x_{t+1}) + \sum_{t=0}^{T}\delta(x_t^*, x_{t+1}^*) - p\cdot x_{t+1} + p\cdot x_{t+1}^*$$
$$\geq -\sum_{t=0}^{T}\delta(x_t, x_{t+1}) + \sum_{t=0}^{T}\delta(x_t^*, x_{t+1}^*) - m.$$

Since $\Pi_G(x_0) \neq \emptyset$, let $\mathbf{x} \in \Pi(x_0)$ be good. We have

$$\sum_{t=0}^{\infty}\delta(x_t^*, x_{t+1}^*) \leq \sum_{t=0}^{\infty}\delta(x_t, x_{t+1}) + \liminf_{T\to+\infty}\sum_{t=0}^{T}[F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)] + m$$
$$\leq \sum_{t=0}^{\infty}\delta(x_t, x_{t+1}) + m < \infty$$

From definition 1.3.1, assertion (i), $\mathbf{x}^*$ is good.

1.4.3 Existence of Optimal Programmes

Let $\gamma : \Pi \to [-\infty, 0]$ be defined by

$$\gamma(\mathbf{x}) = -\sum_{t=0}^{\infty}\delta(x_t, x_{t+1}) \tag{1.4}$$

Proposition 1.4.2. *Assume* **H1-H5**. *Then γ is concave, upper semi continuous in the product topology. If $\Pi_G(x_0) \neq \emptyset$, then γ has a maximizer on $\Pi(x_0)$ which is a good programme. If $\Pi_G(x_0) = \emptyset$, then $\gamma(\mathbf{x}) = -\infty$, for all $\mathbf{x} \in \Pi(x_0)$.*

Proof. Let $\gamma_T(\mathbf{x}) = -\sum_{t=0}^{T}\delta(x_t, x_{t+1})$. For every T, γ_T is concave and continuous in the product topology and $\gamma_T \geq \gamma_{T+1}$. Hence γ is the decreasing limit of continuous concave functions, it is therefore concave, upper semi-continuous. If $\Pi_G(x_0) = \emptyset$, then $\gamma(\mathbf{x}) = -\infty$, for all $\mathbf{x} \in \Pi(x_0)$. If $\Pi_G(x_0) \neq \emptyset$, then γ has a maximizer $\mathbf{x}^*$ on $\Pi(x_0)$ which is compact, non-empty in the product topology. Since $\gamma(\mathbf{x}^*) \neq \infty$, any maximizer is good.

We recall that under **H1-H6G**, from corollary 1.3.2,

$$\Phi(\mathbf{x}) = \lim_{T\to+\infty}\sum_{t=0}^{T}[F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})] \tag{1.5}$$

is well defined ($\Phi(\mathbf{x}) = -\infty$ if $\mathbf{x} \notin \Pi_G(x_0)$) and

$$\Phi(\mathbf{x}) = \gamma(\mathbf{x}) + \bar{p}\cdot\bar{x} - \bar{p}\cdot x_0,\ \mathbf{x} \in \Pi_G(x_0) \tag{1.6}$$

Theorem 1.4.1. *Assume* **H1-H6G** *and that $\Pi_G(x_0) \neq \emptyset$. A programme $\mathbf{x}^* \in \Pi(x_0)$ is optimal iff it is a maximizer of γ or of Φ.*

Proof. Since $\Pi_G(x_0) \neq \emptyset$, from Proposition 1.4.2, γ (hence Φ) has a maximizer $\mathbf{x}^*$ which is good. Let us show that it is optimal. Let $\mathbf{x} \in \Pi_G(x_0)$. We have

$$\lim_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)] = \Phi(\mathbf{x}) - \Phi(\mathbf{x}^*) \leq 0$$

since $\mathbf{x}^*$ is a maximizer of Φ. If $\mathbf{x} \notin \Pi_G(x_0)$, then

$$\lim_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(x'_t, x'_{t+1})] = -\infty$$

Hence $\mathbf{x}^*$ is optimal.
Conversely if $\mathbf{x}^*$ is optimal, then from Proposition1.4.1, it is good. If $\mathbf{x} \in \Pi_G(x_0)$. We have

$$\lim_T \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)] = \Phi(\mathbf{x}) - \Phi(\mathbf{x}^*) \leq 0$$

since $\mathbf{x}^*$ is optimal. Hence $\Phi(\mathbf{x}^*) \geq \Phi(\mathbf{x})$. If $\mathbf{x} \notin \Pi_G(x_0)$, then $\Phi(\mathbf{x}) = -\infty$. Hence $\mathbf{x}^*$ is a maximizer of Φ.

Under weaker assumptions, we obtain a sufficient condition for existence of a weakly maximal programme.

Theorem 1.4.2. *Assume* **H1-H6** *and $\Pi_G(x_0) \neq \emptyset$. If a programme $\mathbf{x}^* \in \Pi(x_0)$ is a maximizer of γ, then it is weakly optimal.*

Proof. Let $\mathbf{x}^*$ be a maximizer of γ. Since $\Pi_G(x_0) \neq \emptyset$, from proposition 1.4.1, $\mathbf{x}^*$ is good. Let $\mathbf{x} \in \Pi_G(x_0)$. From corollary 1.3.1, we have

$$\liminf \sum_{t=0}^{T} [F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)]$$
$$\leq \sum_{t=0}^{\infty} (\delta(x_t^*, x_{t+1}^*) - \delta(x_t, x_{t+1})) + \liminf \left(p \cdot x_{t+1}^* - p \cdot x_{t+1}\right)$$
$$\leq \gamma(\mathbf{x}) - \gamma(\mathbf{x}^*) \leq 0.$$

Hence $\mathbf{x}^*$ is weakly optimal.

Corollary 1.4.1. *Assume* **H1-H6**, *then $\bar{\mathbf{x}} = (\bar{x}, \bar{x}, ..., \bar{x}, ...)$ is a weakly maximal programme from $\bar{x}$. Assume further* **H6G**, *then $\bar{\mathbf{x}}$ is an optimal programme from $\bar{x}$.*

Proof. We have $\gamma(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Pi(\bar{x})$ and $\gamma(\bar{\mathbf{x}}) = 0$. The assertion follows from Theorems 1.4.1 and 1.4.2.

1.4.4 Euler Equation

We now assume that F is C^1 in the interior of graph Γ.

Proposition 1.4.3. *Assume* **H1-H6** *and that F is C^1 in the interior of graph Γ and $\Pi_G(x_0) \neq \emptyset$. Then*

1. *If $\mathbf{x}^*$ maximises γ in $\Pi(x_0)$ and $\forall t$, $(x_t^*, x_{t+1}^*) \in$ int(graph Γ), then*

$$F_2(x_t^*, x_{t+1}^*) + F_1(x_{t+1}^*, x_{t+2}^*) = 0, \quad \textit{for all } t \tag{1.7}$$

2. *Conversely, let us assume that any maximizer of Φ on graph Γ is in int(graph Γ). Then if $\mathbf{x}^* \in \Pi_G(x_0)$ fulfills (1.7), then $\mathbf{x}^*$ maximises γ.*

Proof. Let $\mathbf{x}^*$ maximises γ. Assume that, for all t, $(x_t^*, x_{t+1}^*) \in \text{int(graph } \Gamma)$. We then have:

$$\delta_2(x_t^*, x_{t+1}^*) + \delta_1(x_{t+1}^*, x_{t+2}^*) = 0, \quad \text{for all } t$$

As

$$\delta_2(x_t^*, x_{t+1}^*) = -F_2(x_t^*, x_{t+1}^*) - p$$

and

$$\delta_1(x_{t+1}^*, x_{t+2}^*) = -F_1(x_{t+1}^*, x_{t+2}^*) + p,$$

we obtain (1.7).

Conversely, let $\mathbf{x}^*$ be good and satisfy (1.7). Then $\delta(x_t^*, x_{t+1}^*) \to 0$, hence $\Phi(x_t^*, x_{t+1}^*) \to F(\bar{x}, \bar{x}) = \Phi(\bar{x}, \bar{x})$. Let $(x, x^{'})$ be a cluster point of the sequence (x_t^*, x_{t+1}^*). We then have $\Phi(x, x^{'}) = \Phi(\bar{x}, \bar{x})$, hence $(x, x^{'})$ is a maximizer of Φ. Since $(x, x^{'}) \in \text{int(graph } \Gamma)$ by assumption, we have $\Phi_1(x, x^{'}) = \Phi_2(x, x^{'}) = 0$, hence

$$F_1(x, x^{'}) = p \quad \text{and} \quad F_2(x, x^{'}) = -p \tag{1.8}$$

Let $\mathbf{x} \in \Pi_G(x_0)$ and let $\Delta_T = \sum_{t=0}^{T} -[\delta(x_t^*, x_{t+1}^*) - \delta(x_t, x_{t+1})]$. As δ is concave and $\mathbf{x}^*$ fulfills (1.7), we have

$$\begin{aligned}\Delta_T \quad &\geq \sum_{t=0}^{T} [F_1(x_t^*, x_{t+1}^*)(x_t^* - x_t) + F_2(x_t^*, x_{t+1}^*)(x_{t+1}^* - x_{t+1})] \\ &+ \sum_{t=0}^{T} p.(x_{t+1}^* - x_{t+1}) - \sum_{t=0}^{T} p.(x_t^* - x_t) \\ &= F_2(x_T^*, x_{T+1}^*)(x_{T+1}^* - x_{T+1})] + p.(x_{T+1}^* - x_{T+1})\end{aligned}$$

Passing to the limit on the left hand side, we obtain that for any cluster point $(x, x^{'})$ of the sequence (x_t^*, x_{t+1}^*) and any cluster point y of the sequence (x_t),

$$\gamma(\mathbf{x}^*) - \gamma(\mathbf{x}) \geq (F_2(x, x^{'}) + p) \cdot (x^{'} - y) = 0$$

since $(x, x^{'})$ fulfills (1.8). If $\mathbf{x} \notin \Pi_G(x_0)$, then $\gamma(\mathbf{x}^*) - \gamma(\mathbf{x}) = \infty$. Hence $\mathbf{x}^*$ is a maximizer of γ on $\Pi(x_0)$.

Corollary 1.4.2. *Assume* **H1- H6G** *and that F is C^1 in the interior of graph Γ and $\Pi_G(x_0) \neq \emptyset$. Let $\mathbf{x}^*$ be a good programme from x_0 such that $(x_t^*, x_{t+1}^*) \in int(graph\, \Gamma)$ for all t. Then $\mathbf{x}^*$ is optimal iff it fulfills (1.7).*

1.5 Bellman's Equation, Optimal Policy

1.5.1 Bellman's Equation

Let

$$V(x) = \sup_{\mathbf{x} \in \Pi(x)} \gamma(\mathbf{x})$$

Proposition 1.5.1. *Assume* **H1-H6**. *Then*

1. *$V(\bar{x}) = 0$ and $V(x) = -\infty$ iff $\Pi_G(x) = \emptyset$. V is concave, nondecreasing, negative and upper semi-continuous and satisfies Bellman's equation:*

$$V(x) = \sup_{y \in \Gamma(x)} \{-\delta(x, y) + V(y)\} \tag{1.9}$$

2. *If $\mathbf{x} \in \Pi(x)$ fulfills $V(x) = \gamma(\mathbf{x})$, then $\mathbf{x}$ is weakly maximal. Moreover if $\mathbf{x} \in \Pi_G(x)$, then $lim_{t\to\infty} V(x_t) = 0$. Assume furthermore* **H6G**. *Then $\mathbf{x}$ is optimal iff $V(x) = \gamma(\mathbf{x})$.*
3. *V is the greatest negative solution to (1.9). It is the unique solution to (1.9) such that either $V(x) = -\infty$ or $lim_{t\to\infty} V(x_t) = 0$ for every $\mathbf{x} \in \Pi_G(x)$*

Proof. It follows from Proposition 1.4.2 that

$$V(x) > -\infty \text{ iff } \Pi_G(x) \neq \emptyset$$

The upper semicontinuity of V follows from Berge's theorem [1], page 122 and the fact that the correspondence $x \to \Pi(x)$ has a closed graph and is compact valued and is therefore upper semicontinuous.

If $V(x) > -\infty$, then from Proposition 1.4.2, γ has a maximizer. From Theorem 1.4.2, any maximizer $\mathbf{x}^*$ of γ is weakly optimal. By the usual argument

$$V(x) = \max_{y \in D(x)} \{-\delta(x, y) + V(y)\}$$

If $V(x) = -\infty$, then $V(y) = -\infty$ for every $y \in D(x)$ (if not, there would exists a good programme from x). Hence the Bellman's equation holds also in that case.

Let us prove that $\lim_{t\to\infty} V(x_t) = 0$ for every $\mathbf{x} \in \Pi_G(x)$. Indeed

$$\sum_{t=T}^{\infty} -\delta(x_t, x_{t+1}) \leq V(x_T) \leq 0$$

Since $\mathbf{x} \in \Pi_G(x)$, $\lim_T \sum_{t=T}^{\infty} -\delta(x_t, x_{t+1}) = 0$, hence the sequence $\{V(x_T)\}$ has a limit and $\lim_{T\to\infty} V(x_T) = 0$ proving the desired result.

Let W be another negative solution to (1.9). Let ε be given. There exists $x_1, x_2, \ldots, x_T$ such that $x_1 \in \Gamma(x)$, $x_t \in \Gamma(x_{t-1})$ for $t = 1, \ldots, T$ and

$$-\delta(x, x_1) + W(x_1) > W(x) - \varepsilon$$

$$-\delta(x_1, x_2) + W(x_2) > W(x_1) - \frac{\varepsilon}{2}$$

$$-\delta(x_{T-1}, x_T) + W(x_T) > W(x_{T-1}) - \frac{\varepsilon}{2^{T-1}}$$

Summing up the preceeding inequalities, we obtain with $x_0 = x$, since W is negative

$$\begin{aligned}\sum_{0}^{T-1} -\delta(x_{t-1}, x_t) &\geq \sum_{0}^{T-1} -\delta(x_{t-1}, x_t) + W(x_T) \\ &> W(x) - \varepsilon(1 + \tfrac{1}{2} + \ldots + \tfrac{1}{2^{T-1}})\end{aligned}$$

Thus $V(x) > W(x) - \frac{\varepsilon}{2}$ and therefore $V(x) \geq W(x)$. Hence, if $V(x) = -\infty$, then $W(x) = -\infty$.

Assume now that $V(x) > -\infty$ and let $\mathbf{x}^*$ be such that $V(x) = \gamma(\mathbf{x}^*)$. Then $\mathbf{x}^* \in \Pi_G(x)$. Since W is a solution to (1.9),

$$W(x) \geq \sum_{0}^{T} -\delta(x^*_{t-1}, x^*_t) + W(x^*_{T+1})$$

Since $\lim_{T\to\infty} W(x^*_{T+1}) = 0$, we obtain that $W(x) \geq V(x)$, hence $W(x) = V(x)$.

Remark 1.5.1. Assume **H1-H6G** and let

$$\hat{V}(x) = \sup_{\mathbf{x}\in\Pi(x)} \sum_{t=0}^{\infty} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})]$$

Then $\hat{V}$ satisfies the following Bellman's equation:

$$\hat{V}(x) = \sup_{y\in\Gamma(x)} \{F(x, y) - F(\bar{x}, \bar{x}) + \hat{V}(y)\} \tag{1.10}$$

and $\mathbf{x}$ is optimal iff $\hat{V}(x) = \sum_{t=0}^{\infty} [F(x_t, x_{t+1}) - F(\bar{x}, \bar{x})]$. One easily verifies that

$$\hat{V}(x) = V(x) + p.x - p.\bar{x} \tag{1.11}$$

1.5.2 Optimal Policy

Let us now introduce some more hypotheses.

H7: For $x \geq 0,\ \Gamma(x)$ is strictly convex in $\mathbb{R}^n_+$.

H8: $F(x, \cdot)$ is strictly concave.

Assuming **H8**, we may define for every $x \in$dom$V$,

$$\tau(x) = \text{argmax}_{y \in \Gamma(x)}\{-\delta(x, y) + V(y)\} \tag{1.12}$$

It follows from (1.10) and (1.11) that if **H6G** is also assumed, then τ is also defined by

$$\tau(x) = \text{argmax}_{y \in \Gamma(x)}\{F(x, y) - F(\bar{x}, \bar{x}) + \hat{V}(y)\} \tag{1.13}$$

From the previous section, we thus have

Corollary 1.5.1. *1. Assume* **H1-H6-H8***. Then the sequence $(x, \tau(x), \tau^2(x), \ldots, \tau^n(x), \ldots)$ is weakly maximal from x.*
2. Assume **H1-H6G-H8***. Then there is a unique optimal programme from x and $\mathbf{x}$ is optimal from x iff $\mathbf{x} = (x, \tau(x), \tau^2(x), \ldots, \tau^n(x), \ldots)$.*

We further have

Proposition 1.5.2. *Assume* **H1-H6-H7-H8***. Then V and τ are continuous at any $x \in$domV such that $\Gamma(x)$ has a non-empty interior.*

Proof. Let $x \in$dom$V$ be such that $\Gamma(x)$ has a non-empty interior. Let $x_n \to x$. Assume that $V(x_n) \to V^*$. Let $y \in \text{int}\Gamma(x)$. Then from Lemma 4.1 in [3], $y \in \text{int}\Gamma(x_n)$ for n large enough. Therefore

$$V(x_n) \geq -\delta(x_n, y) + V(y)$$

which implies that

$$V^* \geq -\delta(x, y) + V(y)$$

Since V is upper semicontinuous, $V(x) \geq V^*$. Let us assume that $V(x) > V^*$. Let $y \in \Gamma(x),\ y \neq \tau(x)$. Let $y_\lambda = \lambda y + (1 - \lambda)\tau(x)$. From **H7**, $y_\lambda \in \text{int}\Gamma(x)$, therefore,

$$V^* \geq -\delta(x, y_\lambda) + V(y_\lambda) \tag{1.14}$$

Since $V(x) > V^*$, for λ small enough,

$$\lambda(-\delta(x, y) + V(y)) + (1 - \lambda)V(x) > V^* \tag{1.15}$$

From (1.12), $V(x) = -\delta(x, \tau(x)) + V(\tau(x))$. Using (1.14) and (1.15) and the strict concavity of $-\delta(x, \cdot)$ that follows from **H8**, we obtain

$$V^* \geq -\delta(x, y_\lambda) + V(y_\lambda) > -\lambda(\delta(x, y) + V(y)) + (1 - \lambda)(-\delta(x, \tau(x)) + V(\tau(x)))$$

$$= -\lambda(\delta(x,y) + V(y)) + (1-\lambda)V(x) > V^*$$

hence a contradiction. Therefore $V(x) = V^*$ proving the desired result.

To prove that τ is continuous, let $x_n \to \bar{x}$ and $\tau(x_n) \to \bar{y}$. Let $y \in \text{int}\Gamma(\bar{x})$. Then $y \in \text{int}\Gamma(x_n)$ for n large enough, therefore

$$-\delta(x_n, \tau(x_n)) + V(\tau(x_n)) \geq -\delta(x_n, y) + V(y)$$

As V is upper semicontinuous,

$$-\delta(x,\bar{y}) + V(\bar{y}) \geq -\delta(x,\bar{y}) + \text{limsup} V(\tau(x_n)) \geq -\delta(x,y) + V(y) \quad (1.16)$$

Assume that for some $y \in \Gamma(x)$,

$$-\delta(x,y) + V(y) > -\delta(x,\bar{y}) + V(\bar{y})$$

Let $y_\lambda = \lambda y + (1-\lambda)\bar{y}$. Then since $-\delta(x,\cdot)$ is strictly concave,

$$-\delta(x,y_\lambda) + V(y_\lambda) > -\delta(x,\bar{y}) + V(\bar{y}) \quad (1.17)$$

From **H7**, $y_\lambda \in \text{int}\Gamma(x)$, therefore (1.17) contradicts (1.16).

1.5.3 Examples

Example 1
Let $X = [0,1]$ and $f : [0,1] \to [0,1]$ be a continuously differentiable, strictly concave increasing function fulfilling $f'(0) > 1$. For $x \in X$, let $\Gamma(x) = [0, f(x)]$, and for $(x,y) \in \text{graph}\ \Gamma$, let $F(x,y) = v(f(x)-y)$, where v is a strictly concave increasing function, fulfilling $v(0) = 0$. The optimal stationary point $\bar{x}$ is unique $(f'(\bar{x}) = 1)$. Observe that $\bar{x}$ corresponds to the golden rule, i.e., $f(\bar{x}) - \bar{x}$ is the maximal stationary consumption.

Assumptions **H1-H6G-H7-H8** are satisfied. For any $x_0 > 0$, there exists a unique optimal programme. The Value function V and the optimal policy g are continuous on $]0,1]$.

Example 2 Let us consider a Von Neumann economy. Let K be a convex, compact, non-empty set of $\mathbb{R}^k_+$ constraining the activity levels v. Let A be an $(n \times k)-$ goods input matrix and B be an $(n \times k)-$ goods output matrix. The matrices A, B are non-negative. Let X be a convex, compact set of $\mathbb{R}^n_+$ containing $A(K) \cup B(K)$ where

$$A(K) = \{z \in \mathbb{R}^n : z = Ax,\ x \in \mathbb{R}^k\},$$

and

$$B(K) = \{z \in \mathbb{R}^n : z = Bx,\ x \in \mathbb{R}^k\}.$$

The technology correspondence is:

$$\text{for } x \in X,\ \Gamma(x) = \{y \in X : \exists v \in K, x \geq Av \text{ and } y \leq Bv\}.$$

We assume that, for every i, there exists j such that $b_{ij} > 0$, i.e., all goods are producible.

For $(x, y) \in graph\Gamma$, define

$$F(x, y) = \max\{\tilde{u}(v) : Av \leq x, Bv \geq y\},$$

where $\tilde{u}$ is a strictly concave function from $\mathbb{R}^k_+$ into $\mathbb{R}$. Assume also there exists $v \in K$ such that $Bv >> Av$. We let to reader check that **H1-H6G-H7-H8** are satisfied. Moreover, if $x >> 0$, then there exists $v >> 0$ such that $Av \leq x$ and $Bv >> 0$. Assume that the stationary point $(\bar{x}, \bar{x})$ is in the interior of $graph\Gamma$. Then the utility function F is subdifferentiable at $(\bar{x}, \bar{x})$ and $\Pi_G(x) \neq \emptyset, \forall x >> 0$. Therefore, there exists an optimal solution for any $x >> 0$. The Value function and the optimal policy V are continuous on $x >> 0$.

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2. Optimal Growth Models with Discounted Return

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In this chapter, we provide a unified treatment of a class of optimal growth models by using dynamic programming methods. In the economies we consider in this chapter, a social planner maximizes a discounted sum of utilities which depend on the current and past period states subject to a feasibility constraint. We show that this problem can be brought down to a sequence of static problems by using the value function of the problem and the associated Bellman equation. The Bellman equation allows us to state that
(i)the value function is continuous with respect to the initial data and to the discount factor,
(ii) the optimal trajectory of state variables can be described as a dynamical system (which may be multi-valued)
We first give two examples of optimal growth models.
Example 1
Consider a two-sector economy. At date t, sector 1 produces consumption good c_t by using a capital stock k_t^1 which is produced in sector 2. At date t, sector 2 produces capital stock k_{t+1} which will be used in period $t+1$ by the two sectors. To produce k_{t+1}, sector 2 needs a quantity k_t^2 of capital good. The social planner solves at date 0 the following problem:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t), \beta \in]0,1[,$$

under the constraints:

$$\forall t,\ 0 \le c_t \le f^c(k_t^1),$$
$$0 \le k_{t+1} \le f^k(k_t^2),$$

[1] The author is deeply indebted to Rose-Anne Dana for numerous helpful remarks and observations. All mistakes remain his own.

$$k_t^1 + k_t^2 \leq k_t$$
$$k_t^1 \geq 0, k_t^2 \geq 0,$$

and $k_0 \geq 0$ is given. The functions f^c and f^k are respectively the production functions of the consumption good sector and of the capital good sector.
The reader can check that, if the utility function u and the production functions f^c and f^k are strictly increasing, the initial problem becomes:

$$\max \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}), 0 < \beta < 1,$$

under the constraints:

$$\forall t \geq 0, \; k_{t+1} \in \Gamma(k_t),$$

and $k_0 \geq 0$ is given. The correspondence Γ is defined by $\forall k \geq 0, \Gamma(k) = [0, f^k(k)]$, and the return function V by $V(k_t, k_{t+1}) = u(f^c(k_t - (f^k)^{-1}(k_{t+1})))$, the function $(f^k)^{-1}$ being the inverse function of f^k.
Example 2 (Human capital; Lucas [10]).
We have an one-sector growth model. But the output is a function of physical capital k and of effective labor N^e. Effective labor is the sum of skill-weighted manhours devoted to current production. More explicitly, assume there are N identical workers. Each worker has $h \in [0, +\infty[$ as skill level and devotes a fraction θ of his non-leisure time to current production and the remaining $(1-\theta)$ to human capital accumulation. We thus have $N^e = Nh\theta$. Given k, h, θ, N, the level of output is $Ah^\gamma F(k, Nh\theta)$. The total productivity now is Ah^γ. The term h^γ captures the external effects of human capital while the technology level A is assumed to be constant.
We assume that the rate of growth of human capital depends, through a function G, on the non-leisure time devoted to its accumulation.
The model is as follows:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t), 0 < \beta < 1,$$

under the constraints:

$$\forall t \geq 0, c_t + k_{t+1} - (1-\delta)k_t \leq Ah_t^\gamma F(k_t, Nh_t\theta_t),$$
$$h_{t+1} \leq h_t(1 + G(1-\theta_t))$$
$$h_t \geq 0, k_t \geq 0,$$

and $k_0 \geq 0, h_0 \geq 0$ are given.

Let $x = (k_0, h_0) \in \mathbb{R}_+^2, \; y = (k_1, h_1) \in \mathbb{R}_+^2$. Define the indirect utility V:

$$V(x, y) = \max_{c,\theta} \{u(c)\}$$

under the constraints:

$$c + k_1 \leq Ah_0^\gamma f(k_0, Nh_0\theta) + (1-\delta)k_0,$$

$$h_1 \leq h_0(1 + G(1-\theta)).$$

Define

$$\Phi(h_0, k_0, \theta) = Ah_0^\gamma f(k_0, Nh_0\theta) + (1-\delta)k_0,$$
$$\Psi(h_0, \theta) = h_0(1 + G(1-\theta)).$$

Define also for $x \in \mathbb{R}_+^2$:

$$\Gamma(x) = \{y \in \mathbb{R}_+^2 : \exists \theta \in [0,1], \text{ s.t. } k_1 \leq \Phi(h_0, k_0, \theta), h_1 \leq \Psi(h_0, \theta)\}.$$

The reader can check that the initial problem is equivalent to:

$$\max \sum_{t=0}^{+\infty} \beta^t V(x_t, x_{t+1})$$

under the constraint:

$$\forall t \geq 0, x_{t+1} \in \Gamma(x_t), x_t \in \mathbb{R}_+^2$$

and x_0 is given in $\mathbb{R}_+^2$.

2.1 Bounded from Below Utility

We consider a model where the technology can exhibit a non zero maximal rate of growth. In the first section we assume that the absolute value of the utility is bounded by an affine function. This excludes Cobb-Douglas utility functions with negative elasticities or logarithm. We devote the next section to the case where the utility function can take the value $-\infty$.

The plan of this section is as follows. In subsection 1, we study the general case, i.e., when the utility function may be non-concave, and the technology may have increasing returns. In subsection 2, we assume that the utility function is concave and the technology is convex. Subsection 3 will be devoted to examples.

2.1.1 The General Case

We consider the problem:

$$\max \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$$

under the constraints:

$$\forall t \geq 0,\ x_{t+1} \in \Gamma(x_t),$$

$$x_t \in X,$$

and x_0 is given in X, closed subset of $\mathbb{R}^n$ that contains 0.

The assumptions are the following.

H1 The correspondence $\Gamma : X \to X$ is continuous, with non-empty, compact values. Moreover $0 \in \Gamma(0)$.

H2 There exist $\gamma \geq 0,\ \gamma' \geq 0, \gamma + \gamma' > 0$ such that if $y \in \Gamma(x)$ then $\|y\| \leq \gamma\|x\| + \gamma'$.

H3 The function $F : graph(\Gamma) \to \mathbb{R}$ is continuous. Moreover, there exist $A \geq 0,\ B \geq 0$ such that $A + B > 0$ and $\forall (x, y) \in graph(\Gamma),\ |F(x, y)| \leq A + B(\|x\| + \|y\|)$.

H4 We have $\beta \in]0, 1[$ and if the constant B in assumption **H3** is strictly positive, we assume $\beta\gamma < 1$.

Remark 2.1.1. 1. The assumption $0 \in \Gamma(0)$ in **H1** is less restrictive than $\Gamma(0) = \{0\}$.
2. Assumption **H2** means that the maximal rate of growth is γ. **H2** is satisfied when $\Gamma(x) = [0, f(x)]$ where f is a positive, concave function defined on $\mathbb{R}_+$.
3. Assumption **H3** is fulfilled if the function F is concave and non-negative.
4. When the constant B in **H3** is strictly positive, condition $\beta\gamma < 1$ in **H4** ensures, as we will see below, that the sum $\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$ exists in $\mathbb{R}$ for any sequence $(x_0, x_1, ..., x_t, ...)$ which satisfies $\forall t \geq 0, x_{t+1} \in \Gamma(x_t)$.

Existence of an Optimal Solution

Let us recall that an infinite sequence $(x_0, x_1, ..., x_t, ...)$ of elements in $\mathbb{R}^n$ will be denoted by $\mathbf{x}$.
A sequence $\mathbf{x}$ is *feasible* from $x_0 \in X$, if it satisfies: $\forall t \geq 0, x_{t+1} \in \Gamma(x_t)$. The set of feasible sequences from x_0 is denoted by $\Pi(x_0)$.
In order to prove the existence of solutions to the optimal growth problem, we use the product topology.We show that the function $u(\mathbf{x}) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$ is continuous on $\Pi(x_0)$ which is compact.One concludes that an optimal solution exists.

Lemma 2.1.1. *(i) Assume* **H2**. *Then one has* $\forall x_0 \in X, \forall \mathbf{x} \in \Pi(x_0)$*:*

$$\forall t \geq 1, \|x_t\| \leq \gamma^t \|x_0\| + \gamma' \sum_{j=0}^{t-1} \gamma^j. \tag{2.1}$$

(ii) Assume **H2, H3**. *Then there exists numbers* $c_1 > 0, c_2 \geq 0$ *such that:* $\forall x_0 \in X, \forall \mathbf{x} \in \Pi(x_0)$, *one has:*

$$\forall t, |F(x_t, x_{t+1})| \leq Bc_1(\gamma^t \|x_0\| + \gamma'(1 + \gamma + \gamma^2 + ... + \gamma^{t-1})) + c_2, \tag{2.2}$$

where B *is the constant in Assumption* **H3**.

(iii) Assume **H2**, **H3**, **H4**. *Then for every* $x_0 \in X$, *for every* $\mathbf{x} \in \Pi(x_0)$, *the sum* $\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$ *exists and is finite-valued. Moreover, there exist* $D > 0$ *such that:*

$$\sum_{t=0}^{+\infty} \beta^t |F(x_t, x_{t+1})| \leq Bc_1 \frac{\|x_0\|}{(1-\beta\gamma)} + D, \forall \mathbf{x} \in \Pi(x_0). \tag{2.3}$$

Proof. (i) For $t = 0$, the claim is obviously true. Proceed by induction to obtain the result for $t > 0$.
(ii) Using **H3** and (2.1), we have for $t \geq 0$

$$|F(x_t, x_{t+1})| \leq A + B\gamma' + B(1+\gamma)\gamma'(1+\gamma+...+\gamma^{t-1}) + B(1+\gamma)\gamma^t\|x_0\|.$$

Let $c_1 = \max\{(1+\gamma)\gamma', (1+\gamma)\}$ and $c_2 = A + B\gamma'$, then $c_1 > 0$ and :

$$|F(x_t, x_{t+1})| \leq Bc_1(\gamma^t\|x_0\| + \gamma'(1+\gamma+\gamma^2+...+\gamma^{t-1})) + c_2.$$

(iii) Assume first $\gamma = 1$. Then

$$\sum_{t=0}^{+\infty} \beta^t |F(x_t, x_{t+1})| \leq Bc_1 \frac{\|x_0\|}{(1-\beta)} + Bc_1\gamma' \sum_{t=0}^{+\infty} t\beta^t + c_2 < +\infty,$$

and (2.3) is true. If $\gamma \neq 1$, then

$$|F(x_t, x_{t+1})| \leq Bc_1(\gamma^t\|x_0\| + \gamma'(\frac{1-\gamma^t}{1-\gamma})) + c_2,$$

and (2.3) is also true.

Lemma 2.1.2. *Assume* **H1-H4**. *Then: (i) The set* $\Pi(x_0)$ *is compact for the product topology.*
(ii) Let $u(\mathbf{x}) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$ *for* $\mathbf{x} \in \Pi(x_0)$. *Then* u *is continuous.*

Proof. (i) From Lemma (2.1), if $\Pi(x_0)$ is a bounded set of the product topology. Let us prove that it is closed for this topology.
Indeed, let $\{\mathbf{x^n}\}$ be a sequence in $\Pi(x_0)$ which converges in the product topology to $\mathbf{x}$. Since X is closed, $x_t \in X, \forall t$. Since Γ is continuous, we have $x_{t+1} \in \Gamma(x_t), \forall t$. Hence, $\mathbf{x} \in \Pi(x_0)$.
(ii) Let $\{\mathbf{x^n}\}$ be a sequence of $\Pi(x_0)$ which converges for the product topology to $\mathbf{x} \in \Pi(x_0)$. From (2.2), if $\gamma = 1$, then, $\forall \mathbf{x} \in \Pi(x_0)$, we have

$$\forall t, |F(x_t, x_{t+1})| \leq Bc_1(\|x_0\| + t\gamma') + c_2,$$

and if $\gamma \neq 1$, then

$$\forall t, |F(x_t, x_{t+1})| \leq Bc_1(\gamma^t\|x_0\| + \frac{1-\gamma^t}{1-\gamma}\gamma') + c_2.$$

Hence, for any $\varepsilon > 0$, it follows from Assumption **H4** that there exists T such that for any $\mathbf{x} \in \Pi(x_0)$, for any $T' \geq T$, we have $\sum_{t=T'}^{+\infty} \beta^t |F(x_t, x_{t+1})| \leq \varepsilon$. Fix such a T'. Then $|u(\mathbf{x^n}) - u(\mathbf{x})| \leq \sum_{t=0}^{T'} \beta^t |F(x_t^n, x_{t+1}^n) - F(x_t, x_{t+1})| + 2\varepsilon$. Let n converge to $+\infty$. We get $\lim_{n\to+\infty} |u(\mathbf{x^n}) - u(\mathbf{x})| \leq 2\varepsilon$. Since ε is arbitrary, we have $\lim_{n\to+\infty} u(\mathbf{x^n}) = u(\mathbf{x})$, which proves the continuity of u.

Proposition 2.1.1. *Assume* **H1-H4**. *Then there exists an optimal solution.*

Proof. The problem is equivalent to $\max\{u(\mathbf{x}) : \mathbf{x} \in \Pi(x_0)\}$. Let $\Pi(x_0)$ be endowed with the product topology. Then it is compact. Since, in the product topology, u is continuous, there exists a solution.

Value Function and Optimal Correspondence

Let $V(x_0) = \max\{u(\mathbf{x}) : \mathbf{x} \in \Pi(x_0)\}$ be the *Value Function* of the optimal growth problem.

Proposition 2.1.2. *Assume* **H1-H4**. *Then*
(i) The function V satisfies

$$\sup_{x_0 \in X} \left\{ \frac{|V(x_0)|}{1 + B\|x_0\|} \right\} < +\infty,$$

where B is the constant in **H3**.
(ii) The Value function V satisfies the Bellman equation:

$$\forall x_0 \in X, V(x_0) = \sup_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta V(y)\}.$$

Proof. (i) From (2.3), there exist $A_1 > 0, A_2 \geq 0$ such that, for any $x_0 \in X$, for any $\mathbf{x} \in \Pi(x_0)$, we have

$$\sum_{t=0}^{+\infty} \beta^t |F(x_t, x_{t+1})| \leq B A_1 \|x_0\| + A_2.$$

Hence,

$$|V(x_0)| \leq B A_1 \|x_0\| + A_2 \tag{2.4}$$

and $\sup_{x_0 \in X} \left\{ \frac{|V(x_0)|}{1+B\|x_0\|} \right\} < +\infty$.
(ii) Let $\mathbf{x} \in \Pi(x_0)$ satisfy $V(x_0) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$. Since $(x_2, x_3, ...) \in \Pi(x_1)$, by the definition of V, we have $V(x_0) \leq F(x_0, x_1) + \beta V(x_1)$, and consequently,

$$V(x_0) \leq \sup_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta V(y)\}.$$

Now, let $x_1 \in \Gamma(x_0)$. There exists $(x_2, x_3, ...) \in \Pi(x_1)$ such that $V(x_1) = \sum_{t=1}^{+\infty} \beta^{t-1} F(x_t, x_{t+1})$. This implies:

$$F(x_0, x_1) + \beta V(x_1) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) \leq V(x_0),$$

since $(x_0, x_1, x_2, ...) \in \Pi(x_0)$. Therefore

$$\sup_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta V(y)\} \leq V(x_0).$$

Let us now prove that V satisfies the Bellman equation with the sup replaced by max, i.e.

$$V(x_0) = \max_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta V(y)\}.$$

Let E be the Banach space of functions h from $\mathbb{R}_+^n$ into $\mathbb{R}$ which satisfy $\sup_{x \in X} \left\{ \frac{|h(x)|}{1+B\|x\|} \right\} < +\infty$ endowed with the norm $\|h\| = \sup_{x \in X} \left\{ \frac{|h(x)|}{1+B\|x\|} \right\}$. We prove that V is the unique solution to the Bellman equation in E.

We introduce the following operator T:

$$\forall h \in E, \forall x \in X, Tf(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta h(y)\}.$$

Let us check that T maps E into E.
Let $h \in E$. We have $\forall y \in \Gamma(x_0), \|y\| \leq \gamma \|x_0\| + \gamma'$, hence:

$$\frac{|F(x_0, y) + \beta h(y)|}{1 + B\|x_0\|} \leq \frac{(A + B\gamma') + B(1+\gamma)\|x_0\|}{1 + B\|x_0\|} + \frac{\beta |h(y)|}{1 + B\|y\|} \frac{1 + B(\gamma\|x_0\| + \gamma')}{1 + B\|x_0\|}.$$

Since $\frac{|h(y)|}{1+B\|y\|} \leq \|h\|$, the second member is uniformly bounded above. Therefore $\sup_{x \in X} \left\{ \frac{|Th(x)|}{1+B\|x\|} \right\} < +\infty$. We have proved that $Th \in E$.
We first show a general result in topology that if T^k (the k-iterated of T) is a contraction mapping of E for some integer k, then T has a unique fixed point.

Lemma 2.1.3. *Assume there exist an integer k and a number $\lambda \in]0, 1[$ such that T^k is a λ-contraction of E, i.e.*

$$\forall h \in E, \forall g \in E, \|T^k h - T^k g\| \leq \lambda \|h - g\|.$$

Then T has a unique fixed point in E. Moreover, if v is a fixed point, then $v = \lim_{n \to +\infty} \{T^n h\}$ for any h in E.

Proof. Since T^k is a contraction, it admits a unique fixed point $v \in E$, i.e. $T^k v = v$. We claim that v is a fixed point of T.
Indeed, we have

$$\|Tv - v\| = \|T(T^k v) - T^k v\| = \|T^k(Tv) - T^k v\| \leq \lambda \|Tv - v\|.$$

Thus, $Tv = v$. Since a fixed point of T is a fixed point of T^k, T has a unique fixed point.

Since T^k is a contraction, v is the limit of the sequence $\{(T^k)^n h\}$ for any h in E. But we want to prove that, actually, v is the limit of the sequence $\{T^n h\}$ for any h in E.
Define successively $h_1 = Th, h_2 = T^2 h, ..., h_{k-1} = T^{k-1}h$. The functions h_i belong to E. Let $\varepsilon > 0$ be given. We have:

$$\exists N_i, \text{ for } i = 0, 1, .., k-1, \text{ such that } \forall i, \ \forall n \geq N_i, \|(T^k)^n h_i - v\| \leq \varepsilon.$$

Let $\hat{N} = \max\{N_0, N_1, ..., N_{k-1}\}$. Let $n \geq \hat{N}k + k - 1$. Then there exists N such that $n \in \{Nk, Nk+1, ..., Nk+k-1\}$. Write $n = Nk + j$, with $j \leq k-1$. Obviously, $N \geq \hat{N}$. Therefore:

$$\|T^n h - v\| = \|(T^k)^N h_j - v\| \leq \varepsilon.$$

We have proved the statement :
$\forall \varepsilon > 0, \exists \tilde{N}$ such that, if $n \geq \tilde{N}$, then $\|T^n h - v\| \leq \varepsilon$.
In other words, we have proved that v is the limit of the sequence $\{T^n h\}$.

In the following proposition, we prove (a) that the Value function V is the unique solution in E to the Bellman equation, and, (b) it is also the unique solution in the set of continuous functions (not necessarily in E) which satisfy a transversality condition.

Proposition 2.1.3. *Assume* **H1-H4**. *then*
(i) The Value function V is the unique continuous solution in E to the Bellman equation:

$$\forall x_0 \in X, V(x_0) = \max_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta V(y)\}.$$

(ii) We have $V = \lim_{n \to +\infty} T^n h$, for any $h \in E$.
(iii) The Value function V is the unique continuous solution to the Bellman equation, which satisfies:

$$\forall x_0 \in X, \forall \mathbf{x} \in \Pi(x_0), \lim \beta^t V(x_t) = 0.$$

(iv) We have $V = \lim_{n \to +\infty} T^n h$, for any continuous function h which satisfies the condition:

$$\forall x_0 \in X, \forall \mathbf{x} \in \Pi(x_0), \lim \beta^t h(x_t) = 0.$$

Proof. (i) It is obvious that V is a fixed point of T.
In a first step, we prove that T^k is a λ-contraction where k is an integer and $\lambda \in]0, 1[$. Second, we prove that V is continuous and hence is the unique continuous fixed point of T.
We will show that there exist an integer k and a real number $\lambda \in]0, 1[$ such that T^k is a λ-contraction.
Let $\varphi(x) = 1 + B\|x\|$, for $x \in X$. Take $g \in E, \ h \in E$. Since

$$\forall y \in X, \ g(y) \leq h(y) + \|g - h\|\varphi(y),$$

we have

$$Tg(x_0) \leq Th(x_0) + \beta\|g-h\| \sup_{x_1 \in \Gamma(x_0)} \varphi(x_1)$$

which implies, from Assumption **H2**, that:

$$Tg(x_0) \leq Th(x_0) + \beta\|g-h\|(B\gamma\|x_0\| + 1 + B\gamma').$$

By induction, and using again Assumption **H2**, we get:

$$T^k g(x_0) \leq T^k h(x_0) + \beta^k \|g-h\|(B\gamma^k\|x_0\| + 1 + B\gamma' \sum_{j=0}^{k-1} \gamma^j).$$

Reversing the role of h and g, one obtains:

$$\frac{|T^k g(x_0) - T^k h(x_0)|}{\varphi(x_0)} \leq \|g-h\|\beta^k \frac{B\gamma^k\|x_0\| + 1 + B\gamma' \sum_{j=0}^{k-1}\gamma^j}{\varphi(x_0)}.$$

If $B = 0$, then

$$\|T^k g - T^k h\| \leq \beta^k \|g-h\|.$$

In that case we can take $k = 1$ since $0 < \beta < 1$. In other words T is a β-contraction. If $B > 0$, we have

$$\frac{B\gamma^k\|x_0\| + 1 + B\gamma' \sum_{j=0}^{k-1}\gamma^j}{\varphi(x_0)} \leq \gamma^k + 1 + B\gamma' \sum_{j=0}^{k-1} \gamma^j.$$

Hence,

$$\|T^k g - T^k h\| \leq \lambda \|g-h\|,$$

with $\lambda = \beta^k\gamma^k + \beta^k(1 + \gamma' B \sum_{m=0}^{m=k-1} \gamma^m)$. For k large enough, we have $0 < \lambda < 1$.

We have proved that there exists an integer k and a number $\lambda \in]0,1[$ such that T^k is a λ-contraction. Apply Lemma 2.1.3 to conclude that V is the unique fixed point in E of T.

Since T^k is a contraction, we have $V = \lim_{n\to+\infty}(T^k)^n 0$. By the Maximum Theorem, $T0$ is continuous, and hence,$(T^k)^n 0$ is continuous for any n. Obviously, V is the uniform limit of $(T^k)^n 0$ on any compact set of X. One concludes that V is continuous. We therefore have

$$\forall x_0 \in X, V(x_0) = \max_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta V(y)\}.$$

(ii) From Lemma 2.1.3, $V = \lim_n T^n h$ for any h in E.

(iii) We first check that

$$\forall x_0 \in X, \forall \mathbf{x} \in \Pi(x_0), \lim \beta^t V(x_t) = 0.$$

Indeed, from (2.4), there exist $A_1 > 0, A_2 \geq 0$ such that $\forall x_0 \in X, |V(x_0)| \leq BA_1\|x_0\| + A_2$. Let $\mathbf{x} \in \Pi(x_0)$. From (2.1), we have

$$\forall t, |V(x_t)| \leq BA_1(\gamma^t\|x_0\| + \gamma'(1+\gamma+...+\gamma^{t-1})) + A_2.$$

If $B = 0$, then $\forall t, |V(x_t)| \leq A_2$ and the claim is true. If $B > 0$, since $\beta \in]0,1[, \beta\gamma < 1$, the claim is also true.
Now, let $\hat{V}$ be another solution to the Bellman equation which is continuous and satisfies the condition

$$\forall x_0 \in X, \forall \mathbf{x} \in \Pi(x_0), \lim \beta^t \hat{V}(x_t) = 0.$$

Let $\mathbf{x}$ be an optimal solution from x_0. Then, $V(x_0) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$. Since $\hat{V}$ satisfies the Bellman equation, we have:

$$\hat{V}(x_0) \geq \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T \hat{V}(x_T).$$

Let T converge to infinity. We get $\hat{V}(x_0) \geq V(x_0)$.
We now prove the converse. Since the functions $F, \hat{V}$ are continuous and the correspondence Γ is compact-valued, there exists a sequence $\mathbf{x} \in \Pi(x_0)$ such that:

$$\hat{V}(x_0) = F(x_0, x_1) + \beta\hat{V}(x_1), \hat{V}(x_1) = F(x_1, x_2) + \beta\hat{V}(x_2),$$

and by induction:

$$\forall T, \hat{V}(x_0) = \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T \hat{V}(x_T).$$

Take the limit when $T \to +\infty$. We obtain $\hat{V}(x_0) = u(\mathbf{x})$ and hence $\hat{V}(x_0) \leq V(x_0)$.
(iv) Let h be a continuous function which satisfies the condition

$$\forall x_0 \in X, \ \forall \mathbf{x} \in \Pi(x_0), \ \lim_{t \to +\infty} \beta^t h(x_t) = 0.$$

First observe that, by the Maximum Theorem, $T^n h$ is continuous for all n. Since h, F are continuous, there exists $x_1 \in \Gamma(x_0)$ such that $Th(x_0) = F(x_0, x_1) + \beta h(x_1)$. By induction, we find a sequence $\mathbf{x} \in \Pi(x_0)$ such that

$$\forall N, \ T^N h(x_0) = \sum_{t=0}^{t=N-1} \beta^t F(x_t, x_{t+1}) + \beta^N h(x_N).$$

From (2.3) and since $\lim_{N \to +\infty} \beta^N h(x_N) = 0$, the righthand side has a limit. Thus $\lim T^N h(x_0)$ exists. We obtain

$$\lim_N T^N h(x_0) = u(\mathbf{x}) + \lim_N \beta^N h(x_N) = u(\mathbf{x})$$

and hence, $\lim_N T^N h(x_0) \leq V(x_0)$.
We now prove that $\lim T^N h(x_0) \geq V(x_0)$. Let $\mathbf{x} \in \Pi(x_0)$. By the very definition of T, we have

$$\forall N,\ T^N h(x_0) \geq \sum_{t=0}^{t=N-1} \beta^t F(x_t, x_{t+1}) + \beta^N h(x_N).$$

Take the limits when $N \to +\infty$. We have $\lim_N T^N h(x_0) \geq u(\mathbf{x})$. Since this inequality holds for any $\mathbf{x} \in \Pi(x_0)$, we actually have $\lim_N T^N h(x_0) \geq V(x_0)$. Thus, $\lim_N T^N h(x_0) = V(x_0)$.

Remark 2.1.2. Observe that to prove that the Value function V is the unique solution to the Bellman equation, we use (i) a weighted-norm, and (ii) the fact that the mapping T is a T^k-contraction for some integer k. This "trick" has been used by Duran [6] for recursive utility.

We now introduce the *optimal correspondence*. It is the correspondence G defined by

$$\forall x \in X, G(x) = \mathrm{argmax}_{y \in \Gamma(x)}\{F(x,y) + \beta V(y)\}$$

or equivalently:

$$\forall x \in X, G(x) = \{y \in \Gamma(x) : V(x) = F(x,y) + \beta V(y)\}.$$

If G is single-valued, then we define a mapping g by $G(x) = \{g(x)\}$. The mapping g will be called *optimal policy*.
In the following proposition, we first show that the optimal correspondence is an upper semi-continuous correspondence. Second, we can use an algorithm which gives an approximate value of an optimal point. Hence, one can compute an approximate value of the Value function.

Proposition 2.1.4. *Assume* **H1-H4**. *Then:*
(i) G is an upper semi-continuous correspondence. If G is single-valued, then it is a continuous mapping.
(ii) Let $h \in E$. Define the correspondences G_h and G_h^k for $k = 1, 2, ...$ by:

$$\forall x \in X, G_h(x) = argmax_{y \in \Gamma(x)}\{F(x,y) + \beta h(y)\}$$

$$G_h^k(x) = argmax_{y \in \Gamma(x)}\{F(x,y) + \beta T^k h(y)\}.$$

Consider a sequence $\{y^k\}_{k=1,2,..}$ with $y^k \in G_h^k(x)$, $\forall k$. Then there exists a subsequence $\{y^{k_\nu}\}$ which converges to an element $y \in G(x)$ when ν converges to infinity.

Proof. (i) The statement is a consequence of the Maximum Theorem [4].
(ii) Take $z \in \Gamma(x)$. For every k, we have:

$$F(x, y^k) + \beta T^k h(y^k) \geq F(x,z) + \beta T^k h(z).$$

From Proposition 2.1.3(ii), the function V is the limit of the sequence $\{T^k h\}$ when $k \to +\infty$. Observe also that, since $y^k \in \Gamma(x), \forall k$, there exists a subsequence $\{y^{k_\nu}\}$ which converges to some $y \in \Gamma(x)$.
Take the limits when $\nu \to +\infty$. We get:

$$F(x, y) + \beta V(y) \geq F(x, z) + \beta V(z).$$

Since z is arbitrarily chosen in $\Gamma(x)$, we conclude that $y \in G(x)$.

Properties of Optimal Paths

We can sum up these properties in the following Proposition.

Proposition 2.1.5. *Assume* **H1-H4**. *Let* $\mathbf{x} \in \Pi(x_0)$. *Then*
(i) The sequence $\mathbf{x}$ *is optimal if, and only if:*

$$\forall t \geq 0, V(x_t) = F(x_t, x_{t+1}) + \beta V(x_{t+1}).$$

(ii) The sequence $\mathbf{x}$ *is optimal if, and only if:*

$$\forall t \geq 0, x_{t+1} \in G(x_t).$$

If G *is single-valued and if* g *is the associated optimal policy, we then have:*

$$\forall t \geq 0, x_{t+1} = g(x_t). \tag{2.5}$$

(iii) Assume that the function F *is differentiable on* $int(graph(\Gamma))$ *(the interior of* $graph(\Gamma)$*).*
If $\mathbf{x}$ *is optimal and satisfies* $\forall t, (x_t, x_{t+1}) \in int(graph(\Gamma))$*, then* $\mathbf{x}$ *satisfies Euler equation:*

$$\forall t \geq 0, F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0$$

where F_1, F_2 *denote the derivatives of* F *with respect to the first and the second variables.*

Proof. (i) Let $\mathbf{x}$ be optimal. From the very definition of V, we have

$$V(x_t) \leq F(x_t, x_{t+1}) + \beta V(x_{t+1}), \forall t \geq 0.$$

Now, since $x_{t+1} \in \Gamma(x_t)$ and since V satisfies the Bellman equation, we have $V(x_t) \geq F(x_t, x_{t+1}) + \beta V(x_{t+1}), \forall t \geq 0$.
Now, let $\mathbf{x} \in \Pi(x_0)$ satisfy

$$\forall t \geq 0, V(x_t) = F(x_t, x_{t+1}) + \beta V(x_{t+1}).$$

Then, by induction, we get:

$$\forall T, V(x_0) = \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \beta^{T+1} V(x_{T+1}).$$

From Proposition 2.1.3(iii), $\lim_{T\to+\infty}\beta^{T+1}V(x_{T+1}) = 0$. Hence, $V(x_0) = \sum_{t=0}^{+\infty}\beta^t F(x_t, x_{t+1})$, i.e., the sequence $\mathbf{x}$ is optimal. We have proved statement (i).
(ii) If $\mathbf{x}$ is optimal, then by statement (i) we have $V(x_t) = F(x_t, x_{t+1}) + \beta V(x_{t+1})$. That means that $x_{t+1} \in G(x_t)$.
Conversely, if $x_{t+1} \in G(x_t)$, then $V(x_t) = F(x_t, x_{t+1}) + \beta V(x_{t+1})$. If this relation holds for every t, then by statement (i), the sequence $\mathbf{x}$ is optimal.
(iii) Since $(x_t, x_{t+1}) \in int(graph(\Gamma)), (x_{t+1}, x_{t+2}) \in int(graph(\Gamma))$, we have $(x_t, y) \in int(graph(\Gamma))$, $(y, x_{t+2}) \in int(graph(\Gamma))$ for any y in a neighborhood of x_{t+1}. The sequence $\mathbf{z}$ defined by $z_\tau = x_\tau$ for $\tau \neq t+1$, and $z_{t+1} = y$ belongs to $\Pi(x_0)$. Hence, $u(\mathbf{x}) \geq u(\mathbf{z})$. This implies:

$$F(x_t, x_{t+1}) + \beta F(x_{t+1}, x_{t+2}) \geq F(x_t, y) + \beta F(y, x_{t+2}).$$

This inequality holds for every y in a neighborhood of x_{t+1}. That means that x_{t+1} maximizes in this neighborhood the function $y \to F(x_t, y) + \beta F(y, x_{t+2})$. The result then follows.

On the Continuity of the Value Function and of the Optimal Correspondence with Respect to (β, x_0)

We will write respectively $V(\beta, x_0)$, $G(\beta, x_0)$ instead of $V(x_0), G(x_0)$ for the Value function and for the optimal correspondence. We want to prove that V is continuous and G is upper semi-continuous with respect to the pair (β, x_0). For simplicity, let us assume that the growth parameter γ of assumption **H2** equals 0. We let to the reader check that it is also true when $\gamma > 0$.
Let us go back to the proofs of Propositions 2.1.2 and 2.1.3. Consider the space E and the operator T defined in these proofs. Since $\gamma = 0$, the operator T is a β-contraction of E. Let 0 denote the function equal to zero everywhere. Let $n > m$ and $r = n - m$. Then we have:

$$\|T^n 0 - T^m 0\| \leq \beta^m \|T^r 0\|.$$

But $\|T^r 0\| \leq \|0 - T0\| + \|T0 - T^2 0\| + ... + \|T^{r-1}0 - T^r 0\| \leq (1 + \beta + ... + \beta^{r-1})\|0 - T0\|$. Hence

$$\|T^n 0 - T^m 0\| \leq \frac{\beta^m}{1-\beta}\|T0\|.$$

Fix m and let n go to infinity. Since T^n converges to $V(\beta, .)$, we have

$$\|V(\beta, .) - T^m 0\| \leq \frac{\beta^m}{1-\beta}\|T0\|.$$

Use the definition of $\|\|$ given in the proofs of the mentioned propositions. We have

$$\sup_{x\in X}\frac{|V(\beta, x) - T^m 0(x)|}{1 + B\|x\|} \leq \frac{\beta^m}{1-\beta}\sup_{x\in X}\frac{|T0(x)|}{1 + B\|x\|}.$$

The function $T0$ is independent of β while $T^m 0$ is a continuous function of (β, x) by the Maximum Theorem. Let $\epsilon_1 \geq 0, \epsilon_2 > 0$ satisfy $0 \leq \beta - \epsilon_1, \beta + \epsilon_2 < 1$. Define $\beta_1 = \beta - \epsilon_1, \beta_2 = \beta + \epsilon_2$ and $I = [\beta_1, \beta_2]$. Then, we have

$$\sup_{x \in X} \sup_{\beta' \in I} \frac{|V(\beta', x) - T^m 0(x)|}{1 + B\|x\|} \leq \frac{\beta_2^m}{1 - \beta_1} \sup_{x \in X} \frac{|T0(x)|}{1 + B\|x\|}.$$

This shows that $\frac{V(\beta', x)}{1+B\|x\|}$ is the uniform limit with respect to $(\beta', x) \in I \times X$ of the sequence of continuous functions $\{\frac{T^m 0(x)}{1+B\|x\|}\}$. Hence $V(\beta', x)$ is continuous in $I \times X$.
Since

$$G(\beta, x_0) = \text{argmax}_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta V(\beta, y)\},$$

G is an upper semi-continuous correspondence by the Maximum Theorem. We summarize these results in

Proposition 2.1.6. *The Value function V is continuous with respect to (β, x) in $[0, 1[\times X$. The optimal correspondence G is upper semi-continuous with respect to $(\beta, x) \in I \times X$.*

2.1.2 The Case of a Concave Return Function and a Convex Technology

We assume that $X \neq \{0\}$ and $\Gamma(X) \neq \{0\}$ (if not, the optimal solution is immediate: it is the null sequence $(0, 0, ...0, ...)$. We replace **H2** by **H'2**:
H'2: *graph*(Γ) is convex (i.e., if $y_1 \in \Gamma(x_1), y_2 \in \Gamma(x_2), \lambda \in [0, 1]$, then $\lambda y_1 + (1 - \lambda) y_2 \in \Gamma(\lambda x_1 + (1 - \lambda) x_2)$).
The following Lemma shows that **H1** and **H'2** imply **H2**.

Lemma 2.1.4. *Assume* **H1** *and* **H' 2**. *Then there exist $\gamma' > 0$ and $\gamma \geq 0$ such that: $y \in \Gamma(x) \Longrightarrow \|y\| \leq \gamma' + \gamma \|x\|$.*

Proof. Let $a \in X$, $a \neq 0$ and $\Gamma(a) \neq \{0\}$. Let $y \in \Gamma(x)$ with $\|x\| \geq \|a\|$. Since $(0, 0) \in graph(\Gamma)$, by **H'2** we have $\frac{\|a\|}{\|x\|} y \in \Gamma(\frac{\|a\| x}{\|x\|})$. By **H1** one can define

$$\hat{\gamma} = \max_{z \in \Gamma(X \cap S_a)} \{\|z\|\}$$

where $S_a = \{x \in \mathbb{R}^n : \|x\| = \|a\|\}$. Then $y \in \Gamma(x)$ then $\|y\| \leq \gamma \|x\|$, with $\gamma = \frac{\hat{\gamma}}{\|a\|}$. Let $\gamma' = \max\{\|z\| : z \in \Gamma(x), \|x\| \leq \|a\|\}$. By **H1**, such a γ' exists and $\gamma' \geq \max\{\|z\| : z \in \Gamma(a)\} > 0$.
Summing up, if $y \in \Gamma(x)$, then $\|y\| \leq \gamma' + \gamma \|x\|$.

We now replace **H3** by **H'3**.
H'3: F is a concave, non negative function from *graph*(Γ) into $\mathbb{R}$.
The following Lemma shows that **H'3** implies **H3**.

Lemma 2.1.5. *If* **H'3** *holds, then* **H3** *also holds.*

Proof. For sake of simplicity, we assume that $int(graph(\Gamma))$ is non-empty. Since F is concave, it has subdifferentials everywhere in $int(graph(\Gamma))$. Let $(\hat{x},\hat{y}) \in int(graph(\Gamma))$ and (p,q) be in the subdifferential of F at $(\hat{x},\hat{y})$. We then have:

$$\forall (x,y) \in graph(\Gamma), F(\hat{x},\hat{y}) - F(x,y) \geq p\cdot(\hat{x}-x) + q\cdot(\hat{y}-y),$$

and hence,

$$\begin{aligned} 0 \leq F(x,y) &\leq F(\hat{x},\hat{y}) - p\cdot\hat{x} - q\cdot\hat{y} + p\cdot x + q\cdot y \\ &\leq |F(\hat{x},\hat{y})| + \|p\|\|\hat{x}\| + \|q\|\|\hat{y}\| + B(\|x\|+\|y\|), \end{aligned}$$

where $B = \max\{\|p\|,\|q\|\}$. Take $A = |F(\hat{x},\hat{y})| + \|p\|\|\hat{x}\| + \|q\|\|\hat{y}\|$ to end the proof.

All the results of the previous subsection hold for this subsection. But we have more.

Proposition 2.1.7. *Assume* **H1**, **H'2**, **H'3**, *and* **H4**. *Then*
(i) The Value function V is concave and non-negative.
(ii) It is the unique solution to the Bellman equation which is concave and non-negative.

Proof. (i) Let $\mathbf{x} \in \Pi(x_0)$ and $\mathbf{x}' \in \Pi(x_0')$ with $x_0 \neq x_0'$. Let $\lambda \in [0,1]$. By **H'2**,

$$\forall t, \lambda x_{t+1} + (1-\lambda)x'_{t+1} \in \Gamma(\lambda x_t + (1-\lambda)x'_t).$$

Hence

$$\begin{aligned} V(\lambda x_0 + (1-\lambda)x_0') &\geq \sum_{t=0}^{+\infty} \beta^t F(\lambda x_t + (1-\lambda)x'_t, \lambda x_{t+1} + (1-\lambda)x'_{t+1}) \\ &= \sum_{t=0}^{+\infty} \beta^t F(\lambda(x_t,x_{t+1}) + (1-\lambda)(x'_t,x'_{t+1})) \\ &\geq \sum_{t=0}^{+\infty} \beta^t [\lambda F(x_t,x_{t+1}) + (1-\lambda)F(x'_t,x'_{t+1})] \text{ (by } \mathbf{H'3}) \\ &= \lambda \sum_{t=0}^{+\infty} \beta^t F(x_t,x_{t+1}) + (1-\lambda)\sum_{t=0}^{+\infty} \beta^t F(x'_t,x'_{t+1}) \\ &= \lambda u(\mathbf{x}) + (1-\lambda)u(\mathbf{x}'). \end{aligned}$$

Since the above inequality holds for all the feasible paths starting from x_0 and x_0', it holds also for the optimal paths. Thus

$$V(\lambda x_0 + (1-\lambda)x_0') \geq \lambda V(x_0) + (1-\lambda)V(x_0').$$

Since F is non-negative, V is also non-negative.

(ii) Consider the operator T associated with the Bellman equation. We have shown that there exists an integer k such that T^k is a contraction mapping of the space E of functions h from X into $\mathbb{R}$ which satisfy $\sup_{x\in X} \frac{|h(x)|}{1+B\|x\|} < +\infty$. Moreover, V is a fixed point of T in E. Any concave, non-negative function h on X belongs to E because it satisfies $\sup_{x\in X} \frac{|h(x)|}{1+B\|x\|} < +\infty$. We then conclude that V is the unique solution to the Bellman equation which is concave, non-negative on X.

Proposition 2.1.8. *Assume* **H1**, **H'2**, **H'3**, **H4**, *and F is strictly concave with respect to the second variable. Then the optimal correspondence G is single-valued. The associated optimal policy g satisfies*

$$\forall x \in X, g(x) = Argmax_{y\in\Gamma(x)}\{F(x,y) + \beta V(y)\}.$$

The optimal sequence from $x_0 \in X$ is $\{g^n(x_0)\}_{n=1,\dots,+\infty}$.

Proof. When F is strictly concave in the second variable, it is obvious that the optimal correspondence is single valued. Use (2.5) to end the proof.

The following proposition gives sufficient conditions for a feasible path from x_0 to be optimal. Observe that one of the conditions is that X is a subset of the positive orthant $\mathbb{R}^n_+$.

Proposition 2.1.9 (Mangasarian Lemma).
Assume **H1**, **H'2**, **H'3**, **H4**, *$X \subset \mathbb{R}^n_+$, X contains 0, F is differentiable in $int(graph(\Gamma))$ and $F_2 \le 0$.*
If $\mathbf{x}$ is a feasible path from x_0 which satisfies $\forall t, (x_t, x_{t+1}) \in int(graph(\Gamma))$ and

$$(i)\ \textit{Euler equation: } \forall t, F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0$$

and

$$(ii)\ \textit{Transversality Condition: } \lim_{t\to\infty} \beta^t F_1(x_t, x_{t+1})\cdot x_t = 0$$

then $\mathbf{x}$ is optimal.

Proof. Let $\mathbf{x} \in \Pi(x_0)$ satisfy $\forall t, (x_t, x_{t+1}) \in int(graph(\Gamma))$. Let $\mathbf{x}' \in \Pi(x_0)$. By the concavity of F, we have:

$$\begin{aligned}
\Delta_T &= \sum_{t=0}^{t=T} \beta^t F(x_t, x_{t+1}) - \sum_{t=0}^{t=T} \beta^t F(x'_t, x'_{t+1}) \\
&\ge \sum_{t=0}^{t=T} \beta^t [F_1(x_t, x_{t+1})\cdot(x_t - x'_t) + F_2(x_t, x_{t+1})\cdot(x_{t+1} - x'_{t+1})] \\
&= \sum_{t=0}^{t=T-1} \beta^t (F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}))\cdot(x_{t+1} - x'_{t+1}) \\
&+ \beta^T F_2(x_T, x_{T+1})\cdot(x_{T+1} - x'_{T+1}).
\end{aligned}$$

Since the Euler equation holds and $x'_{T+1} \in \mathbb{R}^n_+$ and $F_2(x_T, x_{T+1}) \leq 0$, we have

$$\Delta_T \geq \beta^T F_2(x_T, x_{T+1}) \cdot x_{T+1} = -\beta^{T+1} F_1(x_{T+1}, x_{T+2}) \cdot x_{T+1}.$$

Hence $\lim_{T \to +\infty} \Delta_T \geq 0$ and $\mathbf{x}$ is optimal.

Let us now give sufficient conditions for the Value function to be differentiable.

Proposition 2.1.10 (Benveniste-Scheinkman). *[3]*
Assume **H1**, **H'2**, **H'3**, **H4** *and that F is differentiable on the interior of $graph(\Gamma)$. Let (x, y) satisfy $y \in G(x)$ and $(x, y) \in int(graph(\Gamma))$. Then $V'(x)$, the derivative of V at x exists and equals $F_1(x, y)$.*

Proof. First, we have $V(x) = F(x, y) + \beta V(y)$. Second, since (x, y) is in the interior of $(graph(\Gamma))$, there exists a neighborhood $\mathcal{U}(x)$ of x such that $y \in \Gamma(x'), \forall x' \in \mathcal{U}(x)$.
Define for $x' \in \mathcal{U}(x)$, $\phi(x') = F(x', y) + \beta V(y)$. We have:

$$\forall x' \in \mathcal{U}(x), V(x') \geq \phi(x').$$

Since x belongs to the interior of X, the subdifferential of V at x is non-empty. Let p be in this subdifferential. Since $V(x) = \phi(x)$, we have:

$$\forall x' \in \mathcal{U}(x), \phi(x) - \phi(x') \geq \ V(x) - V(x') \geq p \cdot (x - x'). \tag{2.6}$$

Inequality (2.6) implies

$$F(x, y) + \beta V(y) - (F(x', y) + \beta V(y)) \geq p \cdot (x - x'),$$

or equivalently,

$$F(x, y) - F(x', y) \geq p \cdot (x - x').$$

This inequality holds for any $x' \in \mathcal{U}(x)$. Hence $p = F_1(x, y)$. This shows that the subdifferential of V at x is a singleton. Therefore, V is differentiable at x and $V'(x) = F_1(x, y)$.

Remark 2.1.3. One can remark that the proof of the differentiability of V is fairly simple. The proof that the optimal policy is differentiable and the Value function V is twice differentiable turns out to be much harder and requires more assumptions on the utility function F (see e.g. Araujo and Scheinkman [2], Araujo [1], Santos [13], Montrucchio [11], Blot and Crettez [5]).

The One Dimension Case
In the one-sector models, the optimal paths always converge to a steady state. In the introduction, we have given an example of a two-sector model that can be transformed into a one dimensional optimal growth model. However the optimal path may be non-monotonic.

Proposition 2.1.11. *Assume* $X = \mathbb{R}_+$ *and* **H1**, **H'2**, **H'3**, **H4**. *Assume also that the function* F *is strictly concave with respect to the second variable. We then have:*
(i) The optimal policy is a continuous function g.
(ii) Assume furthermore: F *is twice continuously differentiable in* $int(graph(\Gamma))$ *and* $\forall x \in \mathbb{R}_+, (x, g(x)) \in int(graph(\Gamma))$. *Then*
(a) If the cross derivative F_{12} *is positive (respectively negative) in* $int(graph(\Gamma))$, *then* g *is increasing (respectively decreasing).*
(b) If moreover, $\gamma < 1$ *in assumption* **H2** *then, if* $F_{12} > 0$, *any optimal path converges to a steady state* x^* *(i.e.* $x^* = g(x^*)$*), and if* $F_{12} < 0, (x, g(x)) \in int(graph\Gamma), \forall x$, *any optimal sequence converges either to a steady state or to a two-period cycle.*

Proof. (i) We have $g(x) = \text{argmax}\{F(x,y) + \beta V(y) : y \in \Gamma(x)\}$. Since $F(x, .)$ is strictly concave, the optimal solution is unique. The optimal policy is a function g which is continuous by the Maximum Theorem.
(ii) Since $\forall x, (x, g(x)) \in int(graph\Gamma)$, from Proposition 2.1.10, we obtain that V is differentiable and $V'(x) = F_1(x, g(x))$. Since $g(x)$ is an interior solution, we have $F_2(x, g(x)) + \beta V'(g(x)) = 0, \forall x \in \mathbb{R}_+$. Assume $F_{12} > 0$ and that there exist $x < x'$, with $g(x) \geq g(x')$. Then:

$$\begin{aligned} 0 = F_2(x', g(x')) + \beta V'(g(x')) &> F_2(x, g(x')) + \beta V'(g(x')) \\ &\geq F_2(x, g(x)) + \beta V'(g(x)) = 0 \end{aligned}$$

(the second inequality follows from the concavity of F and V): a contradiction. Hence $x < x' \Rightarrow g(x) < g(x')$.
The same argument applies when $F_{12} < 0$ to claim that $x < x' \Rightarrow g(x) > g(x')$.
Now assume that $\gamma < 1$ in assumption **H2**. From (2.1), we get that:

$$\forall \mathbf{x} \in \Pi(x_0), \forall t, 0 \leq x_t \leq \gamma^t x_0 + \frac{\gamma'}{1-\gamma} \leq x_0 + \frac{\gamma'}{1-\gamma}.$$

Hence, any optimal path satisfies

$$\forall t, x_t \in [0, \hat{A}] \text{ with } \hat{A} = x_0 + \frac{\gamma'}{1-\gamma}.$$

When $F_{12} > 0$, the optimal sequence $\mathbf{x}$ is monotonic (either decreasing, or increasing), bounded. It converges to a steady state, since g is continuous and since $x_{t+1} = g(x_t), \forall t$.
Consider the case where $F_{12} < 0$. Let $\mathbf{x}$ be an optimal sequence from x_0.
If $x_0 = x_1$, then $x_t = x_0, \forall t$, and the claim is obviously true.
Assume, without loss of generality, that $x_0 < x_1$. Then $x_2 = g(x_1) < x_1 = g(x_0)$ (g is decreasing).
First, assume $x_2 \leq x_0$. Then $x_3 = g(x_2) \geq g(x_0) = x_1$. By induction, we have:

$$x_{2n} \leq x_{2(n-1)} \leq ... \leq x_0 < x_1 \leq x_3 \leq ... \leq x_{2n+1}.$$

Since $x_t \in [0, \hat{A}], \forall t$, the sequence $\{x_{2n}\}$ converges to $\underline{x}$ and $\{x_{2n+1}\}$ converges to $\overline{x}$, with $\overline{x} > \underline{x}$. Since $x_{2n+1} = g(x_{2n}),\ x_{2n} = g(x_{2n-1}), \forall n$, we therefore have $\overline{x} = g(\underline{x})$ and $\underline{x} = g(\overline{x})$. This implies $g^2(\overline{x}) = \overline{x},\ g^2(\underline{x}) = \underline{x}$. The optimal sequence converges to a two-periods cycle.
Now assume $x_0 < x_2$. Then $x_3 = g(x_2) < g(x_0) = x_1$. But, since $x_2 < x_1$, we have $x_3 = g(x_2) > g(x_1) = x_2$. By induction:

$$x_0 < x_2 < x_4 < ...x_{2n} < x_{2n+1} < x_{2n-1} < ... < x_1.$$

The increasing sequence $\{x_{2n}\}_{n=0,1,...}$ converges to $\overline{x}$ and the decreasing sequence $\{x_{2n+1}\}_{n=0,1,...}$ converges to $\underline{x}$. If $\overline{x} = \underline{x}$, then the optimal sequence converges to a steady state. If $\overline{x} < \underline{x}$, then the optimal sequence converges to a two-periods cycle, because $g^2(\underline{x}) = \underline{x}$ and $g^2(\overline{x}) = \overline{x}$.

Remark 2.1.4. We let the reader prove the following:
Assume that F is strictly concave with respect to the second variable, twice continuously differentiable on $int(graph(\Gamma))$.
If $F_{12} \geq 0$ (respectively ≤ 0), then g is non decreasing (respectively non increasing), i.e. $x > x'$ (respectively $x < x'$) $\Rightarrow g(x) \geq g(x')$ (respectively $g(x) \leq g(x')$).

2.1.3 Examples

Example 1 (The Ramsey Model)

Consider the Ramsey Model:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t), \text{ with } 0 < \beta < 1,$$

under the constraints

$$\forall t, c_t + k_{t+1} \leq f(k_t),$$

$$c_t \geq 0, k_t \geq 0, k_0 > 0 \text{ is given.}$$

We assume that the function u is strictly concave, strictly increasing, differentiable, $u(0) = 0$, that the function f is concave, strictly increasing, differentiable, and satisfies $f(0) = 0, f'(+\infty) < 1$.
We know that the problem is equivalent to

$$\max \sum_{t=0}^{+\infty} \beta^t F(k_t, k_{t+1}),$$

$$\text{with } 0 < \beta < 1,\ F(k, y) = u(f(k) - y),$$

under the constraints

$$\forall t, 0 \leq k_{t+1} \leq f(k_t),\ k_0 > 0 \text{ is given.}$$

Under the assumptions on f, there exists a unique point $\bar{k}$ such that $\bar{k} = f(\bar{k}), k > \bar{k} \Rightarrow f(k) < k, k < \bar{k} \Rightarrow f(k) > k$. Let $\hat{A} > \max\{\bar{k}, k_0\}$. Then, if $\mathbf{k} \in \Pi(k_0)$, we have $\forall t, k_t \leq \hat{A}$.
This model satisfies the assumptions **H1, H2, H3, H4**.
First, we can set $X = [0, \hat{A}]$.
In **H1**, the correspondence Γ is defined by $\Gamma(k) = [0, f(k)]$. Obviously, Γ maps X into X and is continuous.
In **H2**, we can take $\gamma = 0, \gamma' = \hat{A}$.
Since F is concave, non negative, it satisfies **H3**.
Obviously, **H4** is satisfied.
Hence there exists an optimal solution. The value function is the unique continuous, concave solution to the Bellman equation, by Proposition 2.1.7. From Proposition 2.1.11, the optimal path is monotonic and converges to a steady state. Recall that if $u'(0) = +\infty$, then the optimal consumptions are strictly positive at each period.

Example 2

We now consider a Ramsey model with a linear utility function. It does not satisfy the Inada condition. We will show that, when the initial capital stock is low, the optimal consumptions will be equal to zero up to some date T_0 and equal the steady state consumption after T_0. In other words, it is optimal to consume nothing during the first T_0 periods.

Consider the problem:

$$\max \sum_{t=0}^{+\infty} \beta^t c_t, \text{ with } 0 < \beta < 1,$$

under the constraints

$$\forall t, c_t + k_{t+1} \leq f(k_t),$$

$$c_t \geq 0, k_t \geq 0, k_0 > 0 \text{ is given.}$$

We assume that the function f is strictly concave, strictly increasing, and satisfies $f(0) = 0$, $f'(0) > \frac{1}{\beta}$, $f'(+\infty) < 1$.
This model is equivalent to

$$\max \sum_{t=0}^{+\infty} \beta^t F(k_t, k_{t+1}),$$

$$\text{with } 0 < \beta < 1,\ F(k, y) = f(k) - y,$$

under the constraints

$$\forall t, 0 \leq k_{t+1} \leq f(k_t),\ k_0 > 0 \text{ is given.}$$

This model satisfies the assumptions of Proposition 2.1.7 and the remark which follows Proposition 2.1.9. Hence there exists an optimal path which is

monotonic and which converges to a steady state. More precisely, let k^s satisfies $f'(k^s) = \frac{1}{\beta}$. We claim that if $f(k_0) \geq k^s$, then the sequence $\mathbf{k} \in \Pi(k_0)$ defined by $k_t = k^s, \forall t \geq 1$ is optimal.
Proof of the claim First assume that $f(k_0) > k^s$. The sequence $\mathbf{k}$ defined above (i) is feasible, (ii) interior, (iii) satisfies Euler equation which is $-1+\beta f'(k_{t+1}) = 0$, and (iv) satisfies the transversality condition. From Proposition 2.1.9, this sequence is optimal.
Now assume $f(k_0) = k^s$. Let g denote the optimal policy. For any $k'_0 > k_0$, from the proof given above, we have $g(k'_0) = k^s$. Since g is continuous (Proposition 2.1.4), by letting k'_0 converge to k_0, we obtain $k^s = g(k_0)$. Hence, the sequence $\mathbf{k}$ defined by $k_t = k^s, \forall t \geq 1$ is optimal from k_0.
We now claim that if $f(k_0) < k^s$, then the sequence $\mathbf{k} \in \Pi(k_0)$ defined by $k_t = f^t(k_0)$ for $t = 0, ..., T$, and $k_t = k^s$, for $t > T$, where T is the first period such that $f^{T+1}(k_0) \geq k^s$, is optimal from k_0.
Proof of the claim Take another feasible path $\mathbf{k}' \in \Pi(k_0)$. Take some integer $N > T$. Let

$$\Delta_N = \sum_{t=0}^{N} \beta^t F(k_t, k_{t+1}) - \sum_{t=0}^{N} \beta^t F(k'_t, k'_{t+1}).$$

By concavity of F, we get

$$\begin{aligned} \Delta_N &\geq (\beta f'(k_1) - 1)(k_1 - k'_1) + ... + \beta^{T-1}(\beta f'(k_T) - 1)(k_T - k'_T) \\ &+ \beta^T(\beta f'(k^s) - 1)(k^s - k'_{T+1}) + ... \\ &+ \beta^{N-1}(\beta f'(k^s) - 1)(k^s - k_N) - \beta^N(k^s - k_{N+1}). \end{aligned}$$

Since, by the definition of T, we have $k_t = f^t(k_0) < k^s, \forall t \leq T$ and hence, $f'(k_t) > f'(k^s) = \frac{1}{\beta}, \forall t \leq T$. We also have $\forall t \leq T, k'_t \leq f^t(k_0) = k_t$. Thus:

$$\Delta_N \geq -\beta^N k^s.$$

Therefore $\lim_{N \to +\infty} \Delta_N \geq 0$. This ends the proof of the claim. Observe that the optimal consumptions is equal to zero up to T.

Example 3: A Two-Sector Model

We now present a two-sector optimal growth model which can be transformed in an one-dimensional growth model with an optimal policy which is not monotonic.

In our economy, there are two goods: (i) a consumption good c produced in sector 1 through a production function $f^c(k^1, l^1)$ using a capital stock k^1 and a quantity of labor l^1, and (ii) a capital good k produced in sector 2 through a production function f^k using k^2 capital and l^2 labor. We assume that the supply of labor is constant over time and equal to 1. The functions f^c and f^k are concave, strictly increasing and satisfy $f^c(0, l) = f^c(k, 0) = f^k(0, l) = f^k(k, 0) = 0, \forall k \geq 0, \forall l \geq 0$.

The consumer solves the problem:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t), \text{ with } 0 < \beta < 1,$$

under the constraints:

$$\forall t \geq 0,\ 0 \leq c_t \leq f^c(k_t^1, l_t^1),$$
$$0 \leq k_{t+1} \leq f^k(k_t^2, l_t^2),$$
$$k_t^1 + k_t^2 \leq k_t,$$
$$l_t^1 + l_t^2 \leq 1,$$
$$l_t^1 \geq 0,\ l_t^2 \geq 0,\ k_t^1 \geq 0,\ k_t^2 \geq 0,$$

and $k_0 > 0$ is given.
Introduce the indirect utility F defined as follows

$$F(k, y) = \max_{k^1, l^1, k^2, l^2} f^c(k^1, l^1),$$

under the constraints

$$0 \leq k^1 + k^2 \leq k,$$
$$0 \leq l^1 + l^2 \leq 1,$$
$$0 \leq y \leq f^k(k^2, l^2),$$
$$k^1 \geq 0,\ k^2 \geq 0,\ l^1 \geq 0,\ l^2 \geq 0.$$

We assume that the utility function is $u(c) = c$. We let to the reader to prove that the initial problem is equivalent to the following one:

$$\max \sum_{t=0}^{+\infty} \beta^t F(k_t, k_{t+1}),$$

under the constraints

$$0 \leq k_{t+1} \leq f^k(k_t, 1),\ \forall t \geq 0,$$

and k_0 is given.
We suppose that $f^c(k, l) = k^{1-\alpha} l^\alpha$, with $1 > \alpha > 0$, and that $f^k(k, l) = \min\{l, \frac{k}{\lambda}\}$, with $1 > \lambda > 0$.
One can check that

$$F(k, y) = (k - \lambda y)^{1-\alpha}(1 - y)^\alpha.$$

The problem becomes

$$\max \sum_{t=0}^{+\infty} \beta^t (k_t - \lambda k_{t+1})^{1-\alpha}(1 - k_{t+1})^\alpha, \text{ such that}$$

$$\forall t \geq 0,\ 0 \leq k_{t+1} \leq \min\{1, \frac{k_t}{\lambda}\},$$

and $k_0 > 0$ is given. Let $X = \mathbb{R}_+$.
In **H1**, the correspondence Γ is defined by $\Gamma(x) = [0, \min\{1, \frac{x}{\lambda}\}]$. Obviously $0 \in \Gamma(0)$.
In **H2**, we take $\gamma = 0$ and $\gamma' = 1$, since $y \in \Gamma(x) \Rightarrow 0 \leq y \leq 1$.
The function F being concave, nonnegative, assumption **H3** is satisfied.
H4 is satisfied since $0 < \beta < 1$ and $\gamma = 0$.
Apply Propositions (2.1.7) and (2.1.11). In particular, the Value function is the unique solution to the Bellman Equation, which is concave, continuous, and the optimal policy is continuous.
But the cross derivative of F is

$$F_{12}(k, y) = \alpha(1-\alpha)(1-y)^{\alpha-1}(k-\lambda)^{-\alpha-1}(\lambda - k).$$

Then $F_{12} > 0$ if $k < \lambda$ and $F_{12} < 0$ if $k > \lambda$. The optimal policy is increasing if $k < \lambda$ and decreasing if $k > \lambda$.

Example 4: A Human Capital Model

We present a simplified version of the Lucas human capital model without physical capital and externality (see Stokey and Lucas [14], p.111).
Assume that at date t, the growth rate of capital is given by the formula

$$\theta_t = \phi(\frac{h_{t+1}}{h_t}),$$

where h_t is the human capital at date t and θ_t is the number of working hours. We assume that $0 \leq \theta_t \leq 1$.
We assume that ϕ is continuous, decreasing and satisfies $\phi(1+\lambda) = 0,\ \phi(1-\delta) = 1$, where $\lambda > 0,\ 0 \leq \delta < 1$. In other words, we assume that without training ($\theta_t = 1$) the human capital depreciates with rate δ and if the worker devotes his whole time for training, his human capital will grow at rate λ. Hence, λ is the maximal rate of growth of human capital.

The consumption good is produced through a production function using only effective labor. At date t, effective labor is $\theta_t h_t N_t$ with N_t denoting the number of workers at date t. We assume that $N_t = 1,\ \forall t$.

The model is

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t),$$

$$\text{such that } \forall t,\ 0 \leq c_t \leq f(\theta_t h_t),$$

$$\theta_t = \phi(\frac{h_{t+1}}{h_t}),\ 0 \leq \theta_t \leq 1,$$

and $h_0 > 0$ is given.

We make the following assumptions:

(i) $u(c) = c^\mu,\ 0 < \mu < 1,$
(ii) $\beta > 0,$
(iii) $f(L) = L^\alpha,\ 0 < \alpha < 1,$
(iv) $\beta(1 + \lambda) < 1.$
It is easy to check that this model is equivalent to:

$$\max \sum_{t=0}^{+\infty} \beta^t [h_t \phi(\frac{h_{t+1}}{h_t})]^{\alpha\mu},$$

under the constraints

$$\forall t \geq 0,\ (1-\delta)h_t \leq h_{t+1} \leq (1+\lambda)h_t,$$

and $h_0 > 0$ is given.

Let $X = \mathbb{R}_+$ and Γ be defined by $\Gamma(x) = [(1-\delta)x,\ (1+\lambda)x]$. For $(x, y) \in graph(\Gamma)$, define the function F by $F(x, y) = [x\phi(\frac{y}{x})]^{\alpha\mu}$, if $x > 0$, and $F(0, y) = 0$ for $y \geq 0$. Since $0 \leq \phi(\frac{x}{y}) \leq 1$, $0 \leq F(x, y) \leq x^{\alpha\mu}$. Thus F is continuous on $graph(\Gamma)$.

The model is obviously of the form:

$$\max \sum_{t=0}^{+\infty} \beta^t F(h_t, h_{t+1})$$

$$\text{such that } \forall t \geq 0, h_{t+1} \in \Gamma(h_t),$$

and $h_0 > 0$ is given.
Assumptions **H1-H4** are satisfied. Hence there exists an optimal solution. The Value function V satisfies the Bellman Equation :

$$\forall h \geq 0,\ V(h) = \max_{y \in [(1-\delta)h,\ (1+\lambda)h]} \{F(h, y) + \beta V(y)\}.$$

We claim that (i) $V(h) = Ah^{\alpha\mu}$ for some constant A, and (ii) there exists $u^* \in [1-\delta,\ 1+\lambda]$ such that the optimal path $\mathbf{h}$ from h_0 is $h_t = (u^*)^t h_0,\ \forall t \geq 0$.
Proof of the claim Let T denote the operator which associates with any continuous function f on $\mathbb{R}_+$ the function $Tf(h) = \max_{y \in [(1-\delta)h,\ (1+\lambda)h]}\{F(h, y) + \beta f(y)\}$. From Proposition 2.1.3, we know that $V = \lim_{n \to +\infty} T^n 0$. Take $h > 0$. We have successively:

$$T0(h) = \max_{y \in [(1-\delta)h,\ (1+\lambda)h]} \{(h\phi(\frac{y}{h}))^{\alpha\mu}\} = h^{\alpha\mu} \max_{u \in [1-\delta,\ 1+\lambda]} \{\phi(u)^{\alpha\mu}\} = A_1 h^{\alpha\mu},$$

$$\begin{aligned} T^2 0(h) &= \max_{y \in [(1-\delta)h,\ (1+\lambda)h]} \{(h\phi(\frac{y}{h}))^{\alpha\mu} + \beta A_1 y^{\alpha\mu}\} \\ &= h^{\alpha\mu} \max_{u \in [1-\delta,\ 1+\lambda]} \{\phi(u)^{\alpha\mu} + \beta A_1 u^{\alpha\mu}\} \ = A_2 h^{\alpha\mu}. \end{aligned}$$

By induction, we have $T^n 0(h) = A_n h^{\alpha\mu}$. Hence $A_n \to A$ when $n \to +\infty$. In other words, $V(h) = Ah^{\alpha\mu}$. We have proved the first part of the claim.
The Value function satisfies the Bellman Equation:

$$\begin{aligned} \forall h > 0,\ V(h) &= \max_{y \in [(1-\delta)h,\ (1+\lambda)h]} \{F(h,y) + \beta A y^{\alpha\mu})\} \\ &= h^{\alpha\mu} \max_{u \in [1-\delta,\ 1+\lambda]} \{\phi(u)^{\alpha\mu} + \beta A u^{\alpha\mu}\} \\ &= h^{\alpha\mu}\{\phi(u^*)^{\alpha\mu} + \beta A (u^*)^{\alpha\mu}\}, \end{aligned}$$

where $u^* \in [1-\delta,\ 1+\lambda]$. Therefore, the optimal policy g is defined by $g(h) = u^* h$ and $\mathbf{h}$ is optimal if, and only if,$h_t = (u^*)^t h_0,\ \forall t \geq 0$. We have proved the second part of the claim.

Example 5: Learning by Doing Model

This example is given in Stokey and Lucas [14] p. 108. Consider a monopolist producing a new product; his production function displays learning by doing. We suppose that the cost function C depends on the production at date t, q_t, and on the cumulative production Q_t, i.e., $Q_t = Q_{t-1} + q_{t-1}$. More precisely, the production cost C_t is $C_t = C(q_t, Q_t)$. We assume that C is convex, continuously differentiable and satisfies $\forall Q \geq 0,\ C(0,Q) = 0,\ 0 < \underline{c} \leq \frac{\partial C}{\partial q}(0,Q) < \overline{c}$. We also assume that given q, the unit-cost function $\frac{C(q,Q)}{q}$ is a decreasing function with respect to Q.
The price is given by an inverse demand function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuously differentiable, strictly decreasing, and such that the income function $q\psi(q)$ is strictly concave. We assume that $\psi(0) > \overline{c}$ and $\psi(+\infty) < \underline{c}$.
The monopolist maximises his intertemporal profit

$$\max \sum_{t=0}^{+\infty} \beta^t [(Q_{t+1} - Q_t)\psi(Q_{t+1} - Q_t) - C(Q_{t+1} - Q_t, Q_t)],$$

$$\text{with } 0 < \beta < 1,$$

under the constraints

$$\forall t \geq 0, Q_{t+1} \geq Q_t,$$

and $Q_0 \geq 0$ is given.
Define the return function F:

$$F(x,y) = (y-x)\psi(y-x) - C(y-x, x),\ \text{for } y \geq x.$$

The model becomes

$$\max \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}),$$

such that

$$x_{t+1} \geq x_t \geq 0,\ \forall t \geq 0,$$

and x_0 is given.
A priori, this problem does not fall in the category of problems we study in this chapter since the technology correspondence Γ is not compact valued ($\Gamma(x) = [x, +\infty[$). We will show that actually we can restrict $\Gamma(x)$ to a compact set.

Let V denote the Value function. First, observe that for any $x_0 \geq 0$, the stationary sequence $(x_0, x_0, ..., x_0, ...)$ is feasible. Thus $V(x_0) \geq 0$. Second, observe that $F(x,y) \leq (y-x)\psi(y-x) - \underline{c}(y-x)$ since, by convexity of C and the fact that $C(0,y) = 0$, we have $C(x,y) = C(x,y) - C(0,y) \geq \frac{\partial C}{\partial q}(0,y)x \geq \underline{c}x$. Consider the function f defined by $f(x) = x\psi(x) - \underline{c}x$. This function is strictly concave, satisfies $f(0) = 0$. Moreover $\frac{f(x)}{x}$ converges to $\psi(0) - \underline{c} > \overline{c} - \underline{c} > 0$ when x converges to zero, which implies $f'(0) > 0$. We also have that $\frac{f(x)}{x}$ converges to $\psi(+\infty) - \underline{c} < 0$ when x converges to $+\infty$. This shows that $f(x)$ converges to $-\infty$ when x converges to $+\infty$. Since, by concavity, $f'(x) \leq \frac{f(x)}{x}$, we have $f'(+\infty) < 0$. Thus, there exists a unique maximum point $\bar{x}$ of f. Let $M = f(\bar{x})$. Since $F(x,y) \leq f(y-x)$, we have

$$\forall x_0 \geq 0,\ 0 \leq V(x_0) \leq \frac{M}{1-\beta}.$$

We also have

$$\begin{aligned}
\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) &= \sum_{t=0}^{+\infty} \beta^t [(x_{t+1} - x_t)\psi(x_{t+1} - x_t) - C(x_{t+1} - x_t, x_t)] \\
&\leq (x_1 - x_0)\psi(x_1 - x_0) - C(x_1 - x_0, x_0) + \frac{\beta M}{1-\beta} \\
&\leq (x_1 - x_0)\psi(x_1 - x_0) - \underline{c}(x_1 - x_0) + \frac{\beta M}{1-\beta} \\
&= f(x_1 - x_0) + \frac{\beta M}{1-\beta}.
\end{aligned}$$

The function $g(x) = f(x) + \frac{\beta M}{1-\beta}$ is strictly concave, satisfies $g(0) = \frac{\beta M}{1-\beta} > 0,\ g'(0) > 0, g(+\infty) = -\infty$. Therefore, there exists a unique point $\hat{x}$ such that $g(x) \geq 0 \Leftrightarrow x \leq \hat{x}$. Since $V(x_0) \geq 0$, we must choose x_1 such that $x_1 - x_0 \leq \hat{x}$. The problem now becomes

$$\max \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$$

under the constraints

$$\forall t \geq 0,\ 0 \leq x_t \leq x_{t+1} \leq x_t + \hat{x},$$

and $x_0 \geq 0$ is given.

We first check that the function F satisfies **H3**. Indeed, we here have $\Gamma(x) = [x\ , x + \hat{x}]$. Let $(x,y) \in graph(\Gamma)$. Since, by assumption, the function $q\psi(q)$ is

concave, we have $F(x,y) \leq (y-x)\psi(y-x) - \underline{c}(y-x) \leq B(y-x) + A$ for some constants A and B. Thus, $F(x,y) \leq |B|(|x|+|y|) + |A|$.
Now, we have $F(x,y) \geq -C(y-x,y)$. Since $C(y-x,y) \leq C(\hat{x},y) \leq C(\hat{x},0)$, one gets $F(x,y) \geq -C(\hat{x},0)$.
Summing up,

$$|F(x,y)| \leq |B|(|x|+|y|) + |A| + C(\hat{x},0).$$

We let the reader check that this model satisfies the remaining assumptions **H1, H2, H4**.

2.2 Unbounded from Below Utility

We again consider the problem:

$$\max \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$$

under the constraints:

$$\forall t \geq 0,\ x_{t+1} \in \Gamma(x_t),$$

$$x_t \in X,$$

and x_0 is given in X, where X is a closed subset of $\mathbb{R}^n_+$ and contains 0.

As in Section 2.1, an infinite sequence $(x_0, x_1, ..., x_t, ...)$ of elements in $\mathbb{R}^n$ will be denoted by $\mathbf{x}$.
A sequence $\mathbf{x}$ is *feasible* from $x_0 \in X$, if it satisfies: $\forall t \geq 0, x_{t+1} \in \Gamma(x_t)$. The set of feasible sequences from x_0 is denoted by $\Pi(x_0)$.
We assume that the utility function F equals $-\infty$ on some subset of $graph(\Gamma)$.

The assumptions are the following.
H1 The correspondence $\Gamma : X \to X$ is continuous, with non-empty, compact values. Moreover $0 \in \Gamma(0)$.
H2 There exist $\gamma \geq 0,\ \gamma' \geq 0$, such that if $y \in \Gamma(x)$ then $\|y\| \leq \gamma\|x\| + \gamma'$.
H3bis The function $F : graph(\Gamma) \to \mathbb{R} \cup \{-\infty\}$ is continuous at any $(x,y) \in graph(\Gamma)$ such that $F(x,y) > -\infty$. If $F(x,y) = -\infty$ and if $\lim_n (x_n, y_n) = (x,y)$, then $\lim_n F(x_n,y_n) = -\infty$. Moreover, there exist $A \geq 0,\ B \geq 0$ such that $\forall (x,y) \in graph(\Gamma),\ F(x,y) \leq A + B(\|x\| + \|y\|)$.
H4 We have $\beta\ \in]0,1[$ and if the constant B in assumption **H3** is strictly positive, we assume $\beta\gamma < 1$.

Remark 2.2.1. (i) Statement (i) of Lemma 2.1.1 still holds. (ii) The sum $\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$ where the sequence $\mathbf{x}$ belongs to the feasible set $\Pi(x_0)$ is meaningful. Indeed, one has

$$\sum_0^T \beta^t F(x_t, x_{t+1}) = \sum_0^T \beta^t F^+(x_t, x_{t+1}) - \sum_0^T \beta^t F^-(x_t, x_{t+1}),$$

where F^+, F^- respectively denote the positive and the negative parts of F. From statement (i) of Lemma 2.1.1, **H3bis**, and **H4**, $\sum_{t=0}^{+\infty} \beta^t F^+(x_t, x_{t+1})$ exists in $\mathbb{R}$. The sum $\sum_0^{+\infty} \beta^t F^-(x_t, x_{t+1})$ exists in $\mathbb{R}_+ \cup \{+\infty\}$. Hence, $\sum_0^{+\infty} \beta^t F(x_t, x_{t+1})$ exists in $\mathbb{R} \cup \{-\infty\}$.

Lemma 2.2.1. *Assume* **H1-H2-H3bis-H4**. *Then, (i) the set $\Pi(x_0)$ is compact for the product topology and (ii) the function u defined for every $\mathbf{x} \in \Pi(x_0)$ by $u(\mathbf{x}) = \sum_0^{+\infty} \beta^t F(x_t, x_{t+1})$ is upper semi-continuous for the product topology.*

Proof. (i) See the proof in Lemma 2.1.2.
(ii) From **H3bis**, we have, for $\mathbf{x} \in \Pi(x_0)$:

$$\sum_0^{+\infty} \beta^t F(x_t, x_{t+1}) \leq \sum_{t=0}^{+\infty} \beta^t (A + B(\|x_t\| + \|x_{t+1})\|))$$

Using statement (i) of Lemma 2.1.1, we obtain that, for any $\varepsilon > 0$, there exists T_0 such that $\forall T \geq T_0, \forall \mathbf{x} \in \Pi(x_0)$, we have $\sum_T^{+\infty} \beta^t F(x_t, x_{t+1}) \leq \varepsilon$.
Now, let $\{\mathbf{x^n}\} \subset \Pi(x_0)$ converge to $\mathbf{x} \in \Pi(x_0)$ for the product topology. For $T \geq T_0$, we have

$$\forall n, u(\mathbf{x^n}) \leq \sum_0^{T-1} \beta^t F(x_t^n, x_{t+1}^n) + \varepsilon.$$

Let $n \to +\infty$. Then $\limsup_n u(\mathbf{x^n}) \leq \sum_0^{T-1} \beta^t F(x_t, x_{t+1}) + \varepsilon$. Let $T \to +\infty$. We obtain: $\limsup_n u(\mathbf{x^n}) \leq u(\mathbf{x}) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we actually have $\limsup_n u(\mathbf{x^n}) \leq u(\mathbf{x})$. The proof is complete.

Let $\Pi'(x_0)$ denote the set of feasible sequences $\mathbf{x}$ such that $u(\mathbf{x}) > -\infty$.

Proposition 2.2.1. *Assume* **H1-H2-H3bis-H4**. *Then there exists an optimal solution.*

Proof. The proof is obvious since the problem is $\{\max u(\mathbf{x}) : \mathbf{x} \in \Pi(x_0)\}$ with u upper semi-continuous and $\Pi(x_0)$ compact.

We say that a function φ from X into $\mathbb{R} \cup \{-\infty\}$ which satisfies $\varphi(x) > -\infty$ if $x \neq 0, \varphi(0) = -\infty$, is continuous in the generalized sense if
(i) it is continuous at any point $x \neq 0$ and
(ii) if a sequence $\{x_n\}$ of points in $X \backslash \{0\}$ converges to 0 then $\varphi(x_n) \to -\infty$.

Proposition 2.2.2. *Assume* **H1-H2-H3bis-H4**. *(i) The Value function V is upper semi-continuous.*
(ii) $\forall x_0 \in X \ \forall \mathbf{x} \in \Pi(x_0), \limsup_T \beta^t V(x_t) \leq 0$.
(iii) For every $x_0 \in X$, and for every $\mathbf{x} \in \Pi'(x_0)$, we have $\lim_{t \to +\infty} \beta^t V(x_t) = 0$.
(iv) Assume moreover $\Gamma(0) = \{0\}, F(0,0) = -\infty$, $\forall x_0 \neq 0, \Pi'(x_0) \neq \emptyset$ and there exists a continuous (in the generalized sense) function φ which satisfies $\forall x_0 \in X, \varphi(x_0) \leq V(x_0)$, $\forall \mathbf{x} \in \Pi'(x_0), \lim_{t \to +\infty} \beta^t \varphi(x_t) = 0$. Then V is continuous in the generalized sense.

Proof. (i) Let $x_0^n \in X \to x_0 \in X$. For every n, let $\mathbf{x^n} \in \Pi(x_0^n)$ satisfy $V(x_0^n) = u(\mathbf{x^n})$. As before, given $\varepsilon > 0$, there exists T_0 such that, $\forall T \geq T_0, \forall n$, $V(x_0^n) \leq \sum_{t=0}^{T} \beta^t F(x_t^n, x_{t+1}^n) + \varepsilon$. We can assume that $\{\mathbf{x^n}\} \to \mathbf{x} \in \Pi(x_0)$. Then, $\limsup_n V(x_0^n) \leq \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \varepsilon$. Now, let $T \to +\infty$. We get $\limsup_n V(x_0^n) \leq u(\mathbf{x}) + \varepsilon \leq V(x_0) + \varepsilon$. We have proved that V is upper semi-continuous.
(ii) Let $\mathbf{x} \in \Pi(x_0)$. For every T, for every $\mathbf{y^T} \in \Pi(x_T)$, the sequence $(x_1, \ldots, x_T, y_1^T, y_2^T, \ldots,)$ belongs to $\Pi(x_0)$. Given $\varepsilon > 0$, for every T large enough, we have $\beta^T F(x_T, y_1^T) + \beta^{T+1} F(y_1^T, y_2^T) + \ldots \leq \varepsilon$. Hence, $\beta^T V(x_T) \leq \varepsilon$ for every T large enough. This implies $\limsup_t \beta^t V(x_t) \leq 0$.
(iii) For every $\mathbf{x} \in \Pi'(x_0)$, we have

$$-\infty < u(\mathbf{x}) \leq \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \beta^{T+1} V(x_{T+1}).$$

Then

$$0 = \lim_T \{u(\mathbf{x}) - \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})\} \leq \liminf_T \beta^{T+1} V(x_{T+1}).$$

Hence $\lim_T \beta^T V(x_T) = 0$.
(iv) Let $x_0^n \in X$ converge to x_0 when n converges to $+\infty$. If $V(x_0) = -\infty$, then $\lim V(x_0^n) = -\infty$ since V is upper semi-continuous. Assume $V(x_0) > -\infty$. Let $\mathbf{x} \in \Pi(x_0)$ satisfy $V(x_0) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$. Let $x_0^n \in X$ converge to x_0 when n converges to $+\infty$. Since Π is lower semi-continuous, there exists $\mathbf{x^n} \in \Pi(x_0^n), \forall n$ which converges to $\mathbf{x}$. Let N be fixed. We have

$$V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t^n, x_{t+1}^n) + \beta^{N+1} V(x_{N+1}^n).$$

We have, for any t, $\Pi'(x_t) \neq \emptyset$. Since $\Gamma(0) = \{0\}, F(0,0) = -\infty$, $\forall x_0 \neq 0, \Pi'(x_0) \neq \emptyset$, we have $x_t \neq 0, \forall t$. Hence for n large enough, $x_{N+1}^n \neq 0$ and $\Pi'(x_{N+1}^n) \neq \emptyset$. Therefore

$$V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t^n, x_{t+1}^n) + \beta^{N+1} \varphi(x_{N+1}^n).$$

and

$$\liminf_n V(x_0^n) \geq \sum_{t=0}^{N} \beta^t F(x_t, x_{t+1}) + \beta^{N+1} \varphi(x_{N+1}).$$

Let $N \to +\infty$:

$$\liminf_n V(x_0^n) \geq V(x_0) + \lim_N \beta^{N+1} \varphi(x_{N+1}) = V(x_0).$$

Since V is upper semi-continuous, we have $\limsup_n V(x_0^n) \leq V(x_0)$ and hence, $\lim_n V(x_0^n) = V(x_0)$.

Let S be the set of upper semi-continuous functions from X into $\mathbb{R} \cup \{-\infty\}$ which satisfy
(i) For all $x_0 \in X$, all $\mathbf{x} \in \Pi(x_0), \limsup_t \beta^t f(x_t) \leq 0$.
(ii) For all $x_0 \in X$, all $\mathbf{x} \in \Pi'(x_0), \lim_t \beta^t f(x_t) = 0$.

Proposition 2.2.3 (Bellman Equation). *Assume* **H1-H2-H3bis-H4**. *Then*
(i) The Value function V satisfies the Bellman equation:

$$\forall x_0 \in X, V(x_0) = \max\{F(x_0, y) + \beta V(y) : y \in \Gamma(x_0).\}$$

(ii) The function V is the unique solution in S to the Bellman equation.

Proof. (i) is standard.
(ii) Let W in S be another solution. There exists $\mathbf{x} \in \Pi(x_0)$ such that

$$\forall T, W(x_0) = F(x_0, x_1) + \beta F(x_1, x_2) + \ldots + \beta^{T-1} F(x_{T-1}, x_T) + \beta^T W(x_T).$$

Let $T \to +\infty$. Since $\limsup_t \beta^t W(x_t) \leq 0$, we have $W(x_0) \leq u(\mathbf{x}) \leq V(x_0)$.
If $\Pi'(x_0) = \emptyset$, then $V(x_0) = -\infty$ and consequently, $W(x_0) = -\infty$.
If $\Pi'(x_0) \neq \emptyset$, take $\mathbf{x} \in \Pi'(x_0)$. We have, by induction,

$$W(x_0) \geq \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \beta^{T+1} W(x_{T+1}).$$

Let $T \to +\infty$. Since $\lim_T \beta^T W(x_T) = 0$ for any $\mathbf{x} \in \Pi'(x_0)$, we have $W(x_0) \geq u(\mathbf{x})$. This inequality holds for any $\mathbf{x} \in \Pi'(x_0)$. Thus, $W(x_0) \geq V(x_0)$.

Proposition 2.2.4 (Optimal policy). *Assume* **H1-H2-H3bis-H4**, *$\Gamma(0) = \{0\}, F(0,0) = -\infty$, $\forall x_0 \neq 0, \Pi'(x_0) \neq \emptyset$ and there exists a continuous (in the generalized sense) function φ which satisfies $\forall x_0 \in X, \varphi(x_0) \leq V(x_0)$, $\forall \mathbf{x} \in \Pi'(x_0), \lim_{t\to+\infty} \beta^t \varphi(x_t) = 0$. Let $G = Argmax\{F(x,y) + \beta V(y) : y \in \Gamma(x)\}$. Then G is an upper semi-continuous correspondence.*

Proof. It is easy and left to the reader.

The proof of the following proposition is also left to the reader.

Proposition 2.2.5. *Assume* **H1-H2-H3bis-H4**.*Let* $\mathbf{x} \in \Pi(x_0)$. *Then*
(i) The sequence $\mathbf{x}$ *is optimal if, and only if:*

$$\forall t \geq 0, V(x_t) = F(x_t, x_{t+1}) + \beta V(x_{t+1}).$$

(ii)The sequence $\mathbf{x}$ *is optimal if, and only if:*

$$\forall t \geq 0, x_{t+1} \in G(x_t).$$

(iii) Assume that the function F is differentiable on $int(graph(\Gamma))$ (the interior of $graph(\Gamma)$).

If $\mathbf{x}$ *is optimal and satisfies* $\forall t, (x_t, x_{t+1}) \in int(graph(\Gamma))$, *then* $\mathbf{x}$ *satisfies Euler equation:*

$$\forall t \geq 0, F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0$$

where F_1, F_2 *denote the derivatives of* F *with respect to the first and the second variables.*

Let T denote the operator defined on the set of functions f which are upper semi-continuous from X into $\mathbb{R} \cup \{-\infty\}$:

$$\forall x_0 \in X, T(f)(x_0) = \max\{F(x_0, y) + \beta f(y) : y \in \Gamma(x_0)\}.$$

Proposition 2.2.6 (Algorithms to compute the Value function). *Assume* **H1-H2-H3bis-H4**. *We have*
(i) $\forall x_0 \in X, V(x_0) = \lim_{n\to+\infty} T^n(a)(x_0)$, *where* a *is a constant number.*
(ii) $\forall x_0 \in X, V(x_0) = \lim_{n\to+\infty} T^n(h)(x_0)$, *where* h *belongs to* S *and satisfies* $Th \leq h$.

Proof. (i) For any $\mathbf{x} \in \Pi(x_0)$, we have

$$\forall n, T^n(a)(x_0) \geq F(x_0, x_1) + \beta F(x_1, x_2) + ... + \beta^n F(x_n, x_{n+1}) + \beta^{n+1} a,$$

hence $\liminf_n T^n(a)(x_0) \geq u(\mathbf{x})$. This implies $\liminf_n T^n(a)(x_0) \geq V(x_0)$. For every n, there exists $\mathbf{x^n} \in \Pi(x_0)$ such that

$$T^n(a)(x_0) = F(x_0, x_1^n) + \beta F(x_1^n, x_2^n) + ... + \beta^{n-1} F(x_{n-1}^n, x_n^n) + \beta^n a.$$

Without loss of generality we can assume that $\mathbf{x^n} \to \mathbf{x} \in \Pi(x_0)$. From **H3bis-H4**, given $\varepsilon > 0$, there exists T_0 such that

$$\forall N \geq T_0, \forall n \geq N, \sum_{t=N}^{t=n} \beta^t F(x_t, x_{t+1}) < \varepsilon.$$

Fix some $N > T_0$. We then have for $n > N$:

$$T^n(a)(x_0) \leq F(x_0, x_1^n) + \beta F(x_1^n, x_2^n) + ... + \beta^{N-1} F(x_{N-1}^n, x_N^n) + \varepsilon + \beta^n a.$$

Let n converge to $+\infty$. We get

$$\limsup_n T^n(a)(x_0) \leq F(x_0, x_1) + \beta F(x_1, x_2) + ... + \beta^{N-1} F(x_{N-1}, x_N) + \varepsilon.$$

Let $N \to +\infty$. Then

$$\limsup_n T^n(a)(x_0) \leq u(\mathbf{x}) + \varepsilon \leq V(x_0) + \varepsilon.$$

This inequality holds for any $\varepsilon > 0$. Hence $\limsup_n T^n(a)(x_0) \leq V(x_0)$.

(ii) Observe that for any x_0, $\lim T^n(h)(x_0)$ exists in $\mathbb{R} \cup \{-\infty\}$, since T is nondecreasing. By the same argument as above, we obtain a sequence $\mathbf{x} \in \Pi(x_0)$ such that, for every N large enough:

$$\lim_n T^n(h)(x_0) \leq F(x_0, x_1) + \beta F(x_1, x_2) + ... + \beta^{N-1} F(x_{N-1}, x_N) + \limsup_N h(x_N),$$

and hence $\lim_n T^n(h)(x_0) \leq V(x_0)$ since $\limsup_N h(x_N) \leq 0$.
If $V(x_0) = -\infty$, then $\lim_n T^n(h)(x_0) = -\infty = V(x_0)$ from the previous inequality. If $V(x_0) > -\infty$, then write $V(x_0) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$, where $\mathbf{x} \in \Pi'(x_0)$. By induction, we obtain:

$$\forall n, T^n(h)(x_0) \geq \sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1}) + \beta^n h(x^n).$$

Since $h \in S$, we have, by taking the limits, $\lim_n T^n(h)(x_0) \geq V(x_0)$.
Summing up, we have proved that $\lim_n T^n(h)(x_0) = V(x_0)$.

We let to the reader to check that, in addition to **H1-H2-H3bis-H4**, if we assume that $graph(\Gamma)$ is convex and F is concave then Proposition 2.1.9, Proposition 2.1.10 and Proposition 2.1.11 still hold.

Remark 2.2.2. We use in this section the approaches, for unbounded from below utilities, of Stokey and Lucas [14], Le Van and Morhaim [8] when the intertemporal utility is separable and Le Van and Vailakis [9] with recursive utilities. Rincon-Zapatero and Rodriguez-Palmero [12] impose more conditions to obtain that the operator T is a local contraction for some distance on the set of continuous functions $C(X^*)$ where $X^* = X \backslash \{0\}$.

Example 6

Consider the model

$$\max \sum_{t=0}^{+\infty} \beta^t Log(c_t),\ 0 < \beta < 1$$
$$\forall t,\ c_t + k_{t+1} \leq k_t^{\alpha},\ 0 < \alpha < 1$$
$$c_t \geq 0,\ k_t \geq 0$$
$$k_0 \geq 0 \ \text{ is given.}$$

This model is equivalent to

$$\max \sum_{t=0}^{+\infty} \beta^t Log(k_t^{\alpha} - k_{t+1}),\ 0 < \beta \leq 1,\ 0 \leq \alpha < 1$$
$$\forall t,\ 0 \leq k_{t+1} \leq k_t^{\alpha},\ \text{and } k_0 \geq 0 \ \text{ is given.}$$

It is easy to check that **H1, H2, H3bis and H4** are satisfied. Moreover, for any $k_0 > 0$, the set $\Pi'(k_0)$ is non-empty. Indeed, if $k_0 > 0$, then there exists $\hat{k}$ such that $\hat{k} < 1$, $\hat{k} < f(k_0)$ and $\hat{k} < f(\hat{k})$. Hence for any $k_0 > 0$, the sequence $\mathbf{k} = (k_0, \hat{k}, \hat{k}, ...)$ is feasible. Obviously,

$$\sum_{t=0}^{+\infty} \beta^t u(f(k_t) - k_{t+1}) = u(f(k_0) - \hat{k}) + \frac{\beta}{1-\beta} u(f(\hat{k}) - \hat{k}) > -\infty.$$

Since the utility function (which is logarithmic) is strictly concave, and the production function f is convex and increasing, there exists a unique optimal solution $\mathbf{k}$.
We now prove that the Value function V is continuous. We use Proposition 2.2.2 (iv). For that, we show that there exists a continuous function ϕ such that $V(k_0) \geq \phi(k_0), \forall k_0$ and if $\mathbf{k} \in \Pi'(k_0)$,then $\lim_t \beta^t \phi(k_t) = 0$.
Take some $\hat{k} \in]0, 1[$. Define

$$\phi(k_0) = \frac{Log(k_0^\alpha - k_0)}{1-\beta} \text{ if } 0 < k_0 < \hat{k}$$

$$\phi(k_0) = Log(k_0^\alpha - \hat{k}) + \frac{\beta}{1-\beta} Log(\hat{k}^\alpha - \hat{k}), \text{ if } k_0 \geq \hat{k}.$$

Obviously ϕ is continuous. Assume $k_0 \geq \hat{k}$. If $k_0 \geq 1$, then $\hat{k} < 1 = f(1) \leq f(k_0)$. Since $\hat{k} < f(\hat{k})$, the sequence $\mathbf{k} = (k_0, \hat{k}, \hat{k}, ...)$ is feasible. If $\hat{k} \leq k_0 < 1$ then $\hat{k} \leq k_0 \leq f(k_0)$ and again the sequence $\mathbf{k} = (k_0, \hat{k}, \hat{k}, ...)$ is feasible. We have: $V(k_0) \geq \phi(k_0) = \sum \beta^t Log(f(k_t) - k_{t+1})$.
We now prove that, for any $\mathbf{k} \in \Pi'(k_0)$, $\lim_t \beta^t \phi(k_t) = 0$. Since $k_t \leq k_0^{\alpha^t}$ for any t, we have

$$\begin{aligned} -\infty < \sum \beta^t Log(k_t^\alpha - k_{t+1}) &\leq \sum \beta^t Log(k_t^\alpha) \\ &\leq \sum (\alpha\beta)^t Log(k_0) < +\infty \end{aligned}$$

and hence $\lim_t \beta^t Log(k_t) = 0$.
We have two cases: either there exists $0 < \tilde{k} < \hat{k}$ such that $k_t \geq \tilde{k} > 0$ for every t, or there exists a subsequence $k_\nu \to 0$. In the first case, we have either

$$\min_{\tilde{k} \leq k \leq \hat{k}} \phi(k) \leq \phi(k_t) \leq \frac{\alpha Log k_t}{1-\beta} \text{ if } k_t \leq \hat{k}$$

or

$$\phi(\hat{k}) \leq \phi(k_t) \leq \alpha Log k_t + \frac{\beta}{1-\beta} Log(\hat{k}^\alpha - \hat{k}), \text{ if } k_t > \hat{k},$$

and thus $\lim_t \beta^t \phi(k_t) = 0$.
In the second case, we have:

$$\beta^\nu \phi(k_\nu) = \beta^\nu \frac{Log(k_\nu^\alpha - k_\nu)}{1-\beta} = \frac{\beta^\nu}{1-\beta}(Log k_\nu^\alpha - Log(1-k_\nu^{1-\alpha})).$$

Since $k_\nu \to 0$, we have $\lim_\nu \beta^\nu \phi(k_\nu) = 0$.
We have proved that V is continuous.
We now want to compute V and the optimal policy. For that, we will apply Proposition 2.2.6 (ii). Define $h(k) = \frac{\alpha}{1-\alpha\beta} Log(k)$. Since for any $\mathbf{k} \in \Pi(k_0)$ we have $k_t \leq k_0^{\alpha^t}, \forall t$, then $V(k_0) \leq h(k_0)$ for any $k_0 \geq 0$. We can find by direct calculation that

$$Th(k_0) = h(k_0) + Log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} Log(\alpha\beta) < h(k_0).$$

Since for any $\mathbf{k} \in \Pi(k_0)$, we have $k_t \leq k_0^{\alpha^t}$, then:

$$\limsup_t \beta^t h(k_t) \leq \lim_t \frac{\beta^t \alpha^{t+1}}{1-\alpha\beta} Log(k_0) = 0.$$

Hence, $V = \lim_n T^n h$. Tedious computations yield

$$T^n h(k_0) = h(k_0) + \frac{1-\beta^n}{1-\beta}[Log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} Log(\alpha\beta)]$$

and then

$$V(k_0) = h(k_0) + \frac{1}{1-\beta}[Log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} Log(\alpha\beta)].$$

To compute the optimal policy we use the Bellman equation

$$V(k_0) = \max_{0 \leq y \leq f(k_0)} \{Log(k_0^\alpha - y) + \beta V(y)\}.$$

We can easily find the optimal value $k_1 = \alpha\beta k_0^\alpha$.

Example 7: The AK Model

We consider the model

$$\max \sum \beta^t \frac{c_t^\theta}{\theta},\ \theta < 0$$
$$\forall t,\ c_t + I_t \leq A k_t,\ 0 < A$$
$$I_t = k_{t+1} - (1-\delta) k_t$$
$$c_t \geq 0,\ k_t \geq 0,\ I_t \geq 0$$
$$k_0 \geq 0 \ \text{ is given.}$$

We suppose $(1-\delta) < \beta(A + 1 - \delta) < 1$ and $A + 1 - \delta > 1$. This model is equivalent to

$$\max \sum \beta^t \frac{((A+1-\delta)k_t - k_{t+1})^\theta}{\theta},\ \theta < 0$$

$$(1-\delta)k_t \leq k_{t+1} \leq (A+1-\delta)k_t$$

$$k_0 \geq 0 \text{ is given.}$$

The reader can check that the assumptions **H1, H2, H3bis and H4** are satisfied. To prove that the Value function V is continuous we observe that for any $k_0 \geq 0$ we have $V(k_0) \geq \phi(k_0) = \frac{(A-\delta)^\theta}{\theta}\frac{1}{1-\beta}k_0^\theta$. We will show that for any $k > 0$ for any $\mathbf{k} \in \Pi'(k_0)$, $\lim_t \beta^t \phi(k_t) = 0$, or equivalently $\lim_t \beta^t k_t^\theta = 0$. But, for $\mathbf{k} \in \Pi'(k_0)$ we have

$$-\infty < \sum \beta^t \frac{((A+1-\delta)k_t - k_{t+1})^\theta}{\theta} \leq \sum \beta^t \frac{A^\theta {k_t}^\theta}{\theta} < 0.$$

Hence $\lim_t \beta^t k_t^\theta = 0$. Apply Proposition 2.2.2 (iv) to conclude that V is continuous.
The reader can check that the sequence $\mathbf{k}^*$ defined by $k_t^* = \lambda^t k_0$ for every t and where $\lambda = (\frac{1}{\beta(A+1-\delta)})^{\frac{1}{\theta-1}}$ is optimal (see Proposition 2.1.9).

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3. Duality Theory in Infinite Horizon Optimization Models

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3.1 Introduction

In intertemporal resource allocation problems with no terminal date, price systems which characterize efficient or optimal allocations have figured prominently since the pioneering contribution by Malinvaud (1953). The method of duality theory that has been developed to study such problems relies on convex analysis and may be viewed as an extension of the corresponding theory for static or finite horizon allocation problems. The purpose of this survey is to introduce the reader to this method by showing how it has been applied in the literature dealing with optimal intertemporal allocation, when future utilities are discounted, which constitutes only a part (although a significant one) of the class of problems referred to above.

A major accomplishment of this literature is the result that, in a very general framework of capital accumulation (often referred to in the literature as a reduced-form model), optimal programs may be characterized by the existence of dual variables, interpreted as "shadow prices", such that at these prices the given program satisfies the so-called "competitive conditions" and the "transversality condition". The competitive conditions are analogous to those in static or finite horizon optimality problems, and involve myopic (generalized) intertemporal profit maximization. The fundamental difference stems from the infinite-horizon nature of the problem, and is captured by the transversality condition.

The usefulness of this central result may be described as follows. Sufficient conditions (in terms of shadow prices) for a program to be optimal can be used to check whether a candidate program is optimal, if one has a good idea of

[1] Discussion over the years with many persons has influenced my understanding of the subject matter covered in this essay. They include David Cass, Swapan Dasgupta, Ali Khan, Mukul Majumdar, Lionel McKenzie, Kazuo Nishimura, Bezalel Peleg, Debraj Ray and Itzhak Zilcha.

shadow prices that support such a program. This makes duality theory a principal alternative to dynamic programming methods in solving for an optimal program. Necessary conditions (in terms of shadow prices) for a program to be optimal can be used to obtain qualitative properties of an optimal program without necessarily solving for an optimal program.

Even though the theory of optimal growth dates back to the seminal contribution of Ramsey (1928), versions of the "price chracterization result", referred to above, were developed almost forty years later, in the papers of Gale (1967), McFadden (1967) and McKenzie (1968). Following Ramsey's lead, the principal concern of these papers was the theory of undiscounted optimal growth in general capital accumulation models. Subsequently, methods of duality theory were applied to the discounted case by Peleg (1970) and Peleg and Ryder (1972). However, it is only with the contribution of Weitzman (1973) that we have a completely satisfactory price characterization result for the discounted case. The setting for his result is a very general and flexible framework of capital accumulation (described here in Section 3.2), and his approach (combining elements of duality theory and dynamic programming) makes the logic of the result (and the assumptions needed for its validity) entirely transparent. We present the basic characterization result, following his approach, in Section 3.3.

Dual variables have been used very effectively in the literature on optimal intertemporal allocation in obtaining another major result, namely the existence of a non-trivial stationary optimal program, supported by "quasi-stationary" shadow prices. Versions of this result appear in Sutherland (1970) and Peleg and Ryder (1974). But, for the general framework described in Section 3.2, the result was developed later by Flynn (1980) and McKenzie (1982). The approach used in these two papers is to establish the existence of a discounted golden-rule (analogous to a golden-rule in the undiscounted case) by a fixed point argument, and then support this discounted golden-rule by appropriate dual variables. We present this theory in Section 3.4.

The fact that there exists a stationary optimal program with quasi-stationary price support allows one to revisit the basic price characterization result (of Section 3.3), and develop an alternative version of it which helps to identify non-optimal competitive programs in a finite number of periods. The transversality condition is an asymptotic condition, and can never be verified in finite time. It turns out that a convenient period-by-period condition can replace the transversality condition in the price characterization theorems, and so a violation of this condition in any period immediately signals non-optimality. Such a period-by-period condition was first proposed and established by Brock and Majumdar (1988) in the undiscounted case, and the theory for the discounted case was developed subsequently in Dasgupta and Mitra (1988). We present this theory in Section 3.5.

Although the transversality condition is both necessary and sufficient for optimality of competitive programs, there is a fairly wide and interesting class of models in which the competitive conditions alone are sufficient to ensure optimality, and the transversality condition is superfluous. That is, programs which are competitive necessarily satisfy the transversality condition and are therefore optimal. These are models which satisfy a "reachability" property, introduced by Dasgupta and Mitra (1999a). Thus, the competitive and transversality conditions are independent restrictions only in models where the reachability property is violated. This finding is presented in Section 3.6.

A framework of optimal growth that has received considerable attention in the literature is one in which utility is derived from consumption alone (referred to as the "consumption model"). It is useful to view this model as a special case of the general framework described in Section 3.2, and apply the results developed for that framework to this particular case. This displays the flexibility of the reduced-form model, and provides an alternative approach to some of the duality results obtained exclusively for the consumption model by Peleg and Ryder (1972, 1974). We present this material in Section 3.7. Of particular interest is the result that, in the consumption model, the competitive condition can be split up into two conditions, one involving purely consumption decisions and the other involving purely production decisions.

Weitzman (1976) showed (in a continuous time optimal growth model) that along an optimal program, the net national product at each instant of time represents the annuity equivalent of its dynamic social welfare from that time onwards. In Section 3.8, as an application of the results on price characterization of optimality in the consumption model, we revisit his interesting economic interpretation of the Bellman equation of dynamic programing. We provide a discrete time analog of his result, displaying the elementary nature of the argument needed to obtain it.

As already indicated in the opening paragraph of this section, the scope of our survey is deliberately limited. To help the reader see some of the connections with the literature that we do not cover, Section 3.9 contains some bibliographic comments on the various sections.

3.2 A General Intertemporal Allocation Model

The framework is described by a triplet (Ω, u, δ), where Ω, a subset of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, is a *transition possibility set*, $u : \Omega \to \mathbb{R}$ is a *utility function* defined on this set, and δ is the *discount factor* satisfying $0 < \delta < 1$. A typical element of Ω is written as an ordered pair (x, y); this means that if the current state is x, then it is possible to be in the state y in one period.

We will be using the following assumptions:

(A.1) (i) $(0,0) \in \Omega$; (ii) $(0,y) \in \Omega$ *implies* $y = 0$.

(A.2) Ω *is* (i) *closed, and* (ii) *convex.*

(A.3) *There is* ξ *such that* "$(x,y) \in \Omega$ *and* $\|x\| > \xi$" *implies* "$\|y\| < \|x\|$".

(A.4) *If* $(x,y) \in \Omega$ *and* $x' \geq x$, $0 \leq y' \leq y$, then (i) $(x',y') \in \Omega$ *and* (ii) $u(x',y') \geq u(x,y)$.

(A.5) u *is* (i) *upper semicontinuous and* (ii) *concave* on Ω.

(A.6) *There is* $\zeta \in \mathbb{R}$, *such that* $(x, y) \in \Omega$ *implies* $u(x, y) \geq \zeta$.

A *program* from $y \in \mathbb{R}^n_+$ is a sequence $\{y(t)\}_0^\infty$ such that $y(0) = y$, and $(y(t), y(t+1)) \in \Omega$ for $t \geq 0$.

A program $\{y(t)\}_0^\infty$ from $y \in \mathbb{R}^n_+$ is an *optimal program* if

$$\sum_{t=0}^{\infty} \delta^t u(y'(t), y'(t+1)) \leq \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1))$$

for every program $\{y'(t)\}_0^\infty$ from y.

The following "boundedness properties" of our model are well-known.

(R.1) *Under assumptions* (A.3) *and* (A.4) (i),

(i) *If* $(x, y) \in \Omega$, *then* $\|y\| \leq \max[\xi, \|x\|]$.

(ii) *If* $\{y(t)\}_0^\infty$ *is a program from* $y \in \mathbb{R}^n_+$, *then* $\|y(t)\| \leq \max[\xi, \|y\|]$ *for* $t \geq 0$.

The existence of an optimal program in this framework is also a standard result.

(R.2) *Under assumptions* (A.1), (A.2), (A.3), (A.4) (i), (A.5) (i) *and* (A.6), *if* $y \in \mathbb{R}^n_+$, *there exists an optimal program from* y.

Given (R.2), there is an optimal program $\{y^*(t)\}_0^\infty$ from each $y \in \mathbb{R}^n_+$. We define

$$V(y) = \sum_{t=0}^{\infty} \delta^t u(y^*(t), y^*(t+1))$$

V is known as the *value function.* By (A.4), V is non-decreasing, and by (A.2) and (A.5), V is concave.

3.3 Characterization of Optimal Programs in Terms of Dual Variables

The principal results on duality theory in infinite horizon optimization models relate optimal programs with programs which are "supported" by dual variables known as shadow prices. At the given shadow prices, the "supported" program maximizes the generalized profit at each date among all feasible activities (pairs (x, y) in the transition possibility set) and is called a competitive program. These results are analogous to the first and second fundamental theorems of welfare economics in general equilibrium theory.

The infinite horizon entails that an additional condition, known as the transversality condition, is involved in relating competitive to optimal programs.

Results which provide sufficient conditions (in terms of shadow prices) for a program to be optimal are often useful in checking that a candidate program is optimal, if one has a good idea of shadow prices that support such a program. Such "price characterization" results make duality theory a principal alternative to dynamic programming methods in solving for an optimal program. Results which provide necessary conditions (in terms of shadow prices) for a program to be optimal are useful in inferring qualitative properties of an optimal program without necessarily solving for an optimal program. Together, these results can be a powerful tool in the hands of an optimal growth theorist to address a variety of problems.

This section provides these "price characterization" results in the context of the general intertemporal allocation model described in the previous section.

A sequence $\{y(t),\, p(t)\}_0^\infty$ is a *competitive program* from $y \in \mathbb{R}_+^n$ if $\{y(t)\}_0^\infty$ is a program from y, $p(t) \in \mathbb{R}_+^n$ for $t \geq 0$, and for all $t \geq 0$ we have

$$\begin{aligned} & \delta^t u(y(t),\, y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq \;& \delta^t u(x,y) + p(t+1)y - p(t)x \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.1}$$

A competitive program $\{y(t),\, p(t)\}_0^\infty$ from $y \in \mathbb{R}_+^n$ is said to satisfy the *transversality condition* if

$$\lim_{t\to\infty} p(t)\, y(t) = 0 \tag{3.2}$$

3.3.1 When Are Competitive Programs Optimal?

Theorem 3.3.1. *If $\{y(t),\, p(t)\}_0^\infty$ is a competitive program from $y \in \mathbb{R}_+^n$ which satisfies the transversality condition, then $\{y(t)\}_0^\infty$ is an optimal program from y.*

Proof. Let $\{y'(t)\}_0^\infty$ be any program from y. Using (3.1), we have for $t \geq 0$:

$$\begin{aligned} & \delta^t[u(y'(t), y'(t+1)) - u(y(t), y(t+1))] \\ & \leq [p(t+1)y(t+1) - p(t)y(t)] - [p(t+1)y'(t+1) - p(t)y'(t)] \end{aligned} \tag{3.3}$$

Summing (3.3) from $t = 0$ to $t = T$:

$$\begin{aligned} & \sum_{t=0}^{T} \delta^t[u(y'(t), y'(t+1)) - u(y(t), y(t+1))] \\ & \leq [p(T+1)y(T+1) - p(0)y(0)] - [p(T+1)y'(T+1) - p(0)y'(0)] \\ & = p(T+1)y(T+1) - p(T+1)y'(T+1) \\ & \leq p(T+1)y(T+1) \end{aligned} \tag{3.4}$$

Since the quantity $\sum_{t=0}^{T} \delta^t u(y'(t), y'(t+1))$ converges as $T \to \infty$, and so does $\sum_{t=0}^{T} \delta^t u(y(t), y(t+1))$, we can take limits on both sides of (3.4), by using (3.2), and we have:

$$\sum_{t=0}^{\infty} \delta^t u(y'(t), y'(t+1)) - \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1)) \leq 0$$

which proves that $\{y(t)\}_0^\infty$ is an optimal program from y.

Remarks:

(i) It is worth noting that the above result does not depend on the convexity of the transition possibility set or the concavity of the utility function.

(ii) In Theorem 3.3.1, the transversality condition (3.2) can be replaced by

$$\liminf_{t\to\infty} p(t)y(t) = 0. \tag{3.5}$$

(iii) The significance of the transversality condition (3.2) was first noted by Malinvaud (1953) in his study of intertemporal *efficiency*. Since then, it has been extensively used in the study of intertemporal optimality (as well as efficiency). The extremely simple method of proof is a variant of Malinvaud's proof in the study of efficiency; it was effectively introduced in the multisectoral optimality literature most notably by Gale (1967).

(iv) Notice that the convergence of the discounted utility sum is not essential to the *method*. Thus, in general, we can define a program $\{y(t)\}_0^\infty$ from y to be optimal (in Brock's (1970) terminology "weakly-maximal") if

$$\liminf_{T\to\infty} \sum_{t=0}^{T} \delta^t [u(y'(t), y'(t+1)) - u(y(t), y(t+1))] \leq 0$$

for every program $\{y'(t)\}_0^\infty$ from y. Then (3.1) and (3.5) lead to the optimality of $\{y(t)\}_0^\infty$ by the same method. The point to be noted is that, in this general form, *no assumptions* are needed on Ω, u, δ.

3.3.2 When Are Optimal Programs Competitive?

The converse to Theorem 3.3.1 relies heavily on the "convex structure" of the model. The proof we report here follows closely the approach of Weitzman (1973): the interesting features of his technique of proof are (a) the use of an induction argument to obtain the "dual variables", and (b) the combination of the dynamic programming approach exploiting the value function, with the duality approach exploiting the separation theorem. These ideas are formalized in Lemmas 1 and 2 below, which are then used to obtain Theorem 3.3.2, the basic result of this subsection.

Lemma 3.3.1. *Suppose $\{\bar{y}(t)\}_0^\infty$ is an optimal program from $\bar{y} >> 0$. Then there is $p(0) \in \mathbb{R}_+^n$ such that:*

$$V(\bar{y}(0)) - p(0)\bar{y}(0) \geq V(y) - p(0)y \ \text{ for all } y \in \mathbb{R}_+^n. \tag{3.6}$$

Proof. Define the sets A and B as follows.

$A = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : V(y) - V(\bar{y}(0)) \geq \alpha,\ (\bar{y}(0) - y) \geq \beta$ for some $y \in \mathbb{R}_+^n\}$

$B = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : (\alpha, \beta) >> 0\}$

Clearly, A and B are non-empty and convex. Also $A \cap B = \phi$. For if there is $(\alpha, \beta) \in A \cap B$, then there is $y \in \mathbb{R}_+^n$, such that $V(y) > V(\bar{y}(0))$

and $\bar{y}(0) >> y$. By the "free-disposal" assumption (A.4), $\bar{y}(0) >> y$ implies $V(\bar{y}(0)) \geq V(y)$, which contradicts $V(y) > V(\bar{y}(0))$.

By a standard separation theorem (see, for example, Theorem 3.5, p. 35 in Nikaido (1968)) there is $(\mu, \nu) \in \mathbb{R}_+^{n+1}$, with $(\mu, \nu) \neq 0$, such that:

$$\mu\alpha + \nu\beta \leq 0 \qquad \text{for all } (\alpha, \beta) \in A \tag{3.7}$$

Thus, using the definition of A, we have:

$$\mu(V(y) - V(\bar{y}(0)) + \nu(\bar{y}(0) - y) \leq 0 \text{ for all } y \in \mathbb{R}_+^n \tag{3.8}$$

We claim that $\mu \neq 0$. For if $\mu = 0$, then $\nu > 0$ and using (3.8),

$$\nu(\bar{y}(0) - y) \leq 0 \text{ for all } y \in \mathbb{R}_+^n \tag{3.9}$$

But since $\bar{y}(0) = y >> 0$, we can pick $y = \bar{y}(0)/2$ to contradict (3.9). This establishes our claim, so that $\mu > 0$.

Define $p(0) = (\nu/\mu)$, and use (3.8) to get

$$[V(y) - V(\bar{y}(0))] + p(0)(\bar{y}(0) - y) \leq 0 \text{ for all } y \in \mathbb{R}_+^n$$

which, after transposition of terms, is (3.6).

We call $\hat{x} \in \mathbb{R}_+^n$ *sufficient* if there is $\hat{y} \in \mathbb{R}_{++}^n$, such that $(\hat{x}, \hat{y}) \in \Omega$.

Lemma 3.3.2. *Suppose $\{\bar{y}(t)\}_0^\infty$ is an optimal program from $\bar{y}$. Suppose, also, that there is some sufficient vector $\hat{x}$ in $\mathbb{R}_+^n$. If there is some $t \geq 0$, and $p(t) \in \mathbb{R}_+^n$ such that:*

$$\delta^t V(\bar{y}(t)) - p(t)\bar{y}(t) \geq \delta^t V(y) - p(t)y \quad \text{for all } \; y \in \mathbb{R}_+^n \tag{3.10}$$

then there is $p(t+1) \in \mathbb{R}_+^n$ such that:

$$\delta^{t+1} V(\bar{y}(t+1)) - p(t+1)\bar{y}(t+1) \geq \delta^{t+1} V(y) - p(t+1)y \quad \text{for all } y \in \mathbb{R}_+^n \tag{3.11}$$

and:

$$\begin{aligned} &\delta^t u(\bar{y}(t), \bar{y}(t+1)) + p(t+1)\bar{y}(t+1) - p(t)\bar{y}(t) \\ &\geq \delta^t u(x, y) + p(t+1)y - p(t)x \quad \text{for all } (x, y) \in \Omega \end{aligned} \tag{3.12}$$

Proof. Since $\{\bar{y}(t)\}_0^\infty$ is an optimal program, we have $V(\bar{y}(t)) = u(\bar{y}(t), \bar{y}(t+1)) + \delta V(\bar{y}(t+1))$. Also, for all $(x, y) \in \Omega$, we have $V(x) \geq u(x, y) + \delta V(y)$. Using these facts in (3.10), we have

$$\begin{aligned} \theta(t+1) &\equiv \delta^t u(\bar{y}(t), \bar{y}(t+1)) + \delta^{t+1} V(\bar{y}(t+1)) - p(t)\bar{y}(t) \\ &\geq \delta^t u(x, y) + \delta^{t+1} V(y) - p(t)x \quad \text{for all } \; (x, y) \in \Omega \end{aligned}$$

Thus,

$$\theta(t+1) - \delta^t u(x, y) + p(t)x \geq \delta^{t+1} V(y) \quad \text{for all } \; (x, y) \in \Omega \tag{3.13}$$

Define two sets A and B as follows:

$A = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : \alpha \leq \delta^{t+1}V(y') - [\theta_{t+1} - \delta^t u(x,y) + p(t)x]$ and $\beta \leq (y - y')$, for some $(x,y) \in \Omega$, and for some $y' \in \mathbb{R}^n_+\}$

$B = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : (\alpha, \beta) >> 0\}$

Clearly, A and B are non-empty and convex (since u is concave on Ω and V is concave on $\mathbb{R}^n_+$). Also, since V is non-decreasing, we can use (3.13) to infer that $A \cap B = \phi$. Hence, by a standard separation theorem (see, for example, p. 35 in Nikaido (1968)), there is $(\mu, \nu) \in \mathbb{R}^{n+1}_+$, with $(\mu, \nu) \neq 0$, such that:

$$\mu\alpha + \nu\beta \leq 0 \quad \text{for all } (\alpha, \beta) \in A \tag{3.14}$$

Using the definition of A and (3.14), we have

$$\begin{aligned} \mu[\theta_{t+1} - \delta^t u(x,y) + p(t)x] - \nu y \quad &\geq \quad \mu\delta^{t+1}V(y') - \nu y' \\ \text{for all } (x,y) \in \Omega \text{ and all } y' \in \mathbb{R}^n_+. & \end{aligned} \tag{3.15}$$

Put $x = \bar{y}(t)$ and $y = \bar{y}(t+1)$ in (3.15) to get:

$$\mu[\delta^{t+1}V(\bar{y}(t+1))] - \nu\bar{y}(t+1) \geq \mu[\delta^{t+1}V(y')] - \nu y' \quad \text{for all } y' \in \mathbb{R}^n_+ \tag{3.16}$$

Put $y' = \bar{y}(t+1)$ in (3.15) to get:

$$\begin{aligned} & \mu[\delta^t u(\bar{y}(t), \bar{y}(t+1)) - p(t)\bar{y}(t)] + \nu\bar{y}(t+1) \\ \geq \quad & \mu[\delta^t u(x,y) - p(t)x] + \nu y \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.17}$$

We claim now that $\mu \neq 0$. For if $\mu = 0$, then by (3.16), $\nu\bar{y}(t+1) \leq \nu y$ for all $y' \in \mathbb{R}^n_+$, and by (3.17), $\nu\bar{y}(t+1) \geq \nu y$ for all y, such that $(x,y) \in \Omega$ for some x. Thus,

$$\nu\bar{y}(t+1) = \nu y \quad \text{for all } y \text{ such that } (x,y) \in \Omega \text{ for some } x \tag{3.18}$$

Since there is a sufficient vector $\hat{x}$, there is $\hat{y} >> 0$, such that $(\hat{x}, \hat{y}) \in \Omega$. Also, $(\hat{x}, 0) \in \Omega$ by "free-disposal" in Ω. Using these facts in (3.18), we have $0 = \nu 0 = \nu\bar{y}(t+1) = \nu\hat{y}$, so that $\nu = 0$. Thus, we get $(\mu, \nu) = 0$, a contradiction. This establishes our claim, and we have $\mu > 0$.

Defining $p(t+1) = (\nu/\mu)$, we can use (3.16) to get (3.11), and (3.17) to get (3.12), establishing the Lemma.

Theorem 3.3.2. *Suppose $\{\bar{y}(t)\}_0^\infty$ is an optimal program from $\bar{y} >> 0$. Suppose, also, that there is some sufficient vector $\hat{x}$ in $\mathbb{R}^n_+$. Then, there is a sequence $\{p(t)\}_0^\infty$ with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that:*

$$\delta^t V(\bar{y}(t)) - p(t)\bar{y}(t) \geq \delta^t V(y) - p(t)y \quad \text{for all } \ y \in \mathbb{R}^n_+ \tag{3.19}$$

and:

$$\begin{aligned} & \delta^t u(\bar{y}(t), \bar{y}(t+1)) + p(t+1)\bar{y}(t+1) - p(t)\bar{y}(t) \\ & \geq \delta^t u(x,y) + p(t+1)y - p(t)x \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.20}$$

and:

$$\lim_{t \to \infty} p(t)\bar{y}(t) = 0 \tag{3.21}$$

Proof. Using Lemmas 1 and 2, there is a sequence $\{p(t)\}_0^\infty$ with $p(t) \in \mathbb{R}_+^n$ for $t \geq 0$,such that (3.19) and (3.20) hold. Putting $y = 0$ in (3.19), we have

$$\delta^t[V(\bar{y}(t)) - V(0)] \geq p(t)\bar{y}(t) \quad \text{for} \;\; t \geq 0 \tag{3.22}$$

Denoting $\max[\xi, \|\bar{y}\|]$ by $B(\bar{y})$ and defining $z = B(\bar{y})e$, where $e = (1, ..., 1)$ in $\mathbb{R}_+^n$, we have $\bar{y}(t) \leq z$ for $t \geq 0$, and so $V(\bar{y}(t)) \leq V(z)$ for $t \geq 0$. Using this in (3.22),

$$\delta^t[V(z) - V(0)] \geq p(t)\bar{y}(t) \geq 0 \quad \text{for} \;\; t \geq 0.$$

Now, since $\delta^t \to 0$ as $t \to \infty$, we have $p(t)\bar{y}(t) \to 0$ as $t \to \infty$, which establishes (3.21).

Remarks:

(i) Peleg (1970) establishes a version of Theorem 3.3.2 by applying the separation theorem in the space of all bounded infinite sequences (of vectors in $\mathbb{R}^n$), known as ℓ_∞^n. This method is also followed in Peleg and Ryder (1972).

(ii) In the statement of Theorem 3.3.2, the initial stock, $\bar{y}$, is assumed to be strictly positive, and it is also assumed that there is some sufficient vector, $\hat{x}$ in $\mathbb{R}_+^n$. Under these assumptions, we note that we can find $0 < \lambda < 1$, such that $\lambda\hat{x} \leq \bar{y}$. Now, since there is $\hat{y} >> 0$, such that $(\hat{x}, \hat{y}) \in \Omega$, we have $(\lambda\hat{x}, \lambda\hat{y}) \in \Omega$, and by free-disposal, $(\bar{y}, \lambda\hat{y}) \in \Omega$. Since $\lambda\hat{y} >> 0$, we see that $\bar{y}$ itself is a sufficient vector. Thus, under the assumptions of Theorem 3.3.2, we have (a) $\bar{y} >> 0$ and (b) $\bar{y}$ is a sufficient vector. On the other hand, if $\bar{y} >> 0$ and $\bar{y}$ is a sufficient vector, then the assumptions used in Theorem 3.3.2 are obvously satisfied.

(iii) Conditions (3.19) and (3.21) in the above result are not "independent". For a competitive program it can be shown that (3.19) is equivalent to (3.21). That (3.19) implies (3.21) is clear from the proof of Theorem 3.3.2. The converse implication can be derived by following the proof of Theorem 3.3.1.

3.3.3 An Example

Theorem 3.3.1 shows that a competitive program satisfying a transversality condition is optimal, and Theorem 3.3.2 shows that an optimal program is competitive and satisfies a transversality condition. This still does not settle the question of whether the transversality condition is needed in the statement of Theorem 3.3.1 to make it valid. It is logically possible, for example, that a competitive program automatically satisfies the transversality condition and is therefore optimal. (For more on this line of thought, see Section 3.6 below). In this subsection, to settle this issue, we give an example of a framework (which is a special case of the one described in Section 3.2) and a competitive program in that framework (with a uniquely defined associated price sequence) which violates the transversality condition and is not optimal. Thus, in general, Theorem 3.3.1 would be invalid if the transversality condition is dropped from its statement.

The framework is the standard aggregative neoclassical growth model, which is described by (f, w, δ), where f is the production function, satisfying:

$$f(x) = 4x^{1/2} \text{ for all } x \geq 0$$

w is the welfare function satisfying:

$$w(c) = 2c^{1/2} \text{ for all } c \geq 0$$

and δ is the discount factor, satisfying $\delta = 1/2$. To convert this model to the framework analyzed in Section 3.2, we can define the transition possibility set by $\Omega = \{(x, y) \in \mathbb{R}^2_+ : y \leq f(x)\}$, and the utility function by $u(x, y) = w(x - f^{-1}(y))$ for all $(x, y) \in \Omega$. [For more on this kind of conversion, see Section 3.7 below].

Define a sequence $\{k(t)\}$ as follows: $k(0) = 2, k(1) = 4\sqrt{2} - 1$, and for $t \geq 0$,

$$k(t + 2) = f(k(t + 1)) - \frac{[f(k(t)) - k(t + 1)]}{k(t + 1)} \tag{3.23}$$

We first verify that (3.23) does, in fact, uniquely define a sequence. To this end, we claim that, if for some $t \geq 0$, we have $(k(t), k(t+1))$ satisfying $k(t) > 1$ and $f(k(t)) > k(t + 1) > k(t)$, then $k(t + 2)$, defined uniquely by (3.23) satisfies:

$$k(t + 1) > 1 \text{ and } f(k(t + 1)) > k(t + 2) > k(t + 1) \tag{3.24}$$

First, we have $k(t + 1) > k(t) > 1$ by hypothesis, so $k(t + 1) > 1$. Second, we have:

$$0 < \frac{[f(k(t)) - k(t + 1)]}{k(t + 1)} < [f(k(t)) - k(t + 1)]$$

so that (3.23) implies that (i) $k(t+2) > f(k(t+1))-[f(k(t))-k(t+1)] > k(t+1)$, since $k(t + 1) > k(t)$ and f is increasing; and (ii) $k(t + 2) < f(k(t + 1))$. This establishes our claim.

Since $(k(0), k(1)) = (2, 4\sqrt{2} - 1)$ satisfies $k(0) > 1$ and $f(k(0)) = 4\sqrt{2} > 4\sqrt{2} - 1 = k(1) > 2 = k(0)$, we can use (3.24) repeatedly to uniquely define $\{k(t)\}$ by (3.23). Further, for all $t \geq 0$, we must have (3.24) holding along such a sequence.

Note that $\{k(t\}$ is monotonically increasing and bounded above by $\xi = 16$, so it must converge to some $k > 1$. Then, using (3.23) and (3.24), we must have $k = f(k) - \{[f(k) - k]/k\}$ and $f(k) \geq k$ respectively, so that $f(k) - k = \{[f(k) - k]/k\}$, and consequently $f(k) = k$ (since $k \neq 1$). Thus $k = \xi = 16$.

Define $\{x(t), y(t), c(t)\}$ as follows: $x(0) = k(0) = 2, y(0) = 4, c(0) = 2$, and for $t \geq 1$,

$$x(t) = k(t), y(t) = f(x(t - 1)), c(t) = y(t) - x(t) \tag{3.25}$$

Note that, by (3.24), we have $c(t) > 0$ for $t \geq 0$, and by (3.23), we have:

$$w'(c(t)) = \delta f'(x(t))w'(c(t + 1)) \text{ for all } t \geq 0 \tag{3.26}$$

the Ramsey-Euler equations for this framework. It is easy to check now that $\{y(t)\}$ is a program from $y(0) = 4$, and that (using the concavity of f and w, and (3.26)), $\{y(t), p(t)\}$ is a competitive program at the uniquely defined price sequence $\{p(t)\}$ given by:

$$p(t) = \delta^t w'(c(t)) \quad \text{for } t \geq 0 \tag{3.27}$$

By (3.23) and $k(t) > 1$ for $t \geq 0$, we have $c(t+1) < c(t)$ for $t \geq 1$, and clearly $c(1) = 1 < c(0) = 2$. Thus, $c(t) \leq 2$ for $t \geq 0$, so that $u(y(t), y(t+1)) \leq w(2) = 2\sqrt{2}$ for all $t \geq 0$. However, the sequence $\{y'(t)\}$ defined by $y'(t) = 4$ for all $t \geq 0$ is clearly a program from $y'(0) = 4$, with $u(y'(t), y'(t+1)) = w(3) = 2\sqrt{3}$ for all $t \geq 0$. Thus, $\{y(t)\}$ is not an optimal program from $y(0) = 4$.

Since $x(t) = k(t) \to 16$ as $t \to \infty$, there is T such that $k(t) \geq 3$, and so $f'(x(t)) \leq (2/3)$ for all $t \geq T$. Using (3.26) and (3.27), we see that:

$$p(t+1) = p(0) / \prod_{s=0}^{t} f'(x(s))$$

and so $p(t) \to \infty$ as $t \to \infty$. Since $y(t) = f(x(t-1)) \to 16$ as $t \to \infty$, we have $p(t)y(t) \to \infty$ as $t \to \infty$, a violation of the transversality condition.

3.4 Duality Theory for Stationary Optimal Programs

3.4.1 Existence of a Stationary Optimal Stock via Duality Theory

The question of existence of a non-trivial stationary optimal stock has been discussed extensively in the literature. Two treatments of the subject can be found in Sutherland (1970) and Khan and Mitra (1986), who use a purely primal approach, and Flynn (1980) and McKenzie (1982), who use the dual variable approach. As an illustration of the power of duality methods, we will provide an exposition of the topic using the latter approach.

An optimal program $\{y(t)\}_0^\infty$ from $y \in \mathbb{R}_+^n$ is a *stationary optimal program* if $y(t) = y(t+1)$ for $t \geq 0$. A *stationary optimal stock* is an element $y \in \mathbb{R}_+^n$, such that $\{y\}_0^\infty$ is a stationary optimal program. It is *non-trivial* if $u(y, y) > u(0, 0)$.

A *discounted golden-rule stock* is a stock $\hat{y}$ with $(\hat{y}, \hat{y}) \in \Omega$, such that:

$$u(\hat{y}, \hat{y}) \geq u(x, y) \quad \text{for all } (x, y) \in \phi(\hat{y})$$

where $\phi(\hat{y}) = \{(x, y) \in \Omega : \delta y - x \geq \delta\hat{y} - \hat{y}\}$.

A *modified golden-rule* is a pair $(\hat{y}, \hat{p})$ with $(\hat{y}, \hat{y}) \in \Omega$, $\hat{p} \in \mathbb{R}_+^n$ such that for all $(x, y) \in \Omega$,

$$u(\hat{y}, \hat{y}) + \delta\hat{p}\hat{y} - \hat{p}\hat{y} \geq u(x, y) + \delta\hat{p}y - \hat{p}x$$

An economy is $\delta - productive$ if there exists $(a, b) \in \Omega$ such that $\delta b >> a$. It is $\delta u - productive$ if there exists $(a, b) \in \Omega$ such that $\delta b >> a$ and $u(\delta b, b) > u(0, 0)$.

It can be shown, by using the Kakutani fixed point theorem, that there exists a discounted golden rule stock (see Theorem 3.4.1 below). Further, when the economy is $\delta - productive$, any discounted golden rule stock, $\hat{y}$, can be supported by a price vector, $\hat{p}$, so that the pair $(\hat{y}, \hat{p})$ is a modified golden-rule, and $\{\hat{y}\}$ is a stationary optimal program from $\hat{y}$ (Theorem 3.4.2). Finally, when the economy is $\delta u - productive$, then a simple consequence of Theorem 3.4.2 is that there exists a non-trivial stationary optimal stock (Corollary 3.4.1).

Theorem 3.4.1. *There exists a discounted golden-rule stock.*

The proof of Theorem 3.4.1 can be obtained from Khan and Mitra (1986), and with some modifications, from McKenzie (1982).

Theorem 3.4.2. *(i) If the economy is $\delta - productive$, and if $\hat{y}$ is a discounted golden-rule stock, then there is $\hat{p} \in \mathbb{R}^n_+$ such that $(\hat{y}, \hat{p})$ is a modified golden-rule.*

(ii) if $(\hat{y}, \hat{p})$ is a modified golden-rule, then $\{\hat{y}\}$ is a stationary optimal program from $\hat{y}$.

Proof. (i) Define the sets A and B as follows:

$$\begin{aligned} A &= \{(\alpha, \beta) \in \mathbb{R}^{n+1} : \alpha \leq u(x, y) - u(\hat{y}, \hat{y}), \text{ and} \\ \beta &\leq (\delta y - x) - (\delta \hat{y} - \hat{y}) \text{ for some } (x, y) \in \Omega\} \end{aligned}$$

$$B = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : (\alpha, \beta) >> 0\}$$

Note that A and B are non-empty, convex sets in $\mathbb{R}^{n+1}$, and they are disjoint, since $\hat{y}$ is a discounted golden-rule stock. Thus, by a standard separation theorem, there is $(\mu, \nu) \in \mathbb{R}^{n+1}_+$ with $(\mu, \nu) \neq 0$, such that:

$$\mu\alpha + \nu\beta \leq 0 \text{ for all } (\alpha, \beta) \in A$$

This implies that for all $(x, y) \in \Omega$,

$$\mu u(x, y) + \nu(\delta y - x) \leq \mu u(\hat{y}, \hat{y}) + \nu(\delta \hat{y} - \hat{y}) \tag{3.28}$$

We claim that $\mu \neq 0$. For if $\mu = 0$, then $\nu \neq 0$, and (3.28) implies that:

$$\nu(\delta y - x) \leq \nu(\delta \hat{y} - \hat{y}) \text{ for all } (x, y) \in \Omega \tag{3.29}$$

Since the economy is $\delta - productive$, there is $(a, b) \in \Omega$ satisfying $\delta b >> a$. Using this in (3.29), we get $0 < \nu(\delta b - a) \leq \nu(\delta \hat{y} - \hat{y}) \leq 0$, a contradiction. Thus, $\mu > 0$, and defining $\hat{p} = (\nu/\mu)$, we see from (3.28) that $(\hat{y}, \hat{p})$ is a modified golden-rule.

(ii) Define a sequence $\{p(t)\}$ by:

$$p(t) = \delta^t \hat{p} \quad \text{for } t \geq 0$$

Then, using the definition of a modified golden-rule it is easy to check that for all $t \geq 0$, we have:

$$\begin{aligned} & \delta^t u(x,y) + p(t+1)y - p(t)x \\ \leq \quad & \delta^t u(\hat{y},\hat{y}) + p(t+1)\hat{y} - p(t)\hat{y} \quad \text{ for all } (x,y) \in \Omega \end{aligned} \tag{3.30}$$

Further, since $\delta \in (0,1)$, we have:

$$\lim_{t \to \infty} p(t)\hat{y} = 0 \tag{3.31}$$

Thus, by Theorem 3.3.1, $\{\hat{y}\}$ is a stationary optimal program from $\hat{y}$.

We now note the basic result on the existence of a non-trivial stationary optimal stock as a simple consequence of the above results.

Corollary 3.4.1. *If the economy is δu − productive, then there exists a non-trivial stationary optimal stock.*

Proof. Using Theorem 3.4.1, there is a discounted golden-rule stock, $\hat{y}$. Since the economy is δ − *productive*, Theorem 3.4.2 can be applied to infer that $\hat{y}$ is a stationary optimal stock. Finally, since the economy is δu − *productive*, we can infer that $\hat{y}$ is a non-trivial stationary optimal stock, by definition of a discounted golden rule.

3.4.2 Quasi-Stationary Price Support for Stationary Optimal Programs

Using Theorems 3.4.1 and 3.4.2, we see that there always exists a stationary optimal program $\{\hat{y}\}$, which is supported (in the sense of (3.30)) by a *quasi-stationary* price sequence; that is, by a price sequence of the form $p(t) = \delta^t \hat{p}$ for $t \geq 0$. It turns out that any stationary optimal program $\{\hat{y}\}$ can be supported by a quasi-stationary price sequence, provided $(\hat{y},\hat{y})$ is in the interior of Ω. That is, compared to Theorem 3.3.2, one can choose the supporting price sequence from a more restricted set when the optimal program happens to be stationary. Our exposition of this result follows Sutherland (1967) and McKenzie (1986).

Theorem 3.4.3. *Suppose $\{\hat{y}\}$ is a stationary optimal program from $\hat{y}$, and $(\hat{y},\hat{y}) \in int\Omega$. Then, there is $\hat{p}$ such that:*

(i) $(\hat{y},\hat{p})$ is a modified golden-rule, and

(ii) defining $\hat{p}(t) = \delta^t \hat{p}$ for $t \geq 0, \{\hat{y},\hat{p}(t)\}$ is a competitive program from $\hat{y}$, which satisfies the transversality condition.

Proof. Using Theorem 3.3.2, we know that there is a sequence $\{p(t)\}$, with $p(t) \in \mathbb{R}^n_+$ such that $\{\hat{y}, p(t)\}$ is a competitive program from $\hat{y}$. Then, for each $t \geq 0$, we have:

$$\begin{aligned} & \delta^t u(x,y) + p(t+1)y - p(t)x \\ \leq \; & \delta^t u(\hat{y},\hat{y}) + p(t+1)\hat{y} - p(t)\hat{y} \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.32}$$

Denoting $p(t)/\delta^t$ by $q(t)$ for each $t \geq 0$, we have:

$$\begin{aligned} & u(x,y) + \delta q(t+1)y - q(t)x \\ \leq \; & u(\hat{y},\hat{y}) + \delta q(t+1)\hat{y} - q(t)\hat{y} \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.33}$$

Since $(\hat{y},\hat{y}) \in int\Omega$, we have $B > 0$, such that $||q(t)|| \leq B$ for all $t \geq 0$.

Averaging the first $(T+1)$ inequalities in (3.33) gives:

$$u(x,y) - u(\hat{y},\hat{y}) \leqq \delta Q(T)(\hat{y} - y) - P(T)(\hat{y} - x) \;\; \text{for all } (x,y) \in \Omega \tag{3.34}$$

where

$$P(T) = \frac{1}{T+1}(q(0) + q(1) + ... + q(T))$$

and

$$\begin{aligned} Q(T) &= \frac{1}{T+1}(q(1) + ... + q(T+1)) \\ &= P(T) + \frac{1}{T+1}(q(T+1) - q(0)) \end{aligned}$$

Clearly, $||P(T)|| \leq B$ for all $T \geq 0$; so, there is a subsequence $\{T_i\}$, $i = 1, 2, ...$, such that $P(T_i) \to \hat{p} \geqq 0$. Then $Q(T_i)$ also converges to $\hat{p}$, and (3.34) yields:

$$u(x,y) - u(\hat{y},\hat{y}) \leqq \delta\hat{p}(\hat{y} - y) - \hat{p}(\hat{y} - x) \;\; \text{for all } (x,y) \in \Omega \tag{3.35}$$

Thus $(\hat{y},\hat{p})$ is a modified golden-rule, establishing (i). The result in (ii) follows by using the proof of (ii) in Theorem 3.4.2.

Remarks:

(i) Suppose $(\hat{y},\hat{y}) \in int\Omega$; then, $\{\hat{y}\}$ is a stationary optimal program if and only if there is $\hat{p} \in \mathbb{R}^n_+$ such that $(\hat{y},\hat{p})$ is a modified golden-rule. This follows from Theorem 3.4.2 (ii) and Theorem 3.4.3 (i).

(ii) Suppose $(\hat{y},\hat{y}) \in int\Omega$, and $\{\hat{y}\}$ is a stationary optimal program. Then, from Theorem 3.4.3 (i), it follows that $\hat{y}$ is also a discounted golden-rule stock.

3.5 Replacing the Transversality Condition by a Period-by-Period Condition

The transversality condition used in the price characterization results of optimality (Theorems 3.3.1 and 3.3.2 of Section 3.3) is an asymptotic condition. It cannot be verified in a finite number of periods, however large the number of periods might be. It is, therefore, of some interest to investigate whether the transversality condition can be replaced in such characterization results by a condition which might convey some information about optimality in a finite number of periods (where the finite number of periods can be arbitrarily "large"). It turns out that, for the class of stationary models we are considering, there is a convenient period-by-period condition which can replace the transversality condition in the price characterization theorems. Our exposition of this result follows Dasgupta and Mitra (1988).

To describe the results of this section, it is convenient to adopt the following convention. If $\{y(t), p(t)\}$ is a competitive program, then we denote the *current value price sequence* associated with it by $\{q(t)\}$, where $q(t) = p(t)/\delta^t$ for $t \geq 0$. If $\{\hat{y}, \hat{q}\}$ is a modified golden-rule, then the *present value price sequence* associated with it is denoted by $\{\hat{p}(t)\}$, where $\hat{p}(t) = \delta^t \hat{q}$ for $t \geq 0$.

If $\{y(t)\}$ is an optimal program, and $(\hat{y}, \hat{q})$ is a modified golden-rule, then Theorem 3.3.2 can be used to show the existence of a price sequence $\{p(t)\}$ such that $\{y(t), p(t)\}$ is a competitive program, and the following inequality holds:

$$(q(t) - \hat{q})(y(t) - \hat{y}) \leq 0 \quad \text{for all } t \geq 0 \tag{3.36}$$

(see Theorem 3.5.1 below). This raises the following question: if $(\hat{y}, \hat{q})$ is a modified golden-rule, and $\{y(t), p(t)\}$ is a competitive program, such that the period-by-period condition (3.36) is satisfied, then is $\{y(t)\}$ an optimal program ? If the stock $\hat{y}$ is "proportionately expansible", then the answer is in the affirmative (see Theorem 3.5.2 below). The results are useful in identifying *non-optimal* competitive programs. That is, if $\{y(t), p(t)\}$ is a competitive program, which is *not* optimal, then it must violate (3.36) for some period. Further, if $\{y(t), p(t)\}$ is a competitive program, for which $\{p(t)\}$ is the unique associated price sequence, and it violates (3.36) for some period t, then it can be pronounced to be non-optimal. Note that this would not be possible by using Theorem 3.3.2.

Theorem 3.5.1. *Suppose there exists a sufficient vector. Let $\{y(t)\}$ be an optimal program from $\bar{y} >> 0$, and let $(\hat{y}, \hat{q})$ be a modified golden-rule. Then, there is a price sequence $\{p(t)\}$, with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that $\{y(t), p(t)\}$ is a competitive program, and:*

$$(q(t) - \hat{q})(y(t) - \hat{y}) \leq 0 \quad \text{for all } t \geq 0$$

Proof. By Theorem 3.3.2, there is a price sequence $\{p(t)\}$, with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that $\{y(t), p(t)\}$ is a competitive program, and:

$$V(y(t)) - q(t)y(t) \geq V(\hat{y}) - q(t)\hat{y} \quad \text{for all } t \geq 0 \tag{3.37}$$

Since $\{\hat{y}, \hat{q}\}$ is a modified golden-rule, $\{\hat{y}, \hat{p}(t)\}$ is a competitive program, and $\hat{p}(t)\hat{y} = \delta^t \hat{q}\hat{y} \to 0$ as $t \to \infty$. Thus, using remark (iii) following Theorem 3.3.2, we have:

$$V(\hat{y}) - \hat{q}\hat{y} \geq V(y(t)) - \hat{q}y(t) \text{ for all } t \geq 0 \tag{3.38}$$

Adding (3.37) and (3.38) and transposing terms yields the desired result.

For the converse result, we first establish some properties that hold for competitive programs (Lemma 3.5.1), and then derive Theorem 3.5.2 from it.

Lemma 3.5.1. *Suppose* $\{y(t), p(t)\}$ *is a competitive program, and* $(\hat{y}, \hat{q})$ *is a modified golden-rule. Then :*

$$(p(t+1) - \hat{p}(t+1))(y(t+1) - \hat{y}) \geq (p(t) - \hat{p}(t))(y(t) - \hat{y}) \quad \text{for all } t \geq 0 \quad (3.39)$$

Further, if (3.36) holds, then:

$$[(p(t+1)-\hat{p}(t+1))(y(t+1)-\hat{y})-(p(t)-\hat{p}(t))(y(t)-\hat{y})] \to 0 \quad \text{as } t \to \infty \quad (3.40)$$

Proof. Since $\{y(t), p(t)\}$ is competitive, we have:

$$\begin{aligned} & \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq \; & \delta^t u(\hat{y}, \hat{y}) + p(t+1)\hat{y} - p(t)\hat{y} \quad \text{for all } t \geq 0 \end{aligned} \quad (3.41)$$

Since $\{\hat{y}, \hat{p}(t)\}$ is competitive, we have:

$$\begin{aligned} & \delta^t u(\hat{y}, \hat{y}) + \hat{p}(t+1)\hat{y} - \hat{p}(t)\hat{y} \\ \geq \; & \delta^t u(y(t), y(t+1)) + \hat{p}(t+1)y(t+1) - \hat{p}(t)y(t) \quad \text{for all } t \geq 0 \end{aligned} \quad (3.42)$$

Adding (3.41) and (3.42) and transposing terms yields (3.39).

Denoting $(p(t) - \hat{p}(t))(y(t) - \hat{y})$ by $\mu(t)$ for $t \geq 0$, we see (from (3.39)) that $\{\mu(t)\}$ is a monotonically non-decreasing sequence. If (3.36) holds, this sequence is bounded above by 0. So, $\mu(t)$ converges as $t \to \infty$. Clearly, this implies that (3.40) must hold.

A stock $y \in \mathbb{R}^n_+$ is called *expansible* if there is $y' >> y$, such that $(y, y') \in \Omega$. It is called *proportionately expansible* if there is $\lambda > 1$ such that $(y, \lambda y) \in \Omega$. Clearly, if y is expansible, it is proportionately expansible. Also, note that if $(y, y) \in int\Omega$, then y is expansible.

Theorem 3.5.2. *Suppose* $(\hat{y}, \hat{q})$ *is a modified golden-rule and* $\hat{y}$ *is proportionately expansible. If* $\{y(t), p(t)\}$ *is a competitive program from* $\bar{y}$*, which satisfies (3.36), then:*

(i) $p(t)\hat{y} \to 0$ *as* $t \to \infty$*, and*

(ii) $\{y(t)\}$ *is an optimal program from* $\bar{y}$*.*

Proof. Since $\{y(t), p(t)\}$ is competitive, we have:

$$\begin{aligned} & \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq \; & \delta^t u(\hat{y}, \lambda\hat{y}) + p(t+1)\lambda\hat{y} - p(t)\hat{y} \quad \text{for all } t \geq 0 \end{aligned} \quad (3.43)$$

Transposing terms, one gets:

$$\delta^t[u(y(t), y(t+1)) - u(\hat{y}, \lambda\hat{y})] + p(t+1)(y(t+1) - \hat{y}) - p(t)(y(t) - \hat{y})$$

$$\geq p(t+1)(\lambda-1)\hat{y} \tag{3.44}$$

Denoting $(p(t)-\hat{p}(t))(y(t)-\hat{y})$ by $\mu(t)$ for $t \geq 0$, we can write:

$$p(t)(y(t)-\hat{y}) = \hat{p}(t)(y(t)-\hat{y}) + \mu(t) \quad \text{for all } t \geq 0 \tag{3.45}$$

Using (3.45) in (3.44), we obtain:

$$\delta^t[u(y(t),y(t+1)) - u(\hat{y},\lambda\hat{y})] + \hat{p}(t+1)(y(t+1)-\hat{y}) - \hat{p}(t)(y(t)-\hat{y})$$

$$+\mu(t+1) - \mu(t) \geq p(t+1)(\lambda-1)\hat{y} \tag{3.46}$$

Denoting $\max\{\xi, ||\bar{y}||\}$ by B, we have $y(t) \leq Be$, where $e = (1,1,...,1)$ in $\mathbb{R}^n$. Thus, $u(y(t),y(t+1)) \leq u(Be,0)$ for all $t \geq 0$. Then, using Lemma 3.5.1, we note that all the terms on the left hand side of (3.46) converge to zero as $t \to \infty$. This establishes (i), since $\lambda > 1$.

By definition of $\mu(t)$ and (3.36), we have:

$$\begin{aligned} p(t)(y(t)-\hat{y}) &= \hat{p}(t)(y(t)-\hat{y}) + \mu(t) \\ &\leq \hat{p}(t)(y(t)-\hat{y}) \\ &\leq \hat{p}(t)y(t) \end{aligned}$$

Thus, we get:

$$p(t)y(t) \leq p(t)\hat{y} + \hat{p}(t)y(t) \tag{3.47}$$

Since $\|y(t)\| \leq \max[\xi, \|\bar{y}\|]$ *for* $t \geq 0$, and $\delta \in (0,1)$, we have $\hat{p}(t)y(t) \to 0$ as $t \to \infty$. Also, $p(t)\hat{y} \to 0$ as $t \to \infty$ by (i). Thus, by (3.47), we must have $p(t)y(t) \to 0$ as $t \to \infty$. By Theorem 3.3.1, $\{y(t), p(t)\}$ is optimal from $\bar{y}$.

3.6 Are Competitive Programs Optimal?

We have seen that in general an infinite horizon competitive program is not optimal (see the example in Section 3.3), and so Theorem 3.3.1 would be invalid if the transversality condition is dropped from its statement. However, this still leaves open the possibility that for some classes of models, the phenomenon observed in the example does not occur, and all competitive programs are optimal. The approach to identify such models has been to specify a class of transition possibility sets and utility functions such that the (myopic) competitive condition itself restricts the rate at which accumulation of stocks can take place; such a class can be conveniently described by some form of a "reachability" condition. This topic has been investigated by Kurz and Starrett (1970), and Dasgupta and Mitra (1999a,b), among others. We base our discussion here on Dasgupta and Mitra (1999a).

Reachability Condition:

There is an expansible stock $\tilde{y}$ such that, given any competitive program $\{y(t), p(t)\}$, there is a program $\{y^0(t)\}$ from $\tilde{y}$ and a positive integer R such that $y^0_R \geq y_R$.

This condition says that, given a competitive program (from an arbitrary initial stock) it is possible, through pure accumulation if need be, to reach the stocks along the given competitive program at some far enough future date, starting from the expansible stock $\tilde{y}$.

Theorem 3.6.1. *Suppose the Reachability Condition is satisfied. If $\{y(t), p(t)\}$ is a competitive program from $\bar{y} \in \mathbb{R}^n_+$, then $\{y(t)\}$ is an optimal program from $\bar{y}$.*

Proof. Since $\tilde{y}$ is expansible, there is $\tilde{z} >> \tilde{y}$ such that $(\tilde{y}, \tilde{z}) \in \Omega$. Denote $(\tilde{z} - \tilde{y})$ by k; then $k >> 0$. Using the competitive condition, we get, for all $t \geq 0$,

$$\begin{aligned} & \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq \; & \delta^t u(\tilde{y}, \tilde{z}) + p(t+1)\tilde{z} - p(t)\tilde{y} \\ = \; & \delta^t u(\tilde{y}, \tilde{z}) + p(t+1)\tilde{y} - p(t)\tilde{y} + p(t+1)k \end{aligned} \tag{3.48}$$

Now consider any $T \geq 2$. From (3.48), we have:

$$\begin{aligned} & \sum_{t=0}^{T-1} \delta^t u(y(t),\, y(t+1)) + p(T)y(T) - p(0)y(0) \\ \geq \; & \sum_{t=0}^{T-1} \delta^t u(\tilde{y}, \tilde{z}) + \sum_{0}^{T-1} p(t+1)k + p(T)\tilde{y} - p(0)\tilde{y} \end{aligned} \tag{3.49}$$

The sequence $\{y''(t), p''(t)\}$ defined by $(y''(t), p''(t)) = (y(T+t), p(T+t))$ for $t \geq 0$ is clearly a competitive program from $y(T)$. By the reachability condition, there is a program $\{y^0(t)\}$ from $\tilde{y}$, and a positive integer R, such that $y^0(R) \geq y''(R) = y(T+R)$. Defining $\{y'(t)\}$ by $y'(t) = \tilde{y}$ for $t = 0, ..., T$, and $y'(t) = y^0(t-T)$ for $t > T$, we see that $\{y'(t)\}$ is a program from $\tilde{y}$, and:

$$y'(T) = \tilde{y} \text{ and } y'(T+R) = y^0(R) \geq y''(R) = y(T+R) \tag{3.50}$$

Applying the competitive condition to $(y'(t),\, y'(t+1)) \in \Omega$ for each $t \geq T$, we have:

$$\begin{aligned} & \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq \; & \delta^t u(y'(t), y'(t+1)) + p(t+1)y'(t+1) - p(t)y'(t) \end{aligned} \tag{3.51}$$

From (3.51), we have:

$$\sum_{t=T}^{T+R-1} \delta^t u(y(t), y(t+1)) + p(T+R)y(T+R) - p(T)y(T)$$
$$\geq \sum_{t=T}^{T+R-1} \delta^t u(y'(t), y'(t+1)) + p(T+R)y'(T+R) - p(T)y'(T) \quad (3.52)$$

From (3.49) and (3.52), we have:

$$\sum_{t=0}^{T+R-1} \delta^t u(y(t), x(t+1)) + p(T+R)y(T+R) - p(0)y(0)$$
$$\geq \sum_{t=0}^{T-1} \delta^t u(\tilde{y}, \tilde{z}) + \sum_{t=T}^{T+R-1} \delta^t u(y'(t), y'(t+1)) + \sum_{t=0}^{T-1} p(t+1)k$$
$$+p(T+R)y'(T+R) - p(T)y'(T) + p(T)\tilde{y} - p(0)\tilde{y} \quad (3.53)$$

Since $p(t) \geq 0$, from (3.50) and (3.53), we have:

$$\sum_{t=0}^{T+R-1} \delta^t u(y(t), y(t+1)) - \sum_{t=0}^{T-1} \delta^t u(\tilde{y}, \tilde{z})$$
$$- \sum_{t=T}^{T+R-1} \delta^t u(y'(t), y'(t+1)) + p(0)(\tilde{y} - y(0))$$
$$\geq \sum_{0}^{T-1} p(t+1)k \quad (3.54)$$

Denoting $\max\{\xi, ||\bar{y}||\}$ by B, we have $y(t) \leq Be$, where $e = (1, 1, ..., 1)$ in $\mathbb{R}^n$. Denoting $\max\{\xi, ||\tilde{y}||\}$ by B', we have $y'(t) \leq B'e$, where $e = (1, 1, ..., 1)$ in $\mathbb{R}^n$. Thus, $u(y(t), y(t+1)) \leq u(Be, 0)$, and $u(y'(t), y'(t+1)) \leq u(B'e, 0)$for all $t \geq 0$. Then, using (A.6) and $\delta \in (0, 1)$, we see that the left hand side of (3.54) is uniformly bounded above regardless of the value of T. Since $k >> 0$ and $p(t) \geq 0$ for all t, it follows that $\sum_0^\infty p(t) < \infty$, and so $p(t) \to 0$ as $t \to \infty$. Since $\|y(t)\| \leq \max[\xi, \|\bar{y}\|]$ *for* $t \geq 0$, we have $\lim_{t\to\infty} p(t)y(t) = 0$ and so, by Theorem 3.3.1, the program $\{y(t)\}$ is optimal from $\bar{y}$.

Given the abstract nature of the reachability condition, a simple multi-sectoral model in which it can be directly verified would be helpful. We describe the production side of such a model by an $n \times n$ non-negative matrix $A = (a_{ij})$, where $i = 1, ..., n$ and $j = 1, ..., n$, and a strictly positive vector $b = (b_1, ..., b_n) >> 0$. Here, a_{ij} and b_j are respectively the amounts of the $i-$th good and labor which are required per unit output of the $j-$th good. The total amount of labor available for production is stationary and is normalized to 1. For each $j = 1, ..., n$, it is assumed that there is some $i = 1, ..., n$ such that $a_{ij} > 0$. Thus, each production process requires a positive amount of labor as well as a positive amount of some produced factor. Further, it is assumed that

A is productive; that is, there is some $\tilde{y} >> 0$ such that $\tilde{y} >> A\tilde{y}$ and $b\tilde{y} \leq 1$. This essentially excludes the economically uninteresting case of a production system which is unable to sustain some positive consumption levels for all of the desired goods. The transition possibility set for this economy is:

$$\Omega = \{(x, y) \in \mathbb{R}^{2n}_{+} : Ay \leq x \text{ and } by \leq 1\}$$

Welfare is derived from consumption, as given by a function $w : \mathbb{R}^n_+ \to \mathbb{R}$, which is continuous, concave and monotone on $\mathbb{R}^n_+$. The (reduced form) utility function is then defined by:

$$u(x, y) = w(x - Ay) \text{ for all } (x, y) \in \Omega$$

Consider any program $\{y(t)\}$ from $\bar{y} \in \mathbb{R}^n_+$. Denoting $(1/\min_j b_j)$ by B, we have $y(t) \leq Be$ for all $t \geq 1$, where $e = (1, ..., 1) \in \mathbb{R}^n$. Since A is productive, we have $A^t \to 0$ as $t \to \infty$, so we can find a positive integer $R \geq 2$, such that:

$$A^R Be << \tilde{y}$$

Now, define the sequence $\{y'(t)\}$ as follows:

$$\left.\begin{array}{ll} y'(0) = \tilde{y} & \\ y'(t) = A^{R-t} y(R) & \text{for } 1 \leq t \leq R - 1 \\ y'(t) = y(t) & \text{for } t \geq R \end{array}\right\}$$

It can be checked (see Dasgupta and Mitra (1999a) for the details) that $\{y'(t)\}$ is a program from $\tilde{y}$. Since $y'(R) = y(R)$, the reachability condition is satisfied.

3.7 Duality Theory in the Consumption Model

A model of optimal growth that has received considerable attention in the literature is one in which utility is derived from consumption alone (referred to as the "consumption model"). In this section we describe the multisectoral version of this model, show how it can be viewed as a special case of the general framework described in Section 3.2, and apply the results developed for that framework to this particular case. In terms of duality theory, the principal difference is that the competitive condition can be split up into two conditions, one involving consumption decisions and the other involving production decisions (see (3.56) and (3.57) below). Our exposition follows Dasgupta and Mitra (1990).

3.7.1 The Model

Consider a framework described by a triplet (Ω, w, δ), where Ω, a subset of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, is the *technology set*, $w : \mathbb{R}^n_+ \to \mathbb{R}$ is the period *welfare function*, and

δ is the *discount factor* satisfying $0 < \delta < 1$. A typical element of Ω is written as an ordered pair (x, y), where x represents the inputs of the n goods, and y the outputs producible with inputs x.

We will need the following assumptions:

(B.1) (i) $(0,0) \in \Omega$; (ii) $(0, y) \in \Omega$ *implies* $y = 0$.

(B.2) Ω *is* (i) *closed, and* (ii) *convex.*

(B.3) *There is* ξ *such that* "$(x, y) \in \Omega$ *and* $\|x\| > \xi$" *implies* "$\|y\| < \|x\|$".

(B.4) *If* $(x, y) \in \Omega$ *and* $x' \geq x$, $0 \leq y' \leq y$, *then* $(x', y') \in \Omega$.

(B.5) w *is* (i) *continuous, and* (ii) *concave.*

(B.6) *If* c, c' *are in* $\mathbb{R}^n_+$, *then* (i) $c' \geq c$ *implies* $w(c') \geq w(c)$, *and* (ii) $c' >> c$ *implies* $w(c') > w(c)$.

A *plan* from $y \in \mathbb{R}^n_+$ is a sequence $\{x(t),\, y(t)\}_0^\infty$ such that

$$y(0) = y;\ 0 \leq x(t) \leq y(t) \text{ and } (x(t), y(t+1)) \in \Omega \text{ for } t \geq 0$$

Associated with a plan $\{x(t), y(t)\}_0^\infty$ from y is a *consumption sequence* $\{c(t)\}_0^\infty$ defined by

$$c(t) = y(t) - x(t) \quad \text{for} \quad t \geq 0$$

A plan $\{\bar{x}(t),\, \bar{y}(t)\}_0^\infty$ from y is an *optimal plan* if

$$\sum_0^\infty \delta^t w(\bar{c}(t)) \geq \sum_0^\infty \delta^t w(c(t)) \tag{3.55}$$

for every plan $\{x(t),\, y(t)\}_0^\infty$ from y.

An optimal plan $\{x(t), y(t)\}_0^\infty$ from y is a *stationary optimal plan* if $(x(t), y(t)) = (x(t+1), y(t+1))$ for $t \geq 0$. In this case we refer to a stationary optimal plan as $\{x, y\}_0^\infty$ with obvious interpretation, and to its associated stationary consumption sequence as $\{c\}_0^\infty$, where $c = y - x$. A *stationary optimal output* is an element $y \in \mathbb{R}^n_+$ such that there is a stationary optimal plan from y. It is *non-trivial* if $w(c) > w(0)$.

A sequence $\{x(t), y(t),\, p(t)\}_0^\infty$ is a *competitive plan* from y if $\{x(t),\, y(t)\}_0^\infty$ is a plan from y, $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, and for $t \geq 0$,

$$\delta^t w(c(t)) - p(t)c(t) \geq \delta^t w(c) - p(t)c \quad \text{for all} \quad c \in \mathbb{R}^n_+ \tag{3.56}$$

and

$$p(t+1)y(t+1) - p(t)x(t) \geq p(t+1)y - p(t)x \quad \text{for all} \quad (x, y) \in \Omega \tag{3.57}$$

A *modified golden-rule equilibrium* is a triple $(\hat{x}, \hat{y}, \hat{p})$ with $(\hat{x}, \hat{y}) \in \Omega$, $\hat{p} \in \mathbb{R}^n_+$, such that denoting $(\hat{y} - \hat{x})$ by $\hat{c}$, we have

(i) $\hat{c} \geq 0$

(ii) $w(\hat{c}) - \hat{p}\hat{c} \geq w(c) - \hat{p}c$ for all c in $\mathbb{R}^n_+$

(iii) $\hat{p}(\delta\hat{y} - \hat{x}) \geq \hat{p}(\delta y - x)$ for all $(x, y) \in \Omega$

3.7.2 Conversion to the Format of the General Model

Our objective, in this subsection, is to show that the consumption model can be viewed as a particular case of the general framework of Section 3.2.

To this end, we define a *feasible input correspondence*, $g : \Omega \to \mathbb{R}^n_+$ by

$$g(a, b) = \{x : (x, b) \in \Omega \text{ and } x \leq a\}$$

Note that for each $(a, b) \in \Omega$, $a \in g(a, b)$ so g is non-empty valued. Also, for each $(a, b) \in \Omega$, $g(a, b)$ is a bounded set (by definition) and a closed set, by (B.2).

Next, we define a *utility function*, $u : \Omega \to \mathbb{R}$ by

$$u(a, b) = Max\,\{w(a - x) : x \in g(a, b)\}$$

Note that for each $(a, b) \in \Omega$, $g(a, b)$ is non-empty, compact, and w is continuous. Thus, defining $h(a, b) = \{\bar{x} : \bar{x} \in g(a, b)$, and $w(a - \bar{x}) \geq w(a - x)$ for all $x \in g(a, b)\}$, we note that h is a non-empty valued correspondence on Ω, and $u(a, b)[\equiv w(a - \bar{x})$ for $\bar{x} \in h(a, b)]$ is well-defined on Ω. It can now be shown that, given (B.1) - (B.6), (Ω, u) satisfies (A.1) - (A.6) of Section 3.2 [see Dasgupta and Mitra (1990) for the details].

Next, we want to consider plans in terms of the general framework of Section 3.2. Note that $\{x(t), y(t)\}_0^\infty$ is a plan from y if and ony if $\{y(t)\}_0^\infty$ is a program from y, and $x(t) \in g(y(t), y(t + 1))$ for $t \geq 0$. Furthermore, if $\{\bar{x}(t), \bar{y}(t)\}_0^\infty$ is an optimal plan from y, then clearly $\bar{x}(t) \in h(\bar{y}(t),\ \bar{y}(t + 1))$ and so $u(\bar{y}(t),$ $\bar{y}(t + 1)) = w(\bar{c}(t))$ for $t \geq 0$. Also, if $\{x(t), y(t)\}_0^\infty$ is a plan from y, then $w(c(t)) = w(y(t) - x(t)) \leq u(y(t), y(t + 1))$. Using these facts, the inequality in (3.55) can be rewritten as:

$$\sum_0^\infty \delta^t u(\bar{y}(t), \tilde{y}(t + 1)) \geq \sum_0^\infty \delta^t u(y(t),\ y(t + 1))$$

for every plan $\{x(t),\ y(t)\}_0^\infty$ from y. In other words, $\{\bar{y}(t)\}_0^\infty$ is an optimal program from y. Conversely, if $\{\bar{y}(t)\}_0^\infty$ is an optimal program from y, then defining $\bar{x}(t) \in h(\bar{y}(t),\ \bar{y}(t + 1))$ for $t \geq 0$, $\{\bar{x}(t),\ \bar{y}(t)\}_0^\infty$ is clearly an optimal plan from y.

3.7.3 Characterization of Optimal Plans in Terms of Dual Variables

An optimal plan can be characterized as a competitive plan satisfying a transversality condition. The standard references for this result are Peleg (1970) and Peleg and Ryder (1972). The (common) technique of proof of these two papers consists in applying a separation theorem in the space of all bounded infinite sequences (of vectors in $\mathbb{R}^n$). Our main objective in presenting this result is to draw attention to the fact that it can be derived as a special case of Theorems 3.3.1 and 3.3.2, which we have noted for the general model.

We now formally state and prove our characterization results, by using the corresponding results for the general model.

Proposition 3.7.1. *If $\{x(t),\, y(t),\, p(t)\}_0^\infty$ is a competitive plan from y, and*

$$\lim_{t\to\infty} p(t)\, y(t) = 0 \tag{3.58}$$

then $\{x(t),\, y(t)\}_0^\infty$ is an optimal plan from y.

Proof. If $\{x(t),\, y(t),\, p(t)\}_0^\infty$ is a competitive plan from y, then using (3.56), (3.57), and $x(t) = y(t) - c(t)$, one gets:

$$\delta^t w(c(t)) + p(t+1)\, y(t+1) - p(t)y(t)$$

$$\geq \delta^t w(c) + p(t+1)y - p(t)(c+x) \quad \text{for all} \quad (x,y) \in, \Omega \text{ and } c \in \mathbb{R}^n_+ \tag{3.59}$$

Note that $x(t) \in g(y(t),\, y(t+1))$, since $(x(t),\, y(t+1)) \in \Omega$, and $x(t) \leq y(t)$. For any $x \in g(y(t),\, y(t+1))$, since $(x(t),\, y(t+1)) \in \Omega$ and $x \leq y(t)$, defining $c = y(t) - x \geq 0$, and using (3.5), $w(c(t)) \geq w(c)$. Thus, $x(t) \in h(y(t),\, y(t+1))$, and $w(c(t)) = u(y(t),\, y(t+1))$.

Let $(a,b) \in \Omega$. Then defining $x \in h(a,b)$, and $c = a - x$, we have $(x,b) \in \Omega$ and $c \geq 0$, so by (3.59),

$$\begin{aligned} &\delta^t u(y(t),\; y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq\;\; &\delta^t u(a,b) + p(t+1)b - p(t)a \quad \text{ for all } (a,b) \in \Omega \end{aligned}$$

Thus, $\{y(t),\, p(t)\}_0^\infty$ is a competitive program from y, satisfying the transversality condition. Hence, by Theorem 3.3.1, $\{y(t)\}_0^\infty$ is an optimal program from y. Since we have already checked that $x(t) \in h(y(t),\, y(t+1))$, we can conclude that $\{x(t),\, y(t)\}_0^\infty$ is an optimal plan from y.

Proposition 3.7.2. *Suppose $\{x(t),\, y(t)\}_0^\infty$ is an optimal plan from $y \in \mathbb{R}^n_{++}$. Suppose, also, that there is some sufficient vector in $\mathbb{R}^n_+$. Then, there is a sequence $\{p(t)\}_0^\infty$ with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that*

(i) $\{x(t),\, y(t),\, p(t)\}_0^\infty$ *is a competitive plan;*

(ii) *For all $y \in \mathbb{R}^n_+$, and $t \geq 0$,*

$$\delta^t V(y(t)) - p(t)y(t) \geq \delta^t V(y) - p(t)y \tag{3.60}$$

and

$$\lim_{t\to\infty} p(t)\, y(t) = 0. \tag{3.61}$$

Proof. Since $\{x(t),\, y(t)\}_0^\infty$ is an optimal plan from y, we have $x(t) \in h(y(t),\, y(t+1))$, and $\{y(t)\}_0^\infty$ is an optimal program from y. Hence, by Theorem 3.3.2, there is a sequence $\{p(t)\}_0^\infty$ such that $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, $\{y(t),\, p(t)\}_0^\infty$ is a competitive program from y, and (3.60), (3.61) hold. It remains to verify (i). This is accomplished by showing that for each $t \geq 0$, the price vector $p(t)$ provides the appropriate price support for both the consumption decision and the production decision.

Given any t, define $\theta_t(c) = \delta^t w(c) - p(t)c$ for all $c \in \mathbb{R}^n_+$, and $\pi_t(x,y) = p(t+1)y - p(t)x$ for all $(x,y) \in \Omega$.

Next, given t, we define the following two sets:

$A(t) = \{\alpha : \text{there exists } c \geq 0, \text{ satisfying } \theta_t(c) - \theta_t(c(t)) > \alpha\}$

$B(t) = \{\alpha : \text{there exists } (x, y) \in \Omega \text{ satisfying } \pi_t(x, y) - \pi_t(x(t), y(t+1)) > -\alpha\}$

We claim that (for each t),

$$A(t) \text{ and } B(t) \text{ are disjoint} \tag{3.62}$$

If (3.62) does not hold (for some t), there is some α which belongs to both $A(t)$ and $B(t)$. Then, there is $(x, y) \in \Omega$ and $c \geq 0$, such that $\theta_t(c) - \theta_t(c(t)) > \alpha$, and $\pi_t(x, y) - \pi_t(x(t),\, y(t+1)) > -\alpha$. Thus, we get:

$$\delta^t w(c) + p(t+1)y - p(t)(x+c) > \delta^t w(c(t)) + p(t+1)y(t+1) - p(t)y(t)$$

Defining $a = (x + c)$, we have $(a, y) \in \Omega$, and $u(a, y) \geq w(a - x) = w(c)$. Thus, $\delta^t u(a, y) + p(t+1)y - p(t)a \geq \delta^t w(c) + p(t+1)y - p(t)(x+c)$. Also, since $x(t) \in h(y(t),\, y(t+1))$, we have $w(c(t)) = w(y(t) - x(t)) = u(y(t),\, y(t+1))$. Hence,

$$\begin{aligned} \delta^t u(a, y) + p(t+1)y - p(t)a \quad &> \quad \delta^t u(y(t),\; y(t+1)) \\ &\qquad + p(t+1)y(t+1) - p(t)y(t) \end{aligned}$$

which contradicts the fact that $\{y(t),\, p(t)\}_0^\infty$ is a competitive program from y. This establishes our claim (3.62).

Next, we note that, by definition of the sets $A(t)$ and $B(t)$,

$$\text{(a) If } \alpha < 0, \text{ then } \alpha \in A(t), \text{ (b) If } \alpha > 0, \text{ then } \alpha \in B(t) \tag{3.63}$$

Now suppose there is some $c \in \mathbb{R}^n_+$, such that $\theta_t(c) > \theta_t(c(t))$. Then by defining $\alpha = \frac{1}{2}[\theta_t(c) - \theta_t(c(t))]$, we have $\alpha > 0$, and $\alpha \in A(t)$. By (3.63), $\alpha \in B(t)$, which contradicts (3.62). Hence $\theta_t(c) \leq \theta_t(c(t))$ for all $c \in \mathbb{R}^n_+$, which is (3.56).

Suppose there is some $(x, y) \in \Omega$ such that $\pi_t(x, y) > \pi_t(x(t),\, y(t+1))$. Then by defining $\alpha = -\frac{1}{2}[\pi_t(x, y) - \pi_t(x(t),\, y(t+1))]$, we note that $(-\alpha) = \frac{1}{2}[\pi_t(x, y) - \pi_t(x(t),\, y(t+1))]$, so $\alpha \in B(t)$, and $\alpha < 0$. By (3.63), $\alpha \in A(t)$, which contradicts (3.62). Thus $\pi_t(x, y) \leq \pi_t(x(t),\, y(t+1))$ for all $(x, y) \in \Omega$, which is (3.57).

We have now shown that $\{x(t),\, y(t),\, p(t)\}_0^\infty$ is a competitive plan from y so that (i) holds. This completes the proof of the proposition.

3.7.4 Existence of a Stationary Optimal Output

The existence of a modified golden-rule equilibrium and a non-trivial stationary optimal stock have been obtained in the literature by Peleg and Ryder (1974) by using duality theory. This result can be obtained as a special case of Theorem 3.4.2 and Corrollary 3.4.1, which we have established for the general framework of Section 3.2.

Call the technology set $\delta - \textit{productive}$ if there exists $(\bar{x}, \bar{y})$ in Ω such that $\delta\bar{y} >> \bar{x}$. Note that if Ω is δ-productive, then with the definition of u given in Section 3.7.1, and assumptions (B.4) and (B.6), (Ω, u, δ) is $\delta u - \textit{productive}$. For $(\delta\bar{y}, \bar{y})$ is clearly in Ω by (B.4), and $\bar{x}$ is in $g(\delta\bar{y}, \bar{y})$. So $u(\delta\bar{y}, \bar{y}) \geq w(\delta\bar{y} - \bar{x}) > w(0) = u(0, 0)$.

Proposition 3.7.3. *If Ω is δ-productive, there is a triple $(\hat{x}, \hat{y}, \hat{p})$ such that $(\hat{x}, \hat{y}, \hat{p})$ is a modified golden-rule equilibrium. Furthermore, $\hat{y}$ is a non-trivial stationary optimal output.*

Proof. Since Ω is δ-productive, it is also δu-productive. So, using Theorem 3.4.2, there is a pair $(\hat{y}, \hat{p})$ such that $(\hat{y}, \hat{p})$ is a modified golden-rule and $\hat{y}$ is a non-trivial stationary optimal stock. That is, $(\hat{y}, \hat{y}) \in \Omega$, $\hat{p} \in \mathbb{R}^n_+$, and for all $(a, b) \in \Omega$,

$$u(\hat{y}, \hat{y}) + \delta\hat{p}\hat{y} - \hat{p}\hat{y} \geq u(a, b) + \delta\, \hat{p}\hat{b} - p\, a \tag{3.64}$$

Let $\hat{x}$ be an element of $h(\hat{y}, \hat{y})$. Then, $(\hat{x}, \hat{y}) \in \Omega$, and denoting $(\hat{y} - \hat{x})$ by $\hat{c}$, we have $\hat{c} \geq 0$ and $w(\hat{c}) = u(\hat{y}, \hat{y})$.

Define $\theta(c) \equiv w(c) - \hat{p}c$ for all $c \in \mathbb{R}^n_+$, and $\pi(x, y) \equiv \delta\hat{p}y - \hat{p}x$ for all $(x, y) \in \Omega$. Now, following the method of proof in Proposition 3.7.2, one can establish that $\theta(c) \leq \theta(\hat{c})$ for all $c \in \mathbb{R}^n_+$, and $\pi(x, y) \leq \pi(\hat{x}, \hat{y})$ for all $(x, y) \in \Omega$. Hence, $(\hat{x}, \hat{y}, \hat{p})$ is a modified golden-rule equilibrium.

Using Proposition 3.7.1, $\{\hat{y}\}_0^\infty$ is a stationary optimal program from $\hat{y}$. Since $\hat{y}$ is a non-trivial stationary optimal stock, it is also a non-trivial stationary optimal output.

3.8 Weitzman's Theorem on the NNP

Weitzman (1976) showed, in a continuous time optimal growth model, that at each instant of time, the present value of the net national product at that instant of time (evaluated at the current supporting prices) equals the maximum discounted sum of utilities the economy is capable of achieving from that time onwards. This is an interesting economic interpretation of the Bellman equation of dynamic programming (in continuous time). We provide here a discrete time analog of Weitzman's observation which, although it does not have the force of his result (discrete-time does not allow us to conclude equality between the two relevant magnitudes), might be of interest. Our approach to this result is an application of the methods of duality theory, discussed in Sections 3.3 and 3.7 above.

Our framework of analysis is the "consumption model" described in Section 3.7. If $\{x(t), y(t), p(t)\}$ is a competitive plan, then the *current value* price sequence $\{q(t)\}$, associated with the plan, is defined by: $q(t) = p(t)/\delta^t$ for $t \geq 0$.

Theorem 3.8.1. *If $\{x(t), y(t)\}_0^\infty$ is an optimal plan from a sufficient vector $y \in \mathbb{R}^n_{++}$, and $\{p(t)\}_0^\infty$ is a sequence with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$ such that*

$\{x(t), y(t), p(t)\}$ is a competitive plan satisfying (3.60) and (3.61). Then, for each $s \geq 1$

$$V(y(s)) \geq \frac{w(c(s)) + q(s)(x(s) - x(s-1))}{(1-\delta)} \tag{3.65}$$

and

$$V(y(s)) \leq \frac{w(c(s)) + q(s-1)(x(s) - x(s-1))}{(1-\delta)} \tag{3.66}$$

Proof. Pick any $s \geq 1$, and use $t = s+1$ and $y = y(s)$ in (3.60) to get

$$V(y(s+1)) - V(y(s)) \geq q(s+1)(y(s+1) - y(s)) \tag{3.67}$$

Use $t = s$ and $(x(s-1), y(s)) \in \Omega$ in (3.57) to get

$$q(s+1)(y(s+1) - y(s)) \geq (q(s)/\delta)(x(s) - x(s-1)) \tag{3.68}$$

Using (3.67) and (3.68),

$$V(y(s+1)) \geq V(y(s)) + (q(s)/\delta)(x(s) - x(s-1)) \tag{3.69}$$

By the principle of optimality, we also have

$$V(y(s)) = w(c(s)) + \delta V(y(s+1)) \tag{3.70}$$

So, using (3.69) in (3.70), we get

$$V(y(s)) \geq w(c(s)) + \delta V(y(s)) + q(s)(x(s) - x(s-1))$$

Transposing terms,

$$(1-\delta)V(y(s)) \geq w(c(s)) + q(s)(x(s) - x(s-1))$$

which yields (3.65).

Following an entirely analogous method, we can use $t = s$ and $y = y(s+1)$ in (3.60) to get

$$V(y(s)) - V(y(s+1)) \geq q(s)(y(s) - y(s+1)) \tag{3.71}$$

Use $t = s-1$ and $(x(s),\ y(s+1)) \in \Omega$ in (3.57) to get

$$q(s)(y(s) - y(s+1)) \geq (q(s-1)/\delta)(x(s-1) - x(s)) \tag{3.72}$$

Using (3.71) and (3.72),

$$V(y(s)) - V(y(s+1)) \geq (q(s-1)/\delta)(x(s-1) - x(s)) \tag{3.73}$$

Transposing terms,

$$V(y(s+1)) \leq V(y(s)) + (q(s-1)/\delta)(x(s) - x(s-1)) \tag{3.74}$$

Using the principle of optimality (3.70), we have:

$$V(y(s)) \leq w(c(s)) + \delta V(y(s)) + q(s-1)(x(s) - x(s-1))$$

Transposing terms,

$$(1-\delta)V(y(s)) \leq w(c(s)) + q(s-1)(x(s) - x(s-1))$$

which yields (3.66).

Remark:

(i) Note that the hypothesis of the Theorem can be seen to be non-vacuous by an appeal to Proposition 3.7.2 in Section 3.7. That is, given an optimal plan $\{x(t), y(t)\}_0^\infty$ from a sufficient vector $y \in \mathbb{R}^n_{++}$, there exists a price sequence $\{p(t)\}_0^\infty$ with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that $\{x(t), y(t), p(t)\}$ is a competitive plan satisfying (3.60) and (3.61).

(ii) Our theorem indicates that the maximum discounted sum of utilities achievable from time s onwards [that is, $V(y(s))$] is trapped between two magnitudes, each of which has some claim to be interpreted as the present-value of the net national product in time period s. The difference between the two magnitudes is the "current" price ($q(s)$ or $q(s-1)$) used to evaluate investment $(x(s) - x(s-1))$ during the time period s.

3.9 Bibliographic Notes

Section 3.2:

The general framework described in this section was introduced into the literature by Gale (1967) and McKenzie (1968) in their contributions on optimal growth when future utilities are *undiscounted*. The framework has great flexibility, and a variety of intertemporal allocation problems can be reduced to this framework. The well-known model, in which utility is derived from consumption alone, is discussed in Section 3.7 as an illustration of this observation. For other intertemporal allocation problems, see the exposition in Mitra (2000).

Section 3.3:

The section describes the basic price characterization results in the discounted case, following the approach of Weitzman (1970). The approach can be adapted to the undiscounted case as well; for this, see Peleg and Zilcha (1977) and McKenzie (1986).

Theorem 3.3.1 does not require convex structures, but under non-convexities it turns out to be not a useful tool for showing that a candidate program is optimal, since in general one will not be able to obtain a price sequence at which the program will satisfy the competitive conditions. When an optimal program is interior, and the utility function is differentiable in the interior of the transition possibility set, a necessary condition of optimality is the Ramsey-Euler equation. In this case, it can also be shown that the optimal program satisfies a suitable transversality condition. However, the Ramsey-Euler conditions together with this transversality condition is not sufficient for optimality in non-convex models.

The results of this section can be generalized to a setting involving changing technology and tastes. For the general theory, see McKenzie (1974); for applications to an aggregative model, see Mitra and Zilcha (1981).

Section 3.4:
The approach, consisting of establishing the existence of a discounted golden-rule, as a step to establishing the existence of a non-trivial stationary optimal stock, is due to Flynn (1980) and McKenzie (1982). Khan and Mitra (1986) showed that this could be accomplished when continuity of the utility function is replaced by upper semicontinuity, thereby making the result more widely applicable. They also showed that duality methods could be completely dispensed with in establishing that the discounted golden-rule stock is a non-trivial stationary optimal stock. The approach of Peleg and Ryder (1974) is somewhat similar, but their method applies only to the "consumption model". The dynamic programming approach of Sutherland (1970) runs into the problem that the stationary optimal stock obtained by the fixed point argument can be trivial, and there is no obvious way to ensure non-triviality of the fixed point, even when the economy is $\delta u - productive$.

Analogous results for the undiscounted case (involving the notion of a golden-rule) are contained in Gale (1967), McKenzie (1968), Brock (1970) and Peleg (1973). However, a major difference is that the existence of a golden-rule can be shown without any use of fixed point methods.

Section 3.5:
The idea of replacing the transversality condition by a period-by-period condition was first proposed by Brock and Majumdar (1988), who established the appropriate result in the undiscounted case, for the "consumption model". For application of the same principle in other settings, see the collection of papers, edited by Majumdar (1992).

Section 3.6:
The reachability condition proposed here is weak. Other related conditions, such as local expandability and local contractability, proposed by Kurz and Starrett (1970), are more restrictive. The example of the simple Leontief model (as described by Gale (1960)) is an instance where the reachability condition can be checked quite easily, but both local expandability and local contractability fail.

Section 3.7:
Viewing the "consumption model" as a special case of the general framework of Section 3.2 has the advantage that many duality results, developed for the consumption model (see, especially, Peleg and Ryder (1972, 1974)), can be obtained by an alternative and simpler route, and the assumptions needed for either approach to work can thereby be compared. For a more complete discussion, see Dasgupta and Mitra (1990).

Section 3.8:
Weitzman's Rule is almost exclusively discussed in the literature in the context of continuous models. The discrete-time analog presented here indicates that the argument involved is quite elementary, and is a good illustration of the essential simplicity of duality methods.

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4. Rationalizability in Optimal Growth Theory

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4.1 Introduction

Before I address rationalizability in optimal growth theory, let me discuss the issue of rationalizability in more general terms. Suppose there exists a class of models, $\mathcal{M}$, such that every model $M \in \mathcal{M}$ describes the economic phenomenon under consideration. Suppose furthermore that, for every $M \in \mathcal{M}$, there exists a (possibly empty) set of 'solutions' or 'equilibria' $H(M) \subseteq \mathcal{H}$, where $\mathcal{H}$ is a fixed set of possible solutions. One has to distinguish between four different problems related to $M \in \mathcal{M}$ and $h \in \mathcal{H}$, respectively.

- The *solution problem* for the model M consists in finding or characterizing one or all elements of $H(M)$.
- The *existence problem* for the model M consists in determining whether $H(M)$ is empty or not.
- The *inverse problem* for the solution h consists in finding or characterizing one or all models $M \in \mathcal{M}$ such that $h \in H(M)$.
- The *rationalizability problem* for the solution h consists in determining whether there exists a model $M \in \mathcal{M}$ such that $h \in H(M)$.

From the above descriptions one can see that the problem of rationalizing a given solution h is related to the inverse problem for h in the same way as the existence problem for the model M is related to the problem of solving M. It is furthermore obvious that every solution h that can be rationalized by the class $\mathcal{M}$ can also be rationalized by any class of models that includes $\mathcal{M}$. Rationalizability of a given solution h by a restricted class of models is therefore a stronger property than rationalizability by a wider class of models. Questions regarding rationalizability have been discussed in a number of different branches of economic theory.[1] An early discussion of the inverse problem in optimal growth theory is the paper by Kurz [7]; see also Chang [3]

[1] A prominent example is the Sonnenschein-Mantel-Debreu theorem. It says that every continuous function, which is homogeneous of degree 0 and satisfies Walras'

and references therein. The issue of rationalizability in optimal growth theory received a strong impetus in the early to mid 1980s, when it was first shown that dynamic macroeconomic models satisfying standard assumptions can generate complicated deterministic dynamics. In other words, this line of research demonstrated the rationalizability of chaotic dynamical systems by certain classes of intertemporal macroeconomic models. For example, in an often cited paper, Boldrin and Montrucchio [1] proved that every twice continuously differentiable function can be the optimal policy function of an infinitely-lived agent model in reduced form, which has the smoothness and convexity properties that are typically assumed by growth theorists. A consequence of this result is that even the most complicated dynamic behavior cannot be ruled out by the standard assumptions of optimal growth theory. It was furthermore argued that the aperiodic fluctuations generated by chaotic dynamical systems can resemble realistic business cycles and that standard optimal growth models are therefore not only consistent with, but can actually explain important stylized facts of the business cycle.[2] However, it soon became clear that the constructive approach used by Boldrin and Montrucchio [1] depends on the choice of unrealistically high rates of time-preference. Sorger [27] provided the first rigorous proof that high time-preference is indeed necessary for the rationalizability of complicated dynamics by optimal growth models or, in other words, that there exist non-trivial discount factor restrictions for the optimality of complicated dynamics. In the present chapter, I survey the literature that has emerged from Boldrin and Montrucchio's and from Sorger's contributions and that addresses the questions of rationalizability and discount factor restrictions in optimal growth models. The rest of this chapter is organized as follows. In section 4.2, I specify three different classes of optimal growth models and I state a few important results about the solutions of such models. This allows me to give a precise definition of the rationalizability problem in optimal growth theory. Section 4.3 summarizes necessary and sufficient conditions for the rationalizability of given functions as optimal policy functions of infinitely-lived agent models. Typically, these conditions are formulated in terms of smoothness properties of the optimal policy functions. For the most comprehensive of the three classes of optimal growth models, I show that every rationalizable function is necessarily continuous and that every Lipschitz-continuous function can be rationalized. Moreover, I illustrate by means of examples that closing the gap between these two conditions is likely to be very difficult. Section 4.4 reviews the literature on discount factor restrictions for the rationalizability of complicated dynamics. The first main result is a simple relation between the discount factor of an optimal growth model and the topological entropy of the

law, is the excess demand function of a static competitive economy with standard properties; see Sonnenschein [24, 25], Mantel [10], and Debreu [4]. In this setting, a model M is described by the number of households, their preferences, and their endowments, and the solution of M is the excess demand function generated by M.

[2] See, e.g., Boldrin and Woodford [2] for a critical discussion of these arguments.

corresponding optimal policy function.[3] This result holds for optimal growth models defined on state spaces of arbitrary high (but finite) dimension. However, it does not provide an exact discount factor restriction, i.e., it does not give the *least* upper bound on the set of discount factors that can be used to rationalize a dynamical system with a given topological entropy. For models defined on one-dimensional state spaces, more precise discount factor restrictions are available. I illustrate this by stating exact discount factor restrictions for the occurrence of period-three cycles and for the optimality of the logistic map and the tent-map, respectively. Finally, section 4.5 discusses possible extensions to alternative classes of optimal growth models. Two technical proofs have been relegated to an appendix.

4.2 Problem Formulation

My first step is to define the class of optimal growth models and to explain what I mean by a solution of such a model. Time is a discrete variable taking values in the domain $\{0, 1, 2, \ldots\}$. The state of the economy at the start of period t is denoted by x_t. The set of all possible states (i.e., the state space of the economy) is an arbitrary non-empty set denoted by X. The state of the system can change once in every period. A transition from state x to state y is feasible if and only if $(x, y) \in \mathbf{T}$, where $\mathbf{T} \subseteq X \times X$ is the transition possibility set. The following assumption says that, from every state $x \in X$, there exists at least one feasible state transition. **A1**: The set $\mathbf{T}_x = \{y \in X \,|\, (x, y) \in \mathbf{T}\}$ is non-empty for all $x \in X$. A sequence $(x_t)_{t=0}^{+\infty}$ is called a feasible path (from x_0) if $(x_t, x_{t+1}) \in \mathbf{T}$ holds for all t. For every $x \in X$, let me denote by $F(x)$ the set of all feasible paths from x. Assumption A1 is a necessary and sufficient condition for $F(x)$ to be non-empty for all $x \in X$. A state transition from x to y generates the instantaneous utility $u(x, y)$, where $u : \mathbf{T} \mapsto \mathbb{R}$ is a given function. The time-preference rate is assumed to be constant and the corresponding discount factor will be denoted by δ. This implies that the total utility generated by a feasible path $(x_t)_{t=0}^{+\infty}$ is given by

$$J\left[(x_t)_{t=0}^{+\infty}\right] = \sum_{t=0}^{+\infty} \delta^t u(x_t, x_{t+1}).$$

I make the following assumption about the preferences. **A2**: (i) The function $u : \mathbf{T} \mapsto \mathbb{R}$ is bounded. (ii) The discount factor δ satisfies $\delta \in (0, 1)$. Assumption A2 is a sufficient but by no means a necessary condition for $J\left[(x_t)_{t=0}^{+\infty}\right]$ to be a well-defined and finite number for all feasible paths $(x_t)_{t=0}^{+\infty}$. The literature often deals with less restrictive assumptions on the utility function u; see, e.g., Stokey and Lucas [32]. However, as I have already mentioned in the introduction, making more restrictive assumptions on the class of models only leads to

[3] See section 4.4 for a precise definition of the topological entropy of a dynamical system.

stronger rationalizability results. A feasible path $(x_t)_{t=0}^{+\infty} \in F(x)$ satisfying the inequality

$$J\left[(x_t)_{t=0}^{+\infty}\right] \geq J\left[(y_t)_{t=0}^{+\infty}\right]$$

for all feasible paths $(y_t)_{t=0}^{+\infty} \in F(x)$ is called an optimal path (from x). The optimal value function $V : X \mapsto \mathbb{R}$ is defined by

$$V(x) = \sup\left\{J\left[(x_t)_{t=0}^{+\infty}\right] \,\middle|\, (x_t)_{t=0}^{+\infty} \in F(x)\right\}$$

for all $x \in X$. Under assumptions A1 and A2, this function is well-defined and finite. If $(x_t)_{t=0}^{+\infty} \in F(x)$ is an optimal path from x, then it follows that $J\left[(x_t)_{t=0}^{+\infty}\right] = V(x)$. This concludes the discussion of the basic structure of the class of models that I consider in the present chapter. Formally, a model M is a triple $(\mathbf{T}, u, \delta)$ consisting of a transition possibility set, an instantaneous utility function, and a discount factor. Models of this type are called optimal growth models in reduced form and they have numerous applications in economics, notably in optimal growth theory; see, e.g., McKenzie [11] and Stokey and Lucas [32]. Assumptions A1 and A2, however, do not impose any non-trivial restrictions on the set of solutions of optimal growth models. For example, if u is a constant function, then every feasible path is also optimal. Thus, every path $(x_t)_{t=0}^{+\infty}$ satisfying $x_t \in X$ for all t can be rationalized by the class of models satisfying A1 and A2. In order to obtain non-trivial rationalizability results, one has to impose more restrictive, structural assumptions on the class of optimal growth models. In the remainder of this section, I therefore discuss a number of such assumptions and I state a few important results about the solutions of optimal growth models satisfying these assumptions. So far, the state space X has been assumed to be an arbitrary non-empty set without any particular structure. In order to be able to impose structural assumptions on the class of optimal growth models, I assume from now on that X is a non-empty, convex, and compact subset of the n-dimensional Euclidean space $\mathbb{R}^n$. Typically, in optimal growth models, the state space represents the set of all feasible vectors of capital stocks, where n is the number of different capital goods. These may include various forms of physical capital as well as human capital. Convexity of X means that convex combinations of two feasible capital vectors are also feasible, and it forms therefore a necessary condition for the convexity of production technologies. The compactness assumption is often justified by the argument that capital stocks must be non-negative and that there exist maximal sustainable levels of all capital stocks. Having put both a linear and a metric structure on the state space X, I can now impose linear and metric properties on the transition possibility set $\mathbf{T}$ and the utility function u. The following assumption is tantamount to saying that the production technologies for capital goods are convex and continuous. **A3**: The transition possibility set $\mathbf{T}$ is a convex and closed set. The utility gain $u(x, y)$ in the reduced form optimal growth model is derived as the maximal utility of consumption that can be derived within one period subject to the constraints that the capital stock at the start of the period is x and the capital stock at the end of the period is y. Thus, the reduced utility function u combines properties of the production technology of consumption goods with properties of the preferences of households or a central planner. I make the following assumption. **A4**: The utility function $u : \mathbf{T} \mapsto \mathbb{R}$ is strictly concave and continuous. Note that compactness of X together with assumptions A3 and A4 implies that assumption A2(i) holds. Strict concavity of u is a strong assumption but, as I have mentioned before, this is no drawback when one is interested in questions of rationalizability. For the purpose of this survey, assumptions A1-A4 summarize the most important structural properties of optimal growth models. To simplify the exposition, I denote by $\mathcal{M}(X)$ the set of all optimal growth models with state space X which satisfy assumptions A1-A4. At some points during the ensuing discussion it will be necessary to impose further assumptions. The following one, for example, deals with monotonicity properties of $\mathbf{T}$ and u and has a natural interpretation in terms of free disposal and monotonicity of preferences. **A5**: (i) If $x \in X$, $x' \in X$, and $x \leq x'$, then it follows that $\mathbf{T}_x \subseteq \mathbf{T}_{x'}$. (ii) The function $u(x, y)$ is non-decreasing with respect to x and non-increasing with respect to y. I denote by $\mathcal{M}_+(X)$ the set of all optimal growth models defined on the state space X which satisfy assumptions A1-A5. The final assumption deals with curvature properties of the utility function u. In order to formulate it, I need to review some basic terminology. A real-valued function f defined on a convex subset of real Euclidean space is called α-concave, if the function $g(x) = f(x) + (\alpha/2)\|x\|^2$ is concave. Here, α is an arbitrary real number. If f is α-concave for some *positive* number α, then f is said to be strongly concave. The function f is called $(-\alpha)$-convex, if the function g defined above is convex. Intuitively, α-concavity implies that f is "at least as concave" as the quadratic form $-(\alpha/2)\|x\|^2$, whereas $(-\alpha)$-convexity implies that it is "at most as concave" as this quadratic form. It is easily seen that every strongly concave function is strictly concave but not vice versa. **A6**: The utility function $u : \mathbf{T} \mapsto \mathbb{R}$ is strongly concave. The curvature property captured by assumption A6 is quite strong and hard to interpret. I denote by $\mathcal{M}_*(X)$ the set of all optimal growth models defined on the state space X which satisfy assumptions A1-A4 and A6. The following two results are well-known; see, e.g., Stokey and Lucas [32].[4]

Proposition 4.2.1. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let M be an optimal growth model in $\mathcal{M}(X)$. Furthermore, let $V : X \mapsto \mathbb{R}$ be the optimal value function of M.*
(i) The function V is bounded, continuous, and strictly concave.
(ii) For all $x \in X$ it holds that

$$V(x) = \max\left\{u(x, y) + \delta V(y) \,|\, y \in \mathbf{T}_x\right\}. \tag{4.1}$$

The function V is the only real-valued and continuous function defined on X which satisfies equation (4.1) for all $x \in X$.

[4] Part (iv) of proposition 4.2.1 follows from a result proved in Montrucchio [16].

(iii) If $M \in \mathcal{M}_+(X)$, then it follows that V is strictly increasing.
(iv) If $M \in \mathcal{M}_*(X)$, then it follows that V is strongly concave. More precisely, if the utility function u is α-concave for some positive number α, then it follows that V is α-concave as well.

The above proposition shows that the optimal value function of an optimal growth model is the unique continuous function satisfying the Bellman equation (4.1). Moreover, this function inherits the properties of boundedness, strict and strong concavity, and monotonicity from the utility function u.

Proposition 4.2.2. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let M be an optimal growth model in $\mathcal{M}(X)$. Furthermore, let $V : X \mapsto \mathbb{R}$ be the optimal value function of M.*
(i) There exists a unique optimal growth path from every $x \in X$.
(ii) There exists a unique function $h : X \mapsto X$ such that the following is true. A feasible path $(x_t)_{t=0}^{+\infty}$ is optimal if and only if it satisfies the difference equation

$$x_{t+1} = h(x_t) \tag{4.2}$$

for all $t = 0, 1, 2, \ldots$.
(iii) The function h from part (ii) is continuous and satisfies the equation

$$V(x) = u(x, h(x)) + \delta V(h(x)) \tag{4.3}$$

for all $x \in X$.

Proposition 4.2.2 establishes the existence and uniqueness of optimal growth paths and shows that these paths can be characterized as the trajectories of the dynamical system (4.2). The function h is called the optimal policy function of M. Part (iii) proves that the optimal policy function is continuous and that $y = h(x)$ is the unique maximizer on the right-hand side of the Bellman equation (4.1). The fact that optimal paths are characterized as the trajectories of a dynamical system defined by the optimal policy function h allows me to regard h as the solution of the model M. The rationalizability problem in optimal growth theory can therefore be formulated in the following way: given a function $h : X \mapsto X$, is there an optimal growth model M in $\mathcal{M}(X)$, $\mathcal{M}_+(X)$, or $\mathcal{M}_*(X)$ such that h is the optimal policy function of M?

4.3 Optimal Policy Functions

Suppose a function $h : X \mapsto X$ is given, where X is a non-empty, compact, and convex subset of real Euclidean space. In the present section I discuss the rationalizability of h as a policy function of an optimal growth model in reduced form. I start by showing that optimal policy functions are necessarily smooth. More precisely, the following theorem shows how different assumptions about the curvature of the utility function u lead to different smoothness properties of the optimal policy function h. Before I formulate the theorem, let me recall the concepts of Hölder-continuity and Lipschitz-continuity. A function $f : X \mapsto \mathbb{R}^\ell$ is called Hölder-continuous of degree $\rho \in (0, 1]$ at point $x \in X$, if there exist positive numbers ε and K such that the inequality

$$\|f(x) - f(y)\| \leq K\|x - y\|^\rho$$

holds for all $y \in X$ satisfying $\|y - x\| \leq \varepsilon$. The function f is said to be Hölder-continuous of degree ρ on a subset of X, if it is Hölder-continuous of degree ρ at every point of this subset. Finally, if f satisfies the above inequality for $\rho = 1$ and all $(x, y) \in X \times X$, then we say that f is (uniformly) Lipschitz-continuous and we call K a Lipschitz-constant for f.

Theorem 4.3.1. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let $h : X \mapsto X$ be a given function.*
(i) If h is the optimal policy function of a model $M \in \mathcal{M}(X)$, then h is continuous.
(ii) If h is the optimal policy function of a model $M \in \mathcal{M}_*(X)$, then h is Hölder-continuous of degree $1/2$ on the interior of the state space.
(iii) Assume that h is the optimal policy function of a model $M \in \mathcal{M}_*(X)$ and that there exists a positive number β such that the optimal value function of M is $(-\beta)$-convex. Then it follows that h is uniformly Lipschitz-continuous on X.

Part (i) of the above theorem iterates a result from proposition 4.2.2(iii). Part (ii) says that models with strongly concave utility functions have optimal policy functions that are not only continuous but even Hölder-continuous of degree 1/2 at every point in the interior of the state space X. This result was first proved by Montrucchio [18]; see also Sorger [29]. Part (iii) of theorem 4.3.1 shows that the optimal policy function is Lipschitz-continuous, if the utility function u is strongly concave and the optimal value function V is $(-\beta)$-convex for some positive number β.[5] This result has been derived in Montrucchio [18] and Sorger [29], whereby the latter paper derives an explicit expression for a Lipschitz-constant of the optimal policy function.[6] Let me point out that the interiority condition in theorem 4.3.1(ii) cannot be omitted, that is, policy functions of optimal growth models in $\mathcal{M}_*(X)$ need not be Hölder-continuous of degree 1/2 at boundary points of the state space. To demonstrate this I consider the following example.

Example 4.3.1. Let $\alpha \in (0, 1/2)$, $\delta \in (0, 1)$, and $\mu \geq (1+\delta)/\delta$ be given numbers and define $X = [0, 1]$, $\mathbf{T} = \{(x, y) \,|\, 0 \leq x \leq 1,\ 0 \leq y \leq x^\alpha\}$, and $u(x, y) =$

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[5] Note that proposition 4.2.1(iv) implies that α-concavity of the utility function u implies α-concavity of the optimal value function V. Thus, the assumptions in theorem 4.3.1(iii) imply that there exist positive numbers α and β such that the optimal value function V is both α-concave and $(-\beta)$-convex. Montrucchio [18] calls optimal growth models satisfying this property 'regular' models.

[6] Montrucchio [16] presents an alternative structural assumption on the class of optimal growth models which ensures that the optimal policy functions are Lipschitz-continuous.

$\mu x-(1/2)(x^2+y^2)$. It is easy to see that the model $M=(\mathbf{T},u,\delta)$ is an element of $\mathcal{M}_+(X)\cap\mathcal{M}_*(X)$. Now note that

$$J\left[(x_t)_{t=0}^{+\infty}\right]=\sum_{t=0}^{+\infty}\delta^t u(x_t,x_{t+1})=\mu x_0-(1/2)x_0^2+\sum_{t=1}^{+\infty}\delta^{t-1}\left[\mu\delta x_t-(1+\delta)x_t^2/2\right].$$

Because $\mu>(1+\delta)/\delta$, it is straightforward to verify that $J\left[(x_t)_{t=0}^{+\infty}\right]$ is strictly increasing with respect to x_t for every $t\geq 1$. This implies that an optimal path from x_0 must have the property that it increases as fast as possible. Using the specification of $\mathbf{T}$ it follows that the optimal policy function of M is given by $h(x)=x^\alpha$. However, because $\alpha<1/2$, this function is not Hölder-continuous of degree $1/2$ at the boundary point $x=0$.

Theorem 4.3.1 demonstrates that making stronger assumptions about the curvature of the utility function u reduces the set of functions that can be rationalized as optimal policy functions. This is what one would expect. Adding the monotonicity assumption A5(ii), however, hardly changes the set of functions that can be rationalized as optimal policy functions. This is formally stated in the following lemma.

Lemma 4.3.1. *Let $X\subseteq\mathbb{R}^n$ be a non-empty, compact, and convex set and let $h:X\mapsto X$ be the optimal policy function of a model $M=(\mathbf{T},u,\delta)\in\mathcal{M}(X)$. If $\mathbf{T}$ satisfies assumption A5(i) and if the utility function u is Lipschitz-continuous on $\mathbf{T}$, then it follows that h can be rationalized by $\mathcal{M}_+(X)$ as well.*

The proof of this lemma is very simple. Suppose that u is Lipschitz-continuous with Lipschitz-constant K. I define a new utility function $\bar{u}:\mathbf{T}\mapsto\mathbb{R}$ by

$$\bar{u}(x,y)=u(x,y)+px-\delta py,$$

where $p=(p_1,p_2,\ldots,p_n)\in\mathbb{R}^n$ is a vector satisfying $p_i\geq K/\delta$. It is easy to see that $\bar{u}$ satisfies the monotonicity assumption A5(ii). Finally, I define the new optimal growth model $\bar{M}=(\mathbf{T},\bar{u},\delta)$. Since M and $\bar{M}$ have the same transition possibility sets, it follows that $F(x)=\bar{F}(x)$ holds for all $x\in X$, where $F(x)$ and $\bar{F}(x)$ denote the sets of feasible paths from x in model M and $\bar{M}$, respectively. Furthermore, one has

$$\sum_{t=0}^{+\infty}\delta^t\bar{u}(x_t,x_{t+1})=px_0+\sum_{t=0}^{+\infty}\delta^t u(x_t,x_{t+1})$$

for all feasible paths $(x_t)_{t=0}^{+\infty}\in F(x_0)=\bar{F}(x_0)$. Since px_0 is a given constant, the unique optimal path from x_0 in model M is also the unique optimal path from x_0 in model $\bar{M}$. This shows that the optimal policy function h of model M must coincide with the optimal policy function of $\bar{M}$ and, hence, that h can be rationalized by $\mathcal{M}_+(X)$. According to lemma 4.3.1, the monotonicity assumption on the utility function , A5(ii), has hardly any effect on the rationalizability problem. It is obvious that one cannot expect a similar result to hold for the monotonicity assumption on the transition possibility set, A5(i). Let me now turn to sufficient conditions for the rationalizability problem. It will be convenient to reformulate some of the optimality conditions stated in propositions 4.2.1 and 4.2.2 in a slightly different form. Consider a model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}(X)$ and let $V : X \mapsto \mathbb{R}$ and $h : X \mapsto X$ be the optimal value function of M and the optimal policy function of M, respectively. Furthermore, define the function $G : \mathbf{T} \mapsto \mathbb{R}$ by

$$G(x, y) = u(x, y) - V(x) + \delta V(y). \tag{4.4}$$

From propositions 4.2.1 and 4.2.2 it follows that

$$G(x, y) \leq G(x, h(x)) = 0 \tag{4.5}$$

holds for all $(x, y) \in \mathbf{T}$. In other words, if h is the optimal policy function of M and if V is the optimal value function of M, then it is necessarily true that conditions (4.4) and (4.5) hold simultaneously. Conversely, let V and h be arbitrary continuous functions such that (4.4) and (4.5) hold and such that V is strictly concave. Then it is easily seen that equations (4.1) and (4.3) hold as well and that $y = h(x)$ is the unique maximizer on the right-hand side of (4.1). Consequently, h must be the optimal policy function of M and V must be the corresponding optimal value function. I summarize these results in the following lemma.

Lemma 4.3.2. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let $M = (\mathbf{T}, u, \delta)$ be an optimal growth model in $\mathcal{M}(X)$. Furthermore, let $h : X \mapsto X$ be a continuous function satisfying $(x, h(x)) \in \mathbf{T}$ for all $x \in X$. The function h is the optimal policy function of M if and only if there exists a strictly concave function $V : X \mapsto \mathbb{R}$ such that conditions (4.4) and (4.5) hold for all $(x, y) \in \mathbf{T}$.*

Lemma 4.3.2 says that the existence of a strictly concave function V satisfying equations (4.4) and (4.5) is a necessary and sufficient condition for h to be the optimal policy function of M.[7] This result suggests the following constructive approach to finding an optimal growth model $M \in \mathcal{M}(X)$ that rationalizes a given function $h : X \mapsto X$:

(i) find a convex and closed set $\mathbf{T} \subseteq X \times X$ such that $(x, h(x)) \in \mathbf{T}$ holds for all $x \in X$;
(ii) find a function $G : \mathbf{T} \mapsto \mathbb{R}$ satisfying (4.5);
(iii) choose a continuous and strictly concave function $V : X \mapsto \mathbb{R}$ and compute the corresponding function $u : \mathbf{T} \mapsto \mathbb{R}$ from equation (4.4);
(iv) check whether u is strictly concave.

[7] Note that only (4.5) is a non-trivial condition because (4.4) simply defines the function G.

Sorger [28, Theorem 3.3] uses this construction to prove the often cited result from Boldrin and Montrucchio [1]. More specifically, he chooses

$$\begin{aligned} \mathbf{T} &= X \times X, \\ G(x,y) &= -(1/2)\|y - h(x)\|^2, \\ V(x) &= -(\alpha/2)\|x\|^2. \end{aligned} \tag{4.6}$$

Obviously, these specifications satisfy the conditions in steps (i)-(iii) of the method described above. According to step (iii), the utility function is given by

$$u(x,y) = -(1/2)\|y - h(x)\|^2 - (\alpha/2)\|x\|^2 + (\alpha\delta/2)\|y\|^2.$$

Now assume that h is a twice continuously differentiable function and denote the Hessian matrix of u evaluated at (x, y) by $S(x, y)$. It follows that

$$S(x,y) = \begin{pmatrix} A(x,y) - \alpha I_n & B(x,y) \\ B(x,y)^T & (\alpha\delta - 1)I_n \end{pmatrix},$$

where I_n denotes the $n \times n$ unit matrix and where

$$\begin{pmatrix} A(x,y) & B(x,y) \\ B(x,y)^T & -I_n \end{pmatrix}$$

is the Hessian matrix of the mapping $(x, y) \mapsto -(1/2)\|y - h(x)\|^2$. Note that the latter Hessian matrix is independent of both α and δ. Fix any number $\gamma \in (0, 1)$. It is straightforward to see that, by choosing α sufficiently large and $\delta = \gamma/\alpha$, one can ensure that all eigenvalues of $S(x, y)$ are negative and uniformly bounded away from 0. Thus, the utility function u is strongly concave and, hence, strictly concave on $X \times X$. This proves the following theorem from Boldrin and Montrucchio [1].

Theorem 4.3.2. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let $h : X \mapsto X$ be a twice continuously differentiable function. Then it follows that h is rationalizable by $\mathcal{M}(X)$, $\mathcal{M}_+(X)$, and $\mathcal{M}_*(X)$.*[8]

Twice continuous differentiability of h is in fact more than is required for the construction based on the specification in (4.6). As has been shown by Neumann et al. [20], it is sufficient to assume that h is differentiable with a Lipschitz-continuous gradient; see also Montrucchio [18]. I skip the discussion of this technical detail because, as will be seen in a moment, there is a much more powerful theorem available according to which every Lipschitz-continuous function is rationalizable. Before I turn to this result, however, let me say a few more words about theorem 4.3.2 and the constructive algorithm that was used in its proof. First, as is apparent from the proof, the discount factor δ

[8] The statement about rationalizability by $\mathcal{M}_+(X)$ follows from lemma 4.3.1 and from the observation that the utility function u constructed above is twice continuously differentiable and, hence, Lipschitz-continuous.

has to be chosen sufficiently small. This means that the construction uses a sufficiently impatient decision maker. In section 4.4 below, I will explore this issue in much greater detail by showing that there exists a negative relation between the complexity of the dynamics generated by h and the discount factors of optimal growth models that can rationalize h. Second, it is clear that the choice of a quadratic function for V and the squared Euclidean distance between y and $h(x)$ for G is not the only choice that makes sense in (4.6). Depending on the characteristics of the function h that one wants to rationalize, other specifications may be more useful. To illustrate this, let me consider the following example.

Example 4.3.2. Let $X = [0, 1]$ *and* $h(x) = 1 - |2x - 1|$*. The function* h *is the so-called tent map, which plays a prominent role in the field of chaotic dynamics. The function is not differentiable and cannot be rationalized using the specification (4.6). However, as shown in Sorger [28, Lemma 3.12], the tent map can be rationalized using the same 4-step procedure outlined above with specifications of* $\mathbf{T}$*,* G*, and* V *that are different from (4.6).*[9] *More specifically, Sorger [28] chooses*

$$\begin{aligned}
&\mathbf{T} = \{(x, y) \,|\, 0 \leq x \leq 1,\ 0 \leq y \leq 2x,\ 0 \leq y \leq 2(1 - x)\},\\
&G(x, y) = (2x - y)(2x - 2 + y),\\
&V(x) = -[2 + 1/(2\delta)]x^2.
\end{aligned}$$

This construction results in a model $M \in \mathcal{M}_*(X)$ *as long as the discount factor* δ *is smaller than* $1/4$*.*

Let me now show that every Lipschitz-continuous function can be rationalized by $\mathcal{M}(X)$, $\mathcal{M}_+(X)$, and $\mathcal{M}_*(X)$. In principle, this could be done using the constructive approach outlined above. It is more transparent, however, to apply the following two-step procedure. I first state a result that gives a sufficient condition for a pair of functions (h, V) to be the optimal policy function and the optimal value function of some model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}_+(X) \cap \mathcal{M}_*(X)$. I will then show that this sufficient rationalizability condition is satisfied if h is Lipschitz-continuous, V is a quadratic polynomial, and δ is sufficiently small. In order to be able to state the general rationalizability condition for pairs (h, V), let me recall the definition of the subdifferential of a real-valued, convex function f defined on a convex set $X^* \subseteq \mathbb{R}^n$. The subdifferential at point $x \in X^*$ is the set $\partial f(x) = \{p \in \mathbb{R}^n \,|\, f(y) \leq f(x) + p(y - x) \text{ for all } y \in X^*\}$. Elements of $\partial f(x)$ are called subgradients of f at x. It is known that $\partial f(x) \neq \emptyset$ holds necessarily for all x in the interior of the domain X^* but may not hold at the boundary of X^*.

Theorem 4.3.3. *Let* $X \subseteq \mathbb{R}^n$ *be a non-empty, compact, and convex set, let* $h : X \mapsto X$ *be a continuous function, and let* $\delta \in (0, 1)$ *be a given number.*

[9] Nishimura et al. [21] use essentially the same specification of $\mathbf{T}$, G, and V as [28, Lemma 3.12] to rationalize a whole family of tent-shaped functions.

(i) Assume that there exist an open set $X^* \subseteq \mathbb{R}^n$ containing X and a strictly concave function $V : X^* \mapsto \mathbb{R}$ such that $\bigcup_{x \in X} \partial V(x)$ is compact. If there exist a number $\alpha > 0$ and, for every $x \in X$, subgradients $p_x \in \partial V(x)$ and $q_x \in \partial V(h(x))$ such that the inequality

$$\begin{aligned} & \delta\Big\{V(h(x)) + q_x[h(y) - h(x)] - V(h(y)) + (\alpha/2)\|h(y) - h(x)\|^2\Big\} \\ < \quad & V(x) + p_x(y - x) - V(y) - (\alpha/2)\|y - x\|^2 \end{aligned} \tag{4.7}$$

holds for all $y \in X$ satisfying $y \neq x$, then it follows that h is the optimal policy function of a model $M \in \mathcal{M}_*(X)$. Furthermore, the restriction of V to X is the optimal value function of M.
(ii) In addition to the conditions in part (i) assume that, for all $y \in X$, the functions $x \mapsto p_y x - (\alpha/2)\|x - y\|^2$ and $x \mapsto q_y x + (\alpha/2)\|x - h(y)\|^2$ are non-decreasing. Then M can be chosen such that $M \in \mathcal{M}_+(X)$.

Theorem 4.3.3 is a variant of Mitra and Sorger [14, theorem 2]. The most important difference between the present theorem and [14, theorem 2] is that the former requires inequality (4.7) to hold for some $\alpha > 0$, while the latter assumes (4.7) with $\alpha = 0$. This makes the assumptions of the present theorem stronger than those of [14, theorem 2]. Because of the stronger assumption, I can prove rationalizability by $\mathcal{M}_*(X)$, whereas Mitra and Sorger [14] prove rationalizability by the class of optimal growth models which satisfy assumptions A1-A3 and which have a strictly concave optimal value function.[10] It has been shown by Mitra and Sorger [14] that inequality (4.7) with $\alpha = 0$ is not only sufficient but also necessary for the property that h is an optimal policy function and V the corresponding optimal value function; see theorem 4.4.1 in section 4.4 below. The proof of theorem 4.3.3 is based on the fact that knowing the optimal policy function and the optimal value function of a model M provides complete information about the value and the slope of the utility function u at every point along the graph of h. Provided that condition (4.7) is satisfied, one can construct a feasible utility function that is consistent with this information and that can be used to rationalize the pair (h, V). Let me go through this argument in more detail. From (4.3) it follows that h and V together completely determine the value of the utility function u along the graph of h, that is, $u(x, h(x)) = V(x) - \delta V(h(x))$ for all $x \in X$. Knowledge of V together with the fact that $y = h(x)$ maximizes the right-hand side of the Bellman equation (4.1) gives me information about the slope of the function $y \mapsto u(x, y)$ at $y = h(x)$. Finally, knowledge of V together with equation (4.1) and the envelope theorem gives me information about the slope of the function $x \mapsto u(x, y)$ at $y = h(x)$. To summarize, if I know h and V, then I know the value and the slope of u along the graph of h. This allows me to construct, for every $z \in X$, a strongly concave function $F_z : X \times X \mapsto \mathbb{R}$ that has the same value and slope as u at the point $(z, h(z))$. The function u that is used to

[10] Because of the slightly different assumptions and conclusions of theorem 4.3.3 and [14, theorem 2], I provide a proof of theorem 4.3.3 in the appendix.

rationalize h is then defined as the lower envelope of the family $\{F_z \mid z \in X\}$. For this construction to work, one has to make sure that the graphs of all functions F_z are located above the curve $u = u(x, h(x)) = V(x) - \delta V(h(x))$ in (x, y, u)-space. In other words, it must hold that $F_z(x, h(x)) \geq V(x) - \delta V(h(x))$ for all $(x, z) \in X \times X$. This is exactly what condition (4.7) guarantees. Theorem 4.3.3 implies that every Lipschitz-continuous function $h : X \mapsto X$ can be rationalized by $\mathcal{M}_*(X)$. This can be seen as follows. Suppose that h is Lipschitz-continuous with Lipschitz-constant K and define $V : \mathbb{R}^n \mapsto \mathbb{R}$ by $V(x) = \mu e_n x - (\beta/2)\|x\|^2$, where $e_n = (1, 1, \ldots, 1) \in \mathbb{R}^n$, and where β and μ are positive constants to be determined later. Obviously, V is strictly concave and continuously differentiable. The latter property along with compactness of X proves that $\bigcup_{x \in X} \partial V(x)$ is compact. More specifically, it holds for all $x \in X$ that $\partial V(x) = \{p_x\}$ and $\partial V(h(x)) = \{q_x\}$ where $p_x = \mu e_n - \beta x$ and $q_x = \mu e_n - \beta h(x)$. It is straightforward to see that this implies

$$V(x) + p_x(y - x) - V(y) - (\alpha/2)\|y - x\|^2 = (\beta - \alpha)\|y - x\|^2/2.$$

Analogously, one can derive

$$V(h(x)) + q_x[h(y) - h(x)] - V(h(y)) + (\alpha/2)\|h(y) - h(x)\|^2 =$$

$$(\beta + \alpha)\|h(y) - h(x)\|^2/2.$$

Condition (4.7) can therefore be written as

$$\delta(\beta + \alpha)\|h(y) - h(x)\|^2 \leq (\beta - \alpha)\|y - x\|^2.$$

Obviously, this condition holds whenever $\beta > \alpha$ and $\delta \leq (\beta - \alpha)/[(\beta + \alpha)K^2]$. Because α can be any strictly positive number, the conditions of theorem 4.3.3(i) can be satisfied for any $\delta < 1/K^2$. Finally, if one chooses the parameter μ sufficiently large, one can always ensure that the monotonicity conditions stated in theorem 4.3.3(ii) are satisfied. Thus, I have proved the following variant of a result by Mitra and Sorger [14].

Theorem 4.3.4. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let $h : X \mapsto X$ be a uniformly Lipschitz-continuous function with Lipschitz-constant K. For every $\delta < 1/K^2$ there exists a model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}_+(X) \cap \mathcal{M}_*(X)$ such that h is the optimal policy function of M.*

Mitra and Sorger [14] prove rationalizability of Lipschitz-continuous functions by models satisfying A1-A3 which have a strictly concave optimal value function. This allows them to choose discount factors smaller than *or equal to* $1/K^2$. In contrast, theorem 4.3.4 requires the discount factor δ to be strictly smaller than $1/K^2$ but ensures rationalizability by strongly concave optimal growth models. In any case one can see that, whenever the Lipschitz-constant K is high, the discount factor δ has to be chosen small. This is in line with the remark made after theorem 4.3.2, namely that rationalizing a given function h typically requires a very small discount factor. I will return to this issue in section 4.4 below. I would like to point out that the construction used in the proof of theorem 4.3.4 uses a quadratic polynomial as optimal value function. This polynomial is $(-\beta)$-convex, where β is a positive number. Combining theorem 4.3.4 with theorem 4.3.1(iii), I obtain the following corollary. It provides a complete characterization of the class of functions that can be rationalized by strongly concave optimal growth models with optimal value functions that are $(-\beta)$-convex for some $\beta > 0$.

Corollary 4.3.1. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let $h : X \mapsto X$ be a continuous function. The following two statements are equivalent.*
(i) The function h is uniformly Lipschitz-continuous on X.
(ii) There exists an optimal growth model $M \in \mathcal{M}_*(X)$ such that h is the optimal policy function of M and such that the optimal value function of M is $(-\beta)$-convex for some $\beta > 0$.

For the more comprehensive and more interesting class $\mathcal{M}(X)$, a complete characterization of the set of corresponding optimal policy functions is not yet available. In the remainder of this section I will illustrate that closing the gap between the necessary condition of continuity (theorem 4.3.1(i)) and the sufficient condition of Lipschitz-continuity (theorem 4.3.4) is likely to be a tricky problem. Throughout this discussion I restrict myself to one-dimensional state spaces. Hewage and Neumann [5] where the first to point out that there exist continuous functions that cannot be rationalized by $\mathcal{M}(X)$. The following version of their result is taken from Mitra and Sorger [15].

Theorem 4.3.5. *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $h : X \mapsto X$ be a continuous function which has the fixed point $x = h(x) \in X$. Assume that there exists $z \in \operatorname{int} X$ such that $x = h(z)$. If $\limsup_{y\to x}[h(y) - h(x)]/(y-x) = +\infty$, then h cannot be the optimal policy function of an optimal growth model in $\mathcal{M}(X)$.*

Theorem 4.3.5 rules out any function as an optimal policy function which has the slope $+\infty$ at a fixed point x which is either in the interior of the state space (this is the case if $z = x$ in theorem 4.3.5) or which can be reached along a trajectory of h emanating from the interior of X. For example, the continuous function

$$h_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

defined on the state space $X = [-1, 1]$ cannot be an optimal policy function of any optimal growth model in $\mathcal{M}(X)$. On the other hand, the continuous function

$$h_2(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ -\sqrt{x} & \text{if } x > 0, \end{cases}$$

which is also defined on $X = [-1, 1]$, can be rationalized by $\mathcal{M}(X)$, as has been shown by Mitra and Sorger [14]. Since this function has the slope $-\infty$

at an interior fixed point, it follows that optimal policy functions need not be Lipschitz-continuous at interior points and that they can have (negative) infinite steepness at interior fixed points. Example 4.3.1 discussed above shows that one cannot omit the interiority requirement from theorem 4.3.5. As a matter of fact, in that example, the optimal policy function has slope $+\infty$ at the fixed point $x = 0$, which cannot be reached along any optimal path starting in the interior of the state space $X = [0, 1]$. Finally, Mitra and Sorger [15] show that the function

$$h_3(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ -1 + \sqrt{x} & \text{if } x > 0 \end{cases}$$

defined on the state space $X = [-1, 1]$ can be rationalized by $\mathcal{M}(X)$. This example demonstrates that optimal policy functions can have the slope $+\infty$ at an interior point that is not a fixed point, here $x = 0$. Because of the above examples, it can be conjectured that there does not exist a simple characterization of the class of functions that can be rationalized by $\mathcal{M}(X)$. As a matter of fact, these examples seem to suggest that it is not only the smoothness of the function h itself that is crucial but also the smoothness of its iterates. To explain this point further, let me introduce the notation $h^{(t)}(x)$ for the t-th iterate of a function $h : X \mapsto X$ evaluated at $x \in X$. In other words, $h^{(0)}(x) = x$ and $h^{(t+1)}(x) = h^{(t)}(h(x))$ for all t and for all $x \in X$. Now note that all iterates of h_1 are infinitely steep at the fixed point $x = 0$ and that the limit function $\lim_{t\to+\infty} h_1^{(t)}$ is even discontinuous at $x = 0$. On the other hand, the functions h_2 and h_3, which have been shown to be rationalizable by $\mathcal{M}(X)$, have the property that their t-th iterate is not only continuous but even constant for all $t \geq 2$. It is tempting to conjecture that the striking difference between the smoothness properties of the iterates of these functions is part of the reason why h_2 and h_3 are rationalizable by $\mathcal{M}(X)$, whereas h_1 is not.

4.4 Discount Factor Restrictions

In the previous section I have already mentioned that constructing an optimal growth model that rationalizes a given function h requires typically the choice of a sufficiently small discount factor. In principle, this could be due to the specific details of the constructions that I have described. However, as I will demonstrate now, there exists a relation between certain properties of the dynamics generated by optimal policy functions and the size of the discount factor of *any* model that can rationalize these functions. In other words, the focus of the present section is on the question of what one can say about the discount factors of optimal growth models that rationalize a given function h. To get started, suppose that a continuous function $h : X \mapsto X$ is the optimal policy function of the model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}(X)$. According to lemma 4.3.2, this is the case if and only if there exist functions $V : X \mapsto \mathbb{R}$ and $G : \mathbf{T} \mapsto \mathbb{R}$ such that V is strictly concave and such that conditions (4.4) and (4.5) hold.

Now let $\bar{\delta}$ be any real number satisfying $\bar{\delta} \in (0, \delta)$. Defining $\bar{u} : \mathbf{T} \mapsto \mathbb{R}$ by $\bar{u}(x, y) = u(x, y) + (\delta - \bar{\delta})V(y)$ it is easy to see that condition (4.4) can be written as

$$G(x, y) = \bar{u}(x, y) - V(x) + \bar{\delta}V(y).$$

Strict concavity of V together with $\bar{\delta} < \delta$ implies that $\bar{u}$ is strictly concave. Applying lemma 4.3.2 again, it follows that h is the optimal policy function of the model $\bar{M} = (\mathbf{T}, \bar{u}, \bar{\delta}) \in \mathcal{M}(X)$. Furthermore, if M is an element of $\mathcal{M}_+(X)$ or $\mathcal{M}_*(X)$, then proposition 4.2.1 implies that the same is true for $\bar{M}$. This proves the following lemma; see also Sorger [30].

Lemma 4.4.1. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let $h : X \mapsto X$ be the optimal policy function of an optimal growth model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}(X)$. For every $\bar{\delta} \in (0, \delta)$, there exists a utility function $\bar{u} : \mathbf{T} \mapsto \mathbb{R}$ such that $\bar{M} = (\mathbf{T}, \bar{u}, \bar{\delta}) \in \mathcal{M}(X)$ and such that h is the optimal policy function of $\bar{M}$. Furthermore, if $M \in \mathcal{M}_*(X)$ or $M \in \mathcal{M}_+(X)$, then $\bar{u}$ can be chosen in such a way that $\bar{M} \in \mathcal{M}_*(X)$ or $\bar{M} \in \mathcal{M}_+(X)$, respectively.*

The above lemma shows that the set of discount factors that can be used to rationalize a given function h is always an interval with left endpoint 0. To get information about the right endpoint, let me state the following theorem from Mitra and Sorger [14]; see also Mitra and Sorger [15, remark 1]. This theorem can be regarded as the converse of theorem 4.3.3 for the case $\alpha = 0$.

Theorem 4.4.1. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let $h : X \mapsto X$ be the optimal policy function of a model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}(X)$. If $V : X \mapsto \mathbb{R}$ is the optimal value function of M, then the following is true. For every $x \in X$ such that $\partial V(x) \neq \emptyset$ and for every $p_x \in \partial V(x)$ there exists $q_x \in \partial V(h(x))$ such that the inequality*

$$\delta\Big\{V(h(x)) + q_x[h(y) - h(x)] - V(h(y))\Big\} < V(x) + p_x(y - x) - V(y) \quad (4.8)$$

holds for all $y \in X$ satisfying $y \neq x$.

In the rest of this section, I will discuss various implications of theorem 4.4.1.[11] More specifically, I will use theorem 4.4.1 to demonstrate that complicated dynamics can only be optimal for small discount factors. To do this, I first have to define what I mean by complicated dynamics. There exist a number of different definitions of complicated dynamics or chaos in the literature on dynamical systems. One of the most important definitions is formulated in terms of the topological entropy of a dynamical system. To explain this concept, suppose that an observer cannot distinguish between two states x and y if $\|x - y\| \leq \varepsilon$. This means that state observations are possible only with finite precision, as measured by ε. Even if two initial states are indistinguishable in this sense, it can be the case that, by observing the dynamical system $x_{t+1} = h(x_t)$ over

[11] Theorem 4.3.5 from the previous section can also be derived from theorem 4.4.1; see Mitra and Sorger [15].

a finite number of periods, say T periods, the trajectories starting in the two initial states can be distinguished. This will be the case if there exists an integer $t \in \{0, 1, 2, \ldots, T-1\}$ such that $\|h^{(t)}(x) - h^{(t)}(y)\| > \varepsilon$.[12] The topological entropy measures the rate at which different trajectories become distinguishable as the number of observations, T, increases. In other words, the topological entropy measures the rate at which information is generated by iterating h. The popular definition of complicated dynamics mentioned above requires the dynamical system under consideration to have positive topological entropy. In this case, one often says that the dynamical system exhibits topological chaos. Let me now state this definition in a more formal way. A subset $B \subseteq X$ is called (T, ε)-separated if, for any two different points x and y in B, there exists $t \in \{0, 1, 2, \ldots, T-1\}$ such that $\|h^{(t)}(x) - h^{(t)}(y)\| > \varepsilon$. Now assume that A is a compact and invariant subset of X.[13] In that case the number

$$s_{T,\varepsilon}(h, A) = \max\{\#B \mid B \subseteq A \text{ and } B \text{ is } (T, \varepsilon)\text{-separated}\}$$

is well-defined and finite. Here, $\#B$ denotes the cardinality of B. The number

$$c(A) = \limsup_{\varepsilon \to 0} \frac{\ln s_{1,\varepsilon}(h, A)}{-\ln \varepsilon}$$

is called the upper capacity of the set A. It measures the growth rate of the number of ε-balls which are required to cover A as ε approaches zero. The upper capacity of A obviously does not depend on the function h. Furthermore, it is clear that $c(A) \leq n$ must hold for every compact set $A \subseteq \mathbb{R}^n$. For any compact set $A \subseteq X$ which is invariant under h, the topological entropy of h on the set A is defined as

$$\kappa(h, A) = \lim_{\varepsilon \to 0} \left[\limsup_{T \to \infty} \frac{\ln s_{T,\varepsilon}(h, A)}{T} \right].$$

I am now ready to state the first important implication of theorem 4.4.1. It is a result due to Montrucchio and Sorger [19] that relates the topological entropy of the dynamical system $x_{t+1} = h(x_t)$ to the discount factor of any model $M \in \mathcal{M}(X)$ that can rationalize h.[14]

Theorem 4.4.2. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set and let $h : X \mapsto X$ be the optimal policy function of a model $(\mathbf{T}, u, \delta) \in \mathcal{M}(X)$. Let A be a compact subset of the interior of the state space X and assume that A is invariant under h. Then it holds that $\delta \leq e^{-\kappa(h,A)/c(A)}$.*

Although theorem 4.4.2 establishes a negative relation between the topological entropy of a function h and the discount factor of any model that can rationalize

[12] Recall that $h^{(t)}$ denotes the t-th iterate of h.

[13] A subset A of the state space X is called invariant (under h), if for every $x \in A$ it holds that $h(x) \in A$.

[14] A similar result appeared already in Montrucchio [18]. The difference between the two results is that Montrucchio and Sorger [19] deal with rationalizability by $\mathcal{M}(X)$, whereas Montrucchio [18] deals with rationalizability by $\mathcal{M}_*(X)$.

h, I have to emphasize that this does not imply that positive topological entropy can only be optimal under strong discounting. This is indeed not the case. Mitra and Sorger [15], for example, demonstrate that, for every $\delta \in (0,1)$, it is possible to construct an optimal growth model $M \in \mathcal{M}_+(X) \cap \mathcal{M}_*(X)$ such that the optimal policy function of M has positive topological entropy on X. Montrucchio [18, remark 2.2] outlines an alternative construction to prove a similar result. What one can conclude from theorem 4.4.2 is therefore only that the topological entropy of the optimal policy function must converge to 0 as the discount factor converges to 1. One drawback of theorem 4.4.2 is the requirement that the set A must be a subset of the *interior* of the state space. In particular, it is not possible to choose $A = X$. The underlying reason for the awkward requirement of interiority is that the subdifferential of V may become empty or unbounded at boundary points of X. Restricting himself to one-dimensional state spaces (i.e., $n = 1$), Mitra [13] derives a version of theorem 4.4.2 which states $\delta \leq e^{-\kappa(h,X)}$ for all optimal policy functions of models in a certain subset of $\mathcal{M}_+(X)$. The crucial assumption which he has to make is that the slope of the utility function with respect to its first argument is bounded at the left endpoint of the state space. In the appendix, I show how theorem 4.4.2 can be derived from theorem 4.4.1. Although the general idea of the proof is the same as in Montrucchio and Sorger [19] and Mitra [13], the specific details of the three proofs differ from each other. In particular, Mitra [13] does not use theorem 4.4.1 directly but a result very similar to the following theorem.

Theorem 4.4.3. *Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex set, let $h : X \mapsto X$ be the optimal policy function of a model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}(X)$, and let $V : X \mapsto \mathbb{R}$ be the optimal value function of M. Furthermore, let x and y be two different elements of the state space X and suppose that $p_x \in \partial V(x)$ and $p_y \in \partial V(y)$. Then there exist subgradients $q_x \in \partial V(h(x))$ and $q_y \in \partial V(h(y))$ such that inequality*

$$\delta(q_x - q_y)[h(y) - h(x)] < (p_x - p_y)(y - x). \tag{4.9}$$

holds.

This result is actually a simple corollary of theorem 4.4.1. Suppose that V has a non-empty subdifferential at two different points $x \in X$ and $y \in X$. I can then interchange the roles of x and y in (4.8) to get

$$\delta\Big\{V(h(y)) + q_y[h(x) - h(y)] - V(h(x))\Big\} < V(y) + p_y(x - y) - V(x).$$

Adding this inequality to (4.8), one obtains (4.9). Mitra [13] also combines his version of theorem 4.4.2 with results relating the topological entropy to the existence of periodic paths. This allows him to obtain non-trivial discount factor restrictions for optimal growth models that admit periodic optimal paths with a period that is not a power of 2. Instead of discussing these results here, I restrict myself to the case of optimal paths of period 3, for which very sharp results are available. Let me start by explaining the importance of periodic paths of period 3. Li and Yorke [8] provided one of the earliest definitions of chaos. Essentially, this definition requires a chaotic dynamical system to admit periodic paths of any period as well as an uncountable scrambled set. Here, a scrambled set is a set of points such that paths starting in those points are not even asymptotically periodic and such that paths starting from two different points in the scrambled set move apart and return close to each other infinitely often. Li and Yorke [8] showed that, in the case where X is an interval on the real line and $h : X \mapsto X$ is a continuous function, the dynamical system $x_{t+1} = h(x_t)$ exhibits chaos in the sense just described if and only if h admits a periodic path of period 3, i.e., a period-three cycle. The latter is a triple $(a, b, c) \in X^3$ such that a, b, and c are mutually different states and such that $h(a) = b$, $h(b) = c$, and $h(c) = a$. The following theorem shows that a one-dimensional map that possesses a period-three cycle can only be rationalized by an optimal growth model with discount factor $\delta < (3 - \sqrt{5})/2$. This result was proved independently by Mitra [12] and Nishimura and Yano [23]. These authors also proved that the number $(3-\sqrt{5})/2$ is the least upper bound on the set of discount factors that can be used to rationalize a one-dimensional map with a period-three cycle. Thus, the theorem provides an exact discount factor restriction for the rationalizability of chaos in the sense of Li and Yorke [8].

Theorem 4.4.4. *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval.*
(i) Suppose there exists a continuous function $h : X \mapsto X$ that admits a period-three cycle and a model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}(X)$ such that h is the optimal policy function of M. Then it follows that $\delta < (3 - \sqrt{5})/2$.
(ii) Conversely, for every $\delta < (3 - \sqrt{5})/2$, there exists a transition possibility set $\mathbf{T} \in X \times X$ and a utility function $u : \mathbf{T} \mapsto \mathbb{R}$ such that $M = (\mathbf{T}, u, \delta) \in \mathcal{M}_+(X) \cap \mathcal{M}_*(X)$ and such that the optimal policy function of M admits a period-three cycle.

Part (ii) of theorem 4.4.4 can be proved by using the construction from Nishimura et al. [21]; see also example 4.3.2 above. To explain one possible proof of part (i), suppose that (a, b, c) is a period-three cycle of h.[15] I may assume without loss of generality that a is the smallest of the three numbers. Then there are two possibilities: $b < c$ or $b > c$. Here, I consider only the first case since the second one can be dealt with analogously. Leaving out some of the technical details, it follows from theorem 4.4.3 that there exist subgradients $p_a = q_c \in \partial V(a)$, $p_b = q_a \in \partial V(b)$, and $p_c = q_b \in \partial V(c)$ such that (4.9) holds for $(x, y) = (a, b)$, $(x, y) = (a, c)$, and $(x, y) = (b, c)$, respectively. From $a < b < c$ and the strict concavity of V it follows that $p_a = q_c > p_b = q_a > p_c = q_b$. Applying (4.9) to $(x, y) = (a, b)$, one obtains

$$\delta < [(p_a - p_b)/(p_b - p_c)][(b - a)/(c - b)]. \tag{4.10}$$

15 The following discussion uses arguments similar to those in Mitra [12]. A completely different proof can be found in Nishimura and Yano [23].

Analogously, applying (4.9) to $(x,y)=(a,c)$ and $(x,y)=(b,c)$, one gets

$$\delta < [(p_a - p_c)/(p_a - p_b)][(c-a)/(b-a)], \tag{4.11}$$
$$\delta < [(p_b - p_c)/(p_a - p_c)][(c-b)/(c-a)]. \tag{4.12}$$

It is obvious that both factors in brackets on the right-hand side of (4.11) are greater than 1 such that (4.11) does not restrict δ in a non-trivial way. Defining $z = (c-b)/(c-a)$ and $q = (p_b - p_c)/(p_a - p_c)$, I can rewrite the non-trivial constraints (4.10) and (4.12) as $\delta < (1/q - 1)(1/z - 1)$ and $\delta < qz$. Note that both $z \in (0,1)$ and $q \in (0,1)$ holds. To summarize, if the optimal policy function of $(\mathbf{T}, u, \delta) \in \mathcal{M}(X)$ admits a period-three cycle (a,b,c) satisfying $a < b < c$ and $(c-b)/(c-a) = z$, then it follows that

$$\delta < \max\left\{\min\{(1/q-1)(1/z-1), qz\} \,|\, q \in (0,1)\right\}.$$

It is straightforward to verify that the right-hand side of this inequality is equal to

$$\bar{\delta}(z) = \frac{2z\sqrt{1-z}}{\sqrt{1-z+4z^2}+\sqrt{1-z}}.$$

Since z can take any value between 0 and 1, a discount factor restriction that holds uniformly for all period-three cycles (a,b,c) with $a < b < c$ can be derived by maximizing $\bar{\delta}(z)$ over $z \in (0,1)$. Using simple calculus, I obtain

$$\max\{\bar{\delta}(z) \,|\, z \in (0,1)\} = \bar{\delta}\left((\sqrt{5}-1)/2\right) = (3-\sqrt{5})/2. \tag{4.13}$$

As I have already mentioned before, period-three cycles (a,b,c) with $a < c < b$ can be dealt with in exactly the same way and yield the same discount factor restriction. This completes the proof of theorem 4.4.4(i). Theorem 4.4.4 is remarkable in a number of ways. Not only is it an exact discount factor restriction for the optimality of period-three cycles, but it also relates the existence of optimal period-three cycles to the so-called golden section or golden ratio, a number that Euclid has already mentioned in the second book of his *Elements* about 300 years before Christ. As a matter of fact, the golden section is given by $\gamma = (\sqrt{5}-1)/2$, which is exactly the square root of the number stated in theorem 4.4.4. Note also that the value of z for which the discount factor restriction for period-three cycles is actually attained is $z = \gamma$; see equation (4.13). This means that all period-three cycles that can be rationalized using the highest possible discount factor have the property that $z = (c-b)/(c-a) = \gamma$. In other words, these cycles have the property that the point b divides the interval (a,c) according to the golden section. In principle, one could try to derive discount factor restrictions for the rationalizability of period-p cycles with $p \neq 3$ using the same strategy as in the proof of theorem 4.4.4(i). I expect that this would lead to non-trivial discount factor restrictions whenever p is not a power of 2, that is, whenever the optimal policy function has positive topological entropy. However, the number of cases to be considered and the analytical complexity of such a proof is daunting and I am not aware of any attempt at carrying out this project. Let me conclude this section by giving exact discount factor restrictions for two important examples. The first example is the tent map $h : [0,1] \mapsto [0,1]$ defined by $h(x) = 1 - |2x - 1|$. I have already discussed this map in example 4.3.2, where I have shown that it can be rationalized by models in $\mathcal{M}_*([0,1])$ with discount factors smaller than 1/4. Rationalizability of the tent map by strongly concave optimal growth models with discount factors smaller than 1/4 follows also from theorem 4.3.4, because the tent map is uniformly Lipschitz-continuous with Lipschitz-constant $K = 2$. Mitra and Sorger [15] show that 1/4 is indeed the least upper bound on the set of discount factors that can be used to rationalize the tent map. To summarize, the following theorem holds.

Theorem 4.4.5. *Let $X = [0,1]$ and define $h : X \mapsto X$ by $h(x) = 1 - |2x - 1|$ for all $x \in X$.*
(i) If there exists a model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}(X)$ such that h is the optimal policy function of M, then it follows that $\delta \leq 1/4$.
(ii) For every $\delta < 1/4$ there exists a transition possibility set $\mathbf{T}$ and a utility function $u : \mathbf{T} \mapsto \mathbb{R}$ such that $(\mathbf{T}, u, \delta) \in \mathcal{M}_*(X)$ and such that h is the optimal policy function of M.

The second example that I want to mention is the logistic map $h(x) = 4x(1-x)$, which is also defined on the state space $X = [0,1]$. Along with the tent map, the logistic map is one of the most prominent examples of chaotic dynamics. Since h is Lipschitz-continuous with Lipschitz-constant $K = 4$, it follows from theorem 4.3.4 that the logistic map can be rationalized by strongly concave optimal growth models with discount factors smaller than 1/16. Also in this case, Mitra and Sorger [15] prove that 1/16 provides the least upper bound on the set of discount factors that can be used to rationalize h.

Theorem 4.4.6. *Let $X = [0,1]$ and define $h : X \mapsto X$ by $h(x) = 4x(1-x)$ for all $x \in X$.*
(i) If there exists a model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}(X)$ such that h is the optimal policy function of M, then it follows that $\delta \leq 1/16$.
(ii) For every $\delta < 1/16$ there exists a transition possibility set $\mathbf{T}$ and a utility function $u : \mathbf{T} \mapsto \mathbb{R}$ such that $(\mathbf{T}, u, \delta) \in \mathcal{M}_*(X)$ and such that h is the optimal policy function of M.

Montrucchio [18] has already shown that rationalizability of the tent map by a regular model, that is, by a model in $\mathcal{M}^*(X)$ that has an optimal value function which is $(-\beta)$-convex for some positive number β, requires a discount factor smaller than 1/4. Analogously, he has shown that rationalizability of the logistic map by the same class of models requires $\delta < 1/16$. The results stated in part (i) of theorem 4.4.5 and theorem 4.4.6, respectively, are stronger, because they are necessary conditions for rationalizability by the larger class $\mathcal{M}(X)$. To summarize the results of the present section, there exists a mathematically rigorous interpretation of the frequently expressed statement that complicated dynamics can be optimal only under heavy discounting.

4.5 Extensions

In the last section of this chapter I want to discuss if and how the results presented above can be generalized to other classes of optimal growth models. First of all, I would like to mention that part of the literature covered by the present survey studies rationalizability by classes of optimal growth models that are defined slightly differently from $\mathcal{M}(X)$, $\mathcal{M}_+(X)$, or $\mathcal{M}_*(X)$. For example, as I have already mentioned before, the class of models satisfying assumptions A1-A3 as well as strict concavity of the optimal value function is often considered. Using that definition, however, is problematic because strict concavity of the optimal value function is a derived property which cannot be checked easily from the primitives $\mathbf{T}$, u, and δ of the model. For this reason, I have focussed on the classes $\mathcal{M}(X)$, $\mathcal{M}_+(X)$, and $\mathcal{M}_*(X)$, which are defined by properties of $\mathbf{T}$, u, and δ only. In any case, it seems that rationalizability results for $\mathcal{M}(X)$ do not differ significantly from corresponding results for optimal growth models satisfying A1-A3 with strictly concave optimal value functions. A more significant departure from the setting of the present study consists in adding more structural properties. Some of the work by Mitra (e.g., [12] and [13]) uses a framework in which the state space is one-dimensional and the structural properties of the model go beyond those postulated in A1-A5. Similarly, Mitra and Sorger [14, section 4] consider optimal growth models with n-dimensional state spaces that satisfy slightly more restrictive properties than A1-A5. In this framework they derive very strong versions of theorems 4.3.3 and 4.4.1. However, all of these papers still use the reduced form of optimal growth models. Only very little work has been done on questions of rationalizability for classes of primitive models, i.e., models with explicit descriptions of the production technologies for consumption goods and capital goods. A possibly incomplete list of papers that study models in primitive form (at least by means of examples) and address questions related to those discussed in the present survey are Boldrin and Montrucchio [1], Majumdar and Mitra [9], Nishimura et al. [21], and Nishimura and Yano [22]. Throughout the present chapter I have assumed that time evolves in discrete periods. Alternatively, one could use a continuous-time formulation. Some of the results stated in earlier sections have continuous-time counterparts, others do not. For example, theorem 4.3.2 about the rationalizability of twice continuously differentiable functions as optimal policy functions has also been proved for continuous-time models; see Montrucchio [17] or Sorger [26]. As for the stronger rationalizability theorems 4.3.3 and 4.3.4, I am not aware of any published version for the continuous-time case, but I conjecture that such results can be proved using essentially the same arguments as in the discrete-time case. The technical details, however, are likely to be more complicated. In a continuous-time framework, the Bellman equation is a partial differential equation and a continuous-time version of the rationalizability condition (4.7) would probably involve second-order partial derivatives of the optimal value function V. Thus, in addition to strict concavity of V, a continuous-time version of theorem 4.3.3 would have to contain sufficient smoothness assumptions for V. This would not pose any problem for the proof of the continuous-time version of theorem 4.3.4, because in that proof one chooses a smooth quadratic polynomial for V anyway. From these arguments it is also clear that a reformulation of the necessary rationalizability theorem 4.4.1 would probably require an additional smoothness condition for the optimal value function V. I do not know whether this would still allow one to derive a continuous-time version of theorem 4.4.2 following the arguments outlined in the appendix, but I guess that the result itself remains valid in continuous time. Theorems 4.4.4-4.4.6 and their proofs, on the other hand, are deeply rooted in the particular properties of one-dimensional maps (i.e., discrete-time dynamical systems on one-dimensional state spaces) which is why there cannot exist analogous results in a continuous-time framework. The reduced-form optimal growth model as defined in section 4.2 above uses the standard model of time-preference, i.e., an objective functional which is additively separable across time and which involves a constant discount factor (geometric discounting). This model of time-preference has been generalized in at least two ways. The first one replaces additive separability by a weaker separability condition but maintains stationarity of preferences (recursive or stationary utility functionals; see Koopmans [6]), whereas the second one maintains additive separability but gets rid of stationarity (non-geometric discounting; see Strotz [33]). Let me briefly discuss the implications of these two generalizations for the results stated in the present survey. First of all, since the standard model of time-preference is a special case of both the class of models with recursive utility functionals and the class of models with general discounting functions, it follows that every function h that can be rationalized by a model in $\mathcal{M}(X)$ can also be rationalized by these more general optimal growth models. Thus, the non-trivial questions regarding these generalizations are (i) whether the more general classes of optimal growth models can be used to rationalize more functions and (ii) whether the discount factor restrictions do still apply in the more general settings. I am not aware of any attempts to answer the first of these questions. As for the second question, however, some work has been done. Sorger [28] discusses issues related to discount factor restrictions in models with general recursive utility functionals. He assumes that the aggregator function, which aggregates instantaneous utility and total future utility, is uniformly Lipschitz-continuous with respect to future utility and he takes the corresponding Lipschitz-constant as a measure of impatience. He is then able to show that rationalizability of a given function h implies, in general, that there are non-trivial restrictions on this measure of impatience. Although he does not deal with the particular discount factor restrictions discussed in section 4.4 of the present survey, I conjecture that all of them have straightforward generalizations to the framework considered in [28]. If this conjecture is true, it means that the additive separability of preferences is not an important assumption for the discount factor restrictions from section 4.4 as long as stationarity of preferences is maintained. The situation is quite different for the second generalization that I mentioned above, namely the assumption of non-geometric discounting. There is overwhelming experimental evidence that discount factors applied by actual decision makers in the immediate future are smaller than those applied by the same individuals in the distant future. As Strotz [33] has pointed out, this may lead to dynamic inconsistency of optimal plans. The most popular solution to the inconsistency problem is to assume that the decision maker plays an intra-personal game between the different selves of which he or she consists and that the implemented solution is a stationary Markov-perfect equilibrium of that game. Sorger [31] shows that one cannot expect any discount factor restrictions to remain valid in this setting. In particular, he derives a version of theorem 4.3.2 in which any discount factor smaller than 1 can be fixed in advance. More specifically, for every given set of short-term and long-term discount factors (different from each other), it holds that any twice continuously differentiable function h that satisfies a certain curvature assumption can be the equilibrium strategy in the aforementioned intra-personal game. Sorger [31] also shows that this curvature assumption does not rule out functions like the logistic map. Thus, one can conclude that the stationarity of preferences does form an important assumption for the discount factor restrictions of section 4.4. In other words, the strategic interaction of different selves (introduced by the requirement of dynamically consistent behavior under non-stationary preferences) allows for a significantly different behavior of the optimal solutions, even if the discount factors for the immediate and the distant future differ from each other only slightly. My final remark deals with stochastic optimal growth models. Such models are described by a transition possibility set, a utility function, a discount factor, and a description of the stochastic process of shocks; see, e.g., Stokey and Lucas [32]. Under appropriate assumptions, solutions of such models can again be characterized as the trajectories of a (stochastic) dynamical system. In discrete time, the optimal policy function maps pairs (x, z), where x is the state of the economy and z is the current shock, into the state space. As far as I know, there do not exist any results for stochastic optimal growth models in discrete time which are similar to those discussed in the present survey. The continuous-time case seems to be more tractable because, in this case, the policy function maps the state space into the control space, i.e., the policy function does not depend on the shock variable. Chang [3] studies the inverse problem in a continuous-time, stochastic framework with a single state variable. His main result is that every twice continuously differentiable and increasing function can be rationalized by a stochastic optimal growth model.

Appendix

Proof of theorem 4.3.3: In order to prove part (i) of the theorem, I will construct a model $M = (\mathbf{T}, u, \delta) \in \mathcal{M}_*(X)$ which rationalizes h and which possesses V as its optimal value function. As transition possibility set I choose $\mathbf{T} = X \times X$. The discount factor δ is assumed to be given. It remains to specify the utility function $u : \mathbf{T} \mapsto \mathbb{R}$. To this end, I first define for every $z \in X$ a function $F_z : \mathbf{T} \mapsto \mathbb{R}$ by

$$F_z(x,y) = V(z) + p_z(x-z) - (\alpha/2)\|x-z\|^2 -$$
$$\delta\Big\{V(h(z)) + q_z[y-h(z)] + (\alpha/2)\|y-h(z)\|^2\Big\}.$$

Note that $F_z(x,y)$ is a $\alpha\delta$-concave, quadratic polynomial with respect to (x,y). Furthermore, because h and V are continuous and because X and $\bigcup_{x \in X} \partial V(x)$ are compact, it follows that $\inf\{F_z(x,y) \,|\, z \in X\}$ is finite for all $(x,y) \in \mathbf{T}$. Let me therefore define the utility function $u : \mathbf{T} \mapsto \mathbb{R}$ by $u(x,y) = \inf\{F_z(x,y) \,|\, z \in X\}$. Because u is the infimum of a family of $\alpha\delta$-concave functions, it is itself $\alpha\delta$-concave and, hence, strongly concave. Thus, the model $M = (\mathbf{T}, u, \delta)$ is a member of the family $\mathcal{M}_*(X)$. I claim that M rationalizes h and I am going to prove this claim by applying lemma 4.3.2. Using condition (4.4) as the definition of the function $G : \mathbf{T} \mapsto \mathbb{R}$, all I need to do is verify (4.5). To this end, first note that

$$\begin{aligned} G(x,y) &= u(x,y) - V(x) + \delta V(y) \\ &\leq F_x(x,y) - V(x) + \delta V(y) \\ &= -\delta\{V(h(x)) + q_x[y-h(x)] - V(y)\} - (\alpha\delta/2)\|y-h(x)\|^2 \\ &\leq 0. \end{aligned}$$

The first line is the definition of G from (4.4), the second line follows from the definition of u, the third line follows from the definition of $F_z(x,y)$ for $z = x$, and the fourth line follows from the strict concavity of V. Thus, I have proved that $G(x,y) \leq 0$ holds for all $(x,y) \in \mathbf{T}$. Now consider the equality in (4.5). It is easy to see that

$$F_x(x, h(x)) - V(x) + \delta V(h(x)) = 0.$$

Condition (4.7) implies that $F_x(y, h(y)) \geq F_y(y, h(y))$. Interchanging the roles of x and y, I obtain

$$F_y(x, h(x)) \geq F_x(x, h(x))$$

for all $(x,y) \in X \times X$. Combining the last two formulas it follows that

$$\begin{aligned} G(x, h(x)) &= u(x, h(x)) - V(x) + \delta V(h(x)) \\ &= \inf\{F_y(x, h(x)) \,|\, y \in X\} - V(x) + \delta V(h(x)) \\ &= F_x(x, h(x)) - V(x) + \delta V(h(x)) \\ &= 0. \end{aligned}$$

Thus, conditions (4.4) and (4.5) hold and lemma 4.3.2 implies that h and V are the optimal policy function and the optimal value function, respectively, of M. This concludes the proof of part (i). Now assume that the additional condition in part (ii) is satisfied. This condition implies immediately that, for all $z \in X$, the function $F_z(x, y)$ is non-decreasing with respect to x and non-increasing with respect to y. Obviously, these monotonicity properties are inherited by u such that assumption A5(ii) holds. Since $\mathbf{T} = X \times X$ trivially satisfies assumption A5(i), it follows that $M \in \mathcal{M}_+(X)$. This proves part (ii) of theorem 4.3.3.

Proof of theorem 4.4.2: I need a few preliminary results. Since A is a compact set in the interior of X, it follows that V must be uniformly Lipschitz-continuous on A. Denoting the Lipschitz-constant by K, I obtain

$$V(x) + p_x(y - x) - V(y) \leq |V(x) - V(y)| + \|p_x\|\|y - x\| \leq 2K\|y - x\|. \quad (4.14)$$

Now let $\varepsilon > 0$ be given and consider the function $f : X \times X \times \mathbb{R}^n \mapsto \mathbb{R}$ defined by $f(x, y, p) = V(x) + p(y - x) - V(y)$. Strict concavity and continuity of V implies that f is continuous and that $f(x, y, p) > 0$ holds for all triples $(x, y, p) \in X \times X \times \mathbb{R}^n$ with $x \neq y$ and $p \in \partial V(x)$. Furthermore, I define the set $S(\varepsilon) = \{(x, y, p) \,|\, x \in A, y \in A, \|y - x\| \geq \varepsilon, p \in \partial V(x)\}$. Compactness of A together with the fact that the correspondence $x \mapsto \partial V(x)$ is compact-valued and upper semicontinuous implies compactness of $S(\varepsilon)$. It follows from these properties that f attains its minimum on $S(\varepsilon)$ and that this minimum is strictly positive. Thus, there exists a number $L(\varepsilon) > 0$ such that

$$V(x) + p_x(y - x) - V(y) > L(\varepsilon) \quad (4.15)$$

holds whenever $x \in A$, $y \in A$, $\|y - x\| \geq \varepsilon$, and $p_x \in \partial V(x)$. Let B be a (T, ε)-separated subset of A. Then there exists for every pair $(x, y) \in B \times B$ with $x \neq y$ a number $t \in \{0, 1, 2, \ldots, T - 1\}$ such that $\|x_t - y_t\| > \varepsilon$, where $x_t = h^{(t)}(x)$ and $y_t = h^{(t)}(y)$. From (4.8) and (4.14)-(4.15) it follows that

$$2K\|y - x\| \geq V(x) + p_x(y - x) - V(y) >$$

$$\delta^t \left[V(x_t) + q_{x_{t-1}}(y_t - x_t) - V(y_t)\right] > \delta^t L(\varepsilon).$$

Because of $t < T$, this implies that $\|y - x\| > \delta^T L(\varepsilon)/(2K)$ which shows that the set B must be $(1, \mu(T, \varepsilon))$-separated where $\mu(T, \varepsilon) = \delta^T L(\varepsilon)/(2K)$. Consequently, we must have $\#B \leq s_{1,\mu(T,\varepsilon)}(h, A)$. This, in turn, implies that $s_{T,\varepsilon}(h, A) \leq s_{1,\mu(T,\varepsilon)}(h, A)$. Defining

$$\kappa_\varepsilon(h, A) = \limsup_{T \to \infty} \frac{\ln s_{T,\varepsilon}(h, A)}{T},$$

I therefore obtain

$$
\begin{aligned}
\kappa_\varepsilon(h,A) &\le \limsup_{T\to\infty}\frac{\ln s_{1,\mu(T,\varepsilon)}(h,A)}{T}\\
&= \limsup_{T\to\infty}\left[\left(\frac{\ln s_{1,\mu(T,\varepsilon)}(h,A)}{-\ln\mu(T,\varepsilon)}\right)\left(\frac{-\ln\mu(T,\varepsilon)}{T}\right)\right]\\
&= \limsup_{\mu\to 0}\frac{\ln s_{1,\mu}(h,A)}{-\ln\mu}\lim_{T\to\infty}\frac{-\ln\mu(T,\varepsilon)}{T}\\
&= -(\ln\delta)c(A).
\end{aligned}
$$

This implies $\kappa(h,A) = \lim_{\varepsilon\to 0}\kappa_\varepsilon(h,A) \le -(\ln\delta)c(A)$ which completes the proof of the theorem.

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5. On Stationary Optimal Stocks in Optimal Growth Theory: Existence and Uniqueness Results

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5.1 Introduction

The concept of a non-trivial stationary optimal stock (SOS) plays a central role in the theory of optimal intertemporal allocation over an infinite horizon. While the optimal policy correspondence describes fully optimal behavior in such models, it is quite difficult to compute it accurately, and it can be solved in explicit form in only a very few highly specialized examples.

However, if non-stationary optimal programs, after a period of transition, are close to a certain stationary program (and the transition period is not very long), then their behavior can be approximately described by the stationary optimal program. Thus, even though it is only by accident that an economy has exactly a stationary optimal stock as its initial stock, a study of the existence, uniqueness and (local and global) stability of stationary optimal programs is of considerable significance.

Furthermore, if one is interested in comparative dynamics in this framework, one observes that it might be very difficult to get definitive results for policy purposes by varying a parameter and seeing the effect of it on the entire optimal policy correspondence. On the other hand, if the stationary optimal program is at least locally stable, then one can often predict the change in the stationary optimal program following a "small" change in a parameter, and this can enable one to conduct local comparative dynamics exercises in this framework.

In this essay, we present the basic results on the existence and uniqueness of (non-trivial) stationary optimal programs. A comprehensive account of the

[1] Our intellectual debt to William Brock and Lionel McKenzie, for our understanding of the subject matter of this survey, should be quite obvious. In writing this survey, we have relied heavily on our collaborative research with Jess Benhabib, Swapan Dasgupta, and Ali Khan.

stability (or turnpike) property of stationary optimal programs is already available in McKenzie (1986), and we refer the reader to his definitive study of this topic.

The existence of a stationary optimal stock (briefly, SOS) in multi-sector optimal growth models has been shown by Sutherland (1970) Hansen and Koopmans (1972), Peleg and Ryder (1974), Cass and Shell (1976), Flynn (1980), McKenzie (1982, 1986) and Khan and Mitra (1986), among others. We follow very closely the approach in Khan and Mitra (1986).

The demonstration of existence typically consists of three separate steps. First, a fixed point argument is used to show the existence of what we call in the sequel, a discounted golden-rule stock. Second, a separation argument in the form of the Kuhn-Tucker theorem is used to provide a "price-support" to the discounted golden-rule stock. Finally, a computation based on the price support property is used to show that the discounted golden-rule stock is optimal among all programs starting from that stock.

This approach, relying on duality theory (in the second and third steps), is followed by Peleg and Ryder (1974), Cass and Shell (1976), Flynn (1980), McKenzie (1982, 1986). An exception to this is Sutherland (1970) who relies on methods of dynamic programming and is able to avoid supporting prices and the Kuhn-Tucker theorem. However, Sutherland does not establish the existence of a *non-trivial* SOS, and as noted by Peleg and Ryder (1974), the null stock is always a SOS in a set-up which allows for the possibility of inaction, and does not allow production of positive outputs from zero inputs.

Khan and Mitra (1986) use a purely primal approach to the existence of a non-trivial SOS, and by a simple computation based on Jensen's inequality, establish that a discounted golden-rule stock is always a SOS. Thus, once the fixed point argument (the first step in the three-step argument indicated above) ensures the existence of a discounted golden-rule stock, the existence of a stationary optimal stock is also assured. This primal approach does not suffer from the shortcoming noted in the dynamic programming method, for it is simple to identify a condition on the economy (known as $\delta - normality$) which ensures that the discounted golden-rule stock (and therefore the corresponding stationary optimal program) is non-trivial.

The existence of a discounted golden-rule stock therefore emerges as a key concept of this subject. The idea is to approach an infinite-horizon optimization problem by solving an appropriate two-period optimization problem.

A direct payoff of the primal approach of Khan and Mitra (1986) is that an assumption frequently used in this literature (known as $\delta - productivity$) can be dispensed with, since its role is simply to ensure that Slater's condition holds when one invokes the Kuhn-Tucker theorem (in the second step of the three-step argument).

Following Khan and Mitra (1986), we also use a purely primal approach to show that a SOS, k, is always a discounted golden-rule stock, provided (k, k) is in the interior of the technology set. This result is proved by McKenzie (1986), relying on duality methods. Again, the proof involves three steps. First, a sequence of prices is found to support the stationary optimal program, following the approach of Weitzman (1970). Second, by an argument due to Sutherland (1967), a "quasi-stationary" price support is obtained from the above sequence of supporting prices. Third, this (quasi-stationary) price support property is used to show that the SOS is a discounted golden-rule stock. In dispensing with support prices, we provide a direct and short proof. We also present an example to show that the result fails when (k, k) is not in the interior of the technology set.

In general, when future utilities are discounted (as we are assuming in our framework throughout), there can be multiple (non-trivial) stationary optimal stocks (even when the utility function of the economy is strictly concave, unlike in the undiscounted case). Examples of economies with more than one non-trivial stationary optimal stock were given by Kurz (1968), Liviatan and Samuelson (1969) and Sutherland (1970). However, for some classes of models, one can provide sufficient conditions under which there can be only one non-trivial SOS.

We present two distinct approaches to the uniqueness issue. First, in an economy in which production is described by a simple linear model involving no joint production, and the utility (derived from consumption alone) satisfies a normality assumption, we show that there is exactly one non-trivial stationary optimal stock, using the methods of convex analysis (and, in particular, duality theory). We also provide an example where the normality assumption is violated and there are multiple non-trivial stationary optimal stocks. These results illustrate the somewhat more general investigations along these lines presented in Brock (1973) and Brock and Burmeister (1976).

Second, using the methods of differential topology, and relying on assumptions on the Jacobian obtained from the Ramsey-Euler equations (which hold for an interior stationary optimal stock in a model in which the utility function is twice continuously differentiable in the interior of the technology set), one can view the uniqueness result for interior stationary optimal stocks in the discounted case as following from the uniqueness result in the undiscounted case. Our approach follows Benhabib and Nishimura (1979), which generalizes a result along these lines by Brock (1973).

5.2 Preliminaries

5.2.1 Notation

Let $\mathbb{N}$ be the set of non-negative integers $\{0, 1, 2, ...\}$, and let n-dimensional Euclidean space be denoted by $\mathbb{R}^n$, where $\|x\|$ denotes the Euclidean norm of any element x in $\mathbb{R}^n$. For any x, y in $\mathbb{R}^n$, we shall write $x \gg y (x \geq y)$ to denote $x_i > y_i (x_i \geq y_i)$ for all coordinates $i = 1, ..., n$; and $x > y$ to denote $x \geq y$ and $x \neq y$. For any set, S, the set of all subsets of S will be denoted by $\mathcal{B}(S)$ and hence we shall write $\phi : X \to \mathcal{B}(Y)$ for any correspondence (set-valued map) ϕ

with domain X and range $\mathcal{B}(Y)$. Finally, let e denote an element of $\mathbb{R}^n_+$, all of whose coordinates are unity.

5.2.2 The Model

The framework is described by a triplet (Ω, u, δ), where Ω, a subset of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, is a *transition possibility set*, $u : \Omega \to R$ is a *utility function* defined on this set, and δ is the *discount factor* satisfying $0 < \delta < 1$. A typical element of Ω is written as an ordered pair (x, y); this means that if the current state is x, then it is possible to be in the state y in one period.

We will be using the following assumptions:

(A.1) (i) $(0,0) \in \Omega$; (ii) $(0,y) \in \Omega$ *implies* $y = 0$.

(A.2) Ω *is* (i) *closed, and* (ii) *convex.*

(A.3) *There is* ξ *such that* "$(x,y) \in \Omega$ *and* $\|x\| \geq \xi$" *implies* "$\|y\| < \|x\|$".

(A.4) *If* $(x,y) \in \Omega$ *and* $x' \geq x$, $0 \leq y' \leq y$, then (i) $(x',y') \in \Omega$ *and* (ii) $u(x',y') \geq u(x,y)$.

(A.5) u *is* (i) *upper semicontinuous and* (ii) *concave* on Ω.

(A.6) *There is* ζ *such that* $(x,y) \in \Omega$ *implies* $u(x,y) \geq \zeta$.

A *program* from $y \in \mathbb{R}^n_+$ is a sequence $\{y(t)\}_0^\infty$ such that $y(0) = y$, and $(y(t),\, y(t+1)) \in \Omega$ for $t \geq 0$.

A program $\{y(t)\}_0^\infty$ from $y \in \mathbb{R}^n_+$ is an *optimal program* if

$$\sum_{t=0}^{\infty} \delta^t u(y'(t), y'(t+1)) \leq \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1))$$

for every program $\{y'(t)\}_0^\infty$ from y.

An optimal program $\{y(t)\}_0^\infty$ from $y \in \mathbb{R}^n_+$ is a *stationary optimal program* if $y(t) = y(t+1)$ for $t \geq 0$. A *stationary optimal stock* is an element $y \in \mathbb{R}^n_+$, such that $\{y\}_0^\infty$ is a stationary optimal program. It is *non-trivial* if $u(y,y) > u(0,0)$.

A *discounted golden-rule stock* k is an element of $\mathbb{R}^n_+$ such that

(i) $(k,k) \in \Omega$

(ii) $u(k,k) \geq u(x,y)$ for all $(x,y) \in \Omega$ such that $\delta y - x \geq (\delta - 1)k$.

It is non-trivial if $u(k,k) > u(0,0)$.

5.2.3 Existence of Optimal Programs and the Principle of Optimality

The following "boundedness properties" of our model are well-known.

(R.1) *Under assumptions* (A.3) *and* (A.4)(i) ,

(i) *If* $(x,y) \in \Omega$, *then* $\|y\| \leq \max[\xi,\, \|x\|]$.

(ii) *If* $\{y(t)\}_0^\infty$ *is a program from* $y \in \mathbb{R}^n_+$, *then* $\|y(t)\| \leq \max[\xi,\, \|y\|]$ *for* $t \geq 0$.

The existence of an optimal program in this framework is also a standard result.

(R.2) *Under assumptions* (A.1), (A.2), (A.3), (A.4) (i), (A.5) (i) *and* (A.6), *if* $y \in \mathbb{R}^n_+$, *there exists an optimal program from* y.

Given (R.2), there is an optimal program $\{y^*(t)\}_0^\infty$ from each $y \in \mathbb{R}^n_+$. We define

$$V(y) = \sum_{t=0}^{\infty} \delta^t u(y^*(t),\, y^*(t+1))$$

V is known as the *value function.*

The following result is standard and is known as the "principle of optimality".

(R.3) If $\{y(t)\}_0^\infty$ is an optimal program from y, then

$$V(y) = \sum_{t=0}^{N} \delta^t u(y(t), y(t+1)) + \delta^{N+1} V(y(N+1)) \quad \text{for} \quad N \geq 0.$$

5.3 Equivalence of Discounted Golden-Rule and Stationary Optimal Stocks

A stationary optimal stock constitutes a solution to an infinite horizon problem. It is a stock such that, if one starts from it, then among all programs starting from it (whether stationary or not), the program which remains stationary at the initial stock is optimal. Yet the stationary nature of the solution makes it plausible to conjecture that one might be able to find it by solving a finite-horizon problem. The equivalence of a discounted golden-rule stock and a stationary optimal stock shows that this is indeed the case, as the discounted golden-rule might be seen as the solution to a problem involving two periods.

Our approach to this equivalence result follows Khan and Mitra (1986). It is "primal" in that it makes no use of supporting prices, unlike most treatments of it in the literature, which rely on duality theory.

Theorem 5.3.1. *Every discounted golden-rule stock* k *is a stationary optimal stock.*

Proof. Let $\{y(t)\}_0^\infty$ be any program from k. We shall show that it does not give a higher discounted utility sum than the stationary program $\{k\}_0^\infty$.

Let $x(T) = \sum_{t=0}^{T-1} (1-\delta)\delta^t y(t)/(1-\delta^T)$ and $z(T) = \sum_{t=0}^{T-1} (1-\delta)\delta^t y(t+1)/(1-\delta^T)$. Given convexity of Ω, certainly $(x(T), y(T)) \in \Omega$ for all $T \geq 1$. We know that $y(t)$ is bounded independently of t. Hence $(\bar{x}, \bar{z}) = \lim_{T\to\infty}(x(T), z(T))$ is well-defined and is an element of Ω.

Now, by the fact that $0 < \delta < 1$, Jensen's inequality yields $u(\bar{x}, \bar{z}) \geq \sum_{t=0}^{\infty} (1-\delta)\delta^t u(y(t),\ y(t+1))$. But $(\bar{x} - \delta\bar{z}) = (1-\delta)[\sum_{t=0}^{\infty} \delta^t y(t) - \sum_{t=0}^{\infty} \delta^{t+1} y(t+1)] = (1-\delta)k$. Since (k, k) is a discounted golden-rule stock, certainly $u(k, k) \geq u(\bar{x}, \bar{z})$, which implies:

$$\sum_{t=0}^{\infty} \delta^t u(k,k) \geq \sum_{t=0}^{\infty} \delta^t u(\bar{x},\bar{z}) = u(\bar{x},\bar{z})/(1-\delta) \geq \sum_{t=0}^{\infty} \delta^t u(y(t),y(t+1))$$

We can now state a converse to Theorem 5.3.1.

Theorem 5.3.2. *Every stationary optimal stock k such that $(k,k) \in$ interior Ω, is a discounted golden-rule stock.*

Proof. Suppose not; then there exists $(x,y) \in \Omega$ such that $\delta y - x \geq \delta k - k$ and $u(x,y) > u(k,k)$. Since u is non-decreasing in the first component, we can assume without any loss of generality that $x = (1-\delta)k + \delta y$. Let $\gamma \equiv u(x,y) - u(k,k) > 0$.

Using (x,y), we shall now construct a program $\{y'(t)\}_0^\infty$ starting from k that gives more discounted sum of utilities than the stationary optimal program $\{k\}_0^\infty$. This furnishes us the required contradiction. Towards this end, for a value of N to be determined later, let:

$$z(q) = (1-\delta^q)k + \delta^q x \quad \text{for all } q = 0, ..., N$$

Then, we have for all $q = 1, ..., N$,

$$\begin{aligned} z(q-1)) &= (1-\delta^{q-1})k + \delta^{q-1}x \\ &= (1-\delta^q)k + \delta^q y \end{aligned}$$

using the fact that $x = (1-\delta)k + \delta y$. Thus, we have:

$$(z(q), z(q-1)) = (1-\delta^q)(k,k) + \delta^q(x,y) \;\; \text{for all } q = 1, ..., N. \tag{5.1}$$

By convexity of Ω, we have $(z(q),\, z(q-1)) \in \Omega$ for all $q = 1, ..., N$. Now let $\{y'(t)\}_0^\infty$ be defined by $y'(0) = k$, $y'(t) = z(N-t+1)$, for $t = 1, ..., N$; $y'(N+1) = z(0) = x$; $y'(t) = 0$ for $t \geq N+2$.

We now show that for large enough N, $\{y'(t)\}_0^\infty$ is a program. For this, it only remains to show that $(k, y'(1)) = (k, z(N)) \in \Omega$. But $(k,k) \in$ *interior* Ω, and so there exists $\alpha > 0$ such that $(k,y) \in \Omega$ for all $y \in S_2 \equiv \{y : k - 2\alpha e << y << k + 2\alpha e\}$. Let $S_1 = \{y : k - \alpha e \leq y \leq k + \alpha e\}$. From the definition of $z(q)$, it is clear that $z(q) - \delta z(q-1) = (1-\delta)k$ for $q = 1, ..., N$, which implies $(z(q) - k) = \delta(z(q-1) - k)$. Since δ is less than 1, certainly $z(q) \to k$ as $q \to \infty$ and hence there exists N_1 such that $z(N) \in S_1$ for all $N \geq N_1$.

Next, we can assert, using the concavity of u, that for all $q = 1, ..., N$,

$$u(z(q), z(q-1)) \geq (1-\delta^q)u(k,k) + \delta^q u(x,y) \geq u(k,k) + \delta^q \gamma.$$

By Mangasarian (7, p. 63), it is also true that

$$\|u(k, z(N)) - u(k,k)\| \leq A \|z(N) - k\| = A\delta^{N+1} \|y - k\|,$$

where $A \equiv (u(k,k) + \hat{\beta})/\alpha, \hat{\beta} = -Min_{y \in W} u(k,y)$ and W is the set of $2n$ vertices of S_1. Hence we have

$$\sum_{t=0}^{N+1} \delta^t[u(y'(t), y'(t+1)) - u(k,k)] \geq -A\delta^{N+1} \|y-k\| + (N+1)\delta^{N+1}\gamma.$$

On adding terms after the time period $(N+1)$, we obtain:

$$\begin{aligned} & \sum_{t=0}^{\infty} \delta^t[u(y'(t), y'(t+1)) - u(k,k)] \\ \geq \quad & \delta^{N+1}((N+1)\gamma - A\|y-k\| + \{\delta u(0,0)/(1-\delta)\} - \delta V(k)). \quad (5.2) \end{aligned}$$

Let N_2 be a value of N such that the right-hand side of (5.2) is positive, and let $N^{'} = Max(N_1, N_2)$. Now any $\{y'(t)\}_0^\infty$ with $N \geq N^{'}$ furnishes us with a contradiction to the fact that $\{k\}_0^\infty$ is a stationary optimal program.

A natural question arises as to whether the interiority hypothesis in Theorem 5.3.2 can be dispensed with. The following example shows this not to be the case.

Example 1:

Let $\Omega = \{(x,y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : Ay \leq x, \;\; ey \leq 3\}$, where:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

and $e = (1,1)$. Let $\delta = 1/2$ and $u(x,y) = ex$. It is clear that this economy satisfies all the assumptions made in Section 5.2. We shall show that $k = (1,0)$ is a stationary optimal stock. To this end, observe that $(k,k) \in \Omega$ and consider any program $\{y(t)\}_0^\infty$ starting from k. Since $(y(t), y(t+1)) \in \Omega$, we have $y(t) \leq (1,0)$ for all t. Hence,

$$\begin{aligned} \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1)) &= \sum_{t=0}^{\infty} \delta^t (ey(t)) \leq \sum_{t=0}^{\infty} \delta^t \\ &= \sum_{t=0}^{\infty} \delta^t (ek) = \sum_{t=0}^{\infty} \delta^t u(k,k). \end{aligned}$$

Now let $x' = (1,1)$, $y' = (1,2)$. Certainly $(x', y') \in \Omega$ and $\delta y' - x' = \delta k - k$. But $u(x', y') = ex' = 2 > ek = u(k,k)$ and thus k is not a discounted golden-rule stock.

5.4 Existence of Discounted Golden-Rule and Stationary Optimal Stocks

Given the equivalence result of Section 5.3, the existence of a stationary optimal stock can be established by showing that there exists a discounted golden-rule stock. Since one can easily impose conditions on the economy to ensure that the discounted golden-rule stock obtained is non-trivial, this approach has the advantage of identifying conditions on the economy sufficient for the existence of a *non-trivial* stationary optimal stock. This advantage is not shared by the dynamic programming approach followed by Sutherland (1967), a shortcoming that was pointed out by Peleg and Ryder (1974).

Lemma 5.4.1. *Let $S = \{x \in \mathbb{R}^n_+ : \|x\| \leq \beta\}$ and ϕ and ψ be mappings from S into $\mathcal{B}(\mathbb{R}^n_+ \times \mathbb{R}^n_+)$ such that for $z \in S$, $\phi(z) = \{(x, y) \in \Omega : \delta y - x \geq \delta z - z\}$ and $\psi(z) = \{(x, y) \in \phi(z) : u(x, y) \geq u(x', y')$ for all $(x', y') \in \phi(z)\}$. Then, ψ is a non-empty, convex-valued, and upper semicontinuous correspondence.*

Proof. Clearly, S is a non-empty, convex, and compact set. Next, we claim that ϕ is a non-empty, compact-valued correspondence. For any $z \in S$, we have $(0, 0) \in \phi(z)$, and, since Ω is convex and closed, $\phi(z)$ is convex and closed. Furthermore, if $(x, y) \in \phi(z)$, then $\|x\| \leq \beta$. This implies, in turn, that if $(x, y) \in \phi(z)$, then $\|y\| \leq \beta$. Thus on defining $S' = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \|y\| \leq \beta\}$, we note that S' is a non-empty, compact set, and for any $z \in S$, $\phi(z)$ is a subset of S'. Since $\phi(z)$ is closed for each $z \in S$, $\phi(z)$ is compact for each $z \in S$.

Since u is an upper semicontinuous function on Ω, and $\phi(z)$ is a non-empty, compact subset of Ω, $\psi(z)$ is non-empty for each $z \in S$. It is also convex by concavity of u and convexity of $\phi(z)$.

Next, we show the upper semicontinuity of ψ. Let z^* be an arbitrary point of S. Consider a sequence $\{z^n\}$, with $z^n \in S$, for $n = 1, 2, 3, ...$, with $z^n \to z^*$ as $n \to \infty$. Let $(x^n, y^n) \in \psi(z^n)$, and $(x^n, y^n) \to (\hat{x}, \hat{y})$. We want to show that $(\hat{x}, \hat{y}) \in \psi(z^*)$. Since Ω is closed, $(\hat{x}, \hat{y}) \in \phi(z^*)$. Suppose $(\hat{x}, \hat{y}) \notin \psi(z^*)$. Then there is some $(x^*, y^*) \in \psi(z^*)$ and an $\varepsilon > 0$ such that $u(x^*, y^*) \geq u(\hat{x}, \hat{y}) + \varepsilon$.

Now, since u is an upper semicontinuous function, $\lim_{n\to\infty} \sup u(x^n, y^n) \leq u(\hat{x}, \hat{y})$. Thus, there is N_1 such that for $n \geq N_1$, $u(x^n, y^n) \leq u(\hat{x}, \hat{y}) + \varepsilon/3$. Consequently, for $n \geq N_1$,

$$u(x^*, y^*) \geq u(x^n, y^n) + 2\varepsilon/3. \tag{5.3}$$

Choose $0 < \lambda < 1$ such that $(1 - \lambda)[u(0, 0) - u(x^*, y^*)] \geq -\varepsilon/3$. We claim that there is an N_2 such that for $n \geq N_2$, $(\lambda x^*, \lambda y^*) \in \phi(z^n)$. To see this, observe that $(0, 0) \in \Omega$ and convexity of Ω imply that $(\lambda x^*, \lambda y^*) \in \phi(\lambda z^*)$. Since $z^n \to z^*$, there is N_2 such that for $n > N_2$, $z^n \geq \lambda z^*$. Thus $\delta\lambda y^* - \lambda x^* \geq (\delta - 1)\lambda z^* \geq (\delta - 1)z^n$, establishing our claim.

Since $(x^n, y^n) \in \psi(z^n)$, for $n \geq N_2$,

$$\begin{aligned} u(x^n, y^n) &\geq u(\lambda x^*, \lambda y^*) \geq \lambda u(x^*, y^*) + (1 - \lambda)u(0, 0) \\ &= u(x^*, y^*) + (1 - \lambda)[u(0, 0) - u(x^*, y^*)] \\ &\geq u(x^*, y^*) - \varepsilon/3. \end{aligned}$$

Using this in (5.3) for $n \geq Max(N_1, N_2)$,

$$u(x^*, y^*) \geq u(x^n, y^n) + 2\varepsilon/3 \geq u(x^*, y^*) + \varepsilon/3,$$

a contradiction, which completes the proof.

Theorem 5.4.1. *There exists a discounted golden-rule stock.*

Proof. Define $Q : S \to \mathcal{B}(\mathbb{R}^n_+)$, where for $z \in S$, $Q(z) = \{x \in \mathbb{R}^n_+ : (x, y) \in \psi(z)\}$. We will show that this correspondence Q satisfies all the requirements of Kakutani's fixed-point theorem.

Lemma 5.4.1 implies that Q is a non-empty, convex-valued correspondence. It also implies that Q is upper semicontinuous. To see this, take an arbitrary $z^* \in X$. Let $z^n \in S$, with $z^n \to z^*$ as $n \to \infty$. Let $x^n \in Q(z^n)$, and $x^n \to \hat{x}$ as $n \to \infty$. We have to show that $\hat{x} \in Q(z^*)$. Since $x^n \in Q(z^n)$, there is y^n such that $(x^n, y^n) \in \psi(z^n)$. This means $(x^n, y^n) \in \phi(z^n)$, and by compactness of $\phi(z^n)$, we can pick a subsequence $(x^{n'}, y^{n'})$ tending to $(\hat{x}, \hat{y}) \in \phi(z^*)$. By the lemma, $(\hat{x}, \hat{y}) \in \psi(z^*)$ and the claim is proved.

Thus, all the conditions of Kakutani's fixed point theorem are fulfilled, and there exists $x^0 \in Q(x^0)$. This means there is some y^0 such that $(x^0, y^0) \in \psi(x^0)$; that is,

$$u(x^0, y^0) \geq u(x, y) \quad \text{for all} \ \ (x, y) \in \phi(x^0).$$

But $(x^0, y^0) \in \phi(x^0)$ implies $x^0 \leq y^0$, and we obtain that $(x^0, x^0) \in \Omega$, and $u(x^0, x^0) \geq u(x^0, y^0) \geq u(x, y)$ for all $(x, y) \in \Omega$, with $\delta y - x \geq \delta x^0 - x^0$. Thus, by definition, x^0 is a discounted golden-rule stock.

An economy (Ω, u, δ) is called $\delta-$*normal* if there exists $(\bar{x}, \bar{y}) \in \Omega$ such that $\bar{x} \leq \delta\bar{y}$ and $u(\bar{x}, \bar{y}) > u(0, 0)$.

Theorem 5.4.2. *If the economy (Ω, u, δ) is δ-normal, there exists (i) a non-trivial discounted golden-rule stock, and (ii) a non-trivial stationary optimal stock.*

Proof. By Theorem 5.4.1, there is a discounted golden-rule stock, x^0. Given δ-normality, there is $(\bar{x}, \bar{y}) \in \Omega$ such that $\delta\bar{y} - \bar{x} \geq 0 \geq \delta x^0 - x^0$, and $u(\bar{x}, \bar{y}) > u(0, 0)$. Thus, by definition of a discounted golden-rule stock, $u(x^0, x^0) \geq u(\bar{x}, \bar{y}) > u(0, 0)$, and hence x^0 is a non-trivial discounted golden-rule stock.

By Theorem 5.3.1, x^0 is a stationary optimal stock, and since we have already checked that $u(x^0, x^0) > u(0, 0)$, it is a non-trivial stationary optimal stock.

Remark:

An economy (Ω, u, δ) is called $\delta-$*productive* if there exists $(\bar{x}, \bar{y}) \in \Omega$ such that $\delta\bar{y} >> \bar{x}$. Flynn (1980) establishes a version of Theorem 5.4.2 under the additional assumption of δ-productivity. This is because, after establishing the existence of a discounted golden-rule, he uses the dual approach to show that the discounted golden-rule stock is a stationary optimal stock, by providing an appropriate price-support. Then $\delta-$*productivity* ensures that Slater's condition is satisfied in the application of the Kuhn-Tucker theorem.

We show now that there exist economies satisfying the hypotheses of Theorem 5.4.2, whose technologies are not δ-productive, and for which there exists a non-trivial SOS.

Example 2:

Let $f(x) = 2x$ for $0 \leq x \leq 1$ and $f(x) = 2 + (x-1)/2$ for $x \geq 1$. Let $\Omega = \{(x, y) \in \mathbb{R}^n_+ : 0 \leq y \leq f(x)\}$, $u(x, y) = 2f(x) - y$ and $\delta = 1/2$.

Now $(\bar{x}, \bar{y}) \equiv (1, 2) \in \Omega$. Certainly $\delta\bar{y} - \bar{x} = 0$ and $u(\bar{x}, \bar{y}) = 2 > 0 = u(0, 0)$. Hence the economy is δ-normal. Also, for any $(x, y) \in \Omega$, $\delta y - x \leq (1/2)f(x) - x \leq 0$, since for $x \geq 1$, $f(x) \leq 2x$. Thus, there cannot exist any $(x, y) \in \Omega$ such that $x << \delta y$ and so the economy is not δ-productive.

Next, we claim that $x^* = 1$ is a discounted golden-rule stock. Pick any $(x, y) \in \Omega$ such that $\delta y - x \geq (\delta - 1)x^*$. Then $y \geq 2x - 1$ and $u(x, y) \leq 2f(x) - 2x + 1$. Now

$$u(x, y) \leq 2(2x) - 2x + 1 \leq 3 \qquad \text{for} \quad 0 \leq x \leq 1$$

and

$$u(x, y) \leq 2(2 + (1/2)(x-1)) - 2x + 1 \leq 3 \qquad \text{for} \quad x \geq 1.$$

In either case, $u(x, y) \leq 3 = u(1, 1) = u(x^*, x^*)$, and our claim is proved.

It should be noted that $x^* = 1$ is an SOS by Theorem 5.3.1, which is non-trivial, since $u(1, 1) = 3 > 0 = u(0, 0)$.

We now present an example of an economy which satisfies all the assumptions of Section 5.2, and which is $\delta -$ *productive*, but which has only a trivial SOS. This economy violates the $\delta -$ *normality* assumption, showing thereby that Theorem 5.4.2 would not be valid if the $\delta -$ *normality* hypothesis is dropped from its statement.

Example 3:

Let $\Omega = \{(x, y) \in \mathbb{R}^2_+ : 0 \leq y \leq 2x^{1/2}\}$, $\delta = 1/2$, and $u(x, y) = x - 2y$. For $(\hat{x}, \hat{y}) = (1/4, 1) \in \Omega$, we have $\delta\hat{y} >> \hat{x}$ and so the economy is $\delta -$ *productive*. For any program $\{k\}_0^\infty$ with $0 < k \leq 4$, we have $\sum_{t=0}^{\infty} \delta^t u(k, k) < 0$, and so it is dominated by the program $\{y(t)\}_0^\infty$ with $y(0) = k$ and $y(t) = 0$ for $t = 1, 2, ...$ Since there is no stationary program $\{k\}_0^\infty$ with $k > 4$, $\{0\}_0^\infty$ is the unique stationary optimal program.

5.5 Uniqueness of Non-trivial SOS

In this section we establish the uniqueness of non-trivial stationary optimal stocks in a framework in which the technology is described by a simple linear model (see Gale(1960)) involving no joint production, and the welfare function, describing the utility derived from consumption (alone), satisfies a *normality* condition[2]. We follow closely the approach, pioneered by Brock (1973), and

[2] Optimal programs in a similar framework, but without the normality condition, have been studied in detail by Dasgupta and Mitra (1999).

developed further in Brock and Burmeister (1976). However, we rely entirely on the methods of convex analysis, and do not make any differentiability assumptions.

5.5.1 Description of the Framework

We describe the production side by an $n \times n$ non-negative matrix $A = (a_{ij})$, where $i = 1, ..., n$ and $j = 1, ..., n$, and a strictly positive vector $b = (b_1, ..., b_n) >> 0$. Here, a_{ij} and b_j are respectively the amounts of the $i-$th good and labor which are required per unit output of the $j-$th good. The total amount of labor available for production is stationary and is normalized to 1. For each $j = 1, ..., n$, it is assumed that there is some $i = 1, ..., n$ such that $a_{ij} > 0$. Thus, each production process requires a positive amount of labor as well as a positive amount of some produced factor. Further, it is assumed that A is productive; that is, there is some $\tilde{y} >> 0$ such that $\tilde{y} >> A\tilde{y}$ and $b\tilde{y} \leq 1$. This essentially excludes the economically uninteresting case of a production system which is unable to sustain some positive consumption levels for all of the desired goods. The fact that A is productive ensures that $(I - A)$ is non-singular, and $(I - A)^{-1} \geq 0$. The transition possibility set for this economy is given by:

$$\Omega = \{(x, y) \in \mathbb{R}_+^{2n} : Ay \leq x \text{ and } by \leq 1\}$$

We will assume in addition to the requirements stated above that A is *indecomposable*; that is, there is no non-empty proper subset J of $\{1, 2, ..., n\}$ such that $a_{ij} = 0$ for $i \notin J,\ j \in J$. In this case, we have the stronger result that $(I - A)^{-1} >> 0$. It is also known that A has a unique Frobenius root, θ, which is positive, and a real Frobenius vector, x^*, which is strictly positive (and taken henceforth to be normalized so that $bx^* = 1$). Since A is productive, we know that $\theta \in (0, 1)$. We make the stronger assumption that:

$$0 < \theta < \delta \tag{DF}$$

where $\delta \in (0, 1)$ is the discount factor. Since $\theta \in (0, 1)$, assumption (DF) will always be satisfied for all discount factors close to 1. But, (DF) gives an explicit lower bound for the discount factor under which the uniqueness theory, to be described below, is valid. Thus, (DF) links the level of impatience, an aspect of intertemporal preferences (δ), with a measure of the productivity of the economy (θ). Under (DF), we have the important result[3] that $(\delta I - A)$ is also non-singular, and:

$$(\delta I - A)^{-1} >> 0 \tag{5.4}$$

Welfare is derived from consumption, as given by a function $w : \mathbb{R}_+^n \to \mathbb{R}$, which is continuous, concave and monotone on $\mathbb{R}_+^n$. In what follows, we normalize $w(0) = 0$, and assume that $w(c) > 0$ if and only if $c >> 0$. We make

[3] All the results relating to the Frobenius theorem that are stated in this paragraph can be found in Nikaido (1968, p.102-108).

stronger assumptions on w when consumption is strictly positive. Specifically, we assume that w is strongly monotone and strictly concave on $\mathbb{R}^n_{++}$.

We now describe the crucial *normality* assumption on w. Suppose $p \in \mathbb{R}^n_{++}$ and $M \in \mathbb{R}_{++}$; consider the optimization problem described by:

$$\left.\begin{array}{cc} Maximize & w(c) \\ subject\ \ to & pc \leq M \\ and & c \geq 0 \end{array}\right\} (P)$$

Clearly, under our assumptions, there is a unique solution $c(p, M)$ to the problem (P).

We assume that this solution is strongly monotone in M. That is, if $p \in \mathbb{R}^n_{++}$, $M \in \mathbb{R}_{++}$ and $M' > M$, then:

$$c(p, M') >> c(p, M) \tag{N}$$

This is known as the *normality* assumption on w, since it is satisfied when all goods are normal goods (in the sense used in standard consumer behavior theory).

Given w, the (reduced form) utility function for our framework is defined by:

$$u(x, y) = w(x - Ay) \ \text{ for all } (x, y) \in \Omega$$

It can be checked that the economy (Ω, u, δ) as defined above satisfies all the assumptions that were stated in Section 5.2.

If $\{y(t)\}$ is a program from y, we will associate with it a consumption sequence $\{c(t)\}$ given by:

$$c(t) = y(t) - Ay(t + 1) \ \ for\ all \ \ t \in \mathbb{N}$$

5.5.2 A Uniqueness Result Under Normality

We now proceed to investigate the nature of stationary optimal stocks in the framework described in the above subsection. To this end, we first summarize in a couple of Lemmas some basic properties of any non-trivial SOS. Then, we establish the uniqueness of non-trivial SOS under the normality assumption (N).

Lemma 5.5.1. *If y is a non-trivial SOS, then (i) $c >> 0$, and (ii) $y >> 0$.*

Proof. Since y is a non-trivial SOS, we have $u(y, y) > u(0, 0) = 0$. Thus, we obtain $w(c) = w(y - Ay) = u(y, y) > 0$, and by our assumption on w, we must have $c >> 0$.

Since $c = y - Ay = (I - A)y$, and $(I - A)$ is non-singular, with $(I - A)^{-1} >> 0$, we have $y = (I - A)^{-1}c >> 0$.

The above lemma allows us to invoke a standard result on duality theory, to provide a price support, q, to a non-trivial SOS, y; the quantity-price pair (y, p) is usually referred to as a modified golden-rule.

Lemma 5.5.2. *If $\bar{y}$ is a non-trivial SOS, then there is $\bar{q} \in \mathbb{R}^n_+$, such that:*

$$w(\bar{c}) - \bar{q}\bar{c} \geq w(c) - \bar{q}c \quad for\ all\ c \geq 0 \tag{5.5}$$

and:

$$\bar{q}(\delta\bar{y} - A\bar{y}) \geq \bar{q}(\delta y - x) \quad for\ all\ (x, y) \in \Omega \tag{5.6}$$

Furthermore, any $\bar{q}$ satisfying (5.5) and (5.6) and $\bar{v} \equiv \bar{q}(\delta\bar{y} - A\bar{y})$ must satisfy:

$$\bar{q}(\delta I - A) = \bar{v}b \tag{5.7}$$

and:

$$(i)\ \bar{q}(\delta\bar{y} - A\bar{y}) > 0,\ \ (ii)\ \bar{q} >> 0 \tag{5.8}$$

And, $\bar{y}$ must satisfy $b\bar{y} = 1$.

Proof. The fact that there exists $\bar{q} \in \mathbb{R}^n_+$ such that (5.5) and (5.6) holds, follows from a standard application of duality theory. We proceed to verify (5.7).

Clearly, we have $\bar{v} \geq 0$, since $(0, 0) \in \Omega$. Define $Y = \{\ y \in \mathbb{R}^n_+ : by = 1\}$. Then, we have, using (5.6), for all $y \in Y$,

$$0 = \bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v} \geq \bar{q}(\delta y - Ay) - \bar{v} = \bar{q}(\delta y - Ay) - \bar{v}by \tag{5.9}$$

Thus, for all $y \in Y$, we have:

$$\bar{q}(\delta y - Ay) - \bar{v}by\ \ \leq 0 \tag{5.10}$$

Now, let y be an arbitrary vector in $\mathbb{R}^n_+$, $y \neq 0$. Then, there is $\lambda > 0$, such that $y' \equiv \lambda y$ is in Y. Applying (5.10) to y', we have:

$$\bar{q}(\delta y' - Ay') - \bar{v}by'\ \ \leq 0$$

and so $\bar{q}(\delta y - Ay) - \bar{v}by\ \ \leq 0$ must hold. This inequality also clearly holds for $y = 0$. So, to summarize, we have now verified that:

$$\bar{q}(\delta y - Ay) - \bar{v}by\ \ \leq 0 \quad for\ all\ y \geq 0 \tag{5.11}$$

Clearly, we have $b\bar{y} \leq 1$, and so:

$$\bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v}b\bar{y} \geq \bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v} = 0 \tag{5.12}$$

Combining (5.11) and (5.12), we obtain:

$$\bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v}b\bar{y} = 0 \tag{5.13}$$

Using (5.11) and (5.13), we conclude that:

$$\bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v}b\bar{y} \geq \bar{q}(\delta y - Ay) - \bar{v}by \quad for\ all\ y \geq 0 \tag{5.14}$$

Since $\bar{y} >> 0$ by Lemma 5.5.1, (5.14) yields (5.7).

We now proceed to verify (5.8). To this end, first note that $\bar{q} \neq 0$. For if $\bar{q} = 0$, then by using (5.5), we must have $w(\bar{c}) \geq w(c)$ for all $c \geq 0$; but since $\bar{c} >> 0$, this inequality would be violated for $c = 2\bar{c}$.

We now claim that (5.8)(i) must hold. For, if it did not hold, then $v = 0$, and (5.7) would yield $\bar{q}(\delta I - A) = 0$. But, since $(\delta I - A)$ is non-singular, we must then have $\bar{q} = 0$, a contradiction.

Using (5.7) and (5.8)(i), we have $\bar{q} = \bar{v}b(\delta I - A)^{-1} >> 0$, since $(\delta I - A)^{-1} >> 0$, thereby establishing (5.8)(ii).

By the definition of $\bar{v}$ and (5.13), we have $\bar{v}b\bar{y} = \bar{v}$, so that $b\bar{y} = 1$, since $\bar{v} > 0$.

Remark:

We note that if $\bar{y}$ is a non-trivial SOS, then by Lemma 5.5.2, $b\bar{y} = 1$, and so non-trivial stationary optimal stocks can never be in the interior of Ω in this framework.

We now turn to the uniqueness result, illustrating the role of the normality assumption on w.

Theorem 5.5.1. *There is only one non-trivial SOS.*

Proof. We know that there exists a non-trivial SOS in this framework, by using Theorem 5.4.2; one can check, using (DF), that $(\delta x^*, x^*) \in \Omega$ satisfies $\delta - normality$, where x^* is the Frobenius vector of A.

To establish uniqueness, suppose on the contrary that there are two non-trivial stationary optimal stocks, y', y'', with $y' \neq y''$. Then, since $(I - A)$ is non-singular, we must have $c' \neq c''$. We now demonstrate that, as a result, q' and q'' must be distinct, where q' and q'' are price supports of y' and y'' respectively, satisfying conditions (5.5), (5.6) of Lemma 5.5.2.

Suppose $q' = q''$. Then, by using (5.5), we have:

$$\begin{aligned} w(c') - q'c' &\geq w((1/2)(c' + c'')) - q'((1/2)(c' + c'')) \\ &> (1/2)[w(c') - q'c'] + (1/2)[w(c'') - q'c''] \end{aligned}$$

the strict inequality following from the fact that w is strictly concave on $\mathbb{R}^n_{++}$ and $c' >> 0$ and $c'' >> 0$ by Lemma 5.5.1. Thus, we must have:

$$w(c') - q'c' > w(c'') - q'c'' \tag{5.15}$$

Similarly, we get from (5.5),

$$w(c'') - q''c'' > w(c') - q''c' \tag{5.16}$$

Clearly, if $q' = q''$, (5.15) and (5.16) cannot both hold. Thus, $q' \neq q''$. Now, it follows from (5.7) of Lemma 5.5.2 that $v' \neq v''$, since $(\delta I - A)$ is non-singular.

Without loss of generality, suppose that $v'' > v'$. Define $\mu = (v''/v')$; then $\mu > 1$, and by (5.7), we must have $q'' = \mu q'$.

Denoting $q'c'$ by M', we note that c' is the unique solution to:

$$\left.\begin{array}{ll} Max & w(c) \\ subject\ to & q'c \leq M' \\ and & c \geq 0 \end{array}\right\} (P')$$

Similarly, denoting $q''c''$ by M'', we note that c'' is the unique solution to:

$$\left.\begin{array}{ll} Max & w(c) \\ subject\ to & q''c \leq M'' \\ and & c \geq 0 \end{array}\right\} (P'')$$

Since $q'' = \mu q'$, it follows that c'' is the unique solution to:

$$\left.\begin{array}{ll} Max & w(c) \\ subject\ to & q'c \leq q'c'' \\ and & c \geq 0 \end{array}\right\} (P''')$$

We can now split up our analysis into three cases (i) $q'c'' = M'$, (ii) $q'c'' > M'$, (iii) $q'c'' < M'$.

In case (i), problems (P') and (P''') are the same and so c' and c'' must both solve (P'), implying $c' = c''$, a contradiction.

In case (ii), we must have $c'' > c'$ by normality of w. Thus, we obtain $(I-A)y'' >> (I-A)y'$, which implies that $y'' >> y'$, since $(I-A)^{-1} >> 0$. But, then, we get a contradiction by noting from Lemma 5.5.2, $1 = by'' > by' = 1$.

The analysis of case (iii) is analogous to that of case (ii).

Thus, the hypothesis that there are two non-trivial stationary optimal stocks must be false, and the theorem is proved.

5.5.3 An Example of Non-uniqueness of SOS

To emphasize the crucial role of normality of the welfare function in the above result, we now present an example, where normality is violated, and there exist two non-trivial stationary optimal stocks. The idea of the example follows the discussion of this issue in Brock (1973) and Brock and Burmeister (1976); however, we are more explicit in our construction, and we ensure that the example of non-uniqueness can be generated by a strictly concave welfare function on consumption vectors.

The technology is described by a 2×2 matrix A and a two-dimensional vector, b, which are specified as follows:

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}; \; b = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

We define the welfare function, w, only on the set $C = \{(c_1, c_2) : c_1 \in [0,1], c_2 \in [0,1]\}$, since the technology does not allow for consumption outside this set on any program after the initial time period. A suitable extension of w from the domain C to $\mathbb{R}^2_+$ can be constructed, preserving the key properties of w on

C, but this is somewhat tedious, and is not included here. The function, w, is defined on C as follows:

$$w(c_1, c_2) = qc_1 - (1/2)rc_1^2 - c_1c_2 + Qc_2 - (1/2)Rc_2^2$$

where $r = 9.8/8 = 1.225$, $R = 9.8/12 = (2/3)r$, $q = 3$ and $Q = 2.41$. A few of the important relations between the parameters may be noted. We have $r < 1.3$, $R < 1$, and $rR = (9.8)^2/96 = 96.04/96 > 1$. Also, $4r + 6R = 9.8 = (49/5)$.

Note that for all $(c_1, c_2) \in C$,

$$w_1(c_1, c_2) = q - rc_1 - c_2 > 0; \;\; w_2(c_1, c_2) = Q - Rc_2 - c_1 > 0$$

so that w is increasing in each component of the consumption vector and:

$$\begin{aligned} w_{11}(c_1, c_2) &= -r, \;\; w_{22}(c_1, c_2) = -R \\ w_{12}(c_1, c_2) &= w_{21}(c_1, c_2) = -1 \end{aligned}$$

so that, using $rR > 1$, w is strictly concave on C.

The discount factor is specified to be $\delta = 0.9$.

We will show that $y' = (0.5, 0.5)$ and $y'' = (0.6, 0.4)$ are both stationary optimal stocks. These are stationary stocks with corresponding consumption vectors $c' = (0.25, 0.3)$ and $c'' = (0.3, 0.24) = (c_1' + \varepsilon, c_2' - (6/5)\varepsilon)$, where $\varepsilon = 0.05$. They are clearly non-trivial. Further, the corresponding input levels are given by $x' = Ay' = (0.25, 0.2)$ and $x'' = Ay'' = (0.3, 0.16)$. There is full-employment of labor for both stocks, since $by' = by'' = 1$.

To verify that y' is a SOS, we use the dual approach, and define:

$$p' = (q - rc_1' - c_2', Q - Rc_2' - c_1') = (w_1(c_1', c_2'), w_2(c_1', c_2'))$$

Then, $p' >> 0$, and by concavity of w on C, we have:

$$w(c') - p'c' \geq w(c) - p'c \;\; for\ all\ c \in C \tag{5.17}$$

Given the definition of p', we see that the relative price $(p_1'/p_2') = (5/4)$. Since this is a crucial fact in our construction, we provide the necessary calculations as follows. We have:

$$(5 - 4r)c_1' + (5R - 4)c_2' = 0.1c_1' + (1/12)c_2' = 0.05$$

and:

$$(5Q - 4q) = 0.05$$

so that:

$$(5 - 4r)c_1' + (5R - 4)c_2' = (5Q - 4q)$$

and by transposing terms:

$$4(q - rc_1' - c_2') = 5(Q - Rc_2' - c_1')$$

Using the fact that $(p'_1/p'_2) = (5/4)$, we have:

$$\begin{aligned} p'(\delta I - A) &= [0.4p'_1, 0.5p'_2] = p'_2[0.5, 0.5] \\ &= (1/2)p'_2 b \end{aligned}$$

Thus, for all $(x, y) \in \Omega$, we have:

$$p'(\delta y - x) \leq p'(\delta I - A)y = (1/2)p'_2 by \leq (1/2)p'_2 = p'(\delta I - A)y' \tag{5.18}$$

Using (5.17) and (5.18), it is straightforward to check that $\{y'\}$ is optimal[4] from y'.

To verify that y'' is a SOS, we define:

$$p'' = (q - rc''_1 - c''_2, Q - Rc''_2 - c''_1) = (w_1(c''_1, c''_2), w_2(c''_1, c''_2))$$

Then, $p'' >> 0$, and by concavity of w on C, we have:

$$w(c'') - p''c'' \geq w(c) - p''c \;\; for\ all\ c \in C \tag{5.19}$$

Given the definition of p'', we see that the relative price $(p''_1/p''_2) = (5/4)$, so that both stationary stocks have price supports, such that the *relative* price is the same. This is important enough to justify providing the necessary calculations. We have:

$$\begin{aligned} (5 - 4r)c''_1 + (5R - 4)c''_2 &= (5 - 4r)c'_1 + (5R - 4)c'_2 \\ &\quad + (5 - 4r)\varepsilon - (5R - 4)(6/5)\varepsilon \end{aligned}$$

Now, $(5 - 4r) - (5R - 4)(6/5) = -(4r + 6R) + (5 + (24/5)) = 0$ and so:

$$(5 - 4r)c''_1 + (5R - 4)c''_2 = (5 - 4r)c'_1 + (5R - 4)c'_2 = 0.05$$

Also, as noted above:

$$(5Q - 4q) = 0.05$$

so that:

$$(5 - 4r)c''_1 + (5R - 4)c''_2 = (5Q - 4q)$$

and by transposing terms:

$$4(q - rc''_1 - c''_2) = 5(Q - Rc''_2 - c''_1)$$

Using the fact that $(p''_1/p''_2) = (5/4)$, we have:

$$\begin{aligned} p''(\delta I - A) &= [0.4p''_1, 0.5p''_2] = p''_2[0.5, 0.5] \\ &= (1/2)p''_2 b \end{aligned}$$

[4] Strictly speaking, the pair (y', p') has not been shown to constitute a modified golden-rule since (5.18) is only shown to hold on C. However, all programs starting from y' must have consumption vectors in *all periods* which belong to C, and so the standard *argument* (which is used to show that the stock associated with a modified golden-rule pair is an SOS) still applies.

Thus, for all $(x, y) \in \Omega$, we have:

$$p''(\delta y - x) \leq p''(\delta I - A)y = (1/2)p_2''by \leq (1/2)p_2'' = p''(\delta I - A)y'' \qquad (5.20)$$

Using (5.19) and (5.20), it is straightforward to check that $\{y''\}$ is optimal from y''.

We can check that normality is violated by w. Denote $p'c'$ by M' and $p''c''$ by M''. Then, using (5.17), and the strict concavity of w on C, we know that the unique solution $c(p', M')$ to the problem:

$$\left.\begin{array}{ll} Max & w(c) \\ subject\ to & p'c \leq M' \\ and & c \in C \end{array}\right\} (P)$$

is given by c'. Consequently, $c(p'/\mu, M'/\mu)$ is also given by c', where $\mu = (p_2'/p_2'')$. But, since $p' = \mu p''$, we have $c(p'', M'/\mu) = c'$; also, of course, $c(p'', M'') = c''$. Now,

$$\begin{aligned} p''c'' &= p_2''[(5/4)c_1'' + c_2''] \\ &= (p_2''/p_2')p_2'[(5/4)c_1' + c_2' + (5/4)\varepsilon - (6/5)\varepsilon] \\ &> (p_2''/p_2')p_2'[(5/4)c_1' + c_2'] \\ &= p'c'/\mu \end{aligned}$$

Thus, we have $M'' > M'/\mu$, but $c_2'' < c_2'$, so that normality of w is violated.

5.6 Uniqueness of Interior SOS for Smooth Economies

When the economy is smooth (the reduced form utility function is twice continuously differentiable in the interior of the transition possibility set), the methods of differential topology can be used to demonstrate uniqueness of *interior* stationary optimal stocks. This is done by establishing a connection (mathematically, a homotopy) between the set of SOS in the discounted case with the set of SOS in the undiscounted case.

When future utilities are undiscounted, the notion of optimality (defined in terms of some version of the overtaking criterion) is somewhat different from the one described in Section 5.2. However, we can avoid getting into a full discussion of the undiscounted case by first stating a purely mathematical result (Lemma 5.6.1), which helps us to effectively make the same connection as is mentioned in the preceding paragraph.[5]

Lemma 5.6.1 is used in two ways. First, it helps us to provide a link between the analysis of SOS (in the discounted case) in Sections 5.3 and 5.4 of this paper with that offered in this section, which is in terms of stationary solutions to

[5] For some discussion of optimality in the undiscounted case, see the bibliographic remarks in Section 5.7 below.

Ramsey-Euler equations (Proposition 5.6.1). Second, it allows us to examine (see Lemma 5.6.2) the set of stationary solutions to Ramsey-Euler equations in the undiscounted case. [Note that this can be done without discussing the relation between these solutions in the undiscounted case and any notion of optimal programs in the undiscounted case].

Lemma 5.6.2 provides the appropriate result to establish the uniqueness theorem (Theorem 5.6.1) for interior SOS in the discounted case, by using the homotopy invariance theorem and the degree theorem from differential topology.

Since we will be dealing now with "smooth economies", we strengthen assumption (A.5) of Section 5.2 as follows:

(A.5+) *u is* (i) *upper semicontinuous and* (ii) *concave* on Ω. Further, u is twice continuously differentiable in the interior of Ω.

Let us define $\Omega^0 = \{(x,y) \in int\ \Omega : ||x|| \ < \xi\}$, where ξ is given by (A.3). Then Ω^0 is an open and bounded subset of $int\ \Omega$. Further, if $(x,x) \in int \Omega$, then $(x,x) \in \Omega^0$ by (A.3). We denote the set $\{x : (x,x) \in \Omega^0\}$ by Λ.

We define a function G from $\Lambda \times [0,1]$ to $\mathbb{R}^n$ by:

$$G(x,\rho) = u_2(x,x) + \rho u_1(x,x) \tag{5.21}$$

In view of (A.5+), the function G is well-defined[6] by (5.21). We denote the Jacobian matrix of G, evaluated at $(x,\rho) \in \Lambda \times [0,1]$, by $J(x,\rho)$, and the determinant of this matrix by $\det J(x,\rho)$. Given $\rho \in [0,1]$, the set of solutions in Λ to the equation $G(x,\rho) = 0$ is denoted by $M(\rho)$.

Lemma 5.6.1. *Suppose* $(k,\rho) \in \Lambda \times [0,1]$ *satisfies:*

$$u_2(k,k) + \rho u_1(k,k) = 0 \tag{5.22}$$

then there is $p \in \mathbb{R}^n_+$ *such that:*

$$u(k,k) + p(\rho k - k) \geq u(x,y) + p(\rho y - x) \quad \textit{for all } (x,y) \in \Omega \tag{5.23}$$

and (k,k) *solves the maximization problem:*

$$\left.\begin{array}{ll} Max & u(x,y) \\ subject\ to & \rho y - x \geq \rho k - k \\ and & (x,y) \in \Omega \end{array}\right\} \tag{5.24}$$

Proof. Define $p = u_1(k,k)$; then $p \in \mathbb{R}^n_+$. Concavity of u implies that for every $(x,y) \in \Omega$,

$$\begin{aligned} u(x,y) - u(k,k) &\leq u_1(k,k)(x-k) + u_2(k,k)(y-k) \\ &= p(x-k) - \rho p(y-k) \end{aligned}$$

[6] The use of ρ rather than δ here is deliberate. The discount factor, δ, has been restricted to be less than 1 in our description of the basic model in Section 5.2. In contrast, we definitely want ρ to take on the value 1, as well as values less than 1.

the last line following from (5.22). Transposing terms yields (5.23). Clearly, (5.24) follows directly from (5.23).

Using Lemma 5.6.1, we see that interior SOS are equivalent to stationary solutions of Ramsey-Euler equations (in the discounted case).

Proposition 5.6.1. *If* $(k, \delta) \in \Lambda \times (0,1)$, *then the following statements are equivalent:*

(i) $u_2(k,k) + \rho u_1(k,k) = 0$.

(ii) k *is a stationary optimal stock.*

Proof. If (i) holds, then we can use Lemma 5.6.1 to obtain $p \in \mathbb{R}^n_+$ such that:

$$u(k,k) + p(\delta k - k) \geq u(x,y) + p(\delta y - x) \quad \textit{for all } (x,y) \in \Omega \tag{5.25}$$

Defining $p(t) = \delta^t p$ for $t \geq 0$, we have for all $t \geq 0$:

$$\delta^t u(k,k) + p(t+1)k - p(t)k \geq \delta^t u(x,y) + p(t+1)y - p(t)x \quad \textit{for all } (x,y) \in \Omega \tag{5.26}$$

and:

$$\lim_{t\to\infty} p(t)k = 0 \tag{5.27}$$

since $\delta \in (0,1)$. Thus by the standard sufficiency result on price characterization of optimality, $\{k\}$ is optimal from k, which establishes (ii).

If (ii) holds, then using the fact that $k \in \Lambda$, we know that k solves the maximization problem:

$$\left.\begin{array}{ll} \textit{Max} & u(k,x) + \delta u(x,k) \\ \textit{subject to} & (k,x) \in \textit{int } \Omega \\ \textit{and} & (x,k) \in \textit{int } \Omega \end{array}\right\} \tag{5.28}$$

Then, we obtain (i) as the necessary first-order condition of the problem (5.28).

To proceed with our analysis, we now impose the condition:

(B.1) There is $(\hat{x}, \hat{x}) \in \Omega^0$, such that $G(\hat{x}, 1) = 0$ and $\det J(\hat{x}, 1) \neq 0$.

Lemma 5.6.2. *Under condition (B.1), the equation* $G(x,1) = 0$ *has exactly one solution for* $x \in \Lambda$.

Proof. By condition (B.1), $\hat{x} \in \Lambda$ is a solution of the equation $G(x,1) = 0$. Suppose $x' \in \Lambda$ is also a solution to $G(x,1) = 0$, with $x' \neq \hat{x}$.

Using Lemma 5.6.1 for $\rho = 1$, we know that $(\hat{x}, \hat{x})$ and (x', x') are both solutions to:

$$\left.\begin{array}{ll} \textit{Max} & u(x,y) \\ \textit{subject to} & y - x \geq 0 \\ \textit{and} & (x,y) \in \Omega \end{array}\right\} \tag{5.29}$$

By convexity of Ω and concavity of u, we know that $(x(\lambda), x(\lambda)) = \lambda(\hat{x}, \hat{x}) + (1-\lambda)(x', x')$ must also solve (5.29) for all $\lambda \in (0,1)$, and for every $\lambda \in (0,1)$, we have $(x(\lambda), x(\lambda)) \neq (\hat{x}, \hat{x})$. But since $\det J(\hat{x}, 1) \neq 0$, the solution $\hat{x}$ is locally unique, and therefore for λ sufficiently close to 1, we get a contradiction. This proves the lemma.

We now impose an additional condition on the set of solutions to $G(x, \rho) = 0$:

(B.2) There is $\delta \in (0, 1)$, and an open set C, such that $\bar{C} \subset \Lambda$, and $M(\rho) \subset C$ for all $\rho \in [\delta, 1]$.

Here $\bar{C}$ is the closure of C. The condition implies that for every $\rho \in [\delta, 1]$, the boundary of C contains no solution to $G(x, \rho) = 0$.

In order to keep our exposition self-contained, we state the two results from differential topology that we will need for the main result of this section. These results, and their complete proofs, can be found in Ortega and Rheinboldt (1970, Chapter 6), who follow the approach of Erhard Heinz (1959) in providing an elementary analytic theory of the degree of a mapping[7].

Homotopy Invariance Theorem:[Ortega and Rheinboldt (1970, Result 6.2.2, p.156)]

Let C be open and bounded and $H : \bar{C} \times [0, 1] \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ a continuous map from $\bar{C} \times [0, 1]$ into $\mathbb{R}^n$. Suppose, further, that $H(x, \rho) \neq 0$ for all $(x, \rho) \in \partial C \times [0, 1]$. Then, $\deg(H(\cdot, \rho), C)$ is constant for all $\rho \in [0, 1]$.

Degree Theorem: [Ortega and Rheinboldt (1970, Result 6.2.9, p. 159)]

Let $g : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on the open set D, and C an open, bounded set such that $\bar{C} \subset D$. For $x \in D$, denote the Jacobian matrix of g at x by $J_g(x)$. If $0 \notin g(\partial C) \cup g(S(\bar{C}))$, where $S(\bar{C}) = \{x \in \bar{C} : J_g(x)$ is singular $\}$, either $\{x \in C : g(x) = 0\}$ is empty and $\deg(g, C) = 0$, or $\{x \in C : g(x) = 0\}$ consists of finitely many points $x^1, ..., x^{m,}$ and:

$$\deg(g, C) = \sum_{j=1}^{m} sgn \;\det\; J_g(x^j)$$

where sgn denotes the sign function[8].

We now state and prove the main result of this section. The approach to the uniqueness result may be indicated as follows. Using the Degree theorem, it is possible to evaluate the degree of $G(\cdot, \rho)$ in a simple case, which is when $\rho = 1$ in our context, since there is a unique solution to $G(x, 1) = 0$ on Λ. To evaluate the degree of $G(\cdot, \rho)$ in the case we are really interested in, namely when $\rho = \delta$, we create a homotopy between the two functions, $G(\cdot, 1)$ and $G(\cdot, \delta)$, and apply the Homotopy Invariance theorem; this is where condition (B.2) is used. This procedure yields one evaluation of the degree of $G(\cdot, \delta)$. However, applying the

[7] That is, in their presentation of degree theory, the results do not involve any concept, not familiar from standard real analysis in Euclidean spaces; nor do their proofs.

[8] That is, sgn is a function from $\mathbb{R}$ to $\{-1, 0, +1\}$, satisfying $sgn(r) = +1$ if $r > 0$, $sgn(r) = -1$ if $r < 0$, and $sgn(r) = 0$ if $r = 0$.

Degree theorem to $G(\cdot,\delta)$, we get another evaluation of the degree of $G(\cdot,\delta)$ in terms of the behavior of the Jacobian of $G(\cdot,\delta)$ at the zeroes of the function. The idea then is to impose a hypothesis restricting this behavior in a way that in turn yields an appropriate restriction on the number of zeroes of the function.

Theorem 5.6.1. *Suppose* $\det J(x,\delta) \neq 0$ *for all* $x \in M(\delta)$*, and further the sign of* $\det J(x,\delta)$ *is the same for all* $x \in M(\delta)$*. Then there is only one interior SOS when the discount factor is* δ*.*

Proof. Under the hypothesis, we need to show that $M(\delta)$ is a singleton. Define $f : \Lambda \to \mathbb{R}$ by $f(x) = G(x,1)$. The Degree theorem then gives us a formula for computing the degree of f on C, where C is given in condition (B.2). Applying the theorem to f on C, we get (in view of Lemma 5.6.2):

$$\deg(f,C) = sgn \det J_f(\hat{x}) \equiv sgn \det J(\hat{x},1) \tag{5.30}$$

where $\hat{x}$ is given by condition (B.1). Thus, the $\deg(f,C)$, the degree of f on C, is either $+1$ or -1.

Define $F : \Lambda \to \mathbb{R}$ by $F(x) = G(x,\delta)$. We now show that $\deg(F,C) = \deg(f,C)$, by establishing a homotopy between f and F. To this end, define $H : \bar{C} \times [\delta,1] \to \mathbb{R}^n$ by:

$$H(x,\rho) = G(x,\rho)$$

and note that C is open and bounded, and H a continuous map from $\bar{C} \times [\delta,1]$ to $\mathbb{R}^n$. Further, by condition (B.2), $H(x,\rho) \neq 0$ for all $(x,\rho) \in \partial C \times [\delta,1]$, where ∂C denotes the boundary of C. Thus, by the Homotopy Invariance theorem, $\deg(H(\cdot,\rho),C)$ is constant for $\rho \in [\delta,1]$. In particular, then, $\deg(F,C) = \deg(f,C)$, and so $\deg(F,C)$ is either $+1$ or -1.

Now, applying the Degree theorem to F on C, we know that $M(\delta)$ consists of finitely many points $x^1, ..., x^m$, and:

$$\deg(F,C) = \sum_{j=1}^{m} sgn\ \det J_F(x^j) \equiv \sum_{j=1}^{m} sgn\ \det J(x^j,\delta) \tag{5.31}$$

The hypothesis of the Theorem ensures that $\det J(x^j,\delta) \neq 0$ for all x^j, and further the sign of $\det J(x^j,\delta)$ is the same for all $j = 1, ..., m$. Since we know that $\deg(F,C)$ is either $+1$ or -1, (5.31) implies that we must have $m = 1$. Thus, $M(\delta)$ is a singleton, and there is only one interior SOS for the discount factor, δ.

Remark:

Brock (1973) showed that if $J(x,\rho)$ is non-singular over $M(\rho)$ for each $\rho \in (\rho_1,1)$, then $M(\rho)$ is a singleton for each $\rho \in (\rho_1,1)$. Benhabib and Nishimura (1979) provided conditions, which appear in the above Theorem, under which $J(x,\rho)$ might be singular for some $\rho \in (\delta,1)$, but $M(\delta)$ is a singleton.

5.7 Bibliographic Remarks

Sections 5.3 and 5.4:
The approach to existence of stationary optimal stocks that we have followed is a primal one, because it is the most direct one, and it economizes on the assumptions used. However, the dual approach provides, in addition, a supporting price vector, and the quantity-price pair is then referred to as a modified golden-rule. The price support is useful in looking at issues related to uniqueness and global asymptotic stability of stationary optimal stocks. This dual approach is surveyed in Mitra (2005).

We have confined our analysis to the case in which future utilities are discounted. In the undiscounted case, programs are compared by using some version of the overtaking criterion. The approach to the existence of stationary optimal stocks in this context is somewhat different. It does not involve the fixed point argument, which is replaced by arguments based on standard constrained optimization theory. The subsequent step of showing that the *golden-rule stock*, found as a solution to the constrained optimization problem, is indeed optimal among all programs starting from that stock, is more complicated, and makes essential use of duality theory and the price support to the golden-rule stock. The complication arises from the fact that the convenient transversality condition (in the discounted case) is not available in the undiscounted case. The reader is referred especially to the contributions by Brock (1970) and Peleg (1973), which are based on the earlier contributions by Gale (1967) and McKenzie (1968).

The price-supported golden-rule is particularly useful in studying long-run dynamic behavior of optimal programs in the undiscounted case. This has been effectively demonstrated in applications of the theory to study the Faustmann solution in the forest management problem (see Mitra and Wan (1986)) and to analyze the choice of technique in development planning (see Khan and Mitra (2005)).

There is no primal approach to the existence problem in the undiscounted case, corresponding to the one presented here for the discounted case. It is of interest to note that it is the dual approach which is employed by Mitra (1991) in establishing existence of stationary optimal stocks in the undiscounted case in models with a *non-convex* transition possibility set, which satisfies a star-shaped property.

Section 5.5:
The approach of this section is based on Brock (1973) and Brock and Burmeister (1976), emphasizing the normality property of the welfare function, based on consumption alone. However, unlike these papers, we emphasize the methods of convex analysis, and refrain from making differentiability assumptions on the welfare function. Stationary optimal stocks turn out to be *not* in the interior of the transition possibility set, making the framework of this section distinctly different from that used in Section 5.6.

Instead of a fixed coefficients Leontief type of technology with no-joint production used in this section, Brock (1973) and Brock and Burmeister (1976) use a non-linear activity analysis model, and appeal to the non-substitution theorem. We have presented the results in the more restrictive framework, because the arguments involved are very transparent in this case. Some of this theory can even be generalized to settings with joint production, provided an approppriate version of the non-substitution theorem holds in that framework; for this theory, see Benhabib and Nishimura (1979).

Section 5.6:
The methods of differential topology were used to address uniqueness problems in general equilibrium theory by Dierker (1972). They were then used in optimal growth models by Brock (1973) and Benhabib and Nishimura (1979).

We have presented this theory so that a reader, familiar only with standard concepts in real analysis, should be able to follow the results without any difficulty. Specifically, concepts and terminology used in differential topology have been avoided.

For smooth economies, it is possible to develop a connection between the normality assumption in Section 5.5, and the hypothesis on the behavior of the Jacobian at the zeroes of the relevant function used in Section 5.6. This is explored in detail in Benhabib and Nishimura (1979).

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6. Optimal Cycles and Chaos

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6.1 Introduction

Optimal growth models have originally been developed in order to analyze the long-run implications of capital accumulation and technological progress. Later on, however, it has been noticed that essentially the same model structure can also be used to shed light on short-run phenomena like the business cycle. The most prominent outcome of this line of research are real-business-cycle (RBC) theories, which assume that business cycles are triggered by exogenous stochastic shocks and which analyze the mechanisms by which these shocks propagate through the economy. The literature surveyed in the present chapter, on the other hand, shows that optimal growth models can generate business cycles even in the absence of exogenous shocks. It is therefore appropriate to refer to these results as endogenous-business-cycle (EBC) theories. An important property common to both RBC and EBC theories is that the business cycles qualify as optimal programs. In other words, the solutions of both RBC and EBC models are Pareto-efficient. As far as the deterministic EBC models are concerned, the most important implication of this fact is that the standard assumptions of optimal growth theory do not rule out intrinsic instability of the economy, an instability that allows for periodic or even chaotic optimal programs.

Following Ramsey (1928), much of the earlier literature in optimal growth theory focused on equal treatment of generations over time, and therefore on the undiscounted case. The analysis of this class of models was brought to maturity in the papers of Gale (1967), McKenzie (1968), and Brock (1970). The treatment of the case in which future utilities were discounted was typically done in the one-sector neoclassical model, where the significant difference between the two cases was not revealed because of the one-to-one conversion of capital to consumption good implicit in its formulation. It was with the examples of Kurz (1968) and Sutherland (1970), in models which did not have this feature, that one recognized that discounting the future in general provided more limited intertemporal arbitrage opportunities; thus, the standard argument for smoothing out cyclical behavior was not valid in such frameworks.

Samuelson (1973) can be considered to provide definitive directions for research towards an understanding of such a phenomenon. On the one hand, he reported an example, due to Weitzman, which showed that cyclical optimal behavior was consistent with interior solutions to Ramsey-Euler equations and therefore would not disappear with assumptions which ruled out boundary solutions to optimal growth problems. On the other hand, he conjectured that, if the utility function was strictly concave, then cyclical optimal behavior of the Weitzman type would not arise, if the planner was sufficiently patient. The second idea was formalized in terms of turnpike theorems for low discount rates in a *Journal of Economic Theory* symposium of 1976, and led to a literature which is comprehensively surveyed in McKenzie (1986). The first idea led Benhabib and Nishimura (1985) to initiate a systematic investigation of the sources of optimal cycles and this, in turn, led to the literature that is surveyed in this chapter.

Section 6.2 sets the stage for our survey by presenting background material on dynamical systems and optimal growth models. Sections 6.3 and 6.4 form the main part of the chapter. In section 6.3 we study the optimality of periodic cycles. Although periodic optimal growth paths cannot be interpreted as realistic business cycles, the characterization of the conditions under which periodic cycles are optimal allows important insights into the mechanisms that can generate non-monotonic optimal growth paths. Section 6.4 then turns to chaotic optimal growth paths. These solutions resemble actual business cycles more closely than periodic ones, but it is somewhat harder to characterize the mechanisms by which they are generated.

6.2 Basic Definitions and Results

This section presents some background material that is necessary to state the main results on optimal cycles and chaos. First we discuss a number of concepts and results that are related to cyclical and chaotic behavior of dynamical systems. Then we formulate the reduced form optimal growth model and show that it encompasses, among other models, a discrete-time version of the two-sector model of Uzawa (1964).

6.2.1 Dynamical Systems

Let X be a non-empty set and let h be a map from X to X. The pair (X,h) is called a *dynamical system*. We refer to X as the *state space* and to h as the *law of motion* of the dynamical system. Thus, if $x_t \in X$ is the state of the system in time period t (where $t = 0,1,2,\ldots$), then $x_{t+1} = h(x_t) \in X$ is the state of the system in time period $t+1$. We write $h^{(0)}(x) = x$ and, for any integer $t \geq 1$, $h^{(t)}(x) = h(h^{(t-1)}(x))$. If $x \in X$, the sequence $\tau(x) = (h^{(t)}(x))_{t=0}^{\infty}$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\omega(x) = \{y \,|\, y = h^{(t)}(x) \text{ for some } t \geq 0\}$.

A point $x \in X$ is a *fixed point* of the dynamical system (X,h), if $h(x) = x$. A point $x \in X$ is called a *periodic point* of (X,h), if there is $p \geq 1$ such that $h^{(p)}(x) = x$. The smallest such p is called the *period* of x. In particular, if $x \in X$ is a fixed point of (X,h), then it is periodic with period 1.

Throughout this chapter we assume that X is a non-empty and compact interval on the real line $\mathbb{R}$. In this case, it makes sense to describe the asymptotic behavior of a trajectory from x by its *limit set*, which is defined as the set of all limit points of $\tau(x)$. The limit set of x is denoted by $\omega_+(x)$. Note that, if $\hat{x} \in X$ is a periodic point, then $\omega_+(h^{(t)}(\hat{x})) = \omega(\hat{x})$ for every $t = 0,1,2,\ldots$. A periodic point $\hat{x}$ is said to be *locally stable*, if there is an open interval $I \subseteq X$ containing $\hat{x}$ such that $\omega_+(x) = \omega(\hat{x})$ for all $x \in I$. In this case we also say that the periodic orbit $\omega(\hat{x})$ is locally stable. If h is continuously differentiable on X and $\hat{x}$ is a periodic point of period p, then a sufficient condition for $\hat{x}$ to be locally stable is that $\left|Dh^{(p)}(\hat{x})\right| < 1$. If $\left|Dh^{(p)}(\hat{x})\right| > 1$, then $\hat{x}$ is not locally stable. A periodic point $\hat{x}$ is said to be *globally stable* (*almost globally stable*), if $\omega_+(x) = \omega(\hat{x})$ holds for all (almost all) initial points $x \in X$.

Suppose that the law of motion h is a non-decreasing function. Obviously, this implies that the trajectory $\tau(x)$ is a monotonic sequence for every $x \in X$. Because X is compact, this sequence must have a unique limit point. It follows therefore for every $x \in X$ that the limit set $\omega_+(x)$ is a singleton. Any form of non-monotonic behavior such as periodic orbits with a period $p \geq 2$ is therefore ruled out when h is non-decreasing. Now suppose that h is non-increasing. This implies that the second iterate $h^{(2)}$ is non-decreasing. Consequently, every limit set of the dynamical system $(X,h^{(2)})$ is a singleton and it follows that every limit set of the original system (X,h) consists of at most two points. A dynamical system (X,h) with a non-increasing law of motion can therefore have periodic points of period 2 but it cannot have periodic points of any period $p > 2$.

Consider the following complete order on the positive integers:

$$\begin{aligned}&3 \prec 5 \prec 7 \prec \ldots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec \ldots \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec 2^2\cdot 7 \prec \\ &\ldots \prec 2^n\cdot 3 \prec 2^n\cdot 5 \prec 2^n\cdot 7 \prec \ldots \prec 2^n \prec \ldots \prec 2^2 \prec 2 \prec 1.\end{aligned}$$

This order is called the *Sarkovskii order*. Sarkovskii (1964) has proved that a dynamical system (X,h), where h is a continuous function, has the following property: if there exists a periodic point of period p and if $p \prec q$, then there exists also a periodic point of period q.

When we say that a dynamical system is history independent, we wish to convey the observation that the long-run (asymptotic) behavior of a typical trajectory is independent of the initial state. Formally, let (X, h) be a dynamical system and let λ be the Lebesgue measure on X. The dynamical system (X, h) is *history independent*, if there exists a subset E of X such that, for λ-almost every $x \in X$, the limit set of x satisfies $\omega_+(x) = E$. The dynamical system (X, h) is *history dependent*, if it is not history independent.

We will often be concerned with a family of dynamical systems, where the members of the family are indexed by a parameter. Formally, let us denote the parameter by $\delta \in D$, where D is taken to be a non-empty interval in $\mathbb{R}$. A *family of dynamical systems* will then be denoted by (X, h_δ), where h_δ maps X to X for each $\delta \in D$. Suppose the dynamical system (X, h_δ) is history independent for every $\delta \in D$. Then, for each $\delta \in D$, we can find a set $E(\delta)$ such that the following is true for Lebesgue almost every x in X: the limit set of the trajectory from x generated by the dynamical system (X, h_δ) is equal to $E(\delta)$. A *bifurcation map* is the correspondence which associates to each $\delta \in D$ its history independent limit set $E(\delta) \subseteq X$. A *bifurcation diagram* is a diagrammatic representation of the graph of the bifurcation map.

We now turn to chaotic behavior. There are several aspects of complicated dynamics that need to be taken into account. One of the most important ones is the so-called sensitivity with respect to initial conditions. A simple way to describe this property is as follows. The dynamical system (X, h) exhibits *geometric sensitivity*, if there exists a number $\gamma > 1$ such that the following is true: for every integer $\tau \geq 0$ there exists $\varepsilon > 0$ such that, for all $(x, y) \in X \times X$ with $\|x - y\| < \varepsilon$ and for all $t \in \{0, 1, 2, \ldots, \tau\}$, it holds that

$$\|h^{(t)}(x) - h^{(t)}(y)\| \geq \gamma^t \|x - y\|.$$

Sensitivity with respect to initial conditions is also captured by the concept of a scrambled set. Let us denote by P the set of all periodic points of the dynamical system (X, h). A subset S of the state space X is called a *scrambled set* for the dynamical system (X, h), if the following two conditions are satisfied. (i) For all pairs (x, y) satisfying $x \in S$ and $y \in S$ it holds that

$$\liminf_{t \to \infty} |h^{(t)}(x) - h^{(t)}(y)| = 0.$$

(ii) For all pairs (x, y) satisfying $x \in S$ and either $x \neq y \in S$ or $y \in P$ it holds that

$$\limsup_{t \to \infty} |h^{(t)}(x) - h^{(t)}(y)| > 0.$$

The dynamical system (X, h) is said to exhibit *topological chaos*, if there exists an uncountable scrambled set and a periodic point of a period that is not a power of 2. Note that due to Sarkovskii's theorem mentioned above, a dynamical system which has a continuous law of motion and which exhibits topological chaos must have infinitely many periodic points of different periods. A famous theorem by Li and Yorke (1975) states that the dynamical system (X, h) exhibits topological chaos, if h is continuous and if there exists a periodic point of period $p = 3$.

One problem with the definition of topological chaos is that the scrambled set can have Lebesgue measure 0. If this is the case, then the chaotic behavior may not be observable. A chaos definition that circumvents this problem is that of ergodic chaos. The dynamical system (X, h) exhibits *ergodic chaos*, if there exists an absolutely continuous (with respect to Lebesgue measure) probability measure μ on X which is invariant and ergodic under h. *Invariance* is the property that $\mu(\{x \in X \,|\, h(x) \in B\}) = \mu(B)$ holds for all measurable sets $B \subseteq X$. The invariant measure μ is *ergodic*, if, for every measurable set $B \subseteq X$ satisfying $\{x \in X \,|\, h(x) \in B\} = B$, it holds that $\mu(B) \in \{0, 1\}$.

Results by Lasota and Yorke (1973) and Li and Yorke (1978) show that the dynamical system (X, h) has geometric sensitivity and ergodic chaos, if there exists $\gamma > 1$ and a point $\tilde{x} \in X$ splitting X into two subintervals (recall that we assume X to be a compact interval on $\mathbb{R}$) such that (i) h is twice continuously differentiable on both subintervals, (ii) h is strictly increasing for $x < \tilde{x}$ and strictly decreasing for $x > \tilde{x}$, and (iii) $|h'(x)| \geq \gamma$ for all $x \in X \setminus \{\tilde{x}\}$.

6.2.2 Optimal Growth Models

We maintain the assumption that the state space X is a non-empty and compact interval on the real line $\mathbb{R}$. A *reduced form optimal growth model* on X is described by a triple (Ω, U, δ), where Ω is the *transition possibility set*, U is the (*reduced form*) *utility function*, and δ is the *discount factor*. The following assumptions on (Ω, U, δ) will be maintained throughout this chapter.

A.1: $\Omega \subseteq X \times X$ is non-empty, closed, and convex.

A.2: $U : \Omega \mapsto \mathbb{R}$ is a continuous and concave function.

A.3: $0 < \delta < 1$.

A *program* from $x \in X$ is a sequence $(x_t)_{t=0}^{\infty}$ satisfying $x_0 = x$ and $(x_t, x_{t+1}) \in \Omega$ for all $t \geq 0$. Let $(x_t)_{t=0}^{\infty}$ be a program from $x \in X$. It is called an *optimal program* from x, if

$$\sum_{t=0}^{\infty} \delta^t U(x_t, x_{t+1}) \geq \sum_{t=0}^{\infty} \delta^t U(y_t, y_{t+1})$$

holds for every program $(y_t)_{t=0}^{\infty}$ from x.

The issues of existence and uniqueness of optimal programs have been well studied; see, e.g., Stokey and Lucas (1989) or Mitra (2000). Under assumptions A.1-A.3 there exists an optimal program from every $x \in X$. Thus, one can define the *value function* $V : X \mapsto \mathbb{R}$ by

$$V(x) = \sum_{t=0}^{\infty} \delta^t U(x_t, x_{t+1}),$$

where $(x_t)_{t=0}^{\infty}$ is an optimal program from x. The value function V is concave and continuous on X. Moreover, for all $x \in X$, the Bellman equation

$$V(x) = \max\{U(x,z) + \delta V(z) \,|\, (x,z) \in \Omega\}$$

holds.

For each $x \in X$, we denote by $h(x)$ the set of all $z \in X$ which maximize $U(x,z) + \delta V(z)$ over all $z \in X$ satisfying $(x,z) \in \Omega$. That is, for each $x \in X$,

$$h(x) = \operatorname{argmax}\{U(x,z) + \delta V(z) \,|\, (x,z) \in \Omega\}\,.$$

A program from $x \in X$ is an optimal program, if and only if $V(x_t) = U(x_t, x_{t+1}) + \delta V(x_{t+1})$ for $t \geq 0$, that is, if and only if $x_{t+1} \in h(x_t)$ holds for all $t \geq 0$. We call h the *optimal policy correspondence*.

Given an initial state $x \in X$, there may be more than one optimal program from x. If, for every $x \in X$, there is a unique optimal program from x, then it follows that the optimal policy correspondence h is a (single-valued) function. It can also be shown that this function is continuous on X. A simple condition that ensures the uniqueness of optimal programs is the strict concavity of the utility function U with respect to its second argument. Whenever the optimal policy correspondence is a single-valued function, we shall refer to it as the *optimal policy function*.

Reduced form optimal growth models arise in many different contexts. For the purpose of the present chapter, the *two-sector optimal growth model* introduced by Uzawa (1964) is the most relevant one. The state variable x_t of this model describes the aggregate capital stock available in the economy at the beginning of period t. There are two production sectors, one producing a consumption good and the other a capital good. Each sector uses the capital good and labor as inputs. The capital good cannot be consumed and depreciates at the rate d, where $d \in [0,1]$. The labor supply is assumed to be constant and equal to 1. Denoting the production functions in the consumption good sector and the capital good sector by $c = F_c(x_c, \ell_c)$ and $y = F_x(x_x, \ell_x)$, respectively, and the utility function by $u(c)$, the two-sector model can be formulated as follows.

$$\begin{aligned}
\text{Maximize} \quad & \sum_{t=0}^{\infty} \delta^t u(c_t) \\
\text{subject to} \quad & c_t \leq F_c(x_{c,t}, \ell_{c,t}) \\
& (1-d)x_t \leq x_{t+1} \leq F_x(x_{x,t}, \ell_{x,t}) + (1-d)x_t \\
& x_{c,t} + x_{x,t} \leq x_t \\
& \ell_{c,t} + \ell_{x,t} \leq 1.
\end{aligned}$$

In order to convert the two-sector optimal growth model into its reduced form, we first determine the transition possibility set Ω. To this end note that the set of all capital stocks that can be reached from the state x within one period is given by $\{z \mid (1-d)x \le z \le F_x(x,1)+(1-d)x\}$. If $d>0$ and if the production function F_x is increasing and concave and satisfies the Inada conditions, then $F_x(x,1)+(1-d)x$ is an increasing and concave function of x. The slope of this function is strictly greater than 1 for small x and strictly smaller than 1 for large x. These properties imply that there exists a unique value $\bar{x}>0$ satisfying $\bar{x}=F_x(\bar{x},1)+(1-d)\bar{x}$. The following properties hold for any pair (x,z), where z is a capital stock that can be reached within one period from x: if $x \le \bar{x}$ then $z \le \bar{x}$, and if $x > \bar{x}$ then $z < x$. In other words, $\bar{x}$ is the maximal sustainable capital stock. For this reason, it is justified to restrict attention to the compact state space $X=[0,\bar{x}]$. The transition possibility set Ω is therefore given by

$$\Omega = \{(x,z) \mid 0 \le x \le \bar{x}\,,\ (1-d)x \le z \le F_x(x,1)+(1-d)x\}$$

and the reduced form utility function is given by $U(x,z)=u(T(x,z))$, where

$$\begin{aligned} T(x,z) \quad &= \quad \max F_c(x_c,\ell_c) \\ \text{subject to} \quad & \quad z \le F_x(x_x,\ell_x)+(1-d)x \\ & \quad x_c + x_x \le x \\ & \quad \ell_c + \ell_x \le 1. \end{aligned}$$

The function T describes, for any given $x \in X$, the trade-off between consumption and capital production.

6.3 Optimal Cycles

Consider a reduced form optimal growth model (Ω,U,δ) on the state space X. Turnpike theory, as developed for example by Scheinkman (1976) or McKenzie (1983, 1986), shows that an optimal program for this model must be convergent provided that certain regularity assumptions are satisfied and the discount factor δ is sufficiently close to 1. In other words, given X, Ω, and U, a sufficiently large time-preference factor δ rules out cyclical or more complicated optimal programs. Whether complicated dynamic patterns can be optimal for small values of the discount factor, however, is left open by turnpike theory and has been the focus of intensive research in the 1980s and 1990s. The present section surveys some important contributions to this literature.

We start by summarizing a few results regarding the monotonicity properties of the optimal policy function. These results provide simple conditions for the non-existence of optimal cycles. The main part of this section, however, is concerned with the question of how long-run optimal behavior is affected by changes in the rate at which the future is discounted. In particular we will observe period-2 cycles being born and changing their amplitude as the discount factor δ falls.

The class of examples that we study in detail allow for period-2 cycles but no more complicated behavior than that, and they indicate an interesting feature about the transition from global asymptotic stability of a (unique) fixed point at high discount factors to global asymptotic stability of cycles at lower discount factors. The family of examples in subsection 6.3.3 constitute variations of the example of Weitzman, as discussed in Samuelson (1973). Weitzman's example has been widely discussed in the literature; see, for example, Scheinkman (1976), McKenzie (1983), Benhabib and Nishimura (1985), and Mitra and Nishimura (2001). For this family of examples there is a critical discount factor, $\hat{\delta}$, such that the following is true. For $\delta > \hat{\delta}$, all optimal programs converge to the unique fixed point $\hat{x}_\delta$ and, for $\delta < \hat{\delta}$, almost all optimal programs converge to a period-2 cycle. The bifurcation diagram also indicates that the amplitude of the period-2 cycle is monotonic in the discount factor.

The class of examples in subsection 6.3.4 constitute variations of the example presented by Sutherland (1970). This example has been discussed in Cass and Shell (1976), Benhabib and Nishimura (1985), and Mitra and Nishimura (2001). For this family of examples, too, there is a critical discount factor, $\hat{\delta}$, such that all optimal programs converge to the unique fixed point, if $\delta > \hat{\delta}$, and that almost all optimal programs converge to a period-2 cycle for $\delta < \hat{\delta}$. The range of discount factors can be further subdivided according to whether the period-2 cycle hits one boundary of the state-space or the other (or both). The global bifurcation diagram reveals that the amplitude of the period-2 cycle is *not* monotonic in the discount factor.

6.3.1 Monotonic Policy Functions

We have seen in subsection 6.2.1 that a dynamical system with a non-decreasing law of motion cannot generate cycles, and that a dynamical system with a non-increasing law of motion can generate cycles of period 2 but no cycles of any period $p > 2$. A first step towards the analysis of optimal cycles is therefore the investigation of the conditions that generate a non-decreasing or a non-increasing optimal policy function, respectively. Very general results in this respect can be derived by the use of the lattice theoretic concepts of super- and submodularity; see, e.g., Topkis (1978) or Ross (1983). The function $U : \Omega \mapsto \mathbb{R}$ is said to be *supermodular* if, for any two pairs $(x, z) \in \Omega$ and $(x', z') \in \Omega$, the following is true: if $x < x'$, $z < z'$, $(x, z') \in \Omega$, and $(x', z) \in \Omega$, then it holds that $U(x, z) + U(x', z') \geq U(x, z') + U(x', z)$. The function U is *submodular* if the inequality holds in reverse, that is, if $U(x, z) + U(x', z') \leq U(x, z') + U(x', z)$. The following theorem is a variant of a result stated in Amir (1996); see also Mitra (2000).

Theorem 6.3.1. *Let (Ω, U, δ) be an optimal growth model satisfying assumptions A.1-A.3 and assume that its optimal programs are described by the optimal policy function h. If U is supermodular (submodular), then h is non-decreasing (non-increasing) locally around every point x satisfying $(x, h(x)) \in \operatorname{int} \Omega$.*

Proof. We present the proof for the case where U is submodular; the case of a supermodular utility function can be dealt with analogously. Let $x \in X$

be a state such that $(x, h(x)) \in \operatorname{int} \Omega$. Since the optimal policy function is continuous, it follows that for every x' sufficiently close to x, the pairs $(x, h(x'))$, $(x', h(x))$, and $(x', h(x'))$ are also contained in Ω. Let us define $z = h(x)$ and $z' = h(x')$. Without loss of generality we may assume $x < x'$. We need to show that $z \geq z'$. Suppose to the contrary that $z < z'$. Because optimal programs are described by an optimal policy function, the maximizer of the right-hand side of the Bellman equation must be unique. Since $z \neq z'$ it follows therefore that $U(x,z) + \delta V(z) > U(x,z') + \delta V(z')$ and $U(x',z') + \delta V(z') > U(x',z) + \delta V(z)$, where V is the value function. Adding these inequalities it follows that $U(x,z) + U(x',z') > U(x,z') + U(x',z)$. This is a contradiction to the assumed submodularity of U, which completes the proof.

Theorem 6.3.1 shows that any interior section of the graph of the optimal policy function of a model with a supermodular (submodular) utility function is a non-decreasing (non-increasing) curve. If we have additional information about the transition possibility set, then it is possible to establish global monotonicity properties of the optimal policy function. This is shown in the following corollary which uses the definitions $\psi(x) = \min\{z \,|\, (x,z) \in \Omega\}$ and $\phi(x) = \max\{z \,|\, (x,z) \in \Omega\}$. Note that assumption A.1 implies that ψ and ϕ are continuous functions on X.

Corollary 6.3.1. *Let (Ω, U, δ) be an optimal growth model satisfying assumptions A.1-A.3 and assume that its optimal programs are described by the optimal policy function h.*
(i) If ψ and ϕ are non-decreasing functions and if U is supermodular, then it follows that h is non-decreasing on X.
(ii) If ψ and ϕ are non-increasing functions and if U is submodular, then it follows that h is non-increasing on X.

An important case in which the functions ψ and ϕ are non-decreasing is the two-sector model discussed in subsection 6.2.2. A simple example in which the functions ψ and ϕ are non-increasing is given by $\Omega = X \times X$.

If U is a twice continuously differentiable function, then supermodularity (submodularity) follows from $U_{12}(x,z) > 0$ ($U_{12}(x,z) < 0$); see, e.g., Ross (1983). This observation can be used to prove the following result due to Benhabib and Nishimura (1985). In order to formulate it, one needs to impose a smoothness condition on the utility function.

A.4: The utility function U is twice continuously differentiable on the interior of Ω with second-order partial derivatives U_{11}, U_{12}, and U_{22}. Moreover, it holds that $U_{11}(x,z) < 0$, $U_{22}(x,z) < 0$, and $U_{11}(x,z)U_{22}(x,z) - U_{12}(x,z)^2 \geq 0$ for all (x,z) in the interior of Ω.

Note that assumption A.4 implies strict concavity of the utility function with respect to its second argument which, in turn, implies that an optimal program from any initial state $x \in X$ is unique. In other words, optimal programs can be described by an optimal policy function.

Theorem 6.3.2. *Let (Ω, U, δ) be an optimal growth model satisfying assumptions A.1-A.4 and let h be its optimal policy function.*
(i) If $(x, h(x)) \in \operatorname{int} \Omega$ and $U_{12}(x, h(x)) > 0$ ($U_{12}(x, h(x)) < 0$), then it follows that h is non-decreasing (non-increasing) locally at x.
(ii) If $(x, h(x)) \in \operatorname{int} \Omega$, $(h(x), h^{(2)}(x)) \in \operatorname{int} \Omega$, and $U_{12}(x, h(x)) > 0$ ($U_{12}(x, h(x)) < 0$), then it follows that h is strictly increasing (strictly decreasing) locally at x.

Proof. Part (i) follows by the same argument that has been used in the proof of theorem 6.3.1 because $U_{12}(x, h(x)) > 0$ ($U_{12}(x, h(x)) < 0$) implies supermodularity (submodularity) of U locally around $(x, h(x))$. To prove part (ii), we simply have to show that $x \neq x'$ implies $h(x) \neq h(x')$. Suppose to the contrary that $h(x) = h(x')$, where x' is sufficiently close to x such that $(x', h(x))$ is in the interior of Ω. It follows that both $(x, h(x), h^{(2)}(x), \ldots)$ and $(x', h(x), h^{(2)}(x), \ldots)$ are optimal paths and, since the three points $(x, h(x))$, $(x', h(x))$, and $(h(x), h^{(2)}(x))$ are in the interior of Ω, the Euler equation $U_2(y, h(x)) + \delta U_1(h(x), h^{(2)}(x)) = 0$ must hold for $y \in \{x, x'\}$. Obviously, this is not possible if $x' \neq x$, x' is sufficiently close to x, and $U_{12}(x, h(x)) \neq 0$.

6.3.2 The Role of Discounting

We now turn to the question of how the existence or non-existence of a period-2 cycle depends on the size of the discount factor δ. This question has been thoroughly investigated by Mitra and Nishimura (2001), and the rest of this section draws heavily from their paper. In order to be able to develop a precise characterization, Mitra and Nishimura (2001) restrict the class of optimal growth models by a number of assumptions. These assumptions ensure that the dynamical system (X, h) is history independent, that optimal programs converge either to fixed points or to period-2 cycles, and that the asymptotic behavior of optimal programs depends in a simple way on the discount factor. We summarize their arguments in the present subsection. Subsections 6.3.3 and 6.3.4 will then illustrate the application of these ideas by means of two important classes of examples.

Mitra and Nishimura (2001) postulate the following strengthened version of assumption A.1.

A.1$^+$: It holds that $X = [0, 1]$ and $\Omega = X \times X$.

In addition to A.1$^+$ and A.2-A.4, they impose strict monotonicity and submodularity of the utility function.

A.5: For all (x, z) in the interior of Ω it holds that $U_1(x, z) > 0$, $U_2(x, z) < 0$, and $U_{12}(x, z) < 0$.

The monotonicity part of assumption A.5 is standard, submodularity is assumed to ensure that optimal programs can exhibit period-2 cycles but no more complicated behavior. The following result is a straightforward consequence of the results stated in the previous subsection.

Lemma 6.3.1. *Let (Ω, U, δ) be an optimal growth model satisfying assumptions A.1$^+$ and A.2-A.5. There exists an optimal policy function h. The optimal policy function is continuous and non-increasing on X. There exists a unique fixed point of the dynamical system (X, h). All optimal programs converge either to the fixed point or to a period-2 cycle.*

At this point it is important to emphasize the fact that the optimal policy function h and, therefore, its fixed point depend on the discount factor. Henceforth, we consider (Ω, U) as fixed and treat δ as a parameter varying between 0 and 1. In order to ensure that the fixed point of (X, h) is in the interior of the state space, Mitra and Nishimura (2001) postulate the following assumption.

A.6: Let the function $\pi : (0,1) \mapsto \mathbb{R}$ be defined by $\pi(x) = -U_2(x,x)/U_1(x,x)$. It holds that $\lim_{x\to 0} \pi(x) = 0$ and $\lim_{x\to 1} \pi(x) > 1$.

It is now possible to prove the following result.

Lemma 6.3.2. *Let Ω and U be given such that assumptions A.1$^+$, A.2, and A.4-A.6 are satisfied. For all $\delta \in (0,1)$ let $h_\delta : X \mapsto X$ be the optimal policy function of (Ω, U, δ) and let $\hat{x}_\delta$ be the unique fixed point of (X, h_δ).*
(i) The inequality $0 < \hat{x}_\delta < 1$ holds for all $\delta \in (0,1)$.
(ii) The fixed point $\hat{x}_\delta$ is continuously differentiable and strictly increasing with respect to δ and it holds that $\lim_{\delta\to 0} \hat{x}_\delta = 0$ and $\hat{x}_1 := \lim_{\delta\to 1} \hat{x}_\delta \in (0,1)$.

The general strategy of Mitra and Nishimura (2001) proceeds now as follows. First, an auxiliary problem is formulated which involves optimization over two periods only and in which the terminal state is restricted to be the same as the initial state. The unique optimal policy function of that problem is denoted by f_δ. Due to the construction of the auxiliary problem, it follows that the fixed point of (X, h_δ) coincides with the fixed point of (X, f_δ) and that interior period-2 cycles of (X, h_δ) coincide with those generated by (X, f_δ). Furthermore, the value of the derivative of f_δ at the fixed point $\hat{x}_\delta$ gives information about the eigenvalues of the Euler equation of (Ω, U, δ) at $\hat{x}_\delta$ which, in turn, determine the local stability (or instability) of $\hat{x}_\delta$ as a fixed point of (X, h_δ). A condition is then imposed on f_δ which ensures that local stability of $\hat{x}_\delta$ implies almost global asymptotic stability of $\hat{x}_\delta$, and that instability of $\hat{x}_\delta$ implies almost global stability of a period-2 cycle of (X, h_δ). Finally, Mitra and Nishimura (2001) impose another condition which ensures that there is only a single switching from local stability of $\hat{x}_\delta$ to instability (and no switch back to local stability) as the discount factor changes from 1 to 0. We shall now briefly explain the most important details of this strategy.

For any given $x \in X$, consider the following auxiliary optimization problem:

$$\begin{array}{ll} \text{Maximize} & U(x,z) + \delta U(z,x) \\ \text{subject to} & z \in X. \end{array}$$

Given our assumptions, this problem has a unique optimal solution which we denote by $f_\delta(x)$. If $(x, f_\delta(x)) \in \operatorname{int}\Omega$, then we have the first-order condition $U_2(x, f_\delta(x)) + \delta U_1(f_\delta(x), x) = 0$. Since $U_{22}(x, f_\delta(x)) + \delta U_{11}(f_\delta(x), x) < 0$, one can apply the implicit function theorem to conclude that f_δ is differentiable at x and that

$$f'_\delta(x) = -\frac{U_{12}(x, f_\delta(x)) + \delta U_{12}(f_\delta(x), x)}{U_{22}(x, f_\delta(x)) + \delta U_{11}(f_\delta(x), x)}. \tag{6.1}$$

Let us denote the second iterate of f_δ by F_δ, that is $F_\delta = f_\delta^{(2)}$. The following so-called history independence condition plays a crucial role in Mitra and Nishimura (2001).

A.7: If a, b, and c are fixed points of (X, F_δ) satisfying $a < b < c$, then it follows that $F'_\delta(b) > 1$.

From (6.1) one can easily see that $f'_\delta(x) < 0$ holds whenever $(x, f_\delta(x)) \in \operatorname{int} \Omega$. The absolute value of $f'_\delta(\hat{x}_\delta)$ contains information about the local stability or instability of the fixed point $\hat{x}_\delta$ with respect to the dynamical system (X, h_δ). Assumption A.7 (history independence) allows one to link the local behavior of (X, h_δ) around $\hat{x}_\delta$ to global properties. The details are summarized in the following theorem.

Theorem 6.3.3. *Let (Ω, U, δ) be an optimal growth model satisfying A.1$^+$ and A.2-A.7.*
(i) If $-1 < f'_\delta(\hat{x}_\delta) < 0$, then $\hat{x}_\delta$ is a locally stable fixed point of (X, h_δ). For all $x \in (0,1)$ it holds that the optimal program from x converges to $\hat{x}_\delta$.
(ii) If $f'_\delta(\hat{x}_\delta) < -1$, then $\hat{x}_\delta$ is an unstable fixed point of (X, h_δ). There exists a period-2 cycle of (X, h_δ) with orbit $\{x^*_\delta, z^*_\delta\}$ such that the following is true: for all $x \in (0, \hat{x}_\delta) \cup (\hat{x}_\delta, 1)$ it holds that the optimal program from x converges to this period-2 cycle, that is, $\omega_+(x) = \{x^*_\delta, z^*_\delta\}$.

The above theorem implies that, under the stated assumptions, the set

$$\{(X, h_\delta) \mid \delta \in (0,1), f'_\delta(\hat{x}_\delta) \neq -1\}$$

is a family of history independent dynamical systems. It makes therefore sense to consider the bifurcation diagram of this family with δ as the bifurcation parameter. In order to construct this diagram, one needs to find out for which δ it holds that $-1 < f'_\delta(\hat{x}_\delta) < 0$ and for which δ it holds that $f'_\delta(\hat{x}_\delta) < -1$. To this end, first note that assumption A.4 implies that $\max\{|U_{11}(\hat{x}_1, \hat{x}_1)|, |U_{22}(\hat{x}_1, \hat{x}_1)|\} \geq |U_{12}(\hat{x}_1, \hat{x}_1)|$, where as before $\hat{x}_1 = \lim_{\delta \to 1} \hat{x}_\delta$. Consider the following slight strengthening of this condition.

A.8: It holds that $\max\{|U_{11}(\hat{x}_1, \hat{x}_1)|, |U_{22}(\hat{x}_1, \hat{x}_1)|\} > |U_{12}(\hat{x}_1, \hat{x}_1)|$.

Under A.4, A.5, and A.8 one has $U_{11}(\hat{x}_1, \hat{x}_1) + U_{22}(\hat{x}_1, \hat{x}_1) < 2U_{12}(\hat{x}_1, \hat{x}_1)$. Since $\hat{x}_\delta$ is continuous with respect to δ, it must therefore hold for all δ sufficiently close to 1 that

$$U_{11}(\hat{x}_\delta, \hat{x}_\delta) + U_{22}(\hat{x}_\delta, \hat{x}_\delta) < (1+\delta)U_{12}(\hat{x}_\delta, \hat{x}_\delta). \tag{6.2}$$

Because of (6.1) this implies $-1 < f'_\delta(\hat{x}_\delta) < 0$. Thus, according to theorem 6.3.3, for δ sufficiently large, the unique fixed point $\hat{x}_\delta$ is almost globally stable. If the inequality in (6.2) is reversed, however, the fixed point loses its stability and a period-2 cycle becomes almost globally stable. A sufficient condition for there to be exactly one switch from stability of the fixed point to its instability as δ decreases from 1 to 0 is the following so-called unique switching condition from Mitra and Nishimura (2001).

A.9: The function $R : (0,1) \mapsto (0,\infty)$ defined by

$$R(x) = \frac{U_2(x,x)U_{11}(x,x) - U_1(x,x)U_{22}(x,x)}{[U_2(x,x) - U_1(x,x)]U_{12}(x,x)}$$

is strictly increasing and satisfies $0 < \lim_{x\to 0} R(x) < 1$.

Indeed Mitra and Nishimura (2001) prove the following result.

Theorem 6.3.4. *For each $\delta \in (0,1)$ let (Ω, U, δ) be an optimal growth model satisfying A.1$^+$, A.2, and A.4-A.7. Furthermore, assume that A.8 and A.9 are satisfied. Then there exists a unique critical discount factor $\hat{\delta} \in (0,1)$ satisfying $R(\hat{x}_{\hat{\delta}}) = 1$. The following properties are true:*
(i) If $\hat{\delta} < \delta < 1$, then the unique fixed point $\hat{x}_\delta$ is almost globally stable.
(ii) If $0 < \delta < \hat{\delta}$, then there exists a period-2 cycle which is almost globally stable.

6.3.3 Variations on Weitzman's Example

In this section we discuss the case in which the utility function U is given by

$$U(x,z) = x^\alpha(1-z)^\beta,$$

where α and β are positive parameters satisfying $\alpha + \beta \leq 1$. Using the Euler equation, it is easy to verify that the unique fixed point of (X, h_δ) is given by $\hat{x}_\delta = \alpha\delta/(\alpha\delta + \beta)$. Furthermore, the function R from assumption A.9 is given by

$$R(x) = [1 - \alpha + (\alpha - \beta)x]/[\alpha - (\alpha - \beta)x]. \tag{6.3}$$

The special case in which $\alpha = \beta = 1/2$ is Weitzman's example (as reported in Samuelson (1973)). For this special case it is known that, for every $\delta \in (0,1)$, the optimal policy function is given by

$$h_\delta(x) = \delta^2(1-x)/[x + \delta^2(1-x)],$$

and that every $x \neq \hat{x}_\delta$ is a periodic point of period 2 of the dynamical system (X, h_δ). This implies in particular that the limit sets $\omega_+(x)$ and $\omega_+(z)$ of any two points $x \in X$ and $z \neq h_\delta(x)$ are different from each other. The dynamical system (X, h_δ) is therefore history dependent. It is also worth pointing out that, in Weitzman's example, the function R from (6.3) is constant and equal to 1, which shows that A.9 fails to be satisfied and underlines the degenerate nature of this example. Slight modifications of Weitzman's example, however, lead to history independent optimal policy functions and can be dealt with by the methods from the previous subsection.

First of all, it is easy to see that the function R from (6.3) satisfies $0 < \lim_{x \to 0} R(x) < 1$, if and only if $\alpha > 1/2$. Because of $\alpha + \beta \leq 1$, this implies that $\alpha > \beta$ and it follows that R is strictly increasing. Thus, in order for assumption A.9 to be satisfied, it is necessary and sufficient to have $\alpha > 1/2$. It can be shown that assumptions A.1$^+$ and A.2-A.8 hold also under this assumption. The function R attains the critical value 1 at the value $\hat{x}_{\hat{\delta}} = (2\alpha - 1)/[2(\alpha - \beta)]$. The corresponding critical value of the discount factor is $\hat{\delta} = (2\alpha - 1)\beta/[\alpha(1 - 2\beta)]$. Let us now distinguish between the two cases $\alpha + \beta = 1$ and $\alpha + \beta < 1$.

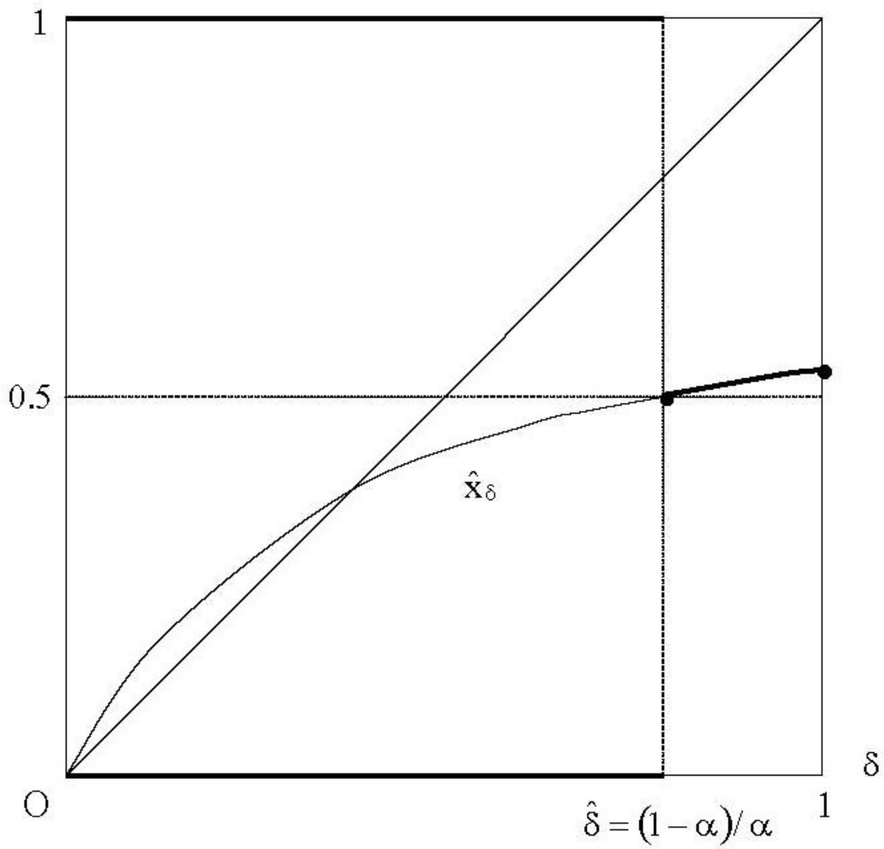

Fig. 6.1. The modified Weitzman example with $1/2 < \alpha = 1 - \beta < 1$.

If $\alpha = 1 - \beta > 1/2$, then we have $\hat{\delta} = (1 - \alpha)/\alpha$ and $\hat{x}_{\hat{\delta}} = 1/2$. For $\delta \in (\hat{\delta}, 1)$, the fixed point $\hat{x}_\delta$ is globally stable. For $\delta \in (0, \hat{\delta})$, optimal programs from all initial stocks other than $\hat{x}_\delta$ converge to the period-2 boundary cycle with orbit

$\{0,1\}$. At the bifurcation point $\delta = \hat{\delta} = (1-\alpha)/\alpha$ one has neutral cycles, that is, starting from any initial stock $x \in X$, the period-2 cycle with orbit $\{x, 1-x\}$ is optimal; see figure 6.1.

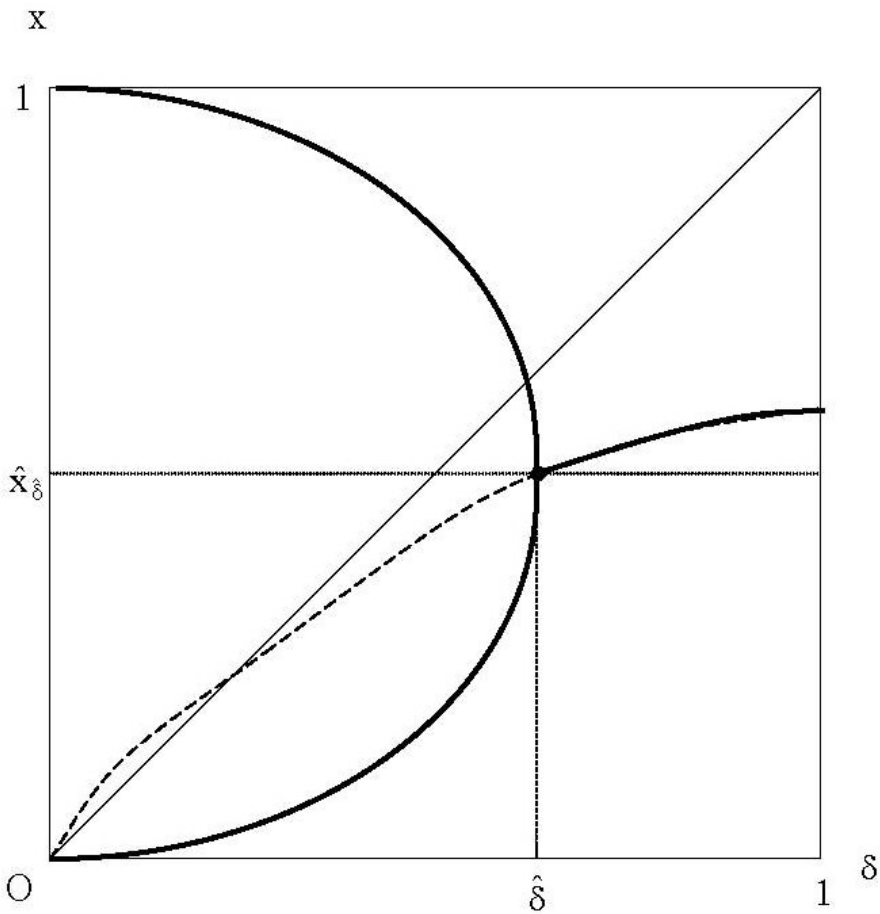

Fig. 6.2. The modified Weitzman example with $1/2 < \alpha < 1 - \beta < 1$.

Now consider the case where $\alpha + \beta < 1$. For $\delta \in (\hat{\delta}, 1)$, the fixed point $\hat{x}_\delta$ is globally stable. For $\delta \in (0, \hat{\delta})$, optimal programs from all initial states other than $\hat{x}_\delta$ converge to a unique period-2 interior cycle. This interior cycle has small amplitude for δ close to $\hat{\delta}$. The amplitude increases as δ falls and, as δ

converges to zero, the orbit of this period-2 cycle approaches $\{0, 1\}$. Thus, we obtain the standard bifurcation diagram of a flip bifurcation; see figure 6.2.

6.3.4 Variations on Sutherland's Example

We now consider the case in which the utility function U is given by

$$U(x, z) = -ax^2 - bxz - cz^2 + dx.$$

It is assumed that a, b, c, and d are positive real numbers satisfying $4ac > b^2$, $b > 2c$, and $2(a + b + c) > d > 2b(a - c)/(b - 2c)$. Note that these parameter restrictions imply that U is strictly concave and that $a > c$, $d > 2a > b$, and $a + c > b$. From the Euler equation one can see that the unique fixed point of (X, h_δ) is given by $\hat{x}_\delta = d\delta/[b + 2c + \delta(2a + b)]$ and that $\hat{x}_\delta \in (0, 1)$. The function R from assumption A.9 is given by

$$R(x) = [2cd + 2b(a - c)x]/[bd - 2b(a - c)x].$$

Given the above mentioned parameter restrictions, assumptions A.1$^+$ and A.2-A.9 are satisfied. The function R attains the critical value 1 at the value $\hat{x}_{\hat{\delta}} = d(b - 2c)/[4b(a - c)]$. The corresponding critical value of the discount factor is $\hat{\delta} = (b-2c)/(2a-b)$. Defining $\delta_1 = 2c/(d-2a)$ and $\delta_2 = b/(d-b)$ it holds that $0 < \delta_1 < \delta_2 < \hat{\delta} < 1$. Mitra and Nishimura (2001) show that the bifurcation diagram of the family (X, h_δ) is given by figure 6.3.

For $\delta \in (\hat{\delta}, 1)$, the fixed point $\hat{x}_\delta$ is globally stable and, consequently, there do not exist any periodic points of period $p \geq 2$. For $\delta \in (\delta_2, \hat{\delta})$, the fixed point $\hat{x}_\delta$ is unstable and there exists $x^*_\delta \in (0, \hat{x}_\delta)$ such that $\{x^*_\delta, 1\}$ is the orbit of a period-2 cycle. All optimal programs starting from $x \neq \hat{x}_\delta$ converge to this period-2 cycle. It holds that $\lim_{\delta \to \delta_2} x^*_\delta = 0$. For $\delta \in (\delta_1, \delta_2)$, the fixed point $\hat{x}_\delta$ is unstable and all optimal programs starting from $x \neq \hat{x}_\delta$ converge to a period-2 cycle with orbit $\{0, 1\}$. Finally, for $\delta \in (0, \delta_1)$, the fixed point $\hat{x}_\delta$ is unstable and there exists $z^*_\delta \in (\hat{x}_\delta, 1)$ such that $\{0, z^*_\delta\}$ is the orbit of a period-2 cycle. All optimal programs starting from $x \neq \hat{x}_\delta$ converge to this period-2 cycle. It holds that $\lim_{\delta \to 0} z^*_\delta = 0$ and $\lim_{\delta \to \delta_1} z^*_\delta = 1$.

6.4 Optimal Chaos

In the early 1980s, the economics profession became aware of the fact that simple economic mechanisms may generate chaotic dynamics; see, e.g., Benhabib and Day (1982) and Day (1982, 1983). It did not take long until it was shown that the occurrence of deterministic chaos does not necessarily rely on market imperfections or on non-standard assumptions. As a matter of fact, the papers by Deneckere and Pelikan (1986) and Boldrin and Montrucchio (1986) demonstrated that chaotic behavior can be optimal in the reduced-form optimal growth model discussed in subsection 6.2.2 above. These results confirmed

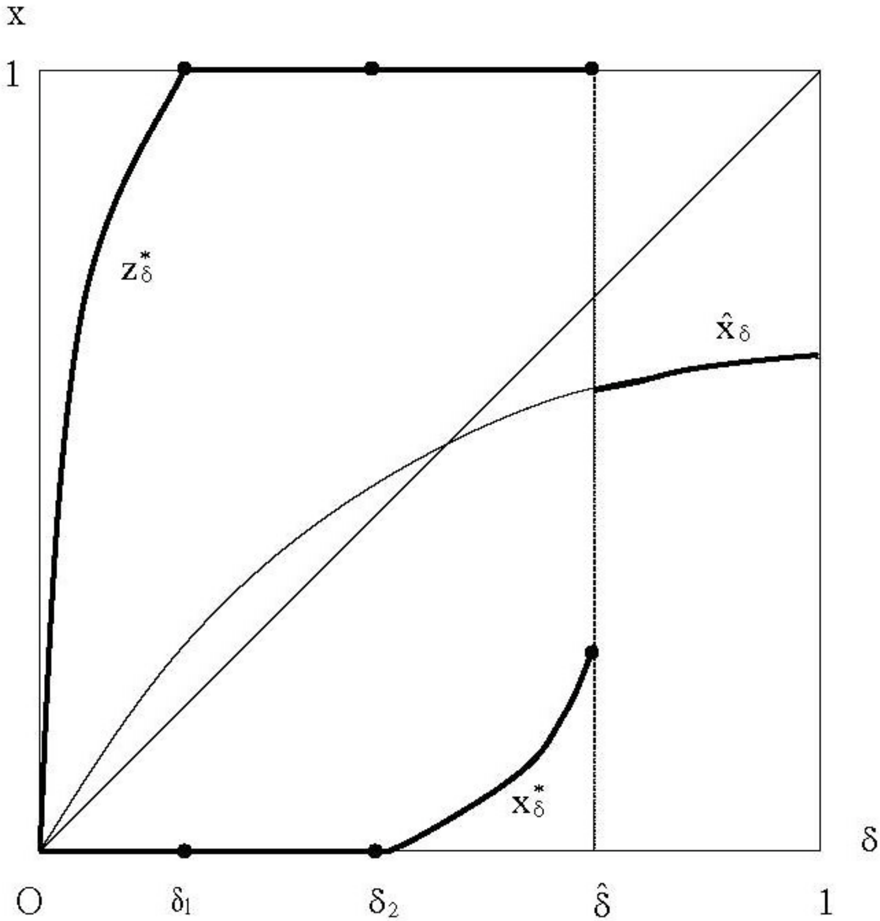

Fig. 6.3. Sutherland's example.

that the standard assumptions of optimal growth theory are logically consistent with endogenously generated business cycles. This insight was very important but it raised also a number of new questions, especially regarding the parameter values for which erratic non-periodic behavior can be optimal.

The only explicit parameter in the reduced form optimal growth model is the discount factor. The research surveyed in chapter 4 of this handbook refined the approach initiated by Boldrin and Montrucchio (1986) and developed discount factor restrictions implied by optimal chaos. Parallel to this development, a number of researchers studied optimal growth models under more detailed structural assumptions on the preferences and the technologies and derived characterizations of the parameter constellations which are consistent with chaos in that framework. By far the most popular framework considered in this literature is the two-sector optimal growth model discussed in subsection 6.2.2 above, but the variations of the Weitzman example already encountered in our discussion of optimal cycles in section 6.3 have also been studied. In the present section we survey the most important contributions to this literature.

6.4.1 Sources of Optimal Chaos

Consider a reduced-form optimal growth model with state space $X = [0, \bar{x}]$ and assume that the transition possibility set takes the form $\Omega = \{(x, z) \,|\, x \in X,\ 0 \le z \le \phi(x)\}$, where $\phi : X \mapsto X$ is a continuous, non-decreasing, and concave function satisfying $\phi(0) = 0$, $\phi(\bar{x}) = \bar{x}$, and $\phi(x) > x$ for all $x \in (0, \bar{x})$. Furthermore, assume that ϕ is continuously differentiable on $(0, \bar{x})$ with derivative ϕ'. It is easy to see that such a transition possibility set can result from a two-sector model with full capital depreciation ($d = 1$) and a production function for the capital good which satisfies $F_x(x, 1) = \phi(x)$ for all $x \in X$. It is also clear that assumption A.1 holds. Furthermore, suppose that assumptions A.2 and A.4 are satisfied. If $U_{12}(x, z)$ is strictly positive for all (x, z) in the interior of Ω, then it follows from corollary 6.3.1(i) that the graph of the optimal policy function h is non-decreasing on X. From the results stated in subsection 6.2.1 we know that all optimal programs must converge to fixed points of (X, h). Thus, chaotic optimal programs are ruled out. A necessary condition for the occurrence of chaotic dynamics in the present situation is therefore that $U_{12}(x, z)$ is negative for some (x, z) in the interior of Ω.

If $U_{12}(x, z) < 0$ holds for all (x, z) in the interior of Ω, then it follows from theorem 6.3.2 that the graph of h is non-increasing whenever it is in the interior of Ω. Nishimura and Yano (1994) elaborate on this observation and describe a method by which one can construct optimal growth models that display topological chaos. The main idea is as follows. First of all, it is assumed that there exists an optimal steady state in the interior of the state space X. Because of the Euler equation, this is tantamount to assuming a solution of the equation $U_2(\hat{x}, \hat{x}) + \delta U_1(\hat{x}, \hat{x}) = 0$ satisfying $0 < \hat{x} < \bar{x}$. Together with the assumption $\phi(x) > x$ for all $x \in (0, \bar{x})$ this implies that the point $(\hat{x}, \hat{x})$ is located on the graph of the optimal policy function and in the interior of Ω. Because the graph of h must be a non-increasing curve whenever it is in the interior of Ω, it follows that there must be $\tilde{x} < \hat{x}$ such that (i) $h(x) = \phi(x)$ for all $x \in [0, \tilde{x}]$ (hence, h is non-decreasing on $[0, \tilde{x}]$), (ii) $h(x) < \phi(x)$ for all $x \in (\tilde{x}, \bar{x}]$, and (iii) h is non-increasing on $[\tilde{x}, \bar{x}]$. The optimal policy function has therefore a tent shape. In a second step one has to make sure that the tent is steep enough in order to generate chaotic dynamics. A sufficient condition for this to be the case is the existence of a period-3 cycle; see Li and Yorke (1975). Nishimura and Yano (1994) construct the period-3 cycle in such a way that two elements of its orbit are in the interval $(0, \tilde{x})$ (that is, they correspond to points on the upper boundary of Ω) while the remaining element is in the interval $(\tilde{x}, \bar{x})$ corresponding to an interior point. Figure 6.4 illustrates the general idea of this construction. The precise conditions under which it is possible are stated in the following theorem in which

$$\begin{aligned}
\Gamma(x,y,z) &= U_2(x,y) + \delta U_1(y,z), \\
\Gamma_{(1)}(x) &= \Gamma(\phi(x), \phi^{(2)}(x), x), \\
\Gamma_{(2)}(x) &= \Gamma(x, \phi(x), \phi^{(2)}(x)) + \delta \Gamma_{(1)}(x)\phi'(\phi(x)), \\
\Gamma_{(3)}(x) &= \Gamma(\phi^{(2)}(x), x, \phi(x)) + \delta \Gamma_{(2)}(x)\phi'(x).
\end{aligned}$$

Theorem 6.4.1. *Consider the optimal growth problem* (Ω, U, δ) *on* $X = [0, \bar{x}]$, *where the transition possibility set is* $\Omega = \{(x,z) \,|\, x \in X,\ 0 \le z \le \phi(x)\}$ *and* ϕ *is continuous, non-decreasing, and concave on* $[0, \bar{x}]$ *and continuously differentiable on* $(0, \bar{x})$. *Suppose that* $\phi(0) = 0$, $\phi(\bar{x}) = \bar{x}$, *and* $\phi(x) > x$ *for all* $x \in (0, \bar{x})$. *Let assumptions A.2-A.4 be satisfied and let* h *be the optimal policy function. If there exists* $x \in (0, \bar{x})$ *and* $x' \in (0, \bar{x})$ *such that* $\Gamma_{(1)}(x) \ge 0$, $\Gamma_{(2)}(x) \ge 0$, $\Gamma_{(3)}(x) \le 0$, *and* $\Gamma_{(3)}(x') > 0$, *then it follows that the dynamical system* (X, h) *exhibits topological chaos.*

Nishimura and Yano (1994) show that the conditions of theorem 6.4.1 can be satisfied in an example, in which the state space is $X = [0, 1]$ and the reduced form utility function is given as in the generalized Weitzman example, that is, $U(x,z) = x^{\alpha}(1-z)^{\beta}$. As we have seen in subsection 6.3.3 above, this utility function allows for period-2 cycles but it does not allow for more complicated dynamics, if $\Omega = X \times X$. For this reason, Nishimura and Yano (1994) have to choose a non-trivial transition possibility set, that is, they have to specify the function ϕ in such a way that $\phi(x) < 1$ holds for all sufficiently small x.

Theorem 6.4.1 traces the occurrence of optimal chaos to the tent shaped optimal policy function. The tent shape arises because the transition possibility set Ω has a non-trivial and strictly increasing upper boundary and because the optimal policy function is steeply decreasing on the interior of Ω. None of these two properties alone is sufficient to generate optimal chaos, but their combination is. The non-trivial upper boundary of Ω (i.e., the fact that $\phi(x) < 1$ holds for all sufficiently small x) corresponds to the assumption that the economic system cannot move instantaneously from very small states to very large states. This, in turn, can be interpreted as a form of 'upward inertia' of the economic system. The steeply decreasing shape of the optimal policy function in the interior of Ω has two sources: submodularity of U and strong discounting. Submodularity of U means that the maximizer of $U(x,z)$ with respect to z is a decreasing function of x. In other words, if the degree of submodularity is sufficiently strong and the decision maker were myopic ($\delta = 0$), he or she would want to permanently oscillate between small and large states. As the discount factor increases, however, this incentive is increasingly dominated by the decision maker's desire to smooth consumption which follows from the

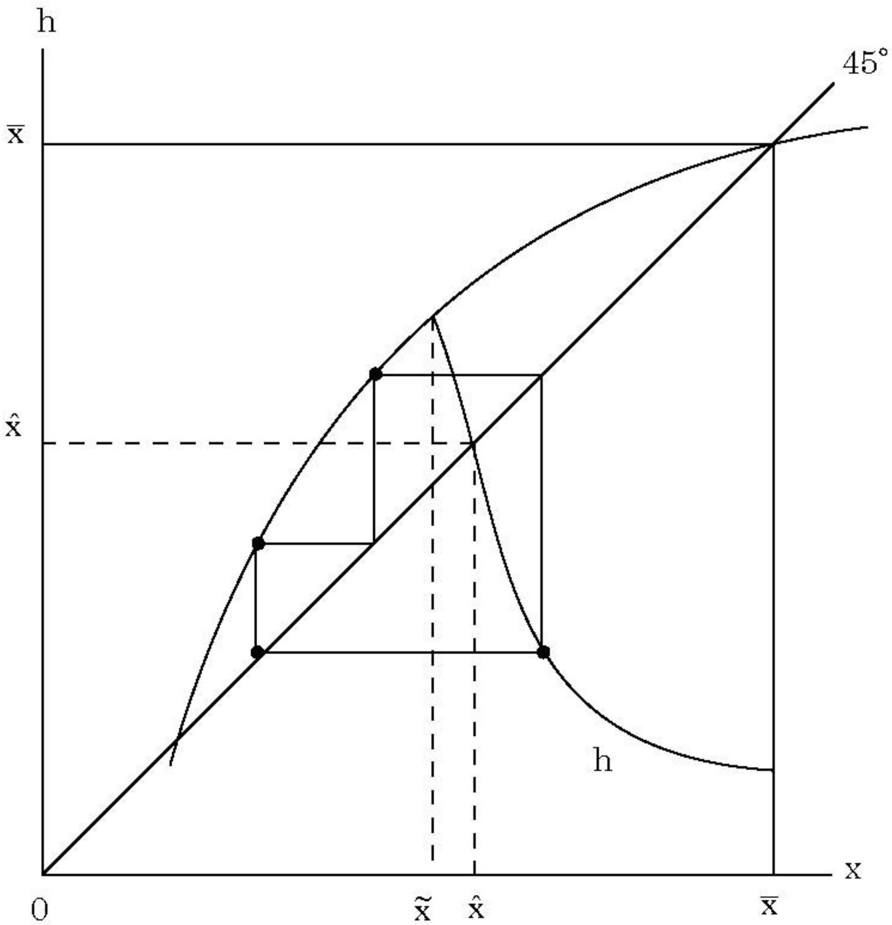

Fig. 6.4. The construction from Nishimura and Yano (1994).

concavity of the utility function. To summarize, Nishimura and Yano (1994) have identified three sources of optimal chaos: upward inertia, submodularity, and strong discounting.

Nishimura and Yano (1995a) replace the assumption of upward inertia of the economic system by 'downward inertia' and show that this can also lead to chaotic optimal programs. They create downward inertia by partial capital depreciation (note that the approach taken by Nishimura and Yano (1994) can be interpreted in the context of the two-sector model provided that capital depreciates fully, i.e., $d = 1$). In the case of partial depreciation the economy cannot move instantaneously from very large states to very small states. Formally, this follows from the fact that the transition possibility set is given by $\Omega = \{(x, z) \,|\, x \in X\,,\ (1-d)x \leq z \leq \phi(x)\}$. Except for the assumption $d < 1$, the approach taken in Nishimura and Yano (1995a) is the same as in Nishimura and Yano (1994). As before it is assumed that the utility function satisfies A.2 and A.4 as well as $U_{12}(x, z) < 0$ for all (x, z) in the interior of Ω. However, in the present case with less than full depreciation, it follows that there must exist two states $\tilde{x}$ and $\tilde{x}'$ in $(0, \bar{x})$ such that the graph of the optimal policy function h coincides with the upper boundary of Ω for $x \in [0, \tilde{x}]$ and with its lower boundary for $x \in [\tilde{x}', \bar{x}]$. In between the two values $\tilde{x}$ and $\tilde{x}'$, the graph of h is in the interior of Ω and is strictly decreasing. The optimal policy function is therefore not tent-shaped but has an interior maximum $\tilde{x}$ and an interior minimum $\tilde{x}'$. The authors then go on and make the graph of h on the interval $(\tilde{x}, \tilde{x}')$ sufficiently steep such that a period-3 cycle exists, which touches the lower boundary of Ω twice whereas the third element of the orbit of the cycle corresponds to a point on the interior section of the optimal policy function. The construction is illustrated in figure 6.5. Conditions very similar to those stated in theorem 6.4.1 are shown to be sufficient for the construction to lead to the desired result. As before, the conditions are shown to be satisfied by a model in which the utility function is given as in the generalized Weitzman example, i.e., $U(x, z) = x^\alpha (1 - z)^\beta$. This time, the upper boundary of Ω can be chosen as $\phi(x) = 1$, because the possibility of chaotic dynamics relies on the assumption of downward inertia (partial depreciation $\psi(x) > 0$) rather than on upward inertia ($\phi(x) < 1$).

The paper by Khan and Mitra (2005) also points to downward inertia created by partial depreciation of capital as a possible source of optimal chaos. Kahn and Mitra (2005) consider a discrete-time version of the Robinson-Solow-Srinivasan model with two production sectors. In the notation introduced in subsection 6.2.2, the model is specified by the discount factor $\delta \in (0, 1)$, the capital depreciation rate $d \in (0, 1)$, and by the functions

$$
\begin{aligned}
F_c(x_c, \ell_c) &= \min\{x_c, \ell_c\}, \\
F_x(x_x, \ell_x) &= \ell_x / \mu, \\
u(c) &= c.
\end{aligned}
$$

In other words, the utility function is linear, the production of one unit of the consumption good requires one unit of capital and one unit of labor, and the production of one unit of capital requires μ units of labor (and no capital). The maximal sustainable capital stock is given by $\bar{x} = 1/(d\mu)$ and the transition possibility set is given by $\Omega = \{(x, z) \,|\, 0 \leq x \leq 1/(d\mu), (1 - d)x \leq z \leq \phi(x)\}$, where $\phi(x) = (1-d)x + (1/\mu)$. Khan and Mitra (2005) first show that, whenever $\delta < \mu$, all optimal programs are described by a continuous optimal policy function h. If in addition to $\delta < \mu$, the parameters μ and d are related in a certain way, then the dynamical system (X, h) is shown to exhibit topological chaos. Finally, Khan and Mitra (2005) prove that, for any value $d \in (0, 1)$, there is some μ such that the aforementioned relation between d and μ is indeed satisfied. That is, whenever the rate of depreciation is positive and smaller than

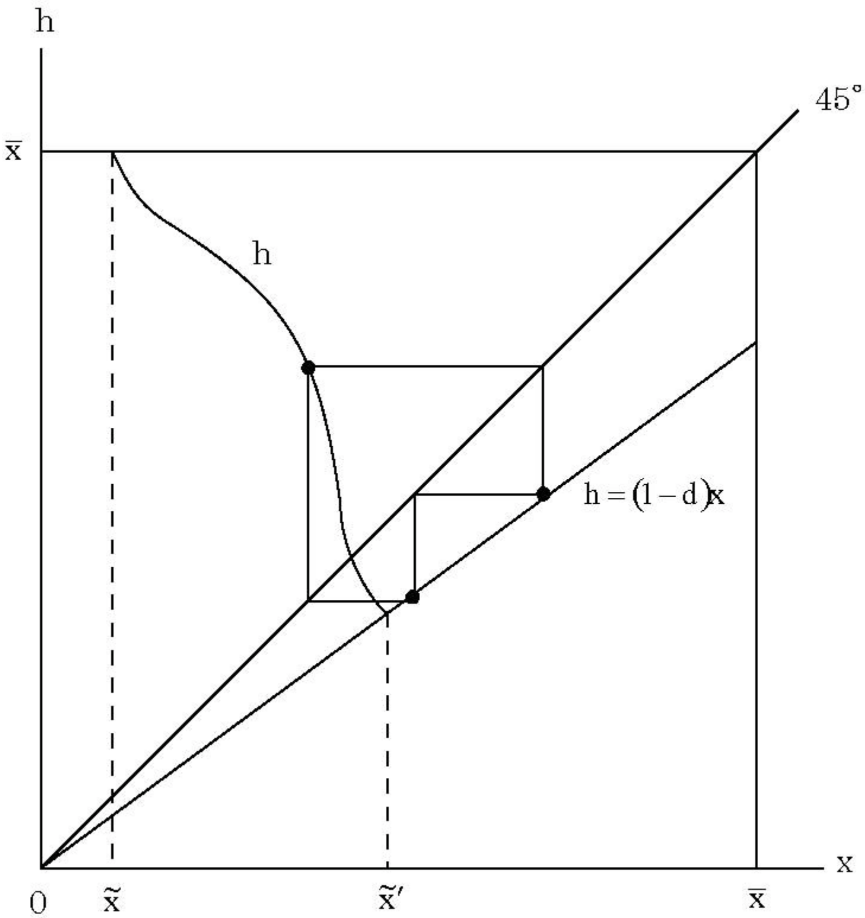

Fig. 6.5. The construction from Nishimura and Yano (1995a).

1, one can find parameters δ and μ such that the model generates topological chaos.

Boldrin and Deneckere (1990) use a combination of analytical and numerical methods to derive interesting insights into the sources of optimal chaos. Their model is a two-sector model with a Cobb-Douglas production function for the consumption good, a Leontief production function for the capital good, and a linear utility function. In the notation of subsection 6.2.2 these assumptions can be written as follows:

$$F_c(x_c, \ell_c) = x_c^\alpha \ell_c^{1-\alpha},$$
$$F_x(x_x, \ell_x) = \mu \min\{x_x, \ell_x/\mu\},$$
$$u(c) = c.$$

This specification gives rise to the transition possibility set $\Omega = \{(x, z) \,|\, 0 \leq x \leq 1, (1-d)x \leq z \leq \phi(x)\}$, where $\phi(x) = \min\{\mu x, 1\}$. The reduced form utility function is

$$U(x, z) = \mu^{-\alpha}[1 + (1-d)x - z]^{1-\alpha}[(1-d+\mu)x - z]^\alpha.$$

As for the parameter values, it is assumed that $\alpha \in (0, 1)$ and $\mu > 1/\delta$. The inequality $\mu > 1/\delta$ implies that the marginal product of capital in the investment good sector covers principal and interest in the steady state (recall that the real interest rate in a steady state is $1/\delta - 1$). This assumption guarantees the existence of an interior steady state. The efficient capital-labor ratio in the investment good sector is fixed at $1/\mu$. The factor substitutability in the consumption good sector implies that both factors will be fully employed. Thus, if the economy-wide capital-labor ratio x exceeds $1/\mu$, then it follows that the consumption good sector must be more capital intensive than the investment good sector. Conversely, if $x < 1/\mu$, then consumption goods are produced with lower capital intensity than investment goods. Thus, this model allows for capital intensity reversal.

Boldrin and Deneckere (1990) derive conditions for the existence of cycles of period 2 and 4. Fixing the values of α, μ, and d and treating δ as a bifurcation parameter, they show by means of numerical simulations that successive bifurcations can lead to topological chaos (period-doubling scenario). For example, when $\alpha = 97/100$, $\mu = 100/9$, and $d = 1$, topological chaos is encountered for discount factors between 0.099 and 0.112. They also show that chaos typically disappears rapidly, if one reduces the depreciation rate d from 100% to smaller values, but reappears, if d takes values of 10% or smaller. This suggests that, for almost full depreciation, chaos is generated by the same mechanism as in Nishimura and Yano (1994) (upward inertia and short-run incentives for oscillations), whereas for small values of d it is generated by the mechanism described by Nishimura and Yano (1995a) (downward inertia and short-run incentives for oscillations).

6.4.2 Optimal Chaos Under Weak Impatience

The parametric examples with chaotic optimal policy functions discussed in the papers by Nishimura and Yano (1994, 1995a) and Boldrin and Deneckere (1990) involve the discount factors $\delta = 0.01$, $\delta = 0.05$, and $\delta \approx 0.1$, respectively. These are unrealistically small numbers if the chaotic fluctuations are interpreted as business cycles. The study by Nishimura et al. (1994) shows that chaotic dynamics can occur for all values of the discount factor. In addition, Nishimura et al. (1994) construct optimal policy functions which are not only chaotic in the sense of topological chaos but also in the sense of ergodic chaos and geometric sensitivity. This is accomplished by proving that, for every $\delta \in (0,1)$ and every γ satisfying $1 < \gamma < \min\{2, \delta^{-2}\}$, there exists an optimal growth model (Ω, U, δ) on the state space $X = [0,1]$ with the optimal policy function

$$h(x) = \begin{cases} \gamma x & \text{if } x \in [0, 1/\gamma], \\ 2 - \gamma x & \text{if } x \in [1/\gamma, 1]. \end{cases}$$

Since $\gamma > 1$, it follows from the results by Lasota and Yorke (1973) and Li and Yorke (1978) mentioned in subsection 6.2.1 that this optimal policy function exhibits ergodic chaos and geometric sensitivity. The transition possibility set is chosen to be $\Omega = \{(x,z) \,|\, 0 \leq x \leq 1, 0 \leq z \leq \phi(x)\}$ with $\phi(x) = \min\{\gamma x, 1\}$. This shows that the increasing part of the optimal policy function coincides with the boundary of Ω. The decreasing part, however, lies in the interior of Ω. The specification of the utility function U, which is the key step in the construction, is based on ideas developed in Sorger (1992). It leads to an optimal value function which is a simple quadratic polynomial. Thus, Nishimura et al. (1994) have proved the following result.

Theorem 6.4.2. *For every $\delta \in (0,1)$ there exists a reduced form optimal growth model which satisfies assumptions A.1 and A.2, has a strictly concave utility function, and has an optimal policy function exhibiting ergodic chaos and geometric sensitivity.*

Nishimura et al. (1994) also demonstrate that the reduced form optimal growth models in theorem 6.4.2 can be thought of as arising from two-sector models in which both production functions are of the Leontief type and in which the utility function reflects a wealth effect.

Nishimura and Yano (1995b) consider the two-sector growth model with Leontief production functions in both sectors and a linear utility function without any wealth effect. They also demonstrate that ergodic chaos and geometric sensitivity can occur for arbitrary small values of the discount rate. The linear structure imposed by the utility function and the production technologies makes it possible to consider the model as a dynamic linear programming problem; see Nishimura and Yano (1996).

In the notation of subsection 6.2.2, Nishimura and Yano (1995b) assume $d = 1$ and

$$\begin{aligned} F_c(x_c, \ell_c) &= \min\{x_c, \ell_c/\alpha\}, \\ F_x(x_x, \ell_x) &= \mu \min\{x_x, \ell_x/\beta\}, \\ u(c) &= c. \end{aligned}$$

This specification implies that the maximal capital stock that can be reached from x within a single period is given by $\phi(x) = \mu \min\{x, 1/\beta\}$. Therefore, the maximal sustainable capital stock is μ/β and it suffices to restrict attention to the state space $X = [0, \mu/\beta]$. The transition possibility set is given by

$\Omega = \{(x, z) \,|\, 0 \le x \le \mu/\beta,\, 0 \le z \le \phi(x)\}$. This two-sector model is fully determined by the technological parameters α, β, and μ and the discount factor δ. We shall refer to the model by $M(\alpha, \beta, \mu, \delta)$. The main result from Nishimura and Yano (1995b) can now be stated as follows.

Theorem 6.4.3. *For every $\delta' \in (0, 1)$ there exist parameters α, β, and μ as as well as an open interval $I \subseteq (\delta', 1)$ such that for all $\delta \in I$, the two-sector model $M(\alpha, \beta, \mu, \delta)$ has the optimal policy function h specified by*

$$h(x) = \begin{cases} \mu x & \text{if } x \le 1/\beta, \\ \mu(1 - \alpha x)/(\beta - \alpha) & \text{if } x \ge 1/\beta, \end{cases} \tag{6.4}$$

and the dynamical system (X, h) exhibits ergodic chaos and geometric sensitivity.

The proof of this result is very technical and will not be presented here. Instead, we will restrict ourselves to a discussion of the main steps of the proof. To begin with, Nishimura and Yano (1995b) assume that the following parameter restrictions are satisfied:

$$\begin{aligned} &\mu > 1/\delta, \\ &\beta > \alpha > 0, \\ &(\beta/\alpha) - 1 < \mu < \beta/\alpha. \end{aligned} \tag{6.5}$$

The condition $\mu > 1/\delta$ ensures the existence of an interior steady state. The inequality $\beta > \alpha > 0$ says that the consumption good sector is more capital intensive than the investment good sector. To explain the meaning of the last parameter restriction in (6.5), it is useful to compute the reduced form utility function

$$U(x, z) = \max\{F_c(x_c, \ell_c) \,|\, x_c + x_x \le x, \ell_c + \ell_x \le 1, F_x(x_x, \ell_x) \ge z\}.$$

In the optimal solution to this program, the constraint $\ell_c + \ell_x \le 1$ is not binding, if $z < f(x)$, and the constraint $x_c + x_x \le x$ is not binding, if $z > f(x)$. Here the function $f : X \mapsto \mathbb{R}$ is defined by $f(x) = \mu(1 - \alpha x)/(\beta - \alpha)$. Because of the assumption $\beta > \alpha > 0$, it follows that f is a strictly decreasing and continuous function. Consequently, its inverse f^{-1} exists and the condition $z < f(x)$ can also be written as $x < f^{-1}(z)$. In other words, in the optimal solution, labor is not fully employed, if the available capital stock is so small that capital input forms a bottleneck. If the available capital stock exceeds $f^{-1}(z)$, on the other hand, then capital is not fully employed. Full employment of both input factors occurs, if and only if $z = f(x)$.

Note that $f(1/\beta) = \phi(1/\beta) = \mu/\beta$. This shows that the graph of f intersects the upper boundary of the transition possibility set in its kink at $x = 1/\beta$. Nishimura and Yano (1995b) want to ensure that the graph of f does not intersect the lower boundary of Ω and that its slope is smaller than -1. This is exactly what the last line of (6.5) achieves.

Simple calculations show that the reduced form utility function is given by

$$U(x,z) = \begin{cases} x - z/\mu & \text{if } x \le f(z), \\ (\mu - \beta z)/(\alpha\mu) & \text{if } x \ge f(z). \end{cases}$$

It is easy to see from this expression that the indifference curves of U are obtained by translating the curve $z = \phi(x)$ parallel in the direction of the line $z = f(x)$. Nishimura and Yano (1995b) argue that, as long as $x < 1/\beta$, it is not possible to employ the entire labor supply because $z < f(x)$ holds for all $(x,z) \in \Omega$ satisfying $x < 1/\beta$ (capital is the bottleneck). In this case it will therefore be optimal to produce as much capital as possible in order to clear the bottleneck. Once the capital stock x has become larger than $1/\beta$, it is possible to fully employ both factors. The optimal activity in this case will be characterized by full employment of both factors, that is, it will be described by a point on the graph of f. This suggests that the optimal policy function is given by (6.4). From the first and the third line in (6.5) it follows that h is a piecewise linear map with a slope (wherever it is defined) that is larger than 1 in absolute value. As we have seen in subsection 6.2.1, these properties ensure that the dynamical system (X,h) exhibits ergodic chaos and geometric sensitivity.

The crux of the proof in Nishimura and Yano (1995b) consists in showing that h is actually the optimal policy function of $M(\alpha,\beta,\mu,\delta)$. This is a non-trivial issue because the reduced form utility function is not strictly concave in any of its arguments. Nishimura and Yano (1995b) address this problem in two steps. First, they prove that a sufficient condition for h to be the optimal policy function is that (i) the critical point of h, that is $x = 1/\beta$, is a periodic point of (X,h) with period $p > 1$ and (ii) the periodic trajectory from $x = 1/\beta$ is the unique optimal program from $1/\beta$. Second, they show that this sufficient condition can be satisfied for an open interval of discount factors arbitrarily close to 1 provided one lets the period p approach $+\infty$ in an appropriate way.

A somewhat undesirable feature of the result in theorem 6.4.3 is that the optimal policy is non-interior. As a matter of fact, the increasing section of the optimal policy function coincides with the upper boundary of the transition possibility set which implies that, whenever $x \le 1/\beta$, optimal consumption is equal to 0. Note that the same property holds also for the models constructed in the proof of theorem 6.4.2 (as well as for the models discussed in Nishimura and Yano (1994, 1995a)). One may therefore wonder whether chaos can be optimal under weak discounting also in models for which the graph of the optimal policy function is in the interior of the transition possibility set (except at the boundary of the state space X). Nishimura et al. (1998) prove that this is indeed the case. However, whereas Nishimura and Yano (1995b) were able to determine the optimal policy function analytically, in Nishimura et al. (1998) an analytical expression of the optimal policy function is not available and the proof of the existence of optimal chaos is based on a continuity argument.

They consider the model $M_\lambda(\alpha,\beta,\mu,\delta)$ defined by $d = 1$ (full depreciation) and

$$F_c(x_c, \ell_c) = \left[(1/2)x_c^{-1/\lambda} + (1/2)(\ell_c/\alpha)^{-1/\lambda}\right]^{-\lambda},$$
$$F_x(x_x, \ell_x) = \mu_\lambda \left[(1/2)x_x^{-1/\lambda} + (1/2)(\ell_x/\beta)^{-1/\lambda}\right]^{-\lambda},$$
$$u(c) = c^{1-\lambda}/(1-\lambda).$$

Here $\lambda \in (0,1)$ and $\mu_\lambda = 2^{-\lambda}(1+\mu^{1/\lambda})$. It is straightforward to see that, as λ approaches 0, the functions F_c, F_x, and u converge to the corresponding functions used to define $M(\alpha,\beta,\mu,\delta)$. It is also quite obvious that, for $\lambda > 0$, the marginal utility at $c = 0$ is infinitely large and, hence, that consumption along any optimal program starting in $x > 0$ must be strictly positive. Finally, because the production functions are concave and the utility function is strictly concave, the reduced form utility function must be strictly concave with respect to its second argument. This implies that all optimal programs of $M_\lambda(\alpha,\beta,\mu,\delta)$ are described by an optimal policy function. Let us denote this function by h_λ.

Nishimura et al. (1998) now show that, as λ approaches 0, the optimal policy function h_λ converges uniformly to the optimal policy function of $M(\alpha,\beta,\mu,\delta)$, that is, to the function h defined in (6.4). Furthermore, they appeal to a result by Butler and Pianigiani (1978), which implies that a small perturbation of h preserves the existence of topological chaos. Thus, the final conclusion derived by Nishimura et al. (1998) is the following theorem.

Theorem 6.4.4. *Consider the two-sector optimal growth model $M_\lambda(\alpha,\beta,\mu,\delta)$ defined above. For every $\delta' \in (0,1)$ there exist parameter values $(\alpha,\beta,\mu,\delta,\lambda)$ such that $\delta \in (\delta',1)$ and such that the optimal policy function of $M_\lambda(\alpha,\beta,\mu,\delta)$ exhibits topological chaos.*

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7. Intertemporal Allocation with a Non-convex Technology

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7.1 Introduction

In his famous article on increasing returns and economic progress, Young (1928) concluded with the following summary:

"In recapitulation of these variations of a theme from Adam Smith there are three points to be stressed. First, the mechanism of increasing returns is not to be discerned adequately by observing the effects of variations in the size of an individual firm or a particular industry, for the progressive division and specialization of industries is an essential part of the process by which increasing returns are related. What is required is that the industrial operations be seen as an integrated whole. Second, the securing of increasing returns depends upon...the economies which are to be had by using labor in roundabout or indirect ways. Third, the division of labor depends upon the extent of the market, but the extent of the market also depends upon the division of labor. In this circumstance lies the possibility of economic progress, apart from the progress which comes as a result of the new knowledge which men are able to gain whether in the pursuit of their economic or the non-economic interests."

Elsewhere, Young (1929) maintained the emphasis on the importance of roundabout methods, and the link between division of labor and the extent of the market but went further in describing a dynamic process with some other elements:

"The use of capital on a large scale in industry came later than its use in commerce, for the reason that not until there were markets which were able to absorb large outputs of standard types of goods was it profitable to make any extensive use of roundabout methods of production. Once established, however, industrial capitalism showed that it had within itself the seeds of its own growth. Cheaper goods, improved means of transport, and the increased advantages of specialization led to larger markets, so that the economies of industrial capitalism grew in a cumulative way. The increasing division of labor...not only invited a larger use of instruments, but also prompted the invention of new types of instrument."

Newman's excellent entry on Young in *New Palgrave* (1998, vol. 4, pp. 937-9) has the second quote, and the thoughtful remarks:

"Apart from an interesting discussion by Marx, Young's article was the first serious advance beyond Adam Smith on the relation between increasing returns and economic growth. However, the problems of formalizing that persuasive vision into a tractable model have proved formidable indeed, the chief technical problems being those of nonconvex technologies and the introduction of new intermediate commodities. So, old as it is, his paper remains important for us precisely because there is not much else."

This survey looks at "something else": it provides a selective and biased review of a relatively recent literature on intertemporal allocation theory which faced up to the "technical problems" of nonconvex technologies that Newman alluded to. As we shall see, substantial analytical progress has been made - within the confines of these models - in developing a "primal" approach to identifying the characteristics of "efficient" or "optimal" programs of allocation. The models sketched below capture an optimization problem faced by a "social planner" or a Central Planning Board of a Lange-Lerner economy. A nonconvex set of feasible plans (programs) precludes a routine application of the "classical" tools of optimization (e.g., separation theorems). One may analyze the complications at several levels. As noted by Dobb (1960), "investment in a planned economy is presumably determined as a policy-decision of the government, and not as the resultant of market forces which the government may seek to influence..but does not control directly". Dobb provided a convenient classification of three types of decisions distinguished by alternative levels of aggregation: (i) determination of the total volume of investment (the choice or trade-off between immediate and future consumption), (ii) its distribution among sectors (and among the industries) and (iii) the technical forms (projects) in which the investment is embodied.[1] Sections 7.2 through 7.6 present results that attempt to resolve the first issue.in an aggregative framework with a S-Shaped production function that made an early appearance in Frank Knight's Ph.D. thesis at Cornell.[2] In Section 7.3, I briefly touch upon the question of efficient intertemporal choice, studied first by Malinvaud (and subsequently by Phelps, Radner, Koopmans, Starrett, Majumdar, McFadden, Peleg, Mitra, Benveniste, Gale, Cass, Yaari, Kurz, Alkan, and others). Next, in Sections 7.4 through 7.6, I move on to optimality criteria involving maximization of one period "returns" or "utilities generated by consumptions" and review both the Ramsey-Weizsacker ("overtaking") approach to the "undiscounted" case and the "discounted" case. A reinterpretation of the model enables one to study problems of renewable resource management and the possible conflict between conservation and profit maximization (see Section

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[1] The documents on planning in the Soviet Union, India or Pakistan give us concrete examples of such a broad classification that practical policy makers found useful.

[2] It should be mentioned the thesis was supervised by Young.

7.6.1). Section 7.7 takes up the multi-sector model of Mitra (1992) which explicitly treats the second type of decisions in Dobb's scheme, but only deals with the overtaking criterion. The tenor and language of discourse in Sections 7.4 through 7.7 has been profoundly influenced by David Gale's work (1967) on multi-sector models [with convex technology] in which he identified three subjects "central to the literature": (1) optimal stationary programs (golden rules); (2) asymptotic properties of good programs (turnpike theorems); (3) dynamic competitive prices (duality theory). Postponing for the moment, my comments on the last subject, I would like to stress that Sections 7.4 through 7.7 develop the first two subjects along with the question of existence of an optimal program. Unfortunately, there is not much to report on the evaluation of investment projects (the third issue in Dobb's scheme) that involve decreasing marginal costs or indivisible goods.[3] In contrast with the "closed" models, Section 7.8 reviews an open model of a "small" economy and explores the pattern and gains from trade. It also indicated how the expansion of the market may determine specialization and welfare gains..

Going back to Young, the need to look at a group of industries was also a clear implication of some other prominent contributions to development economics. The enduring themes in these writings were "externalities", "complementarities" and "intersectoral linkages", and the need for a proper "coordination" of activities to initiate and sustain a development program [for useful assessments see Ray (1998, Chapter 4), Basu (2003, Chapter 2)]. But the models to be reviewed do not offer any insights into the role of knowledge or its diffusion, learning-by-doing, externalities, or indivisibilities (although these may be particularly relevant for understanding the roots of increasing returns). However, – for this review – of particular significance are some observations of Scitovsky (1954) that I would like to recall. Stressing the inadequacy of static equilibrium theory to deal with problems of investments – "which have a delayed effect and - looking ahead to a long future period - should be governed not by what the present economic situation is but by what the future economic situation is expected to be. The proper coordination of investment decisions, therefore, would require a signalling device to transmit information about present plans and future conditions as they are determined by present plans; and the price system fails to provide this. Hence the belief that there is need either for centralized investment planning or some additional communication system to supplement the pricing system as a signalling device".

The experiments involving centralized or large scale national planning suggest the importance of designing an appropriate mechanism (hopefully a decentralized mechanism) that can supplement the price system and help attain desirable allocations when economic agents are making independent decisions with incomplete information about the future. Perhaps this has been the most difficult formal problem for infinite economies. For the "classical" convex mod-

[3] Uncertainty and decreasing marginal costs are at the core of the choice of technology problem studied in Majumdar and Radner (1993) by using a Bayesian dynamic programming method.

els, this was attacked through the duality theory [that was stressed in Gale's comments quoted above] of dynamic efficiency prices and the search for an appropriate transversality condition.[4] The problem seems to be quite elusive when nonconvexity is admitted.

7.2 Optimal Allocation in a Closed Economy

7.2.1 Production

To begin with an aggregative model, consider a technology described by a production function f from R_+ to itself when the *input* x in any period gives rise to output $f(x)$ in the subsequent period. The following assumptions on f are introduced:

(A.1) $f(0) = 0$; (A.2) $f(x)$ is strictly increasing for $x \geq 0$; (A.3) $f(x)$ is twice continuously differentiable for $x \geq 0$; (A.4) f satisfies the following end-point conditions: $f'(\infty) < 1 < f'(0) < \infty$; (A.5) There is a (finite) $b_1 > 0$, such that (i) $f''(b_1) = 0$; (ii) $f''(x) > 0$ for $0 \leq x < b_1$; (iii) $f''(x) < 0$ for $x > b_1$.

In contrast to the present ["non-classical"] model, the traditional aggregative [or, "classical"] framework would replace (A.5) by

(A.5′) f is strictly concave for $x \geq 0$ $(f''(x) < 0$ for $x > 0)$

while preserving (A.1) - (A.4). [In some versions, (A.3) and (A.4) would also be modified to allow $f'(0) = \infty$.] In discussions to follow, we will find it convenient to refer to a model with assumptions (A.1) - (A.4) and (A.5′) as "classical", and to a model with (A.1) - (A.5) as "non-classical."

We define a function, h [representing the *average product function*], as follows:

$$h(x) = [f(x)/x] \quad \text{for} \quad x > 0; \qquad h(0) = \lim_{x \to 0}[f(x)/x]. \tag{7.1}$$

Under (A.1) - (A.5), it is easily checked that $h(0) = f'(0)$; furthermore, there exist positive numbers $k^*, \bar{k}$, b_2 satisfying:

(i) $0 < b_1 < b_2 < k^* < \bar{k} < \infty$; (ii) $f'(k^*) = 1$; (iii) $f(\bar{k}) = \bar{k}$; (iv) $f'(b_2) = h(b_2)$. Also, for $0 \leq x < k^*$, $f'(x) > 1$; and for $x > k^*$, $f'(x) < 1$; for $0 < x < \bar{k}$, $x < f(x) < \bar{k}$, and for $x > \bar{k}$, $\bar{k} < f(x) < x$; for $0 < x < b_2$, $f'(x) > h(x)$, and for $x > b_2$, $f'(x) < h(x)$. Also note that for $0 \leq x < b_2$, $h(x)$ is increasing, and for $x > b_2$, $h(x)$ is decreasing; for $0 \leq x < b_1$, $f'(x)$ is increasing, and for $x > b_1$, $f'(x)$ is decreasing.

[4] This was the main theme of the *Journal of Economic Theory Symposium* (1988, volume 45, no. 2) [see also the Foreword by Malinvaud in Majumdar (1992)].

7.2.2 Programs

A *feasible production program* from $\mathbf{x} > 0$ is a sequence $< x, y > = (x_t, y_{t+1})$ satisfying

$$x_0 = \mathbf{x}; \quad 0 \leq x_t \leq y_t \qquad \text{and} \qquad y_t = f(x_{t-1}) \qquad \text{for} \;\; t \geq 1. \tag{7.2}$$

The sequence $< x > = (x_t)_{t \geq 0}$ is the *input program*, while the corresponding $< y > = (y_{t+1})_{t \geq 0}$ satisfying (7.2) is the *output program*. The *consumption program* $< c >= (c_t)$ generated by $< x, y >$ is defined by:

$$c_t \equiv y_t - x_t \qquad \text{for} \quad t \geq 1. \tag{7.3}$$

We will refer to $< x, y, c >$ briefly as *a program* from $\mathbf{x}$, it being understood that $< x, y >$ is a feasible production program, and $< c >$ the corresponding consumption program.

A program $< x, y, c >$ from $\mathbf{x}$ is called *positive* if $(x_t, y_{t+1}, c_{t+1}) >> 0$ for $t \geq 0$. It is called *interior* if $\inf_{t>0} x_t > 0$. It is a standard exercise to check that for any program $< x, y, c >$ from $\mathbf{x}$, we have $(x_t, y_{t+1}, c_{t+1}) \leq (\hat{k}, \hat{k}, \hat{k})$ for $t \geq 0$, where $\hat{k} = \max(\mathbf{x}, \bar{k})$.

A slight abuse of notation: I shall often specify only the input program $< x >= (x_t)_{t \geq 0}$ from $\mathbf{x} > 0$ to describe a program $< x, y, c >$. It will be understood that $x_0 = \mathbf{x}$ and $0 \leq x_t \leq f(x_{t-1})$ for all $t \geq 1$, and (7.2) and (7.3) hold. Indeed I shall also refer to $< x, y, c >$ as a program from $y_1 > 0$ to mean that it is really a program from the unique $\mathbf{x} > 0$ such that $y_1 = f(\mathbf{x})$. Note that decisions on "consumption today versus consumption tomorrow" begin in period 1.

7.2.3 Evaluation Criteria

A social planner (Lange's Central Planning Board) evaluates alternative programs according to some welfare criterion. The criteria we focus on deal exclusively with the sequences of consumptions generated by programs. This admittedly limits the scope of our analysis, and its appeal to policy makers.

A program $< x', y', c' >$ from $\mathbf{x}$ *dominates* a program $< x, y, c >$ from $\mathbf{x}$, if $c_t^{'} \geq c_t$ for all $t \geq 1$, and $c_t^{'} > c_t$ for some t. A program $< x, y, c >$ from $\mathbf{x}$ is said to be *inefficient* if some program from $\mathbf{x}$ dominates it. It is said to be *efficient* if it is not inefficient.

An alternative criterion involves a *utility function*, u, from R_+ to R, and a *discount factor*, δ, where $0 < \delta \leq 1$, which reflects the planner's time preference. A program $< x^*, y^*, c^* >$ from $\mathbf{x}$ is called *optimal* if

$$\limsup_{T \to \infty} \sum_{t=1}^{T} \delta^{t-1} [u(c_t) - u(c_t^*)] \leq 0 \tag{7.4}$$

for every program $< x, y, c >$ from $\mathbf{x}$.

A program $< x, y, c >$ from $\mathbf{x}$ is *intertemporal profit maximizing* (IPM) if there is a non-zero sequence $< p^* > = (p_t^*)$ of non-negative prices, such that, for $t \geq 0$.

$$p_{t+1}^* y_{t+1} - p_t^* x_t \geq p_{t+1}^* y - p_t^* x \qquad \text{for} \qquad x \geq 0, \, y = f(x). \tag{7.5}$$

A price sequence $< p^* > = (p_t^*)$ associated with an IPM program, for which (7.5) holds, is called a sequence of *support* or *Malinvaud* prices. A program $< x, y, c >$ from $\mathbf{x}$ is *competitive* if there is a non-zero sequence $< p^* > = (p_t^*)$ of non-negative prices such that (7.5) holds for $t \geq 0$; and, for $t \geq 1$.

$$\delta^{t-1} u(c_t) - p_t^* c_t \geq \delta^{t-1} u(c) - p_t^* c, \qquad c \geq 0. \tag{7.6}$$

A price sequence $< p^* >= (p_t^*)$ associated with a competitive program $< x, y, c >$, for which (7.5), (7.6) hold, is called a sequence of *competitive* or *Gale prices*; (7.5), (7.6) are called the *competitive conditions.* I shall often write $< x, y, c; p >$ to denote an IPM program or competitive program depending on the context.

The following assumptions on u will be maintained in Sections 7.4 and 7.5.

(A.6) $u(c)$ is continuous for $c \geq 0$, and twice continuously differentiable at $c > 0$, with $u'(c) > 0$, $u''(c) < 0$ at $c > 0$.

(A.7) $u'(c) \to \infty$ as $c \to 0$.

We normalize $u(0) = 0$.

A positive program $< \bar{x}, \bar{y}, \bar{c} >$ from $\mathbf{x}$ is called an *Euler program* if

$$u'(\bar{c}_t) = \delta f'(\bar{x}_t) u'(\bar{c}_{t+1}) \qquad \text{for} \qquad t \geq 1. \tag{7.7}$$

A program $< x, y, c >$ from $\mathbf{x}$ is *stationary* if $x_t = x_{t+1}$ for $t \geq 0$. An *Euler Stationary Program* (ESP) from $\mathbf{x}$ is a stationary program, which is also an Euler program. An *Optimal Stationary Program* (OSP) from $\mathbf{x}$ is a stationary program, which is also an optimal program.

Lemma 7.2.1. *(i) If $< x^*, y^*, c^* >$ is an optimal program from $\mathbf{x} > 0$, then it is an Euler program. (ii) If $< x, y, c; p >$ is a competitive program from $\mathbf{x} > 0$, then it is an Euler program, and $[f(x_t)/x_t] \geq f'(x_t)$ for $t \geq 0$.*

Proof. To prove (i), note that by (A.7), $c_t^* > 0$ for $t \geq 1$, so $(x_t^*, y_{t+1}^*) >> 0$ for $t \geq 0$. For each $t \geq 1$, the expression $u[f(x_{t-1}^*) - x] + \delta u[f(x) - x_{t+1}^*$ is maximized at $x = x_t^*$ among all $x \geq 0$ satisfying $f(x_{t-1}^*) \geq x$, and $f(x) \geq x_{t+1}^*$. Since the maximum is at the interior point, so $u'(c_t^*) = \delta u'(c_{t+1}^*) f'(x_t^*)$ for $t \geq 1$.

To prove (ii), note that by (7.6), $p_t > 0$ for $t \geq 1$, and by (7.5), $p_0 > 0$. Hence by (7.6) $c_t > 0$ for $t \geq 1$, and $(x_t, y_{t+1}) >> 0$ for $t \geq 0$. Then, using (7.5), $p_{t+1} f'(x_t) = p_t$ for $t \geq 0$; while, by (7.6), $\delta^{t-1} u'(c_t) = p_t$ for $t \geq 1$. Hence, for $t \geq 1$, $u'(c_t) = \delta u'(c_{t+1}) f'(x_t)$. So $< x, y, c >$ is an Euler program. Also, using (7.5), $p_{t+1} f(x_t) - p_{t+1} f'(x_t) x_t \geq p_{t+1} f(x) - p_{t+1} f'(x_t) x$ for $t \geq 0$. So using $x = 0$ in the above inequality, $[f(x_t)/x_t] \geq f'(x_t)$ for $t \geq 0$.

7.3 Characterization of Inefficiency

This section is devoted to finding suitable conditions characterizing the set of efficient programs. To this end, it is useful to look at the function $g(x)$ defined by

$$g(x) = \min[h(x), f'(x)] \qquad \text{for} \qquad x \geq 0. \tag{7.8}$$

We associate, with any program $< x, y, c >$ from $\mathbf{x} > 0$, a sequence (q_t) given by

$$q_0 = 1, \quad q_{t+1} = q_t / g(x_t) \qquad \text{for} \qquad t \geq 0. \tag{7.9}$$

and a sequence (r_t) given by

$$r_0 = 1, \quad r_{t+1} = |r_t / f'(x_t)| \qquad \text{for} \qquad t \geq 0. \tag{7.10}$$

Theorem 7.3.1. *If a program $< x, y, c >$ from $\mathbf{x} \,\epsilon\, (0, \bar{k})$ is inefficient, then*

$$\sum_{t=0}^{\infty} (1/q_t) < \infty. \tag{7.11}$$

Proof. See Majumdar and Mitra (1982)

Theorem 7.3.2. *An interior program $< x, y, c >$ from $\mathbf{x} \,\epsilon\, (0, \bar{k})$ is inefficient if*

$$\sum_{t=0}^{\infty} (1/r_t) < \infty \tag{7.12}$$

Proof. Follow exactly the method of Cass (1972, pp. 218-220), noting that concavity of f is nowhere required

Remark 7.3.1. (1) Suppose a program $< x, y, c >$ from $\mathbf{x}$ satisfies

$$\liminf_{t \to \infty} x_t > k^*,$$

then it is inefficient by Theorem 7.3.2. (See Phelps [1965])

(2) If $\mathbf{x} \geq \bar{k}$, then for a program $< x, y, c >$ from $\mathbf{x}$, either (a) $x_t < \bar{k}$ after a finite number of periods; or (b) $x_t \geq \bar{k}$ for all $t \geq 0$. Clearly, in case (b), $< x, y, c >$ is inefficient. Thus, there is no loss of generality in restricting $\mathbf{x}$ to be in $(0, \bar{k})$, as we have done in Theorems 7.3.1 and 7.3.2.

(3) Cass (1972) established that, in a "classical" model, if an *interior* program $< x, y, c >$ from $\mathbf{x} \,\epsilon\, (0, \bar{k})$ is inefficient, then

$$\sum_{t=0}^{\infty} (1/r_t) < \infty. \tag{7.13}$$

Note that the method of proof used by Majumdar and Mitra (1982) for Theorem 7.3.1, can be used in the "classical" model to show that if a program $< x, y, c >$ from $\mathbf{x} \epsilon (0, \bar{k})$ is inefficient, then (7.13) holds. Thus the method of proof used in Theorem 7.3.1 is a refinement of the proof used in Cass (1972).

(4) The outstanding contribution to the literature on intertemporal efficiency in the "classical" [convex] environment is that of Malinvaud [1953]. See Cass and Majumdar (1979) for a review and an extended list of references.

Given Theorems 7.3.1 and 7.3.2, a natural question is whether we can strengthen either of the theorems to obtain a complete characterization of inefficiency. The answer is in the negative, i.e., one can construct an interior program $< x, y, c >$ from $\mathbf{x} \epsilon (0, \bar{k})$, which is inefficient, and violates (7.12). Hence the converse of Theorem 7.3.2 is not true. Also one can construct an interior program $< x, y, c >$ from $\mathbf{x} \epsilon (0, \bar{k})$, which satisfies (7.11) and is efficient. Hence, the converse of Theorem 7.3.1 is not true either (see Majumdar and Mitra (1982), pp. 112-116 for the examples).

We also note, with an example, that efficient programs are not necessarily intertemporal profit maximizing, so that the celebrated theorem of Malinvaud prices in the "classical" model breaks down. Furthermore, it is not known in this "non-classical" framework, whether efficiency implies some concept of "value-maximization" relative to an appropriate "price system".

Example 7.3.1. This example shows that an efficient program need not be intertemporal profit maximizing. Let $\mathbf{x} = b_1$, and consider the sequence $< x, y, c >$ given by $x_t = b_1$ for $t \geq 0$. Clearly, $< x, y, c >$ is a program from $\mathbf{x}$, and by Theorem 7.3.1 it is efficient. We claim it is not IPM. If it were, then there is a non-null sequence (p_t) of non-negative prices, such that (7.5) holds. Let n be the first period for which $p_n > 0$. Since $x_n > 0$, so $p_{n+1} > 0$ [using $x = 0, y = f(0) = 0$ in (7.5)]. Then (7.5) implies, $p_{n+1} f'(b_1) = p_n$, and $p_{n+1}[f(b_1) - f'(b_1)b_1] \geq 0$, so $f(b_1)/b_1 \geq f'(b_1)$, a contradiction.

7.4 The Ramsey Problem: Undiscounted Utilities

If one wishes to capture the preferences of a Central Planning Board which has serious commitments to the long run prospects of the economy or has genuine concerns for the welfare of distant generations, one naturally considers δ to be 1 or very close to 1 in (7.4).

In this section, we study the questions of existence and turnpike properties of optimal programs, when future utilities are undiscounted; in other words, in the tradition of Ramsey (1928) we take $\delta = 1$ in (7.4). I rely primarily on the exposition of Majumdar and Mitra (1982).

Many of the results of the "classical" model continue to hold: (a) there is a unique ESP, and this is also the (unique) OSP; this program is competitive at a stationary price sequence; (b) optimal programs exist from every positive initial input level; they converge monotonically to the optimal stationary program.

Some results of the "classical" model fail to hold: (a′) in general, it is not known whether an optimal program from every initial stock is unique; (b′) optimal programs are not necessarily competitive, and an example is given to confirm this fact.

I begin with the existence and qualitative properties of *Euler* and *Optimal Stationary Programs* in the first sub-section; non-stationary optimal programs are examined in the second sub-section.

7.4.1 Stationary Programs: The Golden Rule Equilibrium

Consider the set $\mathcal{C} = \{c : c = f(x) - x,\ 0 \le x \le \bar{k}\}$. Clearly $\mathcal{C}$ is compact. Hence, there is c^* in $\mathcal{C}$, such that $c \le c^*$ for all c in $\mathcal{C}$. Since $0 < x < \bar{k}$ implies $f(x) - x > 0$, so $c^* > 0$. Associated with c^* is x^* such that $0 < x^* < \bar{k}$, and $f(x^*) - x^* = c^*$. Since x^* maximizes $[f(x) - x]$ over the set $\{x : 0 \le x \le \bar{k}\}$, and the maximum is attained at an interior point,

$$f'(x^*) = 1.$$

Since k^* is the unique non-negative solution to $f'(x) = 1$, so $x^* = k^*$, and k^* is the unique input level, which maximizes c over the set $\mathcal{C}$.

Consider the program from k^* given by $x_t^* = k^*$, $y_{t+1}^* = f(k^*)$, $c_{t+1}^* = c^* = f(k^*) - k^*$, for $t \ge 0$. Then $< k^*,\ f(k^*),\ c^* >$ is the unique ESP.

We show, next, that $< k^*,\ f(k^*),\ c^* >$ is an Optimal Stationary Program from k^*. For this, we need two preliminary results. The first derives a price p^* supporting the *golden rule* triplet $[k^*,\ f(k^*), c^*]$ in R^3_{++}.

Lemma 7.4.1. *There is $p^* > 0$, such that*

$$u(c^*) - p^*c^* \ge u(c) - p^*c \qquad for \quad c \ge 0; \tag{7.14}$$

$$p^*f(k^*) - p^*k^* \ge p^*f(x) - p^*x \qquad for \quad x \ge 0. \tag{7.15}$$

Proof. Denote $u'(c^*)$ by p^*; then $p^* > 0$. By concavity of u, we have for $c \ge 0$, $u(c) - u(c^*) \le u'(c^*)(c - c^*) = p^*(c - c^*)$. By transposing terms, (7.14) is verified.

By definition of k^*, $f(k^*) - k^* \ge f(x) - x$ for $0 \le x \le \bar{k}$, $f(k^*) - k^* > 0 > f(x) - x$ for $x > \bar{k}$. So for all $x \ge 0$, $f(k^*) - k^* \ge f(x) - x$. Multiplying this inequality by $p^* > 0$, yields (7.15).

The golden rule triplet and its supporting price $[k^*,\ f(k^*),\ c^*,\ p^*]$ constitute the *golden rule equilibrium.* At any $c \ge 0$, define the *consumption value loss* at p^* as

$$\alpha(c) \equiv [u(c^*) - p^*c^*] - [u(c) - p^*c]$$

Similarly, at any $x \ge 0$ define the *loss of intertemporal profit* at p^* as:

$$\beta(x) = [p^*f(k^*) - k^*] - [p^*f(x) - p^*x]$$

using (7.14) and (7.15) we see that $\alpha(c) \geq 0$ for all $c \geq 0$ and $\beta(x) \geq 0$ for all $x \geq 0$. Of particular interest in the sequel is the following *value loss lemma* [that has appeared in many contexts in intertemporal economics]:

Lemma 7.4.2. *Given any $\theta > 0$, there is $\eta > 0$, such that if $0 \leq x \leq \bar{k}$, and $|k^* - x| \geq \theta$ then $\beta(x) \geq \eta$.*

Proof. Suppose, on the contrary, there is a sequence (x_n) such that $0 \leq x_n \leq \bar{k}$ and $|k^* - x_n| \geq \theta$, for $n = 1, 2, 3, ...$, but $[p^* f(k^*) - p^* k^*] - [p^* f(x_n) - p^* x_n] \to 0$ as $n \to \infty$. Consider a subsequence of (x_n) (retain notation) converging to $\hat{x}$. Then $\hat{x}$ is in $[0, \bar{k}]$, and by continuity of f, $[p^* f(k^*) - p^* k^*] = [p^* f(\hat{x}) - p^* \hat{x}]$. Hence $f(\hat{x}) - \hat{x} = f(k^*) - k^*$. Since $|k^* - x_n| \geq \theta$ for each n, $|k^* - \hat{x}| \geq \theta$. But this contradicts the uniqueness property of k^*.

Theorem 7.4.1. *The program $< k^*, f(k^*), c^* >$ is an optimal program from k^*.*

Proof. Suppose on the contrary that there is a program $< x, y, c >$ from k^*, a scaler $\alpha > 0$, and a sequence of periods $T_n (n = 1, 2, 3, ...)$, such that

$$\sum_{t=1}^{T_n} [u(c_t) - u(c^*)] \geq \alpha \qquad \text{for all } n. \tag{7.16}$$

Using Lemma 7.4.1, we have for $t \geq 1$, $u(c_t) - u(c^*) \leq p^*(c_t - c^*) = p^* [f(x_{t-1}) - x_t] - p^* [f(k^*) - k^*]$
$= [p^* f(x_{t-1}) - p^* x_{t-1}] + [p^* x_{t-1} - p^* x_t] - p^* [f(k^*) - k^*] \leq [p^* x_{t-1} - p^* x_t]$.
Hence, for $T \geq 1$, we have:

$$\sum_{t=1}^{T} [u(c_t) - u(c^*)] \leq \sum_{t=1}^{T} [p^* x_{t-1} - p^* x_t] = p^* k^* - p^* x_T. \tag{7.17}$$

Hence for all n, we have, using (7.16), (7.17),

$$p^*(k^* - x_{T_n}) \geq \alpha. \tag{7.18}$$

This means that $(k^* - x_{T_n}) \geq (\alpha / p^*)$ for all n, so by Lemma 7.4.2, there is $\varepsilon > 0$, such that

$$[p^* f(k^*) - p^* k^*] \geq [p^* f(x_{T_n}) - p^* x_{T_n}] + \varepsilon \qquad \text{for all } n. \tag{7.19}$$

Using Lemma 7.4.1 again, and (7.19), we have for $t = T_n + 1$,

$$\begin{aligned} u(c_t) - u(c^*) \leq p^*(c_t - c^*) &= p^*[f(x_{t-1}) - x_t] - p^*[f(k^*) - k^*] \\ &= [p^* f(x_{t-1}) - p^* x_{t-1}] - [p^* f(k^*) - p^* k^*] \\ &+ [p^* x_{t-1} - p^* x_t] \\ &\leq [p^* x_{t-1} - p^* x_t] - \varepsilon. \end{aligned}$$

And for $t \neq T_n + 1$, we have by our previous calculations, $u(c_t) - u(c^*) \leq [p^* x_{t-1} - p^* x_t]$. Hence, for all n,

$$\begin{aligned}\alpha &\leq \sum_{t=1}^{T_n} [u(c_t) - u(c^*)] \leq p^*(k^* - x_{T_n}) - (n-1)\varepsilon \\ &\leq p^* k^* - (n-1)\varepsilon.\end{aligned}$$

For n large, this is a contradiction. Hence, $< k^*, f(k^*), c^* >$ is an OSP.

Remark 7.4.1. The program $< k^*, f(k^*), c^* >$ is the *only* OSP in this model, from a positive initial input. For if there were another, say $< x, y, c >$ from $\mathbf{x} > 0$, then it would be a positive program, and an Euler program. But $< k^*, f(k^*), c^* >$ is the only ESP, so $< x, y, c >$ could not be an OSP. In view of Theorem 7.4.1 one can refer to k^* as an *optimal stationary input.*

7.4.2 Non-stationary Programs

We deal with the question of the existence of an optimal program; on the way, a turnpike theorem is proved.

Estimates of the sums of all value losses along a program can be obtained from the following lemma with a routine calculation.

Lemma 7.4.3. *If* $< x, y, c >$ *is a program from* $\mathbf{x} > 0$ *then for any finite* $T \geq 1$

$$\begin{aligned}\sum_{t=1}^{T} [u(c_t) - u(c^*)] &= \sum_{t=1}^{T} p^*(c_t - c^*) - \sum_{t=1}^{T} \alpha(c_t) \\ &= \sum_{t=1}^{T} [p^*(f(x_{t-1}) - x_{t-1}) - p^*(f(k^*) - k^*)] + p^* x_0 - p^* x_T \quad (7.20) \\ &= -\left[\sum_{t=1}^{T} \alpha(c_t) + \sum_{t=0}^{T-1} \beta(x_t)\right] + p^* x_0 - p^* x_T\end{aligned}$$

We call a program $< x, y, c >$ from $\mathbf{x}$, *good* if there exists $M > -\infty$, such that

$$\sum_{t=1}^{T} [u(c_t) - u(c^*)] \geq M \qquad \text{for all} \qquad T \geq 1.$$

It is *bad* if

$$\sum_{t=1}^{T} [u(c_t) - u(c^*)] \to -\infty \qquad \text{as} \qquad T \to \infty.$$

Lemma 7.4.4. *There exists a good program from every* $\mathbf{x} \epsilon (0, \bar{k})$. *If a feasible program from* $\mathbf{x} > 0$ *is not good, it is bad.*

Proof. Consider the *pure accumulation program* $< x^1, y^1, c^1 >$ from $\mathbf{x}$ defined by: $x_0^1 = \mathbf{x}$, $x_{t+1}^1 = y_{t+1}^1 = f(x_t^1)$ for $t \geq 0$, so that $c_t^1 = 0$ for $t \geq 1$. It is not difficult to see that x_t^1 converges to $\bar{k}$ as $t \to \infty$. Since $\bar{k} > k^*$, $x_t^1 > k^*$ for all sufficiently large t. Let T be the first period such that $x_T^1 \geq k^*$. Consider the program $< x, y, c >$ defined by $x_0 = \mathbf{x}$. $x_t = \min(x_t^1, k^*)$, $y_{t+1} = f(x_t)$ and $c_t = y_t - x_t$ for $t \geq 1$. Since $c_t = c^*$ for all $t > T$, $< x, y, c >$ is a good program from $\mathbf{x}$. Suppose that $< x, y, c >$ is a program from $\mathbf{x} > 0$ which is not good. Then given any real number B, there is some T such that

$$\sum_{t=0}^{T} [u(c_t) - u(c^*)] < B$$

Using the fact that $x_t \leq \bar{k}$ for all $t \geq 0$ and Lemma 7.4.3, we have for all $\tau > T$

$$\sum_{t=T+1}^{\tau} [u(c_t) - u(c^*)] \leq p^* \bar{k}$$

Hence given any real number B, there is some T such that for all $\tau > T$

$$\sum_{t=1}^{\tau} [u(c_t) - u(c^*)] < B + p^* \bar{k}$$

which shows that it is bad.

Now we are led to the celebrated *consumption turnpike theorem*:

Theorem 7.4.2. *If a feasible program* $< x, y, c >$ *from* $\mathbf{x} \in (0, \bar{k})$ *is good, then*

$$(x_t, y_{t+1}, c_{t+1}) \to (k^*, f(k^*), c^*) \qquad \textit{as} \qquad t \to \infty.$$

Proof. Suppose x_t does not converge to k^*. Then there is some $\theta > 0$ and a subsequence of periods for which $|x_t - k^*| \geq \theta$. Using Lemma 7.4.2 there is some $\beta > 0$ such that $\beta(x_t) \geq \beta$ for this subsequence of periods. Now using Lemma 7.4.3, it follows that $< x, y, c >$ is not good. Thus, $x_t \to k^*$ as $t \to \infty$; consequently, continuity of f gives us $y_t = f(x_{t-1}) \to f(k^*)$, and finally, as $t \to \infty$,

$$c_t = f(x_{t-1}) - x_t \to f(k^*) - k^* = c^*.$$

We now establish a convergence property of an appropriately normalized sum of utilities generated by good programs.

Lemma 7.4.5. *If* $< x, y, c >$ *is a good program from* $\mathbf{x} \in (0, \bar{k})$, *then*

$$\lim_{T \to \infty} \sum_{t=1}^{T} [u(c_t) - u(c^*)] \ \textit{ exists.}$$

Proof. Since $< x, y, c >$ from $\mathbf{x}$ is good, there is a real number B such that for all $T \geq 0$

$$\sum_{t=1}^{T}[u(c_t) - u(c^*)] \geq B$$

Using Lemma 7.4.3 we get

$$-\left[\sum_{t=1}^{T}\alpha(c_t) + \sum_{t=0}^{T-1}\beta(x_t)\right] + p^*(x_0 - x_T) \geq B$$

or,

$$\sum_{t=1}^{T}\alpha(c_t) + \sum_{t=0}^{T-1}\beta(x_t) \leq p^*x_T - p^*x_0 - B$$

Since $x_T \leq \bar{k}$, we have

$$\sum_{t=1}^{T}\alpha(c_t) + \sum_{t=0}^{T-1}\beta(x_t) \leq p^*(\bar{k} - x_0) - B$$

Since $\alpha(c_t) \geq 0$ for all $t \geq 1$ and $\beta(x_t) \geq 0$ for all $t \geq 0$, we see that

$$L(x, y, c) = \lim_{T\to\infty}\left[\sum_{t=1}^{T}\alpha(c_t) + \sum_{t=0}^{T-1}\beta(x_t)\right]$$

exists. Now, going back to (7.20), we see that as $T \to \infty$, $x_T \to k^*$, and the right side has a limit. Hence,

$$\lim_{T\to\infty} \sum_{t=0}^{T}[u(c_t) - u(c^*)] = p^*(x_0 - k^*) - L(x, y, c)$$

Finally, we settle the question of existence of optimal programs by proving:

Theorem 7.4.3. *There exists an optimal program from every* $\mathbf{x} \,\epsilon\, (0, \bar{k})$.

Proof. Write $L(\mathbf{x}) = \inf[L < x, y, c >: < x, y, c >$ is a good program from $\mathbf{x}]$. Take a sequence $< x^n, y^n, c^n >$ of good programs from $\mathbf{x}$ such that

$$L < x^n, y^n, c^n > \;\; \leq L(\mathbf{x}) + \frac{1}{n}.$$

Recall that $\hat{k} = \max(\mathbf{x}, \bar{k})$ and for *any* program $< x, y, c >$ from $\mathbf{x}$ one has $\sup_t x_t \leq \hat{k}$, $\sup_t y_t \leq \hat{k}$ and $\sup_t c_t \leq \hat{k}$. Hence, using the continuity of f, and a diagonalization argument there is a program $< x^*, y^*, c^* >$ from $\mathbf{x}$ and a subsequence (retain notation) such that for each $t \geq 0$, as $n \to \infty$

$$(x_t^n, y_{t+1}^n, c_{t+1}^n) \to (x_t^*, y_{t+1}^*, c_{t+1}^*).$$

Using Lemma 7.4.3 one can show that $< x^*, y^*, c^* >$ is a good program. We claim that:

$$L < x^*, y^*, c^* >= L(\mathbf{x})$$

If this claim is false, then we can find a positive integer T and some $\varepsilon > 0$ such that

$$\sum_{t=1}^{T} \alpha(c_t^*) + \sum_{t=0}^{T-1} \beta(x_t^*) \geq L(\mathbf{x}) + \varepsilon$$

But continuity of f and u imply that there is some $\bar{n} > 0$ such that for all $n \geq \bar{n}$

$$\sum_{t=1}^{T} \alpha(c_t^{\bar{n}}) + \sum_{t=0}^{T-1} \beta(x_t^{\bar{n}}) \geq L(\mathbf{x}) + \frac{\varepsilon}{2}$$

But this implies that for all $n \geq \bar{n}$

$$L < x^n, y^n, c^n > \geq L(\mathbf{x}) + \frac{\varepsilon}{2}$$

and we have a contradiction for $n > \max(\bar{n}, 2/\varepsilon)$. This establishes the claim. The optimality of $< x^*, y^*, c^* >$ is obvious.

Since there exists a good program, any optimal program is necessarily good. Consequently, by Lemma 7.4.5, every optimal program $< x, y, c >$ from $\mathbf{x} \,\epsilon\, (0, \bar{k})$ has the property that $(x_t, y_{t+1}, c_{t+1}) \to (k^*, f(k^*), c^*)$ as $t \to \infty$. Furthermore, if $\mathbf{x} = k^*$, then $< k^*, f(k^*), c^* >$ itself is an optimal program by Theorem 7.4.1. If $\mathbf{x} < k^*$, then an optimal program $< x, y, c >$ from $\mathbf{x}$ has (x_t, y_{t+1}, c_{t+1}) monotonically *increasing* for all $t \geq 0$, and $(x_t, y_{t+1}, c_{t+1}) \leq (k^*, f(k^*), c^*)$ for all $t \geq 0$. [This assertion, and the next, follow directly from the argument used in Theorem 7.5.5 of the next section, so we omit the proof here]. Similarly, if $\mathbf{x} > k^*$, then an optimal program $< x, y, c >$ from $\mathbf{x}$ has (x_t, y_{t+1}, c_{t+1}) monotonically *decreasing* for all $t \geq 0$, and $(x_t, y_{t+1}, c_{t+1}) \geq (k^*, f(k^*), c^*)$ for all $t \geq 0$.

We now briefly discuss the differences between the "classical" and "non-classical" models, in the analysis of the problem of undiscounted optimality. It is fairly easy to check that for $\mathbf{x} \geq b_2$, every optimal program is unique. But for $\mathbf{x} < b_2$, it is not known whether the result is true; we believe it is not, and a concrete example would be helpful. Certainly, the standard argument, used in the "classical" model does not go through.

Optimal programs from $\mathbf{x} \geq b_2$ can be shown to be competitive; but those from $\mathbf{x} < b_2$ are, in general, not. Consider, for example, $\mathbf{x} > 0$, such that $\mathbf{x} < f^{-1}(b_1)$. Consider an optimal program $< x, y, c >$ from $\mathbf{x}$. If it is competitive there is a sequence (p_t^*) of non-negative prices such that (7.5), (7.6) hold. Now, by (7.6), $p_t^* > 0$ for $t \geq 1$, and by (7.5), $p_0^* > 0$ also. Also, by (7.5), since $u'(0) = \infty$, $c_t > 0$ for $t \geq 1$, so $(x_t, y_{t+1}) >> 0$ for $t \geq 0$. Using (7.5), then $p_{t+1}^* f'(x_t) = p_t^*$, so that we have

$$p_{t+1}^* f(x_t) - p_{t+1}^* f'(x_t) x_t \geq p_{t+1}^* f(x) - p_{t+1}^* f'(x_t) x \quad \text{for } x \geq 0$$

or

$$f(x_t) - f'(x_t)x_t \geq f(x) - f'(x_t)x \quad \text{for} \quad x \geq 0.$$

Using $x = 0$ in the above inequality, $[f(x_t)/x_t] \geq f'(x_t)$. Since $\mathbf{x} < f^{-1}(b_1)$, so $x_1 \leq f(x) < b_1$, and $[f(x_1)/x_1] < f'(x_1)$, a contradiction.

7.5 Mild Discounting: Comparative Dynamics and Stability

We turn to discounting. The reader interested in models with discounting has to turn to Skiba (1978), Majumdar and Nermuth (1982), Dechert and Nishimura (1983), Mitra and Ray (1984) and Majumdar, Mitra and Nyarko (1989). Note, first, that if the discount factor δ is "small", the results on ESP and OSP that we reviewed above no longer hold. To be precise, suppose δ is such that $\delta f'(b_1) < 1$. Hence, for all $x \geq 0$, $\delta f'(x) \leq \delta f'(b_1) < 1$. Consequently if $< x, y, c >$ is *any* Euler program

$$u'(c_t) = \delta f'(x_t) u'(c_{t+1})$$

which means that "$u'(c_t) < u'(c_{t+1})$" leading to $c_t > c_{t+1}$. Hence that is no ESP, and by Lemma 7.2.1 no OSP and no stationary program that is competitive. Indeed, any optimal $< x^*, y^*, c^* >$ from any $\mathbf{x} \in (0, \bar{k})$ has the property that the sequence (x_t^*), (y_{t+1}^*) (c_t^*) are all monotone decreasing, and

$$\lim_{t\to\infty} x_t^* = 0, \ \lim_{t\to\infty} y_{t+1}^* = 0, \ \lim_{t\to\infty} c_{t+1}^* = 0$$

Leaving the 'intermediate' range of discounting for the next section, we shall review some results on optimal allocation when the discount factor $\delta < 1$ is 'not too small'. More precisely, in this subsection we assume

(A.8) $\quad 1 > \delta > [1/f'(0)]$

We first deal with the question of the existence of an OSP. Next we turn to non-stationary programs and their stability properties. On the way we note an important result on comparative dynamics of optimal programs.

7.5.1 The Modified Golden Rule

In this case there is a unique positive solution to the equation $\delta f'(x) = 1$. Call this K_δ^*. Surely, $k^* > K_\delta^* > k_2$. Hence there is a unique ESP defined by

$$x_t = K_\delta^*, \quad y_{t+1} = f(K_\delta^*) \quad c_{t+1} = f(K_\delta^*) - K_\delta^* \quad \text{for} \quad t \geq 0 \tag{7.21}$$

We shall first show that this program is competitive. To this end, define $p_t = \delta^{t-1}u'(c_t)$ for $t \geq 1$, and $p_0 = p_1 f'(K_\delta^*)$. Then, by concavity of u, we have $\delta^{t-1}u(c) - \delta^{t-1}u(c_t) \leq \delta^{t-1}u'(c_t)(c - c_t) = p_t(c - c_t)$, which yields (7.6). Define $\phi(x) = [f(b_2)/b_2]x$ for $0 \leq x \leq b_2$, and $\phi(x) = f(x)$ for $x \geq b_2$. Then $\phi(0) = 0$;

ϕ is an increasing concave differentiable function for $x \geq 0$. Also $\phi(x) \geq f(x)$ for $x \geq 0$. Hence, for $x \geq 0$, we have

$$\begin{aligned} p_{t+1}[f(x) - f(x_t)] &\leq p_{t+1}[\phi(x) - \phi(x_t)] \\ &\leq p_{t+1}\phi'(x_t)[x - x_t] = p_{t+1}f'(x_t)[x - x_t], \end{aligned}$$

since

$$\phi'(x_t) = \phi'(K_\delta^*) = f'(K_\delta^*) = f'(x_t) \text{ for } t \geq 0.$$

Hence,

$$p_{t+1}f(x) - p_{t+1}f'(x_t)x \leq p_{t+1}f(x_t) - p_{t+1}f'(x_t)x_t.$$

Using the fact that

$$\begin{aligned} p_{t+1}f'(x_t) &= \delta^t u'(c_{t+1})f'(x_t) = \delta^{t-1}u'(c_{t+1})[\delta f'(x_t)] \\ &= \delta^{t-1}u'(c_t)\delta f'(K_\delta^*)] = \delta^{t-1}u'(c_t) = p_t, \text{ for } t \geq 1, \end{aligned}$$

and

$$p_1 f'(x_0) = p_1 f'(K_\delta^*) = p_0$$

we have

$$p_{t+1}f(x) - p_t x \leq p_{t+1}f(x_t) - p_1 x_t \text{ for } t \geq 0,$$

which is (7.5).

Note that $< x, y, c >$ is competitive at the above defined price sequence (p_t); also, $p_t x_t = \delta^{t-1}u'(c_t)x_t = \delta^{t-1}u'[f(K_\delta^*) - K_\delta^*]K_\delta^*$, so that $\lim_{t\to\infty} p_t x_t = 0$. Hence, by a completely standard argument, $< x, y, c >$ is an optimal program from K_δ^*. So $< x, y, c >$ is an OSP. By Lemma 7.2.1, it is the unique competitive program and the unique OSP. To summarize: the modified golden rule triplet $(K_\delta^*, f(K_\delta^*), f(K_\delta^*) - K_\delta^*)$ defines a competitive program at the support prices $p_t = \delta^{t-1}u'[f(K_\delta^*) - K_\delta^*]$. Moreover we have the following:

Theorem 7.5.1. *The stationary program from K_δ^* defined by* (7.21) *is the unique OSP.*

7.5.2 Non-stationary Programs

The *existence* of an optimal program can be settled by a direct compactness argument [Majumdar (1975)]. But it is useful to set up the optimization problem in the framework of dynamic programming. With $\bar{k} = f(\bar{k})$, think of $S = [0, \bar{k}]$ as the *state space* and $A = [0, 1]$ as the action space. At the beginning of each period $t \geq 1$, the planner observes the *state*: total stock $y \in S$, and chooses an *action*: a fraction $a \in [0, 1]$. This determines the input x in period t as $x = ay$, and the consumption $c = (1-a)y$. The *immediate return* is then $u(c)$, and the system moves to the state $f = f(ay)$ which is observed in the next period [$''f''$ is the *law of motion*], and the story is repeated.

A *policy* is a sequence $\underset{\sim}{\pi} = (\pi_t)_{t=1}^{\infty}$ of functions where π_t specifies the action in the t-th period as a function of the entire previous history $\eta_t = (y_1.....a_{t-1}, y_t)$ of the system by associating with each η_t an element a_t of A. Hence, a policy defines a program $< x, y, c >$ from y_1 [or, from the unique $\mathbf{x} = x_0$ such that $f(\mathbf{x}) = y_1$]. Any function $g : S \to A$ defines a policy: whenever observe y, choose $a = g(y)$, independently of the evolution of the states and actions leading to y. The corresponding $\underset{\sim}{\pi} = (g^{(\infty)})$ is a *stationary policy*. Each policy $\underset{\sim}{\pi}$ determines the total value $V_{\underset{\sim}{\pi}}(y) \equiv \sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$ where c_t is the consumption generated by the action that π_t specifies. A policy $\underset{\sim}{\pi}^*$ is *optimal* if $V_{\underset{\sim}{\pi}^*}(y) \geq V_{\underset{\sim}{\pi}}(y)$ for all $y \,\epsilon\, S$.

The following is the basic result from the dynamic programming literature.

Theorem 7.5.2. *There is an optimal stationary policy* $\underset{\sim}{\pi}^* = (\hat{h}^{(\infty)})$ *where* $\hat{h} : S \to A$. *The value function* $V_{\underset{\sim}{\pi}^*}$ *defined by* $\underset{\sim}{\pi}^*$ *is continuous and satisfies the functional equation*

$$\begin{aligned} V_{\underset{\sim}{\pi}^*}(y) &\equiv \max_{a \epsilon A}[u[(1-a)y] + \delta V f(a\,y)] \\ &\equiv u(y - \hat{h}(y) \cdot y) + \delta V f(\hat{h}(y) \cdot y) \end{aligned}$$

Call $h(y) \equiv \hat{h}(y) \cdot y$ an *optimal investment function* and the corresponding $c(y) \equiv [1 - \hat{h}(y)]y$ an *optimal consumption policy function.* The optimal program (x^*, y^*, c^*) from $y_1^* > 0$ (strictly, from $\mathbf{x} \equiv f^{-1}(y_1^*)$) that $\underset{\sim}{\pi}^*$ generates is a positive Euler program.

Suppose $y' > y''$ *and* $(x_t^{'})_{t \geq 1}$ *and* $(x_t^{''})_{t \geq 1}$ *are two optimal input programs from* y' *and* y'' *respectively, i.e.,*

$$x_1^{'} < y^{'}, \quad 0 < x_t^{'} < f(x_{t-1}^{'}) \equiv y_t^{'} \text{ for } t \geqq 2$$

$$x_1^{''} < y'', \quad 0 < x_t^{''} < f(x_{t-1}^{''}) \equiv y_t^{''} \text{ for } t \geqq 2$$

The corresponding optimal consumption programs are respectively determined by: $c_t^{'} = y_t^{'} - x_t^{'}$ for $t \geq 1$, and $c_t^{''} = y_t^{''} - x_t^{''}$ for $t \geq 1$.

We shall repeatedly use the following property: if $(x_t, y_t, c_t)_{t \geq 1}$ *is optimal from* $y_1 > 0$ so is $(x_t, y_t, c_t)_{t \geq T}$ *from* $y_T > 0$ *for any* $T \geq 2$.

Theorem 7.5.3. *Let* $y' > y''$. *Suppose that* $(x_t^{'})_{t \geq 1}$ *and* $(x_t^{''})_{t \geq 1}$ *are optimal input programs from* y' *and* y'' *respectively. Then* $x_1^{'} \geq x_1^{''}$.

Proof. Suppose that $x_1^{'} < x_1^{''}$. Consider two sequences $(\bar{x}_t^{'})_{t\geq 1}$ and $(\bar{x}_t^{''})_{t\geq 1}$ defined as follows: $\bar{x}_t^{'} = x_t^{''}$ for $t \geq 1$; $\bar{x}_t^{''} = x_t^{'}$ for $t \geq 1$. Note that $(\bar{x}_t^{'})$ is an input program from y': $\bar{x}_1^{'} = x_1^{''} < y'' < y'$. Also, $\bar{x}_t^{'} = x_t^{''} < f(x_{t-1}^{''}) = f(\bar{x}_{t-1}^{'})$ for $t \geq 2$. Similarly, $(\bar{x}_t^{''})$ is an input program from y'': $\bar{x}_1^{''} = x_1^{'} < x_1^{''} < y''$. Also $\bar{x}_t^{''} = x_t^{'} < f(x_{t-1}^{'}) = f(\bar{x}_{t-1}^{''})$ for $t \geq 2$. Write $\bar{c}_1^{'} = y' - \bar{x}_1^{'}$, $\bar{c}_1^{''} = y'' - \bar{x}_1^{''}$. From the optimality of $(x_t^{'})_{t\geq 1}$ and $(x_t^{''})_{t\geq 1}$ we get:

$$u(c_1^{'}) + \delta V[f(x_1^{'})] \geq u(\bar{c}_1^{'}) + \delta V[f(x_1^{''})] \tag{7.22}$$

$$u(c_1^{''}) + \delta V[f(x_1^{''})] \geq u(\bar{c}_1^{''}) + \delta V[f(x_1^{'})] \tag{7.23}$$

Adding up (7.22) and (7.23)

$$u(c_t^{'}) + u(c_1^{''}) \geq u(\bar{c}_1^{'}) + u(\bar{c}_1^{''}) \tag{7.24}$$

Now,

$$\bar{c}_1^{'} = y' - \bar{x}_1^{'} > y'' - x_1^{''} = c_1^{''}$$

$$\bar{c}_1^{'} = y' - \bar{x}_1^{'} = y' - x_1^{''} < y' - x_1^{'} = c_1^{'}$$

Hence, there is some θ, $0 < \theta < 1$, such that

$$\bar{c}_1^{'} = \theta c_1^{''} + (1-\theta)c_1^{'}$$

Note that $\bar{c}_1^{'} + \bar{c}_1^{''} = c_1^{'} + c_1^{''}$. Hence, $\bar{c}_1^{'} = c_1^{'} + c_1 - \bar{c}_1^{'} = (1-\theta)c_1^{''} + \theta c_1^{'}$. By the strict concavity of u, we get

$$u(\bar{c}_1^{'}) > \theta u(c_1) + (1-\theta)u(c_t^{'}) \tag{7.25}$$

$$u(\bar{c}_1) > (1-\theta)u(c_1) + \theta u(c_1^{'}) \tag{7.26}$$

Adding up (7.25) and (7.26)

$$u(\bar{c}_1^{'}) + u(\bar{c}_1^{''}) > u(c_1^{'}) + u(c_1^{''}) \tag{7.27}$$

From (7.24) and (7.27) we have a contradiction.

Recall that $h(y)$ is an optimal investment policy function, and $c(y) \equiv y - h(y)$ is the corresponding optimal consumption policy function.

Theorem 7.5.4. *h is strictly increasing, i.e. "$y' > y''$" implies "$h(y') > h(y'')$".*

Proof. Suppose $h(y') = h(y'')$. From Theorem 7.5.3, for all $y \,\epsilon\, [y', y'']$, $h(y) = h(y') = h(y'') = h^*$, say, the Euler condition can be stated as:

$$u'(c(y)) = \delta f'(h(y))h'(c(f(h(y)))) \tag{7.28}$$

For all $y \,\epsilon\, [y', y'']$, the right side of (7.28) is clearly some constant, say, m^* where $m^* = \delta f'(h^*)u'(c(f(h^*)))$. But u being strictly concave, from (7.28) we conclude that for all $y \,\epsilon\, [y', y'']$, $c(y) = c^*$, where $u'(c^*) = m^*$. But this means that for all $y \,\epsilon\, [y', y'']$, $c^* + m^* = y$ a contradiction.

With these two theorems, it is easy to derive some decisive results on comparative dynamics and global stability. First, we have a 'non-crossing' lemma:

Lemma 7.5.1. *Let* $\mathbf{x} > \mathbf{x}'$, *and* $\underset{\sim}{\pi} = (\hat{h}^{(\infty)})$ *be a stationary optimal policy. Consider the optimal input programs* $x_t^*(\mathbf{x})$ *and* $x_t^*(\mathbf{x}')$ *generated by the optimal investment policy function* $h(y) \equiv \hat{h}(y) \cdot y$. *Then for all* $t \geqq 1$,

$$x_t^*(\mathbf{x}) > x_t^*(\mathbf{x}') \tag{7.29}$$

It follows that:

$$y_t^*(\mathbf{x}) > y_t^*(\mathbf{x}') \quad \text{for} \quad t \geq 1 \tag{7.30}$$

Proof. Verify (7.29) for $t = 1$:

$$\mathbf{x} > \mathbf{x}' \ \textit{implies} \ f(\mathbf{x}) > f(\mathbf{x}').$$

Hence, $h[f(\mathbf{x})] > h[f(\mathbf{x}')]$ or, $x_1^*(\mathbf{x}) > x_1^*(\mathbf{x}')$ and use an induction argument.

Now we get the *monotonicity* property displayed by an optimal program. Again, let h be an optimal policy function. Take any $\mathbf{x} > 0$ and consider the three possibilities: [where $y_1 = f(\mathbf{x})$]: (i) $h(y_1) > \mathbf{x}$; (ii) $h(y_1) = \mathbf{x}$; (iii) $h(y_1) < \mathbf{x}$. Only case (1) needs to be discussed in any detail: $y_2 = f[h(y_1)] > f(\mathbf{x}) = y_1$. Hence $h(y_2) > h(y_1)$ leading to $y_3 = f[h(y_2)] > f[h(y_1)] = y_2$. Repeating this argument we get:

$$y_{t+1} > y_t \qquad \text{for all } t \geqq 1.$$

Similar arguments establish that in case (ii), $y_{t+1} = y_t$ for all $t \geq 1$; and, in case (iii) $y_{t+1} < y_t$ for all $t \geq 1$.

The steps towards the result on global stability are relatively simple. Consider the modified golden rule program $< K_\delta^*, f(K_\delta^*), f(K_\delta^*) - K_\delta^* >$: it is the optimal program from K_δ^*. Let h be an optimal investment policy function. Then for any $\mathbf{x} < K_\delta^*$, the optimal input program $(x_t^*)_{t \geq 1}$ generated by h is monotone and satisfies (a) $x_t^* < K_\delta$ for all $t \geq 1$ (b) (x_t^*) is increasing (with $x_0^* = \mathbf{x}$).

To show (b) suppose (x_t^*) satisfies $x_{t+1}^* < x_t^*$. It must converge to some $\hat{x} \geq 0$. If $\hat{x} > 0$, then $0 < \hat{x} < x_0 < K_\delta^* < \bar{k}$ and in the limit $\hat{c} = f(\hat{x}) - \hat{x} > 0$ and using the Euler condition (7.7) and taking the limit, we set

$$\delta f'(\hat{x}) = 1$$

This contradicts the uniqueness property of $K_\delta^* > 0$. On the other hand, if $\hat{x} = 0$, since $\delta f'(0) > 1$, let T be the first period such that $\delta f'(x_t^*) \geq 1$ for all $t \geq T$. Then using (7.7) for $t \geq T$ we get:

$$u'(c_t^*) = \delta f'(x_t^*) u'(c_{t+1}^*) \text{ for all } t \geq T.$$

This readily leads to $c_t^* < c_{t+1}^*$ for $t \geq T$. Hence, if $x_t \to 0$ (so that $f(x_t) \to 0$) we arrive at a contradiction for a sufficiently large t.

Clearly for any $\mathbf{x} < K_\delta^*$, the optimal input program $(x_t^*)_{t \geq 1}$ being monotone increasing must converge to some $\mathbf{x} < \hat{x} \leq K_\delta^*$. Again, by using the Euler condition (7.7) and taking limits, we have $\hat{x} = K_\delta^*$, by uniqueness of K_δ^*. To summarize complete the proof of $x > K_\delta^*$:

Theorem 7.5.5. *If $y'' > y'$, $x_t^*(y'') > x_t^*(y')$ for all $t \geq 1$. Hence, if $\mathbf{x} < K_\delta^*$, an optimal input program $(x_t^*)_{t \geq 1}$ from $\mathbf{x}$ satisfies $x_{t+1}^* > x_t^*$ for $t \geq 1$ and $\lim_{t \to \infty} x_t^* = K_\delta^*$. If $\mathbf{x} > K_\delta^*$, an optimal input program $(x_t^*)_{t \geq 1}$ from $\mathbf{x}$ satisfies $x_{t+1}^* \leq x_t^*$ for $t \geq 1$ and $\lim_{t \to \infty} x_t^* = K_\delta^*$.*

7.6 Discounting: The Linear Utility Function

We now touch upon the "intermediate" case of discounting, in the context of a linear utility function, i.e., $u(c) = c$, and refer to Majumdar and Mitra (1980, 1983) for complete proofs. A program $< x^*, y^*, c^* >$ from $\mathbf{x} > 0$ is then optimal if

$$\sum_{t=1}^{\infty} \delta^{t-1} c_t^* \geq \sum_{t=1}^{\infty} \delta^{t-1} c_t \tag{7.31}$$

for all programs $< x, y, c >$ from $\mathbf{x} > 0$. Recall that for any program $< x, y, c >$ from $\mathbf{x} > 0$, we have for all $t \geq 0$, $c_{t+1} \leq \hat{k}$ where $\hat{k} = \max(\mathbf{x}, \bar{k})$. Hence, by using Majumdar (1975, Theorem 1), one asserts that there exists an optimal program from any initial $\mathbf{x} > 0$. However, when the utility function is linear, an optimal program need not be positive; hence a 'routine' application of many of the arguments in Sections 7.4,7.5 is *not* admissible.

The program $< x, y, c >$ from $\mathbf{x} > 0$ defined as $x_0 = \mathbf{x}$, $x_t = 0$ for $t \geqq 1$ is the *extinction program* from $\mathbf{x} > 0$. Here, the entire output $f(\mathbf{x})$ is consumed in period one, i.e., $c_1 = f(\mathbf{x})$.

7.6.1 An Alternative Interpretation: A Competitive Fishery

An alternative interpretation of the model is that of a competitive fishery (see Clark (1976) Chapter 7). According to this interpretation, x_t is the stock of fish in period t; the function f is the biological reproduction relationship or the "stock recruitment" function. The sequence $< c >= (c_t)$ is the sequence of "harvests". Let the profit per unit of harvesting, denoted by $q > 0$ and the rate of interest $\gamma > 0$ remain constant over time. Consider a firm which has an objective of maximizing the discounted sum of profits from harvesting. A program $< x^* > = (x_t^*)$ of stocks from $\mathbf{x} > 0$ is optimal if

$$\sum_{t=1}^{\infty} \left[\frac{q}{(1+\gamma)^{t-1}} \right] c_t^* \geq \sum_{t=1}^{\infty} \left[\frac{q}{(1+\gamma)^{(t-1}} \right] c_t$$

for every program $< x >$ from $\mathbf{x}$. This is exactly the problem posed above if we set $\delta = 1/(1+\gamma)$. Models of this type have been used to discuss the possible conflict between profit-maximization and conservation of natural resources (see, e.g. Clark (1971), Spence (1975) and Dasgupta-Heal (1979, Chapter V)).

7.6.2 Characterization of Optimal Programs

In the qualitative analysis of optimal programs, the roots of the equation $\delta f'(x) = 1$ play an important role. This equation might not have a non-negative real root at all; if it has a unique non-negative real root, denote it by Z; if it has two non-negative real roots, the smaller one is denoted by z the larger by Z.

The qualitative behavior of optimal programs depends on the value of δ, the discount factor. Three cases need to be distinguished. The first two were analyzed (and, interpreted in the context of a model of profit maximizing fishery) by Clark (1971).

Strong Discounting: $\boldsymbol{\delta f'(b_2) \leq 1}$. This is the case when δ is "sufficiently small" ($\delta \leq 1/[f'(b_2)]$ - in the fishery example, $1 + \gamma \geq f(x)/x$ for all $x > 0$).

Theorem 7.6.1. *The extinction program is optimal from any* $\mathbf{x} > 0$, *and is the unique optimal program if* $\delta f'(\hat{b}_2) < 1$.

Remark 7.6.1. First, if $\delta f'(b_2) = 1$, there are many optimal programs [see Majumdar and Mitra (1983 p. 146)]. Secondly, if we consider the "classical" model [satisfying (A.1) - (A.4) and (A.5′)], it is still true that $\delta f'(0) \leq 1$, the extinction program is the unique optimal program from any $\mathbf{x} > 0$.

Mild Discounting $\boldsymbol{\delta f'(0) \geq 1}$. This is the case where δ is "sufficiently close to 1" ($\delta \geq 1/f'(0)$), and $Z > b_2$ exists (if z exists $z = 0$).

Now, given $\mathbf{x} < Z$, let M be the smallest positive integer such that $x^1_M \geq Z$ in other words, M is the first period in which the *pure accumulation* program from $\mathbf{x}$ (defined in the proof of Lemma 7.4.4) attains Z.

Theorem 7.6.2. *If* $\mathbf{x} \geq Z$, *then the program* $< x^*, y^*, c^* >$ *from* $\mathbf{x}$ *generated by* $x^*_0 = \mathbf{x}$, $x^*_t = Z$ *for* $t \geq 1$ *is the unique optimal program from* $\mathbf{x}$.

Theorem 7.6.3. *If* $\mathbf{x} < Z$, *the program* $< x^*, y^*, c^* >$ *generated by* $x^*_0 = \mathbf{x}$, $x^*_t = x^1_t$ *for* $t = 1, ..., M-1$, $x^*_t = Z$ *for* $t \geq M$ *is the unique optimal program.*

In the corresponding "classical" model of $\delta f'(0) > 1$, there is a unique positive K^*_δ solving $\delta f'(x) = 1$. Theorems 7.6.2 and 7.6.3 continue to hold with Z replaced by K^*_δ (also in the definition of M).

Two Turnpikes and the Critical Point of Departure [$\boldsymbol{(\delta f'(0) < 1 < \delta f'(b_2)}$]. In case (a) the program $< x, y, c >$ generated by $x_t = 0$ for all $t \geq 0$ (b) the Optimal Stationary Program generated by $x_t = Z$ for all $t \geq 0$ serve as the "turnpike" approached by the optimal programs. *Both* the classical and non-classical models share the feature that the *long run behavior* of optimal programs is *independent* of the positive *initial stock.* The "intermediate" case of discounting, namely when

$$1/f'(b_2) < \delta < 1/f'(0)$$

turned out to be difficult and to offer a sharp contrast between the classical and non-classical models. In this case

$$0 < z < b_1 < b_2 < Z < k^*$$

The qualitative properties of optimal programs, are summarized in two steps.

Theorem 7.6.4. *If* $\mathbf{x} \geq Z$*, the program* $< x^*, y^*, c^* >$ *generated by* $x_0^* = \mathbf{x}$*,* $x_t^* = Z$ *for* $t \geq 1$ *is optimal.*

A program $< x, y, c >$ from $\mathbf{x} < Z$ is a *regeneration program* if there is some positive integer $N \geq 1$ such that $x_t > x_{t-1}$ for $1 \leq t \leq N$, and $x_t = Z$ for $t \geq N$. It should be stresses that a regeneration program may allow for positive consumption in *all* period, and need not specify "pure accumulation" in the initial periods. For an interesting example of a regeneration program that allows for positive-consumption and is optimal, the reader is referred to Clark (1971, p. 259).

Theorem 7.6.5. *Let* $\mathbf{x} < Z$*. There is a critical stock* $K_c > 0$ *such that if* $0 < \mathbf{x} < K_c$*, the extinction program from* $\mathbf{x}$ *is an optimal program. If* $K_c < \mathbf{x} < Z$*, then any optimal program is a regeneration program.*

In the literature on renewable resources, K_c is naturally called the "minimum safe standard of conservation" (Clark (1971)). It has been argued that a policy that prohibits harvesting of a fishery the till stock exceeds K_c will ensure that the fisery will not become extinct, even under pure "economic exploitation".

Some conditions on $\mathbf{x}$ can be identified under which there is a unique optimal program. But if $\mathbf{x} = K_c$, then the extinction program and a regeneration program are both optimal.

7.7 A Multi-sector Non-convex Economy: The Undiscounted Case

In this section I turn to a model of an economy with many (n) goods, and sketch the principal results due to Mitra (1992) on the problem of optimal accumulation in the *undiscounted* case. Here the *technology set* Ω is a subset of $R_+^n \times R_+^n$. A pair (x, y) is in Ω if the output vector y is producible in period $t+1$ from the input vector x in period $t (\geq 0)$. A *production program* from $\mathbf{x} \,\epsilon\, R_+^n$ is a sequence (x_t, y_{t+1}) such that $x_0 = \mathbf{x}$, $(x_t, y_{t+1}) \,\epsilon\, \Omega$ for $t \geq 0$, $y_t \geq x_t$ for $t \geq 1$. It generates a *consumption program* (c_t) defined by $c_t = y_t - x_t$ for $t \geq 1$. Let $w : R_+^n \to R$ be the utility function. As before, a program $(x_t^*, y_{t+1}^*, c_{t+1}^*)$ from $\mathbf{x}$ is *optimal* if

$$\limsup_{T \to \infty} \sum_{t=0}^{T} [w(c_t) - w(c_t^*)] \leq 0$$

for all programs (x_t, y_{t+1}, c_{t+1}) from $\mathbf{x}$.

Some of standard assumptions on the technology Ω are:

(T.1) $(0,0) \epsilon \Omega$; "$(0,y) \epsilon \Omega$" implies "$y = 0$".

(T.2) $(x,y) \epsilon \Omega$, "$x' \geq x$, $0 \leq y' \leq y$" implies "$(x', y') \epsilon \Omega$".

(T.3) Ω is closed.

(T.4) There is $C > 0$ such that "$(x,y) \epsilon \Omega$, $\|x\| > C$" implies "$\|y\| < \|x\|$".

(T.5) There is $(\bar{x}, \bar{y}) \epsilon \Omega$ with $\bar{y} >> \bar{x}$.

These enable us to establish a useful boundedness property: if $< x, y, c >$ is a program from $\mathbf{x}$, then for all $t \geq 0$, $\|y_t\| \leq \max[\|\mathbf{x}\|, C]$.

To begin with, assume that the one period return function w satisfies:

(W.1) w is continuous and concave on R^n_+.

(W.2) If $c' \geq c \geq 0$, then $w(c') \geq w(c)$; if $c' >> c$, then $w(c') > w(c)$.

The principal objective and accomplishment of Mitra's paper is to identify a particular property of Ω that admits non-convexity and still allows one to extend the basic result on the existence of an optimal program (Theorem 7.4.3) and the turnpike theorem (Theorem 7.4.2). Interestingly enough the overall strategy of proof remains the same as the one I outlined in Section 7.4 [see the comments in Majumdar and Peleg (1992) on Mitra's paper]. Hence, I shall only indicate in some detail the point of departure of this chain of reasoning: a set of assumptions that guarantees the existence of a *golden rule equilibrium.*

A pair $(\hat{x}, \hat{y})$ in Ω is a *golden rule* if $\hat{y} \geq \hat{x}$ and $w(\hat{y} - \hat{x}) \geq w(y - x)$ for all (x, y) in Ω satisfying $y \geq x$. If $(\hat{x}, \hat{y})$ is a *golden rule* and if there is $\hat{p} \epsilon R^n_+$ such that

$$w(\hat{y} - \hat{x}) - \hat{p}(\hat{y} - \hat{x}) \geq w(c) - \hat{p}c \text{ for all } c \epsilon R^n_+$$

$$\hat{p}(\hat{y} - \hat{x}) \geq \hat{p}(y - x) \text{ for all } (x, y) \epsilon \Omega$$

then $\hat{p}$ is a *price support* of $(\hat{x}, \hat{y})$, and the triplet $(\hat{x}, \hat{y}, \hat{p})$ is a *golden rule equilibrium.*

Consider the set $A = \{(x, y) \epsilon \Omega : y \geq x\}$. Then A is (non-empty) compact. Hence $B = \{c : c = y - x,\ (x, y) \epsilon A\}$, is also (non-empty) compact. It follows that there is some $(\hat{x}, \hat{y})$ which is a golden rule and $w(\hat{y} - \hat{x}) \geq w(\bar{y} - \bar{x}) > w(0)$. To prove the existence of a *golden rule equilibrium*, additional assumptions on Ω and w are introduced. The technology set Ω is *quasi-star shaped* with respect to some golden rule $(\hat{x}, \hat{y})$ if "$(x, y) \epsilon \Omega$" implies that there is some $\lambda(x, y) \epsilon (0, 1]$ such that for $0 < \lambda < \lambda(x, y)$, $(\lambda x + (1 - \lambda)\hat{x},\ \lambda y + (1 - \lambda)\hat{y})$ is in Ω. Make two additional assumptions:

(T.6) [condition QS on Ω] The technology set Ω is quasi star-shaped with respect to some golden rule $(\hat{x}, \hat{y})$, and $\hat{y} >> \hat{x}$.

and

(W.3) [condition S on w] The utility function w is twice continuously differentiable on R^n_{++}.

We can then show:

Lemma 7.7.1. *Under assumptions (T.1) - (T.6), (W.1) - (W.3) there is* $\hat{p} \epsilon R^n_+$ *such that* $(\hat{x}, \hat{y}, \hat{p})$ *is a golden rule equilibrium.*

A slight strengthening of (T.6) has some strong implications. We say that the technology set Ω is *strictly quasi-star shaped* with respect to some golden rule $(\hat{x}, \hat{y})$ if "$(x, y) \epsilon \Omega$, $(x, y) \neq (\hat{x}, \hat{y})$" implies that there is $0 < \lambda(x, y) \leq 1$ such that for $0 < \lambda < \lambda(x, y)$, we have $(\lambda x(1-\lambda)\hat{x}, y') \epsilon \Omega$ with $y' >> \lambda y + (1-\lambda)\hat{y}$. Two remarks are in order. If Ω is strictly quasi-star shaped with respect to some golden rule $(\hat{x}, \hat{y})$, then $(\hat{x}, \hat{y})$ is the *only* golden rule. Secondly, the technology set Ω defined through the production function of Section 7.4 is strictly quasi-star shaped at the unique golden rule input-output pair $(k^*, f(k^*))$ in Section 7.4.1.

Define the *stationary* program $(x_t, y_{t+1}) = (\hat{x}, \hat{y})$ for all $t \geq 0$. It generates a *stationary* consumption program $(c_t = \hat{c} = \hat{y} - \hat{x})$. Clearly, $w(\hat{c}) > w(0)$.

As in Section 7.4, one introduces the notion of *good* programs: a program $(x_t, y_{t+1})_{t \geq 0}$ from $\mathbf{x}$ is good if there is some finite M such that

$$\sum_{t=1}^{T} [w(c_t) - w(\hat{c})] \geq M \qquad \text{for all } T \geq 1$$

One can prove the 'turnpike property' of good programs.

Lemma 7.7.2. *If $(x_t, y_{t+1})_{t \geq 0}$ is a good program from $\mathbf{x}$, then*

$$(x_t, y_{t+1}) \to (\hat{x}, \hat{y}) \text{ as } t \to \infty$$

The proof rests on a multi-sector version of the value-loss lemma of Section 7.4 (Lemma 7.4.2). See Mitra (1992, Proposition 8.1 and Lemma 8.2) for details.

Next, the existence of a good program is addressed to:

Lemma 7.7.3. *Assume that the golden rule pair $(\hat{x}, \hat{y})$ has the following property: for every $0 < \lambda < 1$, there exists $y_\lambda >> \hat{x}$ such that the line-segment $[(\lambda\hat{x}, \lambda y_\lambda), (\hat{x}, y_\lambda)]$ is contained in Ω.*

Then there is a good program from every $\mathbf{x} >> 0$.

For a proof see Majumdar and Peleg (1992).

The final two results on the existence of an optimal program and the "turnpike" property are now stated (under the assumptions that guarantee Lemma 7.7.1-7.7.3).

Theorem 7.7.1. *The stationary program*

$$x_t = \hat{x}, \;\; y_{t+1} = \hat{y}, \;\; c_{t+1} = \hat{c} \qquad \text{for } t \geq 0$$

is the optimal stationary program from $\hat{x}$.

Theorem 7.7.2. *For any $\mathbf{x} >> 0$, there is an optimal program (x_t^*, y_{t+1}^*) from $\mathbf{x}$. Since the optimal program is good,*

$$(x_t^*, y_{t+1}^*) \to (\hat{x}, \hat{y}) \quad \text{as } \; t \to \infty.$$

7.8 Optimal Allocation in a Small Open Economy

7.8.1 A Small Open Economy with Non-convexity

A few papers have dealt with the optimal allocation problem for a small open economy, in which a particular sector has an S-shaped technology. The patterns of trade and the process of capital accumulation can be worked out, and contrasted with "classical" models. In this section I briefly recall somewhat informally the main conclusions of Majumdar and Mitra (1995) after outlining the formal model.

The optimization problem is described by $(f, G, \delta, u, \mathbf{k}, p)$ where f and g are functions from $R_+ \to R_+$, $0 < \delta < 1$, $u(c) = c^{1-\nu}$ [where $0 < \nu < 1$], $\mathbf{k} > 0$ and $p > 0$. The technology of the "capital" or "investment" good sector of our economy is described by the production function $f : R_+ \to R_+$ satisfying
(T.1) $f(0) = 0$, f is twice continuously differentiable on R_+, with $f'(x) > 0$ for $x \geq 0$.
(T.2) f satisfies the end-point conditions: $\lim_{x\to\infty} f'(x) = 0, \lim_{x\to 0} f'(x) > 1$.
(T.3) There is a (finite) $b_1 > 0$ such that $f''(x) < 0$ for $x > b_1$.

The technology of the "consumptions good" sector is described by G, and assume simply that
(T.4) $G(x) = \alpha x$ for some $\alpha > 0$.

The economy is "small": it faces a fixed "world" or international prices for the two goods, and $p > 0$ denotes the relative (international or world) price of the investment good (in terms of the consumption good) assumed *constant over time.*

A program from $\mathbf{k} > 0$ is a non-negative sequence $(k_t, x_t, c_{t+1})_{t\geq 0}$ such that

$$\begin{aligned} k_0 &= \mathbf{k},\ 0 \leq x_t \leq k_t \quad for\ \ t \geq 0 \\ c_{t+1} &\geq 0 \\ p\,k_{t+1} + c_{t+1} &= pf(x_t) + G(k_t - x_t) \quad for\ \ t \geq 0 \end{aligned}$$

Define an *autarkic program* from $\mathbf{k}$ as a program from $\mathbf{k}$ which satisfies

$$c_{t+1} = G(k_t - x_t) \quad for\ \ t \geq 0$$

A *program* $(k_t^*, x_{t+1}^*, c_{t+1}^*)$ from $\mathbf{k} > 0$ is *optimal* if

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t^*) \geq \sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$$

for all programs (k_t, x_t, c_{t+1}) from $\mathbf{k} > 0$. Similarly an autarkic program $(\tilde{k}_t, \tilde{x}_t, \tilde{c}_{t+1})$ from $\mathbf{k} > 0$ is an *optimal autarkic* program if

$$\sum_{t=1}^{\infty} \delta^{t-1} u(\tilde{c}_t) \geq \sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$$

for every autarkic program $(k_t, x_t, c_{t+1})_{t\geq 0}$ from $\mathbf{k}$.

7.8.2 Interpretation

The economy has an initial capital stock $\mathbf{k}$. For simplicity of analysis, it is assumed that the capital good depreciates fully within one period. So $\mathbf{k}$ is allocated [as an input] between the two sectors: $x_0 \geq 0$ being the capital good used in the investment good sector, and $(k_0 - x_0) \geq 0$ the capital good used in the consumption good sector. As a result, the pair $(f(x_0), \alpha(k_0 - x_0))$ describes the stocks of two goods in period 1. At the international prices, the *income* i_1 in period 1 is defined by:

$$i_1 \equiv p\, f(x_0) + \alpha(k_0 - x_0) \tag{7.32}$$

This income is spent on acquiring any non-negative quantities (k_1, c_1) of the two goods satisfying the constraint

$$p\, k_1 + c_1 = i_1 \tag{7.33}$$

Of course, if $k_1 > f(x_0)$, the country is a *net importer* of the capital good whereas if $k_1 < f(x_0)$ is a *net exporter* of that good. Similarly the choice of c_1 determines whether the consumption good is exported or imported. And writing (7.32) and (7.33) as:

$$p[k_1 - f(x_0)] + [c_1 - \alpha(k_0 - x_0)] = 0 \tag{7.34}$$

one notes the balance of trade condition: *the value of exports must equal the value of imports in every period.*

The consumption good generates a utility $u(c_1) \equiv c_1^{1-\nu}$, and the capital good k_1 must be allocated between the two sectors: this allocation in period 1 determines the domestic productions of the two goods, as well as the income i_2 in the subsequent period 2, and the story is repeated.

Thus, a program is a complete specification of the sequence of decisions on the allocation $(x_t, k_t - x_t)$ of the available capital k_t between two sectors as well as the decisions on spending the available income to acquire the two goods (k_{t+1}, c_{t+1}). As a result of these decisions [and the condition (7.34) on the balance of trade], the pattern of trade in each period is also completely specified.

If a country is *not* allowed to trade, it must meet its consumption from domestic production, so that $c_{t+1} = \alpha(k_t - x_t)$ for all $t \geq 0$; and this also means that $k_{t+1} = f(x_t)$, the quantity that is domestically produced as a result of using x_t in the capital good industry. This is the description corresponding to our formal definition of an autarkic program. As usual given the discount factor $\delta(0 < \delta < 1)$, the objective is to maximize the discounted sum of one period utilities derived from consumption.

As in the previous section, $f(\bar{k}) = \bar{k}$, and now, in this section, let $b_2 > 0$ $(b_1 < b_2 < \bar{k})$ is defined by $f'(b_2) = h(b_2)$ [where $h(x) \equiv f(x)/x$, $x > 0$].

7.8.3 Escape from the Poverty Trap

Consider first the situation in which all intertemporal choices are restricted to autarkic programs (the economy is closed). It should come as no surprise that - depending on the value of δ - the closed economy may face a 'poverty trip'.

Theorem 7.8.1. *Let $\delta f'(0) < 1$; there is a critical stock $\mathbf{k}_c > 0$ such that if $0 < \mathbf{k} < \mathbf{k}_c$ then for any optimal autarkic program $(\tilde{k}_t, \tilde{x}_t, \hat{c}_{t+1})_{t\geq 0}$ from $\mathbf{k}$ one has:*

$$\lim_{t\to\infty} \tilde{k}_t = 0, \lim_{t\to\infty} \hat{x}_t = 0, \lim_{t\to\infty} \tilde{c}_{t+1} = 0 \tag{7.35}$$

Let us turn to the open economy, and write

$$\beta \equiv \alpha/p$$

To see the contrast between the closed and the open economy, assume

(E.1) $f'(0) < 1/\delta, \quad \delta/\beta > 1$

When $\beta > h(b)$, to guarantee the existence of an optimal program it is assumed that

(E.2) $\delta\beta^{1-\nu} < 1$

A complete analysis of the optimization problem requires some detailed calculations and subtle steps. Two important cases need to be considered separately. First, let $\beta > h(b)$. In this case, one can summarize the main results informally: When $\beta > h(b)$, the optimal policy for the open economy is not to use the domestic investment good industry at all ($x_t^* = 0$ for all t) no matter what $\mathbf{k} \,\epsilon\, (0, \bar{k})$ is. The optimal capital stock and consumption grow (to infinity) exponentially:

$$\begin{aligned} [k_{t+1}^*/k_t^*] &= (\delta\beta)^{1/\nu} \quad for \quad t \geq 0 \\ c_{t+1}^* &= p[\beta - (\delta\beta)^{1/\nu}]k_t^* \quad for \quad t \geq 0 \end{aligned}$$

thus, in all periods, the country exports the consumption good and imports the investment good.

Next, let $\beta < h(b)$. In this case, from any initial $\mathbf{k} > 0$, the optimal sequences (k_t^*) and (c_t^*) of capitals and consumptions increase to infinity. One can identify three stages through which a capital-poor economy will develop. In the first stage, when the initial capital belongs to an interval $(0, A)$, it does not use the capital good industry at all, but produces and exports the consumption good only. Next, the optimal program displays a reversal of trade pattern as the initial stock moves beyond A. Now the consumption good sector is not used at all, and all the input is allocated to the production of the capital good which is exported. At a higher level of initial stock $B(> A)$, it becomes optimal to produce both goods, although for a while the capital good continues to be exported. Finally, there is a threshold where both industries are used, but the capital good industry reaches a maximum size. The consumption good industry continues to grow (given our assumptions of constant returns) and it becomes the export good for "financing" the growing needs for capital inputs.

The admittedly special structure enables us to 'draw' a clear picture of the export-import patterns and optimal specialization at different stages of development. It also provides a sharp contrast between small 'autarkic' and 'open' economies.[5]

7.9 Some Concluding Comments

I have no reason to believe that the results reviewed in this chapter will satisfy a reader looking for a formal account that capture's Young's vision of economic progress. Capturing varied experiences of societies at different stages of transition or suggesting optimal long run development strategies through tractable models are unenviable tasks. The widely circulated report (1993) on one of the most fascinating stories of successful growth tells us that "there is no single East Asian model." Recognizing and incorporating one important feature (non-convexity) makes the inadequacies of other postulates starker. The limits of deterministic models of an "isolated" [no trade] economy with a time-invariant technology to come to the aid of a policy maker dealing with contemporary problems are perhaps too obvious to harp on.

While writing the review, I revisited parts of the literature that influenced the tone of discussion on major issues of growth and development economics in the sixties. A large part was "pre Newtonian" in analytical approach and often failed to go beyond comparisons of possible equilibria. I am invariably reminded of Koopmans (1957) who noted that very often "the wealth of suggestive terms" appears perceptive and insightful partly because "words are cheap" [p.179], and find myself in substantial agreement with his observations (p. 142):

"The theories that have become dear to us can very well stand by themselves as an impressive and highly valuable system of deductive thought, erected on a few premises that seem to be well-chosen first approximations to complicated reality. In many cases the knowledge these deductions yield is the best we have, either because better approximations have not been secured at the level of the premises, or because comparable reasoning recognized as more realistic has not been completed or has not yet been found possible. Is any stronger defense needed, or even desirable?"

[5] Dasgupta (1998) and Ossella-Durbal (2002) have explored possible extensions of the model in several directions.

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8. Isotone Recursive Methods: The Case of Homogeneous Agents

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8.1 Introduction

A foundation of modern macroeconomics is the stochastic growth model originally introduced in the seminal work of Brock and Mirman[16]. Their original model is an infinite horizon economy with a continuum of identical households, each with access to a complete set of financial markets that insure them against all sources of idiosyncratic risk. There is single sector production that employs capital and labor whose returns are summarized by a stochastic neoclassical production function representing an aggregate convex production set with identical private and social returns to inputs. There is also aggregate risk taking the form of a collection of identically and independently distributed (i.i.d.) random variables, the agents in the economy face no frictions in information acquisition (i.e., there is no learning), labor supply is inelastic, and there are no equilibrium distortions. The authors characterize the unique *Markovian Equilibrium Decision Process* (MEDP) and its associated unique (non-trivial) long-run equilibrium dynamics, in particular, the *Stationary Markovian Equilibrium* (SME). Their methodological approach was pioneering, and relied heavily on recursive methods. Implicitly, it exploits the validity of a second welfare theorem and one can interpret the economic outcomes of the fictional social planner's problem

[1] We are deeply indebted to Len Mirman and Olivier Morand for numerous lengthy discussions concerning many issues discussed in this survey. Many of the results presented in this paper were developed originally in some form during our joint work with Len and Olivier over the last five years. We dedicate this paper to Len Mirman on the occasion of his sixty-fifth birthday. Indeed, this paper would not have been written without Len's ongoing pioneering work on equilibrium growth under uncertainty. We also thank Elena Antoniadou, Hector Chade, John Coleman, Jeremy Greenwood, Seppo Heikkilä, Ken Judd, Tom Krebs, Cuong Le Van, Robert Lucas, Jr., Jianjun Miao, Chris Shannon, John Stachurski, Yiannis Vailakis, Charles Van Marrewijk, Jean-Marie Viaene, Itzhak Zilcha, and especially Robert Becker and Manuel Santos for many helpful conversations over the past years. All mistakes remain our own.

from the perspective of a decentralized economic system. A fully decentralized recursive formulation of the Brock-Mirman framework is put forward by Prescott and Mehra[66] (see also, Stokey, Lucas, with Prescott[76]).

Over the last three decades, extensions of this model have become the foundation for the systematic study of many diverse issues in quantitative dynamic macroeconomic theory. Applications include models of economic fluctuations and business cycles, production-based asset pricing, the positive and normative implications of incomplete financial markets and public goods, the wealth inequality, the dynamic structure of altruistic economies, stochastic life-cycle models, models with physical and human capital, and the role of activist fiscal and/or monetary policy etc. However, many recent applications emphasize economic environments where the second welfare theorem is *not* available. These modifications create serious complications for a systematic study of the underlying structure of the MEDPs and the SME. A prevalent approach is to develop extensive applications of numerical methods to characterize MEDPs and the SME. From a mathematical perspective, many of these approaches have been *ad hoc* as they cannot be developed rigorously without providing characterizations of qualitative structure of the MEDPs and/or the SME.

An important question naturally emerges from this apparent disconnect between mathematical principle and macroeconomic practice: can one provide sharp and *constructive* characterizations of the MEDPs or the SME for generalized Brock-Mirman environments where the second welfare theorem fails? The most significant advance in providing an affirmative answer to this question has been the recent literature on "monotone methods" (also known as "monotone map" methods or "isotone recursive methods"). The pioneering work of Coleman [18][19][20][21], Greenwood and Huffman[34], Datta, Mirman, and Reffett[22] and Morand and Reffett[62] provide the genesis of the study of isotone recursive methods over the last fifteen years (they refer to them, as the "monotone-map" method). These papers present the first set of conditions under which constructive methods can be applied for studying the structure of a decentralized Markovian equilibrium in economies with or without non-classical production technologies.[2] An important generalization of this monotone-map approach is found in Mirman, Morand, and Reffett[59]. Here, a new and more general isotone map approach is presented (with the Coleman-Greenwood-Huffman approach as a special case) and can be applied to a larger collection of dynamic economies with production nonconvexities (in the reduced-form production function). In this setting, sets of sufficient conditions for the existence of semicontinuous, continuous, Lipschitz continuous, and

[2] The literature on monotone map methods is vast, and also includes the papers of Lucas and Stokey [55], Bizer and Judd [14] etc. An interesting alternative monotone method is developed in Becker and Foias [9].

For non-existence of a continuous MEDP, see Santos [73] and Krebs [50]. Mirman, Morand, and Reffett [59] show that although the Santos [73] example is robust to a large class of economies, in many case MEDPs are semi-continuous and isotone.

once-differentiable MEDPs are given. Since sufficient conditions for MEDPs to be differentiable are presented, therefore the error bounds constructed in Santos and Vigo[74] and Santos[72] apply. Finally a theory of *ordered* MEDPs is developed applying the seminal work in operations research on lattice programming and the qualitative study of equilibrium introduced in Veinott[82][83] and Topkis[79][80][81].

The chapter is organized as follows: in the next section, we introduce some useful terminology. Section 8.3 provides a survey of the existing literature on fixed point theory in order spaces. This fixed point theory is critical in the development of isotone recursive methods. In Section 8.4, we consider homogeneous agent economies with classical production technology and infinite horizon. In this section, we develop an "Euler equation" approach to isotone recursive methods. We discuss the case studied in Coleman[19] for nonoptimal homogeneous agent economies. In Section 8.5, we discuss the generalizations found in Mirman, Morand and Reffett[59]. Section 8.6 considers the case of elastic labor supply as in Coleman[20] and Datta, Mirman and Reffett[22]. In section 8.7, we conclude with a brief discussion of new frontiers in monotone recursive methods, to models with heterogeneous agents including the overlapping generations models with stochastic production (e.g., Erikson, Morand and Reffett[31]), models with unbounded stochastic nonoptimal growth (e.g., Morand and Reffett[62]), Ramsey-type models with heterogeneous agents (e.g., Datta, Mirman, Morand and Reffett[23], and the mixed monotone recursive methods discussed in Reffett[68] and Mirman, Reffett and Stachurski[60].

8.2 Preliminaries

8.2.1 Ordered Spaces

We begin with some useful terminology. For a more complete accounting of the ideas in this section, see Birkhoff[13], Veinott[83], and Davey and Priestley[25].

Posets. In our subsequent discussion, we shall respect two notational conventions: (i) we write "$\geq$" in place of "$\geq_X$" when the order relation $\geq_X$: $X \times X \to X$ is clearly implied; and (ii) for two elements of X, say a and b, the order relation "$a \geq b$" can also be written as "$b \leq a$". Let X be a set. We say X is *partially ordered set (or Poset)* if X is equipped with an order relation $\geq_X$: $X \times X \to X$ that is reflexive, antisymmetric and transitive. If every element of a poset X is comparable, then we say X is a *totally ordered set* or *chain.* Every chain has an inherent lattice structure.

Lattices. Let X be a poset equipped with a partial order $\geq$. An *upper (*respectively, *lower) bound* for a set $B \subset X$ is an element x^u(respectively, x^l) $\in B$ such that for any other element $x \in B$, $x \leq x^u$ (respectively, $x^l \leq x$) for all $x \in B$. If there is a point x^u (respectively, x^l) such that x^u is the least element in the subset of upper bounds of $B \subset X$ (respectively, the greatest element in the subset of lower bounds of $B \subset X$), we say x^u (respectively,

$x^{;}$) is the *supremum* (respectively, *infimum)* of B. Clearly if they exist, both the supremum (or, sup) and infimum (or, inf) must be unique. We say X is a *lattice* if for any two elements x and x' in X, X is closed under the operation of infimum in X , denoted $x \wedge x'$, and supremum in X, denoted $x \vee x'$.The former is referred to as "the meet", while the latter is referred to as "the join" of the two points, $x, x' \in X$. A subset B of X is a *sublattice* of X if it contains the sup and the inf (with respect to X) of any pair of points in B. A lattice is *complete* if any subset B of X has a least upper bound $\vee B$ and a greatest lower bound $\wedge B$ in B. If this completeness property only holds for countable subsets X_c, the lattice is $\sigma-$*complete.* If every chain $C \subset X$ is complete, then X is referred to as a *chain complete poset (or equivalent, a complete partially ordered set* or *CPO).* A set C is *countable* if it is either finite or there is a bijection from the natural numbers onto C. If every chain $C \subset X$ is countable and complete, then X is referred to as a *countably chain complete poset.* Finally, a subset A of a set $C \subset P$ is *cofinal* if for each $x \in C$, there is a $y \in A$ such that $x \leq y$.

Ordered Vector Spaces and Cones. A *partially ordered vector space* or *linear semi-ordered space* is a poset X that is real vector space equipped with a partial order $\geq$ that is compatible with the following algebraic structure: (i) if $x \geq x'$, then $x + z \geq x' + z$, for all $z \in X$; (ii) if $x \geq x'$, then $\alpha x \geq \alpha x'$ for all $\alpha \geq 0$. Any partially ordered vector space that is also a lattice is called a *vector lattice.* If the space has a norm $\| x \|_X$ which satisfies whenever $| x | \geq | x' |$ in X, $\| x \| \geq \| x' \|$, we say X has a *lattice norm.* A complete normed vector space is a *Banach space.* A *normed vector lattice* is a vector lattice equipped with a lattice norm. A normed vector lattice X that is complete in the Cauchy sense, and is endowed with a lattice norm is referred to as a *Banach lattice.*

Let X be a topological space. The set $X^+ = \{x \in X, \ x \geq 0\}$ is the *order cone* of X if X is nonempty convex closed set that has the following two properties: *(i)* $x \in X^+ \Longrightarrow \alpha x \in X^+$ for $\alpha \geq 0$; *(ii)* if x and -x in $X^+, x = 0$ where 0 denote the zero of the cone. The partial order induced by the cone structure of X^+ has $x_1 \geq x_2$ if $x_1 - x_2 \in X^+$. Now, assume X is a real Banach space. A cone X^+ of X is *normal* if there exists a constant m such that for any $x_1, x_2 \in X^+, \| x_1 + x_2 \| \geq m, \| x_i \| = 1$ for $i = 1, 2$. Intuitively, the restriction of normality of the cone geometrically bounds the angle between any two unit vectors away from π, so a normal cone cannot become "too large". An increasing sequence in the cone $\{x_t\}_{t=1}^{t=\infty}$, $x_t \in X^+$ is a sequence that satisfies $x_1 \leq x_2 \leq ... \leq x_n \leq ...$ We say a cone X^+ is *regular* if if every increasing and bounded order sequence in X^+ has a limit in X^+. We say X^+ is *fully regular* if every increasing and norm bounded sequence in X^+ has a limit in X^+. A fully regular cone is also regular. A regular cone is normal. (See Guo and Lakshmikantham[35], Theorem 1.2.1). A cone X^+ is *solid* if its interior $\mathring{X}^+$ is nonempty.

Let $[a) = \{x | x \in X, x \geq a\}$ be the *upperset* of a, $(b] = \{x | x \in X, x \leq b\}$ the *lowerset* of b. X is an *ordered topological space* if X is equipped with a partial order and topology that implies $[a)$ and $(b]$ are closed in the topology on X. An *order interval* is defined to be $[a, b] = [a) \cap (b]$, $a \leq b$. Therefore, in an ordered topological space, all $[a, b] \subset X$ (e.g., order intervals) are closed in the topology of X. In our work, we often study fixed point problems where the domain/range is a compact, order interval in a normal and solid cone of positive continuous functions $X^+ = C^+(S)$ endowed with the C^0 uniform norm topology (where each function itself is defined on compactum S). Such a space is not a regular cone. We will often work on a transformation space that is a compact suborder interval in $C^+(S)$ (where compactness will be used to compensate for the loss of regularity in the cone $C^+(S)$).

8.2.2 Mappings

We now define some important properties of mappings, especially those defined on lattices and posets:

Isotone (or Order Preserving) Mappings on a Poset: Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be Posets. A *mapping* is a relational statement between two spaces, say X and Y. We consider both "point-to-point" and "point-to-set" mappings. In the case of a "point-to-point" mapping, we refer to the mapping as a *function (*or equivalently as an *operator*). A function $m : X \to Y$ is said to be *isotone* on X if it is "order-preserving", i.e., $m(x') \geq_Y m(x)$, when $x' \geq_X x$, for $x, x' \in X$. If $m(x') >_Y m(x)$ when $x' >_X x$ for x, $x' \in X$, we say the function m is *increasing.* If $m(x') >_Y m(x)$ when $x' \geq_X x$, $x' \neq x$, we say the function m is *strictly increasing.* We say $m(x)$ is *antitone* (or, *order-reversing*) if $m(x) \geq_Y m(x')$ if $x' \geq_x x$. A function that is either isotone or antitone is *monotone.* When the mapping $m(x)$ is a self-mapping on X, we also refer to $m(x)$ as a *transformation* of X, and the set X as a *transformation set.* If our concern is the fixed points of a transformation $m(x)$ on X, we refer to the transformation set X as the *fixed point space.*

Notions of monotonicity are also available for multifunctions or correspondences. By a *correspondence or multifunction,* we always refer to a nonempty-valued mapping $M : X \to 2^Y$, e.g., a nonempty-valued "point-to-set" mapping. We say a correspondence or multifunction is *ascending* in the set relation S (denoted by $\geq_S$) if $M(x') \geq_S M(x)$, when $x' \geq_X x$ where $(X, \geq_X)$ is a partially ordered space. If this set relation $\geq_S$ induces a partial order on the powerset 2^Y (or, perhaps, $2^Y \backslash \emptyset$), we refer the ascending correspondence also as an *isotone correspondence.*

To make concrete the notion of an isotone versus ascending correspondence, we discuss some particular set relations; some that induce partial orders on 2^Y (or, $2^Y \backslash \varnothing$), others that do not.[3] The set relations we consider are each compatible with pointwise set comparisons, and, therefore, closely related to the sufficient conditions under which correspondences admit isotone selections. We focus primarily on four such set relations. Let Y be a set, and $A, B \in 2^Y$. We define : (i) the *Veinott-Weak Set relation* $\geq_w$on $2^Y \backslash \varnothing$: $A \geq_w B$, if for any

[3] For a more detailed discussion, we refer to the classic references of Smithson [75] and Veinott [83].

$a \in A$, $b \in B$, either $a \wedge b \in B$, or, $a \vee b \in A$; (ii) the *Veinott-Strong Set Order* $\geq_s$ on $2^Y \backslash \varnothing : A \geq_a B$, if for any $a \in A$, $b \in B$, $a \wedge b \in B$ and $a \vee b \in A$; (iii) the *Smithson−Weak Set relation* $\geq_{as}$on $2^Y : A \geq_{as} B$ if we have either (C1) for any $b \in B$, there exists an $a \in A$ such that $a \geq b$; or, (C2) for any $a \in A$, there exists an $b \in B$ such that $a \geq b$; (iv) the *Pointwise Strong Set Order* $\geq_{ss}$on $2^Y \backslash \varnothing : A \geq_{ss} B$ if and only if $a \in A$, $b \in B$, then $a \geq b$ in the partial order structure on A, for all a, b.A final classic partial order on the powerset 2^Y is commonly referred to as set inclusion. We say a subset $A \geq_{SI} B$ under *set inclusion* $\geq_{SI}$ if $B \subset A$.

Fixed points. Let $\mu : X \to 2^X$ be a non-empty valued correspondence for each $x \in X$. The correspondence μ is said to have a *fixed point* if there exists an x such that $x \in \mu(x)$. Therefore, if μ is a function, then a fixed point is an x^* such that $x^* = \mu(x^*)$. A fixed point x^* is *minimal (*respectively, *maximal)* if there does not exist another fixed point, say y^*, such that $y^* \leq x^*$ (respectively, $x^* \leq y^*$). If a fixed point is either minimal or maximal, we say it is *extremal.*

8.3 Fixed Point Theory in Ordered Spaces

In this section, we provide an account of fixed point theory in ordered spaces. For a more extensive discussion, see excellent surveys in Amann[4], Guo and Lakshmikantham[35], Veinott [83], Heikkilä and Lakshmikantham[39] and Jachymski[44].

8.3.1 Existence

First, we discuss the existence and characterization of solutions for two prototypical classes of parameterized fixed point (or, transformation) problems often encountered in economic applications. Consider X is a poset, T is an ordered topological space. The two problems are stated as Problem 1 and Problem 2.

Problem 1: To characterize the fixed points of the mapping,

$$f(x,t) : X \times T \to X \text{ and } f \text{ is isotone on } X \text{ for all } t \in T.$$

Problem 2: To characterize the fixed points of the mapping,

$$F(x,t) : X \times T \to 2^X, F \text{ is ascending } (\geq_{as}) \text{ in (C1) or (C2) on } X \text{ for all } t \in T,$$

Recall that $\geq_{as}$ denotes Smithson's weak set relation on the powerset 2^X. We denote the fixed point correspondence, in either Problem 1 or Problem 2, as $G(t)$.

Lattice Theoretic Fixed Point Theorems. A classical case of Problem 1 occurs when X is a nonempty, complete lattice. This is the case studied in the seminal work of Tarski[77][78] in the early 1940s, see also Kantorovich[47].[4] We say a space Y has a *fixed point property* for isotone functions (or, more compactly, *fpp)* if and only if each isotone transformation of Y, say $f : Y \to Y$, has a fixed point. We state Tarski's theorem adapted to Problem 1:

Proposition 8.3.1. *(Tarski[78], Theorem 1): Fix $t \in T$, and let the mapping $f(x,t) : X \times T \to X$ be isotone in x for each t, where X is a complete lattice. Then the fixed point correspondence $G(t)$ is a nonempty, complete lattice for each $t \in T$.*

We make a few remarks on this result. First, the theorem does not say $G(t)$ is subcomplete in X . In general, it is not. Second, the operator f is assumed to have no continuity properties on X (e.g., we assume no order or topological continuity properties for $f(x,t)$).

Often in economic applications, because of the absence of sufficient concavity in the agent's decision problem along equilibrium trajectories, equilibrium fixed point problems cannot be posed in terms of a single valued operator such as in Problem 1; rather, they must be posed in a more abstract setting of the fixed point of multifunctions, as in Problem 2. For the general case, a key generalization of Tarski was obtained by Veinott[82] in the 1970s, see also Veinott[83] (Chapter 4, Theorem 14).[5]

Proposition 8.3.2. *(Veinott[83]):Let $F(x,t) : X \times T \to 2^X \backslash \varnothing, X$ be a complete lattice, T be a set. For any fixed $t \in T$, assume that $F(x,t)$ is a nonempty, isotone in Veinott's strong set order, closed, and sublattice-valued correspondence on X. If $G(t)$ is the fixed point correspondence for $F(x,t)$ at $t \in T$, then $G(t)$ is a nonempty complete lattice for each $t \in T$.*

Propositions 8.3.1 and 8.3.2 provide sufficient conditions for the existence of a complete lattice of fixed points for an isotone and/or ascending transformations of a complete lattice X. An interesting question is necessity: i.e., can one obtain a complete characterization of a complete lattice using the *fixed point property*? Davis[26] (Theorem 1) provides the converse to Tarski's theorem: a lattice X is complete if and only if every isotone transformation $f : X \to X$ has a fixed point. In the context of Problem 2, the Davis characterization of a complete lattice X is also provided. Smithson[75] (corollary 1.8) proves the following: if X is a lattice and $F(x)$ is a multifunction then X is complete if and only if the correspondence $F(x)$ is (a) ascending in the Smithson-weak set relation (C1) (respectively, ascending in the Smithson-weak set relation (C2)),

[4] Tarski's original result dates from around 1942 and is available in Tarski [77]. It is a generalization of a result he developed with Knaster in 1921 (for isotone correspondences under set inclusion). A related result for semi-ordered linear spaces is in Kantorovich [47].

[5] Zhou [85] proves it independently in Theorem 1.

and (b) the least upper bound $(F(x,t)) \in F(x,t)$ (the greatest lower bound $(F(x,t)) \in F(x,t)$) for all $x \in X, t \in T$, and $G(t)$ is nonempty for each $t \in T$.

Other useful characterizations of complete lattices are available, and we use them in the sequel, as needed. For example, one can characterize a complete lattice X in terms of its interval topology (Frink[32]). Recall, the *interval topology* for a set X takes all the closed intervals $[a,b]$ as a subbasis for the closed sets of X. Frink[32] provides the following characterization of a complete lattice X: X is a complete lattice if and only if X is compact in its interval topology (see also Birkhoff [13], Chapter 10, Theorem 20). Another very useful characterization of a complete lattice is in Davey and Priestly[25] (Theorem 2.31). Their result provides the following characterization of a complete lattice X : let X be a nonempty ordered set; then the following statements are equivalent (i) X is a complete lattice; (ii) for any subset $S \subset X$, $\inf(S) \in X$; and X has a top element and $\inf(S) \in X$ for every nonempty subset of X. These two characterizations of a complete lattice X are used repeatedly in this chapter.

Fixed Point Theory in Complete Partially Ordered Sets. Next, we now consider Problems 1 and 2 when the fixed point space X is *not* a complete lattice. A natural set of regularity conditions for an ordered set X to have the fixed point property turns out to be chain-completeness. Recall a set X is chain complete if for any chain C, $\inf(C)$ and $\sup(C)$ are in X. A set X has a *bottom element* a (respectively, *top element* b) if for every $x \in X$, $a \leq x$ (respectively, $x \leq b$). A set X is a *complete partially ordered set* (or, *CPO*) if and only if (i) X has a bottom element, and (ii) for each directed net $D \subset X$, we have a sup $D \in X$. A set X is a CPO if and only if every chain C in X has a least upper bound, $\sup(C) \in X$. (Davey and Priestley[25], Theorem 8.11). Therefore, the notion of a set X being "chain-complete" is equivalent to the space X being a CP0. We often use this terminology when discussing chain-completeness.

Chain completeness is a natural condition to check in applications. For example, every relatively compact chain C in an ordered topological space has an infimum and a supremum, $\inf(C)$ and $\sup(C)$. See Amann[4], Lemma 3.1. Therefore, every compact, ordered topological space is chain complete (Amann[4], Corollary 3.2). One of the earliest results on the existence of a fixed point for a self map on a poset is obtained in Bourbaki[15]. As a consequence of Zorn's lemma, it is shown that if X is an ordered set such that every chain has an upper bound (respectively, a lower bound), and $f(x)$ on X is increasing in the following sense: for all $x \in X$, $x \leq f(x)$ (respectively, $f(x) \leq x$), then f has at least one fixed point. An improvement on this result is given in Abian and Brown[1] (Theorems 2,3,4) and Pelczar[65].The version of the theorem that we state is due to Amann[4] (see also Zeidler [86], Section 11.9 for a proof):

Proposition 8.3.3. *(Amann[4], Theorem 1.4): Let X be a CPO, $f(x,t) : X \times T \to X$ be isotone in X for each $t \in T$. Suppose that for each $t \in T$, there exists a pair $(x_L(t), x^U(t)) \in X \times X$, $x_L(t) \leq x^U(t)$ such that $x_L(t) \leq f(x_L(t),t)$*

and $f(x^U(t),t) \leq x^U(t)$*. Then* f *has a minimal and a maximal fixed point in* $[x_L(t), x^U(t)]$.

We next consider a converse to this theorem.[6] That is, as in the case of a complete lattice, we ask if one can obtain a characterization of a CPO X using the fixed point property relative to isotone transformations. Clearly, an arbitrary ordered set X does not have a fixed point property; but it turns out that if X is an ordered set, and for each isotone operator $f(x)$ on X, $f(x)$ has a least fixed point, then X is a CPO. Alternatively, if G is the set of fixed points of an isotone self-map $f(x)$, and X is a CPO, then G is a CPO. A collection of remarkable necessity results are found in Markowsky ([56], Theorems 9-11); we present the following result of Markowsky's that is summarized in Davey and Priestley [25]:

Proposition 8.3.4. *(Davey and Priestley[25], Propositions 8.25, 8.26). Let* X *and fix* $t \in T$*. Then we have the following: (i) if every isotone map in* $X, f(x,t) : X \times T \to X$*, has a least fixed point* $x^*(t)$*, then* X *is a CPO; (ii) if* $f(x,t) : X \to X$ *is isotone on* X *for each* $t \in T$*, and* $G(t)$ *denotes the set of fixed points of* $f(x,t)$*at* t*; then if* X *is a CPO,* $G(t)$ *is a CPO.*

Next, we discuss generalizations of Proposition 8.3.3 to the case of multifunctions. Smithson[75] and Muenzenberger and Smithson[64] are seminal references. Let $\mathbf{X}$ and $\mathbf{Y}$ be CPOs, $F(x) : \mathbf{X} \to \mathbf{2}^{\mathbf{Y}} \backslash \varnothing$ be a nonempty correspondence, and $X \subset \mathbf{X}$ a subchain. If for any isotone function $f(x) : X \to \mathbf{Y}$ such that $f(x) \in F(x)$, for $x_0 = \sup X$, we have $f(x_0) \leq y(x_0) \in F(x_0)$, we say the mapping $F(x)$ has the property of *Majorizing Chain Subcompleteness (MCSC).* For correspondences that are ascending in Smithson's weak set relation (C1) or (C2), and that satisfy MCSC, we have the following generalization of Amann[4]:

Proposition 8.3.5. *(Smithson[75], Theorem 1.1): Let* X *be a CPO, and suppose* $F(x,t)$ *is isotone in the Smithson-weak set relation, (C1) and/or (C2), and satisfies Condition MCSC. If for each* $t \in T$ *there is a point* $x_L(t) \in X$ *and a point* $y \in F(x_L,t)$ *such that* $x_L(t) \leq y$*, then* $F(x,t)$ *has a fixed point for each* t*.*

Note that Smithson[75] (Proposition 1.6) obtains a generalization of Abian and Brown's[1] fixed point theorem for the case the X is a CPO. In recent work, Heikkilä and Hu [38] and Heikkilä and Reffett[40] have generalized it further.

[6] An important converse to the Bourbaki fixed point principle (due to Zermelo) related to the fixed point result in the Abian-Brown-Pelczar theorem is in Jachymski[45].

8.3.2 Computational Fixed Point Theory

Recall that an operator $f(x) : X \to Y$ is *order-continuous* if for any countable chain $\{x_n\}$ having a supremum, we have $\sup f(x_n) = f(\sup x_n)$. If operators are order-continuous in Problem 1, we can weaken the conditions on the transformation space X, and also obtain stronger results on computing extremal fixed points by successive approximation on an operator from lower solutions x_L (e.g., a point x_L that has $x_L \leq f(x_L)$) and upper solutions x_U (e.g., a point x_U that has $f(x_U) \leq x_U$). The successive approximations indexed on the natural numbers can be shown to converge to extremal fixed points. If the underlying space is an ordered metric space, numerical implementations of our methods via Krasnosel'skii et al[48] (Chapter 4) can be shown to provide a *posteriori* error bounds in the underlying metric on X. This is particularly useful in our work, as many of the fixed point spaces we use (the economies studied in Sections 8.4-8.6) have uniform metric topologies.

We next discuss a result due to Kantorovich[47]. This result is available in a number of places in the literature (e.g., Dugundji and Granas[30] Theorem 4.2, Vulikh[84] Theorem XII.2.1, and Davey and Priestley[25] Theorem 8.15). We have the following result for a special case of Problem 1:

Proposition 8.3.6. *(Kantorovich[47]): Let X be a poset, $D = [a, b] \subset X$ countably chain complete. Assume for each $t \in T$, $f(x, t) : X \times T \to X$ is order continuous in x, such that $a(t) \leq f(a(t), t)$ and $f(b(t), t) \leq b(t)$. Let $G(t)$ be the fixed point correspondence of $f(x, t)$ for $t \in T$.Then (i) $G(t)$ is nonempty, and (ii) $\lim_n f^n(a(t); t) \to \inf G(t)$ (respectively, $\lim_n f^n(b(t); t) \to \sup G(t)$).*

An alternative setting that is common in economic applications of Problem 1 has the following structure: (i) the domain $D \subset X$ is a compact, order interval in a normal cone of positive continuous functions $C(X)$, where $X \subset R^n$ is also compact, and (ii) the operator $f(x, t)$ continuous and compact (e.g., completely continuous) in x for each $t \in T$. This is true in case of Coleman[19] and Datta et al[22] for the fixed point problem that constructs MEDPs. In this case, one can apply an important theorem due to Amann[3]:

Proposition 8.3.7. *(Amann[3], Theorem 6.1; corollary 6.2): Let X be an ordered Banach space, $[x_L(t), x^U(t)]$ an order interval with $x_L(t), x^U(t) \in X$, $x_L(t) \leq x^U(t)$, $f(x, t) : X \times T \to X$ is isotone on $[x_L(t), x^U(t)]$, compact and continuous in x, such that for each $t \in T$, $x_L(t) \leq f(x_L(t), t)$ and $f(x^U(t), t)) \leq x^U(t)$. Let $G(t)$ be the set of fixed points of $f(x, t)$ at $t \in T$.Then for each $t \in T$, (i) $G(t)$ is nonempty; (ii) $\lim_{n\to\infty} f_n(x_L(t); t) = \inf G(t)$ and $\lim_{n\to\infty} f_n(x^U(t), t) = \sup G(t)$ and the sequences $\{f_n(x_L(t), t)\}_{n=0}^{\infty}$ and $\{f_n(x^U(t), t)\}_{n=0}^{\infty}$ are increasing and decreasing, respectively.*

For Proposition 8.3.6 and Proposition 8.3.7, it is important that we obtain sufficient conditions that allow one to tie directly the computation of extremal fixed points to well-known numerical approximation algorithms in the existing literature (e.g., Krasnosel'skii et al[48] and Judd[46]). In some cases, such indexation on the natural number are not sufficient to show that successive approximation from lower or upper solutions for a particular set of fixed points actually computes an extremal fixed point. See the example in Davey and Priestley[25], section 8.16 or Heikkilä and Lakshmikantham[39], example 1.1.1. In these cases, one can define iterations on well-defined index sets that are subsets of chains. Heikkilä and Lakshmikantham [39] address this issue and deliver a generalized iterative method on a chain. A critical advantage of their approach is that it does not require either the axiom schema of replacement or the axiom of choice.

Proposition 8.3.8. *(Heikkilä and Lakshmikantham[39], lemma 1.1.1): Let D be the set of subsets of P, P a poset with $\varnothing \in D$ and $f : D \to P$, there is a unique well-ordered chain C so that $x \in C$ if and only if $x = f\{y \in C|y < x\}$. If $f(C)$ exists, it is not a strict upper bound of C.*

We discuss the elements in the chain C.Standard transfinite iterations are contained: let $x_0 = f(\varnothing)$, $x_{n+1} = f(\{x_0, x_1, ..., x_n\})$ for $x_n < x_{n+1}$, $x_w = f(\{x_n\}_{n=0}^{n=\infty})$ with x_w a strict upper bound of $\{x_n\}_{n=0}^{n=\infty}$, then x_w is a next successor element of C, and so forth. When establishing conditions in applications under which the generalized iterations of the mapping f can be indexed on countable sets, it is useful to recall that by Zorn's lemma, if each well-ordered chain C in P has an upper bound in P, then P has a maximal element. From Heikkilä and Lakshmikantham[39], Lemma 1.1.2, we know that each chain of any poset contains a well-ordered cofinal chain. Further, by another lemma in Heikkilä and Lakshmikantham[39] Lemma 1.1.4, a well-ordered chain C in a poset P is countable if its subchains possess countable cofinal chains. Finally, a monotone sequence in an ordered topological space X converges if each of its subsequences has a cluster point. A natural question concerns sufficient conditions under which iterations on f from some lower solution x_L converge to fixed points on a countable indexation of iterations. One set of sufficient conditions are as follows:

Proposition 8.3.9. *Heikkilä and Lakshmikantham[39], Lemma 1.1.7; Proposition 1.1.5; Proposition 1.1.6: (i) If a chain C in an ordered topological space X has a separable cofinal subset A, and if each nondecreasing sequence of A has a cluster point in X, then C contains a nondecreasing sequence that converges to $\sup C$; (ii) a well ordered chain of X is countable if the following occurs: (a) X is first countable, and each subchain of C is relatively compact; (b) each subset of C is separable and each nondecreasing sequence of C has a cluster point; (iii) If C is a chain in an ordered metric space X, and if each nondecreasing sequence of C has a cluster point, then C contains a nondecreasing sequence which converges to $\sup C$, and C is countable if each nondecreasing sequence of C has a cluster point.*

8.3.3 Monotone Selections and the Equilibrium Correspondence

In Problem 1 and Problem 2, a natural question to analyze is the existence of monotone comparison theorems on the space of parameters T.[7] Let $G(t) : T \to 2^X \backslash \varnothing$ denote the fixed point correspondence. We say the fixed point Problem 1 or 2 exhibits a *strong comparative structure (SCS)* if the fixed point correspondence $G(t)$ is an isotone correspondence from $T \to 2^X \backslash \varnothing$. We say Problem 1 or 2 exhibits a *weak comparative structure (WCS)* if its fixed point correspondence $G(t)$ admits an isotone selection. First, consider the SCS. Known sufficient conditions for $G(t)$ to be consistent with SCS involve the fixed point space X be a complete lattice, ordering the range of $G(t)$ using Veinott's strong set order on $2^X \backslash \varnothing$, and proving that $G(t)$ has a sublattice structure in $2^X \backslash \varnothing$. For example, if $G(t)$ is isotone from T to $2^X \backslash \varnothing$ in Veinott's strong set order, one immediately has the extremal selections $\sup G(t)$ and $\inf G(t)$ as isotone operators on T . The most general version of the result we discuss is due to Veinott[83] (Chapter 4, Theorem 14) and Topkis[81] (Theorem 2.5.2). The Veinott-Topkis Monotone Selection Theorem is stated as follows (see Topkis[81], Theorem 2.5.2 for a proof):

Proposition 8.3.10. *(Veinott[83]; Topkis[81]): Suppose X is a nonempty complete lattice, T a poset, $F(x,t) : X \times T \to 2^X \backslash \varnothing$ for each $(x,t) \in X \times T$, and assume that the correspondence $F(x,t)$ is isotone in Veinott's strong induced set order on $X \times T$. Let $G(t)$ be the fixed point correspondence of $F(x,t)$ at $t \in T$; then (a) for each $t \in T$, $\sup G(t)$ and $\inf G(t)$ exist; (b) $\sup G(t)$ and $\inf G(t)$ are isotone in $t \in T$; (c) If, in addition, $\sup G(t) < \inf G(t\prime)$ for $t < t\prime$, then $\sup G(t)$ and $\inf G(t)$ are strictly increasing in t on T.*

Second, consider the case of WCS. There are many alternative sufficient conditions under which fixed point problems exhibit WCS. Different forms of sufficient conditions are provided in Veinott[83] and Smithson[75]. We consider some additional isotone selection theorems that prove useful in the study of WCS in economic applications. These theorems apply in cases where the range of the fixed point correspondence does not necessarily possess the sublattice structure required to apply the Veinott-Topkis monotone selection theorem. For the first proposition, instead of assuming that the correspondence is isotone in Veinott's strong set order jointly in (x,t), we assume that $F(x,t)$ is ascending in Veinott's weak set relation in x for each $t \in T$. We also assume that the fixed point correspondence has the following structure: (i) $G(t){:}T \to 2^Y \backslash \varnothing$ is a nonempty and chain subcomplete, and (ii) $G(t)$ is ascending in Veinott's weak set order. We now state Veinott's weak monotone selection theorem:

[7] A well-known reference for monotone comparative statics in economics is Milgrom and Shannon [58]. However, their results built on prior results in operations research and reported in Veinott [82] and Topkis [79]. See Veinott's [83] lecture notes and Topkis [81].

Proposition 8.3.11. *(Veinott[83], Theorem 5) Let X be a lattice, T be a partially ordered set. Assume that $G(t) : T \to 2^X \backslash \varnothing$ is a chain subcomplete correspondence that is ascending in the Veinott's weak set relation. Then, (a) $G(t)$ admits an isotone selection. If, in addition, we assume $G(t)$ is meet- (respectively, join-) sublattice-valued for each $t \in T$, then (b) the isotone selection is $\wedge G(t)$ (respectively, $\vee G(t)$).*

Veinott proves more versions of the above isotone selection theorem assuming stronger hypotheses than (a), e.g., $G(t)$ quasi-sublatticed valued for each $t \in T$, but with weaker hypotheses than assumed for result (b). We present two different set of sufficient conditions for the existence of WCS from Smithson[75].

Proposition 8.3.12. *(Smithson[75], Theorem 1.7): Let X be a partially ordered set, T a set, and let $G(t) : T \to 2^X$ be a nonempty correspondence that is ascending in Smithson's weak set relation (C1) (respectively, (C2)) in (x, t). If, in addition, $\sup G(t) \in G(t)$ (respectively, $\inf G(t) \in G(t)$) for all $t \in T$, the there is an isotone selection, namely $g(t) = \sup G(t)$ (respectively, $g(t) = \inf G(t)$).*

We make a final remark on Proposition 8.3.12. The proof rely heavily on an application of the Axiom of Choice (namely, the Zorn's lemma). In principle, this can be a serious problem for developing constructive methods that address the question of approximating monotone selections. Recently, alternative methods are developed for the results in Smithson [75] that do not rely upon the Axiom of Choice (see, Jachymski[44], Theorem 2.21). Also, Heikkilä and Reffett[40] develop chain methods for computing particular selections that are not based on either the Axiom Schema Replacement or the Axiom of Choice. In addition, they provide new WCS theorems on the fixed point correspondence in Problem 2. These extensions are important if one wants to avoid the nonconstructive nature of the monotone selection results based on applications of the Axiom of Choice.

8.4 An Economy with Classical Technology

We generalize Brock and Mirman[16] to allow for more general "distorted classical" stochastic technologies. In these economies, time is discrete and indexed by $t \in T = \{0, 1, 2, ...\}$. There is a continuum of *ex ante* and *ex post* identical infinitely-lived households. The only form of uninsured risk is an aggregate production function shock. Aggregate production in each state is assumed to be constant returns to scale in private returns. Therefore, the value of all firms is zero in equilibrium. Each period households are endowed with a unit of time which is supplied inelastically in competitive markets. For simplicity, we assume uncertainty comes in the form of a finite state, first-order Markov process denoted by $\theta_t \in \Theta$, with stationary transition probabilities $\chi(\theta, \theta')$.[8] Let the set $\mathbf{K} \subset \mathbf{R}_+$ contain all feasible values for the aggregate endogenous state variable K, i.e., the capital to labor ratio, and define the product space $\mathbf{S} : \mathbf{K} \times \Theta$. Since the household also enters each period with an individual level of the endogenous state variable k, the individual capital to labor ratio, we denote the state of a household by the vector $s = (k, S) \in \mathbf{K} \times \mathbf{S}$.

The preferences are represented by a period utility index $u(c)$, where $c \in \mathbf{K} \subset \mathbf{R}_+$ is period consumption. Letting $\theta^i = (\theta_1, ..., \theta_i)$ denote the history of the shocks until period i, a household's lifetime preference is defined over infinite sequences indexed by date and history $\mathbf{c} = (c_{\theta^i})$ and is,

$$U(\mathbf{c}) = E_0 \left\{ \sum_{i=0}^{\infty} \beta^i u(c_i) \right\},$$

where E_0 is the mathematical expectation with respect to the probability structure of the shocks over the infinite horizon. We impose the following assumption on preferences:

Assumption - P1 :*The utility function* $u : \mathbf{K} \mapsto \mathbf{R}$ *is bounded, twice continuously differentiable, strictly increasing, strictly concave. In addition, marginal utility,* $u'(c)$ *satisfies the standard Inada conditions:*

$$\lim_{c \to o} u'(c) = \infty \text{ and } \lim_{c \to \infty} u'(c) = 0.$$

We assume that the output available to the household in the current period can be represented by the function $F(k, 1, K, 1, \theta, t) = f(k, K, \theta; t)$, where t is a parameter that is possibly infinite dimensional (e.g., a continuous mapping that represents distortions and influences technology). We assume that this production function is evaluated at equilibrium employment levels with $n = N = 1$ and make the following assumptions on technology:

Assumption - T1 *The production function* $F(k, n, K, N, \theta, t)$ *is such that:*

(i) $F(k, n, K, N, \theta; t)$ *is constant returns to scale in* (k, n) *for each* (K, N, θ, t) *such that* $F(0, 1, K, 1, \theta, t) = f(0, K, \theta; t) = 0$ *for all* $K \in \mathbf{K}, \theta \in \Theta$ *and* $t \in \mathbb{T}$.

(ii) f (k, K, θ, t) *is twice continuously differentiable, strictly increasing in* (k, K) *and strictly concave in its first argument.*

(iii) $f_1(K, K, \theta; t)$ *is weakly decreasing (i.e., non-increasing) in* K; *there exists a* $k_0 > 0$ *such that* $f(k_0, k_0, \theta; t) - k_0 > 0$, *and* $\beta \int f_1(k_0, k_0, \theta'; t)\chi(\theta, d\theta') \leq 1$ *for all* (θ, t).

(iv) There exist $\hat{k}(\theta) > 0$ *such that* $f(\hat{k}(\theta), k(\theta), \theta; t) = \hat{k}(\theta)$ *and* $f(k, k, \theta; t) < k$ *for all* $k > \hat{k}(\theta)$ *and for all* $\theta \in \Theta$.

[8] The reader should keep two things in mind while reading the results reported in this paper: (i) the results are valid for the case of deterministic nonoptimal growth by setting the shocks to a constant in all states; and, (ii) stochastic optimal growth is obtained as a special case by setting all equilibrium distortions to zero.

The restrictions on the primitives in Assumptions P1 and T1 are standard. (See, e.g., Coleman [19] for discussion). As we consider some baseline comparative statics issues, we consider the economy studied in Coleman [19]. In this setting, there is a state contingent capital income tax; in addition, we allow for nonconvexities in production in social returns. The distorted reduced-form technology f can be written as follows:

$$f(k, K, \theta; t) = (1 - t_1(K, \theta))g(k, K, \theta) + t_2(K, \theta),$$

where g is also a reduced-from distorted classical production function, the parameters $t_1(K, \theta) : \mathbb{S} \to [0, 1]$ and $t_2(K, \theta)$ can be interpreted as the state-contingent tax and a lump sum transfer, respectively. If we define the standard lexicographic order relation on the set of parameter vectors $t \in \mathbb{T}$ as $t'(K, \theta) \succeq t(K, \theta)$ if either $t_1^{'}(K, \theta) > t_1(K, \theta)$ for all $S \in (K, \theta) \in \mathbf{K} \times \Theta = \mathbf{S}$, or $t_1^{'}(K, \theta) = t_1(K, \theta)$ and $t_2^{'}(K, \theta) \geq t_2(K, \theta)$, then $f(k, K, \theta; t)$ is increasing in t.

We make the following assumption on the nature of distortion:

Assumption - D1: *The functions* $t_1(K, \theta)$ *and* $t_2(K, \theta)$ *are Lipschitz continuous on* $\mathbf{K} \times \Theta = \mathbf{S}$.

In developing our existence arguments, we fix $t \in \mathbf{T}$.(and, for the moment suppress notation). For any given $t \in \mathbf{T}$, define the household's feasible correspondence to be $\Gamma(k, K, \theta)$ where Γ defines the set of actions (c, k') that satisfy the standard budget constraint:

$$c + k' = f(k, K, \theta); \text{ and } c, k' \geq 0.$$

Under Assumption T1, $\Gamma(k, K, \theta)$ is a "well-behaved" nonempty correspondence for each $s = (k, K, \theta) \in \mathbf{K} \times \mathbf{S}$. In particular, as f is continuous and isotone, we conclude that Γ is a non-empty, compact and convex-valued, continuous correspondence for each state s that is ascending in (k, K, θ) for each t in the set inclusion order on $\mathbf{2^{KxK}}$ along an equilibrium restriction where $k = K$ and a balanced budget for the government.

Let $\mathbf{C}(\mathbf{S})$ denote the space of continuous functions $h(S)$:$\mathbf{S} \to \mathbf{K}$ equipped with the standard uniform norm topology (i.e., $\| h \| = \sup_{S \in \mathbb{S}} |h(S)|$) and pointwise Euclidean partial order where $\mathbf{S}$ is a compactum, and let $\mathbf{C}^+(\mathbf{S})$ be its cone. To construct the household's decision problem, consider that aggregate capital-labor ratio evolves according to:

$$K' = h(K, \theta) \in \mathbf{C}^+(\mathbf{S}), 0 \leq h \leq f,$$

where for any given t, $h(S) : \mathbf{S} \to \mathbf{K}$ is continuous in both its arguments, increasing in K for each θ. The household solves the following dynamic program:

$$J(s) = \sup_{(c,k') \in \Gamma(s;t)} \{u(c) + \beta \int_{\Theta} J(s')\chi(\theta, d\theta')\}. \tag{8.1}$$

Standard arguments prove the existence of a $J \in \mathbb{V}$ that satisfies this functional equation, where $\mathbb{V}$ is a space of bounded, continuous, real valued functions with the sup norm (see, for instance, Stokey, Lucas and Prescott[76]). In addition, under assumptions P1-T1, following the argument in Mirman and Zilcha[61] (lemma 1) J is differentiable in k.

We define an recursive equilibrium as follows:

Definition: *A (recursive) competitive equilibrium for this economy consists of a parameter vector* (t_1, t_2), *a value function for the household* $J(s)$, *and the associated individual decisions* c *and* k' *such that: (i)* $J(s)$ *satisfies the household's Bellman equation (8.1), and* c, k' *solve the optimization problem in the Bellman's equation given* t*; (ii) all markets clear: i.e.,* $k' = h(S) = K'$ *and (iii) the government budget balances.*

8.4.1 The Existence of MEDPs

The second welfare theorem does not apply in this economy. Therefore, the social planning approaches to characterizing MEDPs do not suffice. We adopt an alternative strategy, the so-called "Euler equation approach". To facilitate our construction, we consider a stronger version of Amann's theorem in Proposition 8.3.7, Section 8.3. This result is proved in Morand and Reffett [62] and considers Amann's theorem for isotone transformations of equicontinuous fixed point spaces.[9]

Proposition 8.4.1. *Let* E *be an equicontinuous fixed point space of continuous functions, each defined on a compact set* X, *equipped with the* sup *continuous uniform topology and the pointwise partial Euclidean order. Let* $[\overline{y}, \widehat{y}]$ *be a closed order interval in* E. *Suppose that* $A : [\overline{y}, \widehat{y}] \to [\overline{y}, \widehat{y}]$ *is an isotone, continuous map. Then* A *has a maximal fixed point* $\widehat{x}$ *and* $\widehat{x} = \lim_{n\to\infty} A^n \widehat{y}$, *and the sequence* $\{A^n \widehat{y}\}_{n=0}^{\infty}$ *is decreasing.*

Proof: See Morand and Reffett[62], Proposition 2.■

To construct existence of recursive equilibrium, we define a candidate nonlinear operator A whose fixed points coincide with a MEDP. The Euler equation associated with the optimal policy function in Bellman's equation (*8.1*) along an equilibrium trajectory where $k = K$ (appealing to the Mirman-Zilcha envelope condition) generates the following necessary and sufficient condition for a recursive competitive equilibrium: the existence of a function $c^*(K, K, \theta) = c^*(K, \theta)$ such that,

[9] Let $\mathbf{B}(X)$ be a bounded subset of the space of continuous functions $\mathbf{C}(X)$ and $E \subset \mathbf{B}(X)$. We say that E is *equicontinuous at a point* $x_0 \in X$ if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that any x in the δ-neighborhood of x_0 we have $\| h(x) - h(x_0) \| \leq \epsilon$ for every $h \in E$ (it is important that δ is independent of h). We say that E is *equicontinuous* if it is equicontinuous at every point of E.

$$u'(c^*(K,\theta)) = \beta \int_{\Theta} u'[c^*(F(K,\theta){-}c^*(K,\theta),\theta')]r(F(K,\theta){-}c^*(K,\theta),\theta')\chi(\theta,d\theta'). \tag{8.2}$$

Here, $F(K,\theta) = f(K,K,\theta;t)$ and $r(K,\theta) = f_1(K,K,\theta;t)$ for notational simplicity.

Definition: $\mathbf{H}^0$ *is the set of consumption functions* h *such that:*(i) $h : \mathbf{S} \to \mathbf{K}$;(ii) $0 \leq h(K,\theta) \leq F(K,\theta)$ *for all* $(K,\theta) \in \mathbf{S}$;(iii) $0 \leq h(K',\theta) - h(K,\theta) \leq F(K',\theta) - F(K,\theta)$ *for all* $K' \geq K, (K,K') \in \mathbf{K} \times \mathbf{K}$ *and all* θ.

Equip $\mathbf{H}^0$ with the standard sup uniform metric topology; and adopt the Euclidean partial order $\geq$ induced by the cone structure of $\mathbf{C}^+(\mathbf{S})$. That is, $h' \geq h$ if and only if $h'(K,\theta) \geq h(K,\theta)$ for all $(K,\theta) \in \mathbf{S}$. The following lemma summarizes some important properties of the space $\mathbf{H}^0$.

Lemma 8.4.1. *Under assumption T1, (i)* $\mathbf{H}^0$ *is a closed, convex, equicontinuous order interval of continuous function (e.g., a convex, compact, order interval); (ii)* $\mathbf{H}^0$ *is a complete lattice.*

Proof: (i) See Coleman[19] Proposition 3. (ii) See Morand and Reffett[62], Lemma 1.■

To construct a recursive equilibrium, we define a nonlinear operator Ah based on an equilibrium version of the Euler equation. Consider any $h \in \mathbf{H}^0$, $h > 0$, and any (K,θ),

Definition: *The operator* $Ah(K,\theta) = \{y | y : \text{for } h > 0, u'(y) = \beta \int_{\Theta} u'(h(F - y,\theta'),\theta')r(F-y,\theta')\chi(\theta,d\theta'); \text{ if } h = 0 \text{ in any } (K,\theta), \text{ we set } Ah(K,\theta) = 0\}$.

The following lemma lists a few key properties of the operator A:

Lemma 8.4.2. *Under Assumptions P1 ,T1, and D1:(i) For any* $h \in \mathbf{H}^0$*, and any* (k,θ)*, there exists a unique* $Ah(k,\theta)$*;(ii)* A *maps* $\mathbf{H}^0$ *into itself (e.g., is a transformation of* $\mathbf{H}^0$*);(iii)* A *is continuous on* $\mathbf{H}^0$*; (iv) there exists a maximal fixed point* $h^* \in \mathbf{H}^0$ *and the sequence* $\{A^n F\}$ *converges uniformly to* h^**; and, (v) the maximal fixed point is strictly positive.*

Proof: The proofs of (i)-(iii) are in Coleman[19] (Proposition 4). Claim (iv) follows directly from Proposition 8.4.1. Claim (v) follows from a standard dynamic programming argument that is presented in the main theorem in Greenwood and Huffman[34] p 615.■

It is important to note that neither (i), (ii), nor (iii), rely on compactness of the state-space, and are therefore valid under Assumptions P1 and T1 only. We can now state our existence result for MEDPs.

Proposition 8.4.2. *Under Assumptions P1, T1 and D1, there exists a recursive equilibrium.*

Proof: Follows from Lemma 8.4.1 and 8.4.2.■

8.4.2 The Uniqueness of MEDPs

We next consider the uniqueness of MEDPs. Let C^+ be the cone of a real Banach space C, and consider a transformation $A : C^+ \to C^+$. We say an operator $A : C^+ \to C^+$ is *$e-$concave* if there exists non-zero $e \in C^+$, such that (i) for an arbitrary non-zero $c \in C^+$ the inequalities $\alpha e \leq Ac \leq \beta e$, where α and β are positive, are valid and (ii) for every $c \in C^+$ such that $\alpha_1(c)e \leq c \leq \beta_1(c)e$ with $(\alpha_1(c), \beta_1(c)) \gg 0$, and there is a number $\eta(c,t) > 0$ such that $A(tc) \geq (1+\eta)tAc$ for any $t \in (0,1)$. An operator is said to be *pseudo-concave* on C^+ if for all $t \in (0,1), c \in C^+, c > 0, Atc >> tAc$. Let C^+ be a solid cone, the operator $A : C^+ \to C^+$ is *strongly sublinear* if $Atc >> tAc$ for all non-zero $c \in C^+$ and $0 < t < 1$. (See Guo and Lakshmikantham[35], Definition 2.2.2). If A is isotone and strongly sublinear, it is well-known A is e-concave. If an operator that is strongly sublinear on the interior of a solid cone, then it is pseudo-concave. Notice that pseudo-concavity is a weaker condition than $e-$concavity; uniqueness of strictly positive fixed point therefore typically requires stronger conditions on the operator and/or cone. Let $\mathbf{P}$ be a solid cone. Two such related conditions for the operator A to have unique strictly positive fixed points are that of cone compression and k_0-monotonicity. The former is used to guarantee existence of positive fixed points. The latter is used for uniqueness relative to the cone.

An operator $Ah : \mathbf{P} \to \mathbf{P}$ is a *cone compression* on the normal cone $\mathbf{P}$ if their exists a pair of numbers $R, r > 0$ such that

$$\begin{aligned} Ah &\nleq h, \text{for } h \in \mathbf{P}, \|h\| < r, h \neq 0; \\ Ah &\ngeq h, \text{for } h \in \mathbf{P}, \|h\| > R. \end{aligned}$$

Let $H \subset \mathbf{C}^+(\mathbf{S})$ be an compact, order interval where $\mathbf{C}^+(\mathbf{S})$ is the space of positive continuous functions on the compact set $\mathbf{S} = \mathbf{K} \times \Theta$. We say an operator A is *k_0-monotone* on H if it is (i) isotone on H, and (ii) if for any strictly positive fixed point h_1, there exists a $k_0 > 0$, $0 \leq k_1 \leq k_0$ and $h_2 \in H$ such that $h_2 \leq h_1$, for all $k \geq k_1$, and $h_1(k,\theta) \geq Ah_2(k,\theta)$ all $k \geq k_1$, for all θ. Notice if A is k_0-monotone, A is a cone compression.

In our argument, we construct new sufficient conditions for existence of a uniquely strictly interior fixed point. We first construct the operator $\widehat{A}$ as in Coleman[21], but we prove additional properties of this operator that are useful for our argument that are not in Coleman. We then show that the operator is strongly sublinear on its interior and a cone compression (which implies the existence of a strictly positive fixed point. We then show the operator is additionally k_0-monotone, which implies he has a unique strictly positive fixed point.

We first define the set of functions $\mathbf{M}$ as follows

Definition: $\mathbf{M} = \{\ m : \mathbf{R}_+ \times \Theta \to \mathbb{R} |$(i) *$m$ is continuous*, (ii) *for all $(K,\theta) \in \mathbf{R}_+ \times \Theta$, $0 \leq m(K,\theta) \leq F(K,\theta)$ and* (iii) *for any $K = 0$, $m(K,\theta) = 0\}$*

Endow **M** with the standard partial pointwise order and the C^0uniform topology. We note that $\mathbf{H}^0$ and **M** can be directly related to each other by a simple mapping. For $m \in \mathbf{M}$, consider the function $\Psi(m(K,\theta))$ implicitly defined by,

$$u'[\Psi(m(K,\theta))] = \frac{1}{m(K,\theta)}, \text{ for } m > 0, \text{ 0 elsewhere.}$$

Clearly, Ψ is continuous, increasing, $\lim_{m\to 0}\Psi(m) = 0$, and $\lim_{m\to F(K,\theta)}\Psi(m) = F(K,\theta)$.Using the function Ψ, for any $m > 0$, we denote the solution (for y) to the following equation by $\widehat{A}m(K,\theta)$,

$$\widehat{Z}(m,y,K,\theta) = \frac{1}{y} - \beta E_\theta[\frac{H(F(K,\theta)-\Psi(y),\theta')}{m(F(K,\theta)-\Psi(y),\theta')}] = 0,$$

and set $\widehat{A}m = 0$ when $m = 0$. Since $\widehat{Z}(m,y,K,\theta)$ is strictly decreasing and continuous in y and $\lim_{y\to 0}\widehat{Z}(m,y,K,\theta) = \infty$ and $\lim_{y\to F(K,\theta)}\widehat{Z}(m,y,K,\theta) = -\infty$, for each $m(K,\theta) > 0$,with $K > 0$, and $\theta \in \Theta$, there exists a unique $\widehat{A}m(K,\theta)$.

It is easy to show that to each fixed point of the operator A corresponds a fixed point of the operator $\widehat{A}$. Indeed, consider x such that $Ax = x$ and define $y = \frac{1}{u'(x)}$ (or, equivalently $\Psi(y) = x$). It is also easy to verify that $Am \subset \mathbf{M}$ and is monotone on **M**. By definition, for all (K,θ), x satisfies,

$$u'(x(K,\theta)) = \beta E_\theta\{H(F(K,\theta)-x(K,\theta),\theta') \times u'(x(F(K,\theta)-x(K,\theta),\theta'))\}.$$

Substituting the definition of y into this expression yields

$$\frac{1}{y} = \beta E_\theta\{\frac{H(F(K,\theta)-\Psi(y(K,\theta)),\theta')}{y(F(K,\theta)-\Psi(y(K,\theta),\theta'))}\},$$

which shows that y is a fixed point of $\widehat{A}$.

We are now prepared to prove our uniqueness result:

Proposition 8.4.3. *Under Assumptions P1, T1, D1, (i) The operator $\widehat{A}$ is strongly sublinear; (ii) $\hat{A}$ has at most one strictly positive fixed point; and, (iii) there exists a unique recursive equilibrium in* $\mathbf{H}^0$.

Proof: (i) First note both $\mathbf{H}^0$ and **M** are order intervals in solid cones of continuous functions defined on a compact set. Therefore since $\widehat{Z}$ is strictly decreasing in its second argument, a sufficient condition for strong sublinearity of $\hat{A}m$ on the interior of **M** is:

$$\widehat{Z}(tm, t\widehat{A}m, K,\theta) > \widehat{Z}(tm, \widehat{A}tm, K,\theta) = 0. \tag{8.3}$$

By definition,

$$\widehat{Z}(tm, t\widehat{A}m, K, \theta) = \frac{1}{t\widehat{A}m} - \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)), \theta')}{tm(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)), \theta')}\},$$

so that,

$$t\widehat{Z}(tm, t\widehat{A}m, K, \theta) = \frac{1}{\widehat{A}m} - \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)), \theta')}{m(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)), \theta')}\}.$$

Since Ψ is increasing and $H(K', \theta')/m(K', \theta')$ is decreasing in K',

$$\frac{1}{\widehat{A}m} - \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)), \theta')}{m(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)), \theta')}\}$$

$$> \frac{1}{\widehat{A}m} - \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(\widehat{A}m(K,\theta)), \theta')}{m(F(K,\theta) - \Psi(\widehat{A}m(K,\theta)), \theta')}\} = 0,$$

and $\widehat{Z}(tm, t\widehat{A}m, K, \theta) > 0$ so it must be the case that $\hat{A}tm > t\ \hat{A}m$.Therefore, $\hat{A}m$ is strongly sublinear on the interior of its domain.
(ii) As $\hat{A}m$ is strongly sublinear on the interior of its domain, we conclude $\hat{A}m$ is pseudo concave. Given the presence of the Inada condition on technology, a standard argument in Coleman [21] shows $\hat{A}m$ is K_o-monotone. Further by the main theorem in Coleman [21], we conclude that $\hat{A}m$ has at most a single strictly positive fixed point. The last question pertains to existence then of strictly positive fixed points. Not that given the definition of $\hat{A}m$, whenever $m > 0$, necessarily $\hat{A}m\ > 0$; further $\hat{A}m < m$. We therefore have $\hat{A}$ a *cone compression* on the interior of its domain. Then Krasnosel'skii and Zabreiko ([49], Theorem 46.4), $\hat{A}m$ has a strictly positive fixed point. Therefore by the proposition immediately above, $\hat{A}m$ actually has a unique strictly positive fixed point. Finally, (noting the relationship between the orbits of $\hat{A}$ and A discussed earlier in this section) as we have a unique strictly positive fixed point for $\hat{A}$, namely $m^* > 0$, we must have a unique fixed point for A , say $h^* > 0$.Since $h^* > 0$ implies strictly positive consumption, it is a MEDP.
(iii) As the $\hat{A}m$ has a unique strictly positive fixed point in M, by the definition of $\hat{A}m$ and the fact that $\hat{A}[M]$ is isomorphic to $A[\mathbb{H}^0]$, we conclude there is a strictly positive fixed point $h^* \in \mathbf{H}^0$ with consumption positive in all states $K > 0$, each θ. By a standard argument (e.g., see Le Van and Vailakis[52], Section 5) , interiority of consumption and investment (along with the fact $h^* \in \mathbf{H}^0$) is sufficient in this case to support prices in $l^1_+\backslash\{0\}$.■

8.4.3 Monotone Comparison Theorems Using Euler Equation Methods

In this section, we construct monotone comparison theorems using Euler equation methods. The monotonicity of the mapping A in lemma 8.4.2 can be exploited to derive strong comparative statics (SCS) results on the space of deep parameters $t \in T$ using the selection theorems in Section 8.3. The set of equilibrium is a non-empty complete lattice, so, in the absence of the uniqueness result, comparative statics analysis requires defining orders on both the set of parameters and on the set of equilibrium. We show that the set of equilibrium is ascending in the strong set order of Veinott in t, consequently, we conclude that the minimal and maximal fixed points are also monotonic in t.

Proposition 8.4.4. *Suppose that the assumptions of lemma 8.4.2 and Proposition 8.4.2 are satisfied for each mapping A_t belonging to the set $\{A_t : \mathbf{H}^0 \to \mathbf{H}^0,\ t \in T\}$, where $(T, \geq_T)$ is a poset, and $G(t)$ is the fixed point correspondence of A_t. If A_t is isotone in t, that is if $t' \geq_T t$ implies that, for all x in X, $A_{t'}x \geq A_t x$, then $G(t)$ is ascending in Veinott's strong set order $\geq_s$ on $2^{\mathbf{H}^0}$ and the minimal and maximal fixed points (respectively, $\wedge G(t)$ and $\vee G(t)$) of A_t are isotone mappings into $\mathbf{H}^0$ on T.*

Proof: The claims follow from the proof in Morand and Reffett[62], Theorem 2, noting that $G(t)$ is isotone in Veinott's strong set order, a direct implication of Proposition 8.3.10.■

For an application of this result, consider a perturbation in the discount rate β. Since the right side of the Euler equation in (8.2) is increasing in β, as a consequence, the root $y^*(K, \theta, h, t) = A_{t=\beta}(c)$ that defined the operator, is increasing in $\beta \in (0, 1) = T$, where T is endowed with the dual order $\geq_T$ on the real line (i.e., $\beta' \geq_T \beta$ if $\beta' \leq \beta$). By Proposition 8.4.4, the maximal and minimal fixed points increase with t (i.e., decrease with β). By Proposition 8.4.3, the set of MEDPs increase in the pointwise strong set order $\geq_{ss}$ and there is a unique isotone selection.

For another application, consider the tax rate $t \in T$, where T is the set of continuous functions $t(K, z) \in [0, 1]$ that are monotone in K. Endow T with the standard pointwise Euclidean order for a space of functions, i.e., $t' \geq_T t$ if $t'(K, z) \geq t(K, z)$ for all (K, z). Then $A_{t'}c \geq A_t c$ in the order defined on E and the equilibrium set (the set of fixed points of the operator A_t) is isotone in t the strong set order. Again, by Proposition 8.4.3, we can obtain a unique isotone selection on T from the set of MEDPs.

8.5 An Economy with Nonclassical Technology

We now allow for more general versions of bounded nonconvex production technologies, linear preferences, Markov technology shocks and a role for public policy. By "distorted nonclassical" production technologies, we mean two cases: the reduced-form production function $f(k, K, \theta)$ is such that (i) $f_1(k, K, \theta)$ is not decreasing in k when $k = K$, and/or (ii) f is not necessarily constant returns to scale in private inputs. In (ii), that there is an issue with interpreting exit and entry conditions in the industry within the equilibrium model but we ignore the industry dynamics. Uncertainty (and much of the model) is as before. Preferences and technologies are denoted as before, except we now have weaker assumptions:

Assumption - P2: *The utility index* $u(c) \in \mathbf{U}$ *where* $\mathbf{U}$ *consists of all* $u(c) : \mathbf{K} \mapsto R$ *that are bounded, continuous, strictly increasing, and either strictly concave on* $\mathbf{K}$ *or linear on* $\mathbf{K}$.

Assumption - T2: *The aggregate production functions* $f \in F$ *, where* F *consists of isotone functions* $f(k, K, \theta)$:$\mathbf{K} \times \mathbf{K} \times \Theta$, *each space ordered with pointwise Euclidean order,* f *is continuous in* k *and there exists* $\hat{k}(\theta) > 0$ *such that* $f(\hat{k}(\theta), \theta) + (1 - \beta)\hat{k}(\theta) = \hat{k}(\theta)$ *and* $f(k, \theta) < k$ *for all* $k > \hat{k}(\theta)$ *for all* $\theta \in \Theta$; *and* f *is twice differentiable its arguments.*[10]

We impose a joint restriction on the curvature of $u(c)$ relative to the complementarity of the equilibrium distortion in $f(k, K, \theta)$. This restriction is used *only* for our methods when $f \in F$ such that $f(K, K, \theta)$ is not concave in K.(See section 8.6.2 for further discussion of this point, and how this restriction can be eliminated in the case $f(K, K, \theta)$ is concave.)

Assumption - PT1: *The utility index* $u \in U$ *and the aggregate production technology* $f \in F$ *are such that* $u'(\gamma(K))f_1(k, K, \theta)$ *is isotone in* K *for each function* $\gamma(K)$ *where* $\gamma(K)$ *satisfies* $0 \leq \gamma(K') - \gamma(K) \leq f(k, K', \theta) - f(k, K, \theta)$ for $K' \geq K$.[11]

We need a regularity property on the stochastic process of shocks.

Assumption - M1: *The transition matrix* $\chi \in \Xi$ *is an irreducible Markov process that satisfies the standard Feller property.*

When discussing the long-run properties of a Markovian equilibrium (and equilibrium comparative statics on limiting distributions), it is useful to restrict attention to a subset of economies where we can prove Markovian dynamics are jointly monotone in (K, θ). Therefore, we note the following additional assumptions:

Assumption - PT2: *The class* U *and* F *have* $u'(\gamma(\theta))f_1(k, K, \theta)$ *are isotone in* θ *for each* $\gamma(\theta)$ *such that* $0 \leq \gamma(\theta') - \gamma(\theta) \leq f(k, K, \theta') - f(k, K, \theta)$.[12]

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[10] We also refer to an isotone function as a monotone function.

[11] If one is willing to adopt the slightly stronger complementarity condition related to the one mentioned in Hopenhayn and Prescott [43] (i.e., $u''(c)f_1f_2 + u'(c)f_{12} \geq 0$), we can allow $u(c)$ in assumption P2 to be concave (but not necessarily linear).

[12] This assumption includes the case for Markov shocks mentioned (but not studied) in Hopenhayn and Prescott [43] for the optimal growth model.

Assumption - M2: *The measure $\chi \in \Xi$ is stochastically increasing (or equivalently, totally positive of order 2).*[13]

The case of optimal growth under uncertainty is embedded in above assumptions. Our results are more general than those obtained for the optimal growth model with Markov shocks in Hopenhayn and Prescott[43]. Although they claim a more general result, a careful reading of their proofs reveals that Hopenhayn and Prescott can *only* claim sufficient conditions for monotone controls in the optimal growth model with Markov shocks when production functions are the fixed-coefficient, Leontief-type.[14] Note that, we can dispense with assumption M1 or M2 for the optimal growth case. Also, if the class of shocks $\chi \in \Xi$ consists of a collection of independent and identically distributed random variables, then we obtain joint monotonicity for decentralized Markovian equilibrium under weaker conditions. We can completely dispense with Assumption PT2, and we still obtain joint monotonicity of the decentralized MEDPs. For the optimal growth case, we only require $u \in U$ concave, and $f(K,\theta)$ monotone in (K,θ).

8.5.1 The Parameter Space and Household Decision Problems

Consider the existence of MEDPs under the assumptions P2, T2, PT1 and M1. We begin by defining the fixed point space we use to compute MEDPs.

Definition: $\mathbf{C}_1 = \{h| \ 0 \leq h(K,\theta) \leq f(K,K,\theta) \ \forall \ (K,\theta); \ h(K',\theta) - h(K,\theta) \geq 0 \ if \ K^{'} \geq K\}$.

Here $h \in \mathbf{C}_1 \subset \mathbf{B}(\mathbf{S})$, $\mathbf{S}$ is a compact partially ordered topological space with the pointwise Euclidean order (and the usual topology on $\mathbf{R}^n$). $\mathbf{B}(\mathbf{S})$ is the set of bounded functions $\mathbf{S}^{\mathbf{K}}$ endowed with the standard pointwise Euclidean order and C^0 uniform topology, and $\mathbf{C}_1$ consists of all positive functions that are isotone in K, and socially feasible, monotone in K.

Assume that households take as given the recursion h on per-capita aggregate capital stock K, which is used to compute future returns on capital (and, therefore, factor prices),

$$K' = h(K,\theta) \in \mathbf{C}_1, 0 \leq h \leq f.$$

[13] See Topkis [81] for a definition of stochastically increasing.

[14] The problem with application of a key theorem in Topkis [81] (Theorem 2.7.6) also arises in Amir [5]. In this paper, if one follows the proofs, one realizes that the author can *only* claim the existence of monotone controls in the nonclassical optimal multisector growth model when production functions are either (i) Leontief or (ii) defined on domains where the inputs are chained. Our approach using generalized envelopes can be applied in the multisector growth model to obtain more general sufficient conditions for monotone controls in multisector models than found in Amir's work.

If we make additionally assume PT2 and M2, we obtain stronger characterizations of Markovian equilibrium. For that situation, consider the space,

Definition: $\mathbf{C}_2 = \{\ h | h(K,\theta) \in \mathbf{C}_1$ that are jointly isotone in $(K,\theta)\}$.

Clearly $\mathbf{C}_2(\mathbf{S})$ is a closed sublattice of $\mathbf{C}_1 \subset \mathbf{B}(\mathbf{S})$. The spaces $\mathbf{C}_1$ and $\mathbf{C}_2$ are used to find Markovian equilibrium for economies without and with assumptions PT2 and M2, respectively. We next prove a lemma that is useful in constructing a Markovian equilibrium.

Lemma 8.5.1. *Both* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are convex and subcomplete in* $\mathbf{B}(\mathbf{S})$.

Proof: See Mirman, Morand and Reffett[59], lemma 1. ■

Therefore, $\mathbf{C}_1$ (respectively, $\mathbf{C}_2$) is a natural place to pose the existence of MEDP question.

We now characterize the best response mapping of households facing an aggregate environment $h \in \mathbf{C}_1$, under the assumptions P2, T2, PT1 and M1. Consider a household entering the period in state $p = (p_c, \theta) \in P = \mathbf{K} \times \mathbf{K} \times \Theta$, $p_c = (k, K) \in \mathbf{K} \times \mathbf{K}$, facing an aggregate economy with aggregate dynamics (and prices) summarized by the function $h \in \mathbf{C}_1$. Let consumption and investment be given as $a = (c, y) \in A \subset \mathbf{K} \times \mathbf{K}$. The value function for the household is a function $v^*(p, h)$ that is a solution of the functional equation:

$$v^*(p;h) = \sup_{a \in \Gamma(p)} \{u(c) + \beta \int_{\Theta} v^*(y, h(K,\theta), \theta'; h)\chi(\theta, d\theta')\}, \tag{8.4}$$

where the feasible correspondence $\Gamma(p) = \{a | c+y\ \leq f(p), c, y \geq 0\}$. In order to study the existence of a v^* that satisfies the above functional equation, consider the operator B^C:

$$\begin{aligned} B^C v(p;h) \quad = \quad & \sup_{a \in \Gamma(p)} \{u(c) + \\ & \beta \int_{\Theta} v(y, h(K,\theta), \theta'; h)\chi(\theta, d\theta')\}. \end{aligned}$$

Here the operator B^C is defined on the space $V_c = \{v(p;h) : \mathbf{P} \times \mathbf{C}_1 {\rightarrow} \mathbf{R}$, v bounded in (k, K, θ, h), isotone in p for each h, continuous in k for each $(K, \theta, h)\}$. Equip V_c with the standard C^0 topology (and the associated uniform metric) and the pointwise Euclidean partial order. V_c is a complete metric space. Lemma 8.5.2 provides a set of results characterizing the unique function v^* that satisfies (8.4):

Lemma 8.5.2. *Under assumptions P2, T2, PT1 and M1, (i)* $B^C v \subset V_c$*;(ii) there exists a unique* $v^* \in V_c$ *that satisfies the Bellman equation (8.4); and, (iii) the fixed point* v^* *is strictly increasing in* p *for each* $h \in \mathbf{C}_1$.

Proof: A standard argument. See Stokey, Lucas, and Prescott[76].■

We now use lattice programming to further characterize the value function.[15] Define the optimal solution associated with $v^*(p;h)$ by $a^*(p,h)$,

$$a^*(p;h) = \{\arg \sup_{c,y \in \Gamma(p)} \{u(c) + \beta \int_{\Theta} v(y, h(K,\theta), \theta'; h) \chi(\theta, d\theta')\}\}. \tag{8.5}$$

To characterize the optimal solution $a^*(p,h) \subset 2^A$, we define a set of partial orders over choices of consumption c and investment y. The class of partial orders is referred to as "direct value" orders and was pioneered in the work of Antoniadou[7]. To fix ideas, consider the simple two good version of the consumer decision problem. Assume that the relative price is one. Define a collection of direct value orders for unit price for $a = (c,y) \in A \subset \mathbf{K} \times \mathbf{K}$ (denoted by $\geq_{vi}$, where $i \in I$, an index set) as follows: $a, a' \in A$, we say $a' \geq_{vi} a$ if and only if $c' + y' \geq_e c + y$ *and* $a' \geq_{Li} a$. Here $\geq_e$ is referred to as the value quasi-order on A, and $\geq_{Li}$ is the standard lexicographic order defined using the index set $I = \{c,y\}$on $A \subseteq R^2_+$. We use this collection of valuation lattices $(A, \geq_{vi})$ to model the action space for the stochastic growth model $A \subseteq R^2_+$. When indexing the lexicographic order in the valuation order by c, we refer to the resulting lattice, on the commodity space $(A, \geq_{vc})$, as the *consumption value lattice*. We also make reference to the *investment value lattice* when indexing the lexicographic order in the valuation order by investment $(A, \geq_{vy})$. Antoniadou [7] shows that the space $(A, \geq_{vi})$ is (i) a partially ordered set for each $i \in I = \{c,y\}$, and (ii) $\geq_{vi}$ induces a lattice structure on A for each $i = c, y$. Define, $\Gamma(p) = \{a |\ c + y \leq m, c, y \geq 0, m = f(p)\} \subseteq A$ when $(A, \geq_{vi})$ $i = 1,2$.Under assumptions P2, T2, PT1 and M1, and each index $i = c, y$, the feasible correspondence $\Gamma(p)$ is (i) an isotone mapping $P \to 2^A$ in the strong set order $\geq_a$ endowed with either of the partial orders $i = c, y$; and (ii) it is a nonempty, continuous, compact, convex, and complete sublattice for each $p \in P$.

We turn next to a characterization of supermodular functions on the collection $(A, \geq_{vi})$. In the next lemma, we characterize additively separable supermodular objectives on the direct value lattices $(A, \geq_{vi})$. Let $U(x,y)$: $A \to R$ on the lattice $(A, \geq_{vi})$.

Lemma 8.5.3. *Assume $U(x,y) = u_1(x) + u_2(y)$, where each $u_i(.)$ is isotone for $i = 1,2$. Then (i) $U(x,y)$ is supermodular (strictly supermodular) on the x valuation lattice $(A, \geq_{vx})$ if and only if $u_2(y)$ concave (strictly concave), (ii) $U(x,y)$ is supermodular (strictly supermodular) on the collection $(A, \geq_{vI})$ for $I = x, y$ if and only if both $u_1(x)$ and $u_2(y)$ are concave (strictly concave).*

[15] We assume familiarity in this section with the basic terminology of lattice programming (supermodular functions etc.). Important references are Li Calzi and Veinott [53], Veinott [83], and Topkis [81].

Proof: See Mirman, Morand and Reffett[59], Lemma 4. ■

Now, we consider sufficient conditions for monotone controls $a^*(p,h)$ from (8.5) to be isotone in the Euclidean order on A. The parameters of interest are $p_c = (k,K)$ and $h \in \mathbf{C}_1$. A major obstacle to studying the dynamic complementaries in (8.5) is characterizing sufficient conditions for preserving supermodularity under maximization. One set of sufficient conditions for preserving supermodularity under maximization on arbitrary projections to the parameter space is found in Topkis[81] (Theorem 2.7.6). This set of sufficient conditions cannot be applied in growth models with multidimensional parameter spaces as they require the graph of the feasible correspondence to be sublattice valued in the powersets of $A \times P$; a condition not available in growth models unless the production function is Leontief. We, therefore, do not follow this line of argument. We develop results on generalized envelope conditions found in the literature on nonsmooth analysis. See Clarke[17] (chapter 2) and Rockafellar and Wets[71]. This approach is used in Askri and Le Van[8] who study envelope theorems in the multisector optimal growth model with nonclassical technologies. Unfortunately, however, their results only apply to economies for which the optimal solutions are strictly interior. In our framework, their methods cannot be directly applied. We extend Askri and Le Van[8] results to economies without boundary restrictions, such as Inada conditions. Our method is based on Gauvin and Dubeau[33].

Let $p \in P$. Note that P is a convex sublattice. Consider the subspace of value functions $V(p) \subset V_c$ consisting of the $v(k,K,\theta,h) \in V_c$ with the following additional restrictions:

(i) $v(p)$ is supermodular in $p_c = (k,K) \in P_c$ for each θ;

(ii) $v(k,K,\theta,h)$ Lipschitz in k with the Lipschitz constant,

$$L = \sup_{c,k,K,\theta,h} |\{u'(c)f_1(k,K,\theta), u'(0)f_1(k,K,\theta) + \varepsilon|,$$

where $\varepsilon = \beta \int u'(f(k,h(K,\theta),\theta'))f_1(k,h(K,\theta),\theta')\chi(\theta,d\theta') - u'(0)$. The subset V is a closed subset of the complete metric space of functions V_c.Also, recall that supermodularity is closed under pointwise limits (see Topkis[81], lemma 2.6.1). We have the following monotonicity result,

Proposition 8.5.1. *Let us assume P2, T2, PT1 and M1 and let $v \in V(p)$. Then (i) the optimal solution $a^*(h;p)$: $\mathbf{C}_1 \to 2^A$ is ascending in h in the strong set order $\geq_a$on the investment valuation lattice $(A, \geq_{vy})$; and, (ii) the maximal and minimal selections for investment $a_y^u(h;p) = \max_y a^*(h;p)$ and $a_y^l = \min_y a^*(h;p)$ are isotone functions from $\mathbf{C}_1 \to A$.*

Proof: See Mirman, Morand and Reffett[59], Theorem 5. ■

Notice that monotonicity on the investment lattice $(A. \geq_{vy})$ implies that investment monotonicity on the Euclidean lattice $(A, \geq_E)$. Proposition 8.5.1 implies that the extremal selections of the best response map are monotone on the space $\mathbf{C}_1$ for each (k, K, θ).Corollary 8.5.1 shows that the extremal selections form self maps to the space $\mathbf{C}_1$:

Corollary 8.5.1. *Assume P2, T2, PT1 and M1, let $v^* \in V$ in equation 8.4 and for each $\theta \in \Theta$; then for $h \in \mathbf{C}_1$ (i) the optimal solution $a^*(p_c, \theta; h)$ is ascending from $P_c \to 2^A$ in the strong set order $\geq_a$ on the investment valuation lattice $(A, \geq_{vy})$; and, (ii) the minimal and maximal selections for investment $a_y^u(p_c, \theta; h) = \max_y a^*(p_c, \theta; h)$ and $a_y^l = \min_y a^*(p_c, \theta; h)$ are isotone functions from $P_c \to A$. Under additional assumptions PT2 and M2, and for $h \in \mathbf{C}_2$,(iii) the optimal solution $a^*(p; h)$ is ascending from P to 2^A in the strong set order $\geq_a$ on the investment valuation lattice $(A, \geq_{vy})$; and, (iv) the minimal and maximal functions for investment $a_y^u(p) = \max_y a^*(p)$ and $a_y^l = \min_y a^*(p)$ are isotone functions from $P \to A$.*

In the proof of Proposition 8.5.1 and Corollary 8.5.1 in Mirman, Morand and Reffett[59], they also prove a new envelope theorem that generalizes the result in Mirman and Zilcha [61], Amir, Mirman and Perkins[6], and Askri and Le Van[8]. With this envelope, it is straightforward to check that the right side of (8.4) at a solution v^* has all the requisite complementary structure to obtain isotone increasing controls in Veinott's strong set order $\geq_s$ (namely, the requisite increasing differences between the controls and the parameters). Given that this new generalized envelope is of independent interest, we present the argument for its existence.

We need to define some terms. A correspondence $\Gamma(p)$ is said to *uniformly compact near* p if there is a neighborhood $N(p)$ of p such that the closure of $\cup_{p' \in N(p)} \Gamma(p')$ is compact. Given the continuity of f in p for economies $\Delta \in E$, one can prove that the feasible correspondence on (8.4), $\Gamma(p)$, is uniformly compact near p. Rewrite the constraints in (8.4), more generally, as $\Gamma(p) = \{a | g(a, p) \leq 0\}$ where $g(a, p)$ is the set of implicit constraints defined in (8.4). We say a pair $(a, p) \in gr\Gamma(p)$ satisfies the *Mangasarian-Fromowitz regularity conditions (*or, are *MF-regular)* if there exists a direction $r \in R^2$ such that the Jacobian $\nabla_a g(a, p) r < 0, g(a, p) = 0$.[16] Here $gr\Gamma(p)$ is the graph of $\Gamma(p)$. In our problem, the constraints are additively separable with constant gradients in the controls, for any pair $(a, p) \in A \times P$; therefore each point (a, p) is MF regular. Therefore, any optimal solution $(a^*(p, h), p) \in gr\Gamma(p)$ is a MF-regular point. Further, because these coefficients do not change as a function of a, we also note that we have a stronger constraint qualification present, namely that the basis elements $\nabla_a g(a^*(p, h), p)$ are linearly independent. Therefore, our problem also satisfies the so-called "linear independence" *(LI)* constraint qualification discussed in Gauvin and Dubeau[33].

[16] As all constraints are inequalities, we are writing that MF regularity constraint qualifications for a problem with only inequality constraints, i.e, we do not require for all binding constrants, say $h(a, p) = 0$, to satisfy that the direction r is othrogonal to $\nabla_a h(a, p)$ where h is the collection of all the equality constraints.

Next, note a few properties of Bellman operator B^c. Let $v \in V$. We know that the feasible correspondence $\Gamma(p) : P \to 2^A$ is a continuous, strong set order ascending correspondence in $p = (k, K, z)$ for each $h \in \mathbf{C}_1$. Further, for each $p, \Gamma(p)$ is nonempty, compact, convex and subcomplete in $(A, \geq_{vi})$ for $i = c, y$. As $u(c)$ is Lipschitz (as its C^1 with bounded gradient on any neighborhood of $\mathbf{K}$ that is strictly interior), and the sum of two Lipschitz functions is Lipschitz, we conclude that the objective is Lipschitz in (c, y) for each (p, h). By a standard application of Berge's maximum theorem[11] p.116), the value function $B^C v$ is continuous in k, and the optimal solutions $a^*(p, h)$ form a nonempty, compact-valued correspondence for each (p, h). Noting the continuity of the objective, a^* is also upper hemi-continuous correspondence in k. As the order on P pointwise Euclidean, when P is endowed with the standard metric/topology, P is a Banach lattice with a continuous lattice structure. Also note that $(A, \geq_{vi})$ $i = c, y$, A has a continuous lattice structure, and $A = \mathbf{K} \times \mathbf{K}$ is Hausdorff. Therefore, by Debreu[27], the optimal solutions $a^*(p) : P \to 2^A$ is are upper-measurable. (See also Hopenhayn and Prescott[43] for discussion of upper-measurability).

We next prove that the value function is locally Lipschitz under our assumptions in this section. This result is used to prove under our assumptions, $B^c v \in V$.

Proposition 8.5.2. *The Bellman operator, $B^c : P \times \mathbf{C}_1 \to R$, is locally Lipschitz near $k > 0$, for each (K, z, h) and $v^*(k, K, \theta, h)$ is Clarke differentiable in its first argument for each (K, θ, h).*

Proof: We have two cases.

Case 1: The optimal solutions $a^*(p, h)$ are strictly interior in $A = \mathbf{K} \times \mathbf{K}$; i.e., for all $a(p, h) \in a^*(p, h), a(p, h) \in int(R^2_+)$

By a result in Amir, Mirman, and Perkins[6] (lemma 3.3) left and right Dini derivatives exist in k for each (K, z, h) and are bounded. By Rockafellar[70] (Proposition 5), $B^c v$ is therefore locally Lipschitz with a upper estimate of the Lipschitz modulus of $L_v(p, h) = \sup_{p,k>0}\{B^{c+}v, B^{c-}v\} \leq L$ where for example B^{c+} is the right Dini at (p, h), $k > 0$.

Case 2: The optimal solutions $a^*(p, h)$ is such that there is an $a(p, h) \in a^*(p, h)$ not interior.

Using a standard Lagrangian approach, the operator $B^c v$ is given as follows: for $h \in \mathbf{C}_1$,

$$\begin{aligned} B^c v &= \sup_{a,\lambda,\varphi_c,\varphi_y} L(a, p, h) \\ &= \sup_{a,\lambda,\varphi_c,\varphi_y} u(c) + \beta \int v(y, h(K, \theta), \theta')\chi(\theta, d\theta') \\ &\quad +\lambda(f - c - y) + \varphi_c c + \varphi_y y \end{aligned} \tag{8.6}$$

where $\lambda, \varphi_c, \varphi_y$ are the multipliers associated with the respective constraints that define $\Gamma(p) = \{a | c + y \leq f(k, K, z), c \geq 0, y \geq 0\}$. As (i) each element of $(a^*(p,h),p)$ is *MF-regular* such that it also satisfies the condition *(LI)* and (ii) the primitive data of the problem is Lipschitz, by corollary 4.4 in Gauvin and Dubeau[33], $B^c v$ has bounded right and left Dini derivatives in k with $B^{c+}v_k(k,K,z,h) = \max_{a\in a^*(p)} \nabla_k^+ L(a,p,h) \le L$, and $B^{c-}v_k(k,K,z,h) = \max_{a\in a^*(p)} \nabla_k^- L(a,p,h) \le L$ for $k > 0, p \in P$. Then by Gauvin and Dubeau[33] (Theorem 5.1), $B^c v$ is locally Lipschitz in $k > 0$, $p \in P, h \in \mathbf{C}_1$ (see also Rockafellar[70], Proposition 5).■

This generalized Clarke envelope is a critical step: the economies that satisfy assumptions P2, T2, PT1 and M1, the value function $v^*(k,K,\theta,h)$ has increasing differences in $(k;K,h)$ for each θ. If, in addition, we assume PT2 and M2, then we obtain v^* also having increasing differences in $(k;\theta)$.

8.5.2 The Existence of MEDPs

We prove the existence of a complete lattice of Markovian equilibrium. Noting the dependence of best responses on the environment (in the next section we conduct monotone comparative statics on the space of environments), we denote a correspondence,

$$Th(K,\theta) =$$
$$\{a(K,K,\theta;h)|a \text{ any monotone selection for investment in } a_y^* \text{ in } (8.5)\}$$

We state some useful properties of the correspondence Th. In particular, we focus on the sublattice structure of its range:

Lemma 8.5.4. *Under assumptions P2, T2, PT1 and M1, $Th \subset \mathbf{C}_1, Th$ is ascending on $\mathbf{C}_1$ in the strong set order $\ge_a$ to 2^{C_1} and is complete-latticed valued; with additional assumptions PT2 and M2, $Th : \mathbf{C}_2 \to 2^{\mathbf{C}_2}$ is ascending in the strong set order $\ge_a$and T is complete lattice valued.*

Recalling the Veinott[82][83] or Zhou[85] version of Tarski's theorem in Proposition 8.3.2, we obtain our first result (the proof follows directly from Lemma 8.5.4 and Proposition 8.3.2),

Proposition 8.5.3. *Under the assumptions P2, T2, PT1 and M1, the set of fixed points φ_T^* is a nonempty complete lattice in $\mathbf{C}_1$; with additional conditions PT2 and M2, the set of fixed points φ_T^* is a nonempty complete lattice in $\mathbf{C}_2$.*

8.5.3 Monotone Comparison Theorems via Lattice Programming Methods

We first point out straightforward monotone comparison results with respect to changes in the discount rate and shock process. Consider ordered perturbations of the discount rate β and/or uncertainty $\chi \in \Xi$ (where the ordered perturbation of measure χ take place in a setting of first order stochastic dominance). Using variations of existing arguments (e.g., Amir, Mirman and Perkins[6] (Theorem 5.1) and Hopenhayn and Prescott[43] (corollary 7) for perturbations in β and χ, respectively), we obtain a Veinott strong set order monotone comparative statics result in the pointwise Euclidean order from the extremal selections of agent investment decisions for investment $a_y^*(p, h; \beta, \chi)$, under assumptions P2, T2, PT1 and M1. Then by the Veinott-Topkis SCS theorem, we obtain Veinott strong-set order fixed point correspondence comparison with the operator Th by $\varphi_T^*(\beta, \chi)$ and have the SCS via Proposition 8.3.10, Section 8.3. We conclude that the fixed point correspondence $\varphi_T^*(\beta, \chi)$ exhibits strong set order comparative statics, i.e., $\varphi_T^* : (0,1) \times \Xi \to 2^{\mathbf{C}_1}$ is a strong set order increasing correspondence.

To study monotone comparative statics with respect to the space of reduced-form distorted production functions, our argument requires the development of a set of partial orders that is suitable for ordering the envelope conditions for agents' decisions. This partial order involves "gradient monotonicity" conditions. Infinite dimensional single crossing properties relative to a space of payoff functions for a collection of parameterized dynamic programs have been studied by Lovejoy[54]. Consider the order on the space of technologies F : $f' \geq_F f$ when $u(f'(k, K, z)) - u(f(k, K, z))$ is increasing in k, for each (K, z) with $f' - f = 0$, when $k = 0, (k, K, z) \in P$, and P is compact.[17] Observe the following: (a) $(F, \geq_F)$ is a partially ordered space (antisymmetry follows given f vanishes at zero), (b) $f' \geq_F f$ implies $f'(p) \geq f(p)$ for all p in the pointwise Euclidean order, and, (c) $f' \geq_F f$ implies that the gradients, $\partial_k f'(p) \geq \partial_k f$ (p), are pointwise ordered in the Euclidean order.

Proposition 8.5.4 provides some monotone comparison results for MEDPs (and stationary Markovian equilibrium). We have examples of SCS and WCS; namely SCS on the set of MEDPs, and WCS on the set of invariant distributions. We note that so far, the existence of *measurable* MEDPs has not been addressed. We need to address this question prior to discussing the structure of Markov operators used to study invariant distributions. Let $(\mathbf{K} \times \mathbf{Z}, \mathcal{K} \times \mathcal{Z})$ be a Borel measurable space where $\mathcal{K} \times \mathcal{Z}$ is the set of Borel subsets of $\mathbf{K} \times \mathbf{Z}$. Let $\mathcal{C}_1$ (respectively, $\mathcal{C}_2$) be the the space of $h \in \mathbf{C}_1$ (respectively, $h \in \mathbf{C}_2$) such that $h(K, z)$ is jointly measurable. By a standard result (e.g., Halmos [36], Section 20, Theorem A), if X_c is any countable subset of $\mathbf{C}_1$ (respectively, $\mathbf{C}_2$), then $\vee X_c$ and $\wedge X_c$ are in $\mathbf{C}_1$ (respectively, $\mathbf{C}_2$). Therefore, we conclude that $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) are $\sigma-$complete lattices. Using the optimal solutions in (8.5) that solve the agents dynamic programs in (8.4), define operators based upon the extremal selectors; namely, $A^u h = \sup_h a^*(p; h)$ (respectively $A^l h = \inf_h a^*(p, h)$). We remark these are both well-defined iso-

[17] Note that the partial order defined with respect this difference is increasing in each component of p. We fix (K, z), and emphasize the role of k in our discussion below.
Also, similar orders can be developed to obtain monotone controls in consumption, relative to the space of production function by developing the obvious dual argument using the dual order relative to capital.

tone and measurable operators on $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) where isotonicity follows from Proposition 8.5.1 and measurability follows from Hopenhayn and Prescott ([43], Proposition 2) (when restricting the domain of each extremal operator to $h \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$, respectively). We also note that by an argument in Mirman, Morand and Reffett [59], $A^u h$ and $A^l h$ are both order continuous operators on sequences in $\mathcal{C}_1$ (respectively, $\mathcal{C}_2$). Then by Proposition 8.3.6, successive approximations on $A^u(f)$ (respectively, $A^l(0)$ converges in order to the maximal (respectively, minimal) fixed point of Th in (8.6) (when Th is restricted to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively). We can use these extremal fixed points (which are appropriately measurable) to conduct monotone comparative dynamics.

As a prerequisite to stating our comparison results, we define a few terms that are useful in characterizing the order theoretic properties of the random dynamical systems. Let $\mathbf{M}(\mathbf{K}\times\mathbf{Z})$ be the space of finite measures on $\mathbf{K}\times\mathbf{Z}$, endow $\mathbf{M}$ with the stochastic dominance partial order, that is $\lambda' \geq_M \lambda$ if for every monotone, measurable, nonnegative, and bounded function $f : \mathbf{K}\times\mathbf{Z} \to R_+$, $\int f\lambda'(dk \times d\gamma) \geq \int f\lambda(dk \times d\gamma)$. Hopenhayn and Prescott[43] (Proposition 3) show that when this order is restricted to the space of monotone, measurable, bounded, and nonnegative functions, ($\mathbf{M}$, $\geq_M$) is a partially order set under the stochastic dominance order $\geq_M$. When viewed from a topological perspective, Dudley[29] (Proposition 11.3.2) provides a metric under which $\mathbf{M}$ is a compact metric space. Let ($\mathbf{K}\times\mathbf{Z}$,$\mathcal{B}(\mathbf{K}) \times \mathcal{B}(\mathbf{Z})$) be measurable spaces where $\mathcal{B}(.)$ denotes the Borel measurable subsets. Consider the adjoint operator $J(\lambda; h) : \mathbf{M}(\mathbf{K}\times\mathbf{Z})\times\mathcal{C}_2 \to \mathbf{M}(\mathbf{K}\times\mathbf{Z})$ defined as,

$$J(\lambda; h)(A \times B) = \int I_A(h(k, z))\chi(z, B)\lambda(dk \times dz), \tag{8.7}$$

where I_A is the indicator function for a measurable set $A \in \mathcal{B}(\mathbf{K})$, $B \in \mathcal{B}(\mathbf{Z})$**.** For each $h \in \mathcal{C}_2$, define the fixed point correspondence for the operator J $(\lambda; h)$ to be $\Psi^*_J(h) = \{\lambda \in \mathbf{M}, \lambda = J(\lambda, h)\}$. Define $\lambda_m(h) = \min \Psi^*_J(h)$, and let $\varphi^*_J(f)$ be the set of invariant distributions associated with the set of Markovian equilibrium $\varphi^*_T(f)$, for any production function $f \in F$.We have,

Proposition 8.5.4. *Assume P2, T2, PT1 and M1, let $f \in (F, \geq_F)$. Then (i) the correspondence of Markovian equilibrium, $\varphi^*_T(f) : F \to 2^{C_1}$ is ascending in the strong set order $\geq_a$ .Further, with additional assumptions PT2 and M2, (ii) the set of equilibrium invariant distributions $\varphi^*_J(f) : F \to 2^M$ is ascending in both (C1) and (C2) of Smithson's-weak set relation $\geq_{as}$and therefore admits a monotone selection on F; and (iii) the dynamics exhibit monotone comparative dynamics in the Smithson-weak set relations (C1) and (C2).*

Note that standard arguments can be used to prove the existence of an invariant distribution for a Markovian equilibrium in $\varphi^*_T(f)$. The main contribution of Proposition 8.5.4 concerns comparative dynamics results on the space of equilibrium correspondence. The problem of ruling out limiting distributions that do not have ergodic sets on a strictly positive support is nontrivial. We leave further characterization of a stationary Markovian equilibrium for future work. Note that, isotone selections in $\varphi_J^*(f)$ exist as one can easily check the conditions of Smithson's weak isotone selection theorem discussed in Section 8.3, Proposition 8.3.12.

8.6 An Economy with Elastic Labor Supply

We revisit the model with classical technology (Section 8.4) and allow for elastic labor supply. This model is formulated as in Datta, Mirman and Reffett[22]. As in the previous sections, we consider a continuum of household/firms populating the economy. Uncertainty and market structure are also similar to that in Sections 8.4 and 8.5 but the household cares about leisure. For each period and state, preferences are represented by a period utility index $u(c_i, l_i), (c_i, l_i) \in R_+ \times [0,1]$. Letting $\theta^i = (\theta_1, ..., \theta_i)$ denote the history of the shocks until period i, the households lifetime preferences are additively separable and defined over infinite sequences indexed by dates and histories,

$$U(\mathbf{c}, \mathbf{l}) = E_0 \left\{ \sum_{i=0}^{\infty} \beta^i u(c_i, l_i) \right\}.$$

Here E_0 is the expectation with respect to the probability structure of future histories of the shocks θ^i given the transition matrix χ. The period utility function $u : R \times [0,1] \mapsto R$, satisfies,

Assumption - P3: *The period utility index $u(c,l)$ is such that:*

(i) $u(c,l)$ is continuously differentiable, strictly increasing, and strictly concave in (c,l).

(ii) The partial derivatives $u_c(c,l)$ and $u_l(c,l)$ satisfy the Inada conditions:

$$lim_{c \to 0}\ u_c(c,l) = \infty,\ lim_{c \to \infty} u_c(c,l) = 0,\ lim_{l \to 0} u_l(c,l) = \infty.$$

(iii) The second partials satisfy,

$$\frac{u_{cc}}{u_c} \leq \frac{u_{lc}}{u_l}, \frac{u_{ll}}{u_l} \leq \frac{u_{cl}}{u_c}.$$

The assumptions on period utility are standard. (See Datta et al[22] and Le Van and Vailakis[52] for discussion of this assumption). Note that condition P3(iii) can be thought of as "normality". It also means that the marginal rate of substitution $\frac{u_l}{u_c}$ is non-decreasing in c and $\frac{u_l}{u_c}$ is non-increasing in l. And this is slightly stronger than quasi-concavity of the period utility function (we assume it to be strictly concave) because it implies,

$$u_c^2 u_{ll} + u_l^2 u_{cc} \leq 2 u_c u_l u_{cl},$$

which is a necessary condition for quasi-concavity. This condition is automatically satisfied if $u_{cc}(c,l) < 0$, $u_{ll}(c,l) < 0$ and $u_{cl}(c,l) \geq 0$. If the cross-partial is negative, the condition restricts its magnitude.

Each household is endowed with a unit of time, and enters into a period with an individual stock of capital k. We assume a decentralization where firms do not face dynamic decision problems. Households own the firms as well as both the factors of production, and they rent these factors of production in competitive markets. In addition, to allow for externalities in the production process, as in previous sections, we assume that the production technologies of the firms to depend on per capita aggregates. Assume that technology satisfies,

Assumption - T3: *The production function* $f : \mathbf{K}\times[0,1]\times\mathbf{K}\times[0,1]\times\Theta \to R$ *satisfies,*

(i) $f(0,0,K,N,\theta) = 0$ *for all* $(K,N,\theta) \in \mathbf{K} \times [0,1] \times \Theta$,

(ii) $f(k,n,K,N,\theta)$ *is continuous, increasing, differentiable; in addition, it is concave and homogeneous of degree one in* (k,n).

(iii) $f(k,n,K,N,\theta)$ *also satisfies the standard Inada conditions in* (k,n) *for all* $(K,N,\theta) \in \mathbf{K} \times [0,1] \times \Theta$; *i.e.,*

$$\begin{aligned}
\lim_{k\to 0} f_k(k,n,K,N,\theta) &= \infty, \\
\lim_{n\to 0} f_n(k,n,K,N,\theta) &= \infty, \\
\lim_{k\to \infty} f_k(k,n,K,N,\theta) &= 0.
\end{aligned}$$

(iv) There exists a $\hat{k}(\theta) > 0$, *such that* $f(\hat{k}(\theta),1,\hat{k}(\theta),1,\theta)+(1-\beta)\hat{k}(\theta) = \hat{k}(\theta)$ *and* $f(k,1,k,1,\theta) < k$ *for all* $k > \hat{k}(\theta)$, *for all* $\theta \in \Theta$.

Assumption T3 is standard in the stochastic growth literature (see Brock and Mirman[16]). With the initial stock k_0, we can define $\bar{k} = \max\{k_0, \sup_\theta \hat{k}(\theta)\}$ and the state space for the capital stock and output can be defined on the compact set $\mathbf{K}\subseteq [0,\bar{k}]$. Let $\mathbf{K}_+$ denote the set of strictly positive values for k.

8.6.1 The Household Decision and Equilibrium

Imagine a consumer faced with a choice problem of a single good and leisure in the first stage. The objective is to maximize the difference between the level of utility and the expenditure to obtain that level of utility (see Topkis[81]). Normalizing on the price of consumption goods, consumers take the price of leisure $w(K,\theta)$, the level of per capita consumption C, and the per capita leisure level $L(C,K,\theta)$, as given. Here $C \in \mathbf{K}$, $w : \mathbf{K} \to R_{++}$, $L : \mathbf{K}\times\mathbf{S} \to [0,1]$, and L is a continuously once-differentiable function, and as in previous sections, $\mathbf{S} := \mathbf{K} \times \Theta$.Given w, the household solves,

$$v(C,K,L,\theta)) = \sup_{l\in[0,1]} \frac{u(C,l)}{u_c(C,L)} - wl,$$

for each $(C,K,L,\theta) \in \mathbf{K}^2\times [0,1] \times\Theta$. Given the assumption P3, standard arguments using the Theorem of the Maximum, establish that the value function v

is well-defined and continuous (e.g., see Berge[11], p.115). Further, by the strict concavity of period utility in P3, the optimal policy correspondence associated with v is a singleton. The necessary condition for this first-stage maximization problem is,

$$\frac{u_l(C, l^*(C, K, \theta))}{u_c(C, L)} = w(K, \theta).$$

To finish our description of the first stage, we need to determine equilibrium factor prices as functions of the aggregate state variable. We do this from the representative firm's static production problem. Assume that firms maximize profits under perfect competition, i.e., the firms maximize profits subject to given factor prices, say $\bar{r}(K, \theta)$ and $\bar{w}(K, \theta)$, the rental rate for capital and the wage rate, respectively. The factor prices are continuous functions of the aggregate state variable. The representative firm's maximum profit is,

$$\Pi(\bar{r}, \bar{w}, K, N, \theta) = sup_{k,n} f(k, n, K, N, \theta) - \bar{r}k - \bar{w}n$$

where anticipating the standard definition of competitive equilibrium, we set $k = K$ and $n = N(S)$, for $S \in \mathbf{S}$.

In the second-stage, the household solves a dynamic capital accumulation problem. To describe this problem, we parameterize the aggregate economy facing a typical decision maker. Define to be the space of bounded, continuous functions with domain $\mathbf{S}$ and range $\mathbf{R}_+$. To parameterize the household's decision problem, we first describe the aggregate economy.

If the aggregate per capita capital stock is K, then households assume a continuous function for per capita labor supply $0 \leq N(S) \leq 1$, and a recursion of the capital stock K' is given by,

$$K' = h(S);\ h \in \mathbf{C}^+(\mathbf{S}), 0 \leq h \leq f(K, 1 - N(S), \theta)$$

where $\mathbf{C}^+(\mathbf{S})$ is as before the space of positive continuous functions on $\mathbf{S}$ with the uniform topology. Using the solution to the household's first stage decision problem (and, imposing equilibrium on the labor market), define the per capita aggregate labor supply $N(S) = 1 - l^*(C, K, \theta)$. Then the aggregate economy consists of functions $\Omega = (w, r, h, C, N)$ from a space of functions with suitable restrictions needed to parameterize the household's decision problem in the second-stage. Assume that the policy-induced equilibrium distortions have the following standard form,

$$r = [1 - \pi_k(S)]\bar{r},\ w = [1 - \pi_n(S)]\bar{w},$$

where $\pi = [\pi_k, \pi_n]$ is a continuous mapping $\mathbf{S} \to [0, 1) \times [0, 1)$. We assume regularity conditions on the distorted prices,

Assumption - D2 *:The vector of distortions $\pi = [\pi_k, \pi_n]$ is such that the distorted wage $w = (1 - \pi_n(K, \theta))\bar{w}$ and the distorted rental rate $r = (1 - \pi_n(K, \theta))\bar{r}$ satisfy,* *(i)* $w : \mathbf{K} \times \Theta \to \mathbf{R}_+$ *is continuous, at least once-differentiable and (weakly) increasing in* K,

(ii) $r : \mathbf{K}_+ \times \Theta \to \mathbf{R}_+$ *is continuous and decreasing in* K *such that,*

$$\lim_{K \to 0} r(K, \theta) \to \infty.$$

In other words, we assume that the distorted wage and rental rates behave as the non-distorted rates $\bar{w}$, $\bar{r}$ or the marginal products of labor and capital, respectively. Assumptions D2(i) and P3(iii) imply that leisure increases with higher consumption and decreases with larger capital accumulation.

Next define the lump-sum transfer to each agent, $d(S) = \pi_k K + \pi_n N(K, \theta)$. Then household's total income is $y(s) = rk + wN + \Pi + d(s)$ where s is the individual household's state, $s = (k, S) = (k, K, \theta)$ and Π is profit. Note that under assumptions P3, T3 and D2, $y(s)$ is a continuous function. We next define the household's feasible correspondence, $\Psi(s)$, which consists of the set $(c, k') \in \mathbf{R}^2_+$ that satisfy,

$$c + wl^*(C, K, \theta) + k' = y,$$

given $(k, K, \theta) \gg 0$. Notice that $\Psi(s)$ is well behaved. In particular since Π is continuous, Ψ is a non-empty, compact and convex-valued, continuous correspondence.

Next, we state the second stage decision problem for the household. At the beginning of any period the aggregate state for the economy is given by $S \in \mathbf{S}$. Each household enters the period with their individual capital stock $k \in \mathbf{K}$, so their individual state is $s \in \mathbf{K} \times \mathbf{S}$. Then the households dynamic decision problem is summarized by the Bellman equation,

$$v(s) = sup_{(c,k') \in \Psi(s)} u(c, l^*(C, K, \theta)) + \beta \int_{\Theta} v(s') \chi(\theta, d\theta') \tag{8.8}$$

Standard arguments show the existence a $v \in \mathbb{V}$ that satisfies this functional equation, where $\mathbb{V}$ is again the space of bounded, continuous functions with the uniform norm. In addition, since u is strictly concave in c, standard arguments also establish that v is strictly concave in its first argument, k. Once again, from Mirman and Zilcha[61], the strict concavity of v also implies that the envelope theorem applies and the solution v to the Bellman equation is once differentiable in k.

We are now prepared to define equilibrium.

Definition: *A (recursive) competitive equilibrium for this economy consists of sequences functions* r, w, d, *and* κ; *a value function for the household* $v(s) \in \mathbb{V}$ *and the associated individual decisions* $c^*(s)$ *and* $n^*(s)$ *such that (i) given* r, w, d *and* κ , $v(s)$ *satisfies the household's Bellman equation (8.8); (ii)* $c^*(s)$ *solves the right-hand side optimization in the Bellman's equation,* $l^*(s) = 1 - n^*(s)$ *solves the first-stage utility maximization; (iii) all markets clear: i.e.,* $k' = h(S) = K'$, $n^*(s) = N(S)$, $c^*(s) = C(S)$ *and the government budget constraint holds, i.e.,* $d = \pi_k k + \pi_n n^*$

8.6.2 The Existence of Equilibrium

Before we state the existence problem, we define a number of functions. In equilibrium, $c(s) = C(S)$, $k = K$, $n = N(S)$,then $y(s) = F(K,\theta) = f(K, 1 - l^*(C(S), K, \theta), \theta) + (1-\beta)K$. The next period capital stock, in equilibrium, is given as $K' = \ y - C$. Also, for later reference, define $\hat{l}(S)$ as the solution to,

$$\frac{u_l(f(K, 1-\hat{l}(S), \theta), \hat{l}(S))}{u_c(f(K, 1-\hat{l}(S), \theta), \hat{l}(S))} = (1 - \pi_n(S)) f_n(K, 1 - \hat{l}(S), \theta).$$

Notice that $\hat{l}$ is the amount of leisure that is compatible with no household investment in the first-stage utility maximization. At any (aggregate) state S,the maximum possible amount of consumption occurs if $c = f$ and, i.e., if there is no investment. In general, the amount of consumption is less than f and leisure, which is positively related to consumption, is therefore less than $\hat{l}(S)$. That is, for a given state S, $1-\hat{l}(S)$ is the lower bound for the amount of labor supplied. In addition, $\hat{l}(S)$ is differentiable with respect to K, by the implicit function theorem, since the marginal utilities, technology and the distorted wage is differentiable in K. Moreover, for the special case, $u_{cl} \geq 0$, $\hat{l}(S)$ is increasing in K. $\hat{l}(S)$ is also increasing in K, for the case $u_{cl} < 0$, if

$$u_{ll} - f_n u_{cl} < 0, u_{cl} - f_n u_{cc} > 0.$$

The Euler equation, associated with the right side of the Bellman equation (8.7) above, can be rewritten as,

$$u_c(c, l^*(c, K, \theta)) = \beta \int_{\Theta} u_c(c(K', \theta'), l^*(c', K', \theta')) r(K', \theta') \chi(\theta, d\theta'). \qquad (8.9)$$

Here the $'$ notation refers to next period value of the particular variable. Given a candidate function $c(S)$, we rewrite the Euler equation (8.9) in equilibrium as,

$$\begin{aligned} u_c(c, l^*(c, K, \theta)) \quad = \quad & \beta \int_{\Theta} u_c(c(F_c - c, \theta'), l^*(c(F_c - c, \theta'), K', \theta')) \cdot \\ & r(F_c - c, \theta') \chi(\theta, d\theta'), \end{aligned} \qquad (8.10)$$

where $F_c = f(K, 1 - l^*(c(K,\theta), K, \theta), \theta) + (1-\beta)K$. We can use equation (8.10) to define a nonlinear operator that yields a strictly positive fixed point in the space of consumption functions. This fixed point is an equilibrium for the economy.

Define $F^u(S) = F^u(K,\theta) = f(K, 1 - \hat{l}(K,\theta), \theta) + (1-\beta)K$ and consider the following space of functions,

Definition: $\mathbf{H}_l = \{h : \mathbf{S} \to \mathbf{K}$, *h continuous*, $h(S) \in [0, F^u(S)]$ *and h such that* $u_c(h(S), l^*(h(S), S))$ *is decreasing in h*, $u_c(h(S), l^*(h(S), S))$ *is decreasing in K*.$\}$

Equip $\mathbf{H}_l$ with the sup norm. Note that the assumption the marginal utility of consumption is decreasing in h means that the space $\mathbf{H}_l$ differs from the space of consumption functions studied in Coleman[20]. It is easily verified that for the preferences considered in that paper, the restriction u_c decreasing in h is implied. However, since the class of preferences studied in this paper is larger than that studied in Coleman, additional restriction is necessary on the space of consumption functions.

Define the extended real valued mapping $Z : \mathbf{H}_l \times \mathbf{Y} \times \mathbf{K} \times \mathbf{Z} \rightarrow \bar{\mathbf{R}}$ where $\mathbf{Y} \subset \mathbf{R}_+$, as

$$Z(h, \zeta, K, \theta) = \Psi_1(\zeta, K, \theta) - \Psi_2(h, \zeta, K, \theta), \tag{8.11}$$

$$\Psi_1 = u_c(\zeta, l^*(\zeta, K, \theta)), \tag{8.12}$$

$$\Psi_2 = \beta \int_\Theta u_c(h(F_\zeta - \zeta, \theta'), l^*(h(F_\zeta - \zeta, \theta'), F_\zeta - \zeta, \theta')) r(F_\zeta - \zeta, \theta') \chi(\theta, d\theta'). \tag{8.13}$$

Here $F_\zeta = f(K, 1 - l^*(\zeta, K, \theta) + (1 - \beta)K$. Then define the nonlinear operator $A : \mathbf{H}_l \rightarrow \mathbf{H}'$ as follows:

$$Ah(K, \theta) = \{\zeta \text{ such that } Z(h, \zeta, K, \theta) = 0, h > 0; Ah(K, \theta) = 0 \text{ elsewhere}\} \tag{8.14}$$

where $\mathbf{H}'$ at this point is an appropriate Banach space.

We discuss some properties of the operator A as defined by equations (8.11) - (8.14).

Proposition 8.6.1. *Under Assumptions P3, T3 and D2, for any $h \in \mathbf{H}_l$, there exists a unique $Ah = \tilde{h}$ such that $Z(h, \tilde{h}, K, \theta) = 0$, for any (K, θ).*

Proof: Datta, Mirman and Reffett[22], Proposition 1. ■

Proposition 8.6.1 implies that for all states, the operator Ah is well defined and under the continuity assumptions on preferences, technologies, and distorted prices, continuity of Ah is obvious. To study the fixed points of A, we first establish that A is a transformation of $\mathbf{H}_l$: i.e., $A : \mathbf{H}_l \rightarrow \mathbf{H}_l$. It will be convenient to assume

Assumption - P4: *The cross-partial of the utility function is non-negative, that is, $u_{cl} \geq 0$.*

Greenwood and Huffman [34] only consider the case where $u_{cl} = 0$. Coleman [20] allows for $u_{cl} \geq 0$ and also some cases where $u_{cl} < 0$. However, he considers a restricted homothetic class of preferences and, in addition, imposes more restrictions (jointly on utility, production functions and distortions) to study the case of negative cross partials of u. The same case of negative cross-partials of u can be handled in our setting also. At this stage, we are unable to capture more general cases of negative cross partials of u than Coleman [20], therefore, we focus only on the $u_{cl} \geq 0$ case. And, we have the following:

Proposition 8.6.2. *Under assumptions P3, P4, T3 and D2,* $Ah \subset \mathbf{H}_l$.

Proof: Datta, Mirman and Reffett[22], Theorem 1.■

Notice that $\mathbf{H}_l$ is a non-empty, convex subset of a space of continuous, bounded real-valued functions but it not equicontinuous, and is therefore not relatively compact.[18] Since the space of all continuous functions on a compactum X, denoted by $\mathbf{C}(\mathbf{S})$, with the sup-norm metric is a Banach lattice, $\mathbf{H}_l$ is a sublattice in $\mathbf{C}(\mathbf{S})$. Now, a closed subset of continuous, bounded real-valued functions (on a compact domain) equipped with sup-norm metric is compact if and only if it is equicontinuous. The theorem of Arzela and Ascoli (see Dieudonne [28], p.136-137) says that a set of equicontinuous, pointwise compact subset of the continuous functions is relatively compact.

Define the following subset of $\mathbf{H}_l$, **Definition:** $\bar{\mathbf{H}} = \{h \in \mathbf{H}_l$ *such that* $0 \leq | h(K_2,\theta) - h(K_1,\theta) | \leq | F(K_2, l^*(h(K_2,\theta), K_2, \theta) - F(K_1, l^*(h(K_1,\theta), K_1, \theta) |$ *, for all* $K_2 \geq K_1.\}$

A standard argument shows that the space of consumption functions $\bar{\mathbf{H}} \subset \mathbf{H}_l$ is a closed, pointwise compact, and equicontinuous set of functions. Then by a standard application of Arzela-Ascoli, $\bar{\mathbf{H}}$ is a compact, convex, order interval in $\mathbf{H}_l$. Notice that the restriction on consumption in the space $\bar{\mathbf{H}}$ that distinguishes it from $\mathbf{H}_l$ implies that the investment function $K' = F_h - h$ is an increasing functions of the current capital stock K which follows because F_h is increasing in K (since l^* is decreasing in K, the marginal products of capital and labor are positive).

We note some important properties of the operator A and the space $\bar{\mathbf{H}}$,

Proposition 8.6.3. *Under assumptions P3, P4, T3 and D2,* $\bar{\mathbf{H}}$ *is a complete lattice and* A *is a transformation on* $\bar{\mathbf{H}}$*, i.e.,* $Ah \subset \bar{\mathbf{H}}$.

Proof: See Datta, Mirman, and Reffett[22], Lemma 1 and Theorem 2.■

To apply a lattice-theoretic fixed point theorem, we need to verify isotonicity,

Proposition 8.6.4. *Under assumption P3, P4, T3 and D2, A is isotone on* $\mathbf{H}_l$.

Proof: Datta et al[22] ,Theorem 3.■

We now restrict the mapping A to the subspace $\bar{\mathbf{H}}$ (which is well-defined since A is continuous, $\bar{\mathbf{H}}$ is compact, order subinterval in $\mathbf{H}_l$ and apply a version of Amann's theorem,

[18] A set is relatively compact if its closure is compact.

Proposition 8.6.5. *Under assumptions P3, P4, T3 and D2 , the set of fixed points of* $A : \bar{\mathbf{H}} \rightarrow \bar{\mathbf{H}}$ *has a maximal fixed point* $Ah^* \in \bar{\mathbf{H}}$ *such that* $lim_{n\rightarrow\infty} A^n F \rightarrow Ah^* = h^*$*, uniformly..*

Proof: Apply Proposition 8.4.1; see also Datta et al[22], Proposition 2.■

8.6.3 The Uniqueness of Equilibrium

As in the case of classical production with inelastic labor supply, we apply our new approach to existence of strictly positive fixed points once again. First, define a function $f^u(K,\theta) = f(K, 1-\hat{l}(K,\theta),\theta)$ and consider the set of functions $\mathbf{M}$ for the inverse of marginal utility in equilibrium,

Definition: $\mathbf{M}_l = \{m(K,\theta) \mid m : \mathbf{K}\times\Theta \rightarrow \mathbf{K}$ *is continuous*; $0 \leq m(K,\theta) \leq \frac{1}{u_c(f^u(K,\theta),\hat{l}(K,\theta))}$ *for* $K > 0$; $m(K,\theta) = 0$ *for* $K = 0$; *and* $\frac{r(K',\theta)}{m(K',\theta)} \leq \frac{r(K,\theta)}{m(K,\theta)}$ *for* $K' \geq K\}$

By assumption D2, $r(K,\theta)$ is continuous and $\mathbf{K}$ is a compact set, therefore, r is uniformly continuous. As in Section 8.4, one can verify that $\mathbf{M}_l$ is a closed, equicontinuous, pointwise compact subset of the space of continuous functions on a compact topological space, namely $\mathbf{C}^+(\mathbf{S})$. $\mathbf{M}_l$ is, therefore, compact.

We now define a suitable operator on the space $\mathbf{M}_l$ and find a unique strictly positive fixed point of this operator (to prove the uniqueness of recursive equilibrium in $\bar{\mathbf{H}}$). As before, define the function $H(m,K,\theta)$ for each $m \in \mathbf{M}_l$ implicitly as follows (the following lemma makes sure that this definition is meaningful),

$$\begin{aligned} u_c(H(m(K,\theta),K,\theta), l(H(m(K,\theta),K,\theta),K,\theta)) &= \frac{1}{m(K,\theta)}, m > 0; \\ \text{and} \quad H(m,K,\theta) &= 0,\ m = 0. \end{aligned}$$

Note that, $H(m(K,\theta),K,\theta) = h(K,\theta)$, pointwise. The proof of uniqueness takes place in three lemmata.

Lemma 8.6.1. *Assume P3, P4, T3 and D2. Then the mapping* $H(m,K,\theta)$ *is well-defined for each* m, K *and* θ*.*

Proof: Datta, Mirman and Reffett[22], Lemma 2.■

To characterize $H(m,K,\theta)$, take $m' \geq m$ in the pointwise partial order on $\mathbf{M}_l$**.** Define $h_2 = H(m',K,\theta)$ and $h_1 = H(m,K,\theta)$. Notice when $m' \geq m$, we have $h_2 \geq h_1$. We can now show that $f(k, 1-l(H(m,K,\theta),K,\theta)) - H(m,K,\theta)$ is decreasing in m by the definition of $H(m,K,\theta)$. Define

$$\Delta(h, f_h - h, \theta) =$$
$$\beta \int u_c(h(f_h - h, \theta'), l(h(f_h - h, \theta'), f_h - h, \theta'))r(f_h - h, \theta')\chi(\theta, d\theta'))$$

Then for $m' \geq m$, we have the following inequality

$$\begin{aligned} u_c(Ah_1, l(Ah_1, K, \theta) &= \Delta(h_1, f_{Ah_1} - Ah_1, \theta) \\ &\geq \Delta(h_2, f_{Ah_1} - Ah_1, \theta) \end{aligned}$$

Therefore, for such a perturbation of h, the mapping Z used in the definition of Ah is now nonnegative. Therefore, the first term in the definition of Z must decrease and the second term must increase in a solution Ah_2. The latter implies $f_{Ah_2} - Ah_2 \leq f_{Ah_1} - Ah_1$. Consequently, by the definition of $H(m, K, \theta)$, $f(K, 1 - l(H(m, K, \theta), K, \theta)) - H(m, K, \theta) = f_{H(m)} - H(m)$ must be decreasing in m.

Now, define the mapping

$$\hat{Z}(m, \tilde{m}, K, \theta) = \frac{1}{\tilde{m}} - \beta \int_{\Theta} \frac{r(f_{\tilde{m}} - H(\tilde{m}, K, \theta), \theta')}{m(f_{\tilde{m}} - H(\tilde{m}, K, \theta), \theta')} \chi(\theta, d\theta'),$$

where $f_m - H(m, K, \theta) = f(K, 1 - l(H(m, K, \theta), K, \theta), \theta) - H(m, K, \theta)$ and we are ready to define the operator,

$$\hat{A}(m) = \{\tilde{m} \in \mathbf{M}_l \mid \hat{Z}(m, \tilde{m}, K, \theta) = 0, \text{for } m > 0; \text{and, } 0 \text{ elsewhere}\}.$$

Defining the standard partial order on $\mathbf{M}_l$, that is, $m' \geq m$, $m', m \in \mathbf{M}_l$ if and only if $m'(K, \theta) \geq m(K, \theta)$ for all (K, θ).
Finally, if $m'(K, \theta) > m(K, \theta)$, $m, m' \in \mathbf{M}_l$, the mapping H must be such that $u_c(H(m, K, \theta), l(H(m, K, \theta), K, \theta))$ is decreasing in m for each (K, θ). Since $h \in \bar{\mathbf{H}}$, $u_c(c, l(c, K, \theta)$ is decreasing in c, and there exists $h, h' \in \bar{\mathbf{H}}$ such that $h' = H(\frac{1}{u_c(h', l(h', K, \theta))}, K, \theta) = H(m', K, \theta)$ and $h = H(\frac{1}{u_c(h, l(h, K, \theta))}, K, \theta) = H(m, K, \theta)$.
If the operator $\hat{A}m$ is well defined, we are able to relate orbits of the operator $\hat{A}^n m_0 \in \mathbf{M}_l$ to those of the operator $A^n h_0 \in \bar{\mathbf{H}}$ by the following construction. Consider some $h_0 \in \bar{\mathbf{H}}$. For such an h_0, there exists an $m_0 = \frac{1}{u_c(h_0, l(h_0, K, \theta))} \in \mathbf{M}_l$ such that $H(\frac{1}{u_c(h_0, l(h_0, K, \theta))}) = h_0$. By definition,

$$\begin{aligned} \hat{Z}(m_0, \hat{A}m_0, K, \theta) &= \hat{Z}(H(\frac{1}{u_c(h_0, l(h_0, K, \theta))}, K, \theta), \\ \hat{A}H(\frac{1}{u_c(Ah_0, l(Ah_0, K, \theta))}), K, \theta) &= Z(h_0, Ah_0, K, \theta). \end{aligned}$$

Therefore, $h_1 = Ah_0 = H(\frac{1}{u_c(Ah_0, l(Ah_0, K, \theta))}) = H(\hat{A}m_0)$. A similar argument establishes $A^n h_0 = H(\hat{A}^n m_0)$, $n = 1, 2, ...$ We next show that the operator $\hat{A}m$ is well defined.

Lemma 8.6.2. *Under assumptions P3, P4, T3 and D2, the operator $\hat{A}$ is a well-defined transformation on $\mathbf{M}_l$.*

Proof: Datta et al[22], Lemma 3.■

We now provide the last step of our argument.

Lemma 8.6.3. *Under assumptions P3, P4, T3 and D2, if $\hat{A}$ has a strictly positive fixed point then it is unique.*

Proof: Since $\hat{Z}$ is increasing in m, and decreasing in $\tilde{m} = \hat{A}m$, $\hat{A}m_1 \geq \hat{A}m_2$ for $m_1 \geq m_2$. A sufficient condition for strong sublinearity is,

$$\hat{Z}(tm, t\hat{A}m, K, \theta) > \hat{Z}(tm, \hat{A}tm, K, \theta).$$

This inequality follows since $m \in \mathbf{M}_l$, and r decreasing in K. Thus,

$$\hat{Z}(tm, t\hat{A}m, K, \theta) = \frac{1}{\tilde{m}} - \beta \int_{\Theta} \frac{r(f_{\tilde{m}} - H(t\tilde{m}), \theta')}{m(f_{\tilde{m}} - H(t\tilde{m}), \theta')} \chi(\theta, d\theta') > 0,$$

and $\hat{Z}(tm, \hat{A}tm, K, \theta) = 0$. Notice also that by examining the definition of $\hat{A}m$, given the Inada condition, $\hat{A}m$ is K_0-monotone. Therefore, by the same argument in the classical production with inelastic labor supply case, via the extension of a uniqueness theorem in Krasnosel'skii and Zabreiko [49] found in Coleman [19], if $\hat{A}$ has a strictly positive fixed point, it is unique in $\mathbf{M}_l$ (and, therefore, in $\bar{\mathbf{H}}$). ■

Finally, we prove the existence a strictly positive fixed point.

Proposition 8.6.6. *Under assumptions P3, P4, T3 and D2, there is a unique strictly positive MEDP.*

Proof: Note that, as $\mathbf{M}_l$ is an order interval in a solid cone of continuous functions, and $\hat{A}m$ is strongly sublinear on its interior. Also, given the definition of $\hat{A}m$, whenever $m > 0$, necessarily $\hat{A}m \ > 0$; further $\hat{A}m < m$. In particular, k_0-monotonicity implies there is a point $m_0 >> 0$ that maps up. Therefore again, we have $\hat{A}$ a cone compression on the order interval $\mathbf{M}_l$. By Krasnosel'skii and Zabreiko ([49], Theorem 46.4), we conclude $\hat{A}m$ has a strictly positive fixed point. Further as $\hat{A}m$ is additionally K_0-monotone, therefore actually has a unique strictly positive fixed point. Finally, again exploiting the relationship between the orbits of $\hat{A}$ and A discussed earlier in this section before the beginning of this proof, as we have a unique strictly positive fixed point for $\hat{A}$ in $\mathbf{M}_l$, namely $m^* > 0$, we have a unique fixed point for A , say $h^* > 0$ in $\bar{\mathbf{H}}$.Since $h^* > 0$ implies strictly positive consumption, it is a MEDP. ■
Again, we note that $h^* > 0$ is crucial for characterizing prices in $l^1_+\backslash\{0\}$ (e.g., see Le Van and Vailakis[52])

8.7 Concluding Remarks

In this chapter, we survey a new and emerging approach to recursive competitive equilibrium theory that is commonly referred to as isotone recursive methods and we focus on economies with homogenous agents. These methods allow one to unify results on the existence, characterization and computation of MEDPs and the SME for a large class of economies commonly encountered in applied dynamic macroeconomics. Datta, Mirman, Morand and Reffett[23] develop isotone recursive methods to study MEDPs in the stochastic Ramsey models of Becker and Zilcha [10] with heterogeneous agents. They find sufficient conditions for MEDPs to be isotone and Lipschitz continuous and for MEDPs that are just Lipschitz continuous. Another application of isotone recursive methods to the case of heterogeneous agent models is in overlapping generation models. These models form the basis of much work in lifecycle theory on social security. Erikson, Morand and Reffett[31] and Morand and Reffett[63] apply the isotone recursive approach to a class of two period stochastic lifecycle-overlapping generations models with social security, production nonconvexities and public policy (fiscal or monetary). Primarily, they consider the case of i. i. d. shocks but provide some preliminary results with Markov shock. This paper (along with others mentioned below) indicate an important direction for isotone recursive methods in future research; namely, the study of Stationary Markovian equilibrium (SME). In this survey, this question is not addressed. In the existing literature, an SME is often considered to be an invariant distribution (e.g., Hopenhayn and Prescott [43]).[19] In many cases the existence of SME can be established with applications of the fixed point theory for complete partially ordered sets as discussed in section 8.3 though the applications might not be as simple as in the case of continuous MEDPs. For example, consider the existence of a stationary Markov equilibrium for the case of nonconvex production technologies (in addition to Erikson, Morand and Reffett [31], see Hopenhayn and Prescott[43] and Mirman, Morand and Reffett[59]). In general, the extremal MEDPs are only semicontinuous;[20] often, Propositions 8.3.6 and 8.3.7 cannot be applied (as operators are not necessarily order-continuous on and/or appropriately topologically continuous on their respective domains). However, the existence of an extremal limiting distribution can be guaranteed by applying Proposition 8.3.3 or 8.3.5 (along with Proposition 8.3.4). The computational issues for numerical solutions to approximate an SME can be addressed using Propositions 8.3.8 and 8.3.9. Proposition 8.3.8 provides a collection of generalized iterative procedures (that are not necessarily successive approximations). Proposition 8.3.9 provides sufficient conditions for the existence of an underlying set for iterations that is cofinal (because

[19] In other work, an SME is considered to be an ergodic distribution with a nongenerate support. Our remarks apply to this case also.

[20] Erikson, Morand, and Reffett[31] and Mirman, Morand, and Reffett[59] provide sufficient conditions that distinguish the cases of the existence of continuous MEDPs and the existence of semicontinuous MEDPs.

the underlying space of probability measures on a compact Polish space is a compact metric space). Therefore, the existence of monotone iterative methods on a countable indexation is obtained via Heikkilä and Lakshmikantham's[39] generalized iterative procedures. Heikkilä and Salonen[41][42] and Heikkilä[37] provide extensive discussions on implementation of such theoretical construction. As for comparative statics of an SME for economies with semicontinous MEDPs: the space of probability measures defined on a compact Polish space is not necessarily a lattice (e.g., consider probability measures defined over a support that $S \subset R_+^N$ for $N > 1$), therefore, the SCS and WCS monotone selection theorems of Veinott in Propositions 8.3.10 and 8.3.11 do not apply. The space of probability measures on a compact subset of a Polish space is a CPO (as it is a compact metric space). One can apply the WCS conditions in Proposition 8.3.12 to obtain an isotone selection between the space of economies and the set of SME (see Mirman, Morand and Reffett[59] section 3 for a discussion). We feel that similar new and interesting applications of recent work in order theoretic fixed point theory will become paramount in future work that seeks to study MEDPs and SME.

Potentially the most important extension of isotone recursive methods is the so-called "mixed-monotone" recursive methods first presented systematically in Reffett[69], and subsequently applied in Mirman, Reffett and Stachurski[60] to Bewley[12] models with a single asset. The mixed-monotone method build upon the mixed-monotone fixed point theory (also known as "coupled" fixed point theory) that has been developed in the literature on discontinuous differential equations. These methods appear powerful, and deliver MEDPs *on the natural state space of current states* even in situations where MEDPs are not unique. Discussions of mixed monotone fixed point theory are found in Amann[4], Heikkilä and Lakshmikantham[39] and Reffett[68], to name a few. The discovery of mixed-monotone recursive methods appears to be a giant step forward in developing methods based on constructive fixed point theory that can be applied in a wide-array of economic situations. One no longer needs to have isotone operators (nor fixed point spaces) where underlying constructions are based on isotonicity. One problem with this method is that one requires sufficient topological structure relative to the fixed point space for antitone transformations to possess the fixed point property. Preliminary work in a series of recent papers by Reffett[67][68][69], Datta and Reffett [24], and Mirman, Reffett and Stachurski [60] indicate that for many interesting economies, such "mixed monotone" fixed point methods are available. For example, these methods provide successive approximation algorithms for computing Bewley models of the sort studied in Aiyagari[2], Krusell and Smith[51], and Miao[57]. In addition, isotone recursive methods are a special case of mixed monotone recursive methods and can be studied in a "single" step using an isotone operators instead of multi-steps for mixed-monotone operators. Mixed monotone recursive methods unify the existing approaches to characterize MEDPs and the SME by allowing researchers to obtain more general results that relate monotone iterative computational procedures to actual fixed point constructions. As numerical methods described in standard monographs (e.g., Krasnosel'skii et al[48]) can build on explicit operators to obtain error estimates of Santos and Vigo[74] and Santos[72]. In principle, one might be able to obtain a complete set of iterative methods for studying numerically, the quantitative properties of the SME in a large class of macroeconomic models to a specified degree of accuracy, which seems to be the goal of quantitative macroeconomics (e.g., real business cycle studies). Indeed, qualitative methods can provide an essential, first step in obtaining a useful (and, mathematically credible) quantitative theory of macroeconomic fluctuations and long-run growth.

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9. Discrete-Time Recursive Utility

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This chapter focuses on the fundamentals of discrete-time models using recursive utility. We examine the relation between preferences, utility, and aggregator, the existence of optimal paths, and several notions of impatience. In the one-sector model, we characterize optimal paths and derive a turnpike theorem.[1]

Topics beyond the scope of this paper include continuous time recursive utility, models involving uncertainty, the turnpike property in multisector models, and properties of Pareto optima and equilibrium in multisector models.[2]

Section 9.1 discusses the limitations of time additive preferences and some of the benefits of using a more general recursive utility specification. Section 9.2 examines the relation between recursive preferences and the associated aggregator function. A general result on existence of optimal paths is shown in Section 9.3. Sections 9.4 and 9.5 focus on the one-sector model. Existence of optimal paths and dynamic programming is considered in Section 9.4. Section 9.5 characterizes optimal paths via the Euler equations and then goes on to prove a one-sector turnpike theorem. Finally, Section 9.6 takes a brief look at the case where preferences are both homothetic and recursive.

9.1 Why Recursive Utility?

Since Ramsey [37], optimal growth models have primarily focused on the case of time additive separable (TAS) utility. Reasons for its popularity are easy to find. It is intuitively simple: We discount each period's utility at a constant

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[1] For a more comprehensive treatment of the discrete-time case, see the book by Becker and Boyd [5].

[2] Epstein [19] examines when a recursive utility function is also a von Neumann-Morgenstern utility function. Existence and characterization of optimal paths is studied in [6] and [4]. Pareto optima and turnpikes in multisectoral models have been investigated by Epstein ([20], [21]) and Dana and Le Van ([13], [14], [15]).

rate before summing over time. It is often possible to obtain clear-cut analytic results. If a problem is not quite standard, a large amount of theory developed by mathematicians is readily applicable. It allows the use of dynamic programming.

In spite of these advantages, TAS utility also has some shortcomings. In particular, it builds in some assumptions about the marginal rate of substitution between consumption in different periods that may not be desirable. This is most obvious when considering consumption paths that are stationary. In that case, the marginal rate of substitution between consumption today and consumption in the following period is the inverse of the discount factor. It is unaffected by the level of consumption. If there are multiple consumption goods, this stationary marginal rate of substitution is also unaffected by the level of consumption of those other goods.[3]

This constant marginal rate of substitution severely constrains the long-run behavior of economic models. For example, a consumer facing a fixed interest rate will try either to save without limit, or to borrow without limit, except in the knife-edge case where the discount rate equals the interest rate.

This problem is especially severe when there are heterogeneous households. Unless all of the households have the same discount rate, the most patient household ends up with all the capital in the long run, while all other households consume nothing, using their labor income to service their debt (Becker, [3]). Recursive utility allows for upward (or downward!) sloping long-run capital supply curves and non-degenerate long-run wealth distributions.

The constant discount rate hypothesis also creates problems for the calculation of welfare losses arising from capital income taxation. In TAS models, the long-run supply of capital by households will be perfectly elastic at the discount rate. We are entitled to be a bit skeptical of the resulting welfare analysis.

When analyzing growing economies, the special behavior of TAS utility on paths that grow at a constant rate facilitates the construction of tractable models. Interestingly, there are non-TAS utility functions that exhibit the same behavior (Dolmas, [17]; Farmer and Lahiri[24]).

9.2 Recursive Utility and Aggregators

Alternative methods of aggregating a sequence of period utilities have long been proposed. Irving Fisher [25] suggested combining today's utility and tomorrow's utility as if they were two different consumption goods. The result could then be analyzed using indifference curves over present and future utility. Fisher's approach was formalized and axiomatized by Koopmans and his collaborators

[3] The use of TAS utility also requires that the intertemporal elasticity of substitution be equal to the coefficient of relative risk aversion. Epstein and Zin [23] used techniques borrowed from the recursive utility literature to construct Kreps-Porteus preferences that relax that restriction.

in the 1960's and early 1970's (Koopmans,[27] 1960; Koopmans, Diamond, and Williamson, [30] 1964; Koopmans, [28] 1972a; Koopmans, [29] 1972b).

Recursive utility is defined in an infinite horizon context. Time is indexed by $t = 1, 2, \ldots$. Let $\mathbf{c} = (c_1, c_2, \ldots)$ be a sequence of consumption bundles $c_t \in \mathbb{R}^n$ in each time period t. We let S denote the *shift operator* defined by $S(c_1, c_2, \ldots) = (c_2, c_3, \ldots)$ and π the projection onto the first co-ordinate, $\pi\mathbf{c} = c_1 \in \mathbb{R}^n$. Let $\mathcal{S}$ denote the space of sequences in $\mathbb{R}^n$ and $\mathbf{X}$ be a subset of $\mathcal{S}$ such that $S\mathbf{X} \subset \mathbf{X}$. We give $\mathcal{S}$ the product topology. Thus $\mathbf{c}^n \to \mathbf{c}$ if and only if $c^n_t \to c_t$ for every t.

Suppose we have continuous preferences $\succsim$ defined on $\mathbf{X}$. We say these preferences have a *recursive utility representation* if there is a function W (called the *aggregator*) and a subutility function $u\colon \mathcal{S} \to \mathbb{R}$ obeying

$$U(\mathbf{c}) = W(u(c_1), U(S\mathbf{c}))$$

for every $\mathbf{c} \in \mathbf{X}$. A simple example of recursive preferences is the TAS form $U(\mathbf{c}) = \sum_{t=1}^{\infty} u(c_t)$ which has aggregator $W(x, y) = x + \delta y$ and subutility function u. A non-TAS example of a recursive utility function is given by $-\sum_{t=1}^{\infty} \exp -[\sum_{s=1}^{t} v(c_s)]$.[4] This function has subutility v and aggregator $W(x, y) = (-1 + y)\exp(-v(x))$. We refer to it as the *EH aggregator*.

Koopmans's Axioms

Koopmans's axioms are:

(K1) $\succsim$ is a *stationary relation*: $(z, \mathbf{x}) \succsim (z, \mathbf{x}')$ for all $z \in \pi\mathbf{X}$ if and only if $\mathbf{x} \succsim \mathbf{x}'$.

(K2) $\succsim$ exhibits *limited independence*: for all $z, z' \in \pi\mathbf{X}$ and $\mathbf{x}, \mathbf{x}' \in \mathbf{X}$, $(z, \mathbf{x}) \succsim (z', \mathbf{x})$ if and only if $(z, \mathbf{x}') \succsim (z', \mathbf{x}')$.

(K3) $\succsim$ is a *sensitive relation*: there is an $\mathbf{x} \in \mathbf{X}$ and a $z, z' \in \pi\mathbf{X}$ with $(z', \mathbf{x}) \succ (z, \mathbf{x})$.

It is easy to show that any preference order with a recursive representation obeys Koopmans's Axioms. Koopmans [27] showed that if a preference order obeys (K1)–(K3) and has a utility representation U, then the utility representation is recursive. Koopmans also showed that there was a unique aggregator and subutility associated with U.

Potentially, the aggregator gives the possibility of representing preferences in a compact form. However, not any function can be an aggregator. This leads to a new question. Given a would-be aggregator W and subutility u, is there a corresponding recursive utility function? If so, what domain is it defined over?

9.2.1 Construction of Recursive Utility from an Aggregator

Lucas and Stokey [33] provided the first major result concerning construction of the utility function. They used the contraction mapping theorem to construct

[4] This is a discrete-time version of the modified Uzawa [43] utility used by Epstein and Hynes [22]

a utility function defined on all non-negative sequences under some restrictive conditions on W. Define the *Koopmans operator* T_W on the continuous functions on $\mathcal{S}_+$ by

$$T_W(f)(\mathbf{c}) = W(c_1, f(S\mathbf{c})).$$

Under Lucas and Stokey's assumptions, the contraction mapping theorem shows that the Koopmans operator has a unique fixed point which is the desired recursive utility function.

The most important restriction was that W be bounded. This ruled out many commonly used TAS utility functions. For example, the aggregator $W(x,y) = (1-\sigma)^{-1}x^{1-\sigma} + \delta y$, which leads to the commonly used utility function $U(\mathbf{c}) = (1-\sigma)^{-1}\sum_{t=1}^{\infty}\delta^{t-1}c_t^{1-\sigma}$ does not fit into Lucas and Stokey's framework, even when $0 < \sigma < 1$.

Many authors have proposed solutions to this problem. The most notable are the weighted contraction method (Boyd, [10]), the "partial sum" method used by Boyd to handle unbounded aggregators and used for all aggregators by Le Van and Vailakis [32], Streufert's ([41], [42]) biconvergence condition, and the k-local contraction recently used by Rincón-Zapatero and Rodríguez-Palmero ([38], [39]).

We will examine the construction of the utility function from an aggregator in detail. For convenience, we will absorb the subutility into the aggregator for the remainder of the paper. Thus we write $W(x,y)$ rather than $W(u(x),y)$. Nonetheless, to maintain compatibility with Koopmans's Axioms, we will presume our aggregators can be written using a subutility. We will first follow Boyd's approach and then comment on biconvergence. Rincón-Zapatero and Rodríguez-Palmero's method is akin to Boyd's, but is able to cope with weaker bounds than given by (W2) below.

Aggregator

A function $W\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ is an aggregator if:

(W1) W is continuous on $\mathbf{X} \times \mathbf{Y}$ and increasing in both c and y.

(W2) W obeys a Lipschitz condition of order one: there exists $\delta > 0$ such that $|W(x,y) - W(x,y')| \le \delta|y - y'|$ for all x in $\mathbf{X}$ and y, y' in $\mathbf{Y}$.

(W3) $(T_W^N y)(\mathbf{c})$ is concave in $\mathbf{c}$ for all N and all constants $y \in \mathbf{Y}$.

When W is differentiable the Lipschitz bound in (W2) is given by $\delta = \sup W_2(c,y)$. In the TAS case, it coincides with the discount factor. This bound is a strong form of Koopmans, Diamond, and Williamson's [30] concept of time perspective. As viewed from the present, future utilities appear closer and closer together as they are further out in time, just as railroad tracks appear to converge in the distance. The Lipschitz bound δ gives us our first measure of impatience, which we refer to as the *time perspective factor* with corresponding rate $\delta^{-1} - 1$.

The sole purpose of condition (W3) is to ensure concavity of the utility function. It is not required for the existence results. Joint concavity of W is not required for the associated utility function to be concave. Although the EH aggregator is not concave, the corresponding utility function $U(\mathbf{c}) = -\sum_{t=1}^{\infty} \exp[-\sum_{\tau=1}^{t} v(c_\tau)]$ is concave whenever $v'' < 0$. More generally, when the utility function is the limit of the functions $(T_W^N(0))(\mathbf{c})$, (W3) ensures concavity is inherited by U.

Let $\mathcal{C}$ be the space of continuous functions on $\mathcal{S}$. Let $\varphi \in \mathcal{C}$ with $\varphi > 0$. Define the φ-weighted norm by $||f||_\varphi = \sup |f(\mathbf{x})/\varphi(\mathbf{x})|$. The space $\mathcal{C}_\varphi = \{f \in \mathcal{C} : ||f||_\varphi < \infty\}$ is then a Banach space under the φ-norm $|| \cdot ||_\varphi$.

Theorem 9.2.1. Weighted Contraction Mapping Theorem
Suppose $T\colon \mathcal{C}_\varphi \to \mathcal{C}$ *such that:*

1. T *is non-decreasing* ($f \leq g$ *implies* $Tf \leq Tg$).
2. $T(0) \in \mathcal{C}_\varphi$.
3. $T(\xi + A\varphi) \leq T\xi + A\theta\varphi$ *for some constant* $\theta < 1$ *and all* $A > 0$.

Then T *has a unique fixed point.*

Proof. The proof is inspired by Blackwell (1965). Let $f, g \in \mathcal{C}_\varphi$ and consider $||f - g||_\varphi$. Then $-||f - g||_\varphi\varphi \leq f - g \leq ||f - g||_\varphi\varphi$. Rearranging, we find $f \leq g + ||f - g||_\varphi\varphi$ and $g \leq f + ||f - g||_\varphi\varphi$. Using properties (1) and (3), we obtain $Tf \leq Tf + \theta||f - g||_\varphi\varphi$ and $Tg \leq Tg + \theta||f - g||_\varphi\varphi$. Together, these yield $||Tf - Tg||_\varphi \leq \theta||f - g||_\varphi$. This shows T is a strict contraction from $\mathcal{C}_\varphi$ to $\mathcal{C}$.

To show T maps into $\mathcal{C}_\varphi$, set $g = 0$ to obtain $||Tf - T(0)||_\varphi \leq \theta||f||_\varphi$. By (2) $T(0) \in \mathcal{C}_\varphi$ which means $Tf \in \mathcal{C}_\varphi$ with $||Tf||_\varphi \leq ||T(0)||_\varphi + \theta||f||_\varphi$.

As T is a strict contraction on $\mathcal{C}_\varphi$, it has a unique fixed point. □

Before attempting to construct the utility function, we must decide what domain is appropriate. Obviously, the utility function will live on a subset of $\mathcal{S}_+$. The domain ultimately chosen may depend on the problem at hand. One of the motivations for studying recursive utility is to admit non-degenerate equilibria. This demands we use a subset that is appropriate for equilibrium problems, a linear space. If we are focusing on capital accumulation problems we may further restrict the domain. Streufert [41] exploits that fact to sharpen the utility existence theorem.

For $\beta \geq 1$, define the β-norm by $|\mathbf{c}|_\beta = \sup_t ||c_t||/\beta^t\}$ where $|| \cdot ||$ is the Euclidean norm on $\mathbb{R}^n$. Then define the β-weighted ℓ^∞ space by $\ell^\infty(\beta) = \{\mathbf{c} \in \mathcal{S} : |\mathbf{c}|_\beta < \infty\}$. The space $\ell^\infty(\beta)$ is a Banach space under the norm $|\cdot|_\beta$. We refer to the associated topology as the β-topology. Since a sequence that converges in β-norm must converge in each coordinate, the β-topology is stronger than the product topology on $\ell^\infty(\beta)$.

Theorem 9.2.2. Continuous Existence Theorem
Suppose $W\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ *obeys (W1) and (W2),* φ *is continuous on some* $\mathcal{A} \subset \mathcal{S}$ *with* $\pi\mathcal{A} \subset \mathbf{X}$ *and* $S\mathcal{A} \subset \mathcal{A}$. *Suppose further* $W(\pi\mathbf{c}, 0)$ *is* φ*-bounded and* $\delta||\varphi \circ S||_\varphi < 1$. *Then there exists a unique* $U \in \mathcal{C}_\varphi(\mathcal{A})$ *such that* $W(\pi\mathbf{c}, U(S\mathbf{c})) = U(\mathbf{c})$. *Moreover,* $(T_W^N 0)(\mathbf{c}) \to U(\mathbf{c})$ *in* $\mathcal{C}_\varphi$.

Proof. The conditions on $\mathcal{A}$ insure everything makes sense. Since W is increasing in y, the Koopmans operator T_W is increasing. Now

$$\frac{|T_W(0)|}{\varphi(\mathbf{c})} = \frac{|W(c_1,0)|}{\varphi(\mathbf{c})} < \infty$$

because $W(\pi\mathbf{c},0)$ is φ-bounded. Moreover,

$$\begin{aligned} T_W(\xi + A\varphi) &= W(c_1, \xi(S\mathbf{c}) + A\varphi(S\mathbf{c})) \\ &\le W(c_1, \xi(S\mathbf{c})) + A\delta\varphi(S\mathbf{c}) \\ &\le T_W\xi + A\delta||\varphi \circ S||_\varphi \varphi(\mathbf{c}) \end{aligned}$$

by the Lipschitz condition (W2). Applying the Weighted Contraction Mapping Theorem with $\theta = \delta||\varphi \circ S||_\varphi < 1$ shows that T_W has a unique fixed point U.

Now consider $||U(\mathbf{c}) - (T_W^N 0)(\mathbf{c})||_\varphi \le \delta^N ||U(S^N\mathbf{c})||_\varphi \le ||U||_\varphi (\delta||\varphi \circ S||_\varphi)^N$. As the right-hand side converges to zero, $(T_W^N 0)(\mathbf{c}) \to U(\mathbf{c})$. □

A couple of applications will help clarify how the theorem may be used. The general strategy is to pick either $W(x,0)$ or a function bounding it for the weighting function φ. Consider the TAS aggregator $W(x,y) = x^{1-\sigma} + \delta y$ for $0 < \sigma < 1$. Choose β with $\delta\beta^{1-\sigma} < 1$ and set $\mathcal{A} = \ell^\infty(\beta)_+$. Here $W(x,0) = x^{1-\sigma}$. This is not positive, so we add one and compose with the β-norm to get a weighting function. That is, $\varphi(\mathbf{c}) = 1 + |\mathbf{c}|_\beta^{1-\sigma}$. Then

$$W(c_1, 0) = c_1^{1-\sigma} \le |\mathbf{c}|_\beta^{1-\sigma} < \varphi\mathbf{c},$$

so $W(\pi\mathbf{c},0)$ is φ-bounded. Also,

$$\varphi(S\mathbf{c}) = 1 + |S\mathbf{c}|_\beta^{1-\sigma} \le 1 + (\beta|\mathbf{c}|_\beta)^{1-\sigma} \le \beta^{1-\sigma}|\mathbf{c}|_\beta,$$

which implies $\delta||\varphi \circ S||_\varphi < 1$.

The EH aggregator $W(x,y) = (-1+y)e^{-v(x)}$ provides a second example. Here $\mathbf{X} = \mathbb{R}_+$ and $\mathbf{Y} = \mathbb{R}_-$. Suppose v is increasing with $v(0) > 0$. Then W is increasing and obeys a Lipschitz condition with $\delta = e^{-v(0)} < 1$. Since $|W(x,0)| = e^{-v(x)} \le 1$, we set $\varphi = 1$. The existence theorem then shows that the corresponding utility function is continuous and bounded on $\mathcal{A} = \mathcal{S}_+$.

This approach has several limitations. When $W(x,0)$ is not bounded below it becomes impossible to construct an appropriate φ. This difficulty can be handled by first constructing the utility function on a restricted space of sequences that are bounded away from zero (so utility can be bounded below), and then using a limiting argument to remove the lower bound.

For $0 < \gamma \le \beta < \infty$, define ${}_\gamma|\mathbf{c}| = \inf ||c_t||/\gamma^{t-1}$ and $\ell^\infty(\beta,\gamma) = \{\mathbf{c} \in \mathcal{S} : 0 < {}_\gamma|\mathbf{c}| \text{ and } |\mathbf{c}|_\beta < \infty\}$. This is the set of paths with growth factors of at least γ and at most β.

To see how utility can be defined on such a space, consider the partial sums of $U(\mathbf{c}) = \sum_{t=1}^\infty \log c_t$. If $\mathbf{c} \in \ell^\infty(\beta,\gamma)$, ${}_\gamma|\mathbf{c}|\gamma^{t-1} \le c_t \le |\mathbf{c}|_\beta \beta^{t-1}$. Then

$$\begin{aligned}\sum_{t=1}^{T}[(t-1)\log\gamma+\log{}_{\gamma}|\mathbf{c}|] &\le \sum_{t=1}^{T}\delta^{t-1}\log c_t\\ &\le \sum_{t=1}^{T}\delta^{t-1}[(t-1)\log\beta+\log|\mathbf{c}|_{\beta}].\end{aligned}$$

Here the utility partial sums converge because they are squeezed between the partial sums of convergent series. The limit is not uniform, so we cannot conclude it is continuous. A slightly different approach gives us upper semicontinuity. Consider the set

$$\mathbf{X}=\{\mathbf{c}\in\ell^{\infty}(\beta)_{+}:|\mathbf{c}|_{\beta}<A\}.$$

On this set, $\sum_{t=1}^{T}\delta^{t-1}(\log c_t-(t-1)\log\beta-\log A)$ has non-positive terms. As the infimum of upper semicontinuous functions, the limit is upper semicontinuous. It differs from the utility function by a constant, so utility is also upper semicontinuous $\mathbf{X}$. Boyd's "partial summation" method adapts this approach to recursive utility.

Before proceeding, we have to consider the consequences of admitting $-\infty$ as a possible value for utility. The obvious solution to Koopmans's equation may not be the only one. In fact, $U(\mathbf{c})=-\infty$ may satisfy the recursion, as it does in the logarithmic case. However, it does not match up with the solution we derived on $\ell^{\infty}(\beta,\gamma)_{+}$. We will rule out such solutions as unreasonable.

The general strategy is to first derive utility on some well-behaved sequences in $\ell^{\infty}(\beta,\gamma)_{+}$, and then use recursive substitution to extend utility to $\ell^{\infty}(\beta)_{+}$.

Assumption
(W1′) $W\colon\mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$ is increasing in both arguments, upper semicontinuous on $\mathbf{X}\times\mathbf{Y}$ continuous for $x>0$ and $y>-\infty$, and obeys $W(x,-\infty)=W(0,y)=-\infty$ for all $x\in\mathbf{X}$ and $y\in\mathbf{Y}$.

Theorem 9.2.3. Upper Semicontinuous Existence Theorem
Suppose W obeys (W1′) and satisfies the Lipschitz condition (W2) whenever it is finite. Suppose further there are increasing functions g and h with $g(||x||)\le W(x,0)\le h(||x||)$. Set $\varphi(\mathbf{c})=\max\{h(|\mathbf{c}|_{\beta}),-g({}_{\gamma}|\mathbf{c}|)\}$. If $\varphi>0$ with $\delta||\varphi\circ S||_{\varphi}<1$ for some $\beta>\gamma>0$ with $1\le\beta$, then there exists a unique U that is φ-bounded on $\ell^{\infty}(\beta,\gamma)_{+}$, obeys Koopmans's equation $W(\pi\mathbf{c},U(S\mathbf{c}))=U(\mathbf{c})$, and is β-upper semicontinuous on $\ell^{\infty}(\beta)_{+}$.

Proof. We first construct the function on $\ell^{\infty}(\beta,\gamma)_{+}$. Temporarily give $\mathcal{A}=\ell^{\infty}(\beta,\gamma)_{+}$ the discrete topology. All functions are continuous there. Since $W(c,0)$ is clearly φ-bounded, the Continuous Existence Theorem applies and yields a unique φ-bounded recursive utility function Ψ defined on $\ell^{\infty}(\beta,\gamma)_{+}$.

Next let $\mathbf{z}$ be an arbitrary element of $\ell^{\infty}(\beta,\gamma)_{+}$ and define the "partial sums" on all of $\ell^{\infty}(\beta)_{+}$ by replacing the utility of the tail of $\mathbf{c}$ with the utility of the tail of $\mathbf{z}$. Formally,

$$\Psi_N(\mathbf{c};\mathbf{z}) = [(T_W^N\Psi)(S^N\mathbf{z})](\mathbf{c}) = W(c_1, W(c_2,\ldots,W(c_n,\Psi(S^N\mathbf{z}))\cdots)).$$

Now for $\mathbf{z},\mathbf{z}' \in \ell^\infty(\beta,\gamma)_+$,

$$\begin{aligned}|\Psi_N(\mathbf{c};\mathbf{z}) - \Psi_N(\mathbf{c};\mathbf{z}')| &\leq \delta^N|\Psi(S^N\mathbf{z}) - \Psi(S^N\mathbf{z}')| \\ &\leq \delta^N M[\varphi(S^N\mathbf{z}) + \varphi(S^N\mathbf{z}')] \\ &\leq M'(\delta||\varphi\circ S||_\varphi)^N\end{aligned}$$

for some M, M'. The first step uses the Lipschitz bound (W2), the second uses the φ-boundedness of Ψ on $\ell^\infty(\beta,\gamma)_+$, and the third uses the fact that $\varphi(S^N\mathbf{z}) \leq (||\varphi\circ S||_\varphi)^N\varphi(\mathbf{z})$. It follows that if $\lim_N \Psi_N(\mathbf{c};\mathbf{z})$ exists, it must be independent of the choice of $\mathbf{z}$. Note that if $\mathbf{c}\in\ell^\infty(\beta,\gamma)_+$, $\Psi_N(\mathbf{c};\mathbf{c}) = \Psi(\mathbf{c})$, so $\lim_N \Psi_N(\mathbf{c};\mathbf{z})$ exists and is equal to $\Psi(\mathbf{c})$ for $\mathbf{c}\in\ell^\infty(\beta,\gamma)_+$.

We next show $U(\mathbf{c}) = \lim_N \Psi_N(\mathbf{c};\mathbf{z})$ exists and is β-upper semicontinuous on all of $\ell^\infty(\beta)_+$. Consider the ball $\mathbf{B}$ about zero of radius κ. Set $z_t = \kappa\beta^{t-1}$. For $\mathbf{c}\in\mathbf{B}$, $c_t \leq z_t$. It follows that $\Psi_N(\mathbf{c};\mathbf{z})$ is a decreasing sequence. Its limit $U(\mathbf{c})$ not only exists, but is upper semicontinuous on $\mathbf{B}$ as the infimum of a sequence of upper semicontinuous (in $\mathbf{c}$) functions. Since $\mathbf{B}$ was any ball, U is upper semicontinuous on all of $\ell^\infty(\beta)_+$.

The function U is also recursive. If $\pi\mathbf{c} = 0$ or if $U(\mathbf{c}) = -\infty$, our hypotheses imply $W(\pi\mathbf{c}, U(S\mathbf{c})) = -\infty = U(\mathbf{c})$. Otherwise, we may write:

$$\begin{aligned}W(\pi\mathbf{c}, U(S\mathbf{c})) &= W(\pi\mathbf{c}, \lim_N(S\mathbf{c}); S\mathbf{z})) \\ &= \lim_N W(\pi\mathbf{c}, \Psi_N(S\mathbf{c}; S\mathbf{z})) \\ &= \lim_N \Psi_{N+1}(\mathbf{c};\mathbf{z}) \\ &= U(\mathbf{c})\end{aligned}$$

which demonstrates Koopmans's equation for $\mathbf{c}\in\ell^\infty(\beta)_+$.

This leaves uniqueness. Let Φ be a β-upper semicontinuous recursive utility function that is φ-bounded on $\ell^\infty(\beta,\gamma)_+$. Since Ψ is the unique such function on $\ell^\infty(\beta,\gamma)_+$, Φ is an extension of Ψ. Let $z_t = |\mathbf{c}|_\beta\beta^{t-1}$ so that $\mathbf{c}\leq\mathbf{z}$. Thus $\Phi(\mathbf{c}) \leq \Psi_N(\mathbf{c};\mathbf{z})$. Taking the limit shows $\Phi(\mathbf{c}) \leq U(\mathbf{c})$.

Now if $c_t = 0$ for some t, $U(\mathbf{c}) = -\infty = \Phi(\mathbf{c})$ and we are done. So suppose $c_t > 0$ for all t. Now set $z_t = \gamma^{t-1}$ and consider $\mathbf{c}^n = (c_1,\ldots,c_n,z_{n+1},z_{n+2},\ldots)$. By construction, $\Phi(\mathbf{c}^n) = \Psi_n(\mathbf{c};\mathbf{z})$. Since $\gamma<\beta$, $|\mathbf{c}^n - \mathbf{c}|_\beta \to 0$. As Φ is upper semicontinuous, $\Phi(\mathbf{c}) \geq \lim_n \Psi_n(\mathbf{c};\mathbf{z}) = U(\mathbf{c})$. Thus $U = \Phi$, proving uniqueness. □

The sort of situation this applies to is $W(x,y) = (1-\sigma)^{-1}x^{1-\sigma} + \delta y$ when $\sigma > 1$. Then $g(x) = -x^{1-\sigma}$ and $h = 0$ is a good choice. Choose γ large enough that $\gamma^{1-\sigma}\delta < 1$ (note that $\delta > 1$ is ok here) and $\beta > \gamma$ arbitrary. Then set $h = 1$ and use the fact that $g(_\gamma|S\mathbf{c}| \geq \gamma_\gamma|\mathbf{c}|$ to show $\delta||\varphi\circ S||_\varphi < 1$.

Rincón-Zapatero and Rodríguez-Palmero [39] also use a two-stage approach to derive the recursive utility function. The biggest difference is that they use a different type of contraction mapping theorem that allows them to handle a wider variety of aggregators.

Le Van and Vailakis [32] modify the partial sum method to provide a unified approach to existence that works under weaker assumptions.

Streufert [41] followed a different path to existence of recursive utility. He focused on the case where consumption sequences of interest are bounded above. Let $\omega_t > 0$ for every t and consider $[\mathbf{0}, \omega] = \{\mathbf{c} \in \mathcal{S} : 0 \le c_t \le \omega_t \text{ for every } t\}$. A utility function U is *upper convergent* over $[\mathbf{0}, \omega]$ if for every $\mathbf{c} \in [\mathbf{0}, \omega]$:

$$\lim_{T\to\infty} U(c_1, \dots, c_T, S^T\omega) = U(\mathbf{c})$$

while a utility function is *lower convergent* if

$$\lim_{T\to\infty} U(c_1, \dots, c_T, \mathbf{0}) = U(\mathbf{c}).$$

A utility function is *biconvergent* over $[\mathbf{0}, \omega]$ if it is both upper and lower convergent over $[\mathbf{0}, \omega]$. A function $U_1 \colon [\mathbf{0}, \omega] \to [0, \infty)$ equivalent to U is a *general solution* to Koopmans's equation if there is a sequence of subutility functions such that

$$U_t(S^{t-1}\mathbf{c}) = W(c_t, U_{t+1}(S^t\mathbf{c})).$$

Such a solution is *admissible* if for all $\mathbf{c} \in [\mathbf{0}, \omega]$, $U(\mathbf{0}) \le U_1(\mathbf{c}) \le U(\omega)$. Streufert showed that if U is biconvergent, it is the only admissible solution to Koopmans's equation. He was also able to prove a converse under a mild additional hypothesis on the connectedness of the image of U.

9.3 Existence of Optimal Paths

The basic method of showing optimal paths exist is to apply the Weierstrass Theorem, which states that an upper semicontinuous function has a maximum on any compact set. The existence theorems for recursive utility establish upper semicontinuity. We need only show that the feasible set is compact.

As usual in normed spaces, β-bounded sets in $\ell^\infty(\beta)$ are not compact. However, if $\alpha < \beta$, any α-bounded set is pre-compact in $\ell^\infty(\beta)$. If it is product-closed, it is compact.[5] Thus the key to showing existence of optimal paths will be to show that all feasible paths grow at most by a growth factor that is below β.

Lemma 9.3.1. *Suppose* $0 < \alpha < \beta$ *and there is an* $A > 0$ *such that* $||c_t|| \le A\alpha^t$ *whenever* $\mathbf{c} \in \mathbf{X}$. *Then if* $\mathbf{X}$ *is closed in the product topology,* $\mathbf{X}$ *is also compact.*

[5] Since all of the β-topologies are stronger than the product topology, β-closed implies product closed. The converse does not hold, but on α-bounded sets we only need the weaker condition.

Proof. Let $\mathbf{c}^n$ be a sequence in $\mathbf{X}$. Since $\mathbf{X}$ is α-bounded, we can extract a subsequence that converges in the product topology. We also denote the subsequence by $\mathbf{c}^n$ and its product-limit by $\mathbf{c}$.

Let $\epsilon > 0$. Choose T so that $A\alpha^t/\beta^t < \epsilon/2$ for all $t > T$. Now choose N so that $||c_t^n - c_t^m|| < \epsilon$ for all $n, m > N$ and $t = 1, \dots, T$. Then $|\mathbf{c}^n - \mathbf{c}^m|_\beta < \epsilon$ for $n, m > N$. As $\{\mathbf{c}^n\}$ is a Cauchy sequence in the β-topology, it has a limit in $\ell^\infty(\beta)$. Since β-convergence implies product convergence, this limit coincides with the product-topology limit $\mathbf{c}$. As $\mathbf{X}$ is β-closed, $\mathbf{c} \in \mathbf{X}$. □

Proposition 9.3.1. *Suppose U is β-upper semicontinuous on a product-closed and α-bounded set $\mathbf{X}$ with $\alpha < \beta$. Then the problem of maximizing $U(\mathbf{c})$ for $\mathbf{c} \in \mathbf{X}$ has a solution.*

Proof. By the preceding lemma, $\mathbf{X}$ is β-compact. The Weierstrass Theorem then applies to yield a maximum. □

9.4 One-Sector Model with Recursive Utility

The traditional one-sector growth model (Ramsey model) has one all-purpose good available at each point in time. This good is used both for consumption and as an input to production in the next period. Production proceeds under conditions of diminishing returns to scale, and is described by a production function. The planner starts with an initial capital stock, and maximizes utility over all feasible consumption paths.

Let c_t denote consumption in time period t and let k_t denote the capital stock accumulated during period t, used for production in period $t+1$. The initial capital stock is k_0. Consider the sequences of consumption levels, $\mathbf{c} = \{c_t\}_{t=1}^\infty$, and capital stocks, $\mathbf{k} = \{k_t\}_{t=1}^\infty$. Both $\mathbf{c}$ and $\mathbf{k}$ are elements of $\mathcal{S}$.

Let f be a non-decreasing continuous *production function* such that $f(0) \geq 0$. In each time period, income $y_t = f(k_{t-1})$ is freely divided between consumption c_t and capital k_t. Any income that is not accumulated as capital may be consumed.[6] A pair of sequences $(\mathbf{c}, \mathbf{k})$ is *feasible* from k if $c_t, k_t \geq 0$ and $0 \leq k_t + c_t \leq f(k_{t-1})$ for $t = 1, 2, \dots$. The *feasible set* is $\mathbf{Y}(k_0) = \{(\mathbf{c}, \mathbf{k}) : (\mathbf{c}, \mathbf{k})$ is feasible from $k\}$. The sets of feasible capital and consumption programs are $\mathbf{F}(k_0) = \{\mathbf{k} : (\mathbf{c}, \mathbf{k}) \in \mathbf{Y}(k_0)$ for some $\mathbf{c}\}$ and $\mathbf{B}(k_0) = \{\mathbf{c} : (\mathbf{c}, \mathbf{k}) \in \mathbf{Y}(k_0)$ for some $\mathbf{k}\}$, respectively. Feasible consumption paths obey $0 \leq c_t \leq f(k_{t-1}) - k_t$, and $\mathbf{B}(k_0)$ is referred to as the *budget set*.

Proposition 9.4.1. *Both $\mathbf{B}(k_0)$ and $\mathbf{F}(k_0)$ are compact in the product topology.*

[6] We can interpret this as 100 percent depreciation. An alternative interpretation is that f denotes output net of depreciation and investment is reversible.

Proof. Define the t-th iterate of f, f^t, inductively by $f^1(x) = f(x)$, $f^t(x) = f(f^{t-1}(x))$. As f is increasing, $c_t, k_t \le f^t(k_0)$, and so both $\mathbf{B}(k_0)$ and $\mathbf{F}(k_0)$ are contained in $\prod_{t=1}^{\infty}[0, f^t(k_0)]$. This set is compact by Tychonoff's Theorem.

Take feasible $\mathbf{k}^\nu$ with $\mathbf{k}^\nu \to \mathbf{k}$. Then $0 \le k_t^\nu \le f(k_{t-1}^\nu)$. Taking the limit shows $0 \le k_t \le f(k_{t-1})$. Thus $\mathbf{F}(k_0)$ is closed. As a closed subset of a compact set, $\mathbf{F}(k_0)$ is compact.

Now suppose $\mathbf{c}^\nu \in \mathbf{B}(k_0)$ with $\mathbf{c}^\nu \to \mathbf{c}$. Consider the associated $\mathbf{k}^\nu \in \mathbf{F}(k_0)$. Take a convergent subsequence with limit $\mathbf{k}$. Retaining notation, we denote it $\mathbf{k}^\nu$. Then $0 \le k_t^\nu + c_t^\nu \le f(k_{t-1}^\nu)$. Taking the limit in the feasibility constraints, we find $\mathbf{c} \in \mathbf{B}(k_0)$. As a closed subset of a compact set, $\mathbf{B}(k_0)$ is also compact. □

Theorem 9.4.1. Ramsey Model Existence Theorem
Suppose U is β-upper semicontinuous and $f(k) \le a+bk$ with $b \ge 1$ and $b, a \ge 0$. If $b < \beta$ then an optimal path exists.

Proof. All that is really left is to show $\mathbf{B}(k_0)$ is α-bounded for some $\alpha < \beta$. Choose α with $b < \alpha < \beta$. Now $c_t \le f^t(k_0)$ and $f^t(k_0) \le a + a\alpha + \cdots + a\alpha^{t-1} + \alpha^t k_0 = a(\alpha^t - 1)/(\alpha - 1) + \alpha^t k_0$ by induction. Since $\alpha > 1$, $c_t \le [k_0 + a/(\alpha-1)]\alpha^t$, which shows $\mathbf{B}(k_0)$ is α-bounded. □

9.4.1 Dynamic Programming

We again consider the Ramsey problem, the problem of maximizing utility given an initial capital stock k and production function f. This time, instead of the direct method, we approach the optimal growth problem via dynamic programming, using the *value function* $J(k) = \sup\{U(\mathbf{c}) : \mathbf{c} \in \mathbf{B}(k)\}$. The value function always exists, although it may be either $+\infty$ or $-\infty$. If it is a continuous function, we will be able to use it to find optimal paths. We establish Bellman's equation using the Principle of Optimality. The classic statement of the Principle of Optimality is:[7]

The Principle of Optimality
An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The Optimality Principle says once on the optimal path, it is optimal to stay there. Optimal choices are time consistent. Koopmans's Stationarity and Limited Independence Axioms are the key to establishing the Principle of Optimality in the context of the recursive one-sector model.

The idea of time consistency underlies the following proof, although it is a bit obscured due to the use of the supremum rather than a maximum. If the

[7] Bellman ([7], p. 83).

maximum does exist, the epsilons can be dispensed with, making clear how the Principle of Optimality is employed. *Bellman's equation* is the analytical implementation of the Principle of Optimality. The basic methodology of dynamic programming is to solve the optimization problem by finding the solution to Bellman's equation. The value function stores all the relevant information necessary to solve the original problem.[8]

Theorem 9.4.2. Bellman's Equation
In the one-sector model,

$$J(k) = \sup \{W(c, J(f(k) - c)) : 0 \leq c \leq f(k)\} .$$

Proof. Let $\epsilon > 0$, and take a feasible path $\mathbf{c}$ with $U(\mathbf{c}) > J(k) - \epsilon$. The path $\mathbf{c}' = \{c_{t+1}\}_{t=1}^{\infty}$ is feasible from $f(k) - c_1$, and so $U(\mathbf{c}') \leq J(f(k) - c_1)$. Thus

$$J(k) - \epsilon < U(\mathbf{c}) = W(c_1, U(\mathbf{c}')) \leq W(c_1, J(f(k) - c_1)).$$

It follows that $J(k) - \epsilon \leq \sup \{W(c, J(f(k) - c))\}$. Because ϵ was arbitrary, $J(k) \leq \sup\{W(c, J(f(k) - c))\}$.

For step two, fix $\epsilon > 0$. Take any $c \in [0, f(k)]$ and choose $\mathbf{c}$ feasible from $f(k) - c$ with $U(\mathbf{c}) \geq J(f(k) - c) - \epsilon/\delta$. Letting $\mathbf{c}^* = (c, \mathbf{c})$, we obtain $U(\mathbf{c}^*) = W(c, U(\mathbf{c})) \geq W(c, J(f(k) - c)) - \epsilon$ by (W2). As $\mathbf{c}^*$ is feasible from initial stocks k, $\epsilon + J(k) \geq \sup\{W(c, J(f(k) - c))\}$. Since ϵ was also arbitrary, $J(k) \geq \sup\{W(c, J(f(k) - c))\}$. Combining this with the previous paragraph yields Bellman's equation. □

Similar results hold in multisector models. When the optimal path is interior, we can use the Envelope Theorem to show the value function is differentiable and that its derivative is the derivative of U with respect to time 1 consumption evaluated at the optimal path. Let U_t denote $\partial U/\partial c_t$.

Regular Paths
An optimal path of capital accumulation $\mathbf{k}$ is *regular* if $0 < k_t < f(k_{t-1})$ for all $t = 1, 2, \ldots$.

Theorem 9.4.3. Differentiability of the Value Function
The value function J is non-decreasing and concave. If U is differentiable with respect to consumption in period 1, and optimal paths are regular, then J is differentiable and obeys $J'(k) = U_1(\mathbf{c})$ where $\mathbf{c}$ is any optimal path from k.

[8] There are many treatments of dynamic programming. We recommend Stokey and Lucas [40] for an exhaustive treatment of dynamic programming in deterministic and stochastic economic models. Streufert's [42] chapter surveys deterministic and stochastic dynamic programming using his biconvergence technique. More recent literature on dynamic programming in the context of recursive utility includes Durán [18] and Rincón-Zapatero and Rodríguez-Palmero [39]. Alvarez and Stokey [1] and Le Van and Morhaim [31] consider the problem of unbounded returns in a TAS context.

Proof. The value function is increasing because the feasible set grows when the initial stock increases. Concavity follows since U is concave and the production function is concave.

Differentiability is established as follows.[9] Let $h > 0$, $\mathbf{h} = (h, 0, \dots)$ and let $\mathbf{c}$ be an optimal path with initial endowment k so that $J(k) = U(\mathbf{c})$. Clearly, $J(k+h) \geq U(\mathbf{c}+\mathbf{h})$ and thus $J(k+h) - J(k) \geq U(\mathbf{c}+\mathbf{h}) - U(\mathbf{c})$. Dividing by h and taking the limit shows that the right-hand derivative $D^+J(k)$ satisfies $D + J(k) \geq U_1(\mathbf{c})$.[10] Since $\mathbf{c}$ is regular, $c_1 = f(k) - k_1$ is non-zero. We may then repeat this with $-c_1 < h < 0$, to show $D^-J(k) \leq U_1(\mathbf{c}) \leq D^+J(k)$. As J is concave, $D^+J(k) \leq D^-J(k)$, thus $J'(k) = U_1(\mathbf{c})$. □

Corollary 9.4.1. *Under the above conditions,*

$$J'(k) = W_1(c_1, U(S\mathbf{c}))$$

where $\mathbf{c}$ *is any optimal path from* k.

For regular paths, the Euler equations are easy to derive.[11]

Theorem 9.4.4. Euler Equations
Suppose $\mathbf{k}$ *is a regular optimal path from initial stock* k *and that* U_t *exists at every time period. Then* $U_t(\mathbf{c}) = f'(k_t)U_{t+1}(\mathbf{c})$.

Proof. Consider the path $k_t(\epsilon) = k_t$ for $t \neq s$, $k_s(\epsilon) = k_s + \epsilon$. The associated consumption path is $c_t(\epsilon) = c_t$ for $t \neq s, s+1$, $c_s(\epsilon) = c_s - \epsilon$, $c_{s+1}(\epsilon) = f(k_s + \epsilon) - f(k_s) + c_s$. For $|\epsilon|$ small, this will be feasible and we can define $g(\epsilon) = U(\mathbf{c}(\epsilon))$. This attains a maximum when $\epsilon = 0$, so $g'(0) = 0$. Now $g'(0) = U_s(\mathbf{c}) - f'(k_s)U_{s+1}(\mathbf{c})$, which establishes the result. □

The Euler equations can also be written $U_t/U_{t+1} = f'(k_t)$. In the regular case, we can simplify the Euler equations by using the aggregator. The chain rule tells us that

$$U_t(\mathbf{c}) = W_2(c_1, U(S\mathbf{c}))W_2(c_2, U(S^2\mathbf{c})) \times ...W_2(c_{t-1}, U(S^{t-1}\mathbf{c}))W_1(c_t, U(S^t\mathbf{c}))$$

Plugging this into the marginal rate of substitution U_t/U_{t+1}, we find that most of the W_2 terms cancel, leaving us with

$$\frac{U_t(\mathbf{c})}{U_{t+1}(\mathbf{c})} = \frac{W_1(c_t, U(S^t\mathbf{c}))}{W_2(c_t, U(S^t\mathbf{c}))W_1(c_{t+1}, U(S^{t+1}\mathbf{c})}.$$

This motivates the definition of the *marginal rate of impatience* by

$$1 + R(x, y, u) = \frac{W_1(x, W(y, u))}{W_2(x, W(y, u))W_1(y, u)},$$

[9] This method is adapted from Mirman and Zilcha [35], which is simpler in this context than adapting Benveniste and Scheinkman [8].

[10] We use D^+ and D^- to denote the right- and left-hand derivatives.

[11] The case of non-regular paths is more complex, see Boyd [10] for details.

so that

$$\frac{U_t(\mathbf{c})}{U_{t+1}(\mathbf{c})} = R(c_t, c_{t+1}, U(S^{t+2}\mathbf{c})).$$

In the TAS case, R reduces to $\delta^{-1}u'(x)/u'(y) - 1$ and is independent of future utility.

Along stationary paths, we define the *rate of impatience* by $\rho(c) = R(c, c, \Phi(c))$ where $\Phi(c) = U(c, c, \dots)$. Thus $\rho(c) = -1 + 1/W_2(c, \Phi(c))$. In the TAS case, it coincides with the discount rate as $\rho(c) = -1 + \delta^{-1}$, so $\delta = 1/(1 + \rho(c))$.

The Euler equations for the TAS case, $W(x, y) = u(x) + \delta y$ have the usual form $u'(c_t) = \delta f'(k_t)u'(c_{t+1})$. The Epstein-Hynes aggregator $W(x, y) = (-1 + y)\exp(-v(x))$ yields $W_1 = -v'(x)(-1 + y)\exp(-v(x))$ and $W_2 = \exp(-v(x))$. After some simplification,

$$\frac{v'(c_t)}{v'(c_{t+1})} \frac{-1 + U(S^t\mathbf{c})}{U(S^t\mathbf{c})} = f'(k_t)$$

The difference is particularly noticeable on stationary paths. Suppose $c_t = c$ and $k_t = k$. Then the TAS case yields $\delta f'(k) = 1$ while the EH case gives $1 - 1/U(\mathbf{c}) = f'(k)$. Overall utility (or wealth) has no effect in the TAS case, but plays a key role in the EH case.

If we specialize further to the case of a constant interest rate r, so $f(k) = 1 + r$, stationarity for TAS utility requires $\delta(1 + r) = 1$. In the EH case, $1 - 1/U = 1 + r$ or $r = -1/U$ (recall that utility is negative here). Different interest rates lead to different steady state consumption levels for EH utility. For TAS utility, interest rates other than $\delta^{-1} - 1$ do not have stationary solutions. Either the optimal path shrinks toward zero (if r is small) or grows without bound (if r is large).

We can go a bit further. We can find $\Phi(c)$ by solving Koopmans's equation $W(c, \Phi(c)) = \Phi(c)$. Using the expression for W, we find $(-1 + \Phi)\exp(-v(c)) = \Phi$, so $\Phi(c) = [1 - e^{v(c)}]^{-1}$. Thus $\exp v(c) = r$ along stationary paths. Only if r is outside the range of $\exp v(c)$ do we get optimal paths converging to zero or growing without bound.

9.5 Optimal Paths in the One-Sector Model

In this section, we focus on one-sector capital accumulation models with differentiable concave production function and differentiable strictly concave recursive utility U. We will require that the aggregator W obey either (W1) or (W1′). It will also obey (W2) with Lipschitz bound $\delta < 1$. We require $U_t(\mathbf{c}) > 0$ exist whenever $U(\mathbf{c})$ is finite and that $f' > 0$ on $\mathbb{R}_+ +$. Further, assume that the feasible set $\mathbf{F}(k)$ is α-bounded and that U is β-upper semicontinuous for some $\beta > \alpha$.[12] These conditions ensure that optimal paths exist. The concavity

[12] If W has the time additive form $W(x, y) = u(x) + \delta y$, this implies $0 < \delta < 1$ and that $u' > 0$ on $\mathbb{R}_+ +$.

of the production function implies that the feasible set is convex. The strict concavity of U then yields a unique optimal path.

9.5.1 The Inada Conditions

The Euler equations are one of our main tools for investigating the properties of optimal paths. We would like to use them to characterize optimal paths. However, we have only derived the standard Euler equations as necessary conditions when optimal paths are regular. There are two ways to work around this. One is to allow for boundary points by using a Kuhn-Tucker inequality when $k_t = 0$ or $c_t = 0$. Modified Euler equations of this sort are presented in Boyd [10]. The other, and simpler, method is to guarantee interiority by imposing the Inada conditions. The *Inada conditions* come in two parts: the Inada utility condition is $U_t^+(\mathbf{c}) = +\infty$ when $c_t = 0$; the Inada production conditions are $f'(0+) = +\infty$ and $f'(\infty)\sup W_2 < 1$. When U is time additive separable, the Inada utility condition becomes $u'(0+) = +\infty$, and the production condition is $\delta f'(\infty) < 1$. We have:

Lemma 9.5.1. *Suppose the Inada condition for utility is satisfied and $f(0) = 0$ and $f'(k) > 0$ for $k \geq 0$. Whenever $k > 0$ and $J(k) > -\infty$, any optimal path is regular, it obeys $c_t, k_t > 0$ for all t.*

Proof. Let $\mathbf{c}$ be optimal and suppose $c_s = 0$. If $c_{s+1} > 0$, then $f(k_s) - k_{s+1} = c_{s+1} > 0$, so $f(k_s) > k_{s+1}$ and hence $k_s > 0$. Take $\Delta > 0$ small enough that $k_s > \Delta$ and $f(k_s - \Delta) > k_{s+1}$. We try an arbitrage between times s and $s+1$ that accelerates consumption. Increase consumption by Δ at time s by taking the path $\mathbf{c}'$ defined by $c'_t = c_t$ for $t \neq s, s+1$, $c'_s = c_s + \Delta$, and $c'_{s+1} = f(k_s - \Delta) - k_{s+1} = c_{s+1} + f(k_s - \Delta) - f(k_s)$, which is feasible. Now $0 \geq U(\mathbf{c}') - U(\mathbf{c})$. Dividing by Δ, and letting $\Delta \to 0^+$, we find $0 \geq U_s(\mathbf{c}) - U_{s+1}(\mathbf{c})f'(k_s)$. But the right-hand side is $+\infty$. This contradiction shows that $c_{s+1} = 0$ also. Once consumption reaches 0, it must stay there.

Now let s be the earliest time with $c_s = 0$. If $s = 1$, $c_t = 0$ for all t. Thus $J(k) = U(\mathbf{0})$. Of course, this is impossible if $U(\mathbf{0}) = -\infty$, so we may assume $U(\mathbf{0})$ is finite. But then, the path $\mathbf{c}^* = (f(k), 0, 0, \dots)$ is feasible, and yields utility $W(k, U(\mathbf{0})) > W(0, U(\mathbf{0})) = U(\mathbf{0})$. This is also impossible as $\mathbf{c}$ is optimal. Thus $s > 1$.

Note that all of the capital must be used up at s. No more consumption will take place, and we would be made better off by consuming any leftover capital. Try an arbitrage between s and $s-1$ that delays consumption. Let $\Delta > 0$ and define $\mathbf{c}'$ by $c'_t = c_t$ for $t \neq s-1, s$, $c'_{s-1} = c_{s-1} - \Delta$, and $c'_s = f(\Delta)$. This is feasible for small Δ. Again, $0 \geq U(\mathbf{c}') - U(\mathbf{c})$. We again divide by Δ, and let $\Delta \to 0^+$. This yields $0 \geq -U_{s-1}(\mathbf{c}) + U_s(\mathbf{c})f'(0)$. The right-hand side is $+\infty$ by the Inada condition on U. This contradiction shows that there is no s with $c_s = 0$. □

9.5.2 Monotonicity

The basic monotonicity and turnpike results for the recursive one-sector model were established by Beals and Koopmans [2], and under slightly weaker conditions by Magill and Nishimura [34].[13] We prove optimal paths are monotonic and that they cannot cross. Convergence to a steady state (or infinity) then follows.

First let $\mathbf{c}(k)$ and $\mathbf{k}(k)$ denote the optimal paths of consumption and capital stocks, respectively.

Theorem 9.5.1. Monotonicity Theorem
Suppose $\partial R/\partial c_1 \neq 0$. For any initial stock k, $k_t(k)$ is a strictly increasing function of k and the optimal path $\mathbf{k}(k)$ is strictly monotonic.

Proof. The strict concavity means $\mathbf{k}(k)$ is single-valued, hence continuous. Let $k < k'$, and let $\mathbf{k} = \mathbf{k}(k)$, $\mathbf{k}' = \mathbf{k}(k')$ be optimal. Suppose $k_1 = k'_1$. By the Principle of Optimality, $k_t = k'_t$ for $t = 2, 3, \dots$. Further, $c_1 = f(k) - k_1 < f(k') - k'_1 = c'_1$ and $c_2 = f(k_1) - k_2 = c'_2$. Thus $c'_t = c_t$ for $t = 2, 3, \dots$. The Euler equations yield $R(c_1, c_2, \dots) = f'(k_1) = f'(k'_1) = 1 + R(c'_1, c'_2, \dots)$. Since $c'_t = c_t$ for $t = 2, 3, \dots$, $R(c_1, c_2, \dots) = R(c'_1, c_2, \dots)$. But this is impossible since R is decreasing in c_1 and $c_1 < c'_1$. Thus $k_1 \neq k'_1$.

Now suppose $k_1 > k'_1$. Since $k_1(0) = 0 < k'_1 < k_1(k)$, and $k_1(k)$ is continuous, there is a k'' with $0 < k'' < k$ and $k_1(k'') = k'_1$. This is impossible by the preceding argument. Therefore k_1 is strictly increasing. Since $k_t(k)$ is the t-th iterate of k_1, it too is strictly increasing. Further, $\mathbf{k}(k)$ is strictly monotonic by the usual argument. □

Since $\partial R/\partial c_1$ is continuous, it must be either always positive or always negative. Many people consider the requirement that $\partial R/\partial c_1 < 0$ as most natural. R is the marginal rate of substitution between consumption today and consumption tomorrow. As we increase today's consumption, we expect the rate of substitution to fall. In the additive case, it must fall as $\partial R/\partial c_1 = u''(c_1)/\delta u'(c_2)$. In the more general recursive case it is equivalent to requiring $\partial[W_1(c_1, u)/W_2(c_1, u)]/\partial c_1 < 0$. This says that the indifference curves in (c_1, u)-space are convex to the origin.

Monotonicity immediately implies that optimal paths converge either to 0, or to a steady state, or to $+\infty$.

Initial stocks can be divided into three disjoint sets. Let $\mathbf{I}^0 = \{k : k = 0 \text{ or } f'(k) = 1 + \rho(f(k) - k)\}$, $\mathbf{I}^+ = \{k : f'(k) > 1 + \rho(f(k) - k)\}$, and $\mathbf{I}^- = \{k : f'(k) < 1 + \rho(f(k) - k)\}$. For $k \in \mathbf{I}^0$, the Euler equations and transversality condition are clearly satisfied by the stationary path $k_t = k$. Thus every element of $\mathbf{I}^0$ is a steady state. The Euler equations also show that all steady states are in $\mathbf{I}^0$. Accumulation is definitely possible in $\mathbf{I}^+$ since $f'(k) > 1 + \rho(f(k) - k) > 1$. Define $\Psi(k) = \Phi(f(k) - k)$ where $\Phi(c)$ is the utility of the constant path $c_t = c$.

[13] Boyer [12] and Iwai [26] examined recursive utility models with one sector. They utilized dynamic programming ideas and conjectured the presence of multiple steady states in some cases.

Theorem 9.5.2. Recursive Non-Optimality Theorem
Suppose $k \in \mathbf{I}^+$ ($k \in \mathbf{I}^-$) and $k_t \le k$ ($k_t \ge k$) for $t < n$ with $k_t = k$ for $t \ge n$. Then $U(\mathbf{c}) \le \Psi(k)$, and $\mathbf{k}$ is not optimal.

Proof. First suppose $k \in \mathbf{I}^+$. That $U(\mathbf{c}) \le \Psi(k)$ is trivial for $n = 1$. We proceed by induction. Suppose $U(\mathbf{c}) \le \Psi(k)$ when $n = m \ge 1$ and consider a path $\mathbf{k}$ with $k_t \le k$ and $k_t = k$ for $t \ge m+1$. If $k_m = k$, $U(\mathbf{c}) \le \Psi(k)$ by the induction hypothesis, so we may suppose $k_m < k$.

First consider the path $\mathbf{k}'$ defined by $k'_t = k$ for $t \ne m$ and $k'_m = k + \Delta$. Obviously $f'(k) > 1$, so this path will be feasible from k for $\Delta > 0$ small enough. Taking a Taylor expansion shows

$$\begin{aligned} U(\mathbf{c}') - U(\mathbf{c}) &= -U_m(\mathbf{c})\Delta + U_{m+1}(\mathbf{c})\Delta f' + o(\Delta)\Delta \\ &= (W_2)^{m-1}[-W_1 + W_2 W_1 f']\Delta + o(\Delta)\Delta \\ &= W_1 (W_2)^{m-1}[W_2 f' - 1]\Delta + o(\Delta)\Delta \end{aligned}$$

where all derivatives are evaluated at k. Now

$$1 + \rho(f(k) - k) = 1/W_2(k, \Psi(k)) < f'(k)$$

as $k \in \mathbf{I}^+$. So $W_2 f' > 1$ and Δ may be chosen small enough that $U(\mathbf{c}') > \Psi(k)$. Note that remaining at k cannot be optimal.

Now take λ, $0 < \lambda < 1$ with $\lambda(k + \Delta) + (1 - \lambda)k_m = k$. (Here $\lambda = (k - k_m)/(k - k_m + \Delta)$.) Then $\mathbf{k}'' = \lambda \mathbf{k}' + (1 - \lambda)\mathbf{k}$ satisfies the hypotheses of the lemma for $n = m$, so $U(\mathbf{c}'') \le \Psi(k)$ by the induction hypothesis. Now

$$\Psi(k) \ge U(\mathbf{c}'') \ge \lambda U(\mathbf{c}') + (1 - \lambda)U(\mathbf{c}) > \lambda \Psi(k) + (1 - \lambda)U(\mathbf{c}).$$

Thus $\Psi(k) > U(\mathbf{c})$. The inequality holds for all n by induction. Further, since the stationary path $k_t = k$ is feasible and not optimal, $\mathbf{k}$ cannot be optimal.

The case of $k \in \mathbf{I}^-$ is similar. □

Using the Recursive Non-Optimality Lemma, we can prove a turnpike result. Since both $\mathbf{I}^+$ and $\mathbf{I}^-$ are open, they are the countable union of open intervals. The endpoints of these intervals must be in $\mathbf{I}^0$.[14] Now label the endpoints $\bar{k}_i$ such that $\bar{k}_i < \bar{k}_{i+1}$. We allow $+\infty$ as the largest $\bar{k}_i$. If $k \notin (\bar{k}_i, \bar{k}_{i+1})$, the optimal path cannot cross the steady states at the endpoints, so $k_t \in (\bar{k}_i, \bar{k}_{i+1})$. Further, since k_t is monotonic, it must converge to some $\bar{k}$. Taking the limit in the Euler equations shows $f'(\bar{k}) = 1 + \rho(f(\bar{k}) - \bar{k})$. The optimal path converges to one of the endpoints. Similarly, if k is greater than all of the steady states it converges either to the largest steady state, or to ∞. The next theorem shows that $k_t \to \bar{k}_{i+1}$ when $k \in (\bar{k}_i, \bar{k}_{i+1}) \subset \mathbf{I}^+$ and $k_t \to \bar{k}_i$ when $k \in (\bar{k}_i, \bar{k}_{i+1}) \subset \mathbf{I}^-$.

Theorem 9.5.3. Turnpike Theorem
Suppose $\partial R/\partial c_1 < 0$. If $k \in (\bar{k}_i, \bar{k}_{i+1}) \subset \mathbf{I}^+$, the optimal path obeys $k_t \uparrow \bar{k}_{i+1}$; if $k \in (\bar{k}_i, \bar{k}_{i+1}) \subset \mathbf{I}^-$, it obeys $k_t \downarrow \bar{k}_{i+1}$; and if $k \in \mathbf{I}^0$, $k_t = k$ is the optimal path.

[14] Note that $\mathbf{I}^0$ may contain points other than these endpoints.

Proof. When $k \in \mathbf{I}^0$, the path $k_t = k$ satisfies the Euler equations and transversality condition. Thus it is optimal.

Consider the case where $k \in \mathbf{I}^+$. We know that k_t is strictly monotonic. Suppose k_t is decreasing. Take a sequence of feasible paths $\mathbf{k}^\nu$ such that $\mathbf{k}^\nu \to \mathbf{k}$ in the product topology with $k_t^\nu \leq k$ for all t and $k_t^\nu = k$ for large t. (This is possible since $f' > 1$ and $f(k) > k$ on $[k_t^\nu, k]$.) Then $U(\mathbf{k}^\nu) \leq \Psi(k)$ by the Recursive Non-Optimality Lemma. Since U is product continuous on the feasible set, $U(\mathbf{k}) \leq \Psi(k)$, contradicting the fact that $\mathbf{k}$ is optimal. Thus k_t is increasing. By the Monotonicty Theorem, $k_t < \bar{k}_{i+1}$. Taking the limit in the Euler equations shows the limit point is in $\mathbf{I}^0$. It must be $\bar{k}_{i+1}$.

The case $k \in \mathbf{I}^-$ is similar, except that the optimal path may simply be truncated to obtain the desired $\mathbf{k}^\nu$. □

9.6 Homogeneous Recursive Utility and Sustained Growth

Many applications of TAS utility use a homogeneous or logarithmic period utility function. This yields a utility function that is homothetic. Rader ([36], 1981) showed that these are the only homothetic TAS utility functions. Such functions are of interest because they can yield balanced growth paths (given appropriate technology). As a result, they are widely used in macroeconomic models of economic growth.

Interestingly, these are not the only homothetic recursive utility functions. Dolmas [16], [17] was able to characterize homothetic recursive utility both axiomatically and via the aggregator. As usual, if a utility function is homothetic, it is equivalent to a utility function that is homogeneous of degree 1.

Proposition 9.6.1. *Let U be a recursive utility function that is homogeneous of degree γ. Its aggregator obeys $W(u(\lambda c), \lambda^\gamma y) = \lambda^\gamma W(u(c), y)$.*

Proof. This follows from the fact that

$$\begin{aligned}
\lambda^\gamma W(u(c_1), U(S\mathbf{c})) &= \lambda^\gamma U(\mathbf{c}) \\
&= U(\lambda \mathbf{c}) = W(u(\lambda c_1), U(\lambda S\mathbf{c})) \\
&= W(u(\lambda c_1), \lambda^\gamma U(S\mathbf{c})).
\end{aligned}$$

□

A converse can be derived whenever the existence theorems apply. E.g., if the Continuous Existence Theorem applies, $U(\mathbf{c})$ is the limit of $T_W^N(0)(\mathbf{c})$, which is easily seen to be homogeneous of degree γ. It follows that U is homogeneous of degree γ.

Dolmas [17] gave the following example of a class of non-additive homogeneous recursive utility functions. Let u be homogeneous of degree γ and set

$W(x,y) = u(x)w(y/u(x))$. It is easy to see that $W(\lambda x, \lambda^\gamma y) = \lambda^\gamma W(x,y)$, so the resulting utility function is homogeneous of degree γ, provided it exists.

Now specialize to the one-good case with $w(y) = [1+\delta y]^\rho/\rho$ where $0 < \delta, \rho < 1$ and $u(x) = x^\gamma$. Then $W_2 = \delta[1+\delta y]^{\rho-1} \leq \delta$ and $W(x,0) = x^\gamma$, so the Continuous Existence Theorem applies on $\ell^\infty(\beta)_+$ whenever $\beta^\gamma\delta < 1$. It is easy to see this is not equivalent to an additive representation by examining the marginal rates of substitution, which depend on future utility as well as the consumption levels c_t and c_{t+1}.

Now consider a path that grows at a constant rate, so $c_{t+1} = \beta c_t$. Since $W(\lambda c, \lambda^\gamma u)$, $W_1(\lambda c, \lambda^\gamma u) = \lambda^{\gamma-1} W_1(c,u)$ and $W_2(\lambda c, \lambda^\gamma u) = W_2(c,u)$. Then the marginal rate of substitution between consumption in adjacent periods is

$$\begin{aligned}\frac{W_1(c_t, U(S^t\mathbf{c}))}{W_2(c_t, U(S^t\mathbf{c}))W_1(c_{t+1}, U(S^{t+1}, \mathbf{c}))} &= \frac{W_1(c_t, U(S^t\mathbf{c}))}{W_2(c_t, U(S^t\mathbf{c}))W_1(\lambda c_t, \lambda^\gamma U(S^t\mathbf{c}))} \\ &= \frac{1}{\lambda^{\gamma-1} W_2(c_t, U(S^t\mathbf{c}))}\end{aligned}$$

Now $W_2(c_t, U(S^t\mathbf{c})) = W_2(c_1, U(S\mathbf{c}))$, which implies that R is constant along such paths, as noted by Farmer and Lahiri [24]. This constancy allows the possibility of balanced growth paths.

Farmer and Lahiri also note that $W_2(c_1, U(S^t\mathbf{c}))$ is independent of the starting level of consumption since replacing c_t by χc_t (so $U(S\mathbf{c})$ is replaced by $\chi^\gamma U(\mathbf{c})$ leaves W_2 unchanged due to the homogeneity property established above.

In models with a maximum sustainable stock, recursive preferences allow for heterogeneity in discounting while permitting a non-degenerate long-run capital distribution.[15] Farmer and Lahiri conclude that recursive preferences add little flexibility in the case of balanced growth. Either the existing wealth distribution is maintained, or it becomes degenerate in the long run. They propose a generalization of recursive utility that allows for more heterogeneity in discounting while maintaining balanced growth.

It should be noted that TAS utility already allows more flexibility in long-run behavior under sustained growth. Boyd [11] noted that the growth rate could affect whether it was possible for agents with differing discount factors to both hold capital in the long run, and that the growth rate could also affect which one ended up with all of the capital in the degenerate case. These results suggest that the advantages of recursive utility occur primarily in models with a maximum sustainable stock.

[15] Becker [3] finds that the TAS case does lead to a degenerate capital distribution, where only the most patient household owns any capital stock in the steady state.

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10. Indeterminacy in Discrete-Time Infinite-Horizon Models

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10.1 Introduction

Recently there has been an increasing interest in sunspot equilibria as a possible explanation of business cycle fluctuations. In a macroeconomic context, sunspot fluctuations is a topic that dates back to the early work of Shell [25], Azariadis [1] and Cass and Shell [10]. This renewed interest is explained by the fact that during the last decade a variety of economic models that incorporate some degree of market imperfections have been shown to exhibit multiple equilibria and local indeterminacy.[1] As shown by Woodford [29], the existence of sunspot equilibria is closely related to the indeterminacy of perfect foresight equilibrium.

Indeterminacy, or multiple equilibria, is known to occur in dynamic models with small market distortions and generates some coordination problems. Basically, the occurrence of indeterminacy needs a mechanism such that, starting from an equilibrium, if all agents were simultaneously to increase their investment in, say, the capital good, the rate of return on this good would tend to increase, and in turn set off relative price changes that would drive the economy back towards the steady state. In one-sector models, such a mechanism may be associated with external effects in production and increasing returns. However, in a two sector model, the rate of return and marginal product of capital depend not only on factor inputs, but also on the composition of output and thus on the relative factor intensities. An increase of the production and the stock of capital following an increase in its price may well increase its rate of return. Therefore constant aggregate returns at the social level are compatible with indeterminacy if there are minor external effects in some of the sectors.

[1] See Benhabib and Farmer [5] for an extensive bibliography.

In this chapter we will present the main conditions for the occurrence of indeterminacy in one and two-sector optimal growth models extended to include market imperfections based on technological external effects. We will focus almost exclusively on discrete-time models. We will distinguish between different formulations for externalities which will be in general associated with different assumptions concerning the returns to scale at the social level. Following Romer [24], one-sector models are characterized by global external effects coupled with increasing social returns. We will show that indeterminacy of equilibria is fundamentally based on the consideration of endogenous labor demand and externalities coming both from capital and labor. In two-sector models, Benhabib and Farmer [4] have introduced sector-specific external effects. While their initial formulation assumed increasing social returns, most of the papers that followed the contribution of Benhabib and Nishimura [7] are based on constant returns to scale at the social level. We will show that some simple conditions on capital intensity differences across sectors generate some amplification mechanisms that produce the existence of indeterminate equilibria.

The chapter is organized as follows. Section 2 presents one-sector models. Two-sector models with Cobb-Douglas technologies, complete depreciation of capital and sector-specific externalities are analyzed in Section 3. Section 4 is devoted to the presentation of similar two-sector models but with CES production functions. The cases with symmetric and asymmetric elasticities of capital-labor substitution are consecutively considered. In Section 5 we discuss extensions of the two-sector Cobb-Douglas formulation. Firstly, we present how the conditions for local indeterminacy are modified when partial depreciation of capital is assumed. Secondly, we introduce a formulation for intersectoral externalities that is compatible with both sector-specific and global externalities specifications. We will then show how additional intersectoral mechanisms provide new room for local indeterminacy. Finally, in Section 6, other formulations of infinite-horizon models are explored. We first deal with the consideration of aggregate models with capacity utilization in which the speed of capital depreciation is endogenously determined. Then we present two-sector models derived from general technologies.

10.2 One-Sector Models

One-sector discrete-time models with Romer-type [24] global externality and increasing returns at the social level have been considered initally by Kehoe [14] and Boldrin and Rustichini [9]. The aggregate production function is augmented to include a new factor which represents the effect of knowledge on production and productivity:

$$Y_t = F(K_t, L_t, A_t)$$

with A_t the externality at time t which will be equal at the equilibrium to K_t/L_t. For any given A, $F(.,.,A)$ is increasing, concave and homogeneous of degree 1, and labor is inelastic. Under constant population, the intensive formulation for the capital accumulation equation is:

$$k_{t+1} = f(k_t, A_t) - c_t$$

with $f(k,A) = F(k,1,A) + (1-\mu)k_t$ and $\mu \in [0,1]$ the rate of depreciation of capital.

Assumption 1 *$f(k,A)$ is C^2 and such that for any $k, A > 0$, $f_1(k,A) > 0$, $f_{11}(k,A) < 0$ while for any $A > 0$, $f(0,A) = 0$.*

From a standard utility function $u(c)$ which satisfies:

Assumption 2 *$u(c)$ is C^2 and such that for any $c > 0$, $u'(c) > 0$, $u''(c) < 0$, $u(0) = 0$, $u'(0) = +\infty$ and $u'(+\infty) = 0$.*

we define the parameterized maximization program of a representative consumer as

$$\begin{array}{ll} \max\limits_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{+\infty} \delta^t u(c_t) \\ s.t. & k_{t+1} = f(k_t, A_t) - c_t \\ & k_0, \{A_t\}_{t=0}^{+\infty} \text{ given} \end{array}$$

with $\delta \in (0,1]$ the discount factor. Along an equilibrium path, $A_t = k_t$ and the Euler equation easily writes as

$$\delta u'(c_{t+1}) f_1(k_{t+1}, k_{t+1}) - u'(c_t) = 0$$

A steady state k^* is obtained considering $k_{t+1} = k_t$ and $c_{t+1} = c_t$ in the Euler equation, i.e. k^* is a solution of

$$f_1(k,k) = 1/\delta$$

It follows that $c^* = f(k^*, k^*)$. Contrary to the optimal growth framework, existence and uniqueness are no longer ensured under Assumption 1. We will however assume that there exists one locally unique steady state k^*. Linearizing the Euler equation around this steady state easily shows that the sum and product of the characteristic roots satisfy:

$$\begin{array}{rcl} \mathcal{T} & = & 1 + \delta^{-1} + f_2(k^*, k^*) + \dfrac{u'(c^*)}{u''(c^*)}[f_{11}(k^*, k^*) + f_{12}(k^*, k^*)] \\ \mathcal{D} & = & \delta^{-1} + f_2(k^*, k^*) \end{array}$$

Definition 10.2.1. *A steady state k^* is called locally indeterminate if there exists $\epsilon > 0$ such that from any k_0 belonging to $(k^* - \epsilon, k^* + \epsilon)$ there are infinitely many equilibrium paths converging to the steady state.*

If both characteristic roots have modulus less than one then the steady state is locally indeterminate. If a steady state is not locally indeterminate, then we call it locally determinate.

As shown by Kehoe [14], it follows easily from the above expressions that necessary conditions for local indeterminacy are:

$$f_2(k^*, k^*) < 0 \text{ and } f_{12}(k^*, k^*) > 0 \text{ such that } f_{11}(k^*, k^*) + f_{12}(k^*, k^*) > 0.$$

Such conditions imply very strong negative externalities which improves enough the private marginal productivity of capital to destroy concavity at the social level. Obviously they cannot be met by usual Cobb-Douglas or CES technologies. When standard positive externalities are considered, Boldrin and Rustichini [9] then show that the steady state is either saddle-point stable (if $f_{11}(k^*, k^*) + f_{12}(k^*, k^*) < 0$) or totally unstable (if $f_{11}(k^*, k^*) + f_{12}(k^*, k^*) > 0$).[2]

Under standard formulations for the fundamentals, Benhabib and Farmer [3] have shown that local indeterminacy in one-sector models requires the consideration of elastic labor supply and aggregate externalities on capital and labor. They consider a CES separable utility function and a Cobb-Douglas technology such that

$$U(C, L) = \log C - \frac{L^{1-\chi}}{1-\chi}, \;\; F(K, L, \bar{K}, \bar{L}) = K^{\alpha} L^{1-\alpha} \bar{K}^{\alpha\eta} \bar{L}^{(1-\alpha)\eta}$$

with $\chi \leq 0$, $\eta > 0$ and $\bar{K}, \bar{L}$ the economy-wide averages of capital and labor. Denoting $\theta = \delta(1-\mu) \in [0,1]$ the discounted value of capital carried over to the next period per unit of capital used in the current period, standard linearization of the first order conditions around the steady state allows to show that the product and sum of the characteristic roots satisfy[3]

$$\begin{aligned} \mathcal{D} &= \tfrac{1}{\delta}\left[1 - \tfrac{\eta(1-\theta)(1-\chi)}{\theta(1-\alpha)(1+\eta)-(1-\chi)}\right] \\ \mathcal{T} &= 1 + \mathcal{D} - \tfrac{(1-\theta)[1-\alpha(1-\eta)](1-\chi)\left(\frac{1-\theta}{\delta\alpha}-\mu\right)}{\theta(1-\alpha)(1+\eta)-(1-\chi)} \end{aligned}$$

where steady-state conditions imply $(1-\theta)/\delta\alpha - \mu > 0$. In a discrete-time framework, local indeterminacy requires $|\mathcal{D}| < 1$ and $|\mathcal{T}| < 1 + \mathcal{D}$. Assuming that the aggregate share of capital satisfies $\alpha(1+\eta) < 1$, the main conclusion of Benhabib and Farmer is the following: in order to generate multiple equilibria, externalities and thus the degree of increasing returns to scale must be large

[2] In a continuous-time framework, Spear [26] assumes that a positive externality A_t is given by tomorrow's aggregate capital stock k_{t+1} and gives sufficient conditions for the existence of sunspot equilibria in a neighborhood of the steady state.

[3] Benhabib and Farmer deal with a continuous-time model. We consider here the corresponding discrete-time formulation (see also Farmer and Guo [12]) in order to provide in Section 6.1 comparisons with the Wen's [28] extension to variable capacity utilization.

enough to imply that the aggregate labor demand curve should be upward-sloping and steeper than the aggregate labor supply curve, i.e. $(1-\alpha)(1+\eta)-1>-\chi>0$. This is obviously a non-standard configuration for the labor market. More recently, Pintus [23], by considering a general separable utility function $U(C,L)=u(C)-v(L)$ and a general technology $F(K,L)A(\bar{K},\bar{L})$ with constant returns to scale at the private level, show that the conditions of Benhabib and Farmer are not necessary. Local indeterminacy may indeed arise with a standard decreasing equilibrium labor demand function and small externalities provided the elasticity of capital-labor substitution is significantly greater than one.

10.3 Two-Sector Models with Cobb-Douglas Technologies

In order to weaken their conditions for local indeterminacy, Benhabib and Farmer [4] consider a two-sector continuous-time model with Cobb-Douglas technologies and sector-specific rather than aggregate externalities. They provide conditions which are compatible with mild externalities and downward sloping labor demand curves. However they assume that each sector is characterised by the same private technology. Benhabib and Nishimura [7] have extended their results to distinct private Cobb-Douglas technologies and provide some nice conditions in terms of capital intensity differences. Even if they still consider an elastic labor supply in order to provide a version of a standard real business cycles model, similar conditions for local indeterminacy may be obtained with inelastic labor.

We then extend to a framework with externalities the contribution of Nishimura and Yano [22] which study an optimal growth model.[4] We consider a discrete-time two-sector economy having an infinitely-lived representative agent with single period linear utility function, i.e. $u(c)=c$. We assume that the labor supply is inelastic. There are two goods: the pure consumption good, c, and the pure capital good, k. Each good is assumed to be produced with a Cobb-Douglas technology which contains some positive sector specific externalities. We denote by c and y the outputs of sectors c and k, and by e_c and e_y the corresponding external effects:

$$c=K_c^{\alpha_1}L_c^{\alpha_2}e_c(\bar{K}_c,\bar{L}_c),\ y=K_y^{\beta_1}L_y^{\beta_2}e_y(\bar{K}_y,\bar{L}_y)$$

The externalities $e_c(\bar{K}_c,\bar{L}_c)$ and $e_y(\bar{K}_y,\bar{L}_y)$ depend on $\bar{K}_i$, $\bar{L}_i$ which denote the average use of capital and labor in sector $i=c,y$ and will be equal to

$$e_c(\bar{K}_c,\bar{L}_c)=\bar{K}_c^{a_1}\bar{L}_c^{a_2},\ e_y(\bar{K}_y,\bar{L}_y)=\bar{K}_y^{b_1}\bar{L}_y^{b_2} \tag{10.1}$$

with $a_i,b_i\geq 0$, $i=1,2$. We assume that these economy-wide averages are taken as given by individual firms. At the equilibrium, all firms of sector $i=c,y$ being

[4] The proof of the results presented in this section can be found in Benhabib, Nishimura and Venditti [8].

identical, we have $\bar{K}_i = K_i$ and $\bar{K}_i = K_i$. Denoting $\hat{\alpha}_i = \alpha_i + a_i$, $\hat{\beta}_i = \beta_i + b_i$, the *social production functions* are defined as

$$c = K_c^{\hat{\alpha}_1} L_c^{\hat{\alpha}_2}, \ y = K_y^{\hat{\beta}_1} L_y^{\hat{\beta}_2}$$

We assume $\hat{\alpha}_1 + \hat{\alpha}_2 = \hat{\beta}_1 + \hat{\beta}_2 = 1$. The returns to scale are therefore constant at the social level, and decreasing at the private level.[5] Factor intensities may be determined by the coefficients of the Cobb-Douglas functions. The investment (consumption) good sector is capital intensive from the private perspective if and only if $\alpha_1\beta_2 - \alpha_2\beta_1 < (>)0$. The investment (consumption) good sector is capital intensive from the social perspective if and only if $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 < (>)0$.[6]

Labor is normalized to one, $L_c + L_y = 1$, and the total stock of capital is given by $K_c + K_y = k$. We assume complete depreciation of capital in one period so that the capital accumulation equation is $y_t = k_{t+1}$. The consumer's optimization program will be given by:

$$\begin{aligned}
\max_{\{K_{ct}, L_{ct}, K_{yt}, L_{yt}, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \delta^t K_{ct}^{\alpha_1} L_{ct}^{\alpha_2} e_{ct} \\
s.t. \quad & y_t = K_{yt}^{\beta_1} L_{yt}^{\beta_2} e_{yt} \\
& 1 = L_{ct} + L_{yt} \\
& k_t = K_{ct} + K_{yt} \\
& y_t = k_{t+1} \\
& k_0, \{e_c(\bar{K}_{ct}, \bar{L}_{ct})\}_{t=0}^{+\infty}, \{e_y(\bar{K}_{ct}, \bar{L}_{ct})\}_{t=0}^{+\infty} \quad \textit{given}
\end{aligned}$$

Denote by p_t, w_{0t} and w_t respectively the price of the capital good, the wage rate of labor and the rental rate of the capital good at time $t \geq 0$, all in terms of the price of the consumption good. Let $e_{it} = e_i(\bar{K}_{it}, \bar{L}_{it})$, $i = c, y$. For any given sequences $\{e_{ct}\}_{t=0}^{\infty}$ and $\{e_{yt}\}_{t=0}^{\infty}$ of external effects, the Lagrangian at time $t \geq 0$ is:

$$\begin{aligned}
\mathcal{L}_t \;=\; & K_{ct}^{\alpha_1} L_{ct}^{\alpha_2} e_{ct} + w_{0t}\big(1 - L_{ct} - L_{yt}\big) + w_t\big(k_t - K_{ct} - K_{yt}\big) \\
+ \; & p_t \Big[K_{yt}^{\beta_1} L_{yt}^{\beta_2} e_{yt} - k_{t+1}\Big]
\end{aligned} \tag{10.2}$$

Given (k_t, y_t), using $y_t = k_{t+1}$ and solving the first order conditions with respect to $(K_{ct}, L_{ct}, K_{yt}, L_{yt})$ gives input demand functions such that

$$\tilde{K}_c = K_c(k_t, k_{t+1}, e_{ct}, e_{yt}), \quad \tilde{L}_c = L_c(k_t, k_{t+1}, e_{ct}, e_{yt}),$$
$$\tilde{K}_y = K_y(k_t, k_{t+1}, e_{ct}, e_{yt}), \quad \tilde{L}_y = L_y(k_t, k_{t+1}, e_{ct}, e_{yt}).$$

[5] Our formulation is however compatible with constant returns at the private level if we assume that there exists a factor in fixed supply such as land in the technologies. In this case, the income of the representative consumer will be increased by the rental of land.

[6] Notice that under constant social returns $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 = \hat{\alpha}_1 - \hat{\beta}_1 = \hat{\beta}_2 - \hat{\alpha}_2$.

We then define the production frontier as

$$T(k_t, k_{t+1}, e_{ct}, e_{yt}) = \tilde{K}_{ct}^{\alpha_1} \tilde{L}_{ct}^{\alpha_2} e_{ct}$$

Using the envelope theorem we derive:

$$p_t = -T_2(k_t, k_{t+1}, e_{ct}, e_{yt}), \quad w_t = T_1(k_t, k_{t+1}, e_{ct}, e_{yt}) \tag{10.3}$$

where $T_1 = \frac{\partial T}{\partial k_t}$ and $T_2 = \frac{\partial T}{\partial k_{t+1}}$. The first order conditions w.r.t. k_t give the Euler equation

$$-p_t + \delta w_{t+1} = 0$$

From the optimal demand functions defined above together with the external effects (10.1) considered at the equilibrium we may define the equilibrium factors demand fonctions $\hat{K}_i = \hat{K}_i(k_t, k_{t+1})$, $\hat{L}_i = \hat{L}_i(k_t, k_{t+1})$ so that $\hat{e}_c = \hat{e}_c(k_t, k_{t+1}) = \hat{K}_c^{a_1} \hat{L}_c^{a_2}$ and $\hat{e}_y = \hat{e}_y(k_t, k_{t+1}) = \hat{K}_y^{b_1} \hat{L}_y^{b_2}$.[7] From (10.3) prices now satisfy

$$\begin{aligned} p_t(k_t, k_{t+1}) &= -T_2(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1})) \\ w_t(k_t, k_{t+1}) &= T_1(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1})) \end{aligned}$$

and we get the Euler equation evaluated at $\hat{e}_c$ and $\hat{e}_y$:

$$-p(k_t, k_{t+1}) + \delta w(k_{t+1}, k_{t+2}) = 0 \tag{10.4}$$

Any solution $\{k_t\}_{t=0}^{+\infty}$ of (10.4) which also satisfies the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t T_1(k_t, k_{t+1}, \hat{e}_{ct}(k_t, k_{t+1}), \hat{e}_{yt}(k_t, k_{t+1})) = 0$$

is called an equilibrium path. A steady state is defined by $k_t = k^*$, $y_t = y^* = k^*$ and is given by the solving of $\delta\omega(k^*, k^*) - p(k^*, k^*) = 0$. The methodology consists first in approximating the Euler equation (10.4), i.e. the first partial derivatives of $T(k_t, k_{t+1}, e_{ct}, e_{yt})$ for any given (e_{ct}, e_{yt}), using the first order conditions derived from the maximization of the Lagrangian (10.2). Then considering the externalities evaluated at the equilibrium $(\hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1}))$, we compute the steady state and then the partial derivatives of $T_i(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1}))$, $i = 1, 2$, in order to get the characteristic polynomial.[8] The first step gives

Proposition 10.3.1. *There exists a unique stationary capital stock k^* such that:*

$$k^* = \frac{\alpha_1 \beta_2}{\alpha_2 \beta_1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \delta \beta_1} (\delta \beta_1)^{1/\hat{\beta}_2}$$

[7] Since we deal with an example we can show the existence of an equilibrium path together with the local indeterminacy. However if utility and production functions are not specified, then the existence of equilibrium paths is not obvious. For existence proofs in some general cases, see Le Van, Morhaim and Dimaria [15] and Mitra [16].

[8] See Benhabib, Nishimura and Venditti [8] for details.

In the second step, the characteristic polynomial gives the following characteristic roots

Theorem 10.3.1. *The characteristic roots are given by*

$$x_1 = \frac{\alpha_2}{\delta(\alpha_2\beta_1 - \alpha_1\beta_2)}, \ x_2 = \frac{\hat{\alpha}_2\hat{\beta}_1 - \hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2}$$

Remark: Note that x_1 does not depend on external effects while x_2 does. Moreover the sign of x_1 is determined by factor intensity differences at the private level, while the sign of x_2 is determined by factor intensity differences at the social level.

As this was shown in a continuous-time framework by Benhabib and Nishimura [7], a necessary condition for the steady to be locally indeterminate is a capital intensive consumption good from the private perspective. This result also holds in a discrete-time framework.[9] We thus introduce the following restriction

Assumption 3 *The consumption good is capital intensive at the private level.*

Under this assumption notice that x_1 is negative. Local indeterminacy of the steady state may be obtained under slightly stronger conditions.

Theorem 10.3.2. . *Under Assumption 3, let $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$. Then the steady state is locally indeterminate if and only if one of the following sets of conditions is satisfied;*
i) the consumption good is labor intensive from the social perspective;
ii) the consumption good is capital intensive from the social perspective and $\hat{\beta}_1 - \hat{\alpha}_1 > -\hat{\alpha}_2$.

Benhabib and Nishimura [7] have conducted a similar analysis with a two-sector Cobb-Douglas economy in continuous time. They prove that local indeterminacy occurs when there is a capital intensity reversal between the private and social levels: the consumption good needs to be capital intensive from the private perspective, but labor intensive from the social perspective. This corresponds to condition *i*) of Theorem 10.3.2 above. In a discrete-time framework however, such a reversal is not necessary. Even in the case the consumption good is capital intensive from both private and social perspectives, indeterminacy can take place as in *ii*) of Theorem 10.3.2. These results are based on the fact that the characteristic root x_2 may be either less than 1 when positive or greater than -1 when negative for any sign of the capital intensity difference at the social level.[10]

[9] If the investment good is capital intensive at the private level, it is easy to show that $x_1 > 1$ and the steady state is locally determinate.

[10] When the discount factor δ crosses from above the critical value $\delta^* = \alpha_2/(\alpha_1\beta_2 - \alpha_2\beta_1) < 1$, the steady state becomes saddle-point stable, a flip bifurcation occurs and there exist equilibrium period-two cycles either in a right or in a left neighborhood of ρ^*.

10.4 Two-Sector Models with CES Technologies

Until now we have assumed Cobb-Douglas technologies and thus unitary elasticities of capital-labor substitution. In order to question the robustness of indeterminacy with respect to that parameter, we now extend the previous formulation to technologies with constant but non-unitary elasticities of substitution.[11] Consider indeed that each good is produced with a CES technology such that

$$\begin{aligned} c &= \left(\alpha_1 K_c^{-\rho_c} + \alpha_2 L_c^{-\rho_c} + e_c(\bar{K}_c, \bar{L}_c)\right)^{-1/\rho_c} \\ y &= \left(\beta_1 K_y^{-\rho_y} + \beta_2 L_y^{-\rho_y} + e_y(\bar{K}_y, \underline{a}r L_y)\right)^{-1/\rho_y} \end{aligned}$$

with $\rho_c, \rho_y > -1$ and $\sigma_c = 1/(1+\rho_c) \geq 0$, $\sigma_y = 1/(1+\rho_y) \geq 0$ the elasticities of capital/labor substitution in each sector. As previously, the externalities, $e_c(\bar{K}_c, \bar{L}_c)$ and $e_y(\bar{K}_y, \bar{L}_y)$ depend on $\bar{K}_i$, $\bar{L}_i$ which denote the average use of capital and labor in sector $i = c, y$ and will now be equal to

$$e_c(\bar{K}_c, \bar{L}_c) = a_1 \bar{K}_c^{-\rho_c} + a_2 \bar{L}_c^{-\rho_c}, \quad e_y(\bar{K}_c, \bar{L}_c) = b_1 \bar{K}_y^{-\rho_y} + b_2 \bar{L}_y^{-\rho_y}$$

with $a_i, b_i \geq 0$, $i = 1, 2$. At the equilibrium, all firms of sector $i = c, y$ being identical, we have $\bar{K}_i = K_i$ and $\bar{K}_i = K_i$. Denoting $\hat{\alpha}_i = \alpha_i + a_i$, $\hat{\beta}_i = \beta_i + b_i$, the *social production functions* are defined as

$$c = \left(\hat{\alpha}_1 K_c^{-\rho_c} + \hat{\alpha}_2 L_c^{-\rho_c}\right)^{-1/\rho_c} \text{ and } y = \left(\hat{\beta}_1 K_y^{-\rho_y} + \hat{\beta}_2 L_y^{-\rho_y}\right)^{-1/\rho_y}$$

The returns to scale are again constant at the social level, and decreasing at the private level. We will assume in the following that $\hat{\alpha}_1 + \hat{\alpha}_2 = \hat{\beta}_1 + \hat{\beta}_2 = 1$ so that the production functions collapse to Cobb-Douglas in the particular case $\rho_c = \rho_y = 0$.
We follow the same methodology as in the previous section with Cobb-Douglas technologies. We need however to assume the following restriction:

Assumption 4 $\hat{\beta}_1 < (\delta\beta_1)^{\rho_y/(1+\rho_y)}$

For some given β_1, $\hat{\beta}_1$ and δ, Assumption 4 provides an upper bound $\hat{\rho}_y > 0$ for ρ_y. We have indeed

$$\rho_y < \frac{ln\hat{\beta}_1}{ln(\delta\beta_1) - ln\hat{\beta}_1} \equiv \hat{\rho}_y \tag{10.5}$$

Such a restriction is quite standard when CES technologies are considered. It is well-known indeed that when the elasticity of capital/labor substitution is less than 1, Inada conditions are not satisfied and corner solutions cannot be a priori ruled out. Assumption 4 precisely ensures positiveness and interiority

[11] The proof of the results presented in this section can be found in Nishimura and Venditti [21].

of all the steady state values for input demand functions K_c, K_y, L_c and L_y. Throughout the paper we will therefore consider that $\rho_y \in (-1, \hat{\rho}_y)$.

Remark: In the Cobb-Douglas case with $\rho_y = 0$, the Inada conditions are satisfed and Assumption 4 becomes $\hat{\beta}_1 < 1$ which always holds.

Under this restriction we then obtain existence and uniqueness of the steady state k^*:

Proposition 10.4.1. *Under Assumption 4, there exists a unique stationary capital stock $k^* > 0$, such that:*

$$k^* = \frac{\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}}\left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}}-\hat{\beta}_1}{\hat{\beta}_2}\right)^{\frac{1+\rho_y}{\rho_y(1+\rho_c)}}}{1-(\delta\beta_1)^{\frac{1}{1+\rho_y}}\left[1-\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}}\left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}}-\hat{\beta}_1}{\hat{\beta}_2}\right)^{\frac{\rho_y-\rho_c}{\rho_y(1+\rho_c)}}\right]}$$

10.4.1 Symmetric Elasticities of Substitution

In order to start with the simplest case, we will assume that both sectors are characterized by the same elasticity of substitution:[12]

Assumption 5 $\rho_c = \rho_y = \rho$

We may now provide expressions of the characteristic roots.

Theorem 10.4.1. *Under Assumption 4-5, the characteristic roots are given by*

$$x_1 = \left\{(\delta\beta_1)^{\frac{1}{1+\rho}}\left[1-\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho}}\right]\right\}^{-1}$$

$$x_2 = (\delta\beta_1)^{\frac{-\rho}{1+\rho}}\hat{\beta}_1\left[1-\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)^{\frac{\rho}{1+\rho}}\right]$$

Remark: If both technologies are Cobb-Douglas with $\rho = 0$, the roots given in Theorem 10.3.1 are recovered.

Theorem 10.4.1 shows that the stability properties of the steady state will depend, among all the parameters, on the sign of the following differences $\alpha_1\beta_2 - \alpha_2\beta_1$ and $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$. As in the Cobb-Douglas case, it can be easily shown around the steady state that if the elasticities of capital/labor substitution are identical across sectors, the consumption good is capital intensive at the private level if and only if $\alpha_1\beta_2 - \alpha_2\beta_1 > 0$ while it is capital intensive at the social level if and only if $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 = \hat{\alpha}_1 - \hat{\beta}_1 > 0$.

[12] A continuous-time version of this model extended to n sector is studied in Nishimura and Venditti [20].

As in the Cobb-Douglas framework, local indeterminacy will require the consumption good to be capital intensive at the private level.[13] The following theorem extends Theorem 10.3.2 to technologies with non unitary elasticities of capital-labor substitution. It shows that under symmetric substitutability, local indeterminacy may still occur for elasticities significantly different from unity. The only consequence of such a restriction is that extreme values for ρ are excluded:

Theorem 10.4.2. *Under Assumptions 3-5, consider $\hat{\rho}_y$ as defined by equation (10.5) and let $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$. There exist $\underline{\rho} \in (-1, 0)$ and $\bar{\rho} \in (0, \hat{\rho}_y)$ such that the steady state is locally indeterminate for any $\rho \in (\underline{\rho}, \bar{\rho})$ if one of the two following conditions is satisfied:*

i) the investment good is capital intensive at the social level;

ii) the investment good is labor intensive at the social level and $\hat{\beta}_1 - \hat{\alpha}_1 > -\hat{\alpha}_2$.

10.4.2 Asymmetric Elasticities of Substitution

We may consider now the general formulation with asymmetric elasticities of capital-labor substitution.

Theorem 10.4.3. *Under Assumption 4, the characteristic roots are given by*

$$x_1 = \left\{ (\delta\beta_1)^{\frac{1}{1+\rho_y}} \left[1 - \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}} \left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}} - \hat{\beta}_1}{\hat{\beta}_2} \right)^{\frac{\rho_y - \rho_c}{\rho_y(1+\rho_c)}} \right] \right\}^{-1}$$

$$x_2 = (\delta\beta_1)^{\frac{-\rho_y}{1+\rho_y}} \hat{\beta}_1 \left[1 - \frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1} \left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)^{\frac{\rho_c}{1+\rho_c}} \left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}} - \hat{\beta}_1}{\hat{\beta}_2} \right)^{\frac{\rho_y - \rho_c}{\rho_y(1+\rho_c)}} \right]$$

Remark: If both technologies have the same elasticity of substitution, i.e. $\rho_c = \rho_y = \rho$, the roots given in Theorem 10.4.1 are recovered.

Contrary to the case with symmetric elasticities of substitution, the capital intensity differences at the private and social levels are not easily captured by the differences $\alpha_1\beta_2 - \alpha_2\beta_1$ and $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$. They also depend on prices and the parameters ρ_c and ρ_y. We may however obtain the following characterization at the steady state:

Proposition 10.4.2. *Under Assumption 4, at the steady state:*

i) the consumption (investment) good sector is capital intensive from the private perspective if and only if

[13] If the investment good is capital intensive at the private level, it is easy to show that $x_1 > 1$ and the steady state is locally determinate.

$$\left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}}-\hat{\beta}_1}{\hat{\beta}_2}\right)^{\frac{\rho_c-\rho_y}{\rho_y(1+\rho_c)}} < (>) \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}} \tag{10.6}$$

ii) the consumption (investment) good sector is capital intensive from the social perspective if and only if

$$\left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}}-\hat{\beta}_1}{\hat{\beta}_2}\right)^{\frac{\rho_c-\rho_y}{\rho_y(1+\rho_c)}} < (>) \left(\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\right)^{\frac{1}{1+\rho_c}} \left(\frac{\hat{\beta}_2\beta_1}{\hat{\beta}_1\beta_2}\right)^{\frac{\rho_c-\rho_y}{(1+\rho_y)(1+\rho_c)}} \tag{10.7}$$

Remark: If $\rho_c = \rho_y = \rho$, condition (10.6) becomes $\alpha_1\beta_2 - \alpha_2\beta_1 > (<)0$ and condition (10.7) becomes $\hat{\alpha}_1 - \hat{\beta}_1 > (<)0$. Notice also from Theorem 10.3.1 and (10.6) that as in the Cobb-Douglas formulation, the root x_1 is positive if and only if the investment good is capital intensive at the private level. On the contrary, when $\rho_c \neq \rho_y \neq 0$, the sign of the second root x_2 does not directly depend on the sign of the capital intensity difference across sectors at the social level.

In order to simplify the exposition, we will discuss the local stability properties of the steady state depending on the sign of the differences $\alpha_1\beta_2 - \alpha_2\beta_1$ and $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$, and the values of the elasticities of substitution in both sectors. We will only refer to capital intensities when the results are economically interpreted.

We have now to give conditions for local indeterminacy when the consumption good is capital intensive at the private level. We first consider the case $\alpha_1\beta_2 - \alpha_2\beta_1 < 0$ which, as we have shown previously, is known in the Cobb-Douglas framework to imply local determinacy of the steady state.[14] The following theorem shows on the contrary that with asymmetric elasticities of substitution, there is room for local indeterminacy.

Theorem 10.4.4. *Under Assumptions 3-4, let* $\alpha_1\beta_2 < \alpha_2\beta_1$ *and* $(\hat{\alpha}_1/\hat{\alpha}_2)/(\alpha_1/\alpha_2) < \delta\beta_2$. *Then there exist* $\underline{\rho}_c > 0$ *and* $\bar{\rho}_y \in (-1, 0)$ *such that the steady state is locally indeterminate if* $\rho_c > \underline{\rho}_c$ *and* $\rho_y \in (-1, \bar{\rho}_y)$.

Theorem 10.4.4 proves that even in the unusual situation with $\alpha_1\beta_2 < \alpha_2\beta_1$, local indeterminacy may occur provided the consumption good sector has a technology close to a Leontief function while the investment good sector has a technology close to a linear function. A direct inspection of inequality (10.6) from Proposition 10.4.2 shows that when ρ_c is high enough while ρ_y is close to -1, the consumption good is capital intensive at the private level.

We will consider now the converse configuration with $\alpha_1\beta_2 > \alpha_2\beta_1$. As this was already the case in a Cobb-Douglas framework, local indeterminacy requires a slightly stronger restriction concerning these parameters:

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[14] When $\rho_y = \rho_c = 0$, such a restriction implies indeed that the investment good is capital intensive at the private level. Local indeterminacy is thus ruled out. As shown in section 10.4.1, the same result actually holds when $\rho_y = \rho_c = \rho \neq 0$.

Assumption 6 $\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}} > 1 + (\delta\beta_1)^{-1}$

If technologies are either Cobb-Douglas or with identical elasticities of substitution, Assumption 6 implies Assumption 3. Notice also that if $\rho_c = 0$, we get the condition $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$ in Theorem 10.3.2. Assumption 6 actually ensures that $x_1 \in (-1, 0)$ when $\rho_y = 0$.

Let us start with some conditions that cover the case in which the investment good sector has a Cobb-Douglas technology with $\rho_y = 0$. Under Assumption 6 we will introduce additional restrictions to get $x_2 \in (-1, 1)$

Theorem 10.4.5. *Under Assumptions 3-4 and 6, consider $\hat{\rho}_y$ as defined by equation (10.5). If the following condition holds for some given $\rho_c > -1$*

$$\frac{\hat{\beta}_1}{1+\hat{\beta}_1}\left(\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\right) < \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho_c}{1+\rho_c}} \tag{10.8}$$

then there exist $\underline{\rho}_y \in (-1, 0)$ and $\bar{\rho}_y \in (0, \hat{\rho}_y)$ such that the steady state is locally indeterminate for any $\rho_y \in (\underline{\rho}_y, \bar{\rho}_y)$. Moreover the lower bound $\underline{\rho}_y$ is equal to -1 if the following additional restrictions hold:

$$1 \leq \frac{\hat{\beta}_2^{1+\rho_c}}{\beta_2} < \delta\frac{\alpha_1}{\alpha_2} \text{ and } \left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)^{\frac{\rho_c}{1+\rho_c}} < (\delta\beta_1)^{\frac{\rho_c}{1+\rho_c}}\frac{\hat{\alpha}_2}{\hat{\alpha}_1} \tag{10.9}$$

When the additional conditions (10.9) hold, Theorem 10.4.5 shows that for some given $\rho_c > -1$, local indeterminacy is compatible with arbitrarily large elasticities of capital/labor substitution in the investment good sector. Notice that this cannot be the case with symmetric elasticities of substitution.

We may discuss Theorem 10.4.5 depending on the sign of the difference $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$. As in the Cobb-Douglas case, local indeterminacy with CES technologies does not require a capital intensity reversal. We derive indeed from Theorem 10.3.1 and Proposition 10.4.2 that the characteristic root x_2 may be either less than 1 when positive or greater than -1 when negative for any sign of the capital intensity difference at the social level.

Consider first the case $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 < 0$. It is then easy to see that condition (10.8) holds for any $\rho_c \geq 0$, i.e. for any consumption good technology having an elasticity of capital/labor substitution less than unity. It follows from Assumption 6 that if $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$ local indeterminacy will hold for any $\rho_c \geq 0$. It is worth noticing that this covers the case $\rho_c = +\infty$ of a Leontief technology for the consumption good. As already mentioned previously, local indeterminacy also occurs for $\rho_c < 0$ but far enough from -1. The robustness of this result will indeed depend on the CES coefficients at the social level.
Consider now the converse case $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 > 0$. Condition (10.8) may still hold but no clear restriction on the parameter ρ_c can be derived. If the difference $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$ is significantly greater than zero, i.e. for instance if

$$\frac{\hat{\beta}_1}{1+\hat{\beta}_1}\left(\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\right) > 1, \tag{10.10}$$

then local indeterminacy cannot hold when ρ_c is close to zero and will require much lower elasticities of capital/labor substitution in the consumption good sector.

Notice also that since $\alpha_1\beta_2 - \alpha_2\beta_1 > 0$ and $\lim_{\rho_c \to -1} \rho_c/(1+\rho_c) = -\infty$, condition (10.8) cannot hold when ρ_c is close enough to -1. It follows that under the Assumptions of Theorem 10.4.5 there exists $\underline{\rho}_c \in (-1, 0)$ such that local indeterminacy occurs when $\rho_c > \underline{\rho}_c$.

We may finally give conditions which cannot be satisfied when the technology of the investment good is Cobb-Douglas. When condition (10.8) does not hold, local indeterminacy appears while the elasticity of substitution in the investment good sector is less than unity.

Theorem 10.4.6. *Under Assumptions 3-4 and 6, consider $\hat{\rho}_y$ as defined by equation (10.5). If the following condition holds for some given $\rho_c \in (-1, \hat{\rho}_y]$*

$$\frac{\hat{\beta}_1}{1+\hat{\beta}_1}\left(\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\right) > \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho_c}{1+\rho_c}} \tag{10.11}$$

then there exist $\underline{\rho}_y \in (0, \hat{\rho}_y)$ and $\bar{\rho}_y \in (0, \hat{\rho}_y)$ with $\underline{\rho}_y < \bar{\rho}_y$ such that the steady state is locally indeterminate for any $\rho_y \in (\underline{\rho}_y, \bar{\rho}_y)$.

Notice that contrary to condition (10.8) in Theorem 10.4.5, condition (10.11) is now compatible with ρ_c close to -1, i.e. with an arbitrarily large elasticity of capital/labor substitution in the consumption good sector.

We may again discuss Theorem 10.4.6 depending on the sign of the difference $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$. Consider first the case $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 < 0$. Condition (10.11) shows that local indeterminacy cannot occur for any $\rho_c \geq 0$. The same conclusion holds also for $\rho_c < 0$ but close to 0. Local indeterminacy indeed requires a strong enough elasticity of capital/labor substitution in the consumption good sector while that elasticity in the investment good sector is restricted to be less than unity.

Consider finally the case $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 > 0$. Local indeterminacy now becomes compatible with positive values for ρ_c provided the difference $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$ is significantly greater than zero, i.e. if equation (10.10) holds.

10.5 Extensions with Cobb-Douglas Technologies

10.5.1 Partial Depreciation

Until now we have assumed that capital fully depreciates every period. This much criticized assumption has been proved to be quite particular by Baierl, Nishimura and Yano [2] in two-sector optimal growth models. Unlike continuous-time models, introducing depreciation of capital indeed creates additional difficulty in studying dynamical properties of equilibrium paths in discrete time models.[15]

We now extend Baierl, Nishimura and Yano [2] to the case with externalities.[16] We thus assume partial depreciation of capital so that the capital accumulation equation becomes $y_t = k_{t+1} - (1-\mu)k_t$, with $\mu \in [0,1]$. In this case, the envelope theorem provides the following equilibrium prices:

$$\begin{aligned} p_t &= -T_2\left(k_t, k_{t+1}, e_{ct}, e_{yt}\right) \\ \omega_t &= T_1\left(k_t, k_{t+1}, e_{ct}, e_{yt}\right) + (1-\mu)T_2\left(k_t, k_{t+1}, e_{ct}, e_{yt}\right) \end{aligned}$$

and the Euler equation becomes

$$-p_t + \delta[\omega_{t+1} + (1-\mu)p_{t+1}] = 0$$

Existence and uniqueness of the steady state still hold but now depend on the parameter μ. As in Section 2, let $\theta = \delta(1-\mu) \in [0,1]$:

Corollary 10.5.1. *There exists a unique stationary capital stock k^* satisfying:*

$$k^* = \frac{\alpha_1\beta_2(1-\theta)}{\beta_1[\alpha_2(1-\theta)+(\alpha_1\beta_2-\alpha_2\beta_1)\delta\mu]}\left(\frac{\delta\beta_1}{1-\theta}\right)^{\frac{1}{\beta_2}}$$

The characteristic roots depend also on the rate of capital depreciation:

Theorem 10.5.1. *The characteristic roots are given by:*

$$\begin{aligned} x_1 &= \frac{\alpha_2\left(1-\theta\right) + \theta(\alpha_2\beta_1 - \alpha_1\beta_2)}{\delta(\alpha_2\beta_1 - \alpha_1\beta_2)} \\ x_2 &= \frac{\hat{\alpha}_2\hat{\beta}_1 - \hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2(1-\theta) + \theta(\hat{\alpha}_2\hat{\beta}_1 - \hat{\alpha}_1\hat{\beta}_2)} \end{aligned}$$

When capital depreciates slowly, local indeterminacy still requires a capital intensive consumption good at the private level.[17] Moreover, as in the case with full depreciation, a capital intensity reversal is not necessary. Assume first that the investment good is capital intensive at the social level.

Theorem 10.5.2. *Under Assumption 3, let $\hat{\beta}_1 > \hat{\alpha}_1$ and $\bar{\theta} = \alpha_2/[\alpha_2(1-\beta_1) + \alpha_1\beta_2] < 1$. Then the following cases hold:*

i) if $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$, there exists $\hat{\theta} \in]\bar{\theta}, 1[$ such that the steady state is

[15] Baierl, Nishimura and Yano [2] show indeed that around the steady-state, optimal paths become less likely to oscillate in the case of partial depreciation than in that of full depreciation.

[16] The proof of the results presented in this subsection can be found in Nishimura and Venditti [17].

[17] If the investment good is capital intensive at the private level, we easily show that $x_1 > 1$ and the steady state is locally determinate.

locally indeterminate for any $\theta \in [0, \hat{\theta}[\setminus\{\bar{\theta}\}$*;*

ii) if $\alpha_1\beta_2 - \alpha_2\beta_1 < \alpha_2/\delta$*, there exist* $\tilde{\theta}, \hat{\theta} \in]0,1[$*, with* $\tilde{\theta} < \bar{\theta} < \hat{\theta}$*, such that the steady state is locally indeterminate for any* $\theta \in]\tilde{\theta}, \hat{\theta}[\setminus\{\bar{\theta}\}$*.*

Case i) provides an extension to partial depreciation of Theorem 10.3.2*i)* which has been established under $\theta = 0$. Moreover we show that given production functions and a discount factor close to 1, equilibrium paths become less likely to be locally indeterminate in the case of partial depreciation (θ close enough to 1) than in that of full depreciation ($\theta = 0$). Baierl, Nishimura and Yano [2] have obtained a similar result concerning the occurrence of period-two cycles in an optimal growth model.

In case ii), a similar result is obtained. We provide however some new conditions for local indeterminacy that cannot arise under full depreciation. For intermediary values of the depreciation rate, local indeterminacy arises under mild conditions on the capital intensity difference at the private level: the consumption good needs to be only slightly more capital intensive than the investment good.[18]

Assume now that the consumption good is also capital intensive at the social level.

Theorem 10.5.3. *Under Assumption3, let* $\hat{\alpha}_1 > \hat{\beta}_1 > \hat{\alpha}_1 - \hat{\alpha}_2$*,* $\theta^* = 2\hat{\alpha}_2/\hat{\beta}_2 - 1 \in (0,1)$ *and* $\bar{\theta} = \alpha_2/[\alpha_2(1-\beta_1) + \alpha_1\beta_2] < 1$*. Then the following cases hold:*

i) if $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$*, the steady state is locally indeterminate for any* $\theta \in [0, \theta^*[\setminus\{\bar{\theta}\}$*;*

ii) Let $\tilde{\theta} = [\alpha_2(1+\delta\beta_1) - \delta\alpha_1\beta_2]/[\alpha_2(1-\beta_1) + \alpha_1\beta_2] < 1$*. If* $\alpha_1\beta_2 - \alpha_2\beta_1 < \alpha_2/\delta$ *and* $\tilde{\theta} < \theta^*$*, the steady state is locally indeterminate for any* $\theta \in]\tilde{\theta}, \theta^*[\setminus\{\bar{\theta}\}$*.*

Case i) provides an extension to partial depreciation of Theorem 10.3.2*ii)* which has been derived under $\theta = 0$. As in the previous case, we show that given production functions and a discount factor close to 1, equilibrium paths become less likely to be locally indeterminate in the case of partial depreciation (θ close enough to 1) than in that of full depreciation ($\theta = 0$).[19]

In case ii), local indeterminacy is more difficult to obtain than under a capital intensity reversal between the private and the social level. The condition on the capital intensity difference at the private level is less demanding than in case

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[18] When θ crosses $\tilde{\theta}$ from above the steady state becomes saddle-point stable, a flip bifurcation occurs and there exist equilibrium period-two cycles either in a left or in a right neighborhood of $\tilde{\theta}$.

[19] Notice that a flip bifurcation occurs when θ crosses $\hat{\theta}$ from below and the steady state then becomes sadle-point stable while there exist equilibrium period-two cycles either in a right or in a left neighborhood of $\hat{\theta}$.

i), but the restriction $\tilde{\theta} < \theta^*$ does not have a precise economic interpretation.[20]

Remark: Nishimura and Venditti [19] have also extended to partial depreciation the CES formulation with symmetric elasticities of substitution considered in Section 10.4.1. Depending on the value of the elasticity of capital-labor substitution, local indeterminacy may arise for any value $\mu \in [0, 1]$ of the rate of depreciation, for low depreciation with μ close to zero, or for high depreciation with μ close to one. The conclusion that local indeterminacy is less likely in the case of partial depreciation than in that of full depreciation is therefore specific to the Cobb-Douglas formulation.

10.5.2 Intersectoral Externalities

Up to now we have considered sector-specific external effects. Although equilibria become sub-optimal, such a formulation remains quite close to standard optimal growth models since no direct additional intersectoral mechanisms are introduced. This is not the case if we consider the initial formulation of Romer [24] in which the aggregate capital stock is used as global technological externalities.

In order to introduce these additional mechanisms in a simple Cobb-Douglas framework, we assume now that the consumption good production function contains positive intersectoral externalities given by a convex combination of the capital stocks of the two sectors.[21] The production functions are thus:

$$y = K_y^{\beta_1} L_y^{\beta_2}, \quad c = K_c^{\alpha_1} L_c^{\alpha_2} e(\bar{K}_c, \bar{K}_y) \ \text{ with } \ e = \left[\phi \bar{K}_c + (1-\phi)\bar{K}_y\right]^a$$

where $\phi \in [0, 1]$, $a \geq 0$ and $\bar{K}_i$ denotes the average use of capital in sector $i = c, y$. Depending on the value of ϕ, our formulation therefore encompasses the usual assumptions of sector specific externalities ($\phi = 1$), and global external effects ($\phi = 1/2$). We will also consider the case with purely intersectoral externalities ($\phi = 0$). We assume that these economy-wide averages are taken as given by individual firms. At the equilibrium, all firms of sector $i = c, y$ being identical, we have $\bar{K}_i = K_i$ and $\bar{K}_i = K_i$. The *social production function* for the consumption good is therefore

$$c = K_c^{\alpha_1} L_c^{\alpha_2} \left[\phi K_c + (1-\phi) K_y\right]^a$$

We will assume non-increasing returns to scale at the social level in the consumption good sector, i.e. $\alpha_1 + a + \alpha_2 \equiv \hat{\alpha}_1 + \alpha_2 \leq 1$, and constant returns to scale in the investment good sector, i.e. $\beta_1 + \beta_2 = 1$.

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[20] If these conditions hold, endogenous fluctuations again appear through a flip bifurcation: when θ crosses $\tilde{\theta}$ (θ^*) from above (below) the steady state becomes saddle-point stable, and there exist equilibrium period-two cycles either in a left or in a right neighborhood of $\tilde{\theta}$ (θ^*).

[21] The proof of the results presented in this subsection can be found in Nishimura and Venditti [18].

It can be easily shown that if $\alpha_1\beta_2 - \alpha_2\beta_1 < (>)0$ the investment (consumption) good sector is capital intensive from the private perspective. Note that this definition is still valid with intersectoral external effects ($\phi < 1$). If the externalities are sector specific ($\phi = 1$), the condition $\beta_1 > (<)\hat{\alpha}_1$ implies that the investment (consumption) good sector is capital intensive from the social perspective.

We follow the same procedure as in the previous sections. Full depreciation of capital is again assumed for simplicity. The steady state is given by Proposition 10.3.1 with $\hat{\beta}_2 = \beta_2$. We start by assuming that there are only sector specific externalities in the consumption good sector. We show in this case that the steady state is always locally determinate.

Proposition 10.5.1. *If the externalities are sector specific ($\phi = 1$), the steady state k^* is locally determinate.*

Theorem 10.3.2 establishes that if the consumption good is capital intensive from the private perspective, locally indeterminate equilibria may occur when the consumption good is either capital or labor intensive at the social level. However, they assume that there are external effects on capital and labor in both sectors. Proposition 10.5.1 shows however that when only the consumption good technology is affected by external effects, indeterminacy necessarily requires externalities coming from labor. Note that this result does not hold and indeterminacy is still possible if we assume that the investment good sector contains external effects on capital. Our formulation therefore will strongly enlighten the role of intersectoral external effects.

Consider indeed the case in which the externality in the consumption good technology comes only from the capital stock of the investment good sector ($\phi = 0$).

Assumption 7 $min\left\{\dfrac{\beta_1\alpha_2}{\hat{\alpha}_1}, 1, \dfrac{a\alpha_2(1-\alpha_1)}{\alpha_1+\alpha_2}\right\} > \beta_2.$

Given arbitrary $a > 0$, Assumption 12.77 may be satisfied if β_2 is chosen to be sufficiently small. Assumption 12.77 also implies that the investment good is capital intensive at the private level since $\beta_1\alpha_2/\hat{\alpha}_1 > \beta_2$ implies $\beta_1 > \hat{\alpha}_1 > \alpha_1$.

Proposition 10.5.2. *Let $\phi = 0$. Under Assumption 12.77, there exists $\delta_1 < 1$ such that the steady state is locally indeterminate for any $\delta \in]\delta_1, 1]$.*

Contrary to the sector-specific formulation in which local indeterminacy requires the consumption good to be capital intensive at the private level, we show that when pure intersectoral external effects are considered, a continuum of equilibria may arise under a capital intensive investment good at the private level.

From Proposition 10.5.2 it is straightforward to extend the indeterminacy result to intermediary cases with positive values of θ as in the following Theorem:

Theorem 10.5.4. . *Under Assumption 12.77, there exist* $0 < \delta_1 < 1$ *and a function* $\phi^* :]\delta_1, 1] \to]0, 1[$ *such that the steady state is locally indeterminate for each* $\delta \in]\delta_1, 1]$ *and* $\phi \in [0, \phi^*(\delta)[$.

Remark: It can be shown that if externalities coming from capital are also introduced in the investment good sector, Proposition 10.5.2 and Theorem 10.5.4 still hold with some more complicated sufficient conditions which will depend on the external effect parameters of the capital good. Indeterminacy under a capital intensive investment good at the private level is thus a robust property as soon as externalities are intersectoral.[22]

10.6 Other Formulations

10.6.1 Variable Capital Utilization

In Section 10.5.1, we have introduced partial depreciation of capital and we have shown how the occurrence of local indeterminacy is affected by this parameter. Building on the fact that capacity utilization is potentially a powerful driving force behind business cycles, Wen [28] considers a discrete-time extension of the Benhabib and Farmer [3] model in which the speed of capital depreciation is endogenously determined. A representative consumer solves indeed

$$\begin{aligned}
\max_{\{c_t, l_t, u_t, k_{t+1}\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t \left(\log c_t - \frac{l_t^{1-\chi}}{1-\chi} \right) \\
s.t. \quad & c_t + k_{t+1} - (1-\mu_t)k_t = (u_t k_t)^\alpha l_t^{1-\alpha} e_t(\bar{u}_t \bar{k}_t, \bar{l}_t) \\
& \mu_t = \tau u_t^\gamma \\
& k_0, \{e_t\}_{t=0}^{+\infty} \text{ given}
\end{aligned}$$

with $\chi \leq 0$, $\alpha \in (0,1)$, $\tau \in (0,1)$, $u_t \in (0,1)$ the rate of capacity utilization, $\mu_t \in (0,1)$ the rate of capital depreciation defined as an increasing function of capacity utilization, i.e. $\gamma > 0$, and $e_t(\bar{u}_t, \bar{k}_t, \bar{l}_t)$ the externality expressed as a function of the average economy-wide levels of productive capacity and labor, i.e.

$$e_t(\bar{u}_t \bar{k}_t, \bar{l}_t) = (\bar{u}_t \bar{k}_t)^{\alpha\eta} \bar{l}_t^{(1-\alpha)\eta}$$

where $\eta \geq 0$. Variable capital utilization is ensured under $\gamma > 1$ while constant partial depreciation as in Benhabib and Farmer [3] follows from $\gamma \leq 1$. Standard

[22] Under slight additional restrictions, it can also be proved that for any given ϕ close enough to zero, there exists a bound $\delta^*(\phi) \in (0,1)$ such that when δ crosses $\delta^*(\phi)$ from above the steady state becomes saddle-point stable and quasi-periodic cycles appear through a Hopf bifurcation. Notice that endogenous fluctuations are obtained under a capital intensive investment good while Benhabib and Nishimura [6] show in an optimal growth model that a capital intensive consumption good is necessary.

linearization of the first order conditions around the steady state allows Wen to show that the product and sum of the characteristic roots satisfy

$$\begin{aligned}\mathcal{D} &= \tfrac{1}{\delta}\left(1 - \tfrac{\eta(1-\chi)(1-\delta)\tau_l}{\delta(1-\alpha)(1+\eta)\tau_l-(1-\chi)}\right)\\ \mathcal{T} &= 1+\mathcal{D} - \tfrac{(1-\chi)(1-\delta)(\gamma-\alpha)(1-\alpha(1-\chi)\tau_k)\mu/\alpha}{\delta(1-\alpha)(1+\eta)\tau_l-(1-\chi)}\end{aligned}$$

where

$$\tau_k = \frac{\gamma-1}{\gamma-\alpha(1+\eta)},\ \tau_l = \frac{\gamma}{\gamma-\alpha(1+\eta)}$$

In a discrete-time framework, local indeterminacy requires $|\mathcal{D}| < 1$ and $|\mathcal{T}| < 1 + \mathcal{D}$. When compared with the corresponding expressions under constant partial depreciation given in Section 2, we easily derive that multiple equilibria become compatible with much lower increasing returns to scale and a downward sloping aggregate labor demand curve, i.e. $(1-\alpha)(1+\eta)-1 < 0$.

More recently, Guo and Harrison [13] provide an extension of the Wen's capacity utilization model to a discrete-time adaptation of the Benhabib and Farmer's [4] two-sector model with sector-specific externalities. Both sectors have the same Cobb-Douglas technology at the private level with constant returns to scale. Variable capital utilization is introduced into technologies as follows

$$c = (uK_c)L_c^{1-\alpha}e_c(\bar{u}\bar{K}_c, \bar{L}_c),\ y = (uK_y)^\alpha L_y^{1-\alpha}e_y(\bar{u}\bar{K}_y, \bar{L}_y)$$

The externalities $e_c(\bar{u}\bar{K}_c, \bar{L}_c)$ and $e_y(\bar{u}\bar{K}_y, \bar{L}_y)$ depend on the average use of capital and labor services and are equal to

$$e_c(\bar{u}\bar{K}_c, \bar{L}_c) = [(\bar{u}\bar{K}_c)^\alpha \bar{L}_c^{1-\alpha}]^\eta,\ e_y(\bar{u}\bar{K}_y, \bar{L}_y) = [(\bar{u}\bar{K}_y)^\alpha \bar{L}_y^{1-\alpha}]^\eta \tag{10.12}$$

with $\eta > 0$. Returns to scale are therefore increasing at the social level. Guo and Harrison show that local indeterminacy occurs under smaller externalities and thus lower increasing returns to scale than in the Benhabib and Farmer's [4] and Wen's [28] models.

10.6.2 Two-Sector Models with General Technologies

In Section 10.5.2, we have considered a model with intersectoral externalities which is compatible with both sector-specific and global external effects. In order to provide simple conditions, we have assumed Cobb-Douglas technologies. Boldrin and Rustichini [9] also introduce Romer-type [24] global externalities in a two-sector discrete-time model but they consider general technologies.

The labor supply is inelastic with total labor normalised to 1, and the population is constant. The pure consumption good c and the capital good y are produced with constant private returns to scale technologies which also depend on an intersectoral externality A:

$$c = f^0(k_0, l_0, A), \quad y = f^1(k_1, l_1, A)$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq 1$. At the equilibrium, the externality A will equal the aggregate capital stock k.

Assumption 8 *Each production function $f^i(k^i, l^i, A)$, $i = 0, 1$, is C^2, increasing in each argument and, for any $A > 0$, concave, homogeneous of degree one and such that for any $l_i > 0$, $f^i_{11}(., l_i, A) < 0$.*

Externalities are therefore positive and returns to scale are increasing at the social level. For any given (k, y, A), the production frontier $T(k, y, A)$ is defined as

$$\begin{array}{rcl} T(k,y,A) = & \max\limits_{k_0,k_1,l_0,l_1} & f^0(k_0,l_0,A) \\ & s.t. & y \leq f^1(k_1,l_1,A) \\ & & k_0 + k_1 \leq k \\ & & l_0 + l_1 \leq 1 \\ & & k_0,k_1,l_0,l_1 \geq 0 \end{array}$$

Under Assumption 8, for any given $A > 0$, $T(k, y, A)$ is concave. Assuming a linear utility function and full depreciation of capital within one period of time, the maximization program of the representative agent is

$$\begin{array}{cl} \max\limits_{\{k_{t+1}\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{+\infty} \delta^t T(k_t, k_{t+1}, A_t) \\ s.t. & (k_t, k_{t+1}) \in \mathcal{D}(A_t) \\ & k_0, \{A_t\}_{t=0}^{+\infty} \text{ given} \end{array}$$

with

$$\mathcal{D}(A_t) = \left\{(k_t, k_{t+1}) \in \mathbb{R}^2_+ / 0 \leq k_{t+1} \leq f^1(k_t, 1, A_t)\right\}$$

the set of admissible paths for any given A_t. Along an equilibrium path with $A_t = k_t$, the Euler equation is

$$T_2(k_t, k_{t+1}, k_t) + \delta T_1(k_{t+1}, k_{t+2}, k_{t+1}) = 0$$

An equilibrium path also satisfies the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t T_1(k_t, k_{t+1}, k_t) = 0$$

A steady state $k_{t+1} = k_t = k^*$ is a solution of

$$f^1_1(k_1(k, k,), l_1(k, k, k), k) = \delta^{-1}$$

Assuming the existence of a locally unique steady state k^* and linearizing the Euler equation around k^* easily shows that the sum and product of the characteristic roots satisfy

$$\begin{array}{rcl} \mathcal{T} & = & -[\delta(T^*_{11} + T^*_{13}) + T^*_{22}]/\delta T^*_{12} \\ \mathcal{D} & = & \delta^{-1} + T^*_{23}/\delta T^*_{12} \end{array}$$

with $T^*_{ij} = T_{ij}(k^*, k^*, k^*)$ and $T^*_{12} \neq 0$. It follows easily that $T^*_{23}/T^*_{12} < 0$ is a necessary condition for the occurrence of local indeterminacy. As in two-sector optimal growth models, the sign of T^*_{12} is ruled by the capital intensity difference at the private level. However, the sign of T^*_{23} is difficult to establish. Since $-T^*_2$ is equal to the price p of the capital good in terms of the price of the consumption good, we only know that

$$T^*_{23} = -\partial p/\partial A$$

Boldrin and Rustichini [9] provide formal conditions for local indeterminacy but it remains difficult to interpret these conditions in terms of the fundamentals.[23] In particular, although one may conjecture that local indeterminacy is compatible with a capital intensive investment good at the private level, there is no clear picture concerning the requirements on the capital intensity difference.

More recently, Drugeon [11] considers a discrete-time two-sector model with general technologies containing sector-specific and intersectoral external effects. Contrary to Boldrin and Rustichini he assumes constant returns at the private and social levels by using production functions which are linear homogeneous with respect to private factors and homogeneous of degree zero with respect to public factors. Moreover, developing a methodology based on the *equilibrium production frontier*,[24] he provides an expression for the characteristic polynomial in terms of elasticities of factor substitution in each sectors and shares of consumption, investment, wage and profits into national income. While local indeterminacy still requires a capital intensive consumption good at the private level, his main results are the following: with strong sector-specific external effects, local indeterminacy requires strong substitutability in the investment good sector and weak substitutability in the consumption good sector.[25] When strong intersectoral externalities are considered, a continuum of equilibria occurs if substitutability is high in the consumption good sector and low in the investment good sector.

[23] See also Venditti [27] for more detailled conditions on local indeterminacy and local bifurcation of periodic cycles.

[24] This corresponds to the function $T(k_t, k_{t+1}, A_t)$ evaluated along an equilibrium path.

[25] This result is in some sense close to Theorem 10.4.4 in Section 10.4.2.

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11. Theory of Stochastic Optimal Economic Growth

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11.1 Introduction

Stochastic optimal growth involves the study of optimal intertemporal allocation of capital and consumption in an economy where production is subject to random disturbances. The theory traces its roots to the seminal work on deterministic optimal growth by Ramsey [108], Cass [21] and Koopmans [56]. Its influence has been enhanced by research that shows how the convex stochastic growth model can be decentralized to represent the behavior of consumers and firms in a dynamic competitive equilibrium of a productive economy ([104], [117], [15]). This makes the stochastic optimal growth model useful both as a normative exercise and in the development of positive theories of how the economy works. As a consequence, the theory has emerged as one of the central paradigms of dynamic economics. It is based on a simple, yet powerful model that encompasses fundamental questions that are basic to any theory of dynamic economic behavior: What are the characteristics and determinants of optimal policies? What are the economic incentives that govern the optimal intertemporal allocation of resources? What is the transient and long run behavior of variables in the model? Under different assumptions the model admits a rich set of answers to these questions.

Historically, the main focal point of the theory has been issues of aggregate economic growth. At the same time its primary variable, capital, has a flexible interpretation that allows the model and its extensions to represent a wide variety of economic problems ranging from the study of business cycles ([60], [64]) and asset pricing ([14], [15]) to the allocation of renewable natural resources ([78], [83], [84]). Equally important, the model provides a strong theoretical foundation for applied analysis of these problems. The model can be solved numerically and has proved a testing ground for many numerical techniques used today in the analysis of dynamic economic problems.

This chapter provides an overview of key results in the *theory* of discounted stochastic optimal growth in discrete time.[1] The paper begins with an analysis of the classical stochastic growth model of Brock and Mirman [18] for a one-sector economy with a convex technology and utility that depends only on consumption.[2] We then consider extensions of the theory to problems with irreversible investment, increasing returns or a non-convex technology, experimentation and learning, and problems where utility depends on more than consumption alone. We develop the competitive price characterization of optimal policies that can be used to establish the equivalence between optimal and competitive outcomes; our focus, however, is on optimal solutions and their properties. The large literature on dynamic competitive equilibria is, therefore, left to the reader to explore. Likewise, we do not survey the many applications of the stochastic growth model. Instead, we focus on how the theory can be extended in different directions that have proved useful in application. Finally, we provide a glimpse of practical methods for solving the model, but the literature on numerical methods is too large for us to review here.

11.2 The Classical Framework

11.2.1 The One Sector Classical Model: Basic Properties

The stochastic growth model has three essential elements: an exogenous stochastic environment corresponding to random productivity disturbances, the production possibilities that determine the set of feasible allocations for consumption, investment and output, and an instantaneous welfare or utility function that represents the preferences of the agent or economic decision-maker. Productivity shocks at dates $t = 1, 2, ...$, are denoted by $\{r_t\}$, a sequence of i.i.d. real-valued random variables, with common distribution ν on $B(\Phi)$, the Borel σ-field of $\Phi \subset \Re$. In particular, Φ is the support of ν and is assumed to be compact. Associated with this stochastic environment is a measure space $(\Omega, \mathcal{F}, \mu)$, where Ω is the set of all real sequences, $\mathcal{F}$ is the σ-field generated by cylinder sets of the form $\prod_{t=0}^{\infty} A_t$, where A_t belongs to $B(\Phi)$ for all t, and μ is the product distribution induced by ν. The statements: for a.e. ω and μ-a.s. mean "except for a subset of Ω of μ-measure zero". The random variable r_t is simply the t^{th} coordinate function on Ω. In the economy, output of a homogeneous consumption/capital good is produced via a production function that is homogeneous of degree one in capital and labor. This allows the economy to be

[1] There is a large literature on stochastic growth in continuous time that builds on Merton's [79] early work (see also, [16]).

[2] Previous surveys of stochastic growth such as [82] and [6] focus primarily on this case.

represented in per capita terms where c_t, k_t and y_t denote per capita consumption, capital and output at time t. Given a capital stock at time $t-1$ and the productivity disturbance at the beginning of period t, $y_t = f(k_{t-1}, r_t)$, where $f : \Re_+ \times \Phi \to \Re_+$ is the production function. The *feasible set* for consumption and investment is: $\Gamma(y_t) = \{(c_t, k_t) | 0 \leq c_t, 0 \leq k_t, \text{ and } c_t + k_t \leq y_t\}$.

Each period the economic agent receives utility $u(c_t)$, where $u : \Re_+ \to \Re$. The discount factor for future utility is δ, where $0 \leq \delta < 1$. At the beginning of period t the agent observes y_t and chooses c_t and k_t. The productivity disturbance, r_{t+1}, occurs and a new output, y_{t+1}, is produced. The objective of the agent is to maximize the expected discounted sum of utility over time subject to the feasibility constraints on consumption and capital, and the transition equation that maps capital to output in the following period. Given an initial output, y_0, the objective is to:

$$Max \;\; E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right]$$
$$\textit{subject to:} \qquad 0 \leq c_t, 0 \leq k_t, c_t + k_t \leq y_t, y_{t+1} = f(k_t, r_{t+1}), t \geq 0. \quad (11.1)$$

This problem can be formulated as a stochastic dynamic programming problem ([11], [127] and [66]). At date t, the *partial history* is $h_t = \{y_0, c_0, k_0, y_1, ..., c_{t-1}, k_{t-1}, y_t\}$. A *policy*, π, is a sequence $\{\pi_0, \pi_1, ...\}$, where π_t is a conditional probability on $B(\Re_+)$, given h_t, such that $\pi_t(\Gamma(y_t) \mid h_t) = 1$. Let F be the set of all measurable functions ϕ such that $\phi(y) \in \Gamma(y)$ for all $y \in \Re_+$. A policy is *Markovian* if $\pi_t \in F$ for all t. A Markov policy is *stationary* if there exists a Borel measurable function, $\widehat{\pi}(y)$, such that $\pi_t(y) = \widehat{\pi}(y)$ for all t. A policy, π, and an initial state, y, induce a *feasible program*, $(\mathbf{y},\mathbf{c},\mathbf{k})=(y_t, c_t, k_t)_{t=0}^{\infty}$, a stochastic process for output, consumption and capital such that $(c_t, k_t) \in \Gamma(y_t)$ and $y_{t+1} = f(k_t, r_{t+1})$ a.s. for all t. Associated with each policy is an expected discounted sum of utility $V_\pi(y) = E\sum_{t=0}^{\infty} \delta^t u(c_t)$, where $(\mathbf{y},\mathbf{c},\mathbf{k})$ is the feasible program generated by π and f in the obvious manner. A policy, π^*, is optimal if $V_{\pi^*}(y) \geq V_\pi(y)$ for all π and y, and the associated program is called an *optimal program*. The value function $V(y)$ is defined on $\Re_+$ by $V(y) = \sup\{V_\pi(y) \mid \pi$ is a policy$\}$. It follows that π^* is an optimal policy if, and only if, $V_{\pi^*}(y) = V(y)$ for all $y \geq 0$.

Throughout the paper, derivatives are denoted using subscripts, so that u_c represents marginal utility of consumption and so on. The production technology and preferences are assumed to satisfy the following assumptions:

A.1. $f(0, r) = 0$, $f(k, r) > 0$ for all $r \in \Phi$ and all $k > 0$.

A.2. f is continuous on $\Re_+ \times \Phi$ and for each $r \in \Phi$, $f(\cdot, r)$ is continuously differentiable on $\Re_{++}$.

A.3. $f_k(k, r) > 0$ and $\inf_{r \in \Phi} f_k(0, r) > 1$.

A.4. $f(., r)$ is strictly concave on $\Re_+$ for all $r \in \Phi$.

A.5. There exists a $\overline{k} > 0$ such that $f(k, r) < k$ for all $k > \overline{k}$ and all $r \in \Phi$.

A.6. u is continuous on $\Re_+$ and continuously differentiable on $\Re_{++}$.
A.7. $u_c(c) > 0$ on $\Re_{++}$.
A.8. u is strictly concave on $\Re_+$.

Under these assumptions the dynamic optimization problem is well defined, the value is finite from any initial state and it satisfies the functional equation:

$$V(y) = \max_{c \in \Gamma(y)} [u(c) + \delta \int V(f(y-c,r))d\nu(r)]. \tag{11.2}$$

Further, there exist stationary optimal policy functions for consumption, $C(y) = \arg\max_{c\in\Gamma(y)}[u(c) + \delta \int V(f(y-c,r))d\nu(r)]$, and capital, $K(y) = y - C(y)$.[3]

To characterize economic behavior in the model it is important to understand the basic properties of the optimal value and policy functions. Further, such knowledge is necessary to examine how departures from the classical model affect economic outcomes. In the classical model, the feasible set correspondence $\Gamma(y)$ is expanding and has a convex graph,[4]. Using the assumption that the production and utility functions are strictly increasing and strictly concave and the fact that the functional equation (11.2) maps the set of continuous, increasing and strictly concave functions into itself this implies (e.g., [126]):

Lemma 11.2.1. *Under A.1-A.8, $V(y)$ is continuous, strictly increasing and strictly concave.*

The value function is a measure of lifetime economic welfare and to a first order approximation is proportional to traditional measures of GDP. The economic implication of Lemma 11.2.1 is that small increases in output have small effects on welfare, and that welfare increases at a diminishing rate as output increases.

Strict concavity of u, f and V implies that the maximization problem on the right hand side of (11.2), has a unique solution for every $y \geq 0$. Using the Maximum Theorem, one can then show that the optimal policy functions $C(y)$ and $K(y)$ are continuous. Monotonicity properties of $C(y)$ and $K(y)$ are determined by the complementarity[5] between k and y, and c and y, respectively.

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[3] Note that existence and all other results in this section continue to hold for logarithmic or CES utility functions that are unbounded below, though $V(0) = u(0) = -\infty$ (e.g., [118]).

[4] The feasible set is expanding if $y \leq y'$ implies $\Gamma(y) \subseteq \Gamma(y')$ and has a convex graph if $\{(c,k,y) \mid (c,k) \in \Gamma(y)\}$ is a convex set.

[5] Formally this is represented by the concept of supermodularity. Let $y \wedge y' = min[y,y']$ and $y \vee y' = max[y,y']$. A function $F(k,y)$ is supermodular in (k,y) if $F(k \wedge k', y \wedge y') + F(k \vee k', y \vee y') \geq F(k,y) + F(k',y')$. For C^2 functions this is equivalent to $F_{ky} \geq 0$ so that an increase in one argument raises the marginal value or marginal productivity of the other.

Lemma 11.2.2. *Under A.1-A.8, $C(y)$ and $K(y)$ are single-valued, continuous and increasing functions.*[6]

Proof. The fact that $C(y)$ and $K(y)$ are single-valued and continuous follows from the maximum theorem and the strict concavity of u and f. Let $k \in K(y)$ and $k' \in K(y')$ for $y < y'$. Suppose that $k' < k$. Then $k' \in \Gamma(y)$ and $k \in \Gamma(y')$. Further, $0 < u(y-k) + \delta EV(f(k,r)) - [u(y-k') + \delta EV(f(k',r))] < u(y'-k) + \delta EV(f(k,r)) - [u(y'-k') + \delta EV(f(k',r))] < 0$, where the first and last inequalities follow from the principle of optimality and the middle inequality follow from the fact that A.8 implies u is strictly supermodular in (y,k). Hence, it must be that $k' > k$. Next suppose that $c' \leq c$. Then, $0 \leq u(c) + \delta EV(f(y-c,r)) - [u(c) + \delta EV(f(y-c',r))] + u(c') + \delta EV(f(y'-c',r)) - [u(c') + \delta EV(f(y'-c))] < 0$, where the first inequality follows from the principle of optimality and the last inequality is due to the strict concavity of f and V. Hence, $c' > c$.

When the stochastic growth model is representative of aggregate economic behavior, it is natural that consumption and investment should always be in the interior of the feasible set. In disaggregate or microeconomic settings, this may not always be true. Since the interiority of optimal policies facilitates the use of differentiable optimization methods it is common to impose an assumption that guarantees interiority.

A.9. $\lim_{c\downarrow 0} u_c = \infty$.

Lemma 11.2.3. *Under A.1-A.9, $C(y) > 0$ and $K(y) > 0$.*

The condition $\lim_{c\downarrow 0} u_c = \infty$ is known as the Inada [44] condition at zero. In the classical model, the intuition for its use is as follows. To invest y yields finite discounted expected marginal value of investment but an infinite marginal utility from consumption. Hence, one can do better by reallocating some output from investment to consumption. Analogous arguments can be used to rule out investment of zero.

When optimal policies are interior, the value function in the classical model is differentiable.

Lemma 11.2.4. *(Mirman-Zilcha [81], Lemma 1). Under A.1-A.9, $V(y)$ is differentiable for all $y > 0$ and $V_y(y) = U_c(C(y))$.*

Proof. As a concave function, V has left and right-hand derivatives, $V_-(y) \geq V_+(y)$. Let k and c be optimal from y. As $c > 0, k$ is feasible from $y + \epsilon$ and $y - \epsilon$ for sufficiently small $\epsilon > 0$. By optimality, $V(y+\epsilon) - V(y) \geq u(c+\epsilon) +$

[6] Lemma 11.2.2 was first established by Brock and Mirman [18]. The monotonicity of $K(y)$ does not depend on the concavity of u or f and can be generalized to the case where $K(y)$ is a correspondence using the methods of Topkis [129] (see also,[121]).

$\delta EV(f(k,r)) - [u(c) + \delta EV(f(k,r))]$, which implies $V_+(y) \geq u_c(c)$. By a similar argument $V_-(y) \leq u_c(c)$.[7]

When V is differentiable, output has a unique shadow price given by $V_y(y)$. This shadow price is useful in examining the intertemporal tradeoff between consumption and investment, and in showing that the optimization problem can be decentralized.

Proposition 11.2.1. *Let (**c**,**k**) be an optimal program induced by $C(y), K(y)$. Under A.1-A.9, necessary and sufficient conditions for $C(y), K(y)$ to be optimal are:*

$$u_c(c_t) = \delta \int u_c(c_{t+1}(r)))f_k(k_t, r)d\nu(r). \tag{11.3}$$

$$\lim_{t \to \infty} \delta^t E[u_c(c_t)k_t] = 0. \tag{11.4}$$

Proof. The necessary condition (11.3) is typically proved in one of two ways. The first method is a variational approach that assumes period t output and the period t+1 capital stock are optimal. It then examines how a change in period t consumption affects discounted expected utility across the two periods. The second method proceeds as follows. If V is differentiable (Lemma 11.2.4) then maximizing the right hand side of equation (11.2) implies: $u_c(c_t) = \delta \int V_y(f(k_t, r)f_k(k_t, r)dv(r)$. Further, $V_y(y_t) = u_c(c_t)$ by the envelope theorem. Combining these yields (11.3). As commonly used, this approach requires both interior solutions and a differentiable value function; but a more general statement using inequalities is possible in other cases.

A proof of (11.4) is given in [87].[8]

Equation (11.3) is known as the stochastic Ramsey-Euler equation. It is a dynamic optimality condition that equates the marginal utility from consumption to the discounted expected marginal value of investment. The latter can be decomposed into the marginal productivity of investment times the marginal utility from consuming the additional output next period.

Equation (11.4) is the transversality condition. It implies that marginal utility is bounded in expectation.[9] It is also important to note that there may

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[7] An alternative approach in [12] assumes that the disturbance distribution has a C^n density. This smooths out possible points of discontinuity in the derivative of V. The approach has the advantage that it can be used to obtain higher order differentiability of both V and the optimal policy function, the latter via the implicit function theorem. Santos and Vigo-Aguiar[116] contains sufficient conditions for the value and policy functions to be C^2 and C^1, respectively. They use their results to place analytical bounds on the approximation error of a numerical solution.

[8] Kamihigashi [54] establishes the necessity of the transversality condition for optimality in a class of multisector stochastic growth models with single consumption good and bounded or constant relative risk aversion utility.

[9] Mirman and Zilcha [86] show that marginal utilities themselves may be unbounded.

be many non-optimal programs that satisfy the Ramsey-Euler equation. The transversality condition selects an optimal program from among those satisfying (11.3).

One of the most important results in the stochastic growth literature relates to the validity of the fundamental theorems of welfare economics in infinite horizon, stochastic economies. The two basic issues are the existence of prices that support an optimal program and the optimality of a dynamic, competitive equilibrium. In their seminal work Malinvaud [72] and Koopmans [55] make clear that the fundamental welfare theorems do not extend to infinite horizon settings without some additional conditions. The importance of these issues is apparent in [18], [81], [86] and [87] even though prices are often implicit in the necessary and/or sufficient conditions for optimality. Zilcha ([135], [136], [137]) examines the fundamental welfare theorems in a setting in which competitive prices are explicit throughout.

A feasible program ($\mathbf{y}$,$\mathbf{c}$,$\mathbf{k}$) is *competitive* if there exists a sequence $\mathbf{p} = (p_t)_{t=0}^{\infty}$ of discounted prices such that $p_t > 0$ a.s. for all t and:

$$\delta^t u(c_t) - p_t c_t \geq \delta^t u(c) - p_t c \ a.s., \forall \ c \geq 0. \tag{11.5}$$

$$Ep_{t+1} f(k_t, r_{t+1}) - p_t k_t \geq Ep_{t+1} f(k, r_{t+1}) - p_t k \ a.s., \forall k \geq 0 \tag{11.6}$$

Proposition 11.2.2. *A feasible program is optimal if and only if it is competitive and satisfies:*

$$\lim_{t \to \infty} \ Ep_t k_t = 0. \tag{11.7}$$

Proof. See [135].

As in Proposition 11.2.1, the existence of competitive prices alone is not sufficient to guarantee optimality. For that, the transversality condition (11.7) is also required.

The supporting price p_t is the discounted shadow price of the consumption-capital good. Equation (11.5) requires that consumption maximize utility less expenditure for almost every realized path and every time period. Equation (11.6) captures intertemporal (expected) profit maximization. When a competitive program is interior it implies $p_t = Ep_{t+1} f_k(k_t, r_{t+1})$. A primary difference between the deterministic and stochastic models is that in the former prices reflect temporal values, while in the latter prices also reflect values across different random states of nature. As a consequence, prices and the marginal willingness to substitute consumption are an important determinant of economic behavior even in the long run.

11.2.2 Stochastic Steady States and Convergence Properties in the One Sector Classical Model

A central concern of optimal growth theory is the study of the long run dynamics of an economy. The deterministic literature focusses on the existence and stability of non-trivial (strictly positive) optimal steady states and on turnpike properties of optimal capital accumulation paths. An optimal steady state or stationary program is a limit point of an optimal program. If optimal paths from all initial states converge to a steady state then this unique optimal steady state is globally stable and the long run behavior of the economy is independent of initial conditions.

When the evolution of capital stocks is stochastic, an optimal program of capital stocks is a sequence of random variables. The optimal policy, the production function, and the random shock map the probability distribution of current capital stocks to the probability distribution of the next period's capital stock. A stochastic steady state is a fixed point of this mapping or a distribution of capital that is invariant under the optimal policy. The stochastic analogue of a globally stable steady state is a unique invariant distribution to which the stochastic process of capital stocks converges from every initial state. In such a steady state the capital stock is not constant over time. Instead, it exhibits endogenous fluctuations in response to random productivity disturbances.[10]

Turnpike theorems study the conditions under which differences in initial conditions have negligible effects on the process of economic growth over long time horizons.[11] In the deterministic case, this involves analyzing when optimal paths from different initial states approach each other asymptotically. The stochastic analogue is convergence to zero in probability (or sometimes, almost surely) of an appropriately defined distance between the optimal capital stocks in each period.

In the classical one sector stochastic optimal growth model the unique optimal stationary policy generates a Markov process of capital stocks k_t. Recall that the optimal investment function $K(y)$ is a continuous and strictly increasing function on $\Re_+$. Define $H(k,r) \equiv K(f(k,r))$ to be the realized capital stock for the next period under the optimal policy. Then, H is continuous in (k,r) and increasing in k. Let S denote the interval $[0,\overline{k}]$. Given $y_0 \in S$, $k_0 = K(y_0) \in S$, the evolution of optimal capital stocks over time is given by:

$$k_t = H(k_{t-1}, r_t) \tag{11.8}$$

Recall that ν is the common probability distribution of the i.i.d random shocks r_t, with support Φ, a compact subset of $\Re$. Let ν^t be the joint distribution of $r^t \equiv (r_1, ...r_t)$ on the product space Φ^t and define $k^n(k_0, r^n) \equiv H(H(....(H(k_0,r_1),r_2)...,r_n)$. In other words, $k^n(k_0,r^n)$ is the n^{th}-period capital stock k_n, given k_0 and realization $r^n = (r_1,...r_n)$ of random shocks in the

[10] The literature also examines stronger concepts of an optimal steady state [133].
[11] See, [77], [75], [76].

first n periods. For any probability measure μ defined on S (and the Borel $\sigma-$field generated by S), define the probability measure $\nu^n\mu$ on S by the relation

$$\nu^n\mu(B) = \int_S \nu^n(\{r^n \in \Phi^n \mid k^n(k_0, r^n) \in B\})\mu(dk_0)$$

where B is any Borel-subset of S. Thus, $\nu^t\mu$ gives us the probability distribution of k_t, when k_0 is distributed according to the probability measure μ. Let S' be a closed interval in S. Then, S' is said to be $\nu-$invariant if $\nu\{r \in \Phi \mid H(k,r) \in S'$ for all $k \in S'\} = 1$. A probability measure μ on S is said to be an invariant probability measure on S' if the support of μ is a subset of $S'$and for any Borel set B in S,

$$\nu\mu(B) = \mu(B) \tag{11.9}$$

In other words, if k_0 is distributed according to an invariant probability μ, then the distribution of optimal capital stocks in every subsequent period follows the same distribution. The distribution function corresponding to an invariant probability measure is an invariant distribution.

There is a large body of work in the mathematical theory of Markov processes and random dynamical systems that provides sufficient conditions for the existence and stability of invariant distributions for a given stochastic process[12]. Let

$$H_m(k) = \min_{r\in\Phi} H(k,r) \text{ and } H_M(k) = \max_{r\in\Phi} H(k,r)$$

denote the lower and upper envelopes, respectively, of the transition function $H(k,r)$ defining the Markov process (11.8). Note that the continuity of H and the fact that Φ is compact imply that $H_m(k)$ and $H_M(k)$ are well defined and continuous. Further, since H is increasing in k, $H_m(k)$ and $H_M(k)$ are increasing functions.

In addition to the assumptions made in the previous section, the standard proof of existence and global stability of the invariant distribution requires that the production function $f(k,r)$ and the optimal transition function $H(k,r)$ satisfy two additional conditions:

A.10. There does not exist any $k > 0$ and $\widetilde{y} \in S$ such that $\nu\{r \mid f(k,r) = \widetilde{y}\} = 1$.

A.11. There exists an $\epsilon > 0$ such that $H_m(k) > k$ for all $k \in (0, \epsilon)$.

[12] See, among many others, [35], [41], [42], [2], [8], [9]. Models of descriptive stochastic growth (such as the stochastic Solow model) where the consumption and investment rules are exogenously specified have also applied these conditions ([81], [13] and [109]).

A.10 requires that every investment level is associated with some non-trivial uncertainty over output. A.11 is a restriction on the optimal policy. It implies that even if the lowest possible output is realized every period, the optimal program from every initial stock is bounded away from zero. In a deterministic model, the optimal policy satisfies this condition as long as marginal productivity at zero is large enough. This is no longer true in the stochastic model. Mirman and Zilcha [86] develop an example where the production function has infinite slope at zero, yet the optimal program from any initial stock comes arbitrarily close to zero with probability one.[13] One can impose restrictions on the production function and distribution of random shocks to ensure that A.11 is satisfied. For example, if there is a strictly positive probability mass on the "worst" production function in the sense that there exists some $\varrho > 0$ such that $\nu\{r \mid f(k,r) = \min_{r\in\Phi} f(k,r)\} > \varrho, \forall k > 0$, then infinite marginal productivity at zero is sufficient for A.11. For conditions that are applicable to atomless distributions, see [93].

Define the maximal fixed point of H_m by $k_m = \max\{k > 0 \mid H_m(k) = k\}$ and the minimal fixed point of H_M by $k_M = \min\{k > 0 \mid H_M(k) = k\}$. Assumption A.11 implies that $k_m, k_M > 0$.

Lemma 11.2.5. $k_m < k_M$.

Proof. Since $H(k,r)$ is continuous in r, there exists $r_m, r_M \in \Phi$ such that $k_m = H_m(k_m) = H(k_m, r_m)$ and $k_M = H_M(k_M) = H(k_M, r_M)$. Further, $f(k_m, r_m) \leq f(k_m, r)$ for all $r \in \Phi$. From the stochastic Ramsey-Euler equation:

$$\begin{aligned} u'(C(f(k_m, r_m)) &= \delta \int_\Phi u'(C(f(H(k_m, r_m), r)))f'(H(k_m, r_m), r)\nu(dr) \\ &= \delta \int_\Phi u'(C(f(k_m, r)))f'(k_m, r)\nu(dr). \end{aligned}$$

Since u is strictly concave and C is increasing $u'(C(f(k_m, r))) \geq u'(C(f(k_m, r_m))$ for all $r \in \Phi$. Hence, the inequality above yields $1 \leq \delta \int_\Phi f'(k_m, r)\nu(dr)$. Similarly, one can show that $1 \geq \delta \int_\Phi f'(k_M, r)\nu(dr)$ so that $\int_\Phi f'(k_M, r)\nu(dr) \leq \int_\Phi f'(k_m, r)\nu(dr)$. The fact that $k_m \leq k_M$ follows from the strict concavity of f. Finally, if $k_m = k_M$ then $f(k,r)$ is constant in r which violates A.10.

Lemma 11.2.5 implies that the highest fixed point of H_m lies below the smallest positive fixed point H_M. We now state the main result regarding the existence and global stability of the optimal stochastic steady state. For the stochastic process of optimal capital stocks k_t defined by (11.8), let $F_t(k)$ be the distribution function of k_t i.e., $F_t(k) = \nu^t\{r^t \in \Phi^t \mid k_t \leq k\}$.

[13] Mitra and Roy [93] develop general conditions under which $Prob\{\liminf_{t\to\infty} k_t = 0\}$ is 0 and 1.

Proposition 11.2.3. *Assume A.1 - A.11. There exists a unique non-zero invariant distribution $F(k)$ on S and its support is the interval $[k_m, k_M]$. For any initial capital stock $k_0 > 0$, as $t \to \infty, F_t(k)$ converges uniformly in k (on S) to $F(k)$.*

Proof. (Sketch).Instead of giving a full proof, we sketch the main arguments for the simple case of multiplicative shock, $f(k,r) = rf(k)$, which assumes just two possible values a and b, $0 < a < b < \infty$. Then, $H_m(k) = K(af(k))$ and $H_M(k) = K(bf(k))$. The proof consists of the following key arguments. First, for the Markov process (11.8), the set of states $(0, k_m)$ and (k_M, ∞) are transient. With probability one, capital stocks move out of these sets in finite time, never to return. Second, once the process enters the set $[k_m, k_M]$ it remains there with probability one. Further, $[k_m, k_M]$ is the smallest $\nu - invariant$ set. Let $y_m = \min\{k : H_m(k) = k\}$ and $y_M = \max\{k : H_M(k) = k\}$. Then, $0 < y_m \leq k_m < k_M \leq y_M$. From any stock $k \in (0, y_m)$ the optimal capital stocks increase almost surely and reach the set $[y_m, k_M]$ in finite time with probability one. Similarly, from any stock $k \in [y_M, \infty)$ the optimal capital stocks decrease almost surely and reach the set $[k_m, y_M]$ in finite time with probability one. Further, for $k \in [y_m, k_m)$ one can show that the probability that the optimal path from such a stock does not enter $[k_m, k_M]$ in finite time is zero. To move the capital stock y_m to the interval $[k_m, k_M]$ only takes a sufficiently long, but finite run $r_t = b$, such that the realized transition occurs along the function $H_M(k)$. Any such run must occur $\omega-$almost surely as shocks are independent. In fact, no strict subset of $[k_m, k_M]$ is invariant. The next step is to show that a well-known "splitting" condition due to Dubins and Freedman [35] (or some variation/extension) holds on the interval $[k_m, k_M]$. For any $n = 1, 2...$, the probability ν^n is said to split on a $\nu-$invariant subset S' of S if there exists $z \in S'$ and $\eta > 0$ such that:

$$\begin{aligned}
\nu^n\{r^n &\in \Phi^n \mid k^n(k, r^n) \leq z \text{ for all } k \in S'\} > \eta \\
\nu^n\{r^n &\in \Phi^n \mid k^n(k, r^n) \geq z \text{ for all } k \in S'\} > \eta.
\end{aligned}$$

To verify that the splitting condition holds fix any $z \in (k_m, k_M)$. There exists some $N \geq 1$, such that: (i) if $r_t = a, t = 1, ...N$, then $k^N(k_M, r^N) \leq z$, and (ii) if $r_t = b, t = 1, ...N$, then $k^N(k_m, r^N) \geq z$. For $0 < \eta < \min\{(\nu(a))^N, (\nu(b))^N\}$, $n = N$, it is easy to see that the splitting condition is satisfied on $S' = [k_m, K_m]$. Dubins and Freedman [35] then show that this implies there exists a unique invariant distribution F on S' and that $F_t(k)$ converges uniformly in k to $F(k)$.[14] Finally, since the set $S - S'$ is transient

[14] Recent extensions of the result that are applicable to situations where $H(k,r)$ is monotonic but not necessarily continuous and situations where the capital process is multidimensional can be found in [8],[9].

and S' is the smallest $\nu-$invariant set on S, it must be that F is the unique invariant distribution on S and $F_t(k)$ converges uniformly in k to $F(k)$ on S.[15]

The basic results on the existence and global stability of an invariant distribution for the classical one sector stochastic model were originally developed in the pioneering work by Brock and Mirman [18] and subsequently refined by Mirman and Zilcha [81]. Majumdar, Mitra and Nyarko [69] were the first to explicitly use the Dubins-Freedman splitting condition. Versions of this problem have also been analyzed by [126] and [42]. [19] [25] contain similar results for the undiscounted model ($\delta = 1$) where optimality is based on the "overtaking criterion".[16] Donaldson and Mehra [34] extend these results to the case of correlated shocks that enter the production function multiplicatively and follow a stationary process.

When shocks are unbounded, Stachurski [124] shows that there is always a unique globally stable steady state for the special case of a multiplicative shock where r has a density function that is strictly positive everywhere on $\Re_{++}$. With an interior optimal policy, the structure imposed on the random shock ensures that the system moves with positive probability from any positive stock to any interval on $\Re_{++}$ in *one* step.

¿From an empirical point of view one may be interested in the asymptotic statistical properties of the stochastic processes for capital and consumption. For example, if the law of large numbers holds so that sample averages from time series converge to the mean of the limiting steady state distribution, then one can test a model by comparing the sample average over a sufficiently long period with the theoretical prediction. Alternatively, one can forecast the mean of the long run distribution by using the sample average. The central limit theorem or asymptotic normality of the partial sums can be used for inference of likelihood of values in a parameter space. Many of the conditions that guarantee global stability of an invariant distribution also ensure that both the law of large numbers and the central limit theorem hold. In addition, they imply a minimum bound on the rate of convergence.[17]

An important implication of global stability is that the long run behavior of the economy is independent of the initial state. This is also brought out in

[15] A more traditional approach in theory of Markov processes is to directly verify that the process is irreducible on $[k_m, k_M]$, that intervals disjoint from $[k_m, k_M]$ are transient and an equicontinuity condition on the sequence of probability measures for the capital stock (defined through the stochastic kernel of the Markov process). See, [78]. Another approach is to show that the iterated random functions satisfy a Lipschitz condition and are "contracting on the average" (see, [33]). In a framework with multiplicative shocks that are not necessarily i.i.d. and have a positive valued density on $\mathbb{R}_+$, Nishimura and Stachurski [96] use a new approach by defining Foster-Lyapunov functions to characterize stability.

[16] For convergence in a stochastic open economy, see for example, [28].

[17] See, [9]. [125] contains similar results for the case of multiplicative unbounded shocks.

turnpike results that directly examine the conditions under which differences in initial conditions have negligible effects on the process of economic growth over long time horizons. Majumdar and Zilcha [71] establish a "late" turnpike theorem in a model that is far more general than the classical model of Section 2. Their model allows for unbounded expansion of capital and consumption, time varying utility and random shocks that may follow a non-stationary stochastic process. Under a condition that requires the elasticity of marginal product to be bounded away from zero (implying a lower bound on the degree of concavity of the production function), they show that the number of periods for which the relative distance between the optimal capital stocks (from any two initial stocks) exceeds any positive threshold is bounded almost surely, where the bound depends on how far apart the initial states are. In other words, optimal paths from different initial states eventually approach each other with probability one. Note that this result is quite independent of whether there is a globally stable invariant distribution. The condition on the elasticity of marginal product ensures a that a uniform "value-loss" argument (originally due to Radner [105]) holds.[18] Joshi [49] provides similar turnpike results in a one-sector model with recursive preferences and time varying technology.

Apart from convergence, the other important question in economic growth relates to characterization of the properties of the limiting steady state; in particular, the relationship between the preferences and technology underlying the economy and the nature of the invariant distribution to which it converges. In the one sector convex deterministic model of optimal growth, there is fairly rich characterization of the steady state. For example, with a strictly concave production function f, the unique steady state or modified golden rule is a capital stock $\widehat{k}$ that is the unique maximizer of $[\delta f(k) - k]$, where the latter can be interpreted as the net gain from investment. For the no-discounting case, the steady state is the well-known golden rule capital stock that maximizes the level of sustainable consumption $[f(k) - k]$. There are also other decentralized or support price-based characterizations of the optimal steady state. In general, in the deterministic one sector model, it is possible to look at the steady state as a solution to an independent *static* optimization problem that has desirable economic properties. In the multisector deterministic model, it is a solution to a static optimization and a fixed point problem.

Surprisingly, there is very little by way of general qualitative characterization of the limit invariant distribution in the stochastic growth literature. One of the reasons behind this is the fact that, unlike the deterministic model, the

[18] The value loss argument uses support prices of optimal paths to look at the accumulation of shortfalls in values (shadow profits and losses) of input-output combinations along one optimal path relative to another at the other's support prices. Loosely speaking, for two optimal paths that do not approach each other asymptotically, if the value loss is uniformly bounded away from zero over all states and time periods, then the accumulated loss is infinite and that contradicts optimality. [51] contains a turnpike result without requiring uniformity of value-loss across time and states.

steady state is not determined solely by the production function and the discount factor. Both the utility function (and its curvature) and the distribution of the random shocks play important roles. Specific examples show that for the same technology, discount factor and distribution of random shocks, the steady state distribution can change dramatically with variations in the utility function [26].

Further, even for very standard utility and production functions, the limiting distribution can be very sensitive to parameter values when the shock does not have a continuous distribution.[19] For the case with logarithmic utility, Cobb-Douglas production, and a binary multiplicative shock, Mirman and Zilcha [81] show that the invariant distribution can be degenerate for some parameter values and uniform for others. Montrucchio and Privileggi [94] show that the invariant distribution can also be a Cantor function. Mitra, Montrucchio and Privileggi [92] expand on this example to establish precise bounds on the parameters under which Cantor and more general singular invariant distributions can arise as well as bounds under which the distribution is absolutely continuous. Recently, Mitra and Privileggi [91] extend the example to the class of all iso-elastic utility functions and establish sufficient conditions for a Cantor type invariant distribution.

11.2.3 Stochastic Steady States and Convergence Properties in the Multisector Classical Model

In the literature on deterministic models of optimal economic growth, the multisector case has been extensively studied. In particular, the literature has focused on two key issues - the existence of an optimal steady state and turnpike results or the convergence properties of optimal paths.[20] In comparison, the stochastic multisector literature is relatively thin and there is only a small literature on the existence and stability of steady states in the stochastic, multisector case.

In the deterministic literature, it is well recognized that with discounting, the existence of a globally stable optimal steady state and other turnpike results may not hold in the multisector case (even though it always holds under very mild restrictions in the one-sector model).[21] With significant discounting, optimal paths in the multisector model may not be convergent. They may exhibit cyclical and even chaotic dynamics.

A general stochastic multisector optimal growth model with i.i.d. shocks has been analyzed by Brock and Majumdar [17]. The model is a natural extension of the classical one-sector model to the case of m capital goods. For each vector of current capital stocks and realization of the random shock there is a correspondence that defines the set of attainable utilities and capital stocks for

[19] For the case of multiplicative shock with continuous density, Danthine and Donaldson [27] show that the limiting invariant distribution has a continuous density function.

[20] For an excellent review of the basic results see McKenzie [76].

[21] See, [128].

the next period, which in turn can be used to define the set of feasible *programs* from any given initial vector of capital stocks. The objective is to maximize the discounted expected sum of utilities, or for the undiscounted case, a stochastic version of the overtaking criterion. The paper imposes four conditions:

(i) there is a compact set $S' \subset \Re^m_+/\{0\}$ such that for any initial vector of capital stocks lying in S', there exists an *optimal* program such that the stochastic process for capital lies almost surely in S'.

(ii) there exist continuous stationary optimal investment and consumption policies.

(iii) an optimal program is "competitive" relative to a non-trivial price process in a similar sense as in the previous section and satisfies a transversality condition that the expected values of the capital stocks (at the competitive prices) go to zero, for the case of discounting, and are bounded, in the undiscounted case.

(iv) the Hamiltonian system corresponding to the optimal process has "suitable curvature" so that a stochastic value-loss condition is satisfied.

Under conditions (i) - (iv), Brock and Majumdar show that the distance between the probability distributions of $t^{th}-$period optimal capital vectors from two distinct initial capital vectors in S' converges to zero as $t \to \infty$. Further, the difference between the two optimal paths converges to zero in probability. Thus, conditions (i) - (iv) are sufficient to ensure that the optimal paths from alternative capital stocks come close to one another asymptotically and that the long run behavior of optimal paths does not depend on initial conditions. The existence of a unique and globally stable invariant distribution for the stochastic process of optimal capital stocks can also be established under these conditions. Unlike the conditions for global stability of an invariant distribution and other turnpike results in the one-sector stochastic growth model, (i) - (iv) are fairly strong restriction imposed directly on the optimal policy rather than the primitives of the model. Conditions (i) - (iii) are readily satisfied in the one sector stochastic growth model. In the multisector case there are plausible conditions on preferences and technology for (i) and (ii) to hold. For example, Majumdar and Radner [70] consider a stochastic nonlinear activity analysis model in which neoclassical conditions on the technology and preferences are sufficient for (i) and (ii).[22] Condition (iii) is motivated by the equivalence between optimal programs and competitive programs that satisfy a transversality condition (see, [135], [136], [137]). Condition (iv) is a stochastic extension of conditions for asymptotic stability in the deterministic multisector model due to Cass and Shell [22] and Rockafellar [110] that are, in turn, based on the well known "value loss" argument alluded to in the previous subsection (see also, McKenzie [75]). In particular, condition (iv) requires that the Cass-Shell-Rockafeller version of the value-loss restriction holds uniformly for all states of the environment.

[22] See also the discussion in [88].

Chang [24] shows that a weaker version of condition (iv) based on *expected value loss* is actually sufficient and further, that the difference between any two optimal paths converges not only in probability, but almost surely. It is worth noting that in a multisector model condition (iv) involves a strong restriction on the extent of discounting in the model, and unlike the one-sector case, it does not follow directly from a restriction on the curvature of the production function.

Föllmer and Majumdar [39] follow a somewhat different approach using the theory of martingales to show that even if one does not impose a condition such as (iv), a weaker result is possible. That is, for any two optimal paths, the number of periods for which the value loss exceeds any given positive threshold is finite with probability one. Under uniformity of value loss and a specific distance metric, optimal paths approach each other asymptotically almost surely.

For the case of no discounting with the "overtaking" criterion of optimality, global stability of the stochastic steady state and other turnpike results can be established under much less restrictive conditions (see, among others, [46], [25], [136]).[23,24]

11.3 Extensions of the Classical Framework

11.3.1 Sustained (Long Run) Growth

The past two decades have seen a renewed interest in the economics of long run growth where unbounded expansion of output, capital and consumption is possible. In the deterministic convex one-sector model, sustained growth is optimal if the marginal productivity of capital at infinity exceeds the discount rate [47]. Much of the literature on stochastic optimal growth theory focusses primarily on models where the technology exhibits bounded growth that rules out indefinite expansion of consumption and capital and sustained long run growth. An exception is the class of models on optimal intertemporal household savings under uncertainty. A portion of this literature considers a linear production function with a multiplicative shock, $f(k, r) = rk$, so that optimal paths may diverge to infinity (see, Phelps [103], Levhari and Srinivasan [63] and subsequent contributions). A closely related literature on the permanent income hypothesis has examined optimal savings where the wealth next period

[23] Dutta [37] provides sufficient conditions under which as $\delta \to 1$, the optimal policies (and value functions) in the discounted stochastic model converge to the optimal policy using two alternative optimality criteria - the undiscounted overtaking criterion and the long run average reward criterion.

[24] In a stochastic multisector model with a double infinity of time periods and discount factors close to 1, Yano [133] establishes the existence and continuity in the discount factor of a stronger concept of an optimal stationary program (where a stationary program is one where the vector of current capital stocks associated with any infinite realized sequence of past history is time invariant with probability one). See also, [74].

is composed of a deterministic return on current savings (interest income) plus an additive income shock (non-interest income).[25]

De Hek and Roy [31] examine the possibility of sustained long run growth in optimal consumption and capital stocks in a one sector model with i.i.d. shocks and a concave production function that is not necessarily linear. Consider the model in Section 2 without assumption A.5. In particular, suppose that $f(k,r) = rf(k)$ and let $\theta = \lim_{k\to\infty} \frac{f(k)}{k}$. They show that under the following two conditions, optimal capital and consumption diverge to infinity with probability one from every positive initial stock:

(i) $E[\ln(\theta r)] > 0$
(ii) $\inf_{y>0} \delta E[\frac{u_c(rf(sy))rf_k(sy)}{u_c((1-s)y)}] > 1$, where $s = \exp[-E[\ln(\theta r)]]$.

Note that these conditions involve the utility function and its curvature. The possibility of long run growth depend on more that a simple comparison of the discount rate and average marginal product at infinity. Once again, this reflects the general fact that in a stochastic growth model, the utility function and distribution of shocks play important roles in determining the nature of long run behavior of the economy. To illustrate this further we consider a specific example of iso-elastic utility and linear production function for which we can derive the optimal policy explicitly and thus provide an almost exact characterization of the condition for sustained long run growth.

Example 11.3.1. $u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \sigma > 0, \sigma \neq 1, f(k,r) = rk$. One can show that the optimal policy function $K(y)$ is linear and given by $K(y) = [\delta E(r^{1-\sigma})]^{\frac{1}{\sigma}} y$ so that

$$k_{t+1} = \alpha r_{t+1} k_t \text{ where } \alpha = [\delta E(r^{1-\sigma})]^{\frac{1}{\sigma}} \tag{11.10}$$

which implies $\ln k_{t+1} = \ln k_0 + (t+1)[\frac{1}{t+1}\sum_{j=0}^{t} \ln \alpha r_{j+1}]$. Using the law of large numbers, it is easy to show that an "almost" exact condition for $\ln k_{t+1}$ to diverge to infinity with probability one is that $E[\ln(\alpha r)] > 0$ which can be rewritten as $\sigma E(\ln r) + \ln \delta + \ln E(r^{1-\sigma}) > 0$. This indicates that the risk aversion/intertemporal elasticity of substitution parameter of utility, σ, plays an important role in determining whether sustained growth occurs.

11.3.2 Stochastic Growth with Irreversible Investment

In the classical framework analyzed in the previous section, investment is either reversible or the existing capital stock depreciates completely at the end of a period. In reality, it is costly to transform capital into consumption and there are limits to how fast the aggregate capital stock depreciates. The stochastic growth model with irreversible investment was first examined by Sargent [117]. In his setting output can either be consumed or invested, but once invested,

[25] See [132] and [119] for the undiscounted case, [120] and [122] for the discounted case and also [23].

capital cannot be converted for consumption. Individual agents transact in a competitive market for existing capital. This allows individual investment decisions to be reversed while maintaining the irreversibility of investment in the aggregate. As Sargent shows, irreversibility in the aggregate provides the necessary friction for Tobin's q, the relative price of used to new capital, to diverge from unity. This enables aggregate investment to be positively correlated with q. However, this same friction implies that agents' investment decisions are necessarily a function of their expectations about the future which cannot be summarized by q. The implication is that q-theory of investment functions are of little use for econometric policy evaluation.

The analysis in Sargent is based on the properties of the value function. Olson [100] develops an alternative approach that characterizes optimal policies using stochastic Kuhn-Tucker conditions. Let $f(k,r) = F(k,r)+(1-d)k$ where k is the depreciation rate of capital. If λ_t is the Lagrange multiplier on the period t irreversibility constraint, $k_{t+1} \geq (1-d)k_t$, the Ramsey-Euler equation can be written as:

$$u_c(c_t) - \lambda_t = \delta E\left[u_c(c_{t+1}(r)))f_k(k_t, r) - (1-d)\lambda_{t+1}(r)\right]. \tag{11.11}$$

Solving for λ_t and substituting forward this can be expressed as:

$$u_c(c_t) = \delta \sum_{i=1}^{T} \left(\delta(1-d)\right)^{i-1} E\left[u_c(c_{t+i})F_k(k_{t+i-1}, r_{t+i})\right] + \delta^T (1-d)^T E[u_c(c_{t+T})]. \tag{11.12}$$

This derivation uses the fact that eventual depreciation of the entire capital stock is not optimal so there is a uniform upper bound, T, on the number of time periods for which the irreversibility constraint binds. Sargent's point that agent's decisions are a function of expectations about the future is clearly evident from (11.12). Evaluating (11.12) at the minimal and maximal optimal transition functions for capital it can be shown that the support of the limiting distribution under irreversible investment is a subset of the support when investment is reversible.

11.3.3 Stochastic Growth with Experimentation and Learning

The stochastic growth model has been extended to environments where there is learning about productivity or the capital stock itself. This requires expanding the state space to represent the agent's beliefs. The transition equation for beliefs follows Bayes' rule. In this setting, the possibility of learning affects the optimization problem in two important ways. First, even if information signals are exogenous so that learning is passive and not affected by the current action, the mere prospect of learning may alter current period decisions. Second, when the current action affects how much learning occurs, there is an incentive to experiment to obtain better information. Friexas [40] was the first to examine this problem. Assume output is produced by a technology $f(k, \theta, r)$, where θ is an unknown parameter. The distribution of r is known. Given an initial value for y and current beliefs about θ, the agent chooses consumption and investment. Output in the following period is observed and provides information that can be used to update beliefs about θ. Friexas examines how learning and experimentation affect the initial consumption/investment decision. The learning effect depends on whether learning increases or decreases the marginal value of investment. Friexas then uses Blackwell's [10] theorem to assert that if larger investment yields more information then the experimentation effect leads to an increase in investment. Subsequently, it was shown in [5] and [29] that this need not always be true. The reason is that investment affects both state variables in the value function so that Blackwell's theorem does not apply. While higher investment may be more informative, the value of information at higher levels of output may be lower. When the second effect dominates, an expected utility maximizer may prefer to invest less even if it is more informative. These tradeoffs have made it difficult to obtain a general set of verifiable conditions to characterize how information affects consumption and investment in the infinite horizon model. Precise results are limited largely to problems where there are only two relevant decision periods.

Nyarko and Olson [99] examine experimentation and learning in a stochastic growth model where there is imperfect information about the capital stock itself. Consumption is observable, but output and investment are not. Beliefs about the state are summarized by a probability distribution over y. After choosing consumption, an information signal is observed that can be used to update beliefs about y. The mapping from beliefs in period t to beliefs in period $t+1$ is determined jointly by consumption, the information signal and the stochastic production function. Here there is learning about a moving target, in contrast to the case above where the unknown parameter is fixed. Nyarko and Olson show that if $u(0) = -\infty$[26] then the optimal policy is to assume the worst and optimize against that. That is, the initial state is assumed to be the lower bound of the support of the agent's beliefs about output and the transition equation is $\inf_r f(k, r)$. When information alters the lower bound of the support of the agent's beliefs there is an endogenous capital discovery process. When it does not, the problem with learning has an equivalent, deterministic representation. In that case, output and investment are more volatile than consumption and there is excess saving compared to the case where the capital stock is observable. In cases where $u(0) > -\infty$, the solution either corresponds to that above, or the capital stock becomes zero with strictly positive probability.

[26] This assumption holds for the class of all constant relative risk averse utility functions with coefficient at least one.

11.4 Non-classical Models of Optimal Stochastic Growth

The models of optimal economic growth under uncertainty reviewed in the previous section are based on the classical assumptions of convex technology and utility that depends only on consumption. This section reviews some extensions of the theory that allow for non-classical features such as non-convexities and state-dependent utility. These non-classical features imply that even in a one-sector model, continuity and monotonicity properties of optimal policies need not hold and optimal paths need not converge to a unique stochastic steady state. The long run behavior of the economy may depend critically on the initial state.

11.4.1 Stochastic Growth with Non-convex Technology

Non-convexities enter the production technology of an economy through numerous sources, such as fixed costs, threshold effects, increasing returns to scale, economies of scope, and depensation in the reproduction of natural resources. In applications of optimal growth models to areas such as environmental management there is also the need to study the implications of a non-convex technology. A separate chapter of this handbook focuses on optimal growth in non-convex economies. In this subsection, we concentrate on explaining how a non-concave production function (non-convex technology) alters the basic results of the classical stochastic growth literature reviewed in the previous subsections.

Majumdar, Mitra and Nyarko [69] were the first to comprehensively analyze the problem of optimal stochastic growth in a one sector model where the production function, $f(k,r)$, is not necessarily concave, though it exhibits bounded growth.[27] In this framework, the set of feasible programs is not necessarily convex and therefore, the value function for the dynamic optimization problem is not necessarily concave even though the utility function satisfies classical concavity restrictions. This non-convexity means that the maximization problem on the right hand side of the functional equation may have multiple solutions so that instead of a unique optimal policy function, the solution is characterized by a measurable selection from an upper semi-continuous optimal policy correspondence. Further, there need not exist any continuous selection and every policy function may exhibit jump discontinuities on a set that is at most countable. Also, non-convexity in the economy implies that the optimal path is not necessarily decentralizable - in particular, support prices may not exist.

As the value function is not necessarily concave, the expected future marginal value of capital may be increasing in current investment.[28] This, in turn,

[27] Some notable contributions to deterministic optimal growth with a non-convex technology include [68],[32].

[28] The term "marginal" is used loosely here as the value function is not necesarily differentiable no matter how smooth the utility and production functions are.

implies that optimal consumption may actually decline with an increase in output.[29] Indeed, in the deterministic model it has been shown that there may not exist an optimal consumption function that is globally monotonic. The optimal investment policy correspondence is, however, an ascending correspondence. Further, if the utility function is strictly concave, then it can be shown that every measurable selection from this correspondence is non-decreasing and an optimal investment policy function $K(y)$ is always non-decreasing in output.[30]

A central question is whether there exists a globally stable invariant distribution. In the deterministic literature with non-concave production functions, it has been shown that there may be a multiplicity of steady states and the limit of the optimal path of capital stocks may depend on the initial state. For example, with an S-shaped production function, it is quite possible that optimal paths from small stocks converge to zero (extinction), while for initial stocks above a critical level,[31] optimal paths converge to a strictly positive optimal steady state. This initial state dependence can be expected to be true in the stochastic model too.

Consider the model of Section 2 without assumption $A.4$. For any measurable selection from the optimal policy correspondence, the transition function $H(k, r)$ for the Markov process of optimal capital stocks (11.8) is non-decreasing in k, but not necessarily continuous.[32] Recall that k_m, k_M are the largest positive fixed point of the lowest transition function $H_m(k)$ and the smallest positive fixed point of the highest transition function $H_M(k)$, respectively. A critical step in the proof of global stability in Proposition 11.2.3 is Lemma 11.2.5 that showed $k_m < k_M$. Indeed, if $A.10$ and $A.11$ hold and $k_m < k_M$, there exists a globally stable invariant distribution even if assumption A.4 does not hold. However, in the non-convex model it is quite possible that $k_m > k_M$ so that Lemma 11.2.5 does not hold. To see what happens in that case, suppose that

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29 Unlike both the classical stochastic model and the deterministic model with non-concave production function, it is difficult to guarantee that optimal consumption is strictly positive in the stochastic model with non-concave production, even if Inada conditions are imposed on the utility and production functions. An interior optimal policy is ensured in [69] by assuming that $u(0) = -\infty$, which is a very strong restriction on the class of admissible utility functions. More recently, [95] establishes interiority by assuming the Inada condition on utility, sufficiently high marginal productivity at zero, and that the random shock is multiplicative and has a density function so that the maximand on the right hand side of the functional equation of dynamic programming is smooth.

30 If the utility function is concave but not strictly concave, then there may be an optimal investment function that is non-monotonic though, even in that case, there is at least one optimal investment function that is non-decreasing.

31 This critical level is referred to as a safe standard of conservation in the literature on renewable resource economics.

32 An innovative approach to the non-convex model can be found in Amir [1]. It takes advantage of the averaging associated with the random disturbances to derive conditions for the monotonicity of optimal policies and higher order differentiability of the value function. As in [12], differentiability of the optimal policy functions follows from the implicit function theorem.

$A.10$ and $A.11$ hold and optimal policy is interior ($0 < K(y) < y$ for all $y > 0$). As in the sketch of the proof of Proposition 11.2.3, confine attention to the case where $f(k,r) = rf(k)$ where r assumes one of two possible values a, b. As before, let $y_m > 0$ be the smallest positive fixed point of the lowest transition function $H_m(k)$. Then, for all $k \in (0, y_m), H_M(k) > H_m(k) > k$ and $H_M(y_m) > H_m(y_m) = y_m$ so that $y_m < k_M$. Similarly, it is easy to show that $k_m < y_M$, where y_M is the largest fixed point for the highest transition function $H_M(k)$. Thus, $k_m > k_M$ implies $y_m < k_M < k_m < y_M$. It is easy to check that the two disjoint intervals $[y_m, k_M]$ and $[k_m, y_M]$ are both $\nu-$invariant; from any initial state in either interval, the optimal capital process remains in that interval almost surely. For $k_0 \in (0, k_M]$, all optimal paths eventually enter and stay in the interval $[y_m, k_M]$ while for $k_0 \in [k_m, \infty)$, all optimal paths eventually enter and stay in the interval $[k_m, y_M]$. There is no globally stable invariant distribution. Using arguments based on the splitting condition referred to earlier, Majumdar, Mitra and Nyarko [69] show that if $k_m > k_M$, then for all $k_0 \in (0, k_M]$, the distribution of capital stocks converges to the same invariant distribution whose support is $[y_m, k_M]$, while for all $k_0 \in [k_m, \infty)$, the distribution of capital stocks converges to another invariant distribution whose support is $[k_m, y_M]$. For any fixed initial stock in the intermediate range (k_M, k_m), the optimal path may enter either of the two invariant sets and remain there, depending on the realization of random shocks. This last possibility illustrates an aspect of path dependence that has no parallel in the deterministic literature.

In general, non-convexities in production may lead to multiple invariant distributions. However, if production is "sufficiently stochastic", then there exists a globally stable invariant distribution despite the non-convexity [69]. Here, the precise condition that ensures global stability is:

$A.12$. There exists some $\vartheta > 0$ in S such that $\nu(\{r \in \Phi \mid f(k,r) \leq \vartheta$ for each $k \in S\}) > 0$ and $\nu(\{r \in \Phi \mid f(k,r) \geq \vartheta$ for each $k \in S\}) > 0$.

Observe that assumption $A.12$ a condition on the production function, not the transition function for the optimal capital process. It captures the idea that the random output that results from any given investment is sufficiently spread out, i.e., the technology exhibits sufficient variability. Under this condition, if we let $z = K(\vartheta)$,then one can easily verify that the splitting condition described in the proof of Proposition 11.2.3 is immediately satisfied. This ensures global stability. Thus, the possibility of multiple stochastic steady states depends on the stochasticity of the model. This is another instance where the stochastic growth model (with sufficient uncertainty) is qualitatively different from the deterministic analogue. We summarize the above discussion in the next proposition:

Proposition 11.4.1. *Assume* $A.1 - A.7, A.10, A.11$ *and that optimal policy is interior. Then, (i) if* $k_m < k_M$ *or if* $A.12$ *holds, there is a unique invariant distribution on* S *whose support is* $[k_m, k_M]$ *and from every* $k_0 > 0$,*the optimal* *capital stocks converge in distribution (uniformly) to this invariant distribution; (ii) if* $k_m \geq k_M$, *then for all* $k_0 \in (0, k_M]$, *the distribution of optimal capital stocks converges to an invariant distribution whose support is* $[y_m, k_M]$, *while for all* $k_0 \in [k_m, \infty)$, *the distribution of capital stocks converges to another invariant distribution whose support is* $[k_m, y_M]$.

As in Section 3.1, A.11 implicitly imposes restrictions on the technology. For example, in [69] it is obtained from the model primitives by assuming (in addition to a condition for interiority of optimal policy) that the random shock has finite support and that the marginal productivity at zero is infinite. The latter is a rather serious restriction on the class of admissible non-concave production functions. It rules out the S-shaped production function that is a widely used canonical form to capture increasing returns to scale and other threshold effects.[33]

Nishimura, Rudnicki and Stachurski [95] analyze a non-convex model with multiplicative i.i.d. random shock that has a density function that is strictly positive on $\Re_{++}$. Under restrictions on the expectation of the random shock, they show that the Markov process of optimal capital stocks either converges to zero from every initial state or there is a globally stable non-zero steady state (and identify conditions for these events). To place their results in context, their assumption on the density function automatically satisfies the "very stochastic" assumption in [69] discussed above. Their result does not require Inada conditions on the production function and, in fact, allows the marginal product at zero to be less than one with positive probability. In a similar framework, Nishimura and Stachurski [96] use the Euler equation to analyze stability of the stochastic optimal capital process; in particular, they use the marginal utilities as Foster-Lyapunov functions in order to obtain stability.

The literature on non-convex stochastic growth also develops turnpike conditions under which optimal paths approach each other asymptotically. In a model with non-convex and non-stationary technology Joshi [50] uses the monotonicity properties of the optimal policy and a supermartingale process generated by the stochastic Ramsey-Euler equation to show that, under a strong "value loss" condition that is uniform with respect to time and state, the asymptotic distance between optimal paths from two distinct initial states converges to zero with probability one. However, as in the case of turnpike theorems in the stochastic multisector convex models, the uniform value loss condition is not very transparent in terms of its implications for the model primitives.

One of the interesting questions in stochastic growth models with non-convexity is the possibility of extinction where optimal paths converge to zero. This is particularly important in applications of the optimal growth model to problems of renewable resource management where utility reflects the net

[33] Mitra and Roy [93] provide weaker conditions that ensure $A.11$ even when the marginal productivity at zero is finite and the distribution of the random shock is absolutely continuous.

benefit from harvesting and the production function reflects natural biological growth. Assuming a bounded growth production function and i.i.d. shocks that have compact support, Mitra and Roy [93] show that there are only three possibilities: (i) optimal paths from all initial states get arbitrarily close to zero infinitely often with probability one (this includes extinction in finite time), (ii) optimal paths from all initial states are bounded away from zero with probability one, and (iii) there exists a critical capital stock or safe standard above which all optimal paths are bounded away from zero with probability one. They develop sufficient conditions on the preferences and technology that lead to each of these outcomes. In contrast to the deterministic literature, these conditions involve not just the discount factor and marginal productivity, but also marginal utility - one compares the discount rate to expected "welfare-modified" return on investment (marginal productivity) as in the condition in Proposition 11.4.2. Another result on optimal extinction is due to Kamihigashi [53] who shows that if the marginal productivity at zero is finite, then sufficient variability in the random shock implies that *all* feasible programs (including therefore, the optimal program) converge to zero almost surely.

11.4.2 Stochastic Growth with Stock-Dependent Utility

For some important capital theoretic allocation problems welfare depends on both consumption and the beginning of period output, as represented by $u(c, y)$.[34] Utility is assumed to be nondecreasing in y, jointly concave in (c, y) and A.7 is no longer imposed.[35] In the deterministic case stock-dependent utility has two important consequences. The first consequence arises if investment and output are substitutes in utility in the sense that $u(y - k, y)$ is submodular in k and y. In that case, an interior optimal investment policy may be decreasing in output. At the same time, there may be intervals of the state space on which corner solutions are optimal and the optimal transition function coincides with the production function. Combining these two possibilities opens the door for the optimal transition function to be like a tent map, or even more complex. When this happens an optimal program may exhibit nonlinear dynamics including cycles or chaos [3]. The second important consequence is that multiple optimal steady states are possible, even if the utility function is supermodular in k and y and the optimal investment policy is monotone (Kurz [59]). In such cases, the asymptotic behavior of an optimal program depends on the initial state.

The first analysis of stochastic models with stock-dependent utility can be traced to the literature on renewable resource allocation. In that literature, the production function represents biological growth of the renewable

[34] Such models include the allocation of natural capital or renewable resources and the effects of wealth on consumption-savings behavior.

[35] In renewable resource allocation problems welfare declines if consumption exceeds the quantity that equates demand and supply.

resource and the random shock represents the effect of environmental disturbances on resource growth. The state variable is the resource stock (output) at the beginning of the period. Stock-dependent utility arises when the harvest costs depend on the resource stock or when the resource stock has amenity or other social value. Early papers ([45], [107], [123]) focused on the case where $u_{cc}(c,y)+u_{cy}(c,y)=0$. In this case the direct and indirect utility effects of an increase in output offset exactly and investment and output are neither strict complements nor strict substitutes in utility. As a result, the optimal policy is a constant investment policy, which in the presence of fixed costs becomes an (s,S) inventory rule. Mendelssohn and Sobel [78] prove monotonicity of the optimal investment policy under the supermodularity condition $u_{cc}(c,y)+u_{cy}(c,y)\le 0$. Nyarko and Olson [97] show that the optimal consumption policy is nondecreasing when $u_{cy}(c,y)\ge 0$ and u and f are concave. They also use the Dubins and Freedman splitting condition to characterize the convergence of optimal programs to a limiting distribution. Without additional restrictions the invariant distribution may not be unique and the long run behavior of an optimal program may depend on initial conditions. Subsequently, Nyarko and Olson [98] show that additional sufficient conditions for the existence of a unique invariant distribution are: (i) $u_c(c,y)=0$ implies $u_y>0$ for sufficiently large y, and (ii) for all $y>0, c\in\Gamma(y)$ and $\alpha>1$, if $u_c(c,y)>0$ and $u_c(\alpha c,\alpha y)>0$ then $\frac{u_y(c,y)}{u_c(c,y)}\ge\frac{u_y(\alpha c,\alpha y)}{u_c(\alpha c,\alpha y)}$. The last assumption is a complementarity condition that implies that the slope of indifference curves for u decrease as output and consumption increase along a ray through the origin in (c,y) space. Nyarko and Olson provide examples to show that multiple invariant distributions can be optimal when either (i) or (ii) are violated. The existence of a unique invariant distribution is also ensured when there is sufficient divergence between production in the best and worst states. [69] and [98] show that there is more than one way to define sufficient variation in production. The underlying intuition is the same. A model with multiple limiting distributions can be transformed into one with a unique invariant distribution by the mixing that results from increasing the variance in production. On the other hand, if the variability in production is small enough and if $u(y-k,y)$ is submodular in (k,y), then an optimal program may oscillate between cyclic sets [3].

The economic possibilities associated with the stochastic growth model expand considerably when a non-convex production technology is combined with stock-dependent utility. To date this combination has primarily been used to examine the conditions under which capital stocks remain strictly bounded away from zero, issues related to conservation and extinction. In the deterministic model with both non-convex production and stock-dependent utility it is possible for there to be disjoint intervals in the state space from which an optimal program converges to zero. That is, an optimal program starting from intermediate states may remain bounded away from zero, while optimal programs starting from lower or higher states converge to zero [101]. The addition of random productivity disturbances leads to the somewhat surprising possibility that a first-order improvement in the distribution of disturbances can reduce the set of initial states from which optimal output and capital stocks have a positive lower bound.

One useful technique to analyze some questions in the non-convex model is to examine behavior under the convex-hull of the technology. If capital stocks under an optimal program always remain in an interval where the convex-hull coincides with the non-convex technology then the two optimization problems coincide on that interval. This can be used, for example, to provide conditions for the existence of a safe standard of conservation.

Proposition 11.4.2. *Assume $u(y-k,y)$ is supermodular in (k,y), u is increasing in c, and f is concave in k for all r. Let $\underline{f}(k) = \inf_r f(k,r)$. If*

$$\inf_{z\in[0,k]} \delta E\left[\frac{u_c(f(k,r)-z, f(k,r)) + u_y(f(k,r)-z, f(k,r))}{u_c(\underline{f}(k)-z, \underline{f}(k))}\right] > 1 \tag{11.13}$$

then $\liminf y_t \geq k$ *for all* $y_0 \geq k$.

A general version of this result in the model with non-convex technology and stock-dependent utility can be found in Olson and Roy [102], along with other results dealing with conservation or extinction. The conclusions depend on the joint properties of the technology, utility, and the distribution of disturbances. As can be seen above, $\underline{f}(k)$ or productivity under the worst disturbance is an important determinant of conservation or extinction.

11.5 Comparative Dynamics

An important question in stochastic growth theory is the sensitivity of optimal decision rules and paths with respect to preference and technology parameters that describe the underlying economy. In a one sector model, continuity of optimal investment and consumption decisions with respect to various parameters of the model generally holds under far weaker assumptions than those described in Section 2.[36]

The theory of monotone comparative statics using supermodular functions and complementarity developed in Topkis [129] has been extended to stochastic dynamic models (see, for example, [121], [42]). One can apply results from this literature to derive the comparative dynamics of the optimal policy function with respect to various preference and technology parameters by looking at the maximization problem on the right hand side of the functional equation of dynamic programming [43]. Most of these results have been derived in a one sector framework.

[36] See for example, [38]. Conditions for parametric continuity of stationary distributions of Markov processes are discussed, among others, by [126] and [61]. These properties are important for numerical simulations.

Danthine and Donaldson [26] show that an increase in the discount factor increases optimal investment and shifts the distribution of optimal capital stocks to the right and hence, the invariant distribution to which the stocks converge.[37] Moreover, they show that an increase in the curvature of the utility function (loosely speaking, an increase in risk aversion), leads to higher consumption (i.e., lower investment) at low levels of output, and lower consumption (i.e., higher investment) at high levels of output; further, the range of the limiting distribution expands as risk aversion increases.[38]

Another important issue in comparative dynamics is the effect of a change in the degree of riskiness or volatility of the random shocks. This relates to a central concern in macroeconomics about the relationship between riskiness of productive assets and the optimal intertemporal precautionary saving decisions of individuals as well as more aggregative analysis of the relationship between growth and economic fluctuations (see for example, [48]). Unfortunately, there is no general characterization of the effect of a second order stochastic change in the distribution of shocks on the optimal policy.[39]

In the specific case of optimal savings under uncertainty [40] discussed in Example 11.3.1, one can characterize the comparative dynamics of riskiness fairly tightly. ¿From (11.10), we have

$$K(y) = \alpha y, E[\frac{k_{t+1}}{k_t}] = \alpha E(r_{t+1}), \text{ where } \alpha = [\delta E(r^{1-\sigma})]^{\frac{1}{\sigma}}$$

so that the propensity to invest/save and the expected growth rate of capital are both proportional to α and the latter is increasing (decreasing) in riskiness of the random shock if $\sigma > (<) 1$ because $r^{1-\sigma}$ is a convex (concave) function of r in that case. Thus, depending on the curvature of the utility function, an increase riskiness may increase or decrease optimal investment and cause a first order increase or decrease in the distribution of optimal capital stocks. Roughly speaking, if utility is more concave than the logarithmic function, an increase in riskiness of the random shock increases the optimal savings rate and the expected rate of growth. The reverse holds if utility is less concave than the logarithmic function. In the case of log utility, the optimal policy depends only on the average realization of the random shock and not on its higher moments.[41]

[37] Dutta [36] shows that lengthening the time horizon for a fixed discount factor and increasing the discount factor for a fixed time horizon are, in a precise sense, equivalent.

[38] In the case of logarithmic utility, Cobb-Douglas production with multiplicative shock, an increase in the discount factor increases the variance of capital stock and output. Danthine and Donaldson [27] provide sufficient conditions for this to occur. They also characterize conditions under which an increase in the curvature of the utility function has a similar effect.

[39] An exception is the model of optimal dynamic consumption with deterministic linear interest and additive labor income shock. See, for example, [80].

[40] See, among others, [103], [63], [111], [62].

[41] For an extension of this kind of result to a model of endogenous growth see [30].

A less ambitious question relates to a comparison of the moments of the limiting distribution to the steady state in the deterministic model.

Example 11.5.1. $u(c) = \ln c$ and $f(k, r) = rk^{\beta}, 0 < \beta < 1$. For this example Mirman and Zilcha [81] show that the optimal investment policy is given by $K(y) = \beta\delta y$. Danthine and Donaldson [27] use this to analyze the properties of the optimal program for capital:

$$k_t = (\beta\delta)^{1+\beta+\ldots+\beta^{t-1}} k_0^{\beta^t} r_0^{\beta^t} r_1^{\beta^{t-1}} \ldots r_{t-1}^{\beta} r_t.$$

This implies:

$$Ek_t = (\beta\delta)^{\sum_{s=0}^{t-1} \beta^s} k_0^{\beta^t} \prod_{s=0}^{t-1} E(r_{t-s-1}^{\beta s}).$$

Assume $E(r_t) = 1$ and a non-degenerate distribution for r_t. Then taking limits as $t \to \infty$, the first moment of the limiting invariant distribution of capital satisfies: $Ek = (\beta\delta)^{\frac{1}{1-\beta}} L$, where, $L = \lim_{t\to\infty} \prod_{s=0}^{t-1} E(r_{t-s-1}^{\beta s})$. Jensen's inequality implies $L < 1$. Expected consumption and output in the limiting invariant distribution are given by: $Ec = (1-\beta\delta)(\beta\delta)^{\frac{\beta}{1-\beta}} L$, $Ey = (\beta\delta)^{\frac{\beta}{1-\beta}} L$. In the deterministic version of the model (where $r_t = 1$ almost surely), steady state capital, consumption and output are given by: $k = (\beta\delta)^{\frac{1}{1-\beta}}$, $c = (1 - \beta\delta)(\beta\delta)^{\frac{\beta}{1-\beta}}$, $y = (\beta\delta)^{\frac{\beta}{1-\beta}}$. This shows that in the stochastic model, the steady state distribution has smaller average capital stock, output and consumption than in the certainty equivalent version of the model.

In Example 11.5.1, uncertainty only affects the evolution of an optimal program and not the optimal policy function itself. This simplifies the task of characterizing the effect of uncertainty on the limiting distribution. In general, uncertainty will also affect the optimal policy function. As we have seen earlier, in the case of iso-elastic utility and linear production, uncertainty may increase or reduce optimal investment depending on the nature of the utility function. This makes it difficult to compare the moments of the limiting distribution of capital for the stochastic model with its certainty equivalent.

11.6 Solving the Stochastic Growth Model

The stochastic growth model is inherently nonlinear. There is no known general, closed form solution. Instead, analysis of the model with general functional forms aims to qualitatively characterize optimal policies and the resulting implications for economic behavior. There are two main approaches to achieving more specific solutions, all of which require assumptions regarding functional forms for production and utility. The only cases with known closed form analytical solutions are those discussed in Examples 11.3.1 and 11.5.1.

Approximation and numerical methods are the alternative when an analytical solution is not available.[42] By far the most common approximation technique is to linearize the Euler equations around the steady state of the model, an idea pioneered by Magill [65] in continuous time and Kydland and Prescott [60] in discrete time. This approach was subsequently extended by [57], [58], [20] and many others. In a model with Cobb-Douglas technology and CES/CRRA utility, [131] develops central limit and large deviation principles that characterize the manner in which capital trajectories in the stochastic model converge to those in the deterministic case as the standard deviation of the random shock goes to zero. In practice, most approximation methods are not entirely analytical and the approximate solution is analyzed using simulations where the underlying parameters are calibrated to data. Solutions are accurate in the neighborhood of a stochastic steady state with support on a small interior interval. Approximation methods are less useful in situations where the disturbance term has support on a large interval, where the solution is not interior, where second order effects are important, and in the study of transition dynamics.

Numerical dynamic programming can be used to solve parametric specifications of the stochastic growth model. The two most common approaches involve iteration of discrete or parametric approximations to the value or policy functions. Recent surveys of numerical methods can be found in [112], [60], [74], and [113]. Once the model is solved, the policy functions can be used to compute moments for the limiting distributions of the economic variables of interest. The main advantage of numerical dynamic programming is that attention need not be restricted to a neighborhood of the steady state. This allows one to investigate almost any question of interest within the context of a given parametric specification, including a study of global dynamics. The primary disadvantage has to do with robustness to model specification, calibration and choice of numerical method.[43] In general, different numerical procedures can yield substantially different results so care must be exercised in their implementation.

11.7 Conclusion

The literature on optimal stochastic growth theory is over three decades old now and there are many important ways in which the theory has contributed

[42] An early survey and comparison of different methods can be found in [130].

[43] For algorithms generated by a contraction mapping of modulus δ , the approximation error is bounded by $\| V_n - V_{n+1} \| < \epsilon/(1-\beta)$, where ϵ is the tolerance level under the given metric and V_n is the n^{th} iterate of the algorithm. Santos [114] shows how the Euler equation residuals can be used to bound the approximation error for other types of algorithms. Santos and Peralta-Alva [115] examine when the simulated moments from a numerical solution converge to their exact values as the approximation errors converge to zero.

to our understanding of capital accumulation, growth and more generally, optimal intertemporal resource allocation. In this section, we summarize some of the contributions of the stochastic growth literature and we point out where the introduction of uncertainty has done little to alter the conclusions of the deterministic model.

First, stochastic growth theory has provided a different explanation of economic volatility. In contrast to the deterministic case, an optimal program in the stochastic model is a sequence of random variables generated jointly by optimal decisions and random productivity disturbances. Realized capital paths fluctuate even when the optimal policy is time stationary and well-behaved. This way of looking at economic volatility has been successfully utilized by the business cycle literature as a way to capture various stylized facts about economic fluctuations.

Second, in the stochastic growth model the utility function plays a prominent role in determining the long run behavior of the economy. Even in a one-sector model and for the same production technology, discount factor and distribution of random shock, the limiting steady state distribution typically differs with the specification of the utility function. The role of the utility function is also seen in conditions for long run growth and avoidance of extinction. This role is absent in deterministic models.

It is possible to examine how optimal paths and the limiting distribution are affected by changes in the riskiness of productive assets, risk aversion, or the willingness to substitute consumption across time. Unfortunately, not much general analytical characterization is available outside a few examples in the log-linear family. These examples nonetheless serve to illustrate how the qualitative nature of comparative dynamics can depend on the parameters of the utility function.

Third, key qualitative features of optimal policies such as continuity and monotonicity are not significantly altered by the presence of uncertainty in the production technology.

Fourth, extending results on the existence and global stability of an optimal steady state to the stochastic model requires verifying that the transition law for the optimal process satisfies certain conditions, which have been discussed in previous sections. This has necessitated strong technical assumptions that have no counterpart in the deterministic literature. In the multisector case, this difficulty has been more pronounced and the conditions for global stability of a stochastic steady state are only specified in terms of the transition law for the optimal process, making it difficult to evaluate their economic implications.

Fifth, in non-classical one-sector models that generate multiple invariant distributions that act as local attractors, it has been shown that if the volatility of technological disturbances is increased sufficiently, one can establish global convergence of optimal processes to a unique stochastic steady state. Loosely speaking, higher stochasticity in the production technology makes it more likely that realized optimal paths exhibit a high degree of economic fluctuations over time, but it also increases the likelihood that the distribution of optimal capital stocks converges globally to a unique invariant distribution independent of the initial state. In other words, greater production uncertainty may be associated with higher economic volatility and at the same time, may ensure long run "convergence" in probability distribution of economies that differ in their initial states. This is a fundamental insight into the process of growth and fluctuations in an economy.

Finally, the stochastic growth literature has followed the deterministic literature very closely in establishing a set of turnpike results that show how optimal paths approach each other almost surely in the long run.

As for the important theoretical questions that remain unanswered, our survey indicates that a general characterization of the stochastic steady state or invariant distribution, is lacking. Steps toward such a characterization would improve our understanding of the forces that determine long run economic behavior in a convergent stochastic economy. We not only need to understand how complex the limiting distribution can be, but also have some idea of the relationship between the fundamentals of the model and the properties of the limiting distribution. That is, what do technology and preferences imply about the nature of the limiting distribution? Much work remains to be done there.

Other important open questions in the one sector model are: a *complete* characterization of conditions under which optimal paths converge to zero almost surely, to a non-trivial invariant distribution and diverge to infinity almost surely (the existing literature only provides strong sufficient conditions for each of these events); relaxing the conditions for convergence and stability in the non-convex model; and the question of asymptotic convergence in versions of the model with non-monotone optimal investment policy (such as the stock-dependent model). Developing more transparent conditions for convergence and stability in the multisector stochastic model and conditions for sustained long run growth in such models are also problems that remain open to the current generation of growth theorists.

Finally, the methodology of stochastic optimal growth is increasingly applied to other problems of dynamic resource allocation ranging from models of financial markets and macroeconomic fluctuations to the management of natural and environmental assets. These applications often require extensions and modifications to the basic framework in order to suit the stylized facts that characterize these problems. This, in turn, poses new questions for the growth theorist. The development of new applications and extensions of existing ones may well continue to be the most fruitful source of new ideas related to the stochastic growth model.

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12. The von Neumann–Gale Growth Model and Its Stochastic Generalization

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12.1 Introduction

Von Neumann's [60] model of an expanding economy, generalized by Gale [30], was one of the first models in Mathematical Economics that served as the basis for a rich and interesting theory. This theory was developed for the most part in the 1950s and 1960s. Substantial contributions to it were made by such outstanding economists and mathematicians as McKenzie, Radner, Rockafellar, Nikaido, Morishima and others (see, e.g., the monograph by Nikaido [47] and references therein).

The theory of the von Neumann–Gale model, in its classical form, is purely deterministic. It does not reflect the influence of random factors on economic growth. The importance of taking these factors into account was realized early on. First attempts aimed at the construction of stochastic analogues of the von Neumann–Gale model were undertaken in the 1970s by Radner [51, 52]. However, the initial attack on the problem left many questions unanswered. Studies in this direction faced serious mathematical difficulties. To overcome these difficulties, new mathematical techniques were required, that were developed only during the last decade. The main purpose of the present paper is to provide an account of recent achievements in the field.

Along with probabilistic generalizations of the classical results, new applications of the stochastic version of the von Neumann–Gale model will be highlighted: the applications to the analysis of the dynamics of financial markets.

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This marks an unexpected change in focus, as well as a revival of the model because the framework—that originally aimed at the modeling of economic growth—turned out to be very natural in connection with financial issues. The financial aspects pose a number of interesting new questions, which are currently only partially answered and which indicate directions for further work.

The basic mathematical framework of the von Neumann–Gale model is that of set-valued dynamical systems, e.g. Akin [2]. Dynamics of such systems are described by multivalued operators specifying for every state of the system "today" a set of possible states "tomorrow." The characteristic features of the operators associated with the von Neumann–Gale model are certain properties of convexity and homogeneity. A profound mathematical study of such dynamical systems has been carried out by Rockafellar [54], Makarov and Rubinov [41] and others. In the stochastic case, one has to deal with *random* set-valued dynamical systems possessing analogous properties of convexity and homogeneity. For an introduction to the theory of random dynamical systems see Arnold [8].

In the theory of economic growth, the von Neumann–Gale model occupies quite a special position. By and large, models of economic growth prevailing in the current literature belong to one of the following two types. They either assume that the growth rates of economic factors (e.g. labor) are given exogenously, or consider the phenomenon of economic growth from the point of view of endogenous changes in the technology or production functions. Examples of models of the former kind are those proposed by Solow [56] and Ramsey [53]. The analysis of process of growth in such models consists essentially in the study of optimal *proportions* of growth (e.g. expressed in terms of *per capita* quantities). In this context, paths of the system can exogenously be normalized, after which the work basically reduces to the analysis of optimization problems with bounded state variables. The stochastic theory of such models is well developed. Foundations for it were laid in the 1970s and 1980s in the work of Dynkin, Radner, Brock, Mirman, Bewley, Dana, Majumdar, Mitra, Zilcha and others. Results obtained in this field have been reflected in a number of surveys and monographs, e.g. Mirman [44], Dynkin and Yushkevich [19], Arkin and Evstigneev [7], Stokey, Lucas and Prescott [59], Brock and Dechert [13], Olson and Roy [49], containing references to the original papers. For more recent work, see Amir [4], Amir and Evstigneev [5], Mitra, Montrucchio and Privileggi [45] and Stachurski [58].

Models of the second kind—in which endogenous changes in the production function or technology are analyzed—are systematically reviewed, for example, in Aghion and Howitt [1] and Barro and Sala-i-Martin [11]. The theory of stochastic endogenous growth models is still in its infancy. Results in this direction have been obtained by de Hek [14] and de Hek and Roy [15].

The von Neumann–Gale model does not fit into either of the above two classes of models. The "technology" in it is given exogenously, and in the stationary deterministic case, it does not change in time. However, the growth rates are endogenous: they are derived from the model itself. The maximum growth rate (the von Neumann rate) is a solution to an optimization problem whose constraints are determined by the technological restrictions given in the model.

A very important feature of the von Neumann–Gale model is that it focuses—primarily—on the analysis of growth rates. Growth rates are measured in terms of price systems (or objective functions) satisfying some very general assumptions. This is in contrast with many other models presuming the existence of a social planner whose objective, being expressed as the maximization of the sum of discounted utilities, represents the objective of the whole economic system. The von Neumann–Gale model can fit situations where the assumption of the existence of such a social planner seems to be restrictive and does not reflect the content of the economic problem under study. Further, it should be noted that the von Neumann–Gale model admits any number of sectors, while most of the growth models prevailing in the current literature confine to the cases of one, two or at most three sectors. Typically, studies in this field deal with most general situations, rather than analyzing growth in specialized settings. All the above features, combined with the richness of the relevant theory, make the von Neumann–Gale model and its stochastic generalization interesting objects of study. We hope that this survey will attract attention and effort to this fruitful and fascinating area of research.

The paper consists of two parts, dealing with the deterministic and the stochastic versions of the von Neumann–Gale model respectively. Our presentation of the deterministic version of the theory differs in a number of respects from the conventional ones; it is aimed primarily at stochastic generalizations. We first describe the main concepts and results in Part I and then consider their probabilistic analogues in Part II. We set out the deterministic theory by using only elementary means. To make the presentation self-contained, we provide in the Appendix formulations of some fundamental results related to convexity and optimization. Part II, dealing with stochastic models, is more advanced; it assumes that the reader is familiar with basic concepts of Probability and Functional Analysis.

I. THE VON NEUMANN–GALE MODEL: THE DETERMINISTIC CASE

12.2 The Model and the Main Concepts

12.2.1 Basic Definitions

In its classic, deterministic form, the *von Neumann–Gale model* is specified by a family of cones[2] $Z_t \subseteq \mathbb{R}^n_+ \times \mathbb{R}^n_+$, $t = 1, 2, ...$, consisting of pairs of nonnegative

[2] A set in a linear space is called a *cone* if it contains with each vectors x and y the vector $\alpha x + \beta y$, where α and β are any nonnegative numbers.

n-dimensional vectors $z = (x, y)$. A sequence $\{x_t\}_{t=0}^N$ $(N \leq \infty)$ is called a *path* (*trajectory*) in the model if

$$(x_{t-1}, x_t) \in Z_t \tag{12.1}$$

for all t. Such sequences describe possible time evolutions of the state x_t of the system under consideration. Those paths are of primary interest which grow in a sense faster than others. The following definition plays a central role. A path $\{x_t\}_{t=0}^N$ is called *rapid* if there exists a sequence $\{p_t\}_{t=0}^N$ of vectors in $\mathbb{R}_+^n$ such that

$$p_t x_t > 0, \tag{12.2}$$

and

$$\frac{p_t y}{p_{t-1} x} \leq \frac{p_t x_t}{p_{t-1} x_{t-1}} \tag{12.3}$$

for all $(x, y) \in Z_t$ with $p_{t-1}x \neq 0$. (If $p = (p^1, ..., p^n)$ and $x = (x^1, ..., x^n)$ are two vectors, then px stands for the scalar product $px = \sum p^i x^i$.) Condition (12.3) implies that the path $\{x_t\}_{t=0}^N$ maximizes, for each time period $t = 1, 2, ...,$ the *growth rate*

$$\frac{p_t y_t - p_{t-1} y_{t-1}}{p_{t-1} y_{t-1}} \tag{12.4}$$

among all paths $\{y_t\}_{t=0}^N$ with $p_{t-1} y_{t-1} \neq 0$. The growth rate is computed by using the sequence of linear functions $p_t x$ that assign an "aggregate value" to any state $x \in \mathbb{R}_+^n$ of the system at each time t (various interpretations are discussed below). Condition (12.2) is a non-degeneracy assumption requiring a strictly positive valuation $p_t x_t$ of each state x_t of the path $\{x_t\}$. It is easily seen that, if (12.2) holds, we can replace p_t by $p_t / p_t x_t$ and obtain additionally that

$$p_t x_t = 1 \tag{12.5}$$

for all t. Thus rapid paths $\{x_t\}_{t=0}^N$ can be defined as those for which there exists a sequence $\{p_t\}_{t=0}^N$ of vectors in $\mathbb{R}_+^n$ for which conditions (12.3) and (12.5) hold.

Instead of describing the von Neumann–Gale model in terms of a sequence of cones Z_t, we can equivalently specify it in terms of multivalued (set-valued) operators $x \mapsto A_t(x)$, where

$$A_t(x) = \{y : \ (x, y) \in Z_t\}. \tag{12.6}$$

It will be assumed that $A_t(x) \neq \emptyset$ for each $x \in \mathbb{R}_+^n$ which means that the projection of the cone Z_t on the first factor in the product $\mathbb{R}_+^n \times \mathbb{R}_+^n$ coincides with $\mathbb{R}_+^n$. *Paths (trajectories) of the multivalued dynamical system* generated by the sequence of set-valued operators $x \mapsto A_t(x)$ are defined as sequences $\{x_t\}$ satisfying

$$x_t \in A_t(x_{t-1}). \tag{12.7}$$

Clearly, relation (12.1) is equivalent to (12.7), and so the class of such sequences coincides with the class of paths in the von Neumann–Gale model.

Since the graph $Z_t = \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : y \in A_t(x)\}$ of $A_t(\cdot)$ is a cone, the mapping $A_t(\cdot)$ possesses the following properties:

$$\lambda A_t(x) \subseteq A_t(\lambda x), \ \lambda \in [0,\infty), \ x \in \mathbb{R}^n_+; \tag{12.8}$$

$$\theta A_t(x) + (1-\theta) A_t(x') \subseteq A_t(\theta x + (1-\theta) x'), \ x, x' \in \mathbb{R}^n_+, \ \theta \in [0,1]. \tag{12.9}$$

A linear combination of two sets in a vector space is defined as the set of pairwise linear combinations of their elements: $A + A' = \{a + a' : a \in A, a' \in A'\}$ and $\lambda A = \{\lambda a : a \in A\}$. Conversely, if conditions (12.8) and (12.9) hold, then the graph of $A_t(\cdot)$ is a cone.

There are important cases of multivalued dynamical systems of the above type that are defined in terms of single-valued operators. Suppose that

$$A_t(x) = \{y \in \mathbb{R}^n_+ : \ y \leq D_t(x)\}, \ x \in \mathbb{R}^n_+, \tag{12.10}$$

where $D_t : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is an operator satisfying

$$D_t(\lambda x) = \lambda D_t(x), \ \lambda \in [0,\infty), \tag{12.11}$$

$$D_t(x+y) \geq D_t(x) + D_t(y) \tag{12.12}$$

(all inequalities between vectors, non-strict and strict, are understood coordinatewise). In mathematical economics contexts, dynamical systems of this kind have been studied, in particular, by Solow and Samuelson [57] and Nikaido [47]. The analysis of paths of such systems reduces essentially to the analysis of products $D_t \circ D_{t-1} \circ ... \circ D_1$ of the operators D_t $(t = 1, 2, ...)$.

The case where $D_t = D$ is a nonnegative linear operator (independent of t) is perhaps the simplest, but at the same time, a quite important one. The study of such systems reduces to the analysis of the iterates D^t, $t = 1, 2, ...$, of the nonnegative matrix D. In the deterministic case, these questions are studied by using results related to the Perron–Frobenius theorem (see the Appendix, Theorems C.1 and C.2). Non-linear generalizations of the Perron–Frobenius theorem applicable to the analysis of dynamical systems (12.10) are reviewed in the monograph by Nussbaum and Verduyn Lunel [48].

Consider the sets Z^*_t in $\mathbb{R}^{2n}_+$ $(t = 1, 2, ...)$ consisting of pairs of nonnegative vectors (p, q) satisfying

$$qy - px \leq 0 \text{ for all } (x, y) \in Z_t. \tag{12.13}$$

Clearly Z^*_t are closed cones, and the sets

$$A^*_t(p) = \{q : \ (p, q) \in Z^*_t\} \tag{12.14}$$

are non-empty for each $p \in \mathbb{R}^n_+$ (since $0 \in A^*_t(p)$). The cones Z^*_t define the von Neumann–Gale model which is called the *dual* to the original one. The multivalued operators (12.14) define the multivalued dynamical system dual to the one specified by (12.6). Paths of the latter, i.e. sequences $\{p_t\}^N_{t=0}$ satisfying

$$p_t y - p_{t-1} x \leq 0 \text{ for all } (x, y) \in Z_t, \tag{12.15}$$

are called *dual paths.*

It is easily seen that if $p_t x_t = 1$ for all t, then (12.15) is equivalent to (12.3). Consequently, *a trajectory* $\{x_t\}^N_{t=0}$ *is rapid if and only if there exists a dual path* $\{p_t\}^N_{t=0}$ *satisfying the condition* $p_t x_t = 1$ for all t. If this condition is satisfied, we say that the dual path $\{p_t\}^N_{t=0}$ *supports* the trajectory $\{x_t\}^N_{t=0}$.

12.2.2 Nonlinear von Neumann Models

In the applications of the von Neumann–Gale model, the coordinates x_t^i of vectors $x_t = (x_t^1, ..., x_t^n)$, describing states of the system under consideration, may represent various economic or financial variables. In this paper, we consider two basic fields of applications of the von Neumann–Gale model—the theory of economic growth and the analysis of dynamics of financial markets. We will begin with a description of a framework aimed at the analysis of economic growth; in the next subsection, we will consider examples related to finance.

Suppose that the economic system under consideration consists of n components or "economic units," for example, sectors of an economy, regions in a country, or countries—in a model of international trade and/or cooperation. Suppose that the state of the system at time $t = 0, 1, ...$ is characterized by a vector $x_t = (x_t^1, ..., x_t^n)$, whose coordinates x_t^i are construed as *intensities* of operating the ith unit. The numbers x_t^i may represent, in particular, the levels of use of an exogenous production factor (e.g. labor or energy) in unit i, or the level of investment in unit i, or its total income, etc. The set of feasible intensity vectors x_t is denoted by X_t. The set X_t is supposed to be a cone; typically, $X_t = \mathbb{R}_+^n$.

There are m commodities in the economy that are produced and consumed in each of the units $i = 1, 2, ..., n$. (Usually the number of such commodities is supposed to be small and they are understood as aggregates—food, fuel, vehicles, etc.) Two nonnegative vector functions of $x \in X_t$ are given:

$$\Phi_{t+1}(x) = (\Phi_{t+1}^1(x), ..., \Phi_{t+1}^m(x)) \text{ and } \Psi_t(x) = (\Psi_t^1(x), ..., \Psi_t^m(x)).$$

If the units $i = 1, 2, ..., n$ are run at intensities x_t^i, $i = 1, ..., n$, the total *input* (at time t) is described by the vector of commodities $\Psi_t(x_t) = (\Psi_t^1(x_t), ..., \Psi_t^m(x_t))$ and the total *output* (at time $t + 1$) is represented by the commodity vector $\Phi_{t+1}(x_t) = (\Phi_{t+1}^1(x_t), ..., \Phi_{t+1}^m(x_t))$, where $x_t = (x_t^1, ..., x_t^n)$. Outputs obtained at the end of each time period are used as inputs in the next time period, free disposal being allowed, so that

$$\Psi_{t+1}(x_{t+1}) \leq \Phi_{t+1}(x_t),\ t = 0, 1, ... \tag{12.16}$$

We can include the dynamical system at hand into the framework of the von Neumann–Gale model by setting

$$Z_t = \{(x, y) :\ \Psi_t(y) \leq \Phi_t(x)\},\ t = 1, 2, ... \tag{12.17}$$

If Φ_t and $-\Psi_t$ are homogeneous of degree one and concave in each coordinate, then Z_t is a cone. Paths $\{x_t\}$ of the von Neumann–Gale model specified by the cones Z_t, $t = 1, 2, ...$, defined by (12.17) are those and only those sequences $\{x_t\}$ that satisfy (12.16). The properties of homogeneity of Φ_t and Ψ_t express the hypothesis of constant returns to scale. The concavity of Φ_t and convexity of Ψ_t reflect the "advantages of cooperation:" by combining (mixing) activities of different units $i = 1, 2, ..., n$, one can gain in output and reduce input.

In models of this kind, the process of economic growth is treated as the process of time evolution of the intensity vectors x_t, the constraints on possible ways of evolution being given by (12.16) or, equivalently, by (12.17). We are interested primarily in those paths of the system which maximize growth rates (12.4) defined in terms of some vectors $p_t \in \mathbb{R}^n_+$. If the intensities x^i_t are measured in terms of an exogenous factor, then the coordinates p^i_t of the vectors p_t involved in the definition of rapid paths may be construed as ("local," prevailing at unit i) prices of this factor. If the output mapping $\Phi_t(x) = \Phi_t x$ is linear, we can use for comparing growth rates of paths linear functions $p_t x$ of the form $p_t x = q_t \Phi_t x$, expressing the value of the total output in the system for the intensity vector x and the commodity price vector $q_t = (q^1_t, ..., q^m_t) \in \mathbb{R}^m_+$.

One can include consumption into the model as follows. The mapping Φ_t can represent the net-of-consumption output. Alternatively, Ψ_t may be viewed as a "consumption plus production input" vector.

In his seminal paper, von Neumann [60] considered a version of the above model with linear mappings $\Phi_t(x) = \Phi_t x$ and $\Psi_t(x) = \Psi_t x$. In the von Neumann model, the cones Z_t are given by

$$Z_t = \{(x, y) \in \mathbb{R}^{2n}_+ : \ \Psi_t y \leq \Phi_t x\}, \tag{12.18}$$

where $\Phi_t : \mathbb{R}^n \to \mathbb{R}^m$ and $\Psi_t : \mathbb{R}^n \to \mathbb{R}^m$ are nonnegative linear operators. According to the classical von Neumann interpretation, there are $i = 1, 2, ..., n$ *technological processes*. Any such technological process i can be operated at any intensity level $x^i_t \geq 0$. The *technology matrices* Φ_{t+1} and Ψ_t specify the total output $\Phi_{t+1} x_t$ and the total input $\Psi_t x_t$ of the system given the vector of intensities $x_t = (x^1_t, ..., x^n_t)$. In the von Neumann model, the cones Z_t are polyhedral[3]. Gale [30] initiated the study of models with general, not necessarily polyhedral, cones.

Given technology matrices Φ_t and Ψ_t, or their nonlinear counterparts $\Phi_t(\cdot)$ and $\Psi_t(\cdot)$, we can study economic growth by analyzing paths $\{u_t\}$ in the commodity space $\mathbb{R}^m_+$, rather than paths $\{x_t\}$ in the space $\mathbb{R}^n_+$ of intensity vectors. To this end we can define the cones (*technology sets*):

$$W_t = \{(u, v) \in \mathbb{R}^{2m}_+ : \ u \geq \Psi_{t-1}(x), \ v \leq \Phi_t(x) \text{ for some } x \in \mathbb{R}^n_+\} \tag{12.19}$$

$(t = 1, 2, ...)$. If $\{u_t\}$ is a path in the model specified by the cones W_t, i.e., a sequence satisfying $(u_{t-1}, u_t) \in W_t$, then there exists a sequence $\{x_t\}$ for which $u_{t-1} \geq \Psi_{t-1}(x_{t-1})$ and $u_t \leq \Phi_t(x_{t-1})$. These inequalities imply $\Phi_t(x_{t-1}) \geq \Psi_t(x_t)$, and so $\{x_t\}$ is a path in the model defined by (12.17). Conversely, if a sequence $\{x_t\}$ satisfies $\Phi_t(x_{t-1}) \geq \Psi_t(x_t)$, i.e., it is a path in the model (12.17), we can set $u_t = \Psi_t(x_t)$, which will give us a path $\{u_t\}$ in the model (12.19). The approaches based on the considerations of the cones Z_t and W_t are in a number of respects equivalent, but they also have some distinctions—advantages and disadvantages. We will follow, basically, the former because

[3] A set in a linear space is called polyhedral if it is an intersection of a finite family of closed half-spaces.

some of the assumptions we are going to impose are more plausible for the cones Z_t than for W_t.

The dynamical systems defined by (12.17) and (12.19) generalize the classical von Neumann setting in that the mappings $\Phi_t(\cdot)$ and $\Psi_t(\cdot)$ are not supposed to be linear. In this connection, we will call (12.17) and (12.19) *nonlinear von Neumann models*. To distinguish between them, we will refer to (12.17) and (12.19) as the *first* and the *second* nonlinear von Neumann model, respectively.

12.2.3 Modeling Financial Markets

It has recently been observed (Evstigneev and Taksar [28]) that the von Neumann–Gale model can serve as a convenient vehicle for the modeling of financial markets with "frictions," i.e. transaction costs and trading constraints. We will outline a framework covering a number of examples of this kind. In the financial applications, the most interesting questions make sense only in stochastic models, where uncertainty is involved in a non-trivial manner. Therefore we will quite briefly touch the deterministic case in this introductory section, leaving a more comprehensive discussion of financial models for later.

Consider a financial market in which n assets are traded. For each $t = 0, 1, ...$ the vector $S_t = (S_t^1, ..., S_t^n) > 0$ of asset prices is given. The number S_t^i stands for the price of one unit of asset i at time t. Let us assume that vectors $h_t = (h_t^1, ..., h_t^n) \in \mathbb{R}^n$ represent *portfolios* of assets: h_t^i is the ith *position* of portfolio h_t specifying the number of ("physical") units of asset i in the portfolio. Generally, portfolio positions might be both positive and negative. The latter case means a possibility of *short sales* of some of the assets—this is allowed in the theory of frictionless markets. There are various restrictions on short sales in real markets, and therefore it is important to consider those models where these restrictions are taken into account. By contrast with the conventional theory, we will assume in this paper that short sales are not allowed, and so $h_t \in \mathbb{R}_+^n$ for all feasible portfolios h_t. Such models fit quite well the von Neumann–Gale framework.

A *trading (investment) strategy* is a sequence of feasible portfolios $\{h_t\}_{t=0}^N$, where h_t is the portfolio held by the investor at time t. It is supposed that the investor might change his/her portfolio reacting on the changes in the asset prices (which are supposed to be random in the stochastic case). A key role in many aspects of mathematical finance is played by the notion of a *self-financing trading strategy*.[4] This is an investment strategy that can be implemented without outside funding. In each time period, the investor can *rebalance* his/her portfolio, i.e. transform it by buying assets only at the expense of selling some other assets. The analysis of self-financing trading strategies is based on the consideration, for each time period t, of the set

$$K_t = \{(h, g) \in \mathbb{R}_+^{2n} : \text{ portfolio } h \text{ can be rebalanced into portfolio } g\}. \quad (12.20)$$

[4] This notion lies in the basis of fundamental pricing principles for derivative securities such as the Black–Scholes formula (see, e.g., Föllmer and Schied [29]).

If there are no transaction costs (and portfolio positions are measured in terms of physical units of assets), then the set K_t is as follows:

$$K_t = \{(h, g) \in \mathbb{R}^{2n}_+ : \ S_t g \le S_t h\}. \tag{12.21}$$

According to (12.21), one can rebalance a portfolio h into a portfolio g at time t if the value $S_t g$ of g (measured in terms of the price vector S_t) is not greater than the value $S_t h$ of h. Thus self-financing trading strategies are sequences $h_0, h_1, h_2, ...$satisfying

$$S_t h_t \le S_t h_{t-1}, \ t = 1, 2, ... \tag{12.22}$$

These inequalities are equivalent to the inclusions $(h_{t-1}, h_t) \in K_t$ $(t = 1, 2...)$. Clearly the sets K_t defined by (12.21) are cones in $\mathbb{R}^{2n}_+$, and so they specify a von Neumann–Gale model. Self-financing trading strategies are nothing but paths in this model.

It is often more convenient to measure portfolio positions in terms of their market values, rather than in terms of physical units of assets. We can associate with a vector $h_t = (h^1_t, ..., h^n_t)$ (describing a portfolio in terms of units of assets the vector $x_t = (x^1_t, ..., x^n_t)$, where $x^i_t = S^i_t h^i_t$, describing the portfolio in terms of the market values of its positions at time t in the current prices S^i_t. Then the self-financing condition (12.22) takes on the following form: a portfolio $x_{t-1} = (x^1_{t-1}, ..., x^n_{t-1})$ can be rebalanced into $y_t = (y^1_t, ..., y^n_t)$ if and only if

$$\textstyle\sum_{i=1}^n y^i_t \le \sum_{i=1}^n \frac{S^i_t}{S^i_{t-1}} x^i_{t-1}. \tag{12.23}$$

We can write (12.22) as

$$|y| \le R_t x, \tag{12.24}$$

where $|y|$ stands for the sum of absolute values of the coordinates of vector y and $R_t = (R^1_t, ..., R^n_t)$, where $R^i_t = S^i_t / S^i_{t-1}$. The coordinates R^i_t of the vector R_t are (*gross*) *returns* on assets $i = 1, ..., n$ over the time period between $t - 1$ and t.

If the portfolio positions are measured in terms of their current values, then self-financing strategies are sequences $x_0, x_1, ...$ of portfolios satisfying $|x_t| \le R_t x_{t-1}, \ t = 1, 2....$ These inequalities can be written as $(x_{t-1}, x_t) \in Z_t$, $t = 1, 2...$, where

$$Z_t = \{(x, y) \in \mathbb{R}^{2n}_+, \ |y| \le R_t x\}. \tag{12.25}$$

We can consider the von Neumann–Gale model defined by the cones (12.25), and then analyze self-financing strategies as paths in this model.

If transaction costs are present, portfolio rebalancing leads to a loss in wealth, and, in general, one cannot transform a portfolio into another portfolio with the same value. As an illustration, we will briefly describe a way of introducing transaction costs into the model in which portfolio positions are measured in physical units of assets. We will confine the analysis to the case of *proportional* transaction costs. This means that the set K_t (see (12.20)) describing the rebalancing constraints is supposed to contain with each pair (h, g) of vectors all pairs of vectors of the form $(\lambda h, \lambda g)$, where $\lambda \geq 0$. Furthermore, we will assume that if there are two portfolio pairs (h, g) and (h', g') such that h and h' can be transformed into g and g', respectively, then $h + h'$ can be transformed into $g + g'$. This means that the set (12.20) is a cone, and so we are in the von Neumann–Gale framework. The assumptions imposed are fulfilled in most of the transaction cost models considered in the literature. A typical example of K_t is as follows: a pair of portfolios $h, g \geq 0$ belongs to K_t if and only if

$$\sum_{i=1}^{n}(1+\lambda_i^+)\, S_t^i\, (g^i - h^i)_+ \leq \sum_{i=1}^{n}(1-\lambda_i^-)\, S_t^i\, (h^i - g^i)_+, \tag{12.26}$$

where, for a real number a, we denote $a_+ = \max\{a, 0\}$. Inequality (12.26) expresses the fact that, when rebalancing a portfolio h into a portfolio g, purchases of assets are made only at the expense of sales of other assets, the transaction costs being taken into account. The transaction cost rates for buying and selling are given by the numbers $\lambda_i^+ \geq 0$ and $1 > \lambda_i^- \geq 0$, respectively. It is easily verified that the set K_t defined by (12.26) is a cone.

12.2.4 Stationary Models

Let us return to the general von Neumann–Gale framework (see **12.2.1**). In the study of various aspects of the von Neumann–Gale model, the primary focus is on the *stationary* case—where the cones Z_t do not change in time: $Z_t = Z$. The seminal papers of von Neumann [60] and Gale [30] focused on this case. In the stationary setting, the following notion plays an important role. A path $\{x_t\}$ is called *balanced* if x_t is of the form $x_t = \lambda^t x,\ t = 0, 1, ...,$ where λ is a strictly positive number and $x \geq 0$ is a vector with $|x| = 1$. This notion expresses the idea of growth at a constant rate (determined by the factor λ) and with constant proportions (determined by the coordinates of the vector x). Clearly, those and only those pairs λ, x define balanced paths for which

$$\lambda > 0,\ |x| = 1 \text{ and } (x, \lambda x) \in Z. \tag{12.27}$$

A balanced path $x_t = \lambda^t x$ for which the growth factor λ is a maximum is called a *von Neumann ray*. To find those x and λ which determine a von Neumann ray, we have to maximize λ over all pairs λ, x satisfying (12.27).

Those balanced paths are of primary interest which are rapid. Recall that rapid trajectories are those for which there exists a supporting dual path—a sequence of vectors $\{p_t\}$ satisfying (12.3) and (12.5), or equivalently (12.15) and (12.5). A dual path $\{p_t\}$ is called *balanced* if $p_t = \lambda^{-t} p$ for some nonzero vector $p \geq 0$ and some number $\lambda > 0$ (λ^{-1} being the *discount factor*).

A triplet $(\bar{x}, p, \lambda)$, where $\bar{x}, p$ are vectors in $\mathbb{R}^n_+$ and $\lambda > 0$ is a number, is called a *von Neumann equilibrium* if $\{\lambda^t \bar{x}\}$ is a balanced path and $\{\lambda^{-t} p\}$ is a balanced dual path supporting it. One can easily verify that $(\bar{x}, p, \lambda)$ is an equilibrium if and only if the requirements

$$\lambda^{-1} py \leq px \text{ for all } (x, y) \in Z, \ p\bar{x} = 1 \tag{12.28}$$

are satisfied and the conditions listed in (12.27) hold for $x = \bar{x}$. According to the definition of a balanced dual path, the valuation $p_t x$ of any state x of the system is $\lambda^{-t} px$, where the number λ^{-1} is the discount factor. In a von Neumann equilibrium, this number coincides with the inverse of the growth factor λ of the balanced path $\{\lambda^t \bar{x}\}$.

Note that if $(\bar{x}, p, \lambda)$ is a von Neumann equilibrium and the vector p is strictly positive, then λ is the maximum growth factor among all balanced paths, and so $\{\lambda^t \bar{x}\}$ is a von Neumann ray. Indeed, suppose $(x', \lambda x') \in Z$, $|x| = 1$. Then, by setting $x = x'$ and $y = \lambda' x'$, we obtain from (12.24) that $p\lambda' x' / \lambda \leq px'$, which implies $\lambda' \leq \lambda$ because $px' > 0$.

One more comment is in order. Consider the stationary versions of two nonlinear von Neumann models (12.17) and (12.19) in which the mappings $\Phi_t = \Phi$ and $\Psi_t = \Psi$ do not depend on t. The former is defined in terms of the cone $Z = \{(x, y) \in \mathbb{R}^{2n}_+ : \Psi(y) \leq \Phi(x)\}$. The latter is given by the cone $W = \{(u, v) \in \mathbb{R}^{2m}_+ : u \geq \Psi(x), \ v \leq \Phi(x) \text{ for some } x \in \mathbb{R}^n_+\}$. Paths $\{x_t\}$ of the former are sequences of intensity vectors, while paths $\{u_t\}$ of the latter are sequences of commodity vectors. It can be shown under quite general assumptions that the existence of a von Neumann equilibrium $(\bar{x}, p, \lambda)$ in the former model implies the existence of a von Neumann equilibrium $(\bar{u}, q, \lambda)$ in the latter and vice versa.

A proof of the existence of an equilibrium in a stationary version of the von Neumann model (of the second kind, in our classification) with linear Φ and Ψ was the main result of the von Neumann's paper [60]. To comment on the meaning of this result, consider an equilibrium $(\bar{u}, q, \lambda)$ in the model at hand and assume, additionally, that $q > 0$. Then λ may be regarded as the *expansion factor* of the economy—the maximum of those numbers α for which there is a feasible balanced path $\{\alpha^t u\}$ in the commodity space $\mathbb{R}^m_+$. A fundamental implication of the existence of such an equilibrium is the conclusion that the expansion factor λ is equal to μ, where μ^{-1} is the *greatest admissible discount factor*. This is the greatest number for which there exists a commodity price vector $q' > 0$ satisfying

$$\mu^{-1} q' v \leq q' u \text{ for all } (u, v) \in W. \tag{12.29}$$

Observe that if inequality (12.29) does not hold for some price vector q' and some input-output pair $(u, v) \in W$, then the maximum of the *net discounted profit* $\mu^{-1} q' v' - q' u'$ over all $(u', v') \in W$ is infinite. This is so because $(\theta u, \theta v) \in W$ for all $\theta > 0$. To avoid possibilities of getting infinite profits (arbitrage), admissible discount factors must satisfy condition (12.29). Clearly, if $(\bar{u}, q, \lambda)$ is a von Neumann equilibrium with $q > 0$, then (12.29) holds with $\mu = \lambda$ and $q' = q$. On the other hand, if (12.29) holds for some $\mu' > 0$ and $q' > 0$, we obtain, by setting $u = \bar{u}$ and $v = \lambda\bar{v}$ in (12.29), that $\lambda(\mu')^{-1}q'\bar{u} \leq q'\bar{u}$. Consequently, $\lambda \leq \mu'$, which proves that $\mu^{-1} = \lambda^{-1}$ is the greatest admissible discount factor.

Further aspects of the concept of von Neumann equilibrium and related deterministic notions are discussed in Gale [30, 31], Rockafellar [54], Nikaido [47], and Makarov and Rubinov [41].

12.3 Assumptions and Results

12.3.1 Assumptions

The *standard assumptions* on the cones Z_t, defining a von Neumann–Gale model are as follows:

(Z.1) The cone Z_t is closed.

(Z.2) There is a constant M such that $|y| \leq M|x|$ for any $(x, y) \in Z_t$.

(Z.3) If $(x, y) \in Z_t$, $x' \geq x$ and $0 \leq y' \leq y$, then $(x', y') \in Z_t$.

(Z.4) For some $(\check{x}_{t-1}, \check{y}_t) \in Z_t$, we have $\check{y}_t \geq \gamma e$, where $e = (1, 1, ..., 1)$.

These assumptions are imposed for every t for which Z_t is given. Condition (Z.2) says that the ratio $|y|/|x|$ of the norms of the output vector y and the input vector x is uniformly bounded for all $(x, y) \in Z_t$ and $t = 1, 2, ...$ Assumption (Z.3) is a "free disposal" hypothesis. Condition (Z.4) is a non-degeneracy assumption guaranteeing, in particular, the existence of paths $\{x_t\}$ with strictly positive x_t. *Assumptions (Z.1)–(Z.4) will be supposed to hold throughout the paper.*

If the model is stationary, i.e. $Z_t = Z$ does not depend on t, then hypothesis (Z.2) holds if the cone Z is closed and satisfies the following condition.

(Z.2′) If $(0, y) \in Z$, then $y = 0$.

Condition (Z.2′) expresses impossibility of "getting something from nothing."

Clearly, in the stationary case, condition (Z.4) is equivalent to (Z.4′) below.

(Z.4′) There exist $\hat{x}$ and $\gamma > 0$ such that $(\hat{x}, \gamma e) \in Z$.

For the model to work smoothly, assumptions (Z.1)–(Z.4) are usually not sufficient. These assumptions should be complemented by the following condition, which often turns out to be very substantial.[5]

(Z.5) There exists an integer $m \geq 1$ such that, for any $i \in \{1, ..., n\}$ and any $t \in \{0, 1, 2, ...\}$, one can find vectors $y_t \in \mathbb{R}^n_+, ..., y_{t+m} \in \mathbb{R}^n_+$ satisfying

$$y_t = e_i,\ (y_t, y_{t+1}) \in Z_{t+1}, ..., (y_{t+m-1}, y_{t+m}) \in Z_{t+m},\ y_{t+m} \geq \gamma e. \tag{12.30}$$

[5] In the paper by Hulsmann and Steinmetz [34], a delicate counterexample was constructed showing that the conditions (Z.1)–(Z.4) do not guarantee the existence of an equilibrium in a general von Neumann–Gale model. They are sufficient, however, for the existence of an equilibrium in the classical (linear) von Neumann model. An elegant proof of this is given in a paper by Gale [32].

We denote by e_i the vector with the ith coordinate being equal to one and all others being equal to zero.

In the context of the nonlinear von Neumann model (12.17), condition (Z.5) means that if any of the economic units $i = 1, 2, ..., n$ functions at time t with unit intensity, then in m time periods, all the units can be operated with intensities not less than $\gamma > 0$. In the two versions of the nonlinear von Neumann model, (12.17) and (12.19), condition (Z.5) looks much more plausible in the former (dealing with intensity vectors) than in the latter (dealing with commodity vectors). For the system (12.10) defined by a positive matrix $D_t = D$, assumption (Z.5) holds if and only if the mth power D^m of the matrix D is strictly positive. It should be noted that, in the financial applications (see **12.2.3**) requirement (Z.5) appears absolutely non-restrictive, and it typically holds with $m = 1$: one can, by selling asset i today, buy some—perhaps small—positive amounts of all the assets tomorrow.

The most complete theory can be developed (especially in the stochastic case) if the cones Z_t possess certain properties of strict convexity. Two conditions will be employed, in particular, for establishing results regarding the asymptotic behavior of rapid paths. The first expresses a property of strict convexity of Z_t with respect to x and the second with respect to y (these properties hold uniformly in t).

(SC1) For all $\epsilon > 0$, there exists a number $\rho(\epsilon) > 0$ having the following property. For each $(x, y) \in Z_t$, $(x', y') \in Z_t$ satisfying $|x| = |x'| = 1$ and $|x - x'| \geq \epsilon$, there is a vector $w \in \mathbb{R}^n_+$ such that

$$(x + x', y + y' + w) \in Z_t \text{ and } |w| \geq \rho(\varepsilon).$$

(SC2) For all $\epsilon > 0$, there is a number $\tau(\epsilon) > 0$ with the following property. If $(x, y) \in Z_t$, $(x', y') \in Z_t$ satisfy $|x| = |x'| = 1$ and $|y - y'| \geq \epsilon$, one can find a vector $w \in \mathbb{R}^n_+$ for which

$$(x + x', y + y' + w) \in Z_t \text{ and } |w| \geq \tau(\varepsilon).$$

In Section 12.5, we formulate conditions on the mappings Φ_t and Ψ_t of in the nonlinear von Neumann model (12.17) that guarantee the validity of all the assumptions imposed (these conditions are formulated in the more general, stochastic setting).

12.3.2 Finite Rapid Paths

We point to several groups of results that constitute the core of the theory of the von Neumann–Gale model. These results are concerned mainly with properties of rapid paths over finite and infinite time horizons. The emphasis is on growth properties of such paths. Some of the results are obtained in the general, non-stationary framework. However, the stationary case is regarded as a central one, in particular, because a number of notions introduced above (such as the von Neumann equilibrium) pertain only to the stationary version of the theory.

Let us first discuss results concerned with rapid paths defined over finite time intervals. Theorem 12.3.1 below states that such paths can be constructed by maximizing concave strictly monotone utility functions. A function $\psi(x)$, $x \in \mathbb{R}^n_+$, is called *strictly monotone* if $\psi(x') > \psi(x)$ when $x' \geq x$ and $x' \neq x$.

Fix some natural number N and a strictly positive vector $x_0 \in \mathbb{R}^n_+$.

Theorem 12.3.1. *Let $\psi(x)$, $x \in \mathbb{R}^n_+$, be a concave strictly monotone function. Let $\xi = \{x_0, x_1, ..., x_N\}$ be a path with initial state x_0 maximizing $\psi(x_N)$ over all such paths. Then ξ is rapid.*

Proof of Theorem 12.3.1. Consider the following maximization problem:

$$\text{Maximize } F(\theta) = \psi(a_N) \tag{12.31}$$

over all sequences $\theta = \{(a_t, b_t)\}_{t=0}^N$ satisfying

$$b_0 \in \mathbb{R}^n_+,\ (a_{t-1}, b_t) \in Z_t\ (t = 1, ..., N),\ a_N \in \mathbb{R}^n_+, \tag{12.32}$$

$$b_t \geq a_t\ (t = 0, ..., N), \tag{12.33}$$

and $b_0 = x_0$. Observe that the sequence $\bar{\theta} = \left\{\left(\bar{a}_t, \bar{b}_t\right)\right\}_{t=0}^N$ defined by $\bar{a}_t = \bar{b}_t = x_t$ is a solution to the above maximization problem. Indeed, $\bar{\theta}$ satisfies constraints (12.32) and (12.33) because ξ is a path with initial state x_0, and we have $F(\bar{\theta}) = \psi(x_N)$. Further, for any $\theta = \{(a_t, b_t)\}_{t=0}^N$ satisfying (12.32) and (12.33), the sequence $\{x_0, a_1, a_2, ..., a_N\}$ is a path with initial state x_0 (by virtue of hypothesis (Z.3)), and $F(\theta) = \psi(a_N)$. Consequently, $F(\theta) = \psi(a_N) \leq \psi(x_N) = F(\bar{\theta})$, which proves the optimality of $\bar{\theta}$.

We are going to apply the Kuhn–Tucker theorem (see the Appendix, Theorem A.1) to the concave optimization problem (12.31)–(12.33). From (Z.3) and (Z.4), it follows that $(e, \kappa e) \in Z_t$ $(t = 1, 2, ..., N)$ for some $\kappa > 0$. Define $\check{\theta} = \{(\check{a}_t, \check{b}_t)\}_{t=0}^N$, where

$$\check{b}_0 = x_0,\ (\check{a}_{t-1}, \check{b}_t) = \mu^t(e, \kappa e)\ (t = 1, ..., N),\ \check{a}_N = \mu^{N+1} e$$

and $\mu > 0$ is some number satisfying $\mu e < x_0$ and $\mu < \kappa$. Then we have $\check{b}_t > \check{a}_t$ for all $t = 0, ..., N$, and so the Slater condition holds. By virtue of the Kuhn–Tucker theorem, there exist vectors $p_0, ..., p_N \in \mathbb{R}^n_+$ such that

$$\psi\left(a_N\right) + \sum_{t=0}^{N} p_t(b_t - a_t) \leq \psi\left(x_N\right) \tag{12.34}$$

for all $\{(a_t, b_t)\}_{t=0}^N$ satisfying (12.32) and $b_0 = x_0$. From (12.34), we get

$$p_t b_t - p_{t-1} a_{t-1} \leq p_t x_t - p_{t-1} x_{t-1}\ \left(t = 1, 2, ..., N\right), \tag{12.35}$$

$$\psi(a_N) - p_N a_N \leq \psi(x_N) - p_N x_N \tag{12.36}$$

for all $(a_{t-1}, b_t) \in Z_t$ $(t = 1, ..., N)$ and $a_N \in \mathbb{R}^n_+$. Inequalities (12.35) imply

$$p_t x_t - p_{t-1} x_{t-1} = 0 \tag{12.37}$$

and $p_t y - p_{t-1} x \leq 0$, $(x, y) \in Z_t$, because Z_t is a cone. Thus $\{p_t\}$ is a dual path.

By setting $a_N = x_N + e_i$ in (12.36) and using the strict monotonicity of ψ, we find

$$p_N e_i = p_N (a_N - x_N) \geq \psi(a_N) - \psi(x_N) = \psi(x_N + e_i) - \psi(x_N) > 0.$$

Consequently, $p_N > 0$, and so $p_N x_N > 0$, since $x_N \neq 0$ (indeed, $\psi(x_N) \geq F(\check{\theta}) = \psi(\check{a}_N) > \psi(0)$). Therefore, by virtue of (12.37), we have $p_0 x_0 = p_1 x_1 = ... = p_N x_N > 0$. Replacing p_t by $p_t / p_t x_t$ we obtain that $p_t x_t = 1$, and so the dual path $\{p_t\}$ supports $\{x_t\}$. Hence the path $\{x_0, ..., x_N\}$ is rapid, which completes the proof. □

Remark 12.3.1. In Theorem 12.3.1, we considered paths $\xi = \{x_0, ..., x_N\}$ with given x_0 maximizing the function $\psi(x_N)$. Such paths exist if the function $\psi(x)$, $x \in \mathbb{R}^n_+$, is continuous. This is so because the set of all trajectories $\{x_0, x'_1, ..., x'_N\}$ with given x_0 is closed and bounded. The former follows from (Z.1), and the latter from hypothesis (Z.2), implying

$$|x'_t| \leq M^t |x_0|. \tag{12.38}$$

12.3.3 Infinite Rapid Paths: Existence and Quasi-Optimality

Infinite rapid paths can be constructed by passing to the limit from finite paths of length N, as N tends to infinity.

Theorem 12.3.2. *There exists an infinite rapid path $\{x_0, x_1, ...\}$ with initial state x_0, for each $x_0 > 0$.*

Proof. For each $N = 1, 2, ...$, consider a path $\xi^N = \{x_0^N, x_1^N, ..., x_N^N\}$ with $x_0^N = x_0$ maximizing $|x_N|$ among all paths $\{x_0, x_1, ... x_N\}$. By virtue of Theorem 12.3.2, ξ^N is rapid, and so it possesses a supporting dual path $\{p_0^N, p_1^N, ..., p_N^N\}$. It follows from (12.38) that for each $t = 1, 2, ...$, the sequence $\{x_t^N\}$ $(N = t, t+1, ...)$ is bounded. In view of (Z.4) and (Z.3), $(e, \kappa_t e) \in Z_t$ for some $\kappa_t > 0$. Therefore $\{e, \mu_1 e, \mu_2 e, ...\}$, where $\mu_t = \kappa_1 ... \kappa_t$, is a path, and so $p_t^N \mu_t e \leq p_0^N x_0 = 1$, which implies the boundedness of the sequence $\{p_t^N\}$ $(N = t, t+1, ...)$. By using considerations of compactness, we find a sequence $N_1 < N_2 < ...$ such that $x_t^{N_k} \to x_t$, $p_t^{N_k} \to p_t$ for all t, were x_t, $p_t \in \mathbb{R}^n_+$. Since Z_t is closed, we have $(x_{t-1}, x_t) \in Z_t$, and so $\{x_0, x_1, ...\}$ is a path. By passing to the limit in the relations $p_t^{N_k} x_t^{N_k} = 1$ and $p_t^{N_k} y - p_{t-1}^{N_k} x \leq 0$, $(x, y) \in Z_t$, we obtain that $\{p_0, p_1, ...\}$ is a supporting dual path for $\{x_0, x_1 ...\}$. □

Every infinite rapid path $\{x_t\}_{t=0}^{\infty}$ possesses the important property of (asymptotic) *quasi-optimality* stated in Theorem 12.3.3 below.

Theorem 12.3.3. *Let assumption (Z.5) hold. Then for every path* $\{x'_t\}_{t=0}^{+\infty}$, *we have*

$$\sup_t \left(|x'_t| \,/\, |x_t|\right) < \infty. \tag{12.39}$$

Property (12.39) means that no path can grow "infinitely faster" than $\{x_t\}$ in the long run. Clearly, $|\cdot|$ can be replaced in (12.39) by any other norm in $\mathbb{R}^n$.

The proof of Theorem 12.3.3 is based on the following lemma holding under assumptions (Z.1)–(Z.5).

Lemma 12.3.1. *If* $\{x_t\}$ *is an infinite rapid path and* $\{p_t\}$ *is a dual path supporting* $\{x_t\}$, *then*

$$p_t \geq \gamma M^{-m} \left|x_t\right|^{-1} e, \tag{12.40}$$

where M *is the constant described in (Z.2) and* γ, m *are the numbers appearing in (Z.5).*

In view of (12.40), the coordinates of the vectors $p_t \left|x_t\right|$ are bounded away from 0 by a constant independent of t.

Proof of Lemma 12.3.1. Observe that

$$|p_t||x_t| \geq p_t x_t = 1 \text{ and } |x_{t+m}| \leq M^m |x_t|,$$

the latter holds by virtue of (Z.2). In view of (Z.5) and (12.15), we have $p_{t+m} y_{t+m} \leq p_t e_i$, where $y_t, ..., y_{t+m}$ is the sequence specified in (12.30). Consequently,

$$p_t e_i \geq \gamma |p_{t+m}| \geq \gamma |x_{t+m}|^{-1} \geq \gamma M^{-m} |x_t|^{-1},$$

which yields (12.40). □

Proof of Theorem 12.3.3. By virtue of (12.40), $e \leq p_t \gamma^{-1} M^m \left|x_t\right|$. Therefore

$$\frac{|x'_t|}{|x_t|} = \frac{x'_t e}{|x_t|} \leq p_t x'_t \cdot \gamma^{-1} M^m \leq p_0 x_0 \cdot \gamma^{-1} M^m,$$

which proves (12.39). □

Remarkably, if the strict convexity assumption (SC1) holds, *only one* infinite rapid path emanates from each initial state $x_0 > 0$. Of course, this is not the case for finite paths. Given $N < \infty$, by maximizing different functions $\psi(x_N)$ of the terminal state x_N (for some fixed x_0), we can generally obtain different rapid paths. The uniqueness of infinite rapid paths follows from the turnpike theorems we discuss below.

12.3.4 Turnpike Theorems

These theorems express the idea that all rapid paths have an inclination for leaning to essentially the same route (the "turnpike"). In the stationary case, this route is the von Neumann ray. In order to formulate the results regarding the turnpike properties of rapid paths we have to specify how we measure deviations of such paths from each other. To this end we introduce the following definitions. For any $x, x' \in \mathbb{R}^n_+$ such that $|x| > 0$ and $|x'| > 0$, we define

$$d(x, x') = |\frac{x}{|x|} - \frac{x'}{|x'|}|. \tag{12.41}$$

For $(x, y), (x', y') \in Z_t$ with $|x| > 0$, $|x'| > 0$, we set

$$D(x, y, x', y') = |\frac{(x, y)}{|x|} - \frac{(x', y')}{|x'|}|. \tag{12.42}$$

The number $d(x, x')$ is the "angular distance" between the vectors x and x'. For two paths $\{x_t\}$ and $\{x'_t\}$, the number $d(x_t, x'_t)$ shows to what extent the directions of the vectors x_t and x'_t at time t differ from each other. In some applications, it is important to compare not only the directions of trajectories $\{x_t\}$ and $\{x'_t\}$, but also their growth rates measured, for example, by $|x_{t-1}|^{-1}(|x_t| - |x_{t-1}|)$ and $|x'_{t-1}|^{-1}(|x'_t| - |x'_{t-1}|)$. With this view, the function $D(x, y, x', y')$ is introduced. The relation

$$|\frac{|x_t| - |x_{t-1}|}{|x_{t-1}|} - \frac{|x'_t| - |x'_{t-1}|}{|x'_{t-1}|}| \le D(x_{t-1}, x_t, x'_{t-1}, x'_t) \tag{12.43}$$

shows that the growth rates are close to each other if the value of D is small. As is easily seen, the functions d and D are *pseudometrics* (they are nonnegative, symmetric, and satisfy the triangle inequality). Furthermore, we have $d(x, x') \le D(x, y, x', y')$.

The main turnpike results are collected in the following theorem.

Theorem 12.3.4. *(a) Let condition (SC1) hold. For each $\varepsilon > 0$, there exists a number $L = L(\varepsilon)$ such that, for any two rapid paths*

$$\xi = \{x_t\}_{t=0}^N \text{ and } \xi' = \{x'_t\}_{t=0}^{N'} \text{ with } 2L < N \le N' \le \infty, \tag{12.44}$$

we have $d(x_t, x'_t) \le \varepsilon$ for all t within the interval $L \le t \le N - L$.

(b) If, additionally, condition (SC2) holds, then for any two rapid paths (12.44), the inequalities

$$D(x_{t-1}, x_t, x'_{t-1}, x'_t) \le \varepsilon$$

are valid for all t satisfying $L \le t < N - L$.

(c) Fix some constant $\theta > 0$. Suppose the initial vectors x_0 and x'_0 of the rapid paths (12.44) satisfy

$$x_0 = x'_0 \ge \theta\, |x_0|\, e. \tag{12.45}$$

Then, in assertions (a) and (b), the time interval $L \le t < N - L$ can be replaced by $0 \le t < N - L$.

The theorem formulated can be applied to situations when both trajectories $\{x_t\}$ and $\{x'_t\}$ are finite ($N' < \infty$), $\{x_t\}$ is finite and $\{x'_t\}$ is infinite ($N < \infty, N' = \infty$), and both $\{x_t\}$ and $\{x'_t\}$ are infinite ($N = \infty$). If $N < \infty$, then by virtue of assertion (a) of Theorem 12.3.4, the paths $\{x_t\}$ and $\{x'_t\}$ are close to each other for those moments t which are far enough from the ends of time interval $\{0, 1, ..., N\}$. Note that the number L depends only on ε but not on $\{x_t\}, \{x'_t\}$, N and N'. If $N = \infty$, i.e. both $\{x_t\}$ and $\{x'_t\}$ are infinite, then the trajectories $\{x_t\}$ and $\{x'_t\}$ converge to each other in the sense that $D(x_{t-1}, x_t, x'_{t-1}, x'_t) \to 0$ as $t \to \infty$ (we have $N - L = \infty$ if $N = \infty$). This convergence is uniform over the class of all pairs of infinite rapid trajectories. Assertion (b) of Theorem 12.3.4 contains additional information pertaining to the case where the initial vectors x_0 and x'_0 of the paths $\{x_t\}$ and $\{x'_t\}$ coincide and the coordinates of $x_0 \, |x_0|^{-1} (= x'_0 \, |x'_0|^{-1})$ are bounded away from zero by the given constant θ. If $N < \infty$, then, according to (b), significant deviations between $\{x_t\}$ and $\{x'_t\}$ may occur only within the interval $\{N - L, ..., N\}$. The number $L = L(\varepsilon)$ is the same for all pairs of paths $\{x_t\}$, $\{x'_t\}$ satisfying (12.45).

Versions of the results contained in Theorem 12.3.4 can be found, e.g., in the monographs by Nikaido [47] and by Makarov and Rubinov [41]. Excellent surveys of turnpike results are the papers [42] and [43] by McKenzie, one of the founders of the turnpike theory. First turnpike theorems for the stochastic von Neumann–Gale model were obtained by Evstigneev and Kuznetsov [24]. For a proof of Theorem 12.3.4 and its stochastic generalization see Anoulova, Evstigneev and Gundlach [6].

12.3.5 The Stationary Case: von Neumann Ray and von Neumann Equilibrium

This group of results includes the following key facts.

(a) The existence of a von Neumann ray. This fact is proved in the deterministic case quite easily. For a proof, it is sufficient to observe that the set of those $(x, \lambda) \in \mathbb{R}^n_+ \times \mathbb{R}^1_+$ satisfying $|x| = 1$ and $(x, \lambda x) \in Z$ is compact (by virtue of (Z.1) and (Z.2)) and contains a pair (x, λ) with $\lambda > 0$ (by virtue of (Z.3) and (Z.4)).

(b) The existence of a von Neumann equilibrium can be established based on the following assertion.

Theorem 12.3.5. *Let condition (Z.5) hold. Let $\{\lambda^t x\}$ be a von Neumann ray. Then there exists a vector $p > 0$ such that $(\bar{x}, p, \lambda)$ is a von Neumann equilibrium.*

Proof. Consider the sets

$$B = \{y - \lambda x : \ (x, y) \in Z\}, \ C = \{x \in \mathbb{R}^n : \ x > 0\}.$$

Observe that $B \cap C = \emptyset$. Suppose the contrary. Then $y - \lambda x > 0$ for some $(x, y) \in Z$. The last inequality will remain valid if we replace λ by $\lambda' > \lambda$ which is sufficiently close to λ. Then $\lambda' x < y$ and, by virtue of (Z.3), $(x, \lambda' x) \in Z$. Observe that $x \neq 0$: otherwise $y = 0$ by virtue of (Z.2), and so the inequality $y - \lambda x > 0$ cannot hold. By setting $y = x/|x|$, we get $(y, \lambda' y) \in Z$, $|y| = 1$, $\lambda' > \lambda$, which contradicts the definition of the von Neumann growth factor λ.

The sets B and C are convex and disjoint, and by virtue of a separation theorem (see the Appendix, Theorem B.1), there exists $q \in \mathbb{R}^n$, $q \neq 0$, such that $qc \geq qb$ for all $c \in C$, $b \in B$. Since $0 \in B$, we obtain that $qc \geq 0$ for all $c \in C$, and so $q \geq 0$. Since $qb \leq 0$ for all $b \in B$, we have

$$\lambda^{-1} qy \leq qx \text{ for all } (x, y) \in Z. \tag{12.46}$$

Let us show that $q > 0$. By virtue of (Z.5), for each $i = 1, 2, ...n$, there is a path $\{y_0, y_1, ..., y_m\}$ such that $y_0 = e_i$, $y_m \geq \gamma e$. In view of (12.46) and because $q \neq 0$, we have

$$qe_i = qy_0 \geq \lambda^{-1} qy_1 \geq \lambda^{-2} qy_2 \geq ... \geq \lambda^{-m} qy_m \geq \lambda^{-m} \gamma |q| > 0$$

and so $q > 0$.

Define $p = q/q\bar{x}$ (> 0). Then $p\bar{x} = 1$ and $py/\lambda \leq px$ for all $(x, y) \in Z$. Consequently $(\bar{x}, p, \lambda)$ is a von Neumann equilibrium. □

(c) Quasi-optimality of the von Neumann ray. Since a von Neumann ray is a rapid path, it is quasi-optimal (see Theorem 12.3.3). As it was mentioned above, this property is valid for all rapid paths. However, in the stationary context, this property acquires an important additional content. The von Neumann ray is a path of a special structure: it is balanced. Thus we obtain that there is a balanced path that is quasi-optimal among *all*, not necessarily balanced, ones.

(d) The positive matrix case. Consider a stationary model defined by

$$Z = \{(x, y) \in \mathbb{R}^{2n}_+ : \ y \leq Dx\}, \tag{12.47}$$

where D is a nonnegative matrix. For this model, conditions (Z.1)–(Z.3) hold. Hypothesis (Z.4) holds if $D\hat{x} > 0$ for some $\hat{x} \in \mathbb{R}^n_+$ (which is true if and only if every row of D has a nonzero entry). As we have already noticed in **12.3.1**, (Z.5) holds if $D^m > 0$ for some $m \geq 1$.

By virtue of Theorem 12.3.4, in this model there is a von Neumann equilibrium $(\bar{x}, p, \lambda)$ with $p > 0$. We have

$$\lambda^{-1} pDx \leq px \ \text{ for all } x \in \mathbb{R}^n_+ \tag{12.48}$$

and $\lambda\bar{x} \leq D\bar{x}$. Since $p > 0$, the last two inequalities imply $\lambda\bar{x} = D\bar{x}$. Consequently, λ and $\bar{x}$ are the Perron–Frobenius eigenvalue and eigenvector of the matrix D, respectively (see the Appendix, Theorem C.1). This implies in particular that $\bar{x} > 0$.

From (12.48), it follows that $pD/\lambda \leq p$. On the other hand $pD\bar{x}/\lambda = p\bar{x}$, and since $\bar{x} > 0$, we obtain $pD = \lambda p$. Thus p is an eigenvector of the matrix D' (the conjugate of D) with eigenvalue λ. The matrix $(D')^m = (D^m)'$ is strictly positive, and so p is the Perron–Frobenius eigenvector of D'.

Thus we have proved the following theorem.

Theorem 12.3.6. *There is a unique von Neumann equilibrium* $(\bar{x}, p, \lambda)$ *in the model (12.47). The vectors* $\bar{x}$ *and* p *are the Perron–Frobenius eigenvectors of the matrices* D *and* D' *respectively, and* λ *is their common Perron–Frobenius eigenvalue.*

12.3.6 Duality and Reachability

All the results we reviewed were aimed first of all at the analysis of growth properties of paths in the von Neumann–Gale model. The topic we will consider now has a somewhat different focus. Fix some finite time horizon $N < \infty$ and consider two states $x \in \mathbb{R}^n_+$ and $y \in \mathbb{R}^n_+$ of the von Neumann–Gale dynamical system (12.1). Let us say that y can be *reached* from x and write $(x, y) \in \mathcal{Z}$ if there exists a path $x_0, ..., x_N$ such that $x_0 = x$ and $x_N = y$. Of interest is the *reachability problem* dealing with a characterization of the set $\mathcal{Z}$. This characterization is typically given in terms of dual paths, as described in Theorem 12.3.7 below.

Let us say that a dual path $\{p_0, ..., p_N\}$ is *strict* if $p_N > 0$. Assume that conditions (Z.1)–(Z.5) hold.

Theorem 12.3.7. *A state* $y \in \mathbb{R}^n_+$ *can be reached from a state* $x \in \mathbb{R}^n_+$ *if and only if* $p_0 x \geq p_N y$ *for all strict dual paths* $\{p_0, ..., p_N\}$*.*

The main applications of this result are in the area of financial modeling—see **12.2.3** and **12.8.1**.

Proof of Theorem 12.3.7. The "only if" statement follows directly from the definition of a dual path. To prove the converse statement assume that $(x, y) \notin \mathcal{Z}$. Let us show that $p_0 x < p_N y$ for some strict dual path $\{p_0, ..., p_N\}$.

Define $\mathcal{K} = \{(a, b) : a \leq 0, b \geq 0\}$. It follows from (Z.2) that $\mathcal{K} \cap \mathcal{Z} = \{0\}$ and from (Z.1) and (Z.2) that $\mathcal{Z}$ is closed. Since $(x, y) \geq 0$ and $(x, y) \notin \mathcal{Z}$, we have $(x, y) \notin \mathcal{W} := \mathcal{Z} - \mathcal{K}$ by virtue of (Z.3). The cone $\mathcal{W} = \mathcal{Z} - \mathcal{K}$ is closed because $\mathcal{Z}$ and $\mathcal{K}$ are closed and $\mathcal{K} \cap \mathcal{Z} = \{0\}$ (see the Appendix, Theorem B.1). The cone $\mathcal{K}$ is proper, and since the point $v := (x, y)$ does not belong to $-\mathcal{K}$, the cone $\mathcal{K}_v$ spanned on $\mathcal{K}$ and $\{v\}$ is closed and proper (Theorem B.1). Furthermore, $\mathcal{K}_v \cap \mathcal{W} = \{0\}$, and so there exists a linear function $l(a, b) = qb - pa$ whose values on $\mathcal{K}_v \setminus \{0\}$ are strictly positive and whose values on $\mathcal{W} = \mathcal{Z} - \mathcal{K}$ are non-positive (Theorem B.2). The former implies $p > 0, q > 0$ and $qy - px > 0$, while the latter yields $qb - pa \leq 0$ for all $(a, b) \in \mathcal{Z}$.

Consider the problem of maximization of $f(\theta) = qa_N - pb_0$ over all sequences $\theta = \{(a_t, b_t)\}_{t=0}^N$ satisfying (12.32) and (12.33). Since $qb - pa \leq 0$ for all $(a, b) \in \mathcal{Z}$, the maximum value of $f(\theta)$ in the above problem is zero. By applying the Kuhn–Tucker theorem along the same lines as it was done in Theorem 12.3.1 (relaxing the constraints (12.33)), we construct a dual path $\{p_0, ..., p_N\}$ satisfying $p_0 \leq p$ and $q \leq p_N$. These inequalities imply $p_0 x < p_N y$ and $p_N > 0$, so the dual path constructed is strict. □

Remark 12.3.2. In the course of the proof of the above theorem, we established (under the assumptions imposed) that the set of strict dual paths is non-empty. Indeed, we considered any $(x, y) \notin \mathcal{Z}$ and constructed a strict dual path satisfying $p_0 x < p_N y$. We can consider, for example, $(x, y) := (0, e)$; this pair of vectors does not belong to $\mathcal{Z}$ by virtue of (Z.2).

Remark 12.3.3. We have assumed up to now that the cones Z_t $(t = 1, ..., N)$ defining the von Neumann–Gale model under consideration are contained in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, where n does not depend on t. All the assumptions and results in this subsection—dealing with the case of a finite time horizon N—can easily be extended to the setting where $Z_t \subseteq \mathbb{R}^{n_{t-1}}_+ \times \mathbb{R}^{n_t}_+$, where $n_0, ..., n_N$ is any sequence of natural numbers. All the arguments go through with obvious changes.

We have discussed several groups of results that play central roles in the theory of the von Neumann–Gale model. They will serve as reference points in our presentation of its stochastic analogue. Our general objective is to develop a stochastic version of the model in which these key results admit natural generalizations. To this end, we first have to define natural stochastic analogues of the main concepts: rapid paths, von Neumann ray, von Neumann equilibrium, etc. Definitions of these concepts will be given in the beginning of Part II of the paper.

II. STOCHASTIC ANALOGUE OF THE VON NEUMANN–GALE MODEL

12.4 Model Description

12.4.1 General (Non-stationary) Model

Let Ω be a non-empty set, $\mathcal{F}$ a σ-algebra of subsets of Ω, and P a probability measure on $\mathcal{F}$. Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}$ be a non-decreasing sequence of σ-algebras. Sets in $\mathcal{F}_t$ are interpreted as events observable prior to time t. Vector functions of $\omega \in \Omega$ measurable with respect to $\mathcal{F}_t$ are construed as random vectors whose realizations become known by time t. For each $t = 0, 1, 2, ...$, we denote by $L^n_1(t)$ the space $L_1(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ consisting of (equivalence classes of) n-dimensional $\mathcal{F}_t$-measurable vector functions $x(\omega)$ with $E|x| = E\sum_i |x^i| < \infty$. The letter E stands for the expectation with respect to the given probability measure P. We denote by $L^n_\infty(t) = L_\infty(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ the space of essentially bounded functions in $L^n_1(t)$ and by $\mathcal{X}_t$ the cone of nonnegative elements in $L^n_\infty(t)$.

The stochastic analogue of the von Neumann–Gale model is specified by the family of cones $Z_t \subseteq \mathcal{X}_{t-1} \times \mathcal{X}_t$ $(t = 1, 2, ...)$. A sequence of random vectors $\{x_t\}_{t=0}^N$, $x_t \in \mathcal{X}_t$ $(N \leq \infty)$, is called a *path* (*trajectory*) in the model if $(x_{t-1}, x_t) \in Z_t$ for all t. Equivalently, the model can be described by a family of multivalued operators $x \mapsto A_t(x)$ $(t = 1, 2, ...)$, where $A_t(x) = \{y : (x, y) \in Z_t\}$. It will be assumed that $A_t(x) \neq \emptyset$ for each $x \in \mathcal{X}_{t-1}$, which means that the projection of the cone Z_t on the first factor in the product $\mathcal{X}_{t-1} \times \mathcal{X}_t$ coincides with $\mathcal{X}_{t-1}$. As in the deterministic case, it is easily seen that the graph Z_t of the operator $x \mapsto A_t(x)$ is a cone if and only if homogeneity and convexity conditions fully analogous to (12.8) and (12.9) are satisfied. Paths in the multivalued dynamical system under consideration are defined as sequences $\{x_t\}_{t=0}^N$, $x_t \in \mathcal{X}_t$ $(N \leq \infty)$ satisfying $x_t \in A_t(x_{t-1})$ for all t. Clearly this condition is equivalent to $(x_{t-1}, x_t) \in Z_t$.

One can have in mind the following fundamental example of the probabilistic structure underlying the model. Suppose $..., s_{-1}, s_0, s_1, ...$ is a random process with values in some measurable space (s_t is the "state of the world" at time t). Let Ω be the space whose elements are sequences $\omega = (..., s_{-1}, s_0, s_1, ...)$. Denote by $\mathcal{F}$ the σ-algebra defining the product measurable structure on Ω and by P the probability measure on $\mathcal{F}$ induced by the given stochastic process. Then $\mathcal{F}_t$ $(t = 0, 1, ...)$ is defined as the σ-algebra generated by the "history" $s^t(\omega) = (..., s_{t-1}(\omega), s_t(\omega))$ of the process $..., s_{-1}, s_0, s_1, ...$ up to time t. (We write $s_t(\omega)$ for the tth element of the sequence $\omega = (..., s_{-1}, s_0, s_1, ...)$.)

In addition to the assumption that the sets Z_t are cones, we will always suppose that these sets satisfy the following condition: *if* $(x, y), (x', y') \in Z_t$ *and* $\Gamma \in \mathcal{F}_{t-1}$, *then* $\mathbf{1}_\Gamma(x, y) + (1 - \mathbf{1}_\Gamma)(x', y') \in Z_t$, where $\mathbf{1}_\Gamma$ is the indicator function equal to 1 on Γ and 0 outside Γ. If this condition holds, the cone Z_t is called $\mathcal{F}_{t-1}$-*decomposable*. This property expresses a possibility of *choice* between (x, y) and (x', y') depending on information contained in $\mathcal{F}_{t-1}$. A broad class of examples of $\mathcal{F}_{t-1}$-decomposable cones is given by

$$Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : (x(\omega), y(\omega)) \in G_t(\omega) \text{ almost surely}\} \qquad (12.49)$$

where $G_t(\omega)$ $(\omega \in \Omega)$ is an $\mathcal{F}_{t-1}$-measurable random closed cone[6] in $\mathbb{R}_+^{2n}$. Models of the form (12.49) are said to admit a *normal representation* if the cone $G_t(\omega) \subseteq \mathbb{R}_+^{2n}$ satisfies for each $\omega \in \Omega$ the following conditions:

(i) for each $a \in \mathbb{R}_+^n$, the set $\{b : (a, b) \in G_t(\omega)\}$ is non-empty;

(ii) the set $G_t(\omega)$ contains with every (a, b) all (a', b') such that $a \geq a'$ and $0 \leq b' \leq b$;

(iii) the set $G_t(\omega)$ is contained in $\{(a, b) : |b| \leq M|a|\}$ where M does not depend on t and ω.

12.4.2 Stationary Model

The stationary version of the stochastic von Neumann–Gale model is defined as follows. Suppose that, in addition to the above data, we are given a one-to-one

[6] We say that $G(\omega)$ is a $\mathcal{G}$-measurable random closed set in $\mathbb{R}^k$ if the distance from any $x \in \mathbb{R}^k$ to $G(\omega)$ is a $\mathcal{G}$-measurable function of ω.

mapping $T : \Omega \to \Omega$ (the *time shift*). The model is called *stationary* if the following invariance conditions hold:

(**Inv.1**) The mappings T and T^{-1} are $\mathcal{F}$-measurable and preserve the measure P, i.e., $P(\Gamma) = P(T^{-1}\Gamma) = P(T\Gamma)$ for each $\Gamma \in \mathcal{F}$. (If these conditions hold, the transformation T is called an *automorphism* of the probability space $(\Omega, \mathcal{F}, P)$.)

(**Inv.2**) We have

$$\mathcal{F}_{t+1} = T^{-1}\mathcal{F}_t \ (= \{T^{-1}\Gamma : \Gamma \in \mathcal{F}_t\}) \ (t = 0, 1, 2, ...). \tag{12.50}$$

(**Inv.3**) A pair of vector functions $(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t$ belongs to the set Z_t if and only if the pair $(Tx, Ty) \in \mathcal{X}_t \times \mathcal{X}_{t+1}$ belongs to the set Z_{t+1}.

Here and in what follows, the letter T is used to denote both the transformation of Ω and the induced transformation $(Tx)(\omega) = x(T\omega)$ acting on functions of $\omega \in \Omega$. The transformation T may be thought of as a shift of the time scale one unit of time forward. By virtue of (12.50), a random variable ξ is $\mathcal{F}_t$-measurable if and only if $T\xi$ is $\mathcal{F}_{t+1}$-measurable. Condition (**Inv.3**), stated in terms of the cones Z_t, can equivalently be formulated in terms of the operators $A_t(\cdot)$ as follows:

$$A_{t+1}(Tx) = TA_t(x), \ x \in \mathcal{X}_{t-1}. \tag{12.51}$$

An infinite path $\{x_t\}_{t=0}^{\infty}$ is called *balanced* if there exists a random vector $x \in \mathcal{X}_0$ and a random scalar $0 < \alpha \in L^1_\infty(1)$ such that almost surely[7]

$$x_t = \alpha_1\alpha_2...\alpha_t\bar{x}_t \text{ for all } t \geq 1, \ x_0 = x, \text{ and } |x_0| = 1, \tag{12.52}$$

where

$$\alpha_t = T^{t-1}\alpha, \text{ and } \bar{x}_t = T^t x. \tag{12.53}$$

Each component x_t^i $(i = 1, 2, ..., n)$ of a balanced path grows at the same stationary random rate determined by the growth factor $x_t^i(\omega)/x_{t-1}^i(\omega) = \alpha(T^t\omega)$ and with stationary proportions $x_t^i(\omega)/x_t^j(\omega) = x^i(T^t\omega)/x^j(T^t\omega)$. Clearly, a pair (x, α) generates a balanced path if and only if

$$x \in \mathcal{X}_0, \ |x| = 1, \ 0 < \alpha \in L^1_\infty(1), \text{ and } (x, \alpha Tx) \in Z_1. \tag{12.54}$$

A balanced path (12.52) maximizing the expected logarithmic growth rate $E \ln \alpha_t$ (independent of t in view of stationarity) is called a von *Neumann path*. This trajectory is a natural stochastic analogue of a von Neumann ray. Conditions under which it exists will be discussed in Section 12.6. In the deterministic case (when Ω consists of just one element), the above notions coincide with those introduced in Part I of the paper.

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[7] All equalities and inequalities between scalar- and vector-valued functions of ω are supposed to hold almost surely (a.s.) and coordinatewise. We will usually omit "a.s." where this does not lead to ambiguity.

Radner in his pioneering paper [51] considered a similar probabilistic notion of a von Neumann path in a stationary Markovian setting. An outline of related notions and results in a non-Markov model was given in Radner's paper [52, pp. 108-110]. The Markovian approach was further developed by Presman and Slastnikov [50], see also Belenky [12].

Suppose the model is described in terms of a process $..., s_{-1}, s_0, s_1, ...$ of "states of the world" (see **12.4.1**) and the shift operator T is defined by $s_t(T\omega) = s_{t+1}(\omega)$. Then conditions (**Inv.1**) and (**Inv.2**) are fulfilled if the process $\{s_t\}$ is stationary. Condition (**Inv.3**) holds, for example, if the cones Z_t admit a *stationary normal representation*

$$Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \ (x, y) \in G\left(s^t\right) \text{ a.s.}\}, \tag{12.55}$$

where $G(s^t)$ is a closed cone in $\mathbb{R}_+^{2n}$ given for every s^t and satisfying conditions (i)–(iii) in **12.4.1.** It is supposed here that $G(s^t)$ depends measurably on s^t and does not explicitly depend on t.

12.4.3 Rapid Paths

A path $\{x_t\}_{t=0}^N$ ($N \leq \infty$) is said to be *rapid* if there exists a sequence of nonnegative random vectors $\{p_t\}_{t=0}^N$ such that $p_t \in L_1^n(t)$,

$$p_t x_t = 1 \text{ (a.s.)} \tag{12.56}$$

for all $t \geq 0$, and

$$E\left(p_t y / p_{t-1} x\right) \leq 1 \tag{12.57}$$

for all $t \geq 1$ and all $(x, y) \in Z_t$ with $p_{t-1}x > 0$. A rapid path maximizes in each period the expected value of the *growth rate* $(p_t y_t - p_{t-1} y_{t-1})/p_{t-1} y_{t-1}$ among all paths $\{y_t\}_{t=0}^N$ for which $p_{t-1} y_{t-1} \neq 0$. Clearly the above definition generalizes its deterministic version given in **12.2.1**.

The typical interpretation of $\{p_t\}$ in economic contexts is that of prices depending on the (random) state of the economic environment. A rapid path achieves the highest expected growth rate of the aggregate value $p_t x_t$. The fact that $p_t x_t$ is supposed to be equal to 1 is just a matter of convenience; instead of the constant 1 we could take any constant, independent of time and of ω.

There are several equivalent ways to define a rapid path. It can be shown (see Evstigneev and Flåm [22, Proposition 2.2]) that if (12.56) holds, then condition (12.57), involved in the definition of a rapid path, can be replaced by any of the following requirements:

$$E \ln \left(\frac{p_t y}{p_{t-1} x} \right) \leq 0; \tag{12.58}$$

$$E p_t y \leq E p_{t-1} x; \tag{12.59}$$

$$E\left(p_t y \mid \mathcal{F}_{t-1}\right) \leq p_{t-1} x, \tag{12.60}$$

where $t \geq 1$, $(x, y) \in Z_t$ and, additionally, $p_{t-1}x \neq 0$ in (12.58). A rapid path therefore maximizes the expected logarithmic growth rate, and it maximizes the one-period expected gain in aggregate value (both in the sense of conditional and in the sense of unconditional expectation). It follows from (12.60) that, for any dual path $\{p_t\}_{t=0}^N$ and any trajectory $\{y_t\}_{t=0}^N$, the sequence $\{p_t y_t\}_{t=0}^N$ is a *supermartingale* with respect to the filtration $\{\mathcal{F}_t\}_{t=0}^N$.

A sequence $\{p_t\}_{t=0}^N$ of nonnegative random vectors is called a *dual path* if $p_t \in L_1^n(t)$ and any of the equivalent conditions (12.59) and (12.60) holds. (The equivalence of these conditions follows from the $\mathcal{F}_{t-1}$-decomposability of Z_t.) We say that a dual path $\{p_t\}_{t=0}^N$ *supports* the trajectory $\{x_t\}_{t=0}^N$ if $p_t x_t = 1$ for all t. Thus, *a trajectory* $\{x_t\}_{t=0}^N$ *is rapid if and only if there exists a dual path* $\{p_t\}_{t=0}^N$ *supporting it.*

12.5 Key Assumptions and Results in the Non-stationary Case

12.5.1 Assumptions

We introduce the assumptions on the cones Z_t (or, equivalently, on the operators $A_t(x) = \{y \in \mathcal{X}_t : (x, y) \in Z_t\}$) that will be used in the analysis of the stochastic version of the von Neumann–Gale model. When formulating results, we will specify what set of these assumptions is needed for one result or another. In hypotheses (**Z.0**)–(**Z.4**), the subscript t ranges over $\{1, 2, ...\}$. In (**Z.4**) and (**Z.5**), γ stands for some fixed strictly positive number.

(**Z.0**) If $(x, y) \in Z_t$ and λ is a $\mathcal{F}_{t-1}$-measurable random variable with nonnegative real values, then $(\lambda x, \lambda y) \in Z_t$, provided λx and λy are essentially bounded.

(**Z.1**) The set Z_t is closed in $\mathcal{X}_{t-1} \times \mathcal{X}_t$ with respect to a.s. convergence of sequences uniformly bounded in the norm $||\cdot||_\infty = \operatorname{ess\,sup} |\cdot|$.

(**Z.2**) There is a constant M such that $|y| \leq M|x|$ for any $(x, y) \in Z_t$.

(**Z.3**) If $(x, y) \in Z_t$, $x' \in \mathcal{X}_{t-1}$, $y' \in \mathcal{X}_t$, $x' \geq x$ and $y' \leq y$, then $(x, y') \in Z_t$.

(**Z.4**) For some $(\breve{x}_{t-1}, \breve{y}_t) \in Z_t$, we have $\breve{y}_t \geq \gamma e$, where $e = (1, 1, ..., 1)$.

(**Z.5**) There exists an integer $m \geq 1$ such that, for every $i \in \{1, ..., n\}$ and every $t \in \{0, 1, 2, ...\}$, one can find random vectors $y_t \in \mathcal{X}_t, ..., y_{t+m} \in \mathcal{X}_{t+m}$ satisfying

$$y_t = e_i,\ (y_t, y_{t+1}) \in Z_{t+1}, ..., (y_{t+m-1}, y_{t+m}) \in Z_{t+m},\ y_{t+m} \geq \gamma e. \quad (12.61)$$

Assumptions (**Z.1**)–(**Z.5**) are fully analogous to those we discussed in the deterministic case in **12.3.1**. Condition (**Z.0**) is a version of the hypothesis of $\mathcal{F}_{t-1}$-decomposability of the cone Z_t (note that the random variable λ involved in this condition is not necessarily bounded). Assumptions (**Z.0**) and (**Z.1**) hold if Z_t admits a normal representation (12.49).

As in the deterministic case, we will use additional requirements ensuring uniform strict convexity of the cones Z_t.

(**SC1**) For each $\epsilon > 0$, there exists a number $\rho(\epsilon) > 0$ having the following property. For any $(x, y) \in Z_t$, $(x', y') \in Z_t$ and $\Gamma \in \mathcal{F}_{t-1}$ satisfying $|x| = |x'| = 1$ and $|x - x'| \geq \epsilon$ (a.s. Γ), and there is a vector $w \in \mathcal{X}_t$ such that $(x + x', y + y' + w) \in Z_t$ and $|w| \geq \rho(\epsilon)$ (a.s. Γ).

We write "a.s. Γ" if the property indicated holds almost surely on the set Γ.

(**SC2**) For any $\epsilon > 0$, there is a number $\tau(\epsilon) > 0$ with the following property. If $(x, y) \in Z_t$, $(x', y') \in Z_t$ and $\Gamma \in \mathcal{F}_t$ satisfy $|x| = |x'| = 1$ and $|y - y'| \geq \epsilon$ (a.s. Γ), then one can find a vector $w \in \mathcal{X}_t$ for which $(x + x', y + y' + w) \in Z_t$ and $|w| \geq \tau(\epsilon)$ (a.s. Γ).

It is assumed that the numbers $\rho(\epsilon)$ and $\tau(\epsilon)$ involved in (**SC1**) and (**SC2**) do not depend on t and ω.

Consider the stochastic version of the nonlinear von Neumann model (see **12.2.2**) given by

$$Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \ \Psi_t(\omega, y) \leq \Phi_t(\omega, x)\}, \ t = 1, 2, ..., \tag{12.62}$$

where the vector functions

$$\Phi_t(\omega, a) = (\Phi_t^1(\omega, a), ..., \Phi_t^m(\omega, a)) \text{ and } \Psi_t(\omega, b) = (\Psi_t^1(\omega, a), ..., \Psi_t^m(\omega, a)),$$

defined for $a \in \mathbb{R}_+^n$ and taking values in $\mathbb{R}_+^m$, are measurable with respect to $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}_+^n)$ ($\mathcal{B}(\cdot)$ stands for the Borel σ-algebra). We suppose that, for each t, j, ω, the vector functions $\Psi_t^j(\omega, \cdot)$ and $\Phi_t^j(\omega, \cdot)$ satisfy requirements (**N.1**)–(**N.4**) stated below. To formulate these requirements, we fix $\omega \in \Omega$, $j = 1, ..., m$ and $t = 1, 2, ...$ and put, for shortness, $\Psi(b) = \Psi_t^j(\omega, b)$ and $\Phi(a) = \Phi_t^j(\omega, a)$. We define $\sum = \{x \in \mathbb{R}_+^n : |x| = 1\}$.

Hypotheses (**N.1**)–(**N.4**) are supposed to hold for all ω, j and t. The functions $\kappa(r, \epsilon)$, $\delta(\epsilon)$ and the constants c and C involved in these hypotheses do not depend on ω, j and t.

(**N.1**) The functions $\Phi(a)$ and $-\Psi(a)$ ($a \in \mathbb{R}_+^n$) are positively homogeneous of degree 1, concave and continuous; $\Phi(a)$ and $\Psi(a)$ are monotone (coordinatewise).

(**N.2**) There exists a function $\kappa(r, \epsilon) > 0$ of $r \in (0, 1/2]$ and $\epsilon \in (0, \infty)$ such that

$$\Phi(\theta a + (1 - \theta)a') - \theta\Phi(a) - (1 - \theta)\Phi(a') \geq \kappa(r, \epsilon) \tag{12.63}$$

for all $\theta \in [r, 1/2]$ and $a, a' \in \sum$ satisfying $|a - a'| \geq \epsilon$ (uniform strict concavity of Φ on Σ). The analogous condition holds for the function $-\Psi$ (uniform strict convexity of Ψ on Σ).

(**N.3**) There are constants $0 < c < C < \infty$ for which $c|a| \leq \Phi(a) \leq C|a|$, $c|a| \leq \Psi(a) \leq C|a|$ ($a \in \mathbb{R}_+^n$).

(**N.4**) There exists a function $\delta(\epsilon) > 0$ of $\epsilon > 0$ such that $|\Phi(a) - \Phi(a')| < \epsilon$ and $|\Psi(a) - \Psi(a')| < \epsilon$ when $|a - a'| < \delta(\epsilon)$, $a, a' \in \Sigma$.

Proposition 12.5.1. *Under assumptions (**N.1**)–(**N.4**), the model defined by (12.62) satisfies hypotheses (**Z.0**)–(**Z.5**), (**SC1**) and (**SC2**).*

For a proof see Evstigneev and Taksar [27, Proposition A.1].

12.5.2 Finite Rapid Paths

This section shows that finite rapid paths can be constructed by maximizing appropriately defined *logarithmic* functionals of their terminal states. Consider the class $\mathcal{U}_t$ of real-valued functions $\psi(\omega, a) \geq 0$ of $\omega \in \Omega$ and $a \in \mathbb{R}^n_+$ meeting the following requirements:

(ψ.1) For each $a \in \mathbb{R}^n_+$, the function $\psi(\cdot, a)$ is $\mathcal{F}_t$-measurable, and for each $\omega \in \Omega$, the function $\psi(\omega, \cdot)$ is continuous.

(ψ.2) For all $a, a' \in \mathbb{R}^n_+$, we have $\psi(\omega, a + a') \geq \psi(\omega, a) + \psi(\omega, a')$.

(ψ.3) If $\lambda \in [0, \infty)$ and $a \in \mathbb{R}^n_+$, then $\psi(\omega, \lambda a) = \lambda \psi(\omega, a)$.

(ψ.4) There exists a random variable $H(\omega) > 0$ such that $E|\ln H(\omega)| < \infty$ and

$$\psi(\omega, a) \leq H(\omega)|a|, \ a \in \mathbb{R}^n_+. \tag{12.64}$$

(ψ.5) There is a random vector $x_t^* \in \mathcal{X}_t$ for which $E \ln \psi(x_t^*) > -\infty$.

It is convenient to write $\psi(x_t^*)$ in place of $\psi(\omega, x_t^*(\omega))$.

Conditions (ψ.2), (ψ.3) and inequality (12.64) are supposed to hold for every $\omega \in \Omega$. From the nonnegativity of ψ and requirements (ψ.2), (ψ.3), it follows that the function $\psi(\omega, a)$, $a \in \mathbb{R}^n_+$, is concave and monotone:

$$\psi(\omega, a) \leq \psi(\omega, a') \text{ if } a \leq a'. \tag{12.65}$$

Since $\psi(\omega, a)$ is continuous in a and $\mathcal{F}_t$-measurable in ω, this function is jointly measurable in (ω, a). By virtue of (ψ.4), the expectation $E \ln \psi(x)$ is well-defined and takes values in $[-\infty, \infty)$ for any $x \in \mathcal{X}_t$. Furthermore, (ψ.3), (ψ.5) and (12.65) imply

$$E \ln \psi(x) > -\infty \text{ for any } x \in \mathcal{X}_t, \ x >> 0. \tag{12.66}$$

Here, "$x >> 0$" means that every coordinate of the random vector x is greater than some strictly positive non-random constant.

Examples of functions in $\mathcal{U}_t$ can be constructed as follows. Let $q(\omega)$ and $\kappa(\omega)$ be nonnegative $\mathcal{F}_t$-measurable random variables with values in $\mathbb{R}^n$ and $\mathbb{R}$ respectively, $q(\omega)$ being absolutely integrable. Let $\nu(a)$ be any norm in $\mathbb{R}^n$. Define $\psi(\omega, a) = q(\omega)a - \kappa(\omega)\nu(a)$. Denote by $\psi_*(\omega)$ the maximum value of $\psi(\omega, a)$ on $\{a \in R^n_+ : |a| = 1\}$. If $\psi(\omega, a) \geq 0$ for all ω, a and $E \ln \psi_*(\omega) > -\infty$, then, as is easily seen, $\psi \in \mathcal{U}_N$. In particular, the function $\psi(\omega, a) = |a|$ belongs to $\mathcal{U}_t$.

Fix some natural number N and a random vector $x_0 \in \mathcal{X}_0$, $x_0 >> 0$. Denote by $\Pi_0^N(x_0)$ the set of paths $\{y_t\}_{t=0}^N$ with $y_0 = x_0$.

Theorem 12.5.1. *Assume (**Z.0**) and (**Z.4**). Let $\{x_t\}_{t=0}^N$ be a path in $\Pi_0^N(x_0)$ and ψ a function in $\mathcal{U}_N$. Then the following assertions (a) and (b) are equivalent.*

(a) The path $\{x_t\}_{t=0}^N$ maximizes the functional

$$E \ln \psi(y_N) \tag{12.67}$$

over the set of paths $\{y_t\}_{t=0}^N$ *in* $\Pi_0^N(x_0)$.

(b) There exist random vectors $p_0, ..., p_N$ *such that* $\{p_t\}_{t=0}^N$ *is a supporting dual path for* $\{x_t\}_{t=0}^N$ *(and so the path* $\{x_t\}_{t=0}^N$ *is rapid). Furthermore,* $\psi(x_N) > 0$ *and*

$$E(\psi(y)/\psi(x_N) \mid \mathcal{F}_{N-1}) \leq p_{N-1}x \tag{12.68}$$

for all $(x, y) \in Z_N$.

By virtue of this result, one finds that if a path $\{x_t\}_{t=0}^N$ maximizes the functional (12.67), then $\{x_t\}_{t=0}^N$ is rapid. Theorem 12.5.1 does not address the question of existence of such paths. Sufficient conditions for existence are provided in the following result.

Theorem 12.5.2. *Assume* ***(Z.0)***–***(Z.2)*** *and* ***(Z.4)****. For any* $\psi \in U_N$*, the class* $\Pi_0^N(x_0)$ *contains a path maximizing functional (12.67). This path is rapid.*

Theorem 12.5.1 is proved in [22, Theorem 3.1]; for a proof of Theorem 12.5.2 see [22, Theorem 3.2] and [27, Theorem 3.1].

12.5.3 Quasi-Optimality of Infinite Rapid Paths

Infinite rapid paths possess properties of asymptotic quasi-optimality generalizing the corresponding deterministic property (no path can grow "infinitely faster" in the long run).

Theorem 12.5.3. *Let* $\{x_t\}_{t=0}^\infty$ *be an infinite rapid path and* $\{x'_t\}_{t=0}^\infty$ *any infinite path. Under conditions* ***(Z.2)*** *and* ***(Z.5)****, there exists a sequence of non-negative integrable random variables* κ_t *(depending on the paths), such that* κ_t *is a supermartingale with respect to the filtration* $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ...$ *and*

$$|x'_t| / |x_t| \leq \kappa_t, \; t = 0, 1, ... \tag{12.69}$$

This result is a consequence of [22, Proposition 2.5]. It implies the following *quasi-optimality properties* of infinite rapid paths:

$$\sup_t (|x'_t| / |x_t|) < \infty \text{ (a.s.)}; \tag{12.70}$$

$$\sup_t E(|x'_t| / |x_t|) < \infty; \tag{12.71}$$

$$\sup_t E \ln(|x'_t| / |x_t|) < \infty. \tag{12.72}$$

These properties follow from (12.69) and general results on supermartingales—see [46]. Clearly, $|\cdot|$ can be replaced in (12.69)–(12.72) by any other norm in $\mathbb{R}^n$, or by any measurable function $\psi(\omega, a)$ ($\omega \in \Omega$, $a \in \mathbb{R}^n$) such that $c|a| \leq \psi(\omega, a) \leq C|a|$, where $c > 0$ and $C > 0$ are non-random constants.

In [22, section 5], an example was provided showing that (12.70) fails to hold if we replace in the definition of a rapid path the requirement $p_t x_t = 1$ by the assumption that $\{p_t x_t\}$ is a martingale with $Ep_t x_t = 1$ and $p_t x_t > 0$. Such trajectories are not necessarily quasi-optimal in the sense of (12.70), although they maximize the expected growth rate in every period. Paths $\{y_t\}$ might exist that are "infinitely better" than $\{x_t\}$, i.e., such that $\lim_{t\to\infty}(|y_t|/|x_t|) = \infty$ (a.s.).

12.5.4 Turnpike Theorems and Infinite Rapid Paths

This section examines the qualitative behavior of rapid paths. It also establishes an existence theorem for infinite rapid paths in the general, non-stationary, case. The key results, the stochastic turnpike theorems, are analogous to their deterministic counterparts discussed in **12.3.3**.

The formulation of turnpike results requires a specification of how to measure deviations between paths. With this view, for any $x, x' \in \mathcal{X}_t$ such that $|x| > 0$ and $|x'| > 0$, we put

$$\mathbf{d}(x, x') = Ed(x, x'),\ \mathbf{D}(x, y, x', y') = ED(x, y, x', y'), \tag{12.73}$$

where d and D are the pseudometrics defined by (12.41) and (12.42). The following result holds under conditions (**Z.0**)–(**Z.5**), see [6].

Theorem 12.5.4. *All the assertions of Theorem 12.3.4 remain valid in the stochastic case (with pseudometrics* $\mathbf{d}$ *and* $\mathbf{D}$ *in place of* d *and* D*).*

In the statement of this result references to hypotheses (SC1) and (SC2) should be replaced by references to their stochastic counterparts (**SC1**) and (**SC2**).

The main existence result for infinite rapid paths regarding the general, non-stationary, setting is as follows (see [27, Theorem 2.1]).

Theorem 12.5.5. *Let hypotheses **(Z.0)**–**(Z.5)**, **(SC1)** and **(SC2)** hold. Let* x_0 *be a vector function in* $\mathcal{X}_0$ *such that* $x_0 \geq \sigma e$ *for some non-random strictly positive number* σ*. Then there exists a unique infinite rapid path with initial state* x_0*.*

The existence proof relies on Theorem 12.5.1, which ensures the existence of finite rapid paths, and on the turnpike result, Theorem 12.5.4, which is used to construct an infinite rapid path by passing to the limit from finite ones. The question whether the existence part of Theorem 12.5.5 remains valid without assumptions (**SC1**) and (**SC2**) remains open. Another interesting open problem is to obtain "almost sure" versions of turnpike theorems in the spirit of [7, Theorem V.3.2].

12.6 Stationary Models: von Neumann Equilibrium

12.6.1 Von Neumann Equilibrium

Consider the stationary stochastic analogue of the von Neumann–Gale model introduced in **12.4.2**. A central definition is as follows. A triple of nonnegative functions

$$(x, \alpha, p),\ 0 \le x \in L^n_\infty(0),\ 0 < \alpha \in L^1_\infty(1),\ 0 \le p \in L^n_1(0), \tag{12.74}$$

is said to form a *von Neumann equilibrium* if the following conditions hold:

(a) the sequence $x_t = \alpha_1 ... \alpha_t \bar{x}_t$, $x_0 = \bar{x}_0$ (where $\alpha_t = T^{t-1}\alpha$ and $\bar{x}_t = T^t x$) is a balanced path; and

(b) the sequence $p_t = (\alpha_1 ... \alpha_t)^{-1} \bar{p}_t$, $p_0 = \bar{p}_0$ (where $\bar{p}_t = T^t p$) is a dual path supporting $\{x_t\}$.

If the above requirements are met, $\{x_t\}$ is called an *equilibrium path* and $\{p_t\}$ an *equilibrium dual path.* The stationary process $\alpha_1, ..., \alpha_t, ...$ is the sequence of random *equilibrium growth factors.* Dual paths of the form described in (b) are called *balanced.*

Under the assumptions we impose on the cones Z_t, it can be shown [9, Section 3] that a triple (x, α, p) of the form (12.74) is a von Neumann equilibrium if and only if

$$(x, \alpha T x) \in Z_1,\ |x| = 1,\ px = 1,$$

and

$$E\left(\alpha^{-1}(Tp)y \mid \mathcal{F}_0\right) \le px \text{ for all } (x, y) \in Z_1.$$

According to the above definition, a von Neumann equilibrium defines a balanced path growing at a stationary rate and a balanced dual path supporting it and decreasing at the *same* rate.

12.6.2 The Existence Problem for a von Neumann Equilibrium

This problem is central to the theory under consideration. At present, two main results are obtained in this area. The first one assumes that a von Neumann path exists and establishes, based on this, the existence of a von Neumann equilibrium. To state the result denote by $\mathcal{B}$ the class of those pairs (x, α) which generate balanced paths, i.e. satisfy (12.54). Consider the variational problem:

$$(\mathcal{P}) \text{ Maximize } E \ln \alpha \text{ over all } (x, \alpha) \in \mathcal{B}.$$

Clearly the maximum in this problem is attained if and only if a von Neumann path exists. The following assertion is proved in [9, Theorem 1] under assumptions (**Z.0**)–(**Z.5**).

Theorem 12.6.1. *The following properties of* $(x, \alpha) \in \mathcal{B}$ *are equivalent.*

(a) (x, α) *is a solution to problem* $(\mathcal{P})$*.*

(b) There exists a $p \in L^n_1(0)$, $p \ge 0$, *such that* (x, α, p) *is a von Neumann equilibrium.*

Thus, under assumptions (**Z.0**)–(**Z.5**), the existence of a von Neumann path implies the existence of a von Neumann equilibrium. This result may be regarded as a stochastic analogue of Theorem 12.3.5. While the latter is proved quite easily, the proof of the former is based on advanced techniques of convex analysis in spaces of measurable functions.

What can be said about the existence of a von Neumann path (implying the existence of a von Neumann equilibrium)? Currently, the following is known.

Theorem 12.6.2. *Let conditions (**Z.0**)–(**Z.5**), (**SC1**) and (**SC2**) hold. Then the following assertions are valid.*

(a) There exists a unique von Neumann path.

(b) A von Neumann equilibrium exists. If (x, α, p) *and* (x', α', p') *are two equilibria, then* $x = x'$ *and* $\alpha = \alpha'$.

(c) There exists a unique rapid balanced path. It coincides with the von Neumann path.

We note that all the uniqueness assertions in the above theorem (and throughout the paper) are understood up to stochastic equivalence—random variables are supposed to be equivalent if they coincide a.s. The results contained in the above theorem are established in [27, Theorems 2.2, 2.3 and 3.3]. A version of Theorem 12.6.2 was obtained under more stringent assumptions in [21] (see also [23]). It is clear that some assumptions like (**SC1**) and (**SC2**) are needed for the uniqueness of a von Neumann path. The question whether they are essential for its existence remains open.

Theorem 12.6.2 has the following important consequence. Let $\{x_t\}$ be a von Neumann path. By its definition, it is optimal—in terms of the maximization of the expected logarithmic growth rate—in the class of all balanced paths. By virtue of Theorem 12.6.2, $\{x_t\}$ is rapid. Consequently, the quasi-optimality properties (12.70)–(12.72), and in particular the property $\sup_t(|x'_t| \,/\, |x_t|) < \infty$ (a.s.), hold for any path $\{x'_t\}$. This means that $\{x_t\}$ is quasi-optimal almost surely in the class of *all, not necessarily balanced*, paths.

12.6.3 Randomization

We have seen that if a von Neumann path exists, then an equilibrium can be constructed under sufficiently general assumptions—see Theorem 12.6.1. The existence of a von Neumann ray is trivial in the deterministic case (see **12.3.5**), but in the stochastic case it is currently established only under rather strong conditions **(SC1)** and **(SC2)** (Theorem 12.6.2) or in specialized models such as those we will consider in **12.6.4** and **12.7.2**. A well-known way of dealing with existence problems of this kind is to introduce randomization. A classical example is the concept of a Nash equilibrium in mixed strategies. A whole range of similar notions and related results are known in control, optimization and games. We will show that by an appropriate extension of the model at hand—by introducing an auxiliary "sunspot" process—one can establish the existence of a randomized von Neumann equilibrium under general assumptions.

Consider the stationary version of the von Neumann–Gale model described in terms of a process $..., s_{-1}, s_0, s_1, ...$ of "states of the world." Assume that this model admits a stationary normal representation (12.55) in terms of a random closed cone $G(s^t)$. Suppose that, for some non-random $\gamma > 0$, the cone $G(s^t)$ contains $(e, \gamma e)$ for all s^t.

Let us say that a stochastic process $..., \zeta_{-1}, \zeta_0, \zeta_1, ...$ with values in some measurable space is *non-anticipative* (with respect to the given process $..., s_{-1}$, s_0, $s_1, ...$) if, for each bounded measurable function $g(\zeta^t, s^t)$, we have

$$E[g(\zeta^t, s^t)|..., s_{-1}, s_0, s_1, ...] = E[g(\zeta^t, s^t)|s^t], \ t = 0, \pm 1, ..., \tag{12.75}$$

where $s^t = (..., s_{t-1}, s_t)$ and $\zeta^t = (..., \zeta_{t-1}, \zeta_t)$. Equality (12.75) means that if we wish to predict ζ^t based on information about $..., s_{-1}, s_0, s_1, ...$, then what matters is only s^t—the past and the present of the process $\{s_t\}$, the probabilistic prediction being independent of the future $s_{t+1}, s_{t+2}, ...$ of the process $\{s_t\}$. We will need a similar definition of non-anticipativity for processes $\zeta_0, \zeta_1, ...$ indexed by nonnegative integers t, rather than by all integers t. This definition is fully analogous to the previous one, with the only difference that the "history" ζ^t is defined as $(\zeta_0, ..., \zeta_{t-1}, \zeta_t)$.

Denote the given von Neumann–Gale model by $\mathbf{M}$. Let us say that a random process $\xi_0, \xi_1, ...$ with values in $\mathbb{R}^n_+$ is a *randomized path* (or a *randomized trajectory*) in $\mathbf{M}$ if $\xi_0, \xi_1, ...$ is non-anticipative, $(\xi_{t-1}, \xi_t) \in G(s^t)$ a.s. and *ess sup* $|\xi_t| < \infty$. Suppose $..., \zeta_{-1}, \zeta_0, \zeta_1, ...$ is a non-anticipative process with values in some measurable space. Define $\sigma_t = (\zeta_t, s_t)$ and put $\bar{G}(\sigma^t) = G(s^t)$. Assume that the process $..., \sigma_{-1}, \sigma_0, \sigma_1, ...$ is stationary and denote by $\bar{\mathbf{M}}$ the von Neumann–Gale model admitting a stationary normal representation in terms of the process $\{\sigma_t\}$ and the cones $\bar{G}(\sigma^t)$ $(t = 0, 1, ...)$. We will call $\bar{\mathbf{M}}$ the *extension* of the model $\mathbf{M}$ constructed by using the process $\{\sigma_t\}$—a *stationary non-anticipative extension* of the process $\{s_t\}$.

Theorem 12.6.3. *There exists an extension* $\bar{\mathbf{M}}$ *of the model* $\mathbf{M}$ *possessing a von Neumann path.*

According to this theorem, we can find a stationary non-anticipative extension $\{\sigma_t\}$ $(\sigma_t = (\zeta_t, s_t))$ of the process $\{s_t\}$ such that, in the corresponding extension $\bar{\mathbf{M}}$ of the model $\mathbf{M}$, there exists a von Neumann path. It follows from the assumptions imposed on $G(s^t)$ that both models $\mathbf{M}$ and $\bar{\mathbf{M}}$ satisfy conditions (**Z.0**)–(**Z.4**). If hypothesis (**Z.5**) holds for $\mathbf{M}$, then it also holds for $\bar{\mathbf{M}}$. Thus, under these conditions, we obtain the following result.

Theorem 12.6.4. *There exists an extension* $\bar{\mathbf{M}}$ *of the model* $\mathbf{M}$ *having a von Neumann equilibrium.*

This implies, in particular, that the von Neumann path $\{\hat{\xi}_0, \hat{\xi}_1, ...\}$ in the model $\bar{\mathbf{M}}$ is quasi-optimal (in terms of any of the properties (12.70)–(12.72)) among all paths in the model $\bar{\mathbf{M}}$. It is easily seen that any trajectory in $\bar{\mathbf{M}}$, in particular $\{\hat{\xi}_0, \hat{\xi}_1, ...\}$, is a randomized trajectory in $\mathbf{M}$. This fact combined with

(12.72) leads to the following conclusion: *for each randomized path* $\{\xi_0, \xi_1, ...\}$ *in the model* $\mathbf{M}$*, we have* $\sup(E \ln |\xi_t| - E \ln |\hat{\xi}_t|) < \infty$. In this sense, $\{\hat{\xi}_0, \hat{\xi}_1, ...\}$ is quasi-optimal in the class of all randomized paths in $\mathbf{M}$.

For a proof of Theorems 12.6.3, 12.6.4 and related results see [26].

12.6.4 Von Neumann Path, Log-Optimal Investments and the Numeraire Portfolio

We outline an example that shows links between the theory of the von Neumann–Gale model and some fundamental concepts in Finance. In this subsection, we deal with a stationary (non-randomized) setting described in **12.4.2**. As in **12.4.2**, we are given a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}$ on Ω and a transformation $T : \Omega \to \Omega$ satisfying (**Inv.1**) and (**Inv.2**).

Consider a financial market where n assets are traded without transaction costs. Short sales are not allowed (see **12.2.3**). Let $S_t(\omega) = (S_t^1(\omega), ..., S_t^n(\omega)) > 0$ be the vector of asset prices at time $t = 0, 1, ...$ Denote by $R_t(\omega) = (R_t^1(\omega), ..., R_t^n(\omega))$ the vector of asset (gross) returns: $R_t^i = S_t^i / S_{t-1}^i$. We will suppose that the sequence R_t is stationary, i.e. $R_{t+1} = TR_t$. To simplify presentation, we will assume that all R_t^i are bounded away from 0 and $+\infty$. A *self-financing trading strategy* is a sequence of *contingent portfolios* $x_0 \in \mathcal{X}_0, x_1 \in \mathcal{X}_1, ...$ satisfying $|x_t| \leq R_t x_{t-1}$. In the present context, portfolio positions are measured in terms of their market values (see (12.23) and (12.24)). Self-financing strategies are trajectories in the stationary von Neumann–Gale model given by $Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : |y| \leq R_t x\}$. We are interested in balanced paths and, in particular, in the von Neumann path. By virtue of (12.54), a pair (x, α) generates a balanced trajectory if and only if $x \in \mathcal{X}_0$, $|x| = 1$, $0 < \alpha \in L^1_\infty(1)$ and $\alpha \leq R_1 x$. The von Neumann path is determined by that (x, α) for which $\alpha = R_1 x$, where $E \ln R_1 x \geq E \ln R_1 x'$ for all $x' \in \mathcal{X}_0$ with $|x'| = 1$. Thus x is a *log-optimal portfolio* (Kelly, Breiman, Cover and others)—see Algoet and Cover [3], Hakansson and Ziemba [33], Iyengar and Cover [35] and references therein.

The existence of a log-optimal portfolio follows from the fact that the functional $F(y) = E \ln R_1 y$ is concave and continuous with respect to a.s. convergence on the set $\{y \in \mathcal{X}_0 : |y| = 1\}$ (e.g. [7, Appendix III, Theorem 5]).

By setting $p = p_0 = e$, $\alpha_t = T^{t-1}\alpha = T^{t-1}(R_1 x)$ $(t \geq 1)$ and $p_t = (\alpha_t ... \alpha_1)^{-1} e$, we obtain that $\{p_t\}$ is a balanced dual path supporting the von Neumann trajectory $\{x_t\}$ defined by $x_0 = x$, $x_t = \alpha_1 \alpha_2 ... \alpha_t \bar{x}_t$ $(t \geq 1)$, where $\bar{x}_t = T^t x$ and x is the log-optimal portfolio. Indeed, $p_t x_t = e\bar{x}_t = 1$, and we have

$$E \ln \frac{p_t y'}{p_{t-1} x'} = E \ln \frac{|y'|}{\alpha_t |x'|} \leq E \ln \frac{R_t x'}{\alpha_t |x'|} \leq E \ln \frac{R_t x}{\alpha_t} = 0 \qquad (12.76)$$

for all $(x', y') \in Z_t$ with $|x'| > 0$. The fact that $\{p_t\}$ is a dual path supporting $\{x_t\}$ follows from (12.76) and from the equivalence of (12.57) and (12.58). Consequently, (x, α, e) *is a von Neumann equilibrium.*

Finally, by applying property (12.60) to $(e_i, R_1^i e_i) \in Z_1$, we obtain that $E(R_1^i (R_1 x)^{-1} | \mathcal{F}_0) \leq 1$. If $x > 0$, the last inequality implies $E(R_1^i (R_1 x)^{-1} | \mathcal{F}_0) = 1$. This means that x is a *numeraire portfolio* in the sense of Long [39].

12.7 Stochastic Version of the Perron–Frobenius Theorem and Its Applications

12.7.1 Stochastic Perron–Frobenius Theorem

This section introduces a stochastic analogue of the Perron–Frobenius theorem for positive matrices (see Theorem C.1). The theorem provides natural stochastic analogues of an eigenvector and an eigenvalue for positive matrix cocycles (see the definition below). The results are used for the analysis of stochastic von Neumann–Gale dynamical systems of the form (12.10). Also, we consider some applications to the modeling of financial growth.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $T : \Omega \to \Omega$ its *automorphism*, i.e., a one-to-one mapping such that T and T^{-1} are measurable and preserve the measure P. Let $D(\omega)$ be a measurable function taking values in the set of nonnegative $n \times n$ matrices. Define

$$C(t, \omega) = D(T^{t-1}\omega) D(T^{t-2}\omega) ... D(\omega), \; t = 1, 2, ..., \tag{12.77}$$

and $C(0, \omega) = Id$ (the identity matrix). Then we have

$$C(t, T^s \omega) C(s, \omega) = C(t + s, \omega), \; t, s \geq 0, \tag{12.78}$$

i.e., the matrix function $C(t, \omega)$ is a *cocycle* over the dynamical system $(\Omega, \mathcal{F}, P, T)$ (see, e.g., Arnold [8]).

For a matrix $D > 0$, denote by $\kappa(D)$ the ratio of the smallest element of the matrix to its greatest element. Let the following condition hold.

(*) There is a (non-random) integer $m > 0$ for which $C(m, \omega) > 0$ and $E|\ln \kappa(C(m, \omega))| < \infty$.

Theorem 12.7.1. *There exists a measurable vector function $x(\omega) > 0$ and a measurable scalar function $\alpha(\omega) > 0$ such that*

$$\alpha(\omega) x(T\omega) = D(\omega) x(\omega), \;\; |x(\omega)| = 1 \;\; (a.s.). \tag{12.79}$$

The pair of functions $(\alpha(\cdot), x(\cdot)) > 0$ satisfying (12.79) is determined uniquely up to the equivalence with respect to the measure P. If $t \to \infty$, then

$$\frac{C(t, T^{-t}\omega) a}{|C(t, T^{-t}\omega) a|} \to x(\omega) \; (a.s.), \tag{12.80}$$

where convergence is uniform in $a \geq 0$, $a \neq 0$.

The above result may be regarded as a generalization of the Perron–Frobenius theorem on eigenvalues and eigenvectors of positive matrices: $x(\cdot)$ and $\alpha(\cdot)$ play the roles of an “eigenvector” and an “eigenvalue” of the cocycle $C(t,\omega)$. Theorem 12.7.1 is a special case of a result in Evstigneev [20, Theorem 1]; see also Arnold, Gundlach and Demetrius [10, Theorem 3.1].

Remark 12.7.1. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be sub-σ-algebras of $\mathcal{F}$ such that the random matrices $D(T^{-1}\omega), D(T^{-2}\omega), ...$ are $\mathcal{F}_0$-measurable and the random matrices $D(\omega), D(T^{-1}\omega), ...$ are $\mathcal{F}_1$-measurable. By virtue of (12.77) and (12.80), the functions $x(\cdot)$ and $\alpha(\cdot)$ are measurable with respect to the σ-algebras $\mathcal{F}_0$ and $\mathcal{F}_1$ completed by all sets of measure zero. From this it follows that we can select versions of $x(\cdot)$ and $\alpha(\cdot)$ satisfying (12.79) which are $\mathcal{F}_0$- resp. $\mathcal{F}_1$-measurable.

12.7.2 Von Neumann–Gale Systems Defined by Positive Random Matrices

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}$ an increasing sequence of σ-algebras and $T : \Omega \to \Omega$ a one-to-one mapping satisfying (**Inv.1**) and (**Inv 2**). For each $t = 0, 1, ...$, let $D_t(\omega)$ be a nonnegative random $n \times n$ matrix measurable with respect to $\mathcal{F}_t$. Define

$$Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \ y \leq D_t(\omega)x\}, \tag{12.81}$$

and assume that $D_t(T\omega) = D_{t+1}(\omega)$ for all $t \geq 0$. Then (**Inv.3**) holds, and we are in the framework of a stationary von Neumann–Gale model described in **12.4.2**. The model at hand does not satisfy the strict convexity assumptions discussed in **12.5.1**, and so Theorem 12.6.2 is not applicable. Therefore the question of existence of a von Neumann equilibrium is examined by different means—by using Theorem 12.7.1.

Put $D(\omega) = D_1(\omega)$ and suppose that $D(\omega)$ is uniformly bounded and there exists a constant $\gamma > 0$ such that for some $\check{x}_0 \in \mathcal{X}_0$, we have $D\check{x}_0 \geq \gamma e$. Assume that for some $m \geq 1$, the smallest element of the matrix $C(m, \omega)$ (see (12.77)) is greater than γ. Then conditions (**Z.0**)–(**Z.5**) are satisfied and the following theorem is valid.

Theorem 12.7.2. *The model (12.81) possesses a unique von Neumann equilibrium* (x, α, p)*, where* $0 < x \in L^n_\infty(0)$ *and* $0 < \alpha \in L^1_\infty(1)$ *are the solutions to (12.79) and* $0 < p \in L^n_1(0)$ *is the (unique) solution to*

$$E(\alpha^{-1}(Tp)D|\mathcal{F}_0) = p, \ px = 1. \tag{12.82}$$

This result is a consequence of Theorem 12.7.1; for details of the proof see [26].

12.7.3 Volatility-Induced Financial Growth

We present an application of the stochastic Perron–Frobenius theorem to the analysis of the long-run performance of fixed-mix investment strategies in an asset market where prices (after a proper detrending if necessary) fluctuate as stationary stochastic processes. Let $\beta = (\beta_{kj})$ be a matrix with strictly positive non-random elements satisfying $\sum_{k=1}^{n} \beta_{kj} = 1$ for each j. A *fixed-mix* trading strategy $\{h_t\}_{t=0}^{\infty}$ determined by the matrix β (or, for shortness, *β-strategy*) in a financial market with n assets is defined by

$$S_t^k h_t^k = \sum_{j=1}^{n} \beta_{kj} S_t^j h_{t-1}^j, \tag{12.83}$$

where $S_t^j = S_t^j(\omega) > 0$ is the price of asset j at time $t = 0, 1, 2, ...$ and h_t^k is the amount of asset k in the portfolio h_t. The number $\beta_{kj} > 0$ is the share of wealth transferred from the jth position of the portfolio to the kth position $(k, j = 1, ..., n)$. If the assets are currencies, the matrix (β_{kj}) specifies the strategy of currency exchange—see [16].

We analyze a *stationary* market assuming that $S_{t+1}^j(\omega) = S_t^j(T\omega)$, where the operator T is an ergodic[8] automorphism of the underlying probability space $(\Omega, \mathcal{F}, P)$. The price process $\{S_t\}$ is adapted to the given filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}$ satisfying (**Inv.2**). Further, we assume that the prices $S_t^j(\omega)$ are uniformly bounded away from zero and infinity. The evolution of a portfolio that is dynamically rebalanced according to a β-strategy $\{h_t\}$ can be described by products of positive random matrices. Define $D = D(\omega) = (d^{kj}(\omega))$, where $d^{kj}(\omega) = \beta_{kj} S_1^j(\omega)/S_1^k(\omega)$. Then

$$h_t(\omega) = D(T^{t-1}\omega)D(T^{t-2}\omega)...D(\omega)h_0(\omega),\ t = 1, 2, ...$$

To analyze the asymptotic performance of fixed-mix strategies we use the concept of a balanced strategy. An investment strategy $\{h_t\}$ is called *balanced*, if $h_t(\omega) = \alpha(T^{t-1}\omega)...\alpha(\omega)h(T^t\omega)$ (a.s.), $t = 1, 2, ...$, where $h \in \mathcal{X}_0$, $|h| = 1$, $0 < \alpha \in L_\infty^1(1)$. Functions $h \in \mathcal{X}_0$, $0 < \alpha \in L_\infty^1(1)$ generate a balanced β-strategy if and only if

$$\alpha(\omega)h(T\omega) = D(\omega)h(\omega),\ |h(\omega)| = 1 \text{ (a.s.)}. \tag{12.84}$$

The existence and uniqueness of a solution (h, α) to (12.84) follows from Theorem 12.7.1. The corresponding balanced β-strategy is the von Neumann path in the model (12.81), where $D_t = T^{t-1}D$.
We impose the following mild non-degeneracy condition on the price processes $\{S_t^j\}$.

(**ND**) With positive probability, the ratios $S_t^j(\omega)/S_{t-1}^j(\omega)$ are not constant with respect to j.

[8] Ergodicity means that the averages $[\xi(\omega) + ... + \xi(T^t\omega)]/t$ converge a.s. to $E\xi$ for each random variable ξ with $E|\xi| < \infty$.

Theorem 12.7.3. *For any β-strategy with initial portfolio $h_0 \in \mathcal{X}_0$ such that $|h_0(\omega)| > 0$, we have*

$$\lim_{t\to\infty} t^{-1} \ln h_t^k = \lim_{t\to\infty} t^{-1} \ln S_t h_t = E \ln \alpha(\omega) > 0 \ \ (a.s.), \tag{12.85}$$

for each $k = 1, 2, ..., n$.

In (12.85), $\alpha(\omega)$ is the stochastic eigenvalue of $D(\cdot)$, whose existence follows from Theorem 12.7.1. Under the assumptions imposed, we show that $E \ln \alpha(\omega) > 0$. Thus (12.85) implies that the portfolio process $\{h_t\}$ grows in the long run almost surely in every coordinate at an exponential rate! This result might seem counterintuitive at first glance because a fixed mix-strategy is self-financing and the asset prices S_t^j form stationary processes. For establishing the above fact, we use the assumption of non-degeneracy (**ND**) requiring some randomness, or volatility, of the price process. If this assumption is violated, then the market is essentially deterministic and the result ceases to hold. Thus, in the present context, the price volatility may be viewed as an endogenous source of financial growth. For an analysis of this phenomenon and proofs of Theorem 12.7.3 and related results see [16], [17] and [25]. In the papers cited, more general models than the one considered here are examined. In particular the following generalizations are discussed: (a) asset returns, rather than asset prices, are stationary; (b) the price processes can be decomposed into a stationary component and a trend; (c) small transaction costs are present.

12.8 Asset Pricing and Hedging

12.8.1 Model Description

We describe a model of a financial market aimed at asset pricing and hedging under transaction costs and trading constraints. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_N = \mathcal{F}$ $(1 \le N < \infty)$ be a sequence of algebras of subsets of Ω. It is supposed that $\mathcal{F}_t$ contains events observable prior to time t. To simplify presentation we will assume in this section that the set Ω is finite (therefore we speak of algebras rather than σ-algebras) and $P(\{\omega\}) > 0$ for each $\omega \in \Omega$. We will also suppose that $\mathcal{F}_0$ is trivial, i.e. $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and that $\mathcal{F}$ contains all subsets of Ω.

One may think that there is a finite set of "states of the world," and at each time $t = 1, 2, ..., N$, any of these states can be realized. The state of the world which is realized at time t is denoted by s_t. A sequence $\omega = (s_1, ..., s_N)$ is called a *history* (*scenario*) of the market over the time period $1, 2, ..., N$. For each $t = 1, 2, ..., N-1$, a sequence $s^t = (s_1, ..., s_t)$ is called a *partial history* or *partial scenario* (up to time t). We denote by $\mathcal{F}_t$ the algebra of subsets of Ω generated by s^t. This algebra contains information observable prior to time t; functions measurable with respect to it depend on s^t. The algebra $\mathcal{F}_0$ is defined as the trivial one ($\mathcal{F}_0$-measurable functions are constants).

Trading on the market is possible at any of the dates $t = 0, 1, ..., N$. At time t, n_t *assets* (*securities*) $i = 1, 2, ..., n_t$ are traded. A *portfolio* of assets at time 0 is a vector $h_0 \in \mathbb{R}^{n_0}$. A *(contingent) portfolio* of assets at time $t = 1, 2, ..., N$ is a vector function

$$h_t(\omega) = (h_t^1(\omega), ..., h_t^{n_t}(\omega)) \tag{12.86}$$

of dimension n_t measurable with respect to $\mathcal{F}_t$ (i.e., depending on s^t in terms of the interpretation involving states of the world). The coordinate h_t^i of the vector h_t stands for the number of units of asset i in the portfolio h_t. We will assume in this context that admissible portfolios are nonnegative vectors, which means that *short sales* are prohibited. For a more general model, allowing short sales, see [28]. The set of all contingent portfolios (12.86), i.e., the cone of nonnegative $\mathcal{F}_t$-measurable vector functions of ω with values in $\mathbb{R}^{n_t}$, will be denoted by $\mathcal{X}_t$ $(t = 1, 2, ..., N)$. For $t = 0$, we will write $\mathcal{X}_0 = \mathbb{R}_+^{n_0}$.

Let m_0 and m_N be two natural numbers. Put $\mathcal{V}_0 = \mathbb{R}_+^{m_0}$ and denote by $\mathcal{V}_N$ the cone of all nonnegative m_N-dimensional vector functions of ω. Elements of $\mathcal{V}_0$ (m_0-dimensional non-random vectors) are interpreted as *initial endowments* and elements of $\mathcal{V}_N$ (vector functions depending on the market history ω) are construed as *contingent claims*. Generally, both initial endowments and contingent claims can be vectors, rather than scalars, which is the case, for example, when there are several currencies in the market under consideration. An important special case is $m_0 = m_N = 1$; in this case, initial endowments and contingent claims are measured in terms of a single currency (cash).

In the model, a sequence of cones

$$K_t \subseteq \mathcal{X}_{t-1} \times \mathcal{X}_t,\ t = 1, 2, ..., N, \tag{12.87}$$

describing portfolio *rebalancing constraints* is given. Elements of K_t are pairs (h_{t-1}, h_t) of contingent portfolios such that h_t can be obtained at time t by *rebalancing* h_{t-1} without requiring external funds. When rebalancing, one can buy new assets for the portfolio h_t only at the expense of selling some assets contained in h_{t-1}. All transactions, such as buying and selling assets, involve transaction costs. The model at hand allows to take into account *proportional* transaction costs, which is expressed by the assumption that the sets K_t are cones. Sequences $\{h_0, ..., h_N\}$ such that $(h_{t-1}, h_t) \in K_t$ are *feasible* (self-financing) *trading strategies.*

Further, in the model we are given two cones

$$W_0 \subseteq \mathcal{V}_0 \times \mathcal{X}_0 \text{ and } W_N \subseteq \mathcal{X}_N \times \mathcal{V}_N.$$

The cone W_0 describes possibilities of *constructing an initial portfolio* h_0 starting from some initial endowment $v_0 \in \mathcal{V}_0$. It is supposed that an investor with initial endowment v_0 can construct a portfolio h_0 at time 0 if and only if $(v_0, h_0) \in W_0$. The cone W_N describes possibilities of *portfolio liquidation* and *hedging contingent claims.* Given a contingent claim v_N, an investor with contingent portfolio h_N at time N can hedge v_N by liquidating the portfolio h_N if and only if $(v_N, h_N) \in W_N$.

Let S_0 be a (non-random) vector in $\mathcal{X}_0$ and let $S_1(\cdot) \in \mathcal{X}_1, ..., S_N(\cdot) \in \mathcal{X}_N$ be a sequence of vectors $S_t = S_t(\omega)$ specifying asset prices at times $t = 0, 1, ..., N$. In the case of a frictionless market with $m_0 = 1$, we have $W_0 = \{(v_0, h_0) \in \mathcal{V}_0 \times \mathcal{X}_0 : S_0 h_0 \leq v_0\}$. This means that an investor can construct those and only those portfolios of assets at time 0 whose values, expressed in terms of the price vector S_0, do not exceed the initial endowment v_0. As long as the market is frictionless and $m_N = 1$, then $W_N = \{(v_N, h_N) : S_N h_N \geq v_N\}$. (Recall that we write $\geq$ or $\leq$ between two vector functions of ω if the corresponding inequality holds for each ω and coordinatewise.) Finally, if there are no transaction costs (and short sales are not allowed) then $K_t = \{(h_{t-1}, h_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : S_t h_t \leq S_t h_{t-1}\}$, i.e., the cone K_t is defined by the condition of self-financing. An example of rebalancing constraints under proportional transaction costs is given by (12.26). Further examples will be considered later in this section.

12.8.2 Hedging Problem and Duality

The general question we are going to consider is as follows. Suppose a contingent claim $v_N \in \mathcal{V}_N$ is given. How can we characterize the set of initial endowments $v_0 \in \mathcal{V}_0$ which enable an investor, by trading in the financial market, to obtain v_N at time N? If this is possible, we say that the initial endowment v_0 is sufficient for hedging the contingent claim v_N. Specifically, hedging is performed as follows. At time 0, the investor possessing the initial endowment v_0 constructs a portfolio h_0 satisfying $(v_0, h_0) \in W_0$. Then he/she follows some self-financing trading strategy $\{h_0, h_1, ..., h_N\}$ over the course of time. At time N, the portfolio h_N is liquidated, allowing to hedge the contingent claim. The latter operation is possible when $(h_N, v_N) \in W_N$. The above procedure is specified by a sequence $\{v_0, h_0, ..., h_N, v_N\}$ (where $v_t \in \mathcal{V}_t, t = 0, N$ and $h_t \in \mathcal{X}_t, t = 0, ..., N$) satisfying $(v_0, h_0) \in W_0$, $(h_{t-1}, h_t) \in K_t$ $(t = 1, ..., N)$ and $(h_N, v_N) \in W_N$. Such sequences will be called (feasible) *hedging programs*. Thus the problem we are interested in can equivalently be stated as follows: *how can we characterize the set $\mathcal{Z}$ of those pairs $(v_0, v_N) \in \mathcal{V}_0 \times \mathcal{V}_N$ for which there exists a hedging program of the form $\{v_0, h_0, ..., h_N, v_N\}$?*

If initial endowments $v_0 \in \mathcal{V}_0$ are scalars and if the set $\{v_0 : (v_0, v_N) \in \mathcal{Z}\}$ contains the smallest element, this element—the minimum initial endowment needed to hedge the contingent claim v_N—is called the *hedging price* of the contingent claim v_N.

To examine the hedging problem we use the framework of a stochastic von Neumann–Gale model. We observe that hedging programs may be regarded as paths (of length $N + 2$) in the stochastic von Neumann–Gale model defined by the sequence of cones $W_0, K_1, ..., K_N, W_N$. We will give an answer to the question about the characterization of the set $\mathcal{Z}$ in terms of dual paths in this model. In the current context, dual paths correspond to *consistent price systems* (cf., e.g., [55]). A consistent price system is defined as a sequence

$$q_0 \in \mathcal{V}_0, \, p_0 \in \mathcal{X}_0, ..., p_N \in \mathcal{X}_N, \, q_N \in \mathcal{V}_N$$

such that $q_0 v_0 \geq p_0 h_0$ for all $(v_0, h_0) \in W_0$, $Ep_{t-1}h_{t-1} \geq Ep_t h_t$ for all $(h_{t-1}, h_t) \in K_t$ and $Eq_N h_N \geq Eq_N v_N$ for all $(h_N, v_N) \in W_N$. We say that $\{q_0, p_0, ..., p_N, q_N\}$ is a *strictly consistent price system* if $q_N > 0$. Under general assumptions, the last inequality implies the strict positivity of all the components q_t $(t = 0, N)$ and p_t $(t = 0, ..., N)$—see Remark 12.8.1 below.

Suppose that all the cones K_t, W_t satisfy conditions (**Z.0**)–(**Z.4**). Note that convergence a.s. mentioned in (**Z.1**) is equivalent to convergence for each ω because $P(\{\omega\}) > 0$ for all ω. A central result of this section, providing a solution to the hedging problem in terms of strictly consistent price systems, is as follows.

Theorem 12.8.1. *An initial endowment v_0 is sufficient for hedging a contingent claim v_N if and only if $q_0 v_0 \geq Eq_N v_N$ for all strictly consistent price systems $\{q_0, p_0, ..., p_N, q_N\}$.*

Proof. Elements of the cones K_t and W_t are pairs of vector functions of ω, and since Ω is finite, these functions can be identified with finite-dimensional vectors. Thus the model can be reduced to a deterministic one (with portfolio dimensions depending on t—see Remark 12.3.3). In view of this, Theorem 12.8.1 follows directly from its deterministic counterpart, Theorem 12.3.7. □

Remark 12.8.1. In the financial context, it is quite natural to assume that $(e_i, \gamma e) \in K_t$ for some $\gamma > 0$ (one unit of any asset can be exchanged to some positive amounts of all the assets). Suppose this condition holds and assume that the analogous condition holds for W_t. By using these assumptions and (**Z.0**), it can easily be shown that all the components $q_0, p_0, ..., p_N, q_N$ in a strictly consistent price system are strictly positive.

Remark 12.8.2. The existence of consistent price systems is a consequence of the *no-arbitrage hypothesis*, which holds in the present context by virtue of condition (**Z.2**). For general no-arbitrage criteria in terms of consistent price systems and related notions see [28], [37] and [55].

12.8.3 An Example: A Currency Market Without Short Sales

Consider a financial market in which n currencies $i = 1, 2, ..., n$ are traded. For each $t = 1, 2, ..., N$, we are given an $\mathcal{F}_t$-measurable $n \times n$ matrix $(\mu_t^{ij}(\omega))$ with $\mu_t^{ij} > 0$ and $\mu_t^{ii} = 1$. The numbers μ_t^{ij} represent the *exchange rates* of the currencies (including proportional transaction costs). For one unit of currency j, at time t, one can obtain μ_t^{ij} units of currency i. Admissible portfolios h_t at time $t = 0, ..., N$ are $\mathcal{F}_t$-measurable vector functions $h_t(\omega)$ with values in $\mathbb{R}^n_+$. A portfolio of currencies $h_{t-1} = (h_{t-1}^1, ..., h_{t-1}^n)$ can be exchanged to a portfolio $h_t = (h_t^1, ..., h_t^n)$ at time t in a random situation ω if and only if there exists a nonnegative $\mathcal{F}_t$-measurable $n \times n$ matrix $(d_t^{ji}(\omega))$ (an *exchange matrix*) such that

$$h_{t-1}^i(\omega) \geq \sum_{j=1}^n d_t^{ji}(\omega),\ 0 \leq h_t^i(\omega) \leq \sum_{j=1}^n \mu_t^{ij}(\omega) d_t^{ij}(\omega). \tag{12.88}$$

The set of all such portfolio pairs (h_{t-1}, h_t) will be denoted by K_t. Here, d_t^{ji} $(i \neq j)$ stands for the amount of currency i converted into currency j. The amount d_t^{ii} of currency i is left unexchanged. The first inequality in (12.88) is a balance constraint for the currency i: one cannot exchange more of it than one has at time $t-1$ (no borrowing is allowed). The second inequality in (12.88) says that, at time t, the ith position of the portfolio cannot be greater than the sum $\sum_{j=1}^l \mu_t^{ij} d_t^{ij}$ obtained as a result of the exchange.

We define $m_0 = m_N = n$, $\mathcal{V}_t = \mathcal{X}_t$ $(t = 0, N)$, $W_0 = \{(v_0, h_0) \in \mathcal{X}_0 \times \mathcal{X}_0 : v_0 \geq h_0\}$ and $W_N = \{(h_N, v_N) \in \mathcal{X}_N \times \mathcal{X}_N : h_N \geq v_N\}$.

The following theorem contains results regarding the currency market model.

Theorem 12.8.2. *In the model under consideration, strictly consistent price systems are sequences* $\{q_0, p_0, ..., p_N, q_N\}$ *such that* $q_N > 0$, $q_0 \geq p_0$, $p_N \geq q_N$ *and* $p_0, ..., p_N$ $(p_t \in \mathcal{X}_t)$ *is a strictly positive supermartingale satisfying the following condition:*

(π) For every $t = 1, 2, ..., n$, *there exists a strictly positive* $\mathcal{F}_t$*-measurable vector function* $\pi_t(\omega)$ *such that*

$$\mu_t^{ij} p_t^i \leq \pi_t^j,\ t = 1, 2, ..., N,\ i, j = 1, ..., n,$$

and $E(\pi_t \mid \mathcal{F}_{t-1}) \leq p_{t-1}$.

A contingent claim $v_N \in \mathcal{V}_N$ *can be hedged starting from an initial endowment* $v_0 \in \mathcal{V}_0$ *if and only if* $Ep_0 v \geq Ep_N w$ *for any strictly positive supermartingale* $p_0, ..., p_N$ *satisfying condition (π).*

The model considered in this section is a version of the one proposed by Kabanov and Stricker [36], [37] and [38] (in the Kabanov–Stricker setting, short sales and borrowing are allowed). The model we deal with and its versions are examined in detail in [28], where a proof of Theorem 12.8.2 is given. The paper [28] deals with a more general framework, where portfolio positions are not supposed to be necessarily positive (short sales are not ruled out). The paper analyzes various examples that can be included into that setting, in particular, examples where transaction costs are specified by (12.26). Although the framework in [28] is more general, the approach and the structure of the results are similar to those in the present survey, a key role being played by the von Neumann–Gale model.

Appendix

A The Kuhn–Tucker Theorem

Let X be a convex set in $\mathbb{R}^m$, $F(x)$ a real-valued concave function on X, and $G(x)$ a mapping of X into $\mathbb{R}^k$ such that each coordinate $G^i(x)$ $(i = 1, ..., k)$ of the vector $G(x)$ is a concave function. Consider the maximization problem:

(M) Maximize $F(x)$ on the set X subject to $G(x) \geq 0$.

Assume that the following condition holds (*Slater's constraint qualification*):

(S) There is $\check{x} \in X$ such that $G(\check{x}) > 0$.

Theorem A.1. *Let $\bar{x}$ be a vector in X satisfying $G(\bar{x}) \geq 0$. Then the following assertions are equivalent. (i) The vector $\bar{x}$ is a solution to problem (M). (ii) There is a vector $p \in \mathbb{R}^k_+$ such that*

$$F(x) + pG(x) \leq F(\bar{x})$$

for all $x \in X$.

For a proof of the above theorem under assumption (S), see, e.g., [40, Theorem 8.3.1]. Condition (S) is not needed if the functions $F(x)$ and $G^i(x)$ $(i = 1, ..., k)$ are affine and the set X is polyhedral. This follows from the duality theory for linear programming (e.g., [47, Section II.9]).

B Cones and Separation Theorems

A cone X in $\mathbb{R}^n$ is called *proper* if $X \cap (-X) = \{0\}$.

Theorem B.1. *Let X and Y be closed cones in $\mathbb{R}^n$ such that $X \cap (-Y) = \{0\}$. Then $X + Y$ is a closed cone. If, additionally, X and Y are proper, then $X + Y$ is proper.*

This theorem is a consequence of [54, Corollary 9.1.3]. The next result can be deduced from [54, Corollary 11.4.2].

Theorem B.2. *Let X and Y be closed cones in $\mathbb{R}^n$ such that $X \cap Y = \{0\}$. Let Y be proper. Then there exists $l \in \mathbb{R}^n$ such that $lx \leq 0$ for all $x \in X$ and $ly > 0$ for all nonzero $y \in Y$.*

C Positive Matrices

Let D be an $n \times n$ matrix with nonnegative elements.

Theorem C.1. *Let the matrix D^m be strictly positive for some $m \geq 1$. Then there exists a unique vector $\bar{x} \in \mathbb{R}^n_+$ such that*

$$\lambda \bar{x} = D\bar{x}, \ \ |\bar{x}| = 1,$$

for some $\lambda > 0$.

The vector $\bar{x}$ (resp. the number λ) is called the *Perron–Frobenius eigenvector* (resp. *eigenvalue*) of the matrix D. This result constitutes the key content of the Perron–Frobenius theorem. For its proof and a proof of the following fact see, e.g., [47, Sections II.7 and II.8].

Theorem C.2. *Let $\bar{x}$ be the vector described in Theorem C.1. Then $\bar{x} > 0$ and for any $x \in \mathbb{R}^n_+$, $x \neq 0$, we have*

$$\lim_{k \to \infty} \frac{D^k x}{|D^k x|} = \bar{x}.$$

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13. Equilibrium Dynamics with Many Agents

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13.1 Introduction: Ramsey's Steady State Conjecture

Frank Ramsey's seminal article [90] on optimal capital accumulation is now widely regarded as the foundation of macroeconomic dynamics.[1] The original model is cast as an optimal saving problem to be solved by an omniscient central planner acting over an infinite horizon to maximize discounted utility subject to the economy's resource constraints and given initial endowment of capital goods. Ramsey's planner operates in a deterministic world. Modern researchers have reinterpreted this model as one of intertemporal equilibrium.[2] An infinitely-lived representative consumer takes the place of Ramsey's central planner. This household is assumed to maximize lifetime discounted utility over the infinite horizon with perfect foresight regarding the time paths of all relevant prices. Equilibrium profiles satisfy a materials balance constraint at each time so that demand for goods equals their supply at each instant and in addition, the production sector's profits are maximized.[3]

Ramsey did not focus on an equilibrium interpretation of his optimal saving-accumulation framework. However, in the latter part of his paper he did formu-

[1] Ramsey begins his paper with a detailed analysis of an optimal saving-accumulation problem when the planner does **not** discount future utilities. He criticizes the idea of the **planner** discounting future utilities as one of a failure of imagination. However, he also articulates a fully developed optimal saving-accumulation model for the discounted case and it is this version of the theory that has proven so useful in modern macroeconomics. Modern advanced macro texts emphasizing the Ramsey model include Azariadis [3], Farmer [45], and Ljungqvist and Sargent [69].

Ramsey's contributions to mathematics and philosophy are found in [91]. A presentation and assessment of his work in those areas as well as economics can be read in [94]. Both references include biographies of Ramsey's short life.

[2] The connections between Ramsey's theory and alternative representations of intertemporal equilibrium for representative agent models are detailed in [8].

[3] It turns out in the specifications studied in this chapter that present value profits are maximized if and only if current value profits are maximized at each time.

late a model of stationary equilibrium — one with all variables constant over time. That model involved several types of infinitely lived households differentiated by their fixed rates of time preference. He conjectured that the model's solution would take the form of having the most patient consumer enjoying the largest sustainable consumption and in possession of the economy's capital stock while the remaining households consumed only at the minimum level necessary to sustain their lives.[4] This two-class solution suggested a very uneven distribution of consumption and wealth in a stationary state — a distribution entirely driven by the economy's fundamental taste and technology parameters.

Ramsey did not spell out the details of the equilibrium model or exactly what is meant by an equilibrium in his two-class theory. The purpose of this chapter will be to survey one interpretation of Ramsey's multi-agent model, solve for the steady state distribution and examine the models' dynamics within well-specified theories of intertemporal equilibrium.[5] The resulting analysis will show that there are fundamental differences between the dynamics of the representative agent model and one with heterogenous households. Not only will the long-run distribution of income and wealth differ from the representative agent outcome, but so will the dynamics. Indeed, the convergence of the economy to the long-run steady state from arbitrary initial conditions characteristic of Ramsey's optimal accumulation – representative agent equilibrium model will only hold for **some** specifications of preferences and technology in the multi-agent setup. Complicated dynamics at the aggregate level can arise even with very unequal income and wealth distributions evolving over time. The dynamic properties of the heterogenous agent story are thus richer than those of the representative agent model even when the aggregate economic variables tend over time to their long-run steady state values.

13.2 Impatience and the Distribution of Wealth

Ramsey's conjecture that with households having different rates of impatience, the steady state equilibrium would have very unequal income and wealth distributions was not a particularly new idea at the time his paper was published. The notion that time preference differences operating in a market economy might promote long-run differences in income and wealth can be found in the writings of such eminent economists as John Rae in 1834 [89] and in several books by Irving Fisher beginning with his great work on the rate of interest first published in 1907 [46]. This literature is reviewed next along with Stiglitz's [101] "descriptive" model of wealth distribution. His framework does not specify explicit maximizing behavior for the consumption-saving decisions undertaken

[4] Ramsey's savers could achieve a state of *bliss* either by holding the maximum possible capital stock — *capital saturation*, or by consuming at a level giving rise to *utility saturation*.

[5] This chapter reviews only discrete time models. It also omits extensions to international trade as found in [4], [104], and [107].

by the economy's actors. However, it is useful to review his basic results and comment on how they change when optimizing behavior is imposed on the model.

13.2.1 Rae's and Fisher's Time Preference Theories

Formalization of the idea of a consumer's rate of time preference or rate of impatience begins to take shape in John Rae's *New Principles of Political Economy* [89] published in 1834.[6] He was interested in understanding the accumulation of wealth in a society and what drove some individuals to save and others to dissave. He argued that there were differences in the strength of the desire to accumulate among the members of a society. He states that members of a society whose desire to accumulate is smaller than the society's average are gradually reduced to poverty.[7] People with an above average desire to accumulate gradually acquire property. Funds are redistributed from the more impatient consumers to the less impatient ones.

Rae's ideas about the strength or desire to accumulate are formalized in Irving Fisher's monumental writings on the theory of interest. There, more precise notions of time preference are developed and the nature of market interactions between consumers are spelled out in detail. It is clear that Fisher intended his work to apply to the question of distribution of capital and income possessed by the different members in a society.[8] He argues that individuals with relatively high rates of impatience will transfer their wealth to less impatient consumers through capital market transactions. The relatively impatient people borrow for current consumption and draw down their capital while the more patient ones defer some of their consumption and accumulate capital. For Fisher, this process, once started, is both gradual and irreversible. The spendthrift reduces his capital to that which is only represented by his own person (i.e. his labor). It cannot be stressed enough that in Fisher's theory it is access to markets for loans that drives the redistribution process in a population differentiated by their underlying rates of impatience. This point is reenforced in his later

[6] A detailed summary of the classical writings on time preference by Rae and Fisher (among others) can be found in [49]. That article also offers a broader assessment and critique of discounting in intertemporal models.

[7] See [89], Chapter 9 (in particular, page 199).

[8] This goal is clearly stated in his 1907 book, *The Rate of Interest* [46]. Chapter 12 is devoted to the role of interest in economic theory. Fisher argues that the functional distribution of income separated into the categories interest, rent, wages, and profits as understood in his time lead to important misunderstandings regarding the meaning of interest. Throughout Fisher's writings on interest and capital there is the theme that a society's income stream is the fundamental object which leads, through the rate of interest, to its capitalization. Hence, interest is not a part, but the whole of income in his view. It includes wages, profits, and rentals. Rejecting the traditional view of interest held in his time, Fisher refocuses on questions of personal income distribution. This leads him to his time preference based ideas surveyed in this chapter.

revision published as *The Theory of Interest* [48] in 1930.[9] It is important to note that Fisher also thought there would be factors working against the redistribution of income and wealth (e.g. habit formation). However, Fisher clearly argues that differences in rates of time preference are the driving force in redistribution over time even in the presence of dampening factors.

Fisher clearly thought the instability of an equal distribution of income and wealth owing to time preference differences across individuals was a fundamental theoretical result. He even included it in his *Elementary Principles of Economics* [47], published in 1912.[10] Differences in individuals' thriftiness would undo any attempt to establish an equal distribution of income and wealth so long as consumers had access to capital markets where tomorrow's income can be exchanged for today's income and vice versa. The hypothesis that time preference differences promote unequal distributions of income and wealth is a recurring and basic theme in the classical writings on interest, income, and capital theory.

Whether Ramsey was aware of these precursors or not, is in the end, irrelevant. His classic paper proposed a model of savings and accumulation for a representative agent and hinted strongly at what form it might take when it is extended to an economy of many agents. As already noted, I will offer a modern formulation of Ramsey's model where differences in rates of impatience drive the long-run unequal distribution of income and wealth. The analysis also shows us why an equal distribution of income is unsustainable in such economies. Thus, the model developed in this chapter lays out a consistent framework in which to address the classical questions about distribution first raised by Rae, Fisher, and Ramsey.

13.2.2 The Solow-Stiglitz Convergence Hypothesis

Robert Solow's influential model of economic growth [96] described an economy whose technology permitted smooth substitutability of capital and labor along with diminishing marginal returns to each factor. Accumulation was driven by having households consume a fixed proportion of their annual incomes and save the remainder. A form of exogenous technical change was also imposed on the model so that a balanced growth path was possible where all the economy's consumption and capital stocks grew at the same rate. Solow showed that from arbitrary initial conditions that the fixed saving propensity assumption led the economy to evolve towards the balanced growth path.[11] This implied that over

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[9] See [48], pages 338-340.

[10] Modern macroeconomics principles text with one exception do **not** address the role of different rates of time preference in a population as a factor in determining an economy's income and wealth distribution. The lone exception is the text by Klotz [67], which is a dedicated macro text where all the models are built on the principles derived from Ramsey's theory of optimal accumulation.

[11] Put more precisely, the economy's capital stock and flow of consumption evolve so that they converge to the balanced growth path of capital and consumption as the time variable tends to infinity.

the very long-run, two economies differing only in their initial capital stock sizes, would converge to the same balanced growth path. This result has since been dubbed the *convergence hypothesis.* The aggregate capital stock is eventually arbitrarily close to the balanced growth path of capital. Similarly for the economy's consumption flow. It is important to stress that Solow's model did not posit explicit maximizing behavior for finding private agents' consumption and saving decisions. By way of contrast, Ramsey's model of optimal accumulation, suitably modified to share the same technological setup as in Solow's paper, also results in convergence to the balanced growth path (or steady state in case exogenous technical change is not assumed within the model's structure) provided all households are identical, implying there is a representative agent. This convergence is a direct consequence of the representative consumer – central planner's optimizing consumption-savings behavior. It does not hold for some specifications of the heterogeneous agent theory.

The Solow model implicitly assumed all agents were identical. In an important and often overlooked paper, Stiglitz [101] took Solow's model to another level by analyzing how different saver's wealth and income evolved. Stiglitz followed Solow by not assuming private agents solved optimal savings problems to determine their consumption-savings paths. Stiglitz developed a variety of models in which the multiplicity of agents, each of whom follows his or her own private decision rule, leads to the economy's approach to a balanced growth solution. The resulting distribution of income and wealth was equal in the limit as those variables approached their stationary levels. Stiglitz's simplest model considers two private agents who differ only in their initial endowment of capital stocks. Both agents are assumed to have the same constant marginal propensity to consume (or equivalently, save). That is, each household saved the same proportion of his or her income as the other household.[12] Stiglitz's savers received income from supplying labor (assumed homogeneous and perfectly inelastic) and renting capital at the going competitively determined rental rate. If all savers have identical constant marginal propensities to save and all markets clear, then Stiglitz shows the model can be aggregated in such a way that the aggregate capital stocks follow the dynamics of the analogous Solow model. Under these conditions, the aggregate capital stocks converge to a stationary state, just as in Solow's model. This also implies that the households' incomes and wealth are identical in the steady state.[13] Stiglitz showed this basic result was robust to many other exogenous savings specifications. He also examined the evolution of individual wealth and income over time as the economy approached its stationary state.[14]

[12] Stiglitz assumed savings functions were affine functions of income. In the main text I refer to the linear case (Stiglitz's $b = 0$). This leads immediately to Solow's model upon aggregation across agents.

[13] Bliss [26] develops the links between Solow's, Stiglitz's and Ramsey's theories of the long-run interest rate's determination.

[14] Tsuji [103] extends Stiglitz's work by examining the evolution of various income inequality measures along a dynamic equilibrium path.

The Solow-Stiglitz convergence result yielding an equal distribution of steady state income and wealth turns out to not be realized when explicit optimizing agents are admitted. This is a fundamental difference between his theory and ones based on Ramsey's optimal growth model with many optimizing, forward-looking, agents.[15].

13.2.3 General vs. Temporary Equilibrium

Ramsey's steady state conjecture was studied by Rader [85], [86], and [87]. He examined exchange economy versions of this theory in [85] and [87]. In those cases, he assumed that an agent could purchase any consumption sequence whose present value did not exceed the agent's initial wealth level. Time-dated consumption goods were purchased in this market forward and all market trades took place at time zero. He showed the consumers with the lowest fixed (and exogenous) rates of time preference emerged as the *dominant consumers* in the long-run. Each more impatient consumer's consumption approached zero asymptotically as time tended to infinity. Their consumption levels were negligible. The least impatient individuals enjoyed (in the limit) all the economy's available consumption goods. In [86] this result was reenforced in an intertemporal general equilibrium model with production.[16] Again, the most patient consumers dominated in the long-run while the more impatient ones saw their consumption streams approach zero.

Bewley [23] proposed an integration of Walrasian equilibrium analysis and modern turnpike theory. He assumed strictly concave production processes and showed a type of dominant consumer emerged in the limit as time tended to infinity whenever households had differing fixed and exogenous discount rates. Coles elaborated on this result in [35] and [36]. His contributions included extending Bewley's model to constant returns to scale and also proving a new variant of the turnpike theorem based on Yano's paper [106]. Once again, the dominance of the lowest impatience consumers is shown in a fully articulated general equilibrium model. The Rader, Bewley, and Coles models have a crucial common structure. The budget constraint facing each consumer is stated as a single constraint limiting time dated consumption streams to have present value not larger than the consumer's initial wealth (which itself is endogenously determined in an equilibrium path). Thus, these authors have solved Ramsey's problem provided one essentially allows consumers to borrow and lend subject to the constraint that their consumption's present value does not exceed their wealth. The impatient folks are borrowing against their wealth and repaying their loans later while driving their future consumption towards zero.[17] The

[15] This observation is due to Bliss [27]

[16] See Rader's Chapter 6 for details.

[17] Bewley actually proves a sharper result. The consumption of the nondominant consumers is eventually zero. That is, their consumption vanishes after some finite time. Bewley's proof is based on the assumption that a consumer's marginal utility of consumption (at any date) is uniformly bounded. This rules out the standard iso-

long-term consumption distribution of these equilibrium models is also a feature of their parallel optimal growth models with many consumers.[18]

Duran and Le Van [43] and Le Van and Vailakis [68] sharpened the results found by Bewley and Coles by focusing on the properties of a one-sector model. Duran and Le Van [43] prove an equilibrium existence theorem for a heterogeneous agent one-sector model.[19] Le Van and Vailakis [68] demonstrate existence of an equilibrium as well as derive some qualitative properties of the aggregate capital stock. They show it converges to a limit, but that limiting stock is not a steady state. The consumption levels of the more impatient consumers in their model converge to zero, just as in Rader, Bewley, and Coles' stories.

The general equilibrium interpretation of Ramsey's model given above captures one view of his notion that the relatively impatient people are driven towards the minimum consumption to sustain life. However, that minimum is zero in these models. Hence, each of the more impatient household's steady state consumption is zero.[20] Those agents basically disappear from the economy's demand side, but continue supplying their labor services. In effect, they are so indepted that they must use **all** of their labor income to support their debt and can never consume at all.[21]

The emergence of a dominant consumer result in these general equilibrium models is certainly consistent with the underlying logic and market structure. It is also unsatisfying.[22] Why wouldn't the relatively impatient agents threaten

elastic utility functions favored in the optimal growth literature. Coles makes the same assumption in his work. One implication of this condition is that a steady state where the relatively impatient receive nothing corresponds to a weighted welfare maximization problem (for locating a Pareto optimal allocation) in which their welfare weights are zero..

[18] More recently, Hadji and Le Van [57] explored the existence of equilibrium based on Negishi's weighted welfare optimization idea as well as the asymptotic stability of the model's modified golden-rule. This work, as in the earlier paper by Dana and Le Van [37] analyzes the ways in which the initial capital stocks and changes in the initial distribution of those stocks affect the welfare weights. Lucas and Stokey [70] discuss the weighted welfare approach with recursive utility in an exchange economy setup, while Dana and Le Van ([37], [38], and [39]) work out the properties of Pareto optimal growth in a welfare weighted model. Finally, specialized results for two-sector models have been derived by Ghiglino and Olzak-Duquenne [55] and Ghiglino [54].

[19] Their proof is built to simplify one for the representative agent case given by Aliprantis, Border, and Burkinshaw [2]. Existence proofs for one-sector models (and exchange economies) with common discount factors are also given by Kehoe ([61], [62]), Kehoe and Levine ([63], and Kehoe, Levine and Romer ([64], [65], and [66]). The works by Kehoe, et al also study on the determinacy of equilibrium.

[20] This turns on the assumed property of bounded marginal utility of consumption in Bewley and Coles' papers.

[21] Coles ([35],[36]) does a nice job of formally elaborating this idea in the general equilibrium context.

[22] Zero is a special number. Zero consumption literally means no consumption. It is a fundamental assumption in these general equilibrium models that zero is a possible consumption stream in the consumer's consumption set. Of course, real

to withhold their labor unless their debts were renegotiated? The equilibrium need not be time consistent. This critique led me to formulate an alternative equilibrium concept for these general equilibrium models based on the theory of the core for cooperative games.[23] The Ramsey equilibrium model based on Becker [5] imagines a different solution to this problem. The market structure in the general equilibrium case is one of complete forward markets. If this structure is broken so that agents cannot borrow against their future labor income, then it might be possible to generate a different steady state solution — one where agents without capital received payment for their labor services and could consume at that level indefinitely in the economy's steady state. This leads to a model with an *incomplete market structure* that is represented by a sequence of budget constraints, one for each time. A nonnegativity constraint on individual capital stock holdings represents this limitation on borrowing.[24] Agents are otherwise allowed to trade in a sequence of spot markets where they increase or decrease their capital holdings subject to this nonnegativity constraint. Bewley [23] termed this a *temporary equilibrium* framework. I will show this model exhibits the basic feature of Ramsey's stationary state with the proviso that consumers without capital still consume their wage income. This *Ramsey equilibrium model* is presented in Sections 3-5.

13.2.4 Mankiw's Savers-Spenders Model

The Ramsey equilibrium model turns out to have an interesting application. It can supply a microeconomic foundation for Mankiw's [72] savers-spenders theory of fiscal policy. He argues that the standard optimal growth model, suitably reinterpreted as a representative agent competitive equilibrium model, cannot serve as an adequate theory for the analysis of fiscal policy. Likewise, he argues that the standard overlapping generations model fails to provide a suitable foundation for fiscal policy analysis.[25] These models fail to supply theoretical explanations of three stylized facts. First, he notes that consumption smoothing

people cannot live on nothing at all albeit they might survive for some time on extremely low or subsistance rations.

23 This *recursive core* concept is developed in [6] and [10].

24 This formulation of the constraint is called by some writers an **ad hoc borrowing constraint**. The **natural debt limit** is the weaker requirement that an individual's capital holdings are never smaller at any time than the present discounted value of the wages to be received from that time on. The ad hoc constraint prevents households from ever borrowing against their discounted future stream of wage payments. See Ljungqvist and Sargent [69] for a detailed discussion of alternative forms of borrowing constraints. They also emphasize the distinction between a binding constraint and one that does not bind. The former situation arises only when the constraint's Lagrange multiplier or shadow price is positive along a household's optimal program. They show by examples that there are cases where the ad hoc constraint binds and others when it does not, yet the household's capital is zero.

25 See [41] for a complete treatment of the overlapping generations model and its public finance applications.

is far from perfect. Aggregate consumption spending tracks current income far more than if households borrowed or lent capital to smooth their consumption streams. One of his interpretations for this observed behavior is that a substantial portion of the population faces binding borrowing constraints. These people just hold enough capital to buffer their consumption against bad income shocks. He notes that these individuals possess high discount rates and often face borrowing constraints[26] Second, a review of empirical income and wealth distributions in the U.S. implies, according to Mankiw, that wealth is more concentrated than income. Indeed, he argues that the lowest two quintiles of the U.S. income distribution account for 15 percent of the economy's income and hold 0.2 percent of the country's wealth. He infers from this observation that many households lack the financial resources to smooth their consumption streams over time, as assumed in the standard models of fiscal policy. Third, he observes that based on U.S. data, only a small proportion of the population has sufficient wealth to provide their descendants with an inheritance. These individuals save beyond the levels necessary to smooth their consumption profiles. The vast majority of the population leaves minimal or no bequests. Their saving lacks an altruistic motive.

Mankiw concludes that the standard infinitely-lived representative agent theory and the overlapping generations model are inadequate to address fiscal policy in dynamic macroeconomic theory. His proposed alternative microeconomic foundation for fiscal policy analysis is based on a "particular sort of heterogeneity."[27] He argues that an appropriate model should include both low-wealth households who fail to smooth their consumption over time and high-wealth families who smooth consumption not only from year-to-year, but also from generation to generation. Thus, the model should include consumers who plan ahead for themselves and their descendants (via bequests), while others live paycheck-to-paycheck.

Mankiw's theory divides the household sector into two populations. The first, the **savers**, who have an operational bequest motive and infinite horizons, and second, the **spenders**, who consume their entire after-tax labor income in every period. Mankiw thinks of the savers as individual households solving infinite horizon Ramsey styled optimization problems and spenders as ones following simple rules-of-thumb consumption decisions that result in no savings.

The Ramsey equilibrium model surveyed here provides one possible microeconomic foundation for Mankiw's theory. The major difference is that the spenders in my model also solve infinite horizon optimization problems, but *choose* to run their assets down to zero in the steady state and along at least some important realizations of the model's transition dynamics. An example of the spenders-savers theory is given in Section 5.[28]

[26] See ([72], p. 121).

[27] The basic arguments against the standard theories and Mankiw's alternative story are found in ([72], p. 121).

[28] I do not use the model to address fiscal policy analsysis as my focus lies on the underlying properties of the economy's dynamics without an explicit government

13.3 The Ramsey Equilibrium Model

The Ramsey equilibrium model is described below. The basic model is developed for the case of agents with time additively separable utility functions with fixed discount factors. The technology is specified by a one-sector model with a single all purpose consumption–capital good.

The general complete market competitive one-sector model treats budget constraints as restricting the present value of an agent's consumption to be smaller than or equal to the agent's initial wealth defined as the capitalized wage income plus the present value of that person's initial capital. This allows us to interpret the choice of a consumption stream as if the agent is allowed to borrow and lend at market determined present value prices subject to repaying all loans. Markets are complete — any intertemporal trade satisfying the present value budget constraint is admissible at the individual level. The Ramsey equilibrium model changes the budget constraint from a single one reckoned as a present value to a sequence, one for each period. Agents are forbidden to borrow against their future labor income, so they cannot capitalize the future wage stream into a present value. Markets are incomplete. It becomes crucial to track the evolution of each person's capital stock. This is unnecessary in the complete market models when all values entering the budget constraint are present values.

The heterogeneous discount factor, incomplete market economy, will differ in another important respect — the operation of a borrowing constraint in the individual household problems also breaks the possibility of an equilibrium allocation arising as the economy's optimal allocation. The welfare maximization approach favored in the complete market theory is inapplicable.

13.3.1 The Basic Model and Blanket Assumptions

Time is taken in discrete intervals, $t = 1, 2, \ldots$.[29] Upper case letters denote real(vector)-valued sequences; the corresponding terms of the sequence are expressed as lower case letters, e.g. $X = \{x_t\}_{t=1}^{\infty}$. The real-valued *sequence X is nonnegative* if each component is nonnegative, that is, $x_t \geq 0$ for each t.

There are $H \geq 1$ households indexed by $h = 1, \ldots, H$. There is a single commodity available for consumption or investment at each time. At time zero, households are endowed with capital stocks $k^h \geq 0$. Put $\mathbf{k} = \sum_h k^h$ and assume $\mathbf{k} > 0$. Let c_t^h, x_t^h denote the consumption and capital stock of household h at time t. Household h has *felicity function* u_h (also known as the *temporal utility*

sector. Obviously, a full analysis of public policy within my version of the spenders-savers theory would be of some interest. Boyd [29] illustrates the possibilities for the Ramey version of Mankiw's model in a capital tax incidence problem.

[29] Ramsey [90] formulated his model in continuous time. The model presented here is cast in discrete time. This turns out to have some technical advantages over continuous time modeling when developing general existence of equilibrium theorems and analyzing the dynamics of particular equilibrium paths.

function or *single-period return* function); c_t^h is the argument of u_h. Household h discounts future utilities by the factor δ_h with $0 < \delta_h < 1$. Hence, the household's lifetime utility function is specified by $U_h(C_h) = \sum_{t=1}^{\infty} \delta_h^{t-1} u_h(c_t^h)$.

I assume (put $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$):

Assumption I*: For each h, $u_h : \mathbb{R}_+ \to \mathbb{R}$ is $C^{(2)}$ on $\mathbb{R}_{++}$ with $u_h' > 0$, $u_h'' < 0$, and $\lim_{c \to 0} u'(c) = \infty$.*

The model has *common discount factors* when all agents' discount factors are equal, and *heterogeneous discount factors* otherwise. The major focus in this chapter is on the heterogeneous case, so it is formalized in the next assumption. The major results only require two types — one household is the most patient and the others are less patient. This is expressed by assuming the first household's discount factor is larger than all the other households' discount factors. The first household is the *most patient* agent and the others are said to be *less patient* than the first one. Assumption II orders households from the most patient to the least patient.

Assumption II: $1 > \delta_1 > \delta_2 \geq \cdots \geq \delta_H > 0$.

Production takes place using a single capital good. The productive technology turns labor and capital goods into a composite good that can be either consumed or saved as next period's capital input. The amount of labor is fixed in this economy (there will be one unit of labor services per household and all labor services are assumed to be identical). The technology is summarized by a *production function.* The production function is denoted by f; let $y = f(k)$ denote the compositive good y produced from a fixed amount of labor, together with capital input k. Capital is assumed to depreciate completely within the period.[30] Hence, the model is formally one with circulating capital that is consumed within the production period. The production function is derived from a *neoclassical production function*, $F(k, l)$, where l is the labor input. This function is positively homogeneous of degree one, is continuous and concave on its domain, increasing in each variable separately, at least twice continuously differentiable on the interior of its domain, and satisfies $F(0, l) = F(k, 0) = F(0, 0) = 0$ for all positive l and k. The production function f is derived from F according to the formula: $f(k) = F(k, H)$ where $l = H$ is the fixed labor input. The formal properties of f are recorded as Assumption III.

Assumption III: $f : \mathbb{R}_+ \to \mathbb{R}_{++}$, $f(0) = 0$, *f is $C^{(2)}$ on $\mathbb{R}_{++}$*, $f' > 0, \lim_{x \to 0} f'(x) = \infty, \lim_{x \to \infty} f'(x) = 0$, *and* $f'' < 0$.

The conditions $f'(0+) = +\infty$ and $f'(\infty) = 0$ are the production function's *Inada conditions.* This assumption implies there is a maximum sustainable

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[30] This assumption simplifies the presentation. Depreciated at a fixed rate is easily incorporated into the productive technology.

capital stock, denoted b^m, satisfying $b^m = f(b^m) > 0$. If the initial aggregate capital stock $\mathbf{k}$ is smaller than b^m, then all nonnegative sequences of consumption and capital satisfying the *balance condition*, $c_t + k_t = f(k_{t-1})$ for all t with $k_0 = \mathbf{k}$, are bounded from above by b^m.

Assumptions I-III are **blanket assumptions** assumed for the remainder of this survey and sometimes referred to as *(AI)-(AIII)*.

If $H = 1$, then the Ramsey equilibrium model coincides with the standard optimal growth problem. For that reason, the single household version of the Ramsey equilibrium model is sometimes called the *representative household model*. The fundamental welfare theorems tell us that an allocation maximizes the representative agent's utility function if and only if it is a competitive equilibrium allocation. This will not hold in the multi-agent Ramsey equilibrium model due to the borrowing constraint.

The classical view of capital accumulation was that a growing economy would exhibit an increasing wage bill and a declining interest rate (which is the rental rate for capital goods in one sector models). The one household version of the Ramsey equilibrium model turns out to be the major example of this portrait of capital accumulation that Bliss [25] called the *Orthodox Vision of Capital Theory*. Some specifications of the Ramsey equilibrium model will also exhibit this property, but others will not.

The Ramsey Equilibrium model is formally described by the households' optimization problems, the production sector's profit maximization problem, and the relationships between consumers and the production process.

13.3.2 The Households' Problems

Each infinitely-lived household is supposed to perfectly anticipate the future course of rental and wage rates. Competitive markets operate at each date, so households' consumption-savings decisions do not, by themselves, influence prices. However, prices will be set in equilibrium to clear all markets.

Let $\{1 + r_t, w_t\}$ be a sequence of one period rental factors and wage rates, respectively. The sequences $\{1+r_t, w_t\}$ are always taken to be nonnegative and nonzero. Households are competitive agents and perfectly anticipate the profile of factor returns $\{1 + r_t, w_t\}$. Given $\{1 + r_t, w_t\}$, h solves

$$P(h): \ \sup \sum_{t=1}^{\infty} \delta_h^{t-1} u_h(c_t^h)$$

by choice of nonnegative sequences $\{c_t^h, x_t^h\}$ satisfying $x_0^h = k^h$ and

$$c_t^h + x_t^h = w_t + (1 + r_t)x_{t-1}^h, \ (t = 1, 2, \ldots). \tag{13.1}$$

The market structure of this model requires capital assets to be nonnegative eat each moment of time and that agents without capital cannot borrow against the discounted value of their future wage income. This borrowing constraint binds for the nondominant consumers in a stationary equilibrium. These individuals might be tempted to borrow against their future labor income if they were allowed to do so. If this occurred, their consumption would approach zero as time converges to infinity. Indeed, this is the type of outcome for equilibria in the exchange economy modeled by Rader [87],[85], and [86], and the general equilibrium capital theoretic models appearing in Bewley [23], Coles [35], Duran and Le Van [43], and Le Van and Vailakis [68] as noted in Section 2.

The *no arbitrage* necessary conditions for $\{c_t^h, x_t^h\}$ to solve $P(h)$ are $c_t^h > 0$ and

$$\delta_h(1 + r_{t+1})u_h'(c_{t+1}^h) \leq u_h'(c_t^h) \tag{13.2}$$

with equality whenever $x_t^h > 0$. This condition tells us the household cannot improve its utility by a *one-period unreversed arbitrage*. Suppose that the given consumption and capital sequences are optimal for the expected rental and wage profiles. Then, this household cannot increase utility by undertaking the following activity: at time t marginally increase the capital stock to be carried to time $t+1$. This costs the household $u_h^{'}(c_t^h)$ utils on the margin. Now invest this extra capital to earn the total rental income $1+r_t$ from the production sector. Convert this additional income into consumption at $t+1$ worth $u_h^{'}(c_{t+1}^h)$ utils on the margin. This implies the marginal benefit of this incremental investment measured at $t+1$ is $(1+r_t)u_h^{'}(c_{t+1}^h)$. Now discount this by the utility discount factor δ_h to place the marginal benefit at time $t+1$ and marginal cost at time t in comparable utility units. The marginal benefit cannot exceed the marginal cost along an optimal solution to the household's problem. This is formally expressed by the inequality (13.2).[31] If the capital stock owned by this household at time t happens to be positive, then this arbitrage calculation can be repeated for an increase in consumption at time t paid for by lower consumption at time $t+1$. In this case, the inequality in (13.2) is reversed and the *Euler equation* holds:

$$\delta_h(1 + r_{t+1})u_h'(c_{t+1}^h) = u_h'(c_t^h). \tag{13.3}$$

The corresponding *transversality condition* is

$$\lim_{t\to\infty} \delta_h^{t-1}u_h^{'}(c_t^h)x_{t-1}^h = 0. \tag{13.4}$$

This condition expresses the unprofitability of open-ended or unreversed arbitrage opportunities. That is, the household acquires an extra unit of capital at time t and never reverses that position — the marginal unit of capital is held forever. This cannot increase utility if a program is optimal.[32]

[31] The fact that the no arbitrage condition takes the form of an inequality is another manifestation of the borrowing constraint.

[32] See Becker and Boyd [8] Chapter 4 for a detailed development of this interpretation of the transversality condition.

13.3.3 The Production Sector's Objective

Production takes place with a one period lag — capital inputs from $t-1$ combined with labor supplied at time t produce goods available for consumption or additional capital accumulation during time t. This type of production process is called a *point input–point output* production activity.

The production sector is characterized by the one sector neoclassical production function f; inputs precede outputs by one period. Capital is the only variable factor. The technology's properties are described by Assumption III.

All the intertemporal decisions are taken in the household sector. The assumption of a point-input point-output production function and a competitive rental market implies the maximization of discounted profits is equivalent to solving the static or myopic problem defined below for each time period.

Producers are supposed to take the rental rate as given and solve the following myopic profit maximization problem $P(F)$ at each t:

$$P(F): \ \sup[f(x_{t-1}) - (1 + r_t)x_{t-1}]$$

by choice of $x_{t-1} \geq 0$. The residual profit is treated as the wage bill. It is shared equally by the identical households as wages — production is worker owned.

If $0 < 1+r_t < \infty$, then $(AIII)$ implies there is a unique positive stock K_{t-1} which solves *P(F)* at each t; clearly

$$f'(K_{t-1}) = 1 + r_t; \tag{13.5}$$

furthermore, the corresponding $\{w_t\}$ defined by

$$Hw_t = f(K_{t-1}) - (1 + r_t)K_{t-1} \tag{13.6}$$

is positive. I am using the lower case w as the current value of the agent's wages, whereas it represented a present value before. This should cause no confusion as the context is clear.

The expression for the wage bill can also be viewed as a function of the rental rate — invert (13.5) to solve for the capital stock in terms of the rental rate and substitute this into (13.6). The result can be summarized with the function $\phi(1 + r_t)$, which expresses the per-capital wage as a function of the prevailing rental factor, $1 + r_t$. This function also defines the economy's *factor-price frontier* — the combination of wages and rentals reflecting the economy's maximum profit opportunities.

13.3.4 The Ramsey Economy

A collection $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2, \ldots, H)$ satisfying Assumptions I, II-III, and for which $k^h \geq 0$ for each h with $\mathbf{k} = \sum_{h=1}^{H} k^h > 0$, $\mathbf{k} \leq b^m$, is said to be a **Ramsey economy**, or simply, an **economy**. A given economy is thus a collection of primitives on tastes and technology that meets the basic assumptions for households and the production sector. The economy is always assumed to have a positive aggregate capital stock that is also no larger than the maximum sustainable stock. Individual endowments of capital may or may not be positive. However, at least one agent will always possess some capital at time zero.

13.3.5 The Equilibrium Concept

The equilibrium concept is *perfect foresight*. Households perfectly anticipate the sequences of rental and wage rates. They solve their optimization problems for their planned consumption demand and capital supply sequences. The production sector calculates the capital demand at each time and the corresponding total output supply. Rentals are paid to the households for capital supplied and the residual profits are paid out as the total wage bill. An equilibrium occurs when the households capital supply equals the productions sector's capital demand at every point of time. A form of Walras' Law implies that the total consumption demand and supply of capital for the next period equals current output. Thus, in equilibrium, every agent is maximizing its objective function and planned supplies equal planned demands in every market.

Definition 13.3.1. *Sequences* $\{1+r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *constitute a* Ramsey Equilibrium *for a given economy* $\mathcal{E}$ *provided:*

E1. *For each h,* $\{c_t^h, x_{t-1}^h\}$*solves P(h) given* $\{1+r_t, w_t\}$.
E2. *For each t,* K_{t-1} *solves P(F) given* $1+r_t$.
E3. $Hw_t = f(K_{t-1}) - (1+r_t)K_{t-1}$ $(t = 1, 2, \ldots)$.
E4. $\sum_{h=1}^{H} x_{t-1}^h = K_{t-1}$ $(t = 1, 2, \ldots)$, $0 < \mathbf{k} = K_0 \leq b^m$.

Thus, consumers maximize utility (E1) and producers maximize profits (E3). The labor market clearing condition is expressed in (E3). The capital market clearing condition is (E4). The output market balance follows by combining (E1)– (E4). This is a form of Walras' Law that holds at each time. Hence

$$\sum_{h=1}^{H}(c_t^h + x_t^h) = f(K_{t-1}). \tag{13.7}$$

Note that equilibrium consumption and capital sequences are bounded from above by the maximum sustainable stock.

The Orthodox Vision portrays an economy as evolving towards a steady state. When the economy's capital stock is initially smaller than its stationary level there is growth and the rate of return on capital falls over time. This portrait of capital accumulation is consistent with the dynamics of the one sector Ramsey optimal growth — perfect foresight equilibrium model provided there is a representative household whose preferences are taken as the planner's objective.

The Orthodox Vision was attacked by Bliss [25] on grounds that when there are many distinct capital goods a single rate of interest could not be defined and therefore the idea that growth accompanied a declining rate of interest made no sense. Subsequent research has shown that in aggregate capital Ramsey optimal growth models with a well-defined interest rate, the economy might not follow the Orthodox Vision provided there were at least two sectors producing a consumption good distinct from the capital good. The problem was optimal cycles or even chaotic trajectories could emerge with a sufficiently impatient planner.[33] Heterogeneous discount factor models turn out to differ fundamentally from the representative agent and common discount factor models, even in the classical one-sector case. The Orthodox Vision will only apply to *some* economies when there are heterogeneous discount factors.

13.4 Stationary Ramsey Equilibrium Models

A stationary Ramsey equilibrium is one in which all prices and quantities remain constant over time. This is the simplest form of equilibrium and it is interpreted as the model's long-run solution. The case where all agents discount their future utilities at a constant rate is explored first. This is the case that Ramsey [90] considered in his classic paper. The economy satisfies assumptions (AI), (AII), and (AIII), so there is a single most patient household. The steady state income distribution is shown to be determined by the largest discount factor, or equivalently, the lowest discount rate. The household with the lowest discount rate owns all the capital and earns a wage income; all other households receive a wage income. If discount rates are equal between households, then the steady state distribution of income is indeterminate.

The dominant consumer in the stationary Ramsey equilibrium has two sources of income — rental income from capital and wage income from its supply of labor to the production sector. In Ramsey's original conjecture the dominant consumer's consumption equaled "bliss" and the nondominant households consumed at a level just sufficient to sustain their lives. In the version of the model presented here the nondominant consumers have a wage income which is entirely consumed within the period and there is no presumption that it is at the minimal level that just supports their lives.

13.4.1 Heterogeneous Households and Differing Rates of Impatience

A Ramsey equilibrium program is *stationary* for the economy $\mathcal{E}$ provided the equilibrium wage rate, rental rate, the aggregate capital stock, and the allocations of capital and consumption are constant over time. Becker [5] proves the basic theorem of Ramsey equilibrium theory. It proclaims the existence

[33] See [8] for a detailed review of the relevant literature.

of a unique stationary equilibrium in which only the most patient household has capital — all other households have none and live off their wage incomes. Of course, this most patient consumer also has a wage income, so that person achieves a higher consumption level than the others. Hence, there is an unequal distribution of income and wealth in this steady state solution. The model is telling us that this inequality in stationary equilibrium consumption levels and capital holdings is the direct consequence of a single household being the most patient one. This result is also built on the inflexibility of individual rates of time preference, an issue that is reexamined in Section 4.4.

Let k^{δ_1} be the unique solution to the equation $f'(k) = \delta_1$.[34] This capital stock will be the first household's capital and therefore the aggregate capital stock in the equilibrium solution presented in the following theorem. Let $1 + \varrho_h = \delta_h^{-1}$ define agent h's *pure rate of time preference (or, discount rate)*, ϱ_h. Notice that the assumed ordering of household's discount factors in (AII) is equivalent to

$$0 < \varrho_1 < \varrho_2 \leq \cdots \leq \varrho_H.$$

Recall that $w = \phi(1 + r)$ gives the per capita wage as a function of the rental factor via the factor-price frontier.

Theorem 13.4.1. ***(Ramsey's Theorem)***
Let $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2, \ldots, H)$ be a given economy. Then there is a unique stationary Ramsey equilibrium given by

(S1) $(1 + \bar{r}) = \delta_1^{-1}$ *or equivalently* $\bar{r} = \varrho_1$;
(S2) $\bar{w} = \phi(\delta_1^{-1})$;
(S3) $\bar{k}_0^1 = k^{\delta_1}$ *is the first household's initial allocation of capital;*
(S4) $\bar{k}_0^h = 0$ *is the initial allocation of capital for* $h = 2, \ldots, H$;
(S5) $\bar{c}^1 = \phi(\delta_1^{-1}) + \varrho_1 k^{\delta_1}$;
(S6) $\bar{c}^h = \phi(\delta_1^{-1})$ *for* $h = 2, \ldots, H$.

The proof follows from several lemmas. The basic idea is to conjecture (S1)-(S6) identify an equilibrium and verify that for the given wage rate, rental rate, and assignment of initial capital stocks, the resulting optimal solutions for the households' optimization problems are the constant consumption and capital profiles specified while the production sector profit maximization problem's solution is k^{δ_1}. Moreover, the household's capital supplies sum to the aggregate stocks demanded by the production sector, k^{δ_1}. The sufficiency conditions for the solution of each agents' optimization problems given the values of the prices and initial conditions show the conjectured solutions solve the agents' problems given the wage rate, rental rate, and capital endowments. The first household's sufficient condition holds as the conjectured solution satisfies that agent's no arbitrage and transversality conditions. The other households are handled separately in the lemmas. Recall that $P(h)$ refers to a household h's

[34] Othewise, overbars are used to indicate values for endogenously determined variables which are stationary in a Ramsey equilibrium.

individual maximization problem given the wage and rental sequences and the initial capital stock assignment. I will use this shorthand to refer to the household's problems in the following lemmas on the maintained assumption that $w_t = \bar{w}, 1 + \bar{r} = \delta_1^{-1}, \bar{k}^1 = k^{\delta_1}$, and $\bar{k}^h = 0$ for $h = 2, \ldots, H$.

Lemma 13.4.1. *$P(1)$ has the unique solution $\bar{c}_t^1 = \bar{c}^1 = \phi(\delta_1^{-1}) + \varrho_1 k^{\delta_1}$ and $\bar{x}_{t-1}^1 = k^{\delta_1}$ for $t = 1, 2, \ldots$.*

Lemma 13.4.2. *For $h \geq 2$, $P(h)$ has the unique solution $c_t^h = \phi(\delta_1^{-1})$ and $x_{t-1}^h = 0$ for $t = 1, 2, \ldots$.*

Proof. Superscripts labelling agent $h \geq 2$ are suppressed in this proof in order to simplify the notation. I will work with a household satisfying $0 < \delta < \delta_1$.

Let $\{c_t, x_{t-1}\}_{t=1}^{\infty}$ denote a feasible solution for this person's optimization problem. That is, it satisfies

$$c_t + x_t = \phi(\delta_1^{-1}) + \delta_1^{-1} x_{t-1} \text{ for all } t \text{ and } x_0 = 0.$$

Further, assume that this feasible solution is *good* in the sense that

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t) > -\infty.$$

Feasible programs which are not good are *bad*; there is no reason to consider bad paths since they cannot provide the consumer with a larger discounted utility stream. Hence, for the construction of a comparison path in this sufficiency argument only good paths need be considered.

Let $\bar{c} = \bar{w} = \phi(\delta_1^{-1})$. The following inequality must be verified in order to show that this is the household's optimum given the constant wage and rental paths and zero initial capital condition:

$$\sum_{t=1}^{\infty} \delta^{t-1} [u(\bar{w}) - u(c_t)] > 0, \tag{13.8}$$

where $\{c_t, x_{t-1}\}_{t=1}^{\infty}$ denotes a feasible good path and $c_t \neq \bar{w}$ for some t.

By strict concavity of u,

$$u(\bar{w}) - u'(\bar{w})\bar{w} \geq u(c_t) - u'(\bar{w})c_t \tag{13.9}$$

for all $c_t \geq 0$, $t = 1, 2, \ldots$. Moreover, strict inequality holds in (13.9) whenever $c_t \neq \bar{w}$. So rearranging (13.9) for $c_t \neq \bar{w}$ implies that

$$u(\bar{w}) - u(c_t) > u'(\bar{w})(\bar{w} - c_t). \tag{13.10}$$

Therefore, if $c_t \neq \bar{w}$ for some t, then

$$\sum_{t=1}^{\infty} \delta^{t-1} [u(\bar{w}) - u(c_t)] > \sum_{t=1}^{\infty} \delta^{t-1} u'(\bar{w}) (\bar{w} - c_t). \tag{13.11}$$

Feasibility of this comparison path requires that

$$c_t = \bar{w} + \delta_1^{-1} x_{t-1} - x_t \text{ and } x_0 = 0. \tag{13.12}$$

Therefore, the right-hand side of (13.11) becomes

$$\sum_{t=1}^{\infty} \delta^{t-1} u'(\bar{w}) \left[x_t - \delta_1^{-1} x_{t-1} \right], \text{ with } x_0 = 0. \tag{13.13}$$

Use the first household's pure rate of time preference, ϱ_1, to the rewrite the summation in (13.13) as

$$u'(\bar{w}) \sum_{t=1}^{\infty} \delta^{t-1} \left[\Delta x_{t-1} - x_t \right], \text{ with } \Delta x_{t-1} \equiv x_t - x_{t-1}.$$

The indices in (13.13) can be changed to run from $t = 0, 1, 2, \ldots$. Next use the formula for summation by parts (see [58], page 51) to write (13.13) as

$$u'(\bar{w}) \lim_{T \to \infty} \left\{ \sum_{t=1}^{T-1} (\delta^t - \delta^{t+1}) \sum_{s=0}^{t} \Delta x_s + \delta^T \sum_{t=0}^{T} \Delta x_s - \sum_{t=0}^{T} \delta^t \varrho_1 x_t \right\}, \tag{13.14}$$

where

$$\Delta x_s = x_{s+1} - x_s, x_0 = 0 \text{ and } \sum_{s=0}^{t} \Delta x_s = x_{t+1} - x_0 = x_{t+1}.$$

Since $\delta = (1 + \varrho)^{-1}$, (13.14) can be rewritten as

$$u'(\bar{w}) \lim_{T \to \infty} \left\{ \sum_{t=0}^{T-1} \delta^{t+1} \varrho x_{t+1} - \sum_{t=0}^{T} \delta^t \varrho_1 x_t + \delta^T x_{T+1} \right\} =$$

$$u'(\bar{w}) \lim_{T \to \infty} \left\{ (\varrho - \varrho_1) \sum_{t=1}^{T} \delta^t x_t - \delta^0 \varrho_1 x_0 + \delta^T x_{T+1} \right\} \geq 0, \tag{13.15}$$

because

$$\begin{aligned} (\varrho - \varrho_1) &> 0 \text{ if and only if } 0 < \delta < \delta_1, \\ x_0 &= 0, x_t \geq 0 \text{ all } t, \text{ and} \end{aligned}$$

$$\lim_{T \to \infty} \delta^T x_{T+1} = 0.$$

Therefore (13.8) obtains.

Lemma 13.4.3. *There is a unique k^{δ_1}, $0 < k^{\delta_1} < +\infty$, such that $f'(k^{\delta_1}) = \delta_1^{-1}$, hence k^{δ_1} solves*

$$\max_{x \geq 0} \delta_1 f(x) - x. \tag{13.16}$$

Proof. Existence of k^{δ_1} follows from the Inada conditions imposed in (AIII) and the Intermediate Value Theorem; uniqueness stems from the strict concavity of the production function. That k^{δ_1} solves the maximization problem (13.16) also follows from the strict concavity of the production function as $\delta_1 f'(k^{\delta_1}) = 1$ is the first-order condition for (13.16).

Remark 13.4.1. Gale [50] showed that (13.16) holds for the optimal growth model. This corresponds to the case $H = 1$.[35] He shows that (13.16) gives a complete characterization of the stationary state in that situation. Thus, for heterogeneous households, (13.16) holds as in ([50]) but for the most patient household's discount factor.

Remark 13.4.2. Lemmas 3-5 prove that $\bar{w} = \phi(\delta_1^{-1}), 1+\bar{r} = \delta_1^{-1}, k^1 = k^{\delta_1}$, and $k^h = 0$ for $h = 2, 3, \ldots, H$ is a stationary Ramsey equilibrium.

The theorem follows once this equilibrium is shown to be the only stationary solution to the model. The argument proving this is rather technical; a "heuristic" argument is given in the "proof."

Proposition 13.4.1. *The stationary Ramsey equilibrium of Remark 8 is the only stationary Ramsey equilibrium.*

Proof. In any other candidate stationary Ramsey equilibrium, either $(1+\bar{r}) > \delta_1^{-1}$ or $(1+\bar{r}) < \delta_1^{-1}$ must hold.

If $(1+\bar{r}) > \delta_1^{-1}$, then household one has a feasible policy of ever increasing capital accumulation and consumption yielding arbitrarily large stocks; by diminishing returns this cannot be an equilibrium.

If $(1+\bar{r}) < \delta_1^{-1} < \delta_h^{-1}$ for $h = 2, 3, \ldots, H$, then for any initial distribution of capital, it is shown (in the appendix) that it is optimal for every household to decumulate their capital stocks to zero in **finite time** starting from their initial stocks. Thus, $\sum_{h=1}^{H} x_{t-1}^h$, the total capital stock found by each person's solutions to their problems $P(h)$, is eventually zero. Hence, $f'(\sum_{h=1}^{H} x_{t-1}^h)$ becomes arbitrarily large in finite time (by the Inada condition). This violates the maintained assumption that the equilibrium is stationary.

Remark 13.4.3. Lemmas 3-5 and the Proposition prove Ramsey's Theorem.

[35] In the single household case one should reinterpret the production function as the intensive form for $F(x, 1)$. Notice the production function in (13.16) can be rewritten as $F(x, H)$ to show its parametric dependence on the number of households in the economy.

This version of Ramsey's model utilizes a perfect foresight expectations hypothesis to close the model. One can think of the *long-run* as the steady state solution and regard the *short-run* as the time periods during which the economy evolves from its given initial data.[36] The "short-run" represents the transitional period from the start of the model at time zero until it converges to the steady state. Of course, this presumes the economy actually converges to the steady state — a presumption that will turn out to be true only with additional qualifications as will be shown in section 5.3. Even so, it is useful to maintain the distinction between long-run solutions and short-run dynamics. This is so since the model will generate other types of periodic solutions besides the trivial periodic motion embodied in the stationary state.[37] These non-trivial periodic solutions also have legitimate claims to be considered long-run solutions.

An interesting property of long-run equilibrium is that the economy's steady state wage rate and interest rate depends only on the first household's discount factor. In particular, $H\phi(\delta_1^{-1})$ is the maximized present value of wages valued at the present value prices $(\delta_1, 1)$ as displayed in (13.16). As noted there and the following Remark, this is similar to the case of optimal growth where the planner discounts future utilities by the factor δ_1. In the present model, the per capita wage rate, $\phi(\delta_1^{-1})$, is determined by a decentralized economy in long-run competitive equilibrium. This decentralization works to select a particular household's discount factor and corresponding wage bill from the factor-price frontier. Given that $\delta_1^{-1} < \delta_h^{-1}$ for all $h \geq 2$, then $\phi(\delta_1^{-1})$ is the largest per capita wage rate that could be attained if the present value prices are restricted to be chosen from the list $\{(\delta_1, 1), (\delta_2, 1), \ldots, (\delta_H, 1)\}$ and applied to (13.16). Thus,

$$\delta_1 f(k^{\delta^1}) - k^{\delta^1} = \max_{\delta_h} \max_{x \geq 0} \delta_h f(x) - x. \tag{13.17}$$

Equation (13.17) represents a *variational principle* governing the solution of the steady state Ramsey equilibrium problem. According to David Gale, "variational methods show the existence to a problem," such as the existence of a stationary Ramsey equilibrium, "by picking an object that maximizes or minimizes some function. The resulting object is then shown to have the desired property by showing that if it did not, one could 'vary' the object so that the given function would further increase or decrease."[38] The double max function on the right-hand side of (13.17) is a new example of a variational principle in economics. Here, the "variational" argument is based on allowing discount factors above or below δ_1 and showing they are inconsistent with stationary equilibrium — indeed, this is one way to interpret the heuristic argument given to support Proposition 8 since any discount factor different from δ_1 leads to a situation counter to stationarity equilibrium.

[36] Magill [71] discusses this distinction between long and short-run dynamics.

[37] Two period cycles are examples of non-trivial periodic solutions. An example appears in section 6.2.

[38] See Gale ([51], pp. 46-47) for this description of a variational principle.

The last point to note about (13.17) is that it does not depend any agents' felicity function. This is reminiscent of the Dynamic Non-Substitution Theorem which characterizes steady states in no-joint production multisectoral production models — given the long-run interest rate there is a unique choice of technology consistent with the economy being in a stationary state.[39] Indeed, (13.16) is a simple version of the Dynamic Non-Substitution Theorem since there is only one capital stock consistent with the maximization problem expressed in that equation.[40] Since the steady-state interest rate is the same as the steady state rental rate in a one-sector model without joint production, if follows that for each possible steady state interest rate r there is a unique choice of the capital stock which solves $\max_{x \geq 0} f(x) - (1+r)x$. Since $\delta_1^{-1} - 1 = \varrho_1$ is the steady state interest rate for this economy, the capital stock k^{δ_1} is uniquely determined by solving (13.16). The variational principle embodied in (13.17) is a joint statement about the determination of the long-run interest rate and the choice of technique in this economy.[41] Becker and Tsyganov [17] show the same result obtains when there are two sectors and no joint production. Once again, the Dynamic Nonsubstitution Theorem plays a decisive role in determining the economy's capital intensities in each sector given the long-run interest rate is determined by the most patient household. As in the one-sector case, the distribution of capital is the extreme one where only the most patient individual holds capital.

13.4.2 Stationary Strategic Ramsey Equilibria

The competitive steady state described in Ramsey's Theorem has an incentive problem. Why doesn't the most patient agent behave like a monopsonist in the capital market? After all, that agent is the only one willing and able to supply capital in the steady state. There are basically two ways to look at this question. In one, the economy is assumed to have a continuum of agents of each type. This means that there are uncountably many agents with the same preferences and endowments. Assuming that the equilibrium treats equal types equally, one can select a representative agent of each type since almost all agents of the same type receive the same allocation in equilibrium.[42] Ramsey's Theorem can then

[39] Burmeister and Dobell [34], Burmeister [33], and McKenzie [79] are good summaries of dynamic non-substitution theorems.

[40] Brock [31] noted this feature of the solution already arises in the representative agent one-sector Ramsey problem. The long-run interest rate is the planner's pure rate of time preference and it determines the long-run capital stock independently of the planner's one-period felicity (or reward) function.

[41] Of course (13.16) and (13.17) are equivalent in the case of a single representative household. The presence of multiple households with differing rates of impatience opens the possibility that the economy will arrive at a steady state depending on only one of the potential long-run rates on interest from the list of households' pure rates of time preference.

[42] Let the set of agents belonging to a particular type be indexed by the closed unit interval $[0, 1]$ endowed with Lebesgue measure. It is possible that an equilibrium

be reinterpreted as applying to the representative agent of each consumer type present in the economy. The second solution is to simply recognize the strategic possibilities that are available to any monopsonistic household, or even to any oligopsonistic group of agents. The model becomes a dynamic game.

Sorger [99] was the first to argue for the second, game theoretic alternative in a deterministic setting.[43] He proposed two possible strategic form games. The games' differences are traced to whether or not the players recognize their influence in the labor market or merely treat their labor decisions as taken in a competitive marketplace. He focuses on steady states and it is natural to ask if the stark distribution of income and wealth found in Ramsey's Theorem carries over to these strategic environments. Sorger constructed numerical examples where both households in a two person economy hold capital in a steady state equilibrium. He also gave a general proof that the most patient agent's capital stock would be larger than the second, or more impatient, agent's stock in any steady state configuration. Becker [7] completely characterizes the steady state equilibrium for the two-player model with Cobb-Douglas technology. He gives a criterion for determining whether or not both agents hold capital in the steady state, or just the most patient agent holds capital. Indeed, compared to the competitive case, the rate of return in a steady state is larger in the interior case as both agents hold capital, while the aggregate stocks are smaller than in the competitive case. This shows Sorger's example is robust, at least for this technology.

13.4.3 Heterogeneous Households and Identical Rates of Impatience

The case where households felicity functions may differ and their rates of impatience are identical merits comment. Suppose that $\varrho > 0$ is the common pure rate of time preference and let δ denote the common discount factor.

Let $f'(k^{\delta}) = \delta^{-1}$, and define an $(H-1)$ dimensional k^{δ}– simplex by

$$\mathcal{S} = \left\{ (k_1, k_2, \ldots, k_H) : \sum_{h=1}^{H} k_h = k^{\delta} \text{ and } k_h \geq 0, \text{ for all } h \right\}.$$

Let $\mathcal{S}^0$ be the subset of $\mathcal{S}$ with $k_h > 0$ for all households; $\mathcal{S}^0$ represents the possible distributions of capital stocks with everyone holding some capital in a stationary Ramsey equilibrium under the assumption of a common discount factor across households. In particular, for this case, the distribution of capital in a stationary Ramsey equilibrium is *indeterminate* — long-run equilibrium is

might treat a set of measure zero agents of a given type unequally, but their allocations would not matter for the overall economy-wide supply and demand balances.

[43] Sorger [99] also notes, following Sarte [95], that a competitive model with income taxation can also overturn Ramsey's conjectured long-run solution. The tax wedge on capital income is sufficient in some cases to raise the rate of capital's return so that both agents hold capital in a steady state.

consistent with **any** distribution of capital in $\mathcal{S}^0$. Under the common discount factor assumption, each household may hold positive stocks in a long-run equilibrium, but there may be an unequal distribution of capital and hence income, with income identified by consumption. Moreover, any distribution of initial stocks in $\mathcal{S}$ is consistent with long-run equilibrium. Apply the proof of Lemma 4 to agents without capital. This argument shows that setting consumption equal to the constant wage rate for those households without capital implies no alternative consumption sequence satisfying the budget constraint has a higher lifetime utility (the term $\varrho - \varrho_1 = 0$ in the proof as all rates of impatience are equal).

Indeterminacy in this context refers to the existence of a continuum of steady state capital distributions consistent with the condition for long-run equilibrium given (13.16) by when every agent's discount factor is identical. The common discount factor determines the aggregate capital stock according to the variational principle (13.16). That is, the production sector's demand side for capital determines the economy's total capital given the only viable candidate for a long-run interest rate, the common pure rate of time preference. Each household is happy to supply any amount of capital at the prevailing rate of interest. Hence, the economy's supply side for capital is perfectly elastic at the going interest rate.

13.4.4 Stationary Ramsey Equilibria with Flexible Time Preference

One expects the extreme distribution of capital in Ramsey's Theorem changes if agents have flexible rates of impatience that depend on the underlying consumption stream. It is also reasonable to think that the indeterminacy problems discussed above would also be ameliorated in a more flexible time preference framework.

It seems reasonable to focus on models with flexible time preference — the agents discount factor in a steady state depends on the underlying consumption stream. Recursive utility functions are one family of utilities that allow the steady state consumption stream to influence the corresponding discount factor. Recursive utility functions also generalize many properties of the constant discount factor additive models. Hence, it is possible to include the additive case as a special one in the broader recursive class. This means that the extreme distributions of capital and wealth found for the additive model can still arise in the recursive case, but there are also new cases where several agents hold capital.[44]

The basic idea is readily illustrated by an example of a two person economy — one agent has the conventional fixed discount factor and the other has a utility function with flexible time preference and a recursive structure. Assume this second person's utility function takes the form

[44] See Becker and Boyd [8] for a detailed account of recursive utility theory.

$$-\sum_{t=1}^{\infty} \exp\left(-\sum_{s=1}^{t} v(c_s^2)\right), \tag{13.18}$$

where $v : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly concave, increasing, and satisfies $v(0) > 0$. Equation (13.18) is known as the *Epstein-Hynes (EH) utility function* after the continuous time analogue introduced in [44]. They introduced this function to explore the role of flexible time preference on steady state results such as those reported in [5]. The EH utility from $\left(c_{T+1}^2, c_{T+2}^2, \ldots\right)$ appears in the last term of the following expression breaking down the utility over the entire consumption path into segments for the first T periods and the subsequent periods:

$$\begin{aligned} -\sum_{t=1}^{\infty} \exp\left(-\sum_{s=1}^{t} v(c_s^2)\right) &= -\sum_{t=1}^{T} \exp\left(-\sum_{s=1}^{t} v(c_s^2)\right) \\ &\quad + \exp\left(-\sum_{\tau=1}^{T} v(c_s^2)\right) \\ &\quad \times \left[-\sum_{t=T+1}^{\infty} \exp\left(-\sum_{\tau=T+1}^{t} v(c_s^2)\right)\right] \end{aligned}$$

hence, the utility of the tail of the program is just a time-shifted form of the utility of the original program — the identifying characteristic of a recursive utility function based on stationary preferences.

The steady state conditions for this economy are

$$\delta_1^{-1} = f'(\bar{x}) = 1/\exp(v(\bar{c}^2)), \tag{13.19}$$

where δ_1 is the first agent's utility discount factor and $\bar{x}$ is the aggregate steady state capital stock.[45] The allocations of consumption and capital satisfy

$$\bar{c}^1 + \bar{c}^2 = f(\bar{x}) - \bar{x}, \text{ and} \tag{13.20}$$

$$\bar{k}^1 + \bar{k}^2 = \bar{x}, \tag{13.21}$$

where $\bar{k}^h$ is household h's stationary capital stock. Since $\exp(v(0)) > 1$, one can solve $\delta_1 = \exp(v(\bar{c}^2))$ for the second household's consumption level and likewise find $\bar{x}$ by solving the familiar equation $\delta_1 f'(\bar{x}) = 1$ as a long-run equilibrium where both agents own capital is sought. These calculations determine the aggregate capital stock and the second household's stationary consumption level. The allocation equations (13.20) can be used to find the consumption of the first household and its capital stock.[46] This long-run solution is found assuming agents' have equalized their marginal rates of substitution and since

[45] See [8] for calculations of the necessary conditions leading to this equation. The basic idea is to compute the marginal rate of substitution between adjacent periods' consumption and evaluate it at a steady state. This marginal rate of substitution turns out to be equal to $1/\exp(v(\bar{c}^2))$.

[46] This is possible when $f(\bar{x}) > \bar{c}^2$ and $\bar{k}^2 = \bar{w} + \bar{r}\bar{k}^2 - \bar{c}^2 < \bar{x}$.

both possess capital. Hence, the addition of flexible rates of time preference can overturn the extreme distribution of capital found in the fixed discount factor model.

Ben-Gad [19] investigated the indeterminacy problem for agents with identical time preference rates, but in a recursive utility framework. He considered balanced-growth solutions, so he employed the homogeneity restrictions found by Dolmas [42] in the representative agent case. Ben-Gad found that the indeterminacy problem arises in that setup whenever agents' preferences are represented by the same recursive utility function and in some other cases as well. His results limit the prospects for recursive utility to supply a foundation for determinate balanced growth paths and raise corresponding questions for the case of stationary paths.

The preceding discussion has shown that the basic model can be altered in ways that might change the steady state distribution of wealth and income. However, there are other ways to change the steady state distribution result by alteration of the basic model. In the next subsection uncertainty is shown to produce a different steady state allocation even when rates of impatience are constant. This example is presented in some depth since the major part of this chapter is devoted to the fixed rate of impatience theory when (AII) holds. This focus is central since the equilibrium dynamics are mostly developed for that case and it is the differences introduced by household heterogeneity compared to the representative case that forms the crux of my story.

13.4.5 Uncertainty and Stationary Equilibrium

The introduction of uncertainty allows agents to save for precautionary purposes. This new feature of the model allows the stark distribution of capital found in Ramsey's Theorem to be overturned. Suppose that agents face technological shocks on the production side of the economy. When the economy receives a productivity shock limiting output agents earn a correspondingly low wage income. However, agents holding capital stocks get a return on their investment decision and enjoy a higher wage and capital income than agents without capital. Hence, these agents with capital can use their capital holdings as a buffer shock to self-insure against adverse production–wage shocks.

The economy described in this section originated in Becker and Zilcha's paper [16] based on the aggregate production shock model pioneered by Brock and Mirman [32] and elaborated on by Mirman and Zilcha [80].[47] Markets are incomplete and agents are heterogeneous in the sense that they have (at least) distinct utility discount factors. Households are infinitely lived and face borrowing constraints — just as in the deterministic models presented earlier. Households can accumulate capital in order to smooth consumption in the face of uncertain capital and wage income. Technological shocks of the kind in [32]

[47] Marimon [74] developed the stochastic analogue of Bewley's [23] model when there are stochastic technology shocks.

are responsible for the uncertainty in agents' income streams. A stationary stochastic rational expectations equilibrium in which the interest rate, wage rate, aggregate capital stock, and the distribution of wealth are all jointly determined in the presence of individual borrowing constraints is defined and theorems guaranteeing the existence of these equilibria are noted below.[48]

The following discussion, based on [16], is substantially compressed to focus attention on their two agent example in which both parties hold capital with positive probability. The presentation is informal as the technical developments in their paper lie outside this chapter's scope. The interested reader is referred to their paper for those important details.

The Stochastic Framework

The time-dated sequences of wages, rentals, consumption, capital, and output are taken to be random variables defined on a common basic probability space $(\Omega, \mathcal{F}, \mu)$, which represents the environment.[49] Each element of Ω is a sequence $\{\omega_t\}$, where ω_t represents the environment's state at time t. Here $\mathcal{F}$ is the collection of events and μ is the given probability measure. The stationarity of the environment is expressed in terms of a *shift operator* $T : \Omega \to \Omega$ defined componentwise by

$$(T\omega)_t = \omega_{t+1}.$$

This operator, and its inverse, T^{-1}, are assumed to be measure preserving. That is, for any event E, the probability of TE equals the probability of E.[50] This implies that for any integers t, t', and r, the joint distribution of $(\omega_t, \ldots \omega_{t+r})$ is the same as the joint distribution of $(\omega_{t'}, \ldots \omega_{t'+r})$. This shift operator is also taken to be *ergodic*, which means that if E is any invariant event, i.e. $TE = E$, then E has a probability of either 0 or 1. Examples of measure preserving ergodic environments are given by independently and identically distributed random variables or, a Markov chain with a transition probability matrix which is irreducible.

As attention here is focused on stationary economies, it is technically convenient to imagine the environment's history extends infinitely far back into the past when viewed from the starting date $t = 0$ and thereafter. So, the economy is modeled as having functioned for a very long time before the agents take up their equilibrium problems. The formal setup takes Ω to be the set of doubly-infinite sequences $\{\omega_t\}_{t=-\infty}^{\infty}$ with $\omega_t \in [a, b]$, $0 < a < b < \infty$, corresponding to the possible magnitudes of the total factor productivity shocks introduced below. Let Ω be endowed with the product topology, so $\mathcal{F}$ is Ω's Borel $\sigma-$

[48] The material in this section is based on Becker and Zilcha [16]. Formal proofs of the existence theorems stated below can be found in their paper.

[49] The model's underlying stochastic structure is derived from Radner [88] as presented for this many-agent Ramsey problem by Becker and Zilcha [16].

[50] See Neveu [82] for the background on the probability theory machinery used in this model. See Davidson [40] for an excellent overview of stationary processes.

field. Let $\mathcal{F}_t$ denote the σ–field generated by all the cylinder sets of the form $\prod_{m=-\infty}^{\infty} B_m$,where $B_m = [a, b]$ for all $m > t$.

Suppose that ξ is a real-valued function on Ω that depends at most on the states $(\ldots, \omega_{-2}, \omega_{-1}, \omega_0)$ giving the environment's history up through time 0. Assume that ξ is $\mathcal{F}_0$–measurable. This function induces a sequence of functions $Z_t = \xi(T^t\omega)$ that defines a stationary process. It is further assumed that the Z_t functions are $\mathcal{F}_t$–measurable. The usual interpretation of $\mathcal{F}_t$ as an information set applies.

The Stochastic Economy

Consider the following one-sector growth model with heterogeneous consumers and random technology. Denote by s the random shock to the production function and let s assume values in $[a, b]$. The aggregate production function is random. Given input x in period t and some realization of the shock s the resulting output is $f(x, s)$. For example, let $f(x, s) = s\beta x^\alpha$ for some $\beta > 0$ and $0 < \alpha < 1$ and the total factor productivity shock $s \in [a, b]$. The production function is increasing and strictly concave in its capital input argument and increasing in the productivity shock variable. It is twice continuously differentiable and has the property $f(0, s) = 0$ for each s. It also satisfies the Inada condition at the origin — $f_1(0, s) = \infty$ where $f_1 = \partial f/\partial x$. It is also assumed that there is some constant $\overline{k} < \infty$ such that for all $x > \overline{k}$ and all s, $f(x, s) < x$.

There are H infinitely lived consumers in this economy. Consumer h has a one-period utility function $u_h(\cdot)$ defined on consumption within a period. Each agent seeks to maximize the expected discounted stream of future utilities. Assume for each h, u_h is an increasing, twice continuously differentiable and strictly concave function satisfying $u'_h(0) = \infty$. As before, consumer h discounts future utilities by a factor $\delta_h, 0 < \delta_h < 1$. Make Assumption II, so the first household has the lowest rate of time preference.

Denote by $L^t_\infty \equiv L_\infty(\Omega, \mathcal{F}_t, \mu)$ the set of all essentially bounded real-valued functions on Ω which depend only on the history up to date t; $L^{t,+}_\infty$ stands for the nonnegative functions in L^t_∞ with $L_\infty(\Omega, \mathcal{F}_0, \mu) \equiv L_\infty$ and L^+_∞ the corresponding nonnegative functions. Recall that $\xi \in L_\infty$ is an essentially bounded function on Ω, i.e., $\xi \in L_\infty$ then its norm $\| \xi \|_\infty =$ **ess-sup** $| \xi(\omega) | = \inf_D \sup_{\Omega \setminus D} | \xi(\omega) | < \infty$, where D ranges over sets of $\mu-$measure zero. Obviously the same definitions apply to functions in L^t_∞. There is no loss of generality by restricting all the following random variables to be elements of an appropriate L_∞ space since there is a uniform bound on the production sector's capital and output.

Capital stocks and consumption are random variables since production is random as are the wage and rental rate processes. Moreover, at date t the aggregate capital stock $X_t(\omega)$ depends on the history up to date t; hence $X_t(\omega) \in L^{t,+}_\infty$. The consumption and savings at date t of each household h will depend on the interest rates and wages from date t on. Each household is endowed with one unit of labor that is supplied inelastically in each period.

Moreover, the households labor services are identical. Let $X^h_{-1}(\omega)$ be the initial capital owned by h at $t = 0$ with $X^h_{-1} \in L^+_\infty(\Omega, \mathcal{F}_{-1}, \mu)$. As all processes are taken to be stationary, there are nonnegative real-valued essentially bounded functions on Ω expressing each household's capital and consumption as well as the economy's wage rate and rental rate as functions of the environment's state. More formally, agent h's consumption process is $c^h_t(\omega) = c^h(T^t\omega)$; its capital process is $X^h_t(\omega) = X^h(T^t\omega)$ and the aggregate capital process is written as $X_t(\omega) = X(T^t\omega)$, where it is noted that in equilibrium it must be the case that $X(T^t\omega) = \sum_{h=1}^{H} X^h(T^t\omega)$ holds almost surely (abbreviated a.s.) for each time t. The associated rental process is defined for each time utilizing the production sector's profit maximization condition for $t = -1, 0, 1, 2, \ldots$:

$$1 + r_{t+1}(\omega) = 1 + r(T^{t+1}\omega) \tag{13.22}$$
$$= f'(X_t(\omega), \omega_{t+1}) \tag{13.23}$$
$$= f'(X(T^t\omega), T^{t+1}\omega) \text{ a.s.}; \tag{13.24}$$

the corresponding wage process is defined for $t = -1, 0, 1, 2, \ldots$ according to:

$$HW_{t+1}(\omega) = f(X_t(\omega), \omega_{t+1}) - f'(X_t(\omega), \omega_{t+1})X_t(\omega) \tag{13.25}$$
$$= f(X(T^t\omega), T^{t+1}\omega) - f'(X(T^t\omega), T^{t+1}\omega)X(T^t\omega) \text{ a.s.} \tag{13.26}$$

These processes lie in $L^{t+1,+}_\infty$. The stationarity condition implies (13.22) and (13.25) can be rewritten for each t as the equations

$$1 + r(T\omega) = f'(X(\omega), \omega_1) \text{ a.s.} \tag{13.27}$$

and

$$W(T\omega) = f(X(\omega), \omega_1) - f'(X(\omega), \omega_1)X(T\omega) \text{ a.s.} \tag{13.28}$$

Given stationary rental and wage processes, the corresponding aggregate capital process, and the household's initial capital process, a household is assumed to solve the following expected utility maximization problem:

$$\begin{gathered} \sup \;\; E_0 \textstyle\sum_{t=0}^{\infty} \delta_h^t u_h\left(c^h(T^t\omega)\right) \\ \text{subject to } c^h(T^t\omega) \text{ and } X^h(T^t\omega) \text{ in } L^{t,+}_\infty \text{ for for } t = 0, 1, 2, \ldots, \text{ and} \\ c^h(T^t\omega) + X^h(T^t\omega) = W(T^t\omega) + \left(1 + r(T^t\omega)\right) X^h(T^{t-1}\omega) \text{ a.s.} \end{gathered} \tag{13.29}$$

where E_0 is the conditional expectation given the state of the environment at $t = 0$. Assume an optimum in (13.29) exists and denote it by the processes c^{h*} and X^{h*}. Note that the budget constraint implies for any feasible process that

$$c^h(T\omega) + X^h(T\omega) = W(T\omega) + (1 + r(T\omega))X^h(\omega) \text{ a.s.} \tag{13.30}$$

Given an initial capital process X^h_{-1} for $h = 1, 2, \ldots, H$, a *stationary stochastic Ramsey equilibrium* (abbreviated SSRE) is a collection of stochastic processes $(c^{h*}, X^{h*}, W, 1 + r)$ satisfying

SSRE (1) (c^{h*}, X^{h*}) solve the household h's expected utility maximization problem given $(W, 1+r)$ and X^h_{-1};
SSRE (2) (13.27) and (13.28) hold for X^* and for $t = -1, 0, 1, 2, \ldots$

$$X^*(\omega) = \sum_{h=1}^{H} X^{h*}(\omega) \text{ a.s.} \tag{13.31}$$

Condition (13.31) is the stock equilibrium condition. As with the deterministic case, a form of Walras' Law implies the markets for new goods clears almost surely at each date. The profit maximization conditions for the production sector are expressed through equations (13.27) and (13.28).

Becker and Zilcha [16] proved the existence of a stationary stochastic Ramsey equilibrium under a supplementary concavity assumption governing each agent's income process (wages plus capital income). They required that both the wage and capital income processes to be a concave functions of the economy's aggregate capital stock. Their conditions are met whenever production is Cobb-Douglas. The details of their proof lie beyond this chapter's scope. However, their example showing Ramsey's Theorem does **not** carry over to the stochastic case is worth developing in some detail. Toward that end, the standard Euler or no-arbitrage inequalities for each household are recorded below. They are easily derived from a reversed no-arbitrage argument.

In a stationary stochastic Ramsey equilibrium, a necessary condition for each household's expected utility problem is:

$$u_h^{'}(c^h(\omega)) \geq \delta_h E_t[(1 + r(T\omega))u_h^{'}(c^h(T\omega))|\mathcal{F}_t] \text{ a.s.} \tag{13.32}$$

with equality whenever $X^h(T\omega) > 0$ with positive probability.

Example 13.4.1. Certainty vs. Uncertainty

As was shown earlier for the certainty case and in [5], in a steady state Ramsey equilibrium we must have:

$$X^1 = \bar{K} \text{ and } X^h = 0 \text{ for } h = 2, ..., H$$

where $\bar{K}$ solves $\max_K[f(K) - K(1+\overline{r})]$ where $1 + \overline{r} = 1/\delta_1$.[51] This result breaks down in the *stochastic* model.

Consider the following example with $H = 2$. The stochastic production function is given by:

$$f(X, \tilde{\theta}) = \tilde{\theta} X^{\alpha}, \quad \text{where } 0 < \alpha < 1.$$

The utility functions of individuals 1 and 2 are:

$$u_1(c) = \ln c \text{ and}$$

[51] I am using notation that is consistent with the stochastic model's presentation here. Thus, $\bar{K} \equiv k^{\delta_1}$, and so on.

$$u_2(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1.$$

Also let $1 > \delta_1 > \delta_2 > \frac{1}{2}$, and $\alpha < 1/(2\delta_1 - 1)$.

Let $< X_1^*(\omega), X_2^*(\omega), c_1^*(\omega), c_2^*(\omega); W(\omega), 1+r(\omega) >$ be a SSRE in this economy. Assume now that Becker's [5] result for deterministic stationary Ramsey equilibrium holds in the stochastic case as well. Since $\delta_1 > \delta_2$ this implies that $X_2^*(\omega) = 0$ a.s.; therefore the following expressions for W and r are obtained:

$$W(T\omega) = \frac{1}{2}\left[f(X_1^*(\omega), \omega_1) - X_1^*(\omega) f'(X_1^*(\omega), \omega_1)\right] = \frac{1-\alpha}{2}\omega_1 \left[X_1^*(\omega)\right]^\alpha \text{ a.s.} \tag{13.33}$$

$$1 + r(Tw) = \alpha\omega_1 \left[X_1^*(\omega)\right]^{\alpha-1} \text{ a.s.} \tag{13.34}$$

The budget equation for individual 1 can be written in this example as:

$$X_1^*(\omega) + c_1^*(\omega) = \frac{1+\alpha}{2}\omega_0 \left[X_1^*(T^{-1}\omega)\right]^\alpha \text{ a.s.} \tag{13.35}$$

Following the examples in Mirman and Zilcha [81], when the utility function is logarithmic and the production function is Cobb-Douglas, since $X(\omega) = X_1^*(\omega)$ a.s., the optimal consumption policy function of individual 1, $g_1(y)$, is linear in his beginning of period income y. Set $g_1(y) = \lambda y$ and observe the following functional equation must hold for all y (note that this equation is, basically derived from equation (13.32) and $X_1^*(\omega) > 0$ a.s. in this case),

$$\frac{1}{\lambda\frac{1+\alpha}{2}\omega_0\left[X_1^*(T^{-1}\omega)\right]^\alpha} = \delta_1 E_0\left[\frac{\left(\alpha\omega_1\left((1-\lambda)\frac{1+\alpha}{2}\omega_0 X_1^*(T^{-1}\omega)^\alpha\right)^{\alpha-1}\right)}{\lambda\frac{1+\alpha}{2}\omega_1\left[(1-\lambda)\omega_0\frac{1+\alpha}{2}X_1^*(T^{-1}\omega)^\alpha\right]^\alpha}\right].$$

Hence,

$$\omega_0^{-1} X_1^*(T^{-1}\omega)^{-\alpha} = \delta_1\alpha E_0\left[(1-\lambda)\omega_0\frac{1+\alpha}{2}X_1^*(T^{-1}\omega)^\alpha\right]^{-1} \text{ a.s.}$$

which implies that:

$$\lambda = 1 - \frac{2\delta_1\alpha}{1+\alpha}.$$

Since $1 > \delta_1 > \frac{1}{2}$ and $0 < \alpha < [2\delta_1 - 1]^{-1}$ we find that $0 < \lambda < 1$, hence $g_1(y) = \lambda y$ is a consumption policy function.

Consider now the optimization process of individual 2. Since, by our assumption, $X_2^*(\omega) = 0$ a.s. we have

$$c_2^*(\omega) = W(\omega) \text{ a.s.}$$

Namely,

$$c_2^*(\omega) = \frac{1-\alpha}{2}\omega_0 \left[X_1^*(T^{-1}\omega)\right]^\alpha \text{ a.s.}$$

Using the Euler conditions (13.32), when $X_2^*(\omega) = 0$ a.s. i.e., the inequality case,

$$u_2'(c_2^*(\omega)) \geqq \delta_2 E_0 \left[(1 + r(T\omega))\, u_2'\left(c_2^*(T\omega)\right)\right] \text{ a.s.}$$

We find for our case that

$$\left\{\frac{1-\alpha}{2}\omega_0 X_1^*(T^{-1}\omega)^\alpha\right\}^{-\gamma} \geqq$$

$$\delta_2 E_0 \left\{ \begin{array}{c} \alpha\omega_1 \left[(1-\lambda)\frac{1+\alpha}{2}\omega_0 X_1^*(T^{-1}\omega)^\alpha\right]^{\alpha-1} \\ \times \left[\frac{1-\alpha}{2}\omega_1 \left((1-\lambda)\frac{1+\alpha}{2}\omega_0 X_1^*(T^{-1}\omega)^\alpha\right)^\alpha\right]^{-\gamma} \end{array} \right\}$$

Simplifying this inequality we reach:

$$\left[X_1^*(T^{-1}\omega)\right]^{-\alpha\gamma+\alpha-\alpha^2(1-\gamma)} \geqq \delta_2 \delta_1^{\alpha-1-\alpha\gamma} [\alpha]^{\alpha-\alpha\gamma} [\omega_0]^{\alpha+\gamma-1-\alpha\gamma} \cdot E_0 \left[\omega_1^{1-\gamma}\right] \text{ a.s.} \tag{13.36}$$

Let $\epsilon > 0$ be arbitrarily small. Let us choose $0 < a < \epsilon/2,\ b = 1,\ \alpha < 1/2,\ 0 < \gamma < 1$ and the distribution of ω_t on $[a, b]$ such that:

(i) $E\left[\omega_1^{1-\gamma}\right] > 1/4$.

(ii) $\Lambda = -\alpha\gamma + \alpha - \alpha^2(1-\gamma) \geq \alpha/2$.

(iii) $\delta_2 \delta_1^{\alpha-1-\alpha\gamma} [\alpha]^{\alpha-\alpha\gamma} > 4\epsilon^\Lambda$.

Since $\alpha + \gamma - 1 - \alpha\gamma = (\alpha - 1)(1 - \gamma) < 0$ under the above choice of parameters condition (13.36) implies that:

$$X_1^*(T^{-1}\omega) \geq \epsilon \text{ a.s.}$$

However, it is known in the stochastic growth literature (see, for example Brock and Mirman [32]) that when the probability that $\omega_t \in [a, \epsilon]$ is positive, then:

$$\text{Prob}\left\{X_1^* < \in\right\} > 0.$$

Thus condition (13.36) cannot hold with probability 1. This contradiction demonstrates that our assumption that:in this equilibrium $X_2^* = 0$ a.s. cannot be true. Hence it proves the claim that the deterministic result does not carry over to the stochastic model.

The example shows that in contrast to the stark distribution of steady state capital obtained in the deterministic model, the introduction of uncertainty can lead to positive saving for the relatively impatient agent 2 for a set of states with positive measure. The presence of technological shocks means that the impatient agent has a buffer stock motive for holding capital that is absent from the deterministic story. Thus, the conclusion that one consumer will **always** have **all** the economy's capital is false in the stochastic setup. This implies that the stationary state is not determined by a variational principle based on a single agent's dynamic optimization problem as in the deterministic model. Indeed, the Dynamic Nonsubstitution Theorem fails to hold in the uncertainty version of a one-sector model.[52] Each household's single period return function matters as do their interactions with their respective discount factors along with the marginal conditions for profit maximization. Therefore the establishment of a stochastic stationary equilibrium for this economy must take into account the interactions of all the agents optimization problems (including the profit conditions). This naturally leads to a fixed point argument to demonstrate the existence of a stationary equilibrium. The detailed existence proof for a SSRE given in [16] works out the fixed point argument suggested in this last observation.

This example shows that risk aversion matters in determining the properties of a SSRE in a model with heterogeneous agents. However, their example did not yield the existence of a SSRE in which both households hold capital **almost surely**. However, it does seem reasonable to *conjecture* that for the case of two agents, the simultaneous solutions of their Euler **equations** from (13.32) where both hold capital almost surely is possible.

Becker and Zilcha also generalized their basic existence result for a SSRE. They proved a stationary equilibrium existence theorem allowing strategic decisions by households who recognize their saving-consumption choices influence the aggregate wage rate and capital returns. The full analysis of their result lies beyond this chapter's scope, which is focused on deterministic systems. The interested reader is referred to their paper for details.

13.4.6 Comments on Ramsey's Conjecture

Ramsey's conjecture has been demonstrated for a perfect foresight competitive model with heterogeneous infinitely-lived agents subjected to a borrowing constraint. The most patient household owns all the economy's capital in the long run. Yet, this result can be overturned with structural changes in the basic model. Flexible time preference, the introduction of uncertainty, and even a change in capital market structure to an imperfectly competitive one, have all been shown to modify Ramsey's stark conclusion in some instances. However, the various alterations to Ramsey's story do not exclude the possibility of an unequal long-run income or wealth distribution.

Irving Fisher noted that with unequal rates of impatience, a market economy starting from an equal distribution of wealth would evolve towards one that is unequal. Ramsey's steady state offers the most extreme interpretation of this long-run outcome. Along a dynamic equilibrium path at least some households would live, for a while, off their capital. For others, such as John Galsworthy's character, Old Jolyon, one simply does not live on one's capital.[53] The economy

[52] This observation was made by Brock [**31**] for the representative agent, one-sector discounted Ramsey optimal growth model.

[53] See Galsworthy ([52], p. 332) for a description of Old Jolyon and his attachment to his capital.

is populated by savers and spenders. The problem is to work out the model's explicit equilibrium dynamics. Will the long-run solution conjectured by Ramsey emerge from the self-interested consumption-saving decisions taken in the household sector, or will equilibrium paths have other characteristics?

Answers to these basic questions are developed in the next section. The dynamic analysis is confined to the case of heterogeneous agents with fixed rates of impatience. This basic model provides us with a benchmark case to glimpse the rich dynamic possibilities in more complex specifications of heterogeneous agent models.

13.5 Ramsey Equilibrium Dynamics

The stationary Ramsey equilibrium assigns all the economy's capital to the most patient household. Does this allocation of capital stocks represent the economy's long-run equilibrium? Put differently, is it a stable equilibrium? For example, if society's initial capital stock is equally distributed, will it evolve towards the unequal distribution found in the steady state equilibrium? More generally, will any equilibrium program starting from arbitrary initial allocations of capital converge to the steady state? The Ramsey equilibrium model must be thourougly analyzed to solve this stability problem. The analysis of the model includes proving a Ramsey equilibrium exists for an arbitrary initial distribution of capital. Ideally, the uniqueness of the equilibrium path should also be established. Necessary and sufficient conditions characterizing an equilibrium are also required in order to develop an equilibrium's dynamic properties. The stability question is addressable once the economy's equations of motion are firmly established.

The stability problem is the central question in Ramsey equilibrium theory. For the case of a representative agent, the Equivalence Principle implies that an equilibrium program has a monotonic aggregate capital sequence and it converges asymptotically to the economy's long-run steady state — the modified golden-rule capital stock. Does the introduction of additional agents, each a distinct individual with different tastes and endowments, sustain this result? Or, does the introduction of heterogeneous agents complexify the model's dynamics? Do the increased number of agents and their interrelationships generate equilibrium fluctuations — either regular cycles or chaos, or do their joint actions dampen oscillations and promote convergence to the stationary Ramsey equilibrium?[54]

[54] It is interesting to note by way of contrast that in dynamic biological models of competing and/or cooperating populations, increasing the variety and numbers of species may produce more stable ecosystems that dampen potential oscillations. This idea is motivated by empirical and laboratory observations. See May ([77], Chapter 3) for a detailed account of this problem in biological models of ecosystem dynamics. He also notes that there are predator-prey systems which are more stable than more complex ecosystems. Thus, the question of whether or not addi-

If Ramsey's steady state equilibrium is stable, and all households initially hold capital, then it must be the case that all but the most patient consumer run their capital to zero. This could, in principle, happen either asymptotically or **eventually** (that is, in finite time). In either event, the more impatient households's capital holdings are **impermanent**. However, a household always earns its wage payment at each time. Even if that household completely runs its capital to zero, it *always* has the option of saving and reacquiring capital. This point cannot be overemphasized since it lies at the heart of all the major difficulties in analyzing the model's dynamics.[55] Indeed, if the model's dynamics imply that individual capital holdings display **permanence** — starting from positive initial stocks the individual's capital remains positive and bounded away from zero for all time, then it must be the case that the stationary Ramsey equilibrium is unstable! A perturbation of the steady state that gives some capital to each household would never return to that steady state solution. Thus, it is critical to investigate whether or not individual capital holdings exhibit this permanence property (or some weaker, but closely related property).

The purpose of this section is to survey the model's dynamics. The existence of a Ramsey equilibrium is briefly noted. Sufficient conditions for a Ramsey equilibrium program are given that are particularly useful for constructing examples with particular dynamic features.

Once the sufficient conditions are fully articulated, I will turn to the investigation of the impermanence of individual capital holdings. The main result on that score is the Recurrence Theorem. Sufficient conditions for impermanence are also developed in the study of the so-called turnpike property. An example shows the subtle issues involved in untangling the impermanence of equilibrium paths from the general recurrence property. I show that, in general, the steady state is unstable. This result is presented as an example. The economy can generate a path with undamped oscillations in the form of a two-period cycle. Sufficient conditions are then developed for convergence of an equilibrium to the steady state. In the end, the stability of Ramsey equilibrium paths is only shown for a class of economies. The potential for more complex equilibrium dynamics than convergence is subsequently reviewed.

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tional species promotes stability or complexity does not have a single answer. The outcome turns on properties of the underlying dynamical system and may even be sensitive to values of key parameters. The Ramsey equilibrium model has similar characteristics.

[55] It is instructive to compare this situation to one found in biological models of ecosystems with many species. Once a species vanishes, it is permanently extinct and cannot be brought back to life (except in some notorious science fiction books and movies). The model's state space is the positive orthant of a Euclidean space of dimension equal to the number of species initially present. Each coordinate represents a particular species's density or biomass. The coordinate axes are invariant manifolds — once the dynamical system places a state variable on an axis, the system will remain confined to it for all later times. This need *not* be the case in the Ramsey model.

13.5.1 Existence of Ramsey Equilibrium and Sufficient Conditions for a Ramsey Equilibrium Path

Becker, Boyd, and Foias [9] gave general sufficient conditions for the existence of a Ramsey equilibrium. Their theorem included recursive, as well as some non-recursive utility specifications within a one-sector technology framework. Their general result applies to the time additive discounted utility case presented here. Their theorem, specialized to that framework, is stated below for reference.

Theorem 13.5.1. *Let* $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2, \ldots, H)$ *be a given economy. Then a Ramsey equilibrium exists.*

Becker, Boyd, and Foias [9] prove this result by setting up a mapping from a non-empty compact convex subset of the space of all real-valued sequences to itself. This set is $[0, b^m]^\infty$, the Cartesian product of infinitely many copies of the interval $[0, b^m]$. It is compact in an appropriate weak topology by Tychonoff's compactness theorem.[56] Their mapping is motivated by a tâtonnement procedure. They map capital sequences to their corresponding sequence of rental rates by inverting the condition for profit maximization at each time, $1 + r_t = f'(K_{t-1})$. This determines the wage rate, and consumer budget constraints. Consumers then solve their maximization problems. The aggregate of their capital holdings is formed. It is then adjusted to a new aggregate capital stock in $[0, b^m]^\infty$ that raises the returns in periods where demand exceeds supply, and lowers returns when supply exceeds demand. An equilibrium is a fixed point of this mapping. Of course, there are many technical details that must be carefully worked out for this map to be rigorously defined and its fixed points proven to be equilibria.

Necessary and sufficient conditions for a Ramsey equilibrium are useful for identifying suspected solutions and demonstrating the conjectured solution are, in fact, equilibria. The necessary conditions for a Ramsey equilibrium were stated in sections 3.1-3.2. They are the no arbitrage inequality, (13.2), the transversality condition, (13.4), the profit condition, (13.5), the distribution of wages, (13.6), and market clearing (E4).[57] The necessary conditions for a steady state suggested an allocation and pricing structure that would be a good candidate for an equilibrium. I used sufficient conditions for each agent's optimum (given the prices and allocations) to verify the conjectured equilibrium was in fact the equilibrium. This approach is adapted to many other examples of Ramsey equilibria. Following the statement of general sufficient conditions, I focus on a result formalized by Sorger [97] that applies to a two household economy in which the more impatient household never owns capital. His result

[56] Functional analytic concepts follow the terminology in Aliprantis and Border [1].

[57] Strictly speaking, the transversality condition may or may not be a necessary condition. Its interpretation as a type of no arbitrage condition can be found in ([8], Chapter 4).

provides a ready way to manufacture examples.[58] This is illustrated by constructing an example of Mankiw's [72] savers-spenders theory.

General Sufficient Conditions for Ramsey Equilibrium The necessary conditions for equilibrium turn out to be sufficient as well. This result turns on the assumed concavity of f and the u_h functions, as well as the satisfaction of the no arbitrage inequalities for each agent and their respective transversality conditions in combination with a market clearing equation.

Theorem 13.5.2. *(Sorger [97]). Let $\mathcal{E} = (f, \{u_h, \delta_h, k_h\}, h = 1, 2, \ldots, H)$ be a given economy. Assume that there exists a sequence $\{K_{t-1}, c_t^h, x_{t-1}^h\}$ such that the following conditions are satisfied for all $h = 1, 2, \ldots, H$ and all $t \geq 1$:*

1. $c_t^h > 0, x_{t-1}^h \geq 0, K_{t-1} > 0$;
2. $x_0^h = k^h$;
3. $\delta_h f'(K_t) u_h'(c_{t+1}^h) \leq u_h'(c_t^h)$ *with equality whenever* $x_t^h > 0$*;*
4. $c_t^h + x_t^h = \frac{1}{H}[f(K_{t-1}) - f'(K_{t-1})K_{t-1}] + f'(K_{t-1})x_{t-1}^h$*;*
5. $\sum_{h=1}^{H} x_{t-1}^h = K_{t-1}$*;*
6. $\lim_{t\to\infty} \delta_h^{t-1} u_h'(c_t^h) x_t^h = 0$.

   Then it follows that $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ is a Ramsey equilibrium for the economy $\mathcal{E}$ where the sequence of factor prices $\{1 + r_t, w_t\}$ is given by

$$1 + r_t = f'(K_{t-1}) \text{ and } Hw_t = f(K_{t-1}) - f'(K_{t-1})K_{t-1}.$$

This Theorem's proof is a straightforward adaptation of standard arguments for sufficiency of the Euler equation and transversality condition in the one-sector Ramsey optimal growth problem.

Sufficient Conditions for a Class of Two-Agent Economies. The construction of Ramsey equilibrium examples is greatly facilitated by the next result which gives sufficient conditions in a two person economy for sequences of aggregate capital, factor prices, consumption allocations, and individual capital allocations to be an equilibrium where *only the most patient consumer holds capital.* This is a special type of equilibrium and it is typically a dynamic one where the sequences changes over time. Of course, if the initial conditions are set just right, the equilibrium will be the stationary one. The importance of the lemma is that it asserts that given the most patient agent's function satisfying Assumption I, u_1, and discount factor, δ_1, and for any function u_2 satisfying Assumption I, there is a discount factor, δ_2, such that Assumption II holds. Moreover, the given sequences of aggregate capital, factor prices, consumption and individual capital form a Ramsey equilibrium. In this equilibrium, the first household owns the entire capital stock for all time and the second household's initial capital stock is zero.

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[58] This lemma is already implicit in Becker and Foias's [12] construction of a periodic Ramsey equilibrium path.

Lemma 13.5.1. *(Sorger [97]). Let u and f be functions satisfying Assumptions I and III, respectively, and let δ, ε_1, and ε_2 be positive real numbers with $\delta \in (0,1)$. Assume there exists a sequence $\{K_{t-1}, c_t\}$ such that the following conditions hold true for all $t \geq 1$:*

1. $\varepsilon_1 < c_t$, $\varepsilon_1 < K_{t-1} < \varepsilon_2$;
2. $\delta f'(K_t)u'(c_{t+1}) = u'(c_t)$;
3. $c_t = \frac{1}{2}[f(K_{t-1}) + f'(K_{t-1})K_{t-1}] - K_t$.
   For any function u_2 satisfying Assumption 1, and for any sufficiently small discount factor $\delta_2 > 0$ with $\delta_2 < \delta$, it holds that for the economy $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2)$ defined by $u_1 = u, \delta_1 = \delta, k^1 = K_0$, and $k^2 = 0$ has a Ramsey equilibrium $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ given by

$$1 + r_t \quad = \quad f'(K_{t-1}), c_t^1 = c_t \tag{13.37}$$

$$c_t^2 = w_t = \frac{1}{2}[f(K_{t-1}) - f'(K_{t-1})K_{t-1}], x_{t-1}^1 = K_{t-1}, \text{ and } x_{t-1}^2 = 0.$$

Proof. Choose f, u_1, δ_1, k^1, and k^2 as stated in the lemma's hypotheses. Let u_2 be any function satisfying Assumption I. It will be shown that for a suitable choice of δ_2 that the sufficient conditions of Theorem 12 are met for the economy $\mathcal{E}$. Conditions 2), 4), and 5) of Theorem 12 follow immediately from the assumptions of the present lemma and from the definitions of x_{t-1}^h and c_t^h. From assumption 1) of the present lemma, from Assumption III, and from (13.37) it follows that condition 1) of Theorem 12 is satisfied. Condition 3) of Theorem 12 for $h = 1$ follows from the assumption 2) in the present lemma. To verify condition 3) of Theorem 12 for $h = 2$ first observe that the definition of c_t^2 in (13.37) implies that there are positive constants ε_3 and ε_4 such that $\varepsilon_3 \leq c_t^2 \leq \varepsilon_4$ for all $t \geq 1$. From this observation it follows that condition 2) of Theorem 12 holds for $h = 2$ for any function u_2 satisfying Assumption I provided δ_2 is chosen smaller than the supremum of the function

$$g(K, c, \widetilde{c}) = \frac{u_2'(c)}{f'(K)u_2'(\widetilde{c})}$$

over the set $(K, c, \widetilde{c}) \in [\varepsilon_1, \varepsilon_2] \times [\varepsilon_3, \varepsilon_4] \times [\varepsilon_3, \varepsilon_4]$. Since this set is compact and g is continuous and positive, δ_2 can be chosen positive. Finally, condition 5) of Theorem 12 follows from Assumptions I and II and from Assumption 1) in the present lemma.

Savers and Spenders Redux: An Example

Mankiw's [72] savers-spenders model can be formulated as a Ramsey equilibrium for a two person economy. Agent one is the most patient consumer, acts as the bequest motivated individual, and saves. Agent two is the more impatient person and acts as the economy's spender. Agent two never saves and sets consumption equal to wage income in every period. This outcome represents a choice by agent 2 since the option to save is always present so long as the wage rate is positive. The choice of saving or spending for the more impatient household 2 is the only departure of the model from Mankiw's framework, although it is certainly consistent with a close reading of his interpretation of the spending agent's behavior. The model's specification is completed by assuming special functional forms for the production function and the first agent's u_1 function. The preceding lemma takes care of the second agent's preferences. This example draws on an example in Boyd's thesis (pp. 34-37, [28]), reinterpreted for Mankiw's model.

Let production be Cobb-Douglas with $f(K) = AK^{\alpha}$, where $0 < \alpha < 1$ and $A = 2^{1-\alpha}$. The total factor productivity index, A, reflects the inelastic supply of two units of labor from the household sector. The first household's function $u_1(c_t^1) = \ln c_t^1$. Its discount factor is any δ_1, with $\delta_1 \in (0,1)$. Along an equilibrium profile, agent one solves

$$\sup_{\{c_t^1, x_{t-1}^1\}} \sum_{t=1}^{\infty} \delta_1^{t-1} \ln c_t^1$$

subject to:

$$c_t^1 + x_t^1 = \frac{1}{2}[f(K_{t-1}) - f'(K_{t-1})K_{t-1}] + f'(K_{t-1})x_{t-1}^1,$$

with $k^1 = \mathbf{k} = K_0$ given and the usual nonnegativity constraints on consumption and individual capital holdings. The equilibrium is constructed so that $x_t^1 = K_t$ and $x_t^2 = 0$ *for all* t. Thus, the constraint facing the first agent takes the form:

$$c_t^1 + K_t = \frac{1}{2}[f(K_{t-1}) - f'(K_{t-1})K_{t-1}] + f'(K_{t-1})K_{t-1}.$$

Given the Cobb-Douglas technology specification, this constraint can be rewritten as

$$c_t^1 + K_t = BK_{t-1}^{\alpha},$$

where $B = [\frac{1}{2}(1-\alpha) + \alpha]A$ and $A = 2^{1-\alpha}$.[59] Notice that the first agent's optimization problem (given the second agent never holds capital) is just like the well-known canonical logarithmic economy – Cobb-Douglas example of Ramsey's optimum growth problem. The difference is that the constant B appears and it differs from the technology's total factor productivity index, A, to reflect the fact that the second agent is consuming fifty percent of the total wage bill at each time while never saving. The optimal policy functions for this model can be found using Boyd's symmetry technique [30] and are expressed by:

$$\begin{aligned} c_t^1 &= (1-\delta_1\alpha)BK_{t-1}^{\alpha}; \\ K_t &= \delta_1\alpha BK_{t-1}^{\alpha}. \end{aligned}$$

[59] Notice that $B < A$ holds. For example, let $\alpha = 0.5$. Then $B = 3\sqrt{2}/4$ and $A = \sqrt{2}$.

The aggregate capital sequence is found by iteration of the capital policy function with initial seed K_0. The stationary Ramsey equilibrium capital stocks are easily seen to be $\bar{K} = [1/(\delta_1 \alpha B)]^{1/(1-\alpha)}$. If $K_0 < \bar{K}$, then the equilibrium capital stock sequence $K_t \uparrow \bar{K}$ and if $K_0 > \bar{K}$, then $K_t \downarrow \bar{K}$. The first household's consumption shares the same monotonicity property. The second household's consumption is given by

$$c_t^2 = w_t = \frac{(1-\alpha)}{2} A K_{t-1}^{\alpha}.$$

Hence, its consumption is also monotonic in the same manner as the aggregate stock's monotonicity pattern. The transversality condition holds for each household as their consumption levels converge to the stationary equilibrium levels and the aggregate stocks converge as well. Therefore, Lemma 13 implies for any u_2 satisfying Assumption I, there is a $\delta_2 > 0$, with $\delta_2 < \delta_1$, such that the sequences so defined form a Ramsey equilibrium when the economy's endowments are distributed as $k^1 = K_0$ and $k^2 = 0$.[60] The basic convergence story for the Solow, Stiglitz, and Ramsey one-sector models apply to this realization of the savers-spenders theory. In particular, the Orthodox Vision applies to this specification of Mankiw's theory. Thus, there are economies for which the Ramsey equilibria are stable and converge to the stationary equilibrium allocation.

It is interesting to note that this result obtains for any choice of the first agent's discount factor. It can be very very small and there is still a smaller discount factor for the second household that will make this allocation and implicit factor prices into an equilibrium. Mankiw's savers-spenders model only depends on the relative levels of discounting between the agents. Both can have very high discount rates. It is the magnitude of the smallest discount rate (or, equivalently, the largest discount factor) which determines which agent is the saver and which is the spender.

The monotonicity of the aggregate capital sequence and its convergence to the stationary Ramsey equilibrium are interesting results in this context. How general are these properties? Is it true for arbitrary economies that the aggregate capital sequence is monotonic and convergent to the steady state? Does a household with no capital initially remain in the zero capital state forever?

13.5.2 The Recurrence and Turnpike Properties

The relatively impatient households have no physical assets in the stationary equilibrium. So, if that equilibrium is stable, it must be the case that those

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[60] This allocation is feasible. It is easy to verify that the sum of individual consumption levels at each time plus the capital stock at each time equals the total output available from fully employing the previous period's capital stock. The market for consumption and new capital goods clears at each date.

households capital holdings converge to zero. This is *not* always true in a Ramsey equilibrium. It is of fundamental importance to understand why this is so and if there are conditions under which the relatively impatient consumers drive their capital holdings towards zero.

Recurrence Property

There are a number of properties that an individual's capital sequence might satisfy. To keep our ideas general, just consider an arbitrary nonnegative sequence of capital stocks, $\{x_{t-1}\}$. A capital stock $\underline{x} > 0$ is a **threshold stock** if $x_{t-1} \geq \underline{x}$ for all t. That is, if this consumer starts with positive stocks at least as great as $\underline{x}$, then those stocks remain above that level for all time. If a household has a threshold stock, then its capital holdings can be zero in equilibrium (or arbitrarily close to zero) only if that individual starts with initial capital smaller than the threshold stock. An important theme developed below is that except for the most patient individual, Ramsey equilibria do not have threshold stocks at the individual level.

The nonnegative sequence $\{x_{t-1}\}$ is said to have a **positive capital state** at time t provided $x_t > 0$. Likewise, the **zero capital state** occurs at time t whenever $x_t = 0$. A positive capital state is said to be **permanent** if there exists a $\underline{x} > 0$ such that $\liminf_{t\to\infty} x_{t-1} \geq \underline{x}$. It is said to be **impermanent** whenever $\liminf_{t\to\infty} x_{t-1} = 0$. Clearly an impermanent path with $\limsup_{t\to\infty} x_{t-1} = 0$ actually converges to zero, that is, $\lim_{t\to\infty} x_{t-1} = 0$. The path is **persistent** if $\limsup_{t\to\infty} x_{t-1} > 0$ and it is **strongly persistent** if $\liminf_{t\to\infty} x_{t-1} > 0$. The following definition is singled out for its importance in the following analysis.

Definition 13.5.1. *The capital stock x is a* ***recurrent state*** *for the sequence $\{x_{t-1}\}$ if there exists $\{t_n\}_{n=1}^{\infty}$, $t_n < t_{n+1}$, and $x = x_{t_n} (n = 1, 2, \ldots)$.*

The most general property of Ramsey equilibrium paths turns out to be the *recurrence property* — the zero capital state is recurrent for each $h \geq 2$. That is, the relatively impatient individual's achieve a zero capital state infinitely often. Moreover, an example will show that it is possible for an individual to have a positive capital state infinitely often along an equilibrium path as well as realize the zero capital state infinitely often in the same sequence. Thus, it is possible for a path to be impermanent, yet persistent.

Fix the economy $\mathcal{E}$ meeting Assumptions I-III unless otherwise noted. Following Becker and Foias [12], I begin the development of the recurrence property with

Lemma 13.5.2. *If $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ is a Ramsey equilibrium for $\mathcal{E}$, then*

$$\lim \inf_{t\to\infty} \delta_1(1 + r_{t+1}) \leq 1. \tag{13.38}$$

Proof. Applying (13.2) to $h = 1$ yields after iterating, and noting $c_t^1 < b^m$ that

$$\prod_{t=1}^{T}\delta_1(1+r_{t+1}) \le \frac{u_1^{'}(c_1^1)}{u_1^{'}(c_{T+1}^1)} \le \frac{u_1^{'}(c_1^1)}{u_1^{'}(a)}.$$

Therefore,

$$\lim\sup_{T\to\infty}\prod_{t=1}^{T}\delta_1\delta_1(1+r_{t+1}) < \infty, \tag{13.39}$$

which easily implies (13.38).

The previous lemma shows that $\lim\sup_{t\to\infty} K_{t-1} \ge \bar{K} \equiv k^{\delta_1}$. Moreover $\lim\inf_{t\to\infty} K_{t-1} \ge \underline{K}$, where $\underline{K}$ is defined as the solution to $\delta_H f'(K) = 1$. This last fact follows from the next result (see ([12], pp. 177-78):

Proposition 13.5.1. *The assumptions of the previous Lemma imply $K_t \ge \underline{K}$ holds eventually; this is equivalent to $r_t \le f'(\underline{K}) - 1 \equiv \underline{r}$.*

Since $\underline{K} > 0$, this Proposition immediately implies the sequence $\{w_t\}$ must eventually be bounded from below by $\underline{w}(\underline{K}) = \underline{w}$, where $Hw(K) = f(K) - f'(K)K$. This easily implies

Corollary 13.5.1. *Under the hypotheses of Lemma 15,*

$$\lim\sup_{t\to\infty} c_t^h > 0 \ (h = 1, 2, \ldots, H). \tag{13.40}$$

The main recurrence result is given by the next theorem:

Theorem 13.5.3. ***(Recurrence Theorem).*** *If $\{1+r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ is a Ramsey equilibrium for $\mathcal{E}$, then the no capital state is recurrent for each $h \ge 2$.*

Proof. Suppose the no capital state is not recurrent for some $h \ge 2$. Then there is a t_0 such that $x_t > 0$ for all $t > t_0$ (where I drop the h superscript for this $h \ge 2$). Iterating (13.2) for $t_0, \ldots, T$, gives

$$\prod_{t=t_0}^{T}\delta(1+r_{t+1}) = \frac{u'(c_{t_0})}{u'(c_{T+1})}$$

or

$$\left(\frac{\delta}{\delta_1}\right)^{T-t_0}\prod_{t=t_0}^{T}\delta_1(1+r_{t+1}) = \frac{u'(c_{t_0})}{u'(c_{T+1})}$$

with $0 < \delta < \delta_1 < 1$.

But $(\delta/\delta_1)^{T-t_0} \to 0$ as $T \to \infty$ implies that $c_t \to 0$ by Assumption I and by (13.39). This contradicts (13.40).

It is important to note that the Recurrence Theorem holds for many different choices of technology, including the Cobb-Douglas production function. The stability theorem given below restricts the technology to one where the aggregate capital income is an increasing function of the aggregate capital stock. **Thus, the Recurrence Theorem is the most general result in the literature on the properties enjoyed in a dynamic Ramsey equilibrium.**

An interesting consequence of the Recurrence Theorem is

Corollary 13.5.2. *If* $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *is a Ramsey equilibrium for* $\mathcal{E}$*, then*

$$\lim \inf_{t \to \infty} c_t^h > 0 \ (h = 1, 2, \ldots, H).$$

This Corollary's proof is quite technical; the details are in Becker and Foias [12]. The Corollary says that each agents's consumption is strongly persistent — nobody consumes zero or even approaches zero asymptotically. This result distinguishes the Ramsey model with borrowing constraints from its complete market general equilibrium counterparts as found in Bewley [23], Coles ([35], [36]), Duran and Le Van [43], Le Van and Vailakis [68], and Rader ([85], [86],and [87]).

The Recurrence Theorem tells us households $h \geq 2$ achieve the zero capital state infinitely often. Their capital holdings are impermanent (so there is no threshold level). At any time in which such an agent's capital is zero, the agent can always consume less than its current wage income and thereby achieve a positive capital state one period later. It would be nice from an analytical view if once a household achieved a zero capital state, it maintained that state for all remaining times. Unfortunately, that is not the case in general. This fact is illustrated by the following example due to Michael Stern [100]. His example is constructed using the ideas in Becker and Foias's [12] example of an economy with an equilibrium whose aggregate capital stock repeated every other period in a two-cycle. The Becker-Foias example was constructed so that only the first household owned capital. Stern's example shows more. It describes a periodic equilibrium where the most patient household holds capital all the time, and the more impatient has capital infinitely often.

Example 13.5.1. (Stern [100]). There is a two household economy $\mathcal{E}$ and a corresponding Ramsey equilibrium in which the first household's capital stock alternates between two positive levels, x_H and x_L with $0 < x_L < x_H < b^m$, and the second household's capital stock alternates between zero capital and a positive capital stock, x. Moreover, the aggregate stock $K_H = x_H$ and $K_L = x + x_L$ with $K_L < K_H$. This implies that the relatively impatient households can hold positive capital stocks in infinitely many periods and these positive stocks are bounded away from zero. The second agent's stocks can be impermanent, and persistent, while exhibiting the recurrence property.

There are two households. The first, is called the dominant household as its discount factor is the larger of the two agents' factors. The second, is the non-dominant household. The example shows the dominant consumer cycles between a "low" and a 'high" level of capital and the non-dominant consumer cycles between zero capital and positive capital. Thus, the recurrence property cannot be improved without additional assumptions as the more impatient household can, in some circumstances, oscillate between holding capital and not. The example is constructed so that the non-dominant household will hold zero stocks when the dominant consumer's stocks are high, and hold positive capital when the dominant consumer's stocks are low. So, the sequence of aggregate capital stocks will be $\{K_H, K_L, K_H, K_L, \ldots\}$; the dominant consumer's stocks are $\{x_H, x_L, x_H, x_L, \ldots\}$ and the non-dominant consumer's stocks are denoted $\{0, x, 0, x, \ldots\}$. The initial aggregate stock is $K_H = x_H$. The corresponding consumption streams are denoted $\{c_L, c_H, c_L, c_H, \ldots\}$ and $\{c, c_0, c, c_0, \ldots\}$ for the dominant, and non-dominant consumers, respectively. The no-arbitrage conditions that must be satisfied are:

$$\delta_1 f'(K_L)u_1'(c_H) = u_1'(c_L) \tag{13.41}$$
$$\delta_1 f'(K_H)u_1'(c_L) = u_1'(c_H); \tag{13.42}$$
$$\delta_2 f'(K_H)u_2'(c) \leq u_2'(c_0) \tag{13.43}$$
$$\delta_2 f'(K_L)u_2'(c_0) = u_2'(c). \tag{13.44}$$

The budget balance conditions for each household are:

$$c_L + x_L = w_L + f'(K_H)x_H \tag{13.45}$$
$$c_H + x_H = w_H + f'(K_L)x_L; \tag{13.46}$$
$$c_0 = w_H + f'(K_L)x \tag{13.47}$$
$$c + x = w_L, \tag{13.48}$$

where

$$w_L = \frac{1}{2}\left[f(K_H) - f'(K_H)K_H\right] \tag{13.49}$$
$$w_H = \frac{1}{2}\left[f(K_L) - f'(K_L)K_L\right]; \tag{13.50}$$
$$K_H = x_H \text{ and } K_L = x_L + x. \tag{13.51}$$

Now, take the piecewise linear production function defined by:

$$f(K) = \begin{cases} 10 + 5K,\ 0 \leq K \leq 10; \\ 52 + 0.8K,\ 10 \leq K. \end{cases} \tag{13.52}$$

Here, the maximum sustainable stock is $b^m = 260$.

Let $K_L = 8$, $K_H = 12$, $x_L = 7.8$, and $x = 0.2$. Then (13.49) and (13.52) imply $w_L = 26$ and $w_H = 5$. The budget balance conditions (13.45) imply for these values that $c_L = 27.8, c_H = 32, c_0 = 6$, and $c = 25.8$. In order to satisfy (13.41), it is only necessary that

$$(\delta_1)^2 f'(K_L)f'(K_H) = 1,$$

which implies that $\delta_1 = 0.5$. So, any $u_1'(c_L) > 0$ will do and $u_1'(c_H) = 0.4u_1'(c_L)$. Also,

$$(\delta_2)^2 f'(K_L)f'(K_H) = 1$$

implies that $\delta_2 \le 0.5$. But it must also be the case that $u_2'(c) < u_2'(c_0)$ implies $\delta_2 < 0.5$. So, let $\delta_2 = 0.1$ and take

$$u_1'(27.8) = 1, \quad u_1'(32) = 0.4;$$
$$u_2'(6) = 2.4, \quad u_1'(25.8) = 1.2.$$

Any felicity functions that obey the above restrictions and the usual smoothness assumptions meeting Assumption I will work for this example. Also note that the transversality conditions are trivially satisfied. The production function (13.52) can be smoothed without a problem in order to satisfy Assumption III so long as $f'(8)$, $f(8)$, $f'(12)$, and $f(12)$ stay the same. The economy's stationary Ramsey equilibrium stock will fall in the interval $[8, 12]$ with $f'(k^{\delta_1}) = 2$. The felicity functions, production functions, capital endowments, and paths of consumption and capital clearly pass Sorger's sufficiency test for a Ramsey equilibrium.

This example shows something more. A Ramsey equilibrium's aggregate capital stock need not converge to the stationary equilibrium capital stock. In particular, an equilibrium path of aggregate capital can oscillate in a two-period cycle, also called a 2-cycle. Moroever, every choice of K_H and K_L in a neighborhood of 8 and 12, respectively, gives rise to an equilibrium with a two-cycle for a suitable choice of felicity functions and discount factors. On the other hand, the example of a two-person Savers-Spenders model, has monotonic paths of aggregate capital accumulation and clearly converges to the long-run steady state solution in the limit. Moreover, in that case, the more impatient individual, starting from the zero capital state, remains there for all time. What economic condition within the model is the source of fluctuations in one case, but absent in the other, stable case?

The Turnpike Property

The **turnpike property** obtains if every $h \ge 2$ eventually reaches a no capital position and maintains that state thereafter. This property expresses the steady state capital position of the relatively impatient households. Stern's example shows that without additional assumptions on technology and/or preferences, the turnpike property does not obtain. Yet, the Savers-Spenders example shows it does hold for some economies. Therefore, it is of some interest to work out sufficient conditions for it. Two types of results are available. In the first, it is shown that the turnpike property holds whenever each household $h \ge 2$ is sufficiently myopic in comparison to the first household's discount factor. The second type of result is based on showing the turnpike property obtains whenever the equilibrium aggregate capital stock sequence is convergent and that limit must be the steady state stock. Thus, if the steady state stock is stable (in the sense that it is the limit of the economy's equilibrium capital stock sequence), then it must be true that each household $h \geq 2$ necessarily achieves a zero capital state in finite time and maintains it thereafter. The option to return to a positive capital state is never exercised by those agents.

The first type of result is found in Becker and Tsyganov's paper (Lemma 4.4),[17]).[61]

Proposition 13.5.2. *Make Assumptions I-III. Let* $\{1+r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *be a Ramsey equilibrium for an economy* $\mathcal{E}$ *. If* $\delta_h << \delta_1$*, then eventually* $k_t^h = 0$ *for each* $h \geq 2$*.*

The second type of result, detailed in Becker and Foias (Propositions 4 and 5, [12]), is given below.

Proposition 13.5.3. *Make Assumptions I- III. Let* $\{1+r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *be a Ramsey equilibrium for an economy* $\mathcal{E}$ *. If* $\lim_{t\to\infty} K_t = K_\infty$ *exists, then* $K_\infty = k^{\delta_1}$ *and* $x_t^h = 0$ *for each* $h \geq 2$ *and all* t *large enough. Alternatively, if* $K_t \leq k^{\delta_1}$ *for all* t *large enough, then* $\lim_{t\to\infty} K_t = K_\infty$ *exists and the turnpike property obtains.*

The Savers-Spenders example satisfies the conditions of the latter proposition. Hence, it illustrates how convergence of the aggregate stocks to the steady state stock is linked with the turnpike property. Stern's example of a two-cycle fails this sufficient condition for the turnpike property. This takes us closer to answering when the economy converges and when convergence might fail.

On the Source of Periodic Equilibria

Stern's example shows equilibrium paths of capital accumulation need not converge to the steady state solution. As previously noted, his example is a variation on one developed by Becker and Foias [12] with the express purpose of showing equilibrium can exhibit a two-cycle.[62] Stern's example, and the one found in Becker and Foias [12], can be used to show how cycles can emerge as equilibria without further restrictions on the production function. Becker and

[61] Their result is derived for a two-sector model, but applies to one-sector models upon assuming both sectors have indentical technologies.

[62] The idea that borrowing constraints may be the source of cycles in dynamic competitive models arises in Bewley [24]. His model treats labor supply differently than the Ramsey model. Agents provide labor every other period. This provides consumers with a motive for smoothing consumption. He constructs an example with a periodic cycle in the one-period interest rate (or, rental rate on capital) as agents adapt to their fluctuating labor incomes. His model applies to an economy where all agents have the same discount factor. The agents save in periods when they have labor income, but do not save otherwise. The borrowing constraint can bind in the latter situation and a periodic solution is then shown to be possible.

Foias's example shows that the aggregate stock can fluctuate in a two-cycle while the second household consumes its wage and never saves. The turnpike property holds in their example. The first household owns the aggregate capital stock. Yet, the equilibrium fails to converge to the steady state solution. Their example's details are worked out in a similar way to the details of Stern's example, so those calculations are not included here. It is important to note that in both examples, the cyclic equilibrium stocks oscillate around the steady state stocks. In the case where capital increases from one period to the next, the marginal product of capital falls, capital income declines, the wage rate rises, and the total income received by the first household declines. The case where capital falls from one period to the next has a similar interpretation.

In the familiar representative household Ramsey optimal growth model, total income, $f(K)$, is monotonically increasing in capital; this income is the sum of the wage bill, $f(K) - f'(K)K$, and capital income, $f'(K)K$. In contrast, with many households and the situation of the Becker-Foias example, the total income of the first household differs from the economy wide income by an amount equal to the wage payments made to the noncapital holding agents. In this case, total income of household one need not be increasing in capital. Indeed, as the first household increases its capital, the total income received by it fell: capital income declined by more than the increase in the wage received by the household. The wage increased with capital accumulation due to the negative slope of the factor-price frontier. In the next period, this household naturally saves less than before thereby inducing a subsequent increase in its income. The fluctuations in income are perfectly foreseen together with the fluctuations in factor prices. The decline in capital income as capital increases is the key to their example. This decline is technologically based; it depends on f exhibiting a (variable) elasticity of substitution less than one. The first household is also sufficiently impatient so that incentives to smooth the fluctuations in consumption by arbitrage across periods do not exist. Thus, a decline in capital income as capital increases combines with the relative impatience of the first household to provide a cyclic equilibrium capital sequence.[63]

13.5.3 Equilibrium Dynamics with Capital Income Monotonicity

A sufficient condition for capital income monotonicity is that the production function's elasticity of substitution is greater than or equal to one. This condi-

[63] Woodford [105] considers a two class borrowing constrained model consisting of capitalists and workers. The former optimize their consumption-saving decision over an infinite horizon. They save, but do not work. The latter just consume their wages. The representative capitalist's income is the economy's capital income. It need not be a monotonic function of the aggregate capital stock. Woodford shows periodic and even chaotic trajectories are possible equilibria. Also, see the presentation in Guesnerie and Woodford ([56], pp. 301-311). The Ramsey equilibrium model has one fundamental difference with Woodford's model. The households all work and even the most impatient can always choose to become a capitalist by saving in the Ramsey equilibrium model.

tion is satisfied by the Cobb-Douglas production function. It turns out that this technological property combines with the other assumptions on the tastes and technology to yield a convergence theorem. The formal assumption is stated as:

Assumption IV: *For all* $K > 0$, $(d/dK)(f'(K)K) > 0$.

Becker and Foias's [12] convergence theorem is formally stated below. Its proof rests on showing that either the aggregate capital sequence is nonincreasing, or it is eventually smaller than the steady state stocks. In either case, it is then possible to show convergence as a consequence of Proposition 22. Formal details are found in their paper.

Theorem 13.5.4. ***(Convergence Theorem).*** *Make Assumptions I-IV. For an economy* $\mathcal{E}$ *let* $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *be a Ramsey equilibrium. Then the sequence of aggregate capital stocks,* $\{K_{t-1}\}_{t=1}^{\infty}$*, converges (eventually monotonically) to the stationary Ramsey equilibrium aggregate stocks and the turnpike property obtains.*

This theorem applies to the Savers-Spenders theory example. More generally, for any economy with a Cobb-Douglas production function, every Ramsey equilibrium converges to the steady state. Note that the theorem does not tell us equilibrium paths are unique, or even determinate. Moreover, the convergence is eventually monotonic.[64] Hence, the Orthodox Vision eventually holds. The theorem gives reasonable conditions for Ramsey's long-run conjecture to hold as the economy's short-run dynamics take it towards that stationary equilibrium, even when all agents start off with positive capital. In particular, just as Irving Fisher thought, an economy starting with an equal distribution of capital would eventually diverge from equal wealth as more impatient agents seek ever more current consumption at the expense of future consumption.

It is interesting to note that Becker and Foias' theorem also extends to some classes of two-sector models. Becker and Tsyganov [17] show this provided capital income monotonicity holds. However, the proof depends on which sector is more capital intensive than the other. In the case where the consumption goods sector is more capital intensive than the capital goods sector, then there are limitations on just how far the two-sectors' capital intensity can differ and a convergence theorem demonstrated, even with Cobb-Douglas production functions. Indeed, they show two-cycles can exist for a Cobb-Douglas economy provided the agents discount factors are sufficiently small. This result has parallels in the representative agent theory (see Nishimura and Yano [83] for the single agent theory). It is important to note that a cyclic equilibrium arises in the case where capital income monotonicity holds. This is one way in which the two-sector story differs from the one-sector model's properties. When the capital goods sector is more capital intensive than the consumption goods sector, Becker and Tsyganov [17] prove a convergence theorem as the two-sector

[64] Of course, the theorem includes other production functions so long as (AIV) holds.

analogue of the Becker and Foias [12] result provided capital income monotonicity holds (which is shown to hold in any two-sector model with Cobb-Douglas production functions).

Ramsey equilibria have also been studied in the case where all agents have a common discount factor. Hernandez [59] proves a version of the Becker-Foias convergence theorem under a capital income monotonicity condition. His result does use a different proof technique than Becker and Foias employed.

13.5.4 Special Ramsey Equilibria

The Recurrence and Convergence Theorems form the main general results available in the Ramsey equilibrium model. However, other dynamic possibilities have been exhibited by considering special solutions in which the turnpike property is assumed to hold from the model's start. That is, agents $h \geq 2$ have no capital endowment and the equilibrium path is constructed in such a way that only the most patient household holds capital. The resulting properties of the model are deduced by examining this special case where the aggregate capital stock and the first household's stocks are one and the same. The resulting path of aggregate stocks and consumption for the first household, together with the assignment of the per capita wage to the more impatient households always expresses an equilibrium for **some economy**. That is, the felicity functions of the other households and their discount factors can always be chosen to support the specially constructed path as an equilibrium. The results summarized below are based on this idea.

Special Monotonic Capital Sequences

The convergence theorem for economies satisfying the capital income monotonicity hypothesis applies to every equilibrium path. However, it is possible there can be more than one equilibrium, or that equilibrium might even be indeterminate. Examination of special equilibrium configurations sheds light on this problem. Becker and Foias ([13], [15]) consider an economy in which the turnpike property holds in the manner described above. If the capital monotonicity assumption holds, then they show that there is a dynamical system derived from the first agent's no arbitrage condition and the budget balance condition (where all other agents consume the wage rate at each time). This dynamical system is given by a second-order nonlinear difference equation. They consider the equation's linear approximation system at the stationary Ramsey equilibrium and show that it is a saddle point in the phase space consisting of current capital stocks and next period's capital stocks. In [15], they apply a stable manifold theorem to show there is a local stable manifold. Moreover, it can be extended to a global invariant manifold. That is, there is a continuous function defined on $(0, b^m)$ whose graph is the invariant manifold. Therefore, given a value of the initial capital stock, there is a unique value of the endogenously determined next period's stock such that this pair of stocks is a member of the graph. This process can be continued by iteration. The invariant manifold property says that the resulting capital sequence is in this set for all time. Becker and Foias [15] also show that this sequence converges monotonically to the stationary Ramsey equilibrium.[65] In this special case, the equilibrium is deteminate — given the initial aggregate stocks held entirely by the first household, there is one equilibrium path satisfying the turnpike property for all time. This result does not exclude other equilibria from existing, but suggests that it is reasonable to conjecture equilibrium is determinate when capital income monotonicity obtains.

Special Periodic Equilibria and Chaos

Special equilibria have already been noted to exist where the aggregate capital stock sequences exhibit two-cycles. Capital monotonicity fails in these situations. Becker and Foias [14] considered a dynamical system governing the first player's capital stock evolution under the assumption that all other agents began without capital and maintained that position thereafter. They showed without the capital monotonicity property, the stationary equilibrium could fail to be a saddle point. Indeed, they found a flip bifurcation. As the production function's elasticity of substitution falls below one, the stationary state's saddle point property is lost. A locally stable two-cycle is created.

Sorger [97] constructs examples of two person economies where there are equilibria with periodic solutions of every odd period. He also shows there is a two person economy in which there are multiple equilibria. His example shows that without capital income monotonicity, the steady state is not unique in the class of **all** Ramsey equilibria starting from that initial state. In his example, there is another equilibrium sequence of capital stocks from the same initial distribution of capital as in the stationary equilibrium. This alternative equilibrium is necessarily periodic. This nonstationary equilibrium has an odd period at least as great as three. His proof basically follows the type of constructions used in Becker and Foias's [12] cycle proof (and as illustrated in Stern's example).

Equilibria are locally unique if there are no other equilibria in any sufficiently close neighborhood of any given equilibrium. Local uniqueness represents determinacy. Given the predetermined value of the aggregate capital stock (when the turnpike property holds from the beginning), there is a locally unique choice of the endogenously determined value of next period's stock which gives rise to an equilibrium aggregate capital sequence. Otherwise, equilibrium is said to be indeterminate. Sorger [97] shows there is an economy which has an in-

[65] Becker and Foias [15] actually prove more. They give a theoretical iterative procedure by which the invariant manifold could, in principle, be constructed. They solve a functional equation for the function whose graph is the invariant manifold. Becker and Chen [11] implement this procedure numerically as well as employ projection methods based on Judd [60] to find the invariant manifold's equation, at least approximately.

determinate Ramsey equilibrium. This economy fails to satisfy capital income monotonicity and the turnpike property applies to agents $h \geq 2$. His proof is based on constructing a locally asymptotically stable two-cycle. That is, there is an economy with a two-cycle Ramsey equilibrium such that for any initial nearby aggregate capital stock assigned to the first household, there is an open set of choices for the next period's stock such that the resulting equilibrium aggregate stocks converge asymptotically to this two-cycle Ramsey equilibrium.

The indeterminacy of equilibrium opens the possibility that sunspot equilibria might also emerge in a Ramsey equilibrium for an economy without the capital income monotonicity property. Sorger [97] constructs an example of a rational expectations equilibrium which is a non-trivial stochastic process. That is, the deterministic Ramsey equilibrium model can have, in some economies, stochastic equilibria based on agents' expectations that the forecast wage and rental sequences form a stochastic process. These expectations are based on something other than the model's deep taste and technology parameters. Hence, the term sunspots — uncertainty that lies outside the economy's fundamentals. As in all sunspot models, agents merely have to believe there are random influences, from whatever source, and there can be an equilibrium where those beliefs are self-justifying. He argues that this feature makes the Ramsey equilibrium model similar to overlapping generations models that exhibit sunspots.

A chaotic Ramsey equilibrium has also been constructed by Sorger [98]. His example is based on an example of the second-order difference equation in the aggregate capital stocks derived from the first agent's no arbitrage and budget equations when the other agents satisfy the turnpike property for all time. The example verifies Marotto's [75] *snap-back repeller* condition for a self-mapping of $\mathbb{R}^n$. Intuitively speaking, a snap-back repeller is a fixed point of the transformation with the property that its Jacobian matrix at that point has all its characteristic roots on or outside the unit circle (the fixed point is said to be an *expanding repeller*). This implies any trajectory starting sufficiently nearby will **eventually** leave this neighborhood. The map must also have at least one trajectory starting close to the fixed point, which after leaving the small neighborhood in finite time, "snaps back" onto the fixed point exactly. Marotto [75] showed that a differentiable self map on $\mathbb{R}^n$ with a snap-back repeller also had periodic solutions for all sufficiently large periods, there would be an uncountable invariant set contained in $\mathbb{R}^n$ containing no periodic points such that trajectories initiated in that set showed sensitive dependence on initial conditions (adapted to the n-dimensional setting).[66] Thus, by giving an example of an economy with a snap-back repeller, Sorger [98] shows there are chaotic Ramsey equilibria. This result is troubling. A perfect foresight chaotic equilibrium's sensitive dependence on initial conditions implies that real households, restricted to making approximate calculations and forecasts, would be taken by the famous unseen hand to a time path of capital holdings, consumption,

[66] See Marotto [75] for details and connections with the literature on chaos in one-dimensional maps. His paper is basically an attempt to adapt the one-dimensional case to $\mathbb{R}^n$.

wage rates, and rental rates, that none had foreseen. Ultimately, whether the given Ramsey economy exhibits a stable solution (the Convergence Theorem holds), or one of the cyclic, or even chaotic equilibria emerges, turns on the economy's elasticity of substitution in production. This is, at least, a technological parameter. However, there is at least some empirical evidence that it is smaller than one. So, it is possible that the complicated dynamics found in the Ramsey equilibrium literature may yet prove to be the more interesting feature of the model than the convergence theory.[67]

13.6 Conclusion

Time preference influences intertemporal allocations. Ramsey's many agent model provides us with a framework for seeing how individual tastes can influence an economy's development and the distribution of its produce. The ways in which it differs from the representative agent theory may, with further research, provide us with a foundation for macrodynamic models with many agents where there interactions influence the level of macroeconomic activity and the conduct of macroeconomic policy.

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14. Dynamic Games in Economics

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14.1 Introduction

This section provides a general idea of the contents and organization of this survey of a large body of research in economics and related fields loosely defined by the adoption of the common methodology of dynamic games. We note at the outset that this class of games has also been referred to in various contexts as stochastic games[2], state-space games, sequential games, Markov games and difference (or differential) games. Given the breadth of this task and the long time span of the relevant strands of literature, some omission is inevitable.

The paradigm of dynamic games has long appeared natural and appealing in economic modeling, and has been adopted in many different subfields of the discipline. Various factors have prevented an even more widespread use of this theory, including in particular the complexity of this class of games, the difficulty of proving existence of usable and plausible equilibrium points, and the fact that closed-form solutions are possible only under very few specific functional forms.

We begin by describing the intended goals and limitations of this survey, the confines and special features of stochastic games in economics, and the general organization of this survey.

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[2] Shapley coined the term "stochastic games" by analogy to "stochastic processes", thus implicitly capturing the presence of dynamics. Since most applications actually involve models with deterministic transitions, this may appear somewhat misleading here, and "dynamic game" seems more appropriate. This is particularly true of studies considering open-loop equilibria, an essentially meaningless concept for games with chance moves.

14.1.1 Purpose and Scope of the Survey

This chapter provides a general survey of applications of stochastic games in economics and related fields. We identify clusters of studies according to methodological considerations (such as reliance on open-loop equilibrium, or perfect information, or computational simplicity), or to disciplinary catagories (such as industrial organization or capital theory). The primary concern was to come up with convenient and natural categories that would be consistent with the general purpose of this volume while appealing to a diverse readership.

Coverage may be somewhat detailed and self-contained, or in the form of a brief summary with a listing of references, depending on how broadly used the particular framework under consideration has been, and on space considerations. In particular, for literature strands defined by a common methodological framework, a summary of the main results is provided. As the survey is organized both along methodological and subject lines, it is inevitable that some overlap will appear across different sections. Such occurences are mentioned where appropriate so as to establish links between otherwise separate sections.

This survey will not encompass the continuous-time case, or differential games[3], except in some cases where the results have direct qualitative analogs in discrete-time, or are otherwise of relevance to issues raised here. Likewise, although some links exist with the repeated games literature and with the standard two-stage game framework, these will not be dealt with here.

The survey is targeted more at potential users of the theory of dynamic games and young researchers in economics, rather than to game theorists or expert users. As a result, it seemed appropriate to review in some detail some important definitions, game-theoretic notions and frequently invoked facts from the theory of dynamic games. In particular, the presentation highlights the fact that these useful facts follow in a straightforward way from standard results in dynamic programming theory, which by now are familiar to most economists (see e.g. Stokey, Lucas and prescott, 1989). On the other hand, for the sake of brevity, the reader is referred to other more self-contained or original sources for more detailed treatments and discussions, as well as for most proofs.

14.1.2 Special Features of Economic Applications of Dynamic Games

In relating the present survey to the rest of this volume, one must keep in mind that a number of motivations and widely held beliefs among economists have in large part shaped the nature and the focus of the studies invoking the theory of dynamic games in economics[4]. A brief account of these beliefs is now given.

[3] Basar and Olsder (1999) and Dockner et. al. (2000) are authoritative monographs on this related paradigm.

[4] Having said that, it is also true that different subfields of economics have been influenced by different disciplines (such as systems theory, or operations research, or mathematical game theory), and thus may reflect somewhat divergent practices and beliefs.

1. *Discounted payoffs.* As situations where anything that takes place in finite-time is irrelevant are unnatural in economics, only models with discounted payoffs have been considered. The presence of a positive rate of interest is ubiquitous in economic life. Thus, for economic models, dynamic games with undiscounted payoffs can be relevant only as a robustness check on a model with discounting.

2. *Pure strategies.* Due to a lack of compelling universal interpretation and to their inherent ex-post regret property, mixed strategies have enjoyed limited acceptance in economics in general, and this area is no exception. Mixed strategies have been considered only in a limited number of cases, such as situations where pure-strategy equilibria fail to exist. Another reason for the avoidance of mixes strategies is the computational difficulties associated with numerical procedures when the action space is uncountable.

3. *Uncountable state and action sets.* Owing mostly to the prevalence of calculus-based methods, there is a continuing tradition in economics of working with uncountable spaces, although the theory of stochastic games is much more complete for the case of finite state and action spaces, and reality is sometimes also more conform to the latter case (e.g. discrete units for prices). On the other hand, probably due to the latter two features, there is a recent trend of model-building utilizing finite state spaces in the area of industry dynamics.

4. *Simplicity.* To avoid fixed-point arguments in function spaces and complex systems of functional equations, several studies rely on specific functional forms that allow closed-form equilibrium strategies, such as the linear-quadratic and the myopic models. A key advantage of this approach, in addition to the obvious computational appeal, is that it allows for clear-cut comparative statics conclusions, otherwise a rare luxury in dynamic games. Another simplicity-inspired choice is the nature of the strategies allowed, with many models being limited to open-loop behavior often without compelling contextual economic justification.

5. *Predictive power of models.* Since applications are typically motivated by the search for clear-cut conclusions, only highly structured and relatively aggregated models of stochastic games (typically with scalar state and action sets) have been studied. This is also due to the relative complex nature of this class of games. Also history-dependent behavior and folk-theorem type outcomes have generally been avoided in applications, with some exceptions.

14.1.3 Organization of the Survey

The presentation is divided into nine sections, some of which are defined along methodological lines while others are devoted to particular subfields. The survey proceeds as follows. Section 14.2 provides a summary of the properties of open-loop and Markovian equilibria, and a list of references using the former concept. Section 14.3 considers special classes of dynamic games, such as linear-quadratic games, ubiquitous in economics and systems theory, and games with myopic equilibria. Section 14.4 deals with dynamic games of capital theory/resource extraction. Section 14.5 presents general results on the existence of subgame-perfect equilibrium in dynamic games, both in pure and behavioral strategies. Section 14.6 is devoted to applications in industrial organization, including in particular models of industry dynamics with entry and exit. Section 14.7 considers the class of dynamic games of perfect inofrmation. Section 14.8 gives a brief account of dynamic games with a continuum of players and some applications. Section 14.9 lists some work in experimental economics. Section 14.10 covers some work from the computational literature on dynamic games, some field-specific aspects of which are integrated in the appropriately related sections. Finally, Section 14.11 provides a summary of the basic notions and results from supermodularity analysis used in Sections 14.4, 14.5, and 14.7.

14.2 Open-Loop vs Markovian Equilibrium

Most of the literature on dynamic games in economics considers either open-loop or Markovian strategies. This section wil review the basic definitions and properties of the Nash equilibria resulting from players adopting these two types of strategies. A few studies though do consider more complex, partly history-dependent, behavior, as will be described in later sections.

14.2.1 Open-Loop Strategies in Deterministic Dynamic Games

Open-loop strategies were widely used in deterministic dynamic games (i.e. those with no chance moves) early on. This subsection provides an overview of the main properties of open-loop equilibrium, and refers throughout to a Markov dynamic game with deterministic transitions, i.e. one for which the reward and transitions functions depend on calendar time and current state and actions, but not on past values of states and actions.

An open-loop strategy is defined as a sequence of actions depending only on the initial state and on the date (or period). An open-loop strategy is thus a sequence of length $T+1$, where T is the last period in the horizon, possibly infinite. Open-loop behavior rests on the premise that the players simultaneously commit at the beginning of the game to a completely specified list of actions to be played without any possibility of revision during the entire course of the game. Hence, no contingency planning of any sort is possible.

Several important properties of open-loop equilibria are discussed next. To begin with, for *deterministic* Markov one-person dynamic programs, there always exists an optimal open-loop strategy under minor regularity conditions, so restricting oneself to open-loop policies results in no loss of value compared to using more sophisticated behavior. This fact is certainly intuitive, as is its failure in the presence of chance moves or stochastic transitions.

The game-theoretic analog of the above fact is perhaps less intuitive: In deterministic dynamic games, an open-loop equilibrium remains an equilibrium when the strategy spaces are expanded to include Markovian or history-dependent strategies. The reason is that if all of a given player's rivals are using open-loop strategies, the player cannot achieve a higher payoff by using more sophisticated strategies than open-loop. This follows directly by invoking the aforementioned fact for the player's best-response problem which, given the open-loop strategies of the rivals, is a *deterministic* Markov dynamic program.

Open-loop equilibria are generally not subgame-perfect.[5] By contrast, open-loop optima in one-player deterministic problems clearly satisfy the principle of optimality, since the optimal Markovian and open-loop policies lead to the same actions and states at every period.

Open-loop equilibria are typically much simpler to analyze than Markovian equilibria. In particular, the usually difficult question of existence of pure-strategy equilibrium is most often straightforward in the open-loop case, where it amounts to using Brouwer's fixed-point theorem with the action set viewed as a subset of R^∞ (with the product topology), under standard regularity conditions on the primitives. This relative simplicity is at the heart of the widespread use of open-loop strategies in the early stages of the adoption of dynamic games, despite the broad consensus that the commitment to a completely specified course of action over the indefinite future is not a realistic behavioral postulate in most cases of interest. The simultaneous presence of explicit long-term dynamics and of restricted static-like behavior seems contradictory. Furthermore, subgame perfection is broadly viewed as a desirable property of equilibrium behavior. Consequently, focus has markedly shifted towards Markovian strategies.

14.2.2 Open-Loop Equilibrium in Economic Models

This subsection provides a list of some of the studies in economics relying on open-loop behavior. Open-loop equilibrium originated and has been extensively analyzed in systems theory: See Basar and Olsder (1999) for a detailed account. For problems with a linear-quadratic structure (covered in Section 14.4), open-loop equilibria are easily computed and characterized.

A class of applications that is of interest both from an economic and from a methodological point of view deals with continuous-time patent races. This class includes Loury (1979), Lee and Wilde (1980), and Reinganum (1981) among others. These papers a priori postulate differential games with stochastic duration corresponding to the occurence of a success in an R&D project (i.e. a patent). The probability of a success for a firm follows an exponential distribution with parameter depending on the R&D expenditure of the firm. Due to the special structure of the model, in particular to the memoryless property of

[5] A generally overlooked fact is that for subgame perfection to be well-defined, it is clear that one needs to assume that the action sets are essentially independent of the state.

the exponential distribution, using Markovian strategies leads to an open-loop equilibrium, so that these games actually boil down to simple static games.

In the economics of natural resource exploitation and sustainability, studies that rely on the open-loop information structure tend to be older. They include, among many others, Salant (1976), Lewis and Schmalensee (1980), and Dasgupta and Heal (1979).

Various intrinsically dynamic problems in industrial organization were considered with open-loop strategies early on. Spence (1979) deals with investment in a new market, Spence (1981) and Fudenberg and Tirole (1983) propose models of the learning curve, Flaherty (1980a) studies dynamic limit pricing. Flaherty (1980b) and Spence (1984) are early attempts to model the effects of long run strategic process-R&D. A recent study is Athey and Schmutzler (2001).

The above list is far from complete, but can provide the reader with a flavor of the various approaches to, and results in, dynamic strategic competition relying on open-loop interaction.

14.2.3 Markovian Equilibrium

Throughout this subsection, we consider Markov and Markov-stationary dynamic games with deterministic or stochastic transitions. While the former class of games was defined in Section 14.2.1, the latter is characterized by the property that the reward function and transition law depend only on current state and actions, but not on calendar time or past values of states and actions. In the systems theory (Basar and Olsder, 1999), Markovian strategies are usually referred to as as feedback strategies, and sometimes as closed-loop (no-memory) strategies. The early macroeconomics literature relying on dynamic games has often adopted the terminology from systems theory as well.

A Markov strategy is defined as a sequence of functions, $\{\sigma_0, \sigma_1, ..., \sigma_T\}$, each mapping the state space into the action space, where T is the last period in the horizon, possibly infinite. Thus players are allowed to condition their current actions on calendar time as well as the current state, but not on past values of the state or actions. In other words, players condition their actions only on payoff-relevant variables. A Markov-stationary strategy is a Markov strategy $\{\sigma_0, \sigma_1, ..., \sigma_T\}$ for which $\sigma_i = \sigma_j$, for all i, j. Thus under such a strategy, players base their actions only on the value of current state.

Several important properties of Markov and Markov-stationary equilibria are discussed next. Recall that under minor regularity conditions, every Markov dynamic program has a Markov optimal policy and every Markov-stationary infinite-horizon dynamic program has a Markov-stationary optimal policy.

The following widely used fact is a simple consequence of the above argument. A Markovian (resp. Markov-stationary) equilibrium of a Markov (resp. Markov-stationary infinite-horizon) dynamic game remains an equilibrium when a broader class of strategies (e.g. depending on part of the history of the game) is allowed. Indeed, with all of a player's rivals playing Markovian (resp. Markov-stationary) strategies, the player's best-response problem is a Markov (resp. Markov-stationary) dynamic program, for which there exists a Markov (resp. Markov-stationary) optimal policy. This argument is equally valid in the presence of chance moves (i.e., stochastic transitions). These important justifying arguments, as well as the so-called one-shot deviation principle often invoked in the theory of repeated games, thus follow directly from the theory of dynamic programming in elementary ways.

An important property of Markov and Markov-stationary equilibria is that they are always subgame-perfect in a strong sense: Uniformly in the starting state. This is highly desirable from the point of view of applications.

The remainder of this survey is dedicated almost entirely to a presentation of economic studies employing the Markov equilibrium approach along with a summary of some general methodological results.

14.2.4 On Open-Loop Versus Markovian Equilibria

Here, a non-exhaustive list of remarks are noted on the comparative use of the two different types of strategies at hand for use in economic models. A consensus has formed quite some time ago around the fact that Markovian strategies are much more appropriate than open-loop strategies to approximate economic behavior, on various grounds and for most economic applications. Yet, given the relative simplicity of using open-loop equilibrium, some authors continue to adopt this notion. Historically, the notion of open-loop strategy originated in the systems theory literature, and gained prominence due to its equivalence with Markovian strategies in the context of one-player *deterministic* dynamic optimization problems.[6]

Another frequently adopted option in various applications, both in economics and systems theory, is to elect Markovian strategies, but adopt specific combinations of functional forms for the reward and state transitions that are known to give rise to convenient closed-form solutions (see Section 14.3). For further discussion of various aspects of the appropriateness of the different types of strategies and of their theoretical foundations, the reader is referred to Basar and Olsder (1999), Fudenberg and Tirole (1986, 1991), Reinganum and Stokey (1985) and Maskin and Tirole (2001).

An alternative way of thinking about open-loop strategies is as Markovian strategies where at each stage players use only constant functions of the current state. With open-loop strategies, a game may thus be viewed as a static game with sequences of length $T+1$ as strategy spaces.

A ubiquitous framework of analysis in industrial economics consists of modelling competing firms as making two decisions each, e.g. R&D levels and then prices or outputs. This can be done in a one-shot framework (with two decisions per firm), or in a two-stage game where R&D levels are chosen in the first stage, and outputs or prices are then chosen in the second stage, conditional on the

[6] When using Pontryagin's Maximum Principle to solve such problems, in continuous or discrete-time, one naturally considers open-loop policies.

observed R&D decisions. These two different timing structures can be viewed as relying on open-loop[7] and Markovian (or closed-loop strategies), respectively. For instance, Brander and Spencer (1983) provide a comparison of the two cases in a study of oligopolistic R&D. They find that, for their symmetric model, the open-loop equilibrium yields higher payoffs than the Markovian equilibrium. Apart from such contextual comparisons, no general results are known about the comparison of equilibrium payoffs under open-loop and Markovian behavior.

14.3 Special Classes of Dynamic Games

This section reviews two well-known classes of dynamic games characterized by very specific structures that give rise to simple equilibrium solutions. The first of these is the so-called linear-quadratic formulation, where linear refers to the state equation and quadratic to the one-period reward function. This class is characterized by linear equilibrium strategies and quadratic value functions. The second is the simple class of games characterized by separability assumptions that give rise to a myopic equilibrium, i.e. one for which equilibrium play consists of taking a constant action throughout the game.

14.3.1 Linear-Quadratic Dynamic Games

In a general linear-quadratic game, player i's objective functional is given by,

$$\max \sum_{t=1}^{T} \frac{1}{2} \left\{ s_{t+1}^{'} Q_{t+1}^{i} s_{t+1} + \sum_{j \in N} a_t^{j'} R_t^{ij} a_t^{j} \right\}$$

and the state equation is

$$s_{t+1} = A_t s_t + \sum_{j \in N} B_t^j a_t^j, \text{ for } t = 1, 2, ..., T,$$

where[8] s_t and a_t^j denote respectively the state vector (an element of $\Re^n$) and Player j's action vector (an element of $\Re^{l_j}$), at time t; A_t, B_t^j, Q_{t+1}^i, and R_t^{ij} are matrices with appropriate dimensions, R_t^{ij} is negative definite, and Q_{t+1}^i is symmetric and negative semi-definite.

A Markov equilibrium can be given in closed-form as follows. Let P_t^i be matrices satisfying, for $i = 1, 2, ..., N; t = 1, 2, ..., T$,

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[7] In such models, the use of open-loop strategies is much easier to justify as approximating real behavior, as it simply amounts to assuming a firm does not get to observe its rivals' new technology before choosing its output level.

[8] Matrices are denoted by capital letters, vectors by lower-case letters and the transpose operation by a 'prime' sign. Further details may be found in Basar and Olsder (1999).

$$[R_t^{ii} + B_t^{i'} Z_{t+1}^i B_t^i]P_t^i + B_t^{i'} Z_{t+1}^i \sum_{j \neq i} B_t^j P_t^j = B_t^{i'} Z_{t+1}^i A_t, \tag{14.1}$$

where the Z_t^i are defined recursively by

$$Z_t^i = F_t^{'} Z_{t+1}^i F_t + \sum_{j \in N} P_t^{j'} R_t^{ij} P_t^j + Q_t^i \text{ , with } \ Z_{T+1}^i = Q_{T+1}^i, \tag{14.2}$$

$$\text{and } F_t \triangleq A_t - \sum_{i \in N} B_t^i P_t^i$$

There is a unique Markov equilibrium if and only if (14.1) and (14.2) have a unique solution set $\left\{P_t^{j*}\right\}$, with equilibrium strategies (specifying player i's action vector at time t in a T-period horizon problem) and value function for Player i from stage t onwards given by, for $i = 1, 2, ..., n; t = 1, 2, ..., T$,

$$\gamma_t^{i*} = -P_t^{i*} s_t, \text{ and } V_t^i(s_t) = \frac{1}{2} s_t^{'}(Z_t^i - Q_t^i) s_t,$$

Some extensions of this class of games are now noted: (i) the state equation or the payoff functions may include additional linear terms. (affine-quadratic games.) The resulting equilibrium strategies are then affine functions of the state, (ii) exact conditions for P_t^{j*} to exist and be unique can be given in terms of invertibility of a composite matrix formed from the primitives of the problem, (iii) uncertainty in the form of an additive Gaussian vector (i.i.d. across time) in the state equation is easily incorporated, resulting in no qualitative changes in the solution, and (iv) the open-loop equilibrium is also easily computed.

Next, consider the infinite-horizon *undiscounted* stationary version of the game, obtained by letting $T = \infty$ and A, B^i, Q, R^{ij} be time-invariant. Sufficient conditions on the primitives that guarantee existence are not known at this point[9]. Nonetheless, the following partial answer (involving assumptions on derived objects) is known. Consider the following matrix equations, which are clearly limits of (14.1) and (14.2):

$$[R^{ii} + B^{i'} \overline{Z}^i B^i]\overline{P}^i + B^{i'} \overline{Z}^i \sum_{j \neq i} B^j \overline{P}^j = B^{i'} \overline{Z}^i A, i = 1, 2, ..., N, \tag{14.3}$$

where Z^i is defined by

$$\overline{Z}^i = \overline{F}^{'} \overline{Z}^i \overline{F} + \sum_{j \in N} \overline{P}^{j'} R^{ij} \overline{P}^j + Q^i, \text{ and } \overline{F} \triangleq A - \sum_{i \in N} B^i \overline{P}^i. \tag{14.4}$$

[9] By contrast, nice sufficient conditions are available in the one-player case (e.g. Bertsekas, 1976, pp. 73-80) and in the zero-sum case (Basar and Bernhardt, 1995).

Proposition 14.3.1. *Suppose there exist two N-tuples of matrices* $\{\overline{Z}^i, \overline{P}^i\}$ *satisfying (14.3), (14.4). Let* $\overline{F}_i \triangleq A - \sum_{j\neq i} B^j \overline{P}^j$ *and* $\overline{Q}_i \triangleq Q^i + \sum_{j\neq i} \overline{P}^{j'} R^{ij} \overline{P}^j$*. If the pair* $(\overline{F}_i, B^i)$ *is stabilizable* [10] *and the pair* $(\overline{F}_i, \overline{Q}_i)$ *is detectable*[11]*, then:*
(i) there is a Markov-stationary equilibrium where player i's strategy is $\overline{\gamma}^{i*}(s) = -\overline{P}^i s$ *and his (finite) payoff is* $\frac{1}{2} s_1' \overline{Z}^i s_1$*, and*
(ii) the equilibrium system dynamics $s_{t+1} = \overline{F} s_t$ *is stable (i.e.* $\lim_{t\to\infty} D^t = 0$*).*

While (14.3) and (14.4) can be viewed as the limit of (14.1) and (14.2) as $T \to \infty$, (14.3) and (14.4) can have other solutions that are not related to the finite-horizon solution. Under the above assumptions of stabilizability and detectability, the latter would also constitute equilibria of the infinite-horizon game.

There is an extensive literature in various areas of economics analysing models that constitute either a special case or a variant of the above framework (some in continuous-time). Furthermore, all infinite-horizon models used in economic models have discounted rewards.

A very partial list of references follows. Fershtman and Kamien (1987), Reynolds (1987, 1991), and Beggs and Klemperer (1992) develop various models of dynamic oligopolistic competition. Lindsey (1989) deals with natural resources. Pindyck (1977), Kydland and Prescott (1977), Barro and Gordon (1983), Cohen and Michel (1988) and Jensen and Lockwood (1998) are contributions to the literature on macroeconomic policy games. Some more examples are given in Section 14.6.

14.3.2 Dynamic Games with Myopic Equilibrium

A stochastic game is said to have a myopic equilibrium if a static game, which is usually not the one-period game, can be constructed from the primitives of the stochastic game with the property that the infinite repetition of an equilibrium of the static game constitutes an equilibrium for the stochastic game. We provide sufficient conditions on the reward and transition functions ensuring the existence of a myopic equilibrium for a discounted stochastic game, and then list some applications of this approach. Our presentation follows Heyman and Sobel (1984).

Proposition 14.3.2. *Assume that a stochastic game is such that:*
(i) the reward function is additively separable: $r_i(s,a) = K_i(a) + L_i(s), \forall s, a, i.$
(ii) the transition law is state-independent:

[10] This is defined as follows: The matrix $[B^i, \overline{F}_i B^i, \overline{F}_i^2 B^i, ..., \overline{F}_i^{n-1} B^i]$ has full rank. Intuitively, this ensures there is a pair of strategies that will drive the state to 0 in finite time.

[11] This is defined by $(\overline{F}_i', \overline{Q}_i')$ being stabilizable.

$$\Pr ob(s_{t+1} = s^{'}/s_t = s, a_t = a) = p(s^{'}/a) \quad or \quad s_{t+1} \sim \xi(a_t).$$

(iii) the one-shot game where player i's action set is A_i and his payoff is

$$\gamma_i(a) = K_i(a) + \beta_i E\{L_i[\xi(a)]\}$$

has a pure-strategy equilibrium a^*.
(iv)$P(\xi(a^*) \in \{s : a^* \in A_s\}) = 1$ (a^* is feasible next stage for any current state. Then the strategy where, at every stage t and state s, player i plays a_i^* if a_i^* is feasible and any feasible action otherwise is a Markov-stationary equilibrium of the infinite-horizon game.

While this set of sufficient conditions is obviously very restrictive, there are quite a few economic settings where this sort of myopic behavior constitutes a reasonable approximation of economic behavior. Furthermore, in some settings, this class of games may be appropriately viewed as providing a bridge between static and dynamic analysis.

There are several applications in economics and management science for which this class of games provides a natural framework of analysis. For an early attempt at bringing quantity and price competition together in an oligopoly model with inventory and uncertain demand, see Kirman and Sobel (1974). Different one-player inventory control models have myopic optimal policies: see [Heyman and Sobel (1984), Chapter 3] for references. In the context of fisheries, see Sobel (1982). Noncooperative advertising models with this special structure have also been analyzed by Monahan and Sobel (1994). A simple model of dynamic R&D competition with myopic equilibrium investment strategies is developed by Blonski (1999).

14.4 Common-Property Productive Assets

This is one of the areas of economics that has witnessed a high level of research activity involving dynamic games as the key methodological approach. The main model considered is a strategic version of the well-known discrete-time one-sector optimal growth model. In view of the central role played by the latter model in economic dynamics, in particular in macroeconomics and in resource economics, it should come as no surprise that the strand of literature reported in this section has enjoyed some prominence relative to other areas relying on the dynamic games paradigm.

Consider two agents who jointly own a productive asset (or natural resource) and who consume some amount of the available stock at each stage in order to maximize their (individual) discounted sum of utilities. The payoff and feasible set of (say) Agent 1 is

$$\sum_{t=0}^{T}(1-\lambda_1)\lambda_1^t u_1(a_t)$$

and the state transition law is given by

$$s_{t+1} = f(s_t - a_t - b_t),, t = 0, 1...$$

where s_t is the asset stock level; a_t and b_t are the consumption levels of Agents 1 and 2 at time t; $u_i : [0,\infty) \to [0,\infty)$ is Agent i's one-period utility function, with λ_i being his discount factor; and f is a natural growth law (or production function) mapping "savings" into the next stock. Without extraction capacities, this is a generalized game in that the two actions are time t are jointly constrained in a natural way by

$$a_t + b_t \leq s_t, \quad a_t \geq 0 \text{ and } b_t \geq 0.$$

14.4.1 The Beginnings of this Literature

The seminal paper of Levhari and Mirman (1980) considers a specific version of the above model obtained by letting $u_1(a_t) = \log a_t$ and $f(s_t) = s_t^{\alpha}, 0 < \alpha < 1$.

Using standard induction, Levhari and Mirman show for this "Great Fish War" that (i) for finite horizon, there is a unique Markovian equilibrium with linear consumption strategies and logarithmic value functions[12], (ii) the limits of these strategies as the horizon tends to infinity, which for Agent 1 (say) is $\frac{\alpha\lambda_2(1-\alpha\lambda_1)s}{1-(1-\alpha\lambda_1)(1-\alpha\lambda_2)}$, form a Markov-stationary equilibrium of the infinite-horizon game[13], (iii) a tragedy of the commons prevails in both cases, in that the given equilibria are not Pareto-optimal and lead to over-consumption of the resource (relative to a Pareto-optimal path), and (iv) the equilibrium resource stock converges to a unique globally stable steady-state level $\overline{s} = \{\frac{1}{\alpha\lambda_1} + \frac{1}{\alpha\lambda_2} - 1\}^{\frac{\alpha}{\alpha-1}}$.

Cave (1987) termed the "Cold Fish War" the situation where the two agents, observing the entire history of play, employ trigger (history-dependent) strategies. Specifically, agents coordinate on cooperative extraction paths secured by the threat of reversion to the Markov-stationary strategies in case of defection. Assuming equal discount rates, Cave characterizes the resulting open set of equilibria, which are clearly subgame-perfect. A simple necessary and sufficient condition is given for this set to include a Pareto-optimal extraction path.

The Levhari-Mirman analysis has been extended to more complex resource dynamics, including interactive fish species by Fisher and Mirman (1994, 1997), as well as to market interactions by Datta and Mirman (1999).

[12] Interestingly, the complementary choices of functional forms for the utility and biological growth functions in this model produce the same convenient qualitative results as in the linear-quadratic case. Here, due to the linearity of the equilibrium strategies, the value functions inherit the log nature of the utility function.

[13] Uniqueness of equilibrium in the infinite-horizon game remains an open question to date. There may exist infinite-horizon equilibria that are not necessarily limits of the finite-horizon equilibrium strategies.

14.4.2 General Functional Forms

The above model, with general utility and growth functions, is easily formulated in its stochastic version wherein the transition law becomes (here p is a transition probability mapping savings into distributions on the next stock)

$$s_{t+1} \sim p(\cdot / s_t - a_t - b_t)$$

and each player's payoff is the expectation of the present value of utility over an infinite horizon. The model is a familiar one in economic dynamics, as the one-player version of this game is the standard optimal growth model under uncertainty (Brock and Mirman, 1972).

The Symmetric Case. The above problem is considered with identical agents and the following assumptions (here, F is the distribution function associated with p):

(A1) *Agent symmetry:* $u_1 = u_2$ *and* $\lambda_1 = \lambda_2$.

(A2) u_i *is strictly increasing and strictly concave.*

(A3) *(i)* $F(s'/\cdot)$ *is weakly continuous, with* $F(0/0) = 1$.
(ii) $F(s'/\cdot)$ *is strictly decreasing for every* s'.

The meaning of **(A1)** is clear. **(A2)** is standard. **(A3)** states that the distribution of the next state first-order stochastically increases in the savings.

Under these assumptions (plus some minor regularity conditions), Dutta and Sundaram (1991) establish existence of a symmetric Markov-perfect equilibrium with consumption strategies and value functions respectively in:

$$\widetilde{\Sigma} \triangleq \left\{ \sigma : [0, \infty) \to [0, \infty) : \sigma(0) = 0 \text{ and } \frac{\sigma(s') - \sigma(s)}{s' - s} \leq 1, \forall s, s' \right\}.$$

and

$$\widetilde{\Lambda} = \{ v : [0, \infty) \to [0, \infty) : v \text{ is bounded and nondecreasing} \}$$

Their approach relies crucially on symmetry and cannot be extended to asymmetric settings. The properties of this equilibrium have been investigated in detail in Dutta and Sundaram (1992, 1993).

Finally, numerical methods with rigorous lattice-theoretical grounding were developed by Datta et. al. (2004, 2002) for strategic settings including the symmetric common-property game as well as dynamic general equilibrium settings.

The Asymmetric Case. Amir (1996a) considers the above problem without symmetry (i.e. **(A.1)**) but with the following additional assumptions (**(A5)** is stated say for agent 1):

(A4) $F(s'/\cdot)$ is strictly convex for every s'.

(A5) $a_t \leq K_1(s_t)$, with $K_1(\cdot)$ continuous and uniformly bounded, with $K_1(0) = 0$, $0 \leq K_1(s') - K_1(s) \leq s' - s$ for all $s' > s$, and $K_1(s) + K_2(s) < s$, for all s.

(A4) is a a strong stochastic convexity assumption on the growth process, which together with **(A3)** (ii) has a natural economic interpretation: The probability that the next state exceeds a given level s', i.e. $1 - F(s'/\cdot)$, is increasing at a decreasing rate in the savings. Nevertheless, it is fairly restrictive in that it rules out the deterministic case and requires the effective state space to be all of $[0,\infty)$: see Amir (1996a-b) for details. As to **(A5),** it is natural in many contexts and serves to rule out trivial equilibria with stock exhaustion.

The effective spaces of value functions and consumption strategies are

$$\Lambda \triangleq \{v : [0,\infty) \to [0,B] : v \text{ is continuous and nondecreasing}\}$$

and

$$\Sigma \triangleq \left\{\sigma : [0,\infty) \to [0,\infty) : \sigma(0) = 0 \text{ and } 0 \leq \frac{\sigma(s') - \sigma(s)}{s' - s} \leq 1, \forall s', s\right\}$$

The main result in Amir (1996a) is now given.

Theorem 14.4.1. *Under Assumptions (A2)-(A3), we have:*
(a) The infinite-horizon discounted stochastic game has a Markov-stationary equilibrium, with strategies in Σ and corresponding value functions in Λ.
(b) For every finite-horizon T $(t = 0, 1, ..., T-1)$, there exists a unique Markov equilibrium in Σ^T and corresponding value functions in Λ^T.

Exploiting supermodularity and diagonal dominance arguments in ways similar to the proof of Theorem 14.5.1 in Section 14.5, one can show that the best-response to a strategy in Σ is unique and lies in Σ, so that the best response mapping, from $\Sigma \times \Sigma$ (with the topology of uniform convergence) to itself, has a fixed-point. The details are not presented here.

14.5 General Existence Results

This section summarizes the literature dealing with the abstract existence question for subgame-perfect equilibrium in dynamic games with simultaneous moves. Existence for dynamic games with perfect information is reviewed in Section 14.7.

While the literature on existence of mixed (or actually behavioral) strategy equilibrium is extensive, spanning over half a century starting with the seminal paper by Shapley[14] (1953), the literature dealing with existence of pure-strategy

[14] In order to fully appreciate the contribution of this seminal paper, it is worthwhile to point out that many of the basic results behind the theory of Markov-stationary dynamic programming (such as the contraction property in value function space and the optimality of Markov-stationary policies) were already unequivocally laid out in Shapley's (1953) seminal paper, albeit in the framework of finite states and actions, over a decade before being rediscovered again (Blackwell, 1965 and Denardo, 1967).

equilibrium is much more recent and quite a bit more sparse. While the former literature involved mostly mathematical game theorists, the latter was largely the work of economists.

14.5.1 Existence of Mixed-Strategy Markov Equilibrium

The theory of stochastic games has been an active field of research in pure game theory. The state of the art of this field is covered in a recent comprehensive volume edited by Neyman and Sorin (2003). The main issue that the purely mathematical literature has dealt with is the existence of equilibrium in behavioral strategies for various classes of games distinguished by the following features: zero or nonzero sum, finite or uncountable state and/or action spaces, etc[15]... To summarize the main results in this literature, we will not formally extend the previous definitions of Markov, Markov-stationary pure strategies and expected discounted payoffs, as these are easily adapted to the case of *mixed strategies* considered here.

Shapley (1953) showed that every two-player zero-sum stochastic games with finite state and action spaces admits a value as well as Markov stationary optimal strategies. After many extensions of the work of Shapley, the list of which we skip here for the same of brevity (see Mertens, 2002 or Neyman and Sorin, 2003 for a review), Mertens and Parthasarathy (2003) establish existence of a subgame-perfect equilibrium in strategies that are (partly) history-dependent, assuming the transition law is norm-continuous in the actions[16], in addition to other standard regularity conditions. Shifting focus away from Nash equilibrium, Nowak and Raghavan (1992) and Harris, Reny and Robson (1995) show existence of a type of correlated equilibrium using the strong norm-continuity assumption described above (see also Duffie et.al., 1988). Recently, Nowak (2003) established the existence of Markov-stationary equilibrium for a class of games characterized by a transition law formed as the linear combination of finitely many fixed measures on the state space.

It is fair to say that the important results derived in this literature have not been directly invoked in the economics literature making use of the dynamic games paradigm. As noted earlier, the main reasons for this is that the results discussed in this subsection deal with mixed-strategy equilibrium, which in addition is generally not Markov-stationary.

14.5.2 Existence of Pure-Strategy Markov Equilibrium

The difficulties encountered in establishing existence of pure-strategy equilibrium are markedly different from those associated with behavioral-strategy

[15] There is also an extensive literature dealing with the existence of Nash equilibrium in behavioral strategies for stochastic games with finitely many states and actions and undiscounted payoffs. This literature is not covered in the present survey, and the interested reader is referred to Neyman and Sorin (2003).

[16] Continuity in the variation norm is a strong assumption that rules out many potential economic applications of interest.

equilibrium. As pure strategies are broadly viewed as more appropriate in most settings in economics, we cover the associated literature in more detail, based on the work of Curtat (1996), which we now summarize.

With p denoting the transition probability from $S \times A$ to S, let F be its associated cumulative distribution function. The following assumptions are in effect throughout this section (see Appendix for definitions of new concepts).

(A1) The distribution function $F(\cdot/s, a)$ and the reward functions $r_i(s, a)$ are all twice continuously differentiable in (s, a), for all $i = 1, ..., n$.

(A2) F and r_i are supermodular in a_i and have increasing differences in $(a_i; a_{-i}, s)$.

(A3) F satisfies a dominant diagonal condition in $(a_i; a_{-i})$, and r_i satisfies a strong dominant diagonal condition in $(a_i; a_{-i})$, for all i.

(A4) F is increasing in (s, a) in the sense of first-order stochastic dominance, and r_i is increasing in (s, a_{-i}), for all i.

Amir (2002) provides a detailed analysis of the scope and limitations of these assumptions. As in Amir (1996a), they rule out the case of deterministic transitions and, though atoms are allowed, their location is severely restricted. Let $C(S, R)$ be the Banach space of continuous functions from S to R with the sup norm, to be denoted $\|\cdot\|$. By Assumption **(A1)** and the compactness of S and A_i, there exists $K > 0$ such that $r_i(s, a) \leq K$, $\forall\ i, s, a$. Hence, all feasible payoffs in this game are also $\leq K$. Denote by $CM_K(S, R)$ the subset of the ball of radius K in $C(S, R)$ consisting of nondecreasing functions. The main results in this section are in

Theorem 14.5.1. *Under Assumptions (A1)-(A4), we have:*
(a) The infinite-horizon discounted stochastic game has a pure-strategy Markov-stationary equilibrium, with strategies and corresponding value functions that are nondecreasing and Lipschitz-continuous in the state vector.
(b) For any finite-horizon T, there exists a unique pure-strategy Markov equilibrium, with strategy components and corresponding value functions that are nondecreasing and Liptschitz-continuous in z. Moreover this is also the unique Markov equilibrium in behavioral and correlated strategies, and the game is dominance-solvable.

Curtat (1996) developed the above framework and established Part (a). The elaboration given in Part (b) is due to Amir (2002). Curtat also proved a comparative dynamics result: The first-period equilibrium actions in the infinite-horizon problem are higher than the equilibrium actions of the one-stage game. He then concludes with several applications to economic models.

Due to space constraints, we provide a self-contained outline of the proof of Theorem 14.5.1 but omit some lengthy details of a technical nature. The argument proceeds in several steps, via the analysis of auxiliary games defined here as follows. Let $v = (v_1, ..., v_n) \in CM_K(S, R)^n$ be an n-vector of continuation values, and consider an n-person one-shot game G_v parametrized by the state variable, where Player i has action set A_i and payoff function

$$\Pi_i(v, s, a_i, a_{-i}) \triangleq (1 - \lambda_i) r_i(s, a_i, a_{-i}) + \lambda_i \int v_i(s') dF(s'/s, a_i, a_{-i}) \quad (14.5)$$

With z fixed, let the above game be denoted by G_v^z.

Lemma 14.5.1. *For any $v = (v_1, ..., v_n) \in CM_K(S, R)^n$, the game G_v has a unique Nash equilibrium $a^v(s) = (a_1^v(s), ..., a_n^v(s))$. Furthermore, each $a_i^v(s)$ is nondecreasing, and Lipschitz-continuous in s uniformly in v.*

Proof. By Theorem 14.11.1 and Assumptions **(A.2)**, since v is nondecreasing, $\int v_i(z') dF(s'/s, a_i, a_{-i})$ is supermodular in a_i and has nondecreasing differences in (a_i, a_{-i}). From Assumption **(A.3),** it also satisfies a dominant diagonal condition in (a_i, a_{-i}). Since supermodularity, increasing differences and dominant diagonals are preserved under addition, it follows from Assumptions **(A2)-(A3)** that Π_i is supermodular in a_i and has increasing differences and dominant diagonals in $(a_i; a_{-i})$. Then, since the A_i's are compact, it follows in particular that G_v^z is a supermodular game for each z. Existence of a pure-strategy equilibrium $a^v(s) = (a_1^v(s), ..., a_n^v(s))$ is a consequence of Theorem 14.11.4. Uniqueness of the Nash equilibrium $a^v(s)$ then follows in a standard way from Π_i satisfying the dominant diagonal condition (see Rosen, 1965).

Π_i also has increasing differences in (s, a_i). Hence, by Theorem 14.11.5, each $a_i^v(s)$ is nondecreasing in z (due to uniqueness, the maximal and minimal equilibria clearly coincide.) The fact that each $a_i^v(s)$ is Lipschitz-continuous in z uniformly in v (i.e. the Liptschitz constant D can be chosen independently of v) follows from the compactness of S and A_i, Assumptions **(A1)** and **(A3)**, Theorem 14.11.3 (some omitted lengthy details can be found in Curtat, 1996 p. 188.)

Lemma 14.5.2. *Given $v = (v_1, ..., v_n) \in CM_K(S, R)^n$, the (unique) equilibrium payoff for Player i, $\Pi_i^*(v, s) \triangleq \Pi_i(v, s, a^v)$ is in $CM_K(S, R)$ and is Lipschitz continuous in z uniformly in v.*

Proof. Continuity of $\Pi_i^*(v, s)$ in z follows directly from Lemma 14.5.1 and the structure of the payoffs in (14.5). Monotonicity of $\Pi_i^*(v, s)$ in s follows from Assumption **(A4)**. To show the uniform Lipschitz continuity, consider

$$\Pi_i^*(v, s) = (1 - \lambda_i) r_i(s, a^v(z)) + \lambda_i \int v_i(s') dF(s'/s, a^v(s))$$

Hence, by Taylor's theorem, for any s_1, s_2 in S, there are constants C_1, C_2, C_3, C_4 such that

$$|\Pi_i^*(v, s_1) - \Pi_i^*(v, s_2)| \leq (1 - \lambda_i)(C_1 + D.C_2)\,\|s_1 - s_2\| +$$

$$\lambda_i(C_3 + D.C_4)\left\{\int_S |v_i(t)dt|\right\}\|s_1 - s_2\|$$

where use is made of Assumptions **(A1)**, the compactness of S and A_i, the Liptschitz continuity of $a^v(s)$ from Lemma 14.5.1, and standard facts about composition of functions, and integrals. With

$$M \triangleq (1 - \lambda_i)(C_1 + kC_2) + \lambda_i(C_3 + D.C_4)K\int_S dt \tag{14.6}$$

being independent of v, it follows that

$$\|\Pi_i^*(v, s_1) - \Pi_i^*(v, s_2)\| \leq M\,\|s_1 - s_2\|\,,$$

which concludes the proof.

Let $\Pi^*(v, s) \triangleq (\Pi_1^*(v, s), ..., \Pi_n^*(v, s))$. We now define a single-valued operator mapping continuation values to equilibrium payoffs as follows.

$$\begin{array}{cccc} T: & CM_K(S, R)^n & \rightarrow & CM_K(S, R)^n \\ & v(\cdot) & \rightarrow & \Pi^*(v, \cdot) \end{array}$$

The rest of the proof consists of showing that the operator T has a fixed-point $\overline{v} = T\overline{v}$, in which case the associated equilibrium strategies $(a_1^{\overline{v}}(s), ..., a_n^{\overline{v}}(s))$ clearly constitute a Markov-stationary equilibrium of the infinite horizon discounted stochastic game.

Lemma 14.5.3. *T is continuous in the topology of uniform convergence.*

Proof. Let $\Rightarrow$ denote uniform convergence. We have to show that if $v_i^k(\cdot) \Rightarrow v_i(\cdot)$ for all i, then $\Pi_i^*(v^k, \cdot) \Rightarrow \Pi_i^*(v, \cdot)$ for all i. With $v_i^k(\cdot) \Rightarrow v_i(\cdot)$, it follows from the well-known property of upper hemi-continuity of the equilibrium correspondence in the game G_v^s that, for each fixed s and each i, $a_i^{v^k}(s) \rightarrow a_i^v(s)$ in R. In other words, we have pointwise convergence of the functions $a_i^{v^k}(s)$ to the limit $a_i^v(s)$. Since these functions are all Liptschitz-continuous (Lemma 14.5.1), the convergence is actually uniform. The pointwise, and thus uniform convergence of $\Pi^*(v^k, \cdot)$ to $\Pi^*(v, \cdot)$ in view of Lemma 14.5.2, follows from standard results on the composition of continuous functions.

We are now ready to conclude the overall proof.

Proof of Theorem 14.5.1.

(a) In order to invoke Shauder's fixed-point theorem for T, we need to show that there exists a convex and norm-compact subset Φ of $CM_K(S,R)^n$ such that $T(\Phi) \subset \Phi$. To this end, define the following subset of $CM_K(S,R)^n$:

$$\Phi \triangleq \{v \in CM_K(S,R)^n : \|v_i(s_1) - v_i(s_2)\| \leq M \|s_1 - s_2\| \text{ for all } i, s_1, s_2\}$$

where M is as defined in (14.6). It follows from that Lemma 14.5.2 that $Tv \in \Phi$ whenever $v \in \Phi$. Since all the functions in Φ are uniformly Lipschitz-continuous, Φ is an equi-continuous set of functions, so that its compactness in the sup-norm follows from the Arzela-Ascoli theorem. Hence, by Shauder's fixed-point theorem, T has a fixed-point $\overline{v} = T\overline{v}$ in Φ. Then, from standard results in discounted dynamic programming, the associated equilibrium strategies $(a_1^{\overline{v}}(s), ..., a_n^{\overline{v}}(s))$ in the game $G_{\overline{v}}$ clearly constitute a Markov-stationary equilibrium.

(b) Uniqueness of a pure-strategy Markov equilibrium for every finite horizon T follows simply by iterating $v_n = T(v_{n-1})$ starting from $v_0 \equiv 0$, for $n = 1, 2, ..., T$, and invoking Lemma 14.5.1 at every iteration. The rest then follows directly from Theorem 14.11.5, applied to the games G_v^s for each s. □

Although this set-up cannot be formally viewed as encompassing the framework of Amir (1996a), it has essentially the same mathematical structure – characterized the conjunction of strategic complementarity and diagonal dominance – and can thus be analysed along a very similar line of reasoning[17].

The proof makes it clear that at each iteration of the finite-horizon algorithm, the right hand side of the Bellman equation reflects the payoffs to a supermodular game parametrized by the state. Yet, the infinite-horizon payoffs are not supermodular in any way in the stationary strategies of the players. Similar remarks apply to the model analyzed by Amir (1996a). For more on this point, see Echenique (2001b).

Curtat (1996) also illustrates the applicability of this set-up with strategic dynamic models of search with learning, price competition with durable goods, and quantity competition with learning by doing.

14.6 Dynamic Games in Industrial Organization

In addition to the previously mentioned studies of dynamic oligopolistic competition using dynamic games, this section describes other papers in industrial organization[18], including in particular the extensive literature on industry dynamics and some structural empirical literature with explicit dynamics.

[17] Note also that the scope for strategic complementarities may be much broader here than known so far: See Echenique (2001b) for more on this point.

[18] There is a very large body of literature in industrial economics dealing with two-stage games where firms typically make simultaneous long-term decisions in the firms stage (such as R&D level, capacity, entry, or advertizing, etc...), and, upon observing the outcome of the first stage, the firms make short-term decisions in the product market (price or output levels) in the second stage. While such games can

14.6.1 Dynamic Competition with a Fixed Number of Firms

Among the models with truly dynamic strategic competition, there is one class characterized by price competition and some form of inertia on the part of consumers. Rosenthal (1982) pioneered this literature with Bertrand duopoly competition under complete consumer loyalty, with one firm's market share as the natural state variable. He characterized a Markov-stationary equilibrium where prices remained above marginal costs indefinitely. By contrast, under less-than-complete consumer loyalty, Rosenthal (1986) produces an ϵ-equilibrium in Markov-stationary strategies where prices converge with probability one to marginal costs. A distinctive feature of these papers, as well as of the follow-up piece by Chen and Rosenthal (1996), is that they considered *mixed-strategy* equilibrium, and actually exhibited one in closed-form.

A closely related strand of literature deals with long-run price competition when consumers face costs for switching between different sellers: see Farrell and Shapiro (1988), Beggs and Klemperer (1992), and Padilla (1995). The latter also considers mixed-strategy equilibrium.

Inter-firm racing models, which may be viewed to some extent as discrete-time extensions or analogs of the patent race models discussed in Section 14.2.2, have been investigated by Harris and Vickers (1985, 1987) and Athey and Schmutzler (2001), among others.

Learning-by-doing in Arrow's sense, whereby firms' production costs fall with production experience, also naturally gives rise to interesting phenomena of a dynamic character. Cabral and Riordan (1994) characterize the long-term consequences of this feature in a Markov-stationary framework with firms' cumulative sales as the natural state variables.

Finally, dynamic games of capacity expansion between ex ante identical firms, aimed at explaining the emergence and persistence of inter-firm heterogeity, have received renewed attention by Doraszelski and Besanko (2004).

14.6.2 Dynamic Competition with Entry and Exit

One of the most prominent strands of literature using dynamic games is the theory of industry dynamics. This class of models is distinguished at the outset by the rather peculiar feature, from a game theory standpoint, that the number of players is endogenously determined. The main features that are common to all the models in this class may be inclusively summarized as follows. Time is discrete and firms are ex ante identical. Each firm's objective is to maximize its discounted profits over an infinite horizon. At any time period, there is a set of incumbent firms in the industry and a set of potential entrants waiting to enter. Each incumbent must decide whether to remain active in the industry, in which

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generally be translated into the framework of (finite-horizon) stochastic games, we do not cover here the numerous examples available (see e.g. Amir, Evstigneev and Wooders, 2003 for one such example and some related discussion).

case it may decide on an investment level (such as process R&D, quality or capacity), or to exit upon receiving its fixed scrap value. Each potential entrant must decide at each period whether to pay a fixed set-up cost and enter the industry. At each period, all active firms in the industry perceive a one-period reduced form profit that may correspond to a specified form of competition, such as Cournot, Bertrand, or perfect competition. The state transition law is typically quite complex in that it contains both idiosyncratic and industry-wide elements. Specifically, each incumbent has its own firm-specific state reflecting the firm's technological characteristic, which evolves as a function of the firm's investment and some idiosyncratic shock. The industry-wide state includes an aggregation of the firms' idiosyncratic states and a list of which firms are current incumbents, and evolves in a way governed by aggregate shocks, such as demand-side fluctuations.

One can classify this literature into two main categories. The first of these deals with perfectly competitive (nonstrategic) models, with price-taking firms typically represented by the unit interval. Jovanovic (1982), Hopenhayn (1992), Lambson (1992), among others, are well-known contributions along these lines. For more on this, see Section 14.8 dealing with games with a continuum of players.

The second category considers long run strategic interaction between a variable but finite number of firms. This strand was pioneered by Ericson and Pakes (1995) whose model incorporates investment decisions and endogenous state variables exactly as described above. As their existence proof for Markov perfect equilibrium was found to be incomplete, Dorazselski and Satterthwaite (2003) reconsider a variant of their model wherein each firm's entry cost and scrap value are private information and independant draws from the same pair of distributions. This modification is essentially sufficient to restore existence of a Markov perfect equilibrium. Gowrisankaran (1999) uses another variant of the Ericson-Pakes model allowing for endogenous patterns of mergers in addition to entry and exit to provide a dynamic perspective on the incentives for, and the market performance of, horizontal mergers.

Another strategic model of industry dynamics that fits the above general description but has no idiosyncratic shocks or firm investment, hence without idiosyncratic states, is studied in Amir and Lambson (2003). With only aggregate shocks and ex post identical firms, this paper provides a constructive argument for the existence of a simple (s, S)-type Markovian equilibrium of entry and exit. In addition, it establishes a tendency for excessive entry and insufficient exit, and shows via counterexamples that some key implications of the competitive version of the model (in Lambson, 1992) do not carry over to the strategic model with an explicit integer constraint on the number of firms.

14.6.3 Empirical and Computational Work on Industry Dynamics

In recent years, the structural estimation of dynamic economic models has been a very active field of research, coupled with the development of computational algorithms for solving for approximate solutions (see e.g. Rust, 1994). Given that this survey is primarily concerned with theoretical work, and that empirical research in this field is currently in flux, no attempt will be made here to be exhaustive.

In industrial organization, the model by Ericson and Pakes (1995) has provided the initial basic framework for much of the empirical literature on industry dynamics. On the computational side, Pakes and Ericson (1998) and Pakes and McGuire (1994, 2001) develop numerical procedures for solving such game-theoretical models, along the lines of policy improvement routines. Doraszelski and Satterthwaite (2003) adapt the approximation scheme devised by Whitt (1980) to compute Markov-perfect equilibria in their modification of the Ericson-Pakes model.

There is also some literature dealing with structural estimation of explicitly dynamic models with a fixed number of firms (i.e. no entry and exit), with or without fully-fledged strategic interaction. For instance, Slade (1998) estimates firms' price adjustment costs in dynamic monopolistic competition while Slade (1999) consider similar issues in a strategic dynamic game.

14.7 Dynamic Games of Perfect Information

In some subfields of economics, another class of dynamic games that has been used with some frequency is characterized by perfect information: Players move sequentially, with each player knowing the history of play including the previous move. Perfect information results in many simplifying features, an important one being the possibility of much more general existence results.

This section summarizes three separate though closely related strands of economic literature dealing with dynamic games of perfect information. The first is a well-known problem in the framework of overlapping generations.

14.7.1 Games of Strategic Bequests

Introduced by Phelps and Pollack (1968), the main version of the model posits an infinite sequence of identical generations in a one-good economy, each of whom decides on a consumption level c out of the capital stock x inherited from the previous generation, with the residual $x - c$ forming the bequest to the next generation. With stochastic production, the next stock is determined according to the c.d.f. $F(\cdot/x - c)$, and the payoff to a generation is then

$$\int U[c, h(t)]dF(t/x - c) \ , \ c \in [0, x] \ ,$$

where U is the (common) utility function, and h is next generation's consumption strategy. Here, the Markov assumption takes the form that each generation is interested only in the welfare of their immediate offspring (in addition to their own). The next result deals with the existence of a stationary equilibrium (the notation and assumptions are from Section 14.4):

Proposition 7.1. *Let $U(c_1, c_2)$ be strictly increasing and supermodular in (c_1, c_2) and stricly concave in c_1. Then a Markov-stationary equilibrium exists, with strategies and value functions respectively in*
(a) $\widetilde{\Sigma}$ and $\widetilde{\Lambda}$ *if F satisfies Assumption (A3),*
and in
(b) Σ and Λ *if F satisfies Assumptions (A3)-(A4).*

Leininger (1986) and Bernheim and Ray (1983) independently proved Part (a) in the deterministic production case, while Amir (1996b) proved Part (b). The argument for going from (a) to (b) is similar to that of Section 14.4.

Lane and Leininger (1984) and Bernheim and Ray (1987) study the properties of Markov equilibria. In addition, Amir (1996b) shows that if U and F are twice continuously differentiable, the equilibrium consumption strategy will be continuously differentiable.

Despite some apparent similarities between these dynamic games and those analyzed in Section 14.4.2, such as the fact that they share the same space of consumption strategies, it is important to observe that the presence of perfect information is a key distinctive feature, which in particular makes the existence question fundamentally different and much easier for the class of bequest games. Indeed, for the finite-horizon version of the problem, the existence of a Markov equilibrium is essentially trivial, as it simply follows from a backward induction argument. Furthermore, the limit of such an equilibrium is shown to be a Markov (but not necessarily Markov-stationary) equilibrium of the infinite-horizon game (Leininger, 1987). In other words, the difficulty with existence for the infinite-horizon problem lies only in the stationarity restriction on the Markov strategies. By contrast, for the dynamic games of Section 14.4, the simultaneous-move nature of the game leads to similar difficulties being faced whether one deals with the finite or the infinite horizon version of the game.

14.7.2 A Class of Games with Alternating Moves

An early application to duopoly is by Cyert and DeGroot (1970), who model long-term competition with firms moving alternately, each being committed to its choice in the off-period. This work has inspired the following well-known alternating-move dynamic game from Maskin and Tirole (1988a-b). Firm 1 (firm 2) chooses an action in odd-numbered (even-numbered) periods, each firm remaining committed to its action for 2 periods (so for all k, $a^1_{2k+2} = a^1_{2k+1}$ for firm 1, and $a^2_{2k+1} = a^2_{2k}$ for firm 2). Firm i's payoff is

$$\sum_{t=0}^{\infty}(1-\lambda)\lambda^t \Pi^i(a^1_t, a^2_t), a^1_t, a^2_t \in A,$$

where Π^i is a reduced-form for a per-period (static equilibrium) payoff in price, quantity or other type of competition. A pair of "reaction" functions (R^1, R^2)

forms a Markov-stationary equilibrium if $a^2_{2k} = R^2(a^1_{2k-1})$ maximizes firm 2's payoff at any time $2k$ given a^1_{2k-1} and assuming that, henceforth, firm i will follow $R^i, i = 1, 2$, with an anologous condition for firm 2. Thus, an equilibrium can be described by a triplet (R^i, V^i, W^i) for firm i, such that (say) for firm 1:

$$\begin{aligned} V^1(a^2) &= \max\left\{(1-\lambda)\Pi^1(a^1, a^2) + \lambda W^1(a^1) : a^1 \in A\right\} \\ &= (1-\lambda)\Pi^1(R^1(a^2), a^2) + \lambda W^1(R^1(a^2)) \end{aligned}$$

and

$$W^1(a^1) = \Pi^1(a^1, R^2(a^1)) + \lambda V^1(R^2(a^1)).$$

Maskin and Tirole (1988a-b) prove that in a Markov equilibrium, each of the R^is is nonincreasing (nondecreasing) if the Π^i's have decreasing (increasing) differences. (This can be obtained as an application of Topkis's monotonicity theorem to the above functional equations.) Then they use this framework to provide a new look at various well-known key issues in quantity and price competition, including natural monopoly, kinked demand curve, and strategic excess capacity. They conclude that this new framework is more suitable than the traditional approaches for the analysis of some of these issues.

Asychronous repeated games may also be viewed as part of this class of games: See e.g. Lagunoff and Matsui (1997) and references therein.

14.7.3 General Existence Results

A general framework for dynamic games with perfect information, with uncountable action sets, has been developed and existence of pure-strategy subgame-perfect equilibrium proved in Harris (1986), Hellwig and Leininger (1987), generalizing the classical result erroneously attributed to Zermelo (see Schwalbe and Walker, 2001).

It is important to stress that the existence question, with regard to pure-strategy subgame-perfect equilibrium, is fundamentally different and much simpler under perfect information than under simultaneous moves. Indeed, in the former case, the problem essentially amounts to satisfying Weirstrass's Theorem on the existence of a maximum, while in the latter case, the issue typically boils down to satisfying the conditions of a fixed-point theorem.

14.8 Dynamic Games with a Continuum of Players

Our presentation here follows Bergin and Bernhardt (1992, 1995). With a continuum of players, each player is identified by a characteristic, $\alpha \in \Lambda$, with α evolving stochastically over time. In addition, there is aggregate uncertainty, modeled as a Markov sequence of shocks over time $\{\theta_t\}_{t=1}^{\infty}, \theta_t \in \Theta$: at each period in time, an aggregate shock θ_t is realized in the per period state space Θ. The full process is modeled as a joint distribution, ν , on the sequences of aggregate shocks, Θ^∞. At time t, given a history of states $\theta^t = (\theta_1, ..., \theta_t) \in \Theta^t$, the conditional distribution on Θ^∞ given θ^t is $\nu(\cdot/\theta^t)$.

Each player $\alpha \in \Lambda$ chooses an action, a, from a common action space A. In the stage game, a distributional strategy for the population is a joint distribution over players and actions — $\tau \in M(\Lambda \times A)$, the set of probability measures on $\Lambda \times A$. Preferences of a player at time t are given by a (uniformly bounded) function $r(\alpha, a, \tau_t, \theta_t)$. The characteristic of a player α evolves stochastically over time according to a transition kernel $P(d\alpha/\alpha, a, \tau_t, \theta_t)$. For the distribution on characteristics, a current distribution τ_t implies that the next distribution on characteristics is given by $\mu_{t+1}(X) = \int P(X/\alpha, a, \tau_t, \theta_t) d\tau_t$, so that τ_{t+1} must have marginal distribution μ_{t+1}. Players seek to maximize the present discounted value of payoffs.

Under continuity assumptions on the payoff functions and transition kernels (the latter in the weak* topology), a Markov equilibrium is shown to exist with the state variable being the triplet $(\mu,\ \theta,\ v)$ where $v : \Lambda \to R$ is a continuous function[19]. In equilibrium, at any state $(\mu,\ \theta,\ v)$, the value of the distributional strategy depends only on these variables, where $v(\alpha)$ gives the expected payoff of α in the remainder of the game, and in equilibrium this is the actual payoff. (The role of v is similar to the role of sunspots as an alternative coordinating device in the state space, used to achieve existence of Markov equilibrium.)

The above discussion is in the context of an environment where the aggregate distribution evolves deterministically, conditional on the value of θ_t. The second result uses a reformulation of aggregate uncertainty and focuses on the case where the aggregate distribution evolves stochastically, but where aggregate uncertainty is not explicitly separated.

This class of games constitutes a natural fully-fledged game-theoretical framework for analysing dynamic perfect competition. There are two strands of economic literature that consider models related in one form or another to this framework. The first deals with dynamic market games: see Karatzas, Shubik and Sudderth (1994, 1998) and Shubik and Whitt (1971). The second deals with perfectly competitive industry dynamics (relying on a price-taking assumption in partial equilibrium), with entry and exit over time: See Section 14.6.2.

14.9 Computational Methods

As the theory of dynamic games relies heavily on the theory of dynamic programming, it is natural to expect the well-known computational algorithms developed for the latter to have counterparts for the former. Indeed, there are many different strands of literature dealing with various computational aspects of dynamic games. Numerical procedures were typically derived for dynamic

[19] When the data of the game is history-dependent, existence of a (nonMarkov) equilibrium is also shown.

games with finite state and actions spaces, or else proceed via discretization of those spaces, as performed e.g. in Whitt (1980).

There are many other studies dealing with the computation of equilibrium in dynamic games in operations research: See e.g. von Stengel, van den Elzen and Talman (2002) and some of the references in Raghavan et. al. (1991).

There is an early literature in behavioral strategies in operations research aimed at the computation of Markov-stationary equilibrium in stochastic games: See Raghavan et. al. (1991) and references therein. A more recent contribution is von Stengel, van den Elzen and Talman (2002). Another computational approach for stationary behavioral-strategy equilibrium, based on a homotopy approach, is developed by Herings and Peeters (2004). Related work by Haller and Lagunoff (2000) proves that, generically (in a measure-theoretic sense), discounted stochastic games with finite state and action spaces possess a finite set of Markov-perfect equilibria in behavioral strategies.

There is also a more field-specific literature on computational methods. Studies dealing with industry dynamics are listed in Section 14.6 while those related to games of capital accumulation are listed in Section 14.4.

14.10 Experimental Research on Dynamic Games

Last but not least, some experimental studies based on stochastic games have been conducted in recent times. There are also experimental studies testing the ability of laboratory subjects to play in dynamic games. In a simple oligopoly market game, Keser (1994) finds little support for theoretical predictions based on a unique finite horizon Markov equilibrium. On the other hand, Herr, Gardner and Walker (1997) find the theoretical solutions quite well confirmed by laboratory behavior in a common-property resource game. Keser and Gardner (1999) report a good fit for subgame perfect equilibrium predictions at the aggregate, but not at the individual, level.

Another strand of experimental/empirical work uses field data for the purpose of testing theories. Using data from Wimbledon tennis matches, Walker and Wooders (2001) test the minmax hypothesis in a zero-sum binary Markov game model of tennis serves. Justifying an equilibrium over a stream of serves as the repetition of single-serve equilibria, they report better support for the minmax hypothesis than earlier work based on laboratory data. In this context, this means that the server and the receiver in professional tennis matches randomize their binary choices of going left or going right according to to the indifference principle governing mixed-strategy equilibria. Nonetheless, they also find excessive serial correlation relative to the theoretical predictions. This work has started a trend of research testing game theory in sports.

14.11 Appendix

A brief summary of the lattice-theoretic notions and results, used in Sections 14.4 and 14.5, is presented here. Throughout, S will denote a partially ordered set and A a lattice, and all cartesian products are endowed with the product order.

A function F: $A \to R$ is (strictly) supermodular if

$$F(a \vee a') + F(a \wedge a') \geq (>) F(a) + F(a'), \forall a, a' \in A.$$

If $A \subset R^m$ and F is twice continuously differentiable, F is supermodular if and only if $\frac{\partial^2 F}{\partial a_i \partial a_j} \geq 0, \forall i \neq j$. A function $G : A \times S \to R$ has (strictly) increasing differences in s and a if for $a_1 (>) \geq a_2$, $G(a_1, s) - G(a_2, s)$ is (strictly) increasing in s . If $A \subset R^m$, $S \subset R^n$ and G is smooth, this is equivalent to $\frac{\partial^2 G}{\partial a_i \partial s_j} \geq 0$, for all $i = 1, ..., m$ and $j = 1, ..., n$.

A set I in R^n is increasing if $x \in I$ and $x \leq y \Rightarrow y \in I$. With $S \subset R^n$ and $A \subset R^m$, a transition probability F from $S \times A$ to S is supermodular in a (has increasing differences in s and a) if for every increasing set $I \subset R^n$, $\int 1_I(t) dF(t/s, a)$ is supermodular in a (has increasing differences in s and a) where 1_I is the indicator function of I. A characterization of these properties, using first-order stochastic dominance, follows (see Athey, 2001-2002 for extensive related work):

Theorem 14.11.1. *(Topkis, 1968, 1998) A transition probability F from $S \times A$ to $S \subset R^n$ is supermodular in s (has increasing differences in s and a) if and only if for every integrable increasing function $v : S \to R$, $\int v(t) dF(t/s, a)$ is supermodular in s (has increasing differences in s and a).*

Let $L(A)$ denote the set of all sublattices of A. A set-valued function $H : S \to L(A)$ is ascending if for all $s \leq s'$ in S, $a \in A_s, a' \in A_{s'}$, $a \vee a' \in A_{s'}$ and $a \wedge a' \in A_s$. Topkis's main result follows (also see Milgrom and Shannon, 1994):

Theorem 14.11.2. *(Topkis, 1978) Let $F : S \times A \to R$ be upper semi-continuous and supermodular in a for fixed s, and have (strictly) increasing differences in s and a, and $H : S \to L(A)$ be ascending. Then the maximal and minimal (all the) selections of* $\arg\max \{F(s, a) : a \in H(s)\}$ *are increasing functions of s.*

With $S \subset R^n$ and $A \subset R^m$, a function F: $A \to R$ satisfies (strong) diagonal dominance if $\sum_{j=1}^{m} \frac{\partial^2 F}{\partial a_i \partial a_j} (<) \leq 0$ for each $i \in \{1, 2, ..., m\}$. A transition probability F from A to S satisfies strong diagonal dominance in a if $\int 1_I(t) dF(t/a)$ has the same property, for every increasing set $I \subset R^n$, or equivalently, if for every increasing function $v : S \to R$, $\int v(t) dG(t/a)$ satisfies that same property.

Theorem 14.11.3. *(Curtat, 1996) Assume that $F : S \times A \to R$ is upper semi-continuous and supermodular in a for fixed s, has increasing differences in s and a, and satisfies SDD in a. Then* $\arg\max\{F(s,a) : a \in A\}$ *is an increasing and Lipschitz-continuous (single-valued) function of s.*

A game with action sets that are compact Euclidean lattices and payoff functions that are u.s.c. and supermodular in own action, and have increasing differences in (own action, rivals' actions) is a supermodular game. By Theorem 14.11.2, such games have minimal and maximal best-responses that are monotone functions, so that a pure-strategy equilibrium exists by (see also Vives, 1990):

Theorem 14.11.4. *(Tarski, 1955) An increasing function from a complete lattice to itself has a set of fixed points that is itself a nonempty complete lattice.*

The last result deals with comparing equilibria.

Theorem 14.11.5. *(Milgrom and Roberts, 1990)*
Consider a parametrized supermodular game where each payoff has increasing differences in the parameter (assumed real) and own action. Then the maximal and minimal equilibria are increasing functions of the parameter.

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